University of Copenhagen – Individual Study Project

Bootstrapping $d = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory in the presence of a $\frac{1}{2}$-BPS boundary defect

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Abstract: We present a concise introduction to the bootstrap program for superconformal field theories with defects, using $\frac{1}{2}$-BPS operators in $d = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory with a $\frac{1}{2}$-BPS boundary defect as a central example and following the work of Liendo and Meneghelli [1]. Starting with an analysis of the constraints imposed by conformal symmetry on the correlation functions of primary operators, we then set up the conformal bootstrap by requiring OPE associativity. Building on this foundation, we extend the discussion to superconformal field theories in the presence of extended operators, introducing the superspace formalism to describe such setups. We then derive the superblock expansion for two-point functions of $\frac{1}{2}$-BPS operators, along with the corresponding superconformal Ward identities. Additional details regarding analytical and numerical techniques employed throughout this project are provided in the appendices.

These notes are aimed at graduate students seeking an introduction to the bootstrap approach for superconformal field theories with defects. Only a foundational understanding of quantum field theory is assumed, while relevant background material on conformal field theory is reported in the appendix. The report arises from an Individual Study Project conducted at the Niels Bohr Institute during the Fall/Winter semester of 2024/25, under the supervision of Dr. Adam Chalabi and Prof. Charlotte Kristjansen.

Submitted on 10 January 2025.

Contents

1 Introduction

A fundamental challenge in theoretical physics is the delineation of the space of consistent quantum field theories (QFTs). *Conformal field theory* (CFT) [2–4] offers a powerful framework in which to attempt this task. CFTs are characterized by the absence of a dimensionful scale and describe the critical behavior near fixed points of renormalization group (RG) flow [5], providing insights into the evolution of QFTs across energy scales. For a pedagogical and comprehensive treatment of CFT, see [6]. In this work, we exploit the properties of CFTs, which can be defined axiomatically through their core observables, correlation functions, without requiring a Lagrangian description. These correlation functions satisfy consistency conditions such as conformal symmetry, unitarity, and crossing symmetry. The program that leverages these conditions to determine fixed-point data, such as critical exponents and operator-product expansion (OPE) coefficients, is known as the *conformal bootstrap* [7–11]. See [12] and references therein for a review of recent progress.

Beyond their applications in condensed matter and particle physics, supersymmetric extensions of CFTs, called *superconformal field theories* (SCFTs), are also of interest for their formal connections to the AdS/CFT correspondence [13]. This duality relates SCFTs to gravity theories in AdS spacetimes, with $d = 4$, $\mathcal{N} = 4$, $SU(N)$ Super Yang-Mills (SYM) arising as the worldvolume theory describing a stack of N D3-branes in type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. Notably, $\mathcal{N} = 4$ SYM is superconformal for any value of the coupling g [14]. Its field content is reviewed e.g. in [15]. In this work, we focus on $\frac{1}{2}$-BPS operators in $\mathcal{N} = 4$ SYM, defined as symmetric-traceless combinations of its six real scalar fields ϕ_I:

$$\mathcal{W}_p(x,u) \equiv \mathrm{Tr}_{SU(N)}\left[(u\cdot\phi(x))^p\right] = \left[(u^I\phi_I(x))^p\right]^a{}_a \in B_1\bar{B}_1[0;0]^{[0,2p,0]}_{2p}, \quad p \geq 1, \tag{1.1}$$

where the complex null vector $u \in \mathbb{C}^6$, $u^2 = 0$ is used to contract R-symmetry indices. These operators belong to short multiplets annihilated by half the supercharges and the notation will be made explicit in the main text. Their scaling dimensions are protected from quantum corrections, ensuring that two- and three-point functions remain uncorrected, while nontrivial dynamics emerge at the four-point level [16–22], see also the reviews [23, 24].

While much attention in SCFTs has been on local operators, extended objects preserving part of the bulk supersymmetry, called *defects*, offer a richer class of observables. They provide new ways to partially or completely break symmetries in a controlled manner, probing new physics. This work focuses on $\frac{1}{2}$-BPS boundary defects, which are holographically modeled as D-branes [25–30]. Superconformal boundary conditions in $\mathcal{N} = 4$ SYM were systematically explored in [31] and an account of (super)conformal defects given in [32].

Outline We review the foundational aspects of conformal kinematics and the bootstrap program in Section 2. Section 3 extends these ideas to superconformal setups, focusing on defect theories, the superspace formulation, and the decomposition of correlation functions into superblocks. Section 4 presents the derivation of the bootstrap equations, detailing the construction of bulk and boundary superblocks, contributing supermultiplets, and the role of superconformal Ward identities. We conclude in Section 5. The appendices provide additional technical details, including elements of conformal field theory, the differential equations governing superblocks, and the analysis of contributing representations.

2 Conformal Kinematics and Bootstrap

We start by reviewing the bootstrap program of conformal field theories, following [33–36]. While we assume familiarity with conformal field theory (CFT) in the main text, the concepts and the notation that are central to this work are reviewed in Appendix A: *conformal transformations*, *radial quantization*, *conformal weight*, *conformal primaries* and *descendants*, *shortening conditions*, *state-operator map*... In this section, we turn our attention to constraints imposed by conformal invariance on observables of correlation functions of primary operators. While this can be done rigorously solving Ward identities for the conformal group, we adopt here a simpler approach: the *embedding space formalism* [2, 37–39].

Embedding Space Let $x^\mu = (x^1, ..., x^d)$ be coordinates of d-dimensional Euclidean space. The action of the Euclidean conformal algebra $\mathfrak{so}(d+1,1)$ is most elegantly realized by embedding $\mathbb{R}^d$ in $(d+2)$-dimensional Minkowski space as a null projective cone with coordinates $P^M = (P^+, P^-, P^1, ..., P^d)$ and line element

$$ds^2 = -dP^+dP^- + \delta_{\mu\nu}dP^\mu dP^\nu, \quad \mu,\nu = 1, ..., d.$$

The null projective cone corresponds to the subspace

$$\mathcal{C} \equiv \Big\{ P \in \mathbb{R}^{1,d+1} | \left(P \cdot P \equiv P^M P_M = 0 \right) \wedge \left(P^M \sim \lambda P^M \, \forall \lambda \in \mathbb{R}^+ \right) \Big\} . \tag{2.1}$$

We define the *Poincaré section* $\subset \mathcal{C}$ by $P^M = (1, x_\nu x^\nu, x^\mu)$, which we can choose whenever $P^+ \neq 0$ (this is just the point at infinity). The action of a $(d+2)$-dimensional Lorentz transformation $\Lambda \in \mathrm{SO}(d+1,1)$ is linear and takes $P^M \mapsto (\Lambda P)^M$. By construction,

$$P^M = (1, x^2, x^\mu) \mapsto \frac{(\Lambda P)^M}{(\Lambda P)^+} = (1, x'^2, x'^\mu)$$

is a non-linear map $\mathbb{R}^d \to \mathbb{R}^d$ sending $x^\mu \mapsto x'^\mu$, which can be shown to amount to a conformal transformation in d dimensions [33, 40]. As a result, we are free to realize conformal transformations in $\mathbb{R}^d$ by simpler linear transformations in $\mathbb{R}^{1,d+1}$ instead. The scaling property

$$\mathcal{O}_\Delta(x) \mapsto \Omega(x)^{-\Delta} (R(x) \cdot \mathcal{O}_\Delta)(x)$$

of primary operators $\mathcal{O}_\Delta$ under conformal transformations sending $g_{\mu\nu}(x) \mapsto \Omega(x)^2 g_{\mu\nu}(x)$ is translated to the requirement that the corresponding field O_Δ in embedding space be homogeneous of degree $-\Delta$ in P^A [33],

$$O_\Delta(\lambda P) = \lambda^{-\Delta} O_\Delta(P), \quad \mathcal{O}_\Delta(x^\mu) \equiv O_\Delta(1, x^2, x^\mu). \tag{2.2}$$

The embedding space will be helpful in deriving simpler differential expressions for Casimir operators later on. We may readily derive constraints on low-point correlators:

- translation invariance forces the 1-point function to be a constant. For (2.2) to hold,

$$\langle O_\Delta(P) \rangle = \begin{cases} 1, & O \equiv \mathbb{1} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

  All 1-point correlators thus vanish in a CFT unless the theory is defined on a space with boundaries or other specific features. Moving on,

- the 2-point correlation function of scalar primaries of weight Δ_1, Δ_2 must write

$$\langle O_{\Delta_1}(P_1) O_{\Delta_2}(P_2)\rangle = \frac{c_{12}}{(P_1 \cdot P_2)^{\Delta_1}} \delta_{\Delta_1 \Delta_2}, \quad c_{12} \text{ a normalization constant,}$$

which is the most general Lorentz-invariant expression consistent with (2.2).[1] Noting that $-2P_1 \cdot P_2 = (x_1 - x_2)^2$ the two-point function is projected onto physical space to

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2)\rangle \propto \frac{c_{12}}{(x_1 - x_2)^{2\Delta_1}} \delta_{\Delta_1 \Delta_2}. \tag{2.4}$$

- Applying the same logic, 3-point correlation functions of scalar primaries write

$$\langle O_{\Delta_1} O_{\Delta_2} O_{\Delta_3}\rangle = \frac{\lambda_{123}}{(-2P_1 \cdot P_2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} (-2P_1 \cdot P_3)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} (-2P_2 \cdot P_3)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \tag{2.5}$$

which is the most general expression for a $SO(d+1,1)$ singlet homogeneous of degrees $-\Delta_1$, $-\Delta_2$ and $-\Delta_3$ in P_1, P_2 and P_3 respectively. The λ's are *structure constants.*

- Recall that the conformal group leaves correlation functions invariant under global transformations. This implies that the functional dependence of the latter must reduce to combinations of invariants formed from the spacetime coordinates. With four operator insertions, the two ratios

$$u \equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^2 \quad \text{and} \quad v \equiv \left(\frac{x_{14}x_{23}}{x_{13}x_{24}}\right)^2, \quad x_{ij} \equiv x_i - x_j \tag{2.6}$$

turn out to be conformally invariant as well [40]. This means that the most general form of the 4-point correlation function of scalar operators is only constrained up to an arbitrary function $F(u,v)$ multiplying the kinematic factor,

$$\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) O_{\Delta_3}(x_3) O_{\Delta_4}(x_4)\rangle = F(u,v) \prod_{i<j}^{4} x_{ij}^{\frac{\Delta}{3} - \Delta_i - \Delta_j}, \quad \Delta \equiv \sum_i^4 \Delta_i. \tag{2.7}$$

CFTs enjoy one further property allowing to simplify the calculation of correlation functions: the *operator product expansion* (OPE).

OPE The OPE can be straightforwardly derived from considering the insertion of two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$ inside of a sphere in radial quantization. Let $|\Psi\rangle$ be the state generated at the surface of the sphere, $|\Psi\rangle = \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle$. As a state in a CFT, $|\Psi\rangle$ must admit a decomposition along a basis of conformal primaries and descendants, so

$$|\Psi\rangle = \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_\Delta C_{12}^\Delta(x-y, P)|\Delta\rangle = \sum_\Delta C_{12}^\Delta(x-y, \partial_y)\mathcal{O}_\Delta(y)|_{y=0}|0\rangle$$

and we effectively read off[2]

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_\Delta C_{12}^\Delta(x-y, \partial_y)\mathcal{O}_\Delta(y).$$

[1] The 2-point function of scalar primary operators is usually normalized to 1, $c_{12} \equiv 1$.
[2] OPE statements are always implicitly assumed to hold within radially-ordered correlation functions.

At the level of representations, the OPE amounts to the decomposition of the tensor product of conformal representations. Enforcing conformal invariance and expanding $C^{\Delta}_{12}(x-y,\partial_y) = C^{\Delta}_{12} + ...$ into a series in ∂_y, one finds that the only free parameters that remain unfixed are the overall constant coefficients C^{Δ}_{12} for the primary states [33],

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_{\Delta} C^{\Delta}_{12} \frac{\mathcal{O}_{\Delta}(y)}{|x-y|^{\Delta_1+\Delta_2-\Delta}} + \underbrace{...}_{\text{descendants}} . \tag{2.8}$$

We now return to the 3-point function of scalar primaries. Using (2.8) on $\mathcal{O}_{\Delta_1}$ and $\mathcal{O}_{\Delta_2}$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \sum_{\Delta_k} \frac{C^{\Delta_k}_{12}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_k}} \langle \mathcal{O}_{\Delta_k}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle + ...$$

Inserting (2.4), expanding the result to leading order in x_{12} and inspecting (2.5) we find that $\lambda_{123} \propto C^{\Delta_3}_{12}$, where the proportionality factor depends on c_{12} only. Thus knowing

$$\{\text{spectrum } (\Delta, \text{ spin } J), \text{ OPE coefficients } (\lambda, c)\} = \text{CFT data}$$

is enough to compute any n-point correlation function in a CFT, recursively reducing it to lower-point and finally to 2-point functions using (2.8). The OPE is a convergent expansion.[3] This is an important use feature of CFTs which allows to define a QFT without any reference to a Lagrangian, and also the primary building block of the *conformal bootstrap*.

Conformal Bootstrap The OPE is *associative*, meaning that the reduction of a n-point correlation function to a 2-point correlator may be done along different OPE *channels* but must yield the same answer, allowing to constrain the CFT data. OPE associativity, or *crossing symmetry*, is nothing but a reflection of the various ways of formally inserting a projector $\mathbb{P}_{\Delta}$ onto conformal multiplets to single out pairs of operators before taking the tensor product of their representations. For the 4-point function of identical scalar primaries of weight Δ, we write

$$\begin{aligned}
\langle \mathcal{O}_{\Delta}(x_1)\mathcal{O}_{\Delta}(x_2)\mathcal{O}_{\Delta}(x_3)\mathcal{O}_{\Delta}(x_4)\rangle &= \sum_{\Delta'} \langle \mathcal{O}_{\Delta}(x_1)\mathcal{O}_{\Delta}(x_2)\mathbb{P}_{\Delta'}\mathcal{O}_{\Delta}(x_3)\mathcal{O}_{\Delta}(x_4)\rangle \\
&\overset{(2.8)}{=} \sum_{\mathcal{O}_{\Delta'}} \lambda_{12\mathcal{O}_{\Delta'}}\lambda_{34\mathcal{O}_{\Delta'}} \underbrace{\left[C^{\mathcal{O}_{\Delta'}}_{12}(x_{12},\partial_y) C^{\mathcal{O}_{\Delta'}}_{34}(x_{34},\partial_z) \langle \mathcal{O}_{\Delta'}(y)\mathcal{O}_{\Delta'}(z)\rangle \Big|_{y,z=0} \right]}_{\text{CPW}} \\
&\equiv \sum_{\mathcal{O}_{\Delta'}} \lambda_{12\mathcal{O}_{\Delta'}}\lambda_{34\mathcal{O}_{\Delta'}} \frac{f_{\Delta'}(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} \overset{(2.7)}{=} \frac{F(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} ,
\end{aligned} \tag{2.9}$$

where we used (2.8) to expand the correlator in terms of *conformal partial waves* (CPWs) built out of *conformal blocks* $f_{\Delta'}$.[4] We carried out the CPW expansion along the channel (12)(34), but could have considered (14)(23) or (13)(24) as well; all should match at the end. The conformal bootstrap starts here at 4-point. In the next section, we explain how these concepts can be extended to *superconformal field theories* (SCFTs) containing *defects*.

[3] The OPE $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ will converge as long as x_1 is closer to x_2 than any other operator insertions.
[4] These have the same transformation properties under the conformal group as (2.4) [33].

3 Superconformal Bootstrap with Defects

3.1 Elements of Superconformal Field Theory

Supersymmetry (SUSY) allows one to circumvent the Coleman-Mandula theorem and enhance the Lorentzian conformal algebra $\mathfrak{so}(d,2)$ by adding $\mathcal{N}$ anticommuting symmetry generators Q and S to the algebra, called *supercharges* and *superconformal generators* respectively and obeying

$$\{Q,Q\} \sim P, \quad \{S,S\} \sim K \quad \text{schematically, so } \Delta_Q = \frac{1}{2} \text{ and } \Delta_S = -\frac{1}{2}. \tag{3.1}$$

Both Q and S are fundamental Lorentz spinors which can also be rotated among each other by an internal bosonic $\mathfrak{u}(\mathcal{N})_R$ R-symmetry.[5] $\mathcal{N} > 1$ is called *extended* SUSY. The superconformal algebra may thus be understood as supermatrices of the form

$$\left(\begin{array}{c|c} \text{conformal algebra} & \text{supercharges} \\ \hline \text{supercharges} & \text{R-symmetry} \end{array}\right)$$

with the *bosonic subalgebra* corresponding to the direct sum of the diagonal blocs above. Since conformal algebra $\subset$ superconformal algebra, the properties of conformal multiplets (see Appendix A) are retained in SCFTs, though the newly added structure means that all local operators in a SCFT are now also labelled by their R-symmetry ($R_{\mathcal{O}}$) besides their $\mathfrak{so}(d)$ Lorentz weight $J_{\mathcal{O}}$ and $\mathfrak{so}(2)$ scaling dimension $\Delta_{\mathcal{O}}$; we thus denote *superconformal multiplets* (supermultiplets) by $[J_{\mathcal{O}}]^{(R_{\mathcal{O}})}_{\Delta_{\mathcal{O}}}$. In this work we are mostly interested in bosonic subalgebras with $\mathcal{N} = 4$ and for spacetime dimensions [41] (Lorentzian signature)

$$d = 3: \quad \mathfrak{so}(3,2) \oplus \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \subset \mathfrak{osp}(4|4), \quad \text{representation: } [2l]^{[2r_+,2r_-]}_{\delta} \tag{3.2a}$$

$$d = 4: \quad \mathfrak{so}(4,2) \oplus \mathfrak{su}(4)_R \subset \mathfrak{psu}(2,2|4), \quad \text{representation: } [2j;2\bar{j}]^{[r_1,r_2,r_3]}_{\Delta}, \tag{3.2b}$$

with $r_\pm$ two $\mathfrak{su}(2)_\pm$ spins, l a $\mathfrak{so}(3) \subset \mathfrak{so}(3,2)$ spin, $(r_1,r_2,r_3) \in \mathbb{Z}^3_{\geq 0}$ the $\mathfrak{su}(4)_R$ Dynkin labels, $(j,\bar{j})$ two $\mathfrak{so}(4) \subset \mathfrak{so}(4,2)$ spins and δ, Δ the weight in $d = 3, 4$ respectively.

Superprimary Just as in the pure bosonic case, every irreducible superconformal multiplet admits an operator/state of lowest scaling dimension, which is a conformal primary and annihilated by all S's, called *superconformal primary*–or superprimary (SP) for short. Descendant operators are obtained by acting on the latter with all the Q's at most once, generating a finite set of conformal primaries along the way and thereby providing a finite reorganization of an infinite amount of data into *supermultiplets* (SMs). Enforcing unitarity now leads to a richer hierarchy of long and short SMs, depicted in Figure 1. The decomposition of correlation functions of SPs in superconformal primary waves follows through as in (2.9), except that the sum is now also taken over the different R-symmetry labels (R)

[5] In $d = 4$, the Lorentz algebra is $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The Q's and S's thus split in pairs of chiral and anti-chiral Weyl spinors Q^A_α, $\bar{Q}_{B\dot{\alpha}}$ and S^A_α, $\bar{S}_{B\dot{\alpha}}$, with $sl(2)$-indices $\alpha, \dot{\alpha} = 1, 2$ and with $u(\mathcal{N})_R$-indices $A, B = 1, ..., \mathcal{N}$. In $d = 3$, there is only one set of supercharges Q and superconformal generators S. For more details on the precise form of the algebra we refer the reader to [36, 41, 42].

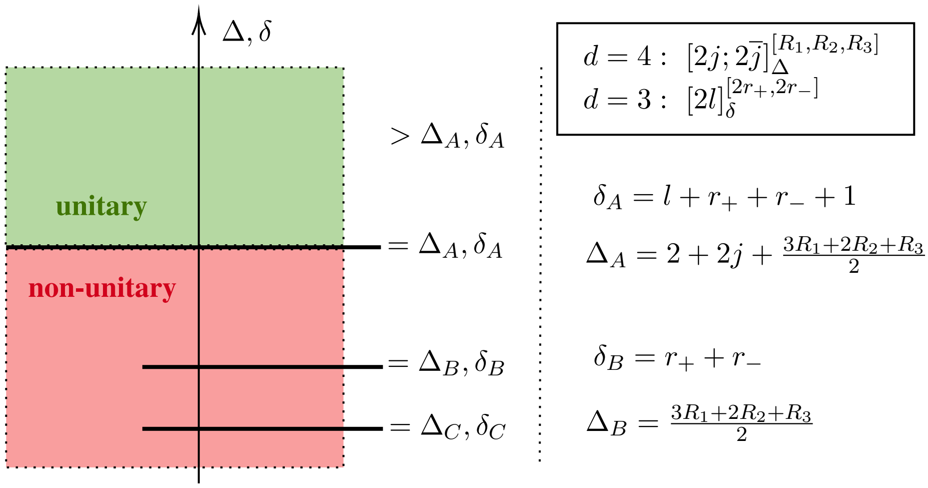

Figure 1. Unitarity structure (left) and unitary bounds (right) of supermultiplets (SMs) in $\mathcal{N} = 4$ and for $d = 3, 4$. *Long* SMs have $\Delta, \delta > \Delta_A, \delta_A$ and are always unitary; they do not have any null states. *Short* SMs saturate one of the bounds labelled A, B, C and may contain null states that should be removed from the Hilbert space. Here Δ_B, δ_B lead to $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS SMs, see [41].

entering a given supermultiplet, and that the *superconformal blocks* (superblocks) $\mathfrak{F}$ are also acted upon by the internal R-symmetry as well. Schematically,

$$\mathfrak{F} \sim \sum_{\Delta,(R)} \underbrace{\#}_{\substack{\text{coefficients} \\ \& \text{ couplings}}} \times \underbrace{\mathfrak{f}_{\Delta}(u,v)}_{\text{conformal block, cf.(2.9)}} \times \underbrace{\mathfrak{h}_{(R)}(U,V)}_{\text{R-symmetry block}} \quad (\equiv \text{superblock}) \tag{3.3}$$

for a 4-point correlation function, where we introduced R-symmetry invariant ratios

$$U \equiv \frac{u_{12}^2 u_{34}^2}{u_{13}^2 u_{24}^2}, \quad \text{and} \quad V \equiv \frac{u_{14}^2 u_{23}^2}{u_{13}^2 u_{24}^2} \tag{3.4}$$

built out of four internal R-symmetry coordinates u_i. Like (2.6), they arise naturally from the constraints imposed by the the R-symmetry part of the superconformal symmetry group on correlation functions. The existence of such invariants will be rigorously motivated from superspace considerations in Section 3.3 where the extension of the formalism to include the fermionic (supersymmetric) and R-symmetry degrees of freedom is made explicit. In the remainder of this section, we discuss the effect of defect operators on SCFTs, eventually narrowing our scope to a $\frac{1}{2}$-BPS boundary in $\mathcal{N} = 4$ Super Yang-Mills in $d = 4$.

3.2 Defect Superconformal Field Theories

So far, we defined (S)CFTs in terms of local operators and their correlation functions. There exists, however, a much richer set of observables which are *extended* in space and time that can also be consistently included in such theories, such as heavy impurities, boundaries or interfaces. These are referred to collectively as *(superconformal) defects* and break part or all of the superconformal symmetry in a controlled manner, see [40, 43–46] and references therein for an overview of the theoretical frameworks and methodologies employed in the study of conformal defects (and their supersymmetric analogues).

Boundary Defect We restrict here our focus to $d = 4$, $\mathcal{N} = 4$ SUSY with a boundary (codimension-1), which means that the relevant superconformal algebras are (3.2a) on the $3d$ boundary and (3.2b) in the bulk. Since the defect breaks translation invariance in the direction transverse to the boundary, $\mathfrak{so}(5,1) \to \mathfrak{so}(4,1)$ and it is obvious from the first equation of (3.1) that (at least) part of SUSY will also be killed. There are however specific physical configurations where one may hope to preserve some fraction of the original supercharges. In this work we consider a $\frac{1}{2}$-BPS boundary, where "$\frac{1}{2}$" means that exactly half of the Q's we started with survive. Let $(\vec{x}, x_\perp) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ be our coordinates, with the boundary inserted by means of suitable boundary conditions at $x_\perp = 0$ and $\vec{x}$ coordinates on the defect. This is implemented on the projective null cone (2.1) by introducing the constant vector $V^M = (0,0,0,0,0,1)$ and restricting ourselves to Lorentz transformations in $\mathbb{R}^{1,5}$ that leave V invariant when constructing correlation functions. With only two insertion points $P_i = (1, x_i^2, \vec{x}_i, x_{i,\perp}) \in \mathbb{R}^{1,4} \times \mathbb{R}_{\geq 0}$ $(i = 1,2)$ these are $P_1 \cdot P_2$, $V \cdot P_1$, $V \cdot P_2$, so 2-point correlators of SPs (or descendants thereof) are no longer fully fixed by conformal symmetry but written

$$\begin{aligned}\langle O_1(P_1, u_1) O_2(P_2, u_2)\rangle &\equiv \mathcal{F}_{D_1 D_2}(P_1, P_2) \times \mathcal{H}_{R_1 R_2}(u_1, u_2) \\ &= \frac{F(\xi)}{(2V \cdot P_1)^{D_1} (2V \cdot P_2)^{D_2}} \times (u_1 \cdot \bar{u}_1)^{\frac{D_1}{2}} (u_2 \cdot \bar{u}_2)^{\frac{D_2}{2}} H(\sigma, \bar{\sigma}), \end{aligned} \tag{3.5}$$

with $\mathfrak{F} \equiv F(\xi) H(\sigma, \bar{\sigma})$ mimicking (3.3) and where both $O_i \equiv \mathcal{O}_i \in [0;0]_{D_i}^{(R_i)}$ or $O_i \equiv \hat{\mathcal{O}}_i \in [0]_{D_i}^{(R_i)}$ are scalar operators in the bulk/boundary respectively. We introduced the quantity

$$\xi \equiv -\frac{P_1 \cdot P_2}{2(V \cdot P_1)(V \cdot P_2)} \overset{\text{Poincaré}}{\to} \frac{|x_1 - x_2|^2}{4 x_{1,\perp} x_{2,\perp}}, \quad x_i \equiv (\vec{x}_i, x_{i,\perp}),\ i = 1,2 \tag{3.6}$$

as the conformal/Lorentz invariant ratio and specified the internal R-symmetry coordinates u_1, u_2 of the operators as well. Since $\mathfrak{su}(4)_R \cong \mathfrak{so}(6)$ in our case, the latter can be thought of as complex null vectors $u \in \mathbb{C}^6$ $(u^2 = 0)$, cf. (1.1). $\mathcal{F}$ is constrained by conformal symmetry in a similar way as (2.4), while $\mathcal{H}$ is constrained by R-symmetry and involves an arbitrary function $H(\sigma, \bar{\sigma})$ of the two independent R-symmetry invariants

$$\sigma \equiv \frac{u_1 \cdot u_2}{\sqrt{u_1 \cdot \bar{u}_1}\sqrt{u_2 \cdot \bar{u}_2}}, \quad \text{and} \quad \bar{\sigma} \equiv \frac{u_1 \cdot \bar{u}_2}{\sqrt{u_1 \cdot \bar{u}_1}\sqrt{u_2 \cdot \bar{u}_2}}, \tag{3.7}$$

where we employ the notation $\bar{u} \equiv (u^1, u^2, u^3, -u^4, -u^5, -u^6)$. We argue for the existence of $\xi, \sigma, \bar{\sigma}$ in Section 3.3. Finally, while we explained that the fusion of two bulk or two boundary operators takes the form (3.5), when bulk excitations are brought close to boundary, they become indistinguishable and may fuse too. We must therefore also allow for mixed bulk-boundary correlators

$$\langle \mathcal{O}(\vec{x}, x_\perp) \hat{\mathcal{O}}(\vec{y})\rangle = \frac{\mu_{\mathcal{O}\hat{\mathcal{O}}}}{\left(|\vec{x} - \vec{y}|^2 + x_\perp^2\right)^\delta (2x_\perp)^{\Delta - \delta}}, \ \mathcal{O} \in [0;0]_\Delta^{[r_1, r_2, r_3]},\ \hat{\mathcal{O}} \in [0]_\delta^{[2r_+, 2r_-]}, \tag{3.8}$$

where the constant $\mu_{\mathcal{O}\hat{\mathcal{O}}}$ is called the *bulk-boundary coupling*. In particular,

$$\langle \mathcal{O}(x) \hat{\mathbb{1}}\rangle = \langle \mathcal{O}(x)\rangle = \frac{a_{\mathcal{O}}}{(2x_\perp)^\Delta}, \quad \text{for } \mathcal{O} \in [0;0]_\Delta^{[r_1, r_2, r_3]} \tag{3.9}$$

with $a_{\mathcal{O}} \equiv \mu_{\mathcal{O}\hat{1}} \neq 0$ in general. This can be viewed as a consequence of the breaking of translation symmetry along $x_\perp$ which no longer requires the 1-point functions to be constant (and vanishing) like in (2.3). The OPE (2.8) of two bulk or boundary scalar operators is unaffected. With (3.8) at hand, there is however now a new way to express bulk operators in terms of boundary operators: this is the *boundary OPE* (BOE)

$$\mathcal{O}(\vec{x}, x_\perp) = \sum_{\hat{\mathcal{O}}} \mu_{\mathcal{O}\hat{\mathcal{O}}} \frac{\hat{\mathcal{O}}(\vec{x})}{(2x_\perp)^{\Delta-\delta}} + \ldots, \quad \mathcal{O} \in [0;0]_\Delta^{[r_1,r_2,r_3]},\ \hat{\mathcal{O}} \in [0]_\delta^{[2r_+,2r_-]}, \tag{3.10}$$

whose interpretation at the level of representation theory is given by the *branching* of bulk supermultiplets into boundary ones (hence according to $\mathfrak{psu}(2,2|4) \to \mathfrak{osp}(4|4)$ here).

3.3 Defect Superspace Setup

So far, we considered SUSY being realized on *ordinary* fields (i.e. functions of spacetime) by transformations that mix bosons and fermions. This however, turns out to be "as cumbersome and inconvenient as doing vector calculus component by component" [47]. Fortunately, there exists an alternative, more efficient way to realize SUSY by packaging the same degrees of freedom in *superfields* living on *superspace*, an extension of ordinary spacetime (with coordinates x) to include extra anticommuting Grassmann coordinates in the form of $\mathcal{N}$ two-component Weyl spinors $\theta, \bar{\theta}$.[6] The whole point of introducing such an apparently exotic space is that many properties of SUSY field theories now become manifest: one finds for instance that the SUSY algebra is represented by translations and rotations involving both x and θ, called *supertranslations* and *superrotations*.

Analytic Superspace The formal definition of superspace depends on the amount $\mathcal{N}$ of supersymmetry considered. We refer the reader to [42] for the construction of $\mathcal{N}=1$ superspace as a coset, while we focus on $\mathcal{N}>1$ in this work. We explained in Section 3.1 that theories with extended SUSY also feature R-symmetry. This is reflected in the construction of superspaces for $\mathcal{N}>1$, which now also include R-symmetry, or *harmonic* coordinates parameterizing a coset space of the R-symmetry group. The spaces of such superfields are known as *harmonic superspaces* and have been introduced in [48, 49], see also the reviews [50, 51]. In this work, we further restrict our attention to *analytic superspace*, a subspace of harmonic superspace where the superfields depend on fewer Grassmann variables, specifically those that are compatible with chirality or analyticity conditions. As such, this formalism is well-suited for the study of correlation functions of $\frac{1}{2}$-BPS operators [52–56]. This point of view will also allow for the derivation of superconformal Ward identities by imposing the absence of harmonic singularities later on, see Section 4.3 and [54, 57].

Adding a Boundary We now specify our analytic superspace to describe $d=4$, $\mathcal{N}=4$ SCFTs in the presence of a flat $\frac{1}{2}$-BPS boundary as done in [1]. Coordinates on superspace are schematically of the form

$$X = \left(\begin{array}{c|c} x^{\alpha\dot{\alpha}} & \theta^{\alpha\dot{a}} \\ \hline \bar{\theta}^{a\dot{\alpha}} & y^{a\dot{a}} \end{array}\right) \tag{3.11}$$

[6] A generic superfield $\Psi \equiv \Psi(x,\theta)$ may then be expanded in a finite Taylor series with respect to θ since $\theta^2 = 0$; the coefficients obtained in this way are precisely the aforementioned ordinary fields on spacetime.

for $\alpha, a = 1,2$ and $\dot\alpha, \dot a = 1,2$ $\mathfrak{sl}(2)$ spinor indices. This block decomposition reflects the superspace structure, with the bispinor $x^{\alpha\dot\alpha}$ describing $d = 4$ Euclidean space $\mathbb{R}^4_x$, the R-symmetry coordinate y viewed as a copy $\mathbb{R}^4_y$ thereof and where the off-diagonal blocks $\theta, \bar\theta$ correspond to left- and right-handed fermionic superspace coordinates. We saw in (3.2b) that the $d = 4$, $\mathcal{N} = 4$ superconformal algebra is $\mathfrak{psu}(2,2|4)$, which corresponds to the (complexified) group $PSL(4|4)$. The action of a general conformal transformation g on X will thus be [1]

$$g \circ X = (AX + B)(CX + D)^{-1}, \quad \text{where} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in PSL(4|4). \tag{3.12}$$

As seen in (3.2a), the introduction of a boundary breaks the superconformal algebra to $\mathfrak{osp}(4|4)$, which exponentiates to the 3d $\mathcal{N} = 4$ superconformal group $OSP(4|4)$. In Section 3.2 we introduced the coordinates $(\vec{x}, x_\perp) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ to describe this situation. We consider here two bulk operators in superspace with superspace coordinates X_1 and X_2, which we now split in a similar fashion, setting $X_i \equiv X_{i,\mathrm{b}} + X_{i,\mathrm{d}}$ for $i = 1,2$ with

$$X_{i,\mathrm{b}} \equiv \begin{pmatrix} x_{i,\mathrm{b}}^{\alpha\beta} & \pi_i^{a\beta} \\ \pi_i^{b\alpha} & \epsilon^{ab} y_{i,\mathrm{b}} \end{pmatrix} \quad \text{and} \quad X_{i,\mathrm{d}} \equiv \begin{pmatrix} \epsilon^{\alpha\beta} x_{i,\mathrm{d}} & \chi_i^{a\beta} \\ -\chi_i^{b\alpha} & y_{i,\mathrm{d}}^{ab} \end{pmatrix}$$

the boundary and distance superspace coordinates respectively, where $x_\mathrm{b}^{\alpha\beta} = x_\mathrm{b}^{\beta\alpha}$, $y_\mathrm{d}^{ab} = y_\mathrm{d}^{ba}$ and up to a $U(1)$ action parametrizing the embedding $OSP(4|4) \subset PSL(4|4)$. In [1], the authors also show explicitly how supertranslations and superrotations are realized. In particular, it is argued that one may use (3.12) to choose a frame such that

$$(X_{1,\mathrm{b}}, X_{1,\mathrm{d}}) = (0, \tilde{X}_{1,\mathrm{d}}), \quad (X_{2,\mathrm{b}}, X_{2,\mathrm{d}}) = (0, \tilde{X}_{2,\mathrm{d}}). \tag{3.13}$$

The authors also state that a superconformal transformation may then allow to further set the fermionic coordinates of both supermatrices to zero. In the frame (3.13) this is

$$X_i \equiv \tilde{X}_{i,\mathrm{d}} = \begin{pmatrix} 0 & x_i & 0 & 0 \\ -x_i & 0 & 0 & 0 \\ 0 & 0 & y_i^{11} & y_i^{12} \\ 0 & -0 & y_i^{12} & y_i^{22} \end{pmatrix}, \quad i = 1,2 \tag{3.14}$$

where we denote $x_i \equiv x_{i,\mathrm{d}}$ and $y_i \equiv y_{i,\mathrm{d}}$ above for notational simplicity. Clearly, the leftover degrees of freedom above match our expectations for the configuration space for the spacetime and R-symmetry coordinates of an operator centered on the defect, with one spacetime coordinate describing the distance to the boundary and three R-symmetry variables combining into a 3-vector describing $SO(3)$ rotations around a line in $\mathbb{R}^4_y$ [1].

Superconformal Invariants An account of the general constraints imposed by superconformal symmetry on 2-point correlation functions of SPs in superspace in the presence of a defect is given in [1], where superspace analogues of (3.5), (3.8) and (3.9) are reported. Just as conformal cross-ratios are the natural invariants of the conformal group, superconformal cross-ratios arise as invariants under the superconformal group. This implies that the functional dependence of correlation functions in superspace must reduce to combinations of invariants formed from the superspace coordinates we just introduced.

Here, we focus on justifying the existence of the superconformal invariants (3.6) and (3.7) entering the 2-point correlation function in the presence of a defect. To do so, we use the results of [54, 57] where 4-point superconformal invariants of $\frac{1}{2}$-BPS primaries are derived. We then translate these to the present problem by resorting to the prescription of [58] to relate correlation functions of n operators inserted away from a boundary in a defect CFT to correlation functions of $2n$ operators in an ordinary CFT: the *method of images*. There are two ways to derive 4-point superconformal invariants according to [54, 57]:

1. The first approach consists in fixing part of the superconformal symmetry, before identifying the remaining independent combinations of coordinates. The 4-point conformal and R-symmetry invariants z_1, z_2 and w_1, w_2 are built in this way in [54] by using conformal symmetry to pick the frame

$$x_1^\mu = \left(1 + \frac{z_1+z_2}{2}, \frac{z_1-z_2}{2}, 0, 0\right), \quad x_2^\mu = (1,0,0,0), \quad x_3^\mu = \frac{x_4^\mu}{x_4^2} = 0, \tag{3.15a}$$

in $\mathbb{R}^4_x$ and R-symmetry to set

$$y_1^i = \left(1 + \frac{w_1+w_2}{2}, \frac{w_1-w_2}{2}, 0, 0\right), \quad y_2^i = (1,0,0,0), \quad y_3^i = \frac{y_4^i}{y_4^2} = 0, \tag{3.15b}$$

in $\mathbb{R}^4_y$. We effectively get down from four sets of coordinates (x_j^μ, y_j^i) to one,

$$x^\mu \equiv x_{12}^\mu = \left(\frac{z_1+z_2}{2}, \frac{z_1-z_2}{2}, 0, 0\right), \quad y^i \equiv y_{12}^i = \left(\frac{w_1+w_2}{2}, \frac{w_1-w_2}{2}, 0, 0\right). \tag{3.15c}$$

The invariants above are related to the u, v of (2.6) and U, V of (3.4) by

$$z_1 = \frac{1}{2v}\left(1 - u - v \pm \sqrt{\Delta}\right), \quad z_2 = \frac{1}{2v}\left(1 - u - v \mp \sqrt{\Delta}\right), \tag{3.16a}$$

where $\Delta = (1-u-v)^2 - 4uv$, and

$$U = \frac{w_1 w_2}{(1+w_1)(1+w_2)}, \quad V = \frac{1}{(1+w_1)(1+w_2)}. \tag{3.16b}$$

However, z_1, z_2 and w_1, w_2 must be augmented with Grassmann coordinates to form fully-fledged superconformal invariants, which may involve subtleties due to potential singularities in the R-symmetry coordinates.

The authors of [54] carefully carry out this task by exploiting conformal and R-symmetry to pick a frame like (3.14) and (3.15c) in superspace such that $x^{\alpha\beta} \equiv \mathrm{diag}(z_1, z_2)$, $y^{ab} \equiv \mathrm{diag}(w_1, w_2)$ and where 3/4 of the SUSY transformations allow to eliminate all the odd variables but $\theta_1, \bar\theta_1 \equiv \theta, \bar\theta$, leaving only one matrix of the form (3.11) to deal with, $\tilde{X}$. In this frame, the residual 1/4 SUSY symmetry acts on $\tilde{X}$ as

$$\delta \begin{pmatrix} \theta^{11} & \theta^{12} \\ \theta^{21} & \theta^{22} \end{pmatrix} = \begin{pmatrix} (w_1 - z_1)\epsilon^1{}_1 & (w_2 - z_1)\epsilon^1{}_2 \\ (w_1 - z_2)\epsilon^2{}_1 & (w_2 - z_2)\epsilon^2{}_2 \end{pmatrix} + (\epsilon\theta\theta), \tag{3.17a}$$

$$\delta \begin{pmatrix} \bar\theta^{11} & \bar\theta^{12} \\ \bar\theta^{21} & \bar\theta^{22} \end{pmatrix} = \begin{pmatrix} (w_1 - z_1)\bar\epsilon^1{}_1 & (w_2 - z_1)\bar\epsilon^1{}_2 \\ (w_1 - z_2)\bar\epsilon^2{}_1 & (w_2 - z_2)\bar\epsilon^2{}_2 \end{pmatrix} + (\bar\epsilon\bar\theta\bar\theta), \tag{3.17b}$$

as well as (where $\epsilon, \bar{\epsilon}$ are the $\mathfrak{sl}(2)$ invariant tensors)

$$\delta z_i = \epsilon^i{}_a \bar{\theta}^{ai} + \theta^{ia} \bar{\epsilon}^a{}_i, \quad \delta w_i = -\bar{\epsilon}^2{}_\alpha \theta^{\alpha i} + \bar{\theta}^{i\alpha} \epsilon^\beta{}_i, \quad i = 1, 2. \tag{3.17c}$$

Using (3.17) one then finds that the quantities

$$\hat{z}_1 \equiv z_1 - \frac{\theta^{11}\bar{\theta}^{11}}{w_1 - z_1} - \frac{\theta^{12}\bar{\theta}^{21}}{w_2 - z_1} + \mathcal{O}((\theta)^4) \tag{3.18a}$$

$$\hat{z}_2 \equiv z_2 - \frac{\theta^{21}\bar{\theta}^{12}}{w_1 - z_2} - \frac{\theta^{22}\bar{\theta}^{22}}{w_2 - z_2} + \mathcal{O}((\theta)^4) \tag{3.18b}$$

$$\hat{w}_1 \equiv w_1 - \frac{\theta^{11}\bar{\theta}^{11}}{w_1 - z_1} - \frac{\theta^{21}\bar{\theta}^{12}}{w_1 - z_2} + \mathcal{O}((\theta)^4) \tag{3.18c}$$

$$\hat{w}_2 \equiv w_2 - \frac{\theta^{12}\bar{\theta}^{21}}{w_2 - z_1} - \frac{\theta^{22}\bar{\theta}^{22}}{w_2 - z_2} + \mathcal{O}((\theta)^4) \tag{3.18d}$$

satisfy $\delta\hat{z}_i = \delta\hat{w}_i = 0$ for $i = 1, 2$. These are thus the (linearized) superconformal completions of the bosonic ratios z_1, z_2, w_1, w_2.

2. The second approach for constructing 4-point superconformal invariants was pointed out in [57], where the authors identified z_1, z_2, w_1, w_2 as the eigenvalues of the 4-point superspace convariant matrix

$$\mathcal{X} \equiv X_{12} X_{23}^{-1} X_{34} X_{41}^{-1},$$

whose superconformal completions (3.18) are then obtained as perturbations around z_1, z_2, w_1, w_2; see Appendix A of [54] for a review.

The two approaches above are strictly equivalent and we thus interchangeably use both point of views in the following to derive the 2-point superconformal invariants in a SCFT with defects. For this we use the *method of images* [58] to regard a 2-point correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ in the presence of a boundary as the 4-point correlator in an *ancillary CFT* obtained by mirroring the operators along the defect, see Figure 2. This situation corresponds to setting $x_{3,\perp} = -x_{1,\perp}$ and $x_{4,\perp} = -x_{2,\perp}$ in (2.6), which in turn yields $z_1 = z_2 \equiv z$ in (3.16a). The relevant set of invariants to consider for the 2-point function of primary operators in the presence of a boundary are thus z, w_1, w_2.[7] Note that these are equivalent to the quantities $\xi, \sigma, \bar{\sigma}$ of (3.6) and (3.7) upon performing the change of variables [59]

$$\xi = \frac{(z-1)^2}{4z}, \quad \sigma + \bar{\sigma} = \frac{1}{2}\left(\sqrt{w_1 w_2} + \frac{1}{\sqrt{w_1 w_2}}\right), \quad \bar{\sigma} - \sigma = \frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} + \sqrt{\frac{w_2}{w_1}}\right). \tag{3.19}$$

The bosonic invariants z, w_1, w_2 are also obtained in [1] as the eigenvalues of the matrix

$$\mathcal{Z} \equiv (1 - \mathcal{Y}^{+1})(\mathcal{Y}^{-1} - 1), \quad \mathcal{Y}|_{(3.14)} = X_2 X_1^{-1}. \tag{3.20}$$

Again, these should be augmented with Grassmann variables to be fully-fledged superconformal invariants. In the next section, we turn our attention to the consequences of adding a defect operator to our theory for the bootstrap program.

[7] We still have two R-symmetry invariants since the boundary corresponds to a line defect in $\mathbb{R}^4_y$ [1].

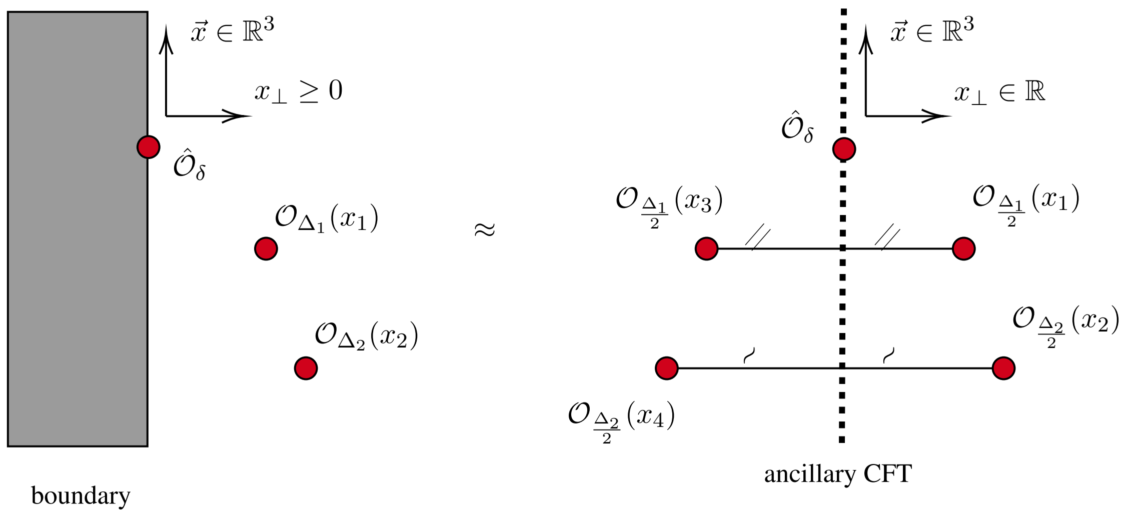

Figure 2. The method of images for a boundary CFT (BCFT) with two bulk and one boundary operators. The BCFT correlator on upper half-space $x_\perp \geq 0$ (left) is equivalent to the corresponding ancillary CFT correlator on the full space (right).

3.4 Revisiting the Bootstrap

While (2.9) required at least 4-point correlators to motivate the ordinary bootstrap, we see that there now already exists two different channels that can be used to bootstrap 2-point correlation functions of bulk SPs in a SCFT with boundary: the *bulk channel* and the *boundary channel*, both depicted in Figure 3. Spelling out the bulk channel schematically,

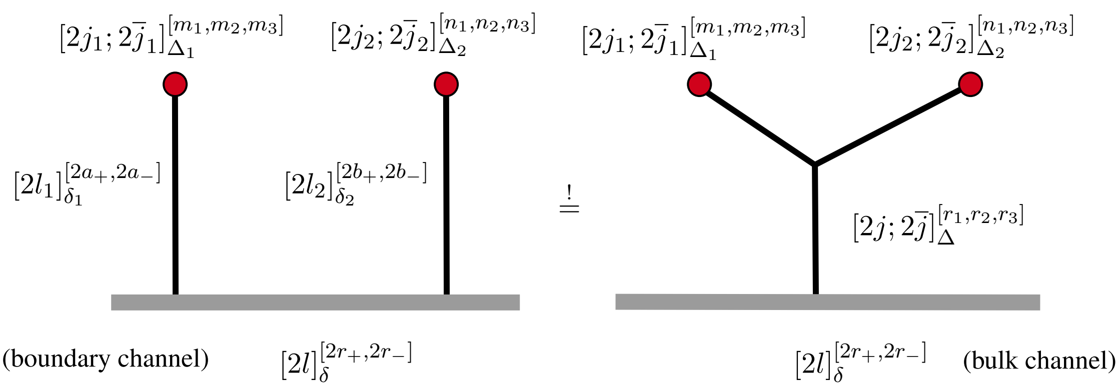

Figure 3. Diagrammatic representation of the defect bootstrap equation, with the boundary channel (left) and bulk channel (right). Each dot represents a local bulk operator while the thick grey line represents the boundary. The superconformal representations match the notation of (3.2).

this is

$$[2j_1; 2\bar{j}_1]^{[m_1,m_2,m_3]}_{\Delta_1} \otimes [2j_2; 2\bar{j}_2]^{[n_1,n_2,n_3]}_{\Delta_2} \stackrel{\text{blk OPE}}{\rightarrow} \bigoplus [2j; 2\bar{j}]^{[r_1,r_2,r_3]}_{\Delta} \stackrel{\text{blk 1pt}}{\rightarrow} \bigoplus [2l]^{[2r_+,2r_-]}_{\delta} \tag{3.21a}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \rightsquigarrow \sum_{\chi_{\text{blk}}} \lambda_{12\mathcal{O}} \langle \mathcal{O}(y) \rangle \rightsquigarrow \sum_{\chi_{\text{blk}}} \lambda_{12\mathcal{O}} a_{\mathcal{O}}, \tag{3.21b}$$

where the sum runs over all SPs (and descendant thereof) $\mathcal{O} \in [2j;2\bar{j}]_\Delta^{[r_1,r_2,r_3]}$ in a set χ_{blk} of suitable SMs to be determined shortly. For the boundary channel,

$$
\begin{aligned}
&[2j_1;2\bar{j}_1]_{\Delta_1}^{[m_1,m_2,m_3]} \otimes [2j_2;2\bar{j}_2]_{\Delta_2}^{[n_1,n_2,n_3]} \\
&\qquad \overset{\text{BOE}^2}{\rightarrow} \left(\bigoplus [2l_1]_{\delta_1}^{[2a_+,2a_-]}\right) \otimes \left(\bigoplus [2l_2]_{\delta_2}^{[2b_+,2b_-]}\right) \overset{\text{bdy 2pt}}{\rightarrow} \bigoplus [2l]_{\delta}^{[2r_+,2r_-]}
\end{aligned} \tag{3.22a}
$$

$$
\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \rightsquigarrow \langle \sum_{\chi_{\text{bdy}}} \mu_{1\hat{\mathcal{O}}_1} \hat{\mathcal{O}}_1 \otimes \sum_{\chi_{\text{bdy}}} \mu_{2\hat{\mathcal{O}}_2} \hat{\mathcal{O}}_2 \rangle \rightsquigarrow \sum_{\chi_{\text{bdy}}} \mu_{1\hat{\mathcal{O}}_1} \mu_{2\hat{\mathcal{O}}_2} \mathcal{F}_{\delta_1\delta_2} \mathcal{H}_{[a_+,a_-][b_+,b_-]} \tag{3.22b}
$$

where the sum runs this time over all SPs (and descendants thereof) $\hat{\mathcal{O}}_1 \in [2l_1]_{\delta_1}^{[2a_+,2a_-]}$ and $\hat{\mathcal{O}}_2 \in [2l_2]_{\delta_2}^{[2b_+,2b_-]}$ in a set χ_{bdy} of suitable SMs. Clearly, χ_{blk} will be the set of all $\mathfrak{psu}(2,2|4)$ SMs admitting states with a non-zero 1-point function such that the second arrow of (3.21) holds. These are found to be [1]

$$
\chi_{\text{blk}} \in \left\{ \mathbb{1}, \quad B_1\bar{B}_1[0;0]_{2n}^{[0,2n,0]}, \quad B_1\bar{B}_1[0;0]_{2n+4m}^{[2m,2n,2m]}, \quad L\bar{L}[0;0]_{\Delta}^{[2m,2n,2\bar{m}]} \right\} \tag{3.23}
$$

for $n, m, \bar{m} \in \mathbb{N}^*$. Besides the identity $\mathbb{1}$, these are $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and long $\mathfrak{psu}(2,2|4)$ SMs respectively [41]. Along the boundary channel, the representations are contrained to descend from $\mathfrak{osp}(4|4)$ SPs that can result from the branching of $\mathfrak{psu}(2,2|4) \rightarrow \mathfrak{osp}(4|4)$ such that the first arrow of (3.22) holds. These are [1]

$$
\chi_{\text{bdy}} \in \left\{ \hat{\mathbb{1}}, \quad B_1[0]_k^{[2k,0]}, \quad B_1[0]_k^{[0,2k]} \quad B_1[0]_{k_++k_-}^{[2k_+,2k_-]}, \quad L[0]_\delta^{[2k_+,2k_-]} \right\} \tag{3.24}
$$

where $k, k_\pm \in \mathbb{Z}_{>0}$ and $k_+k_- \neq 0$. Δ and δ are constrained by the unitarity bounds. Besides $\hat{\mathbb{1}}$, these correspond to two $\frac{1}{2}$-BPS, one $\frac{1}{4}$-BPS and a long $\mathfrak{osp}(4|4)$ SM respectively [41]. Sets (3.23) and (3.24) were also derived in [1] with the tools introduced in Section 3.3.

3.5 Superblock Decomposition

We are now ready to apply all these concepts to derive the superblock decomposition of the 2-point correlation function of two scalar $\frac{1}{2}$-BPS operators of the form (1.1) in $d = 4$, $\mathcal{N} = 4$ SYM with a $\frac{1}{2}$-BPS boundary defect. Starting in the bulk, we use the 2-point function (3.5) as well as (3.21) and (3.23) to write

$$
\langle \mathcal{W}_{\Delta_1} \mathcal{W}_{\Delta_2} \rangle = \sum_{\chi_{\text{blk}}} \sum_{(\Delta,[r_1,r_2,r_3])} c_{(\Delta,[r_1,r_2,r_3])} a_{(\Delta,[r_1,r_2,r_3])} \mathcal{F}_{\Delta_1\Delta_2}^{\text{blk},\Delta} \mathcal{H}_{\Delta_1\Delta_2}^{\text{blk},[r_1,r_2,r_3]}, \tag{3.25}
$$

where we introduced the shorthand notation $(\Delta,[r_1,r_2,r_3]) \equiv [0;0]_\Delta^{[r_1,r_2,r_3]}$ for representations belonging to each SM of (3.23), $c_{(\Delta,[r_1,r_2,r_3])}$ are coefficients to be determined and

$$
\mathcal{F}_{\Delta_1\Delta_2}^{\text{blk},\Delta}(x_1,x_2) \equiv \frac{1}{(2x_{1,\perp})^{\Delta_1}(2x_{2,\perp})^{\Delta_2}} \times \xi^{-\frac{\Delta_1+\Delta_2}{2}} \mathfrak{f}_\Delta^{\text{blk}}(\xi), \tag{3.26a}
$$

$$
\mathcal{H}_{\Delta_1\Delta_2}^{\text{blk},[r_1,r_2,r_3]}(u_1,u_2) \equiv (u_1\cdot\bar{u}_1)^{\frac{\Delta_1}{2}} (u_2\cdot\bar{u}_2)^{\frac{\Delta_2}{2}} \times \sigma^{\frac{\Delta_1+\Delta_2}{2}} \mathfrak{h}_{[r_1,r_2,r_3]}^{\text{blk}}(\sigma,\bar{\sigma}), \tag{3.26b}
$$

with ξ as in (3.6), $\sigma, \bar{\sigma}$ as in (3.7), $\mathfrak{f}^{\text{blk}}_{\Delta}, \mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]}$ conformal and R-symmetry blocks respectively for the representation $\mathcal{O} \in (\Delta, [r_1, r_2, r_3])$, and the factors of ξ and σ taken out for convenience. Moving on to the boundary channel, we use (3.22) and (3.24) to write

$$\langle \mathcal{W}_{\Delta_1} \mathcal{W}_{\Delta_2} \rangle = \sum_{\chi_{\text{bdy}}} \sum_{(\delta,[r_+,r_-])} c_{(\delta,[r_+,r_-])} \mu_{\Delta_1(\delta,[r_+,r_-])} \mu_{\Delta_2(\delta,[r_+,r_-])} \mathcal{F}^{\text{bdy},\delta}_{\Delta_1\Delta_2} \mathcal{H}^{\text{bdy},[r_+,r_-]}_{\Delta_1\Delta_2} \tag{3.27}$$

where we introduced the shorthand notation $(\delta, [r_+, r_-]) \equiv [0]^{[2r_+,2r_-]}_{\delta}$ for representations belonging to each SM of (3.24), $c_{(\delta,[r_+,r_-])}$ are coefficients to be determined and

$$\mathcal{F}^{\text{bdy},\delta}_{\Delta_1\Delta_2}(x_1, x_2) \equiv \frac{\mathfrak{f}^{\text{bdy}}_{\delta}(\xi)}{(2x_{1,\perp})^{\Delta_1}(2x_{2,\perp})^{\Delta_2}}, \tag{3.28a}$$

$$\mathcal{H}^{\text{bdy},[r_+,r_-]}_{\Delta_1\Delta_2}(u_1, u_2) \equiv (u_1 \cdot \bar{u}_1)^{\frac{\Delta_1}{2}} (u_2 \cdot \bar{u}_2)^{\frac{\Delta_2}{2}} \mathfrak{h}^{\text{bdy}}_{r_+}(\sigma, \bar{\sigma}) \mathfrak{h}^{\text{bdy}}_{r_-}(\sigma, \bar{\sigma}). \tag{3.28b}$$

Above $\mathfrak{f}^{\text{bdy}}_{\delta}$ is the conformal block for $\hat{\mathcal{O}} \in (\delta, [r_+, r_-])$, while the R-symmetry blocks split into $\mathfrak{h}^{\text{bdy}}_{r_\pm}$ since the boundary R-symmetry is $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$. They only depend on the invariants ξ and $\sigma, \bar{\sigma}$ respectively. Having set up the conceptual tools needed to bootstrap our theory, we derive the ingredients entering the bootstrap equations next.

4 Derivation of the Bootstrap Equations

4.1 Solving for the Superblocks

Our task is now to specify all the building blocks of (3.25) and (3.27) before requiring that both match. We start by deriving expressions for the superblocks of (3.26) and (3.28).

4.1.1 Bulk Superblocks

The bulk conformal group is $SO(5,1)$. A differential equation for $\mathfrak{f}^{\text{blk}}_{\Delta}$ can be derived by acting with the $SO(5,1)$-Casimir on (3.26a) in embedding space,

$$\begin{aligned} C^{(1,2)}_{SO(5,1)} \mathcal{F}(P_1, P_2) &= \frac{1}{2} \left(L^{(1)MN} + L^{(2)MN} \right) \left(L^{(1)}_{MN} + L^{(2)}_{MN} \right) \mathcal{F}(P_1, P_2) \\ &= \frac{1}{2} \left(P_1^M \frac{\partial}{\partial P_{1,N}} - P_1^N \frac{\partial}{\partial P_{1,M}} + P_2^M \frac{\partial}{\partial P_{2,N}} - P_2^N \frac{\partial}{\partial P_{2,M}} \right) \\ &\quad \times \left(P_{1,M} \frac{\partial \mathcal{F}}{\partial P_1^N} - P_{1,N} \frac{\partial \mathcal{F}}{\partial P_1^M} + P_{2,M} \frac{\partial \mathcal{F}}{\partial P_2^N} - P_{2,N} \frac{\partial \mathcal{F}}{\partial P_2^M} \right) \end{aligned} \tag{4.1}$$

where we abbreviated $\mathcal{F}^{\text{blk},\Delta}_{\Delta_1\Delta_2}(P_1, P_2) \equiv \mathcal{F}(P_1, P_2)$ for convenience. This computation is sketched in Appendix B.1. We find the ordinary differential equation (ODE)

$$\begin{aligned} 4\xi^2(1+\xi)\partial^2 f(\xi) + 2\xi \left[2(1+\xi)(1+\Delta_1+\Delta_2) - 4)\right] \partial f(\xi) \\ + \left[(\Delta - \Delta_1 - \Delta_2)(\Delta + \Delta_1 + \Delta_2 - 4) - 4\Delta_1\Delta_2\xi\right] f(\xi) = 0 \end{aligned}$$

for $f(\xi) \equiv \xi^{-\frac{\Delta_1+\Delta_2}{2}} \mathfrak{f}^{\text{blk}}_{\Delta}(\xi)$, with solution

$$\mathfrak{f}^{\text{blk}}_{\Delta}(\xi) = (4\xi)^{\frac{\Delta}{2}} {}_2F_1\left(\frac{\Delta + \Delta_1 - \Delta_2}{2}, \frac{\Delta - \Delta_1 + \Delta_2}{2}, \Delta - 1; -\xi \right), \tag{4.2}$$

where ${}_2F_1$ is the *hypergeometric function*. We proceed in a similar fashion to determine $\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]}$. The bulk R-symmetry is $SO(6)$ so we act with its Casimir on (3.26b),

$$C^{(1,2)}_{SO(6)}\mathcal{H}(u_1,u_2) = \frac{1}{2}\left(L^{(1)MN}+L^{(2)MN}\right)\left(L^{(1)}_{MN}+L^{(2)}_{MN}\right)\mathcal{H}(u_1,u_2) \tag{4.3}$$

for $\mathcal{H}(u_1,u_2) \equiv \mathcal{H}^{\text{blk},[r_1,r_2,r_3]}_{\Delta_1\Delta_2}(u_1,u_2)$. The computation is akin to that of Appendix B.1 and performed in Appendix B.2. We find the ODE

$$\left[\left(\sum_{i=1}^{2} w_i(w_i-1)^2\partial^2_{w_i}\right)+k(w_1,w_2)\partial_{w_1}+k(w_2,w_1)\partial_{w_2}\right]\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]} = C_{[r_1,r_2,r_3]}\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]} \tag{4.4}$$

where

$$k(w_1,w_2) = \left(\frac{w_1(w_1-1)}{w_1-w_2}+\frac{w_1-1}{w_1w_2-1}-2\right)(w_1-1),$$
$$C_{[2m,2n,2m]} = 2\left(n^2+2n(m+1)+m(2m+3)\right) \quad \text{for } R=[2m,2n,2m].$$

Note that (4.4) admits a closed-form solution for $R=[0,2n,0]$ or $[2m,0,2m]$ only [1].

4.1.2 Boundary Superblocks

Next, we move on to the boundary channel where the Euclidean conformal group is $SO(4,1)$. Like in the bulk, a differential equation for $\mathfrak{f}^{\text{bdy}}_{\delta}$ is obtained by inserting the $SO(4,1)$-Casimir where our OPE projector was, i.e. this time *between* the two operators,

$$\langle \mathcal{W}_{\Delta_1}(P_1,U_1)C^{(P_2)}_{SO(4,1)}\mathcal{W}_{\Delta_2}(P_2,U_2)\rangle.$$

Acting with $C^{(P_2)}_{SO(4,1)}$ on (3.28a) seen as a function of P_2 only, we compute

$$C^{(P)}_{SO(4,1)}\mathcal{F}(P) = \frac{1}{2}\left(P^A\frac{\partial}{\partial P_B}-P^B\frac{\partial}{\partial P_A}\right)\left(P_A\frac{\partial\mathcal{F}(P)}{\partial P^B}-P_B\frac{\partial\mathcal{F}(P)}{\partial P^A}\right) \tag{4.5}$$

where $\mathcal{F}(P) \equiv \mathcal{F}^{\text{bdy},R}_{\delta_1\delta_2}(P_1,P)$ above for brevity. This is carried out in great detail in Appendix B.3. We find the ODE

$$\xi(1+\xi)\partial^2\mathfrak{f}^{\text{bdy}}_{\delta}(\xi)+\frac{3}{2}(1+2\xi)\partial\mathfrak{f}^{\text{bdy}}_{\delta}(\xi)-\delta(\delta-1)\mathfrak{f}^{\text{bdy}}_{\delta}(\xi)=0$$

with suitable solution given by the hypergeometric function

$$\mathfrak{f}^{\text{bdy}}_{\delta}(\xi) = (4\xi)^{-\delta}{}_2F_1\left(\delta,\delta-1,2\delta-2;-\xi^{-1}\right). \tag{4.6}$$

Lastly, since the boundary R-symmetry is $SO(4)\cong SO(3)_+\oplus SO(3)_-$ we may solve for either of $\mathfrak{h}^{\text{bdy}}_{r_\pm}(\sigma^\pm)$ by making the change of variables

$$(\sigma,\bar\sigma)\mapsto(\sigma^+,\sigma^-)\equiv(\sigma+\bar\sigma,\sigma-\bar\sigma) \tag{4.7}$$

and using the corresponding $SO(3)_\pm$-Casimir on (3.28b) (seen as a function of u_2 only). As shown in Appendix B.4, this yields the ODE

$$\left(w^2\partial^2_w+\frac{2w^2}{w-w^{-1}}\partial_w\right)\mathfrak{h}^{\text{bdy}}_k(w) = k(k+1)\mathfrak{h}^{\text{bdy}}_k(w)$$

whose solution is

$$\mathfrak{h}^{\text{bdy}}_k(w) = w^{-k}{}_2F_1\left(\frac{1}{2},-k;\frac{1}{2}-k;w^2\right).$$

4.2 Contributing Supermultiplets

Next, we need to figure out which representations $(\Delta, [r_1, r_2, r_3])$ and $(\delta, [r_+, r_-])$ to include in the sums of (3.25) and (3.27) respectively. We follow [41, 60] to generate the SMs of (3.23) and (3.24), employing the Racah-Speiser algorithm[8] to decompose tensor products of superconformal representations and restricting to scalars. In [1] the authors employ instead the superconformal characters of Dolan [61, 62]. We find that while RS spits out more contributing representations, there is no apparent tension between the two approaches since all the excess coefficients that arise from RS are set to 0 by the superconformal Ward identities of Section 4.3 (see Appendices C.3 and D).

4.2.1 Bulk Channel

Starting in the bulk channel, we generate the full set of superconformal representations of a given SM of (3.23) by acting with the Q's and $\bar{Q}$'s on its superprimary and imposing the relevant shortening conditions at each level. Out of all the representations descending from a given superprimary, we only need to retain *diagonal representations* for the OPE of two chiral operators [60], which arise from the action of an equal number of Q's and $\bar{Q}$'s.[9] Since (1.1) are scalar operators, we also further restrict to scalar representations ($j \equiv \bar{j} \equiv 0$). This procedure is outlined in Appendix C.1 and shows that (3.25) features the representations

$$[0;0]_{\Delta}^{[r_1,r_2,r_3]} \in \{(2n,[0,2n,0]),(2n-4,[0,2n+4,0])\} \quad \text{for } \chi_{\text{blk}} = B_1\bar{B}_1[0;0]_{2n}^{[0,2n,0]}. \quad (4.8)$$

The cases $\chi_{\text{blk}} = B_1\bar{B}_1[0;0]_{2n+4m}^{[2m,2n,2m]}$ or $L\bar{L}[0;0]_{\Delta}^{[2m,2n,2\bar{m}]}$ are more involved; we refer the reader to [1]. In the interest of completeness, we also describe in Appendix C.2 how to use the `Mathematica` package LieART [63] to construct the full $B_1\bar{B}_1[0;0]_2^{[0,2,0]}$ supermultiplet.

4.2.2 Boundary channel

Moving on to the boundary channel, we repeat the same steps as for the bulk channel, acting this time with all the Q's on the superprimaries of all SMs in (3.24), enforcing the relevant shortening conditions at each level and selecting only scalars ($l \equiv 0$) at the end. This is done in great detail in Appendix C.3 and shows that (3.27) features the representations

$$\begin{aligned}
[0]_{\delta}^{[2r_+,2r_-]} &\in \left\{[0]_k^{[2k,0]},[0]_{k+1}^{[2k-2,2]},[0]_{k+2}^{[2k-4,0]}\right\} \quad \text{for } \chi_{\text{bdy}} = B_1[0]_k^{[2k,0]}, \\
[0]_{\delta}^{[2r_+,2r_-]} &\in \left\{[0]_k^{[0,2k]},[0]_{k+1}^{[2,2k-2]},[0]_{k+2}^{[0,2k-4]}\right\} \quad \text{for } \chi_{\text{bdy}} = B_1[0]_k^{[0,2k]},
\end{aligned}$$

and we refer the reader to Appendix C.3 for the cases $\chi_{\text{bdy}} = B_1[0]_{k_++k_-}^{[2k_+,2k_-]}$ and $L[0]_{\delta}^{[2k_+,2k_-]}$.

4.3 Superconformal Ward Identities

We now have knowledge of all the ingredients needed to write out the full bulk and boundary channel expansions (3.25) and (3.27); it only remains to fix the relative coefficients

[8]We refer the reader to Appendix A.3 of [41] for a summary of the Racah-Speiser algorithm.

[9]The need for diagonal representations stems from the requirement that the corresponding operators be uncharged under the $U(1)_Y$ parametrizing the embedding of $\mathfrak{osp}(4|4)$, mentioned in Section 3.3 [1].

$c_{(\Delta,[r_1,r_2,r_3])}$ and $c_{(\delta,[r_+,r_-])}$ for the representations obtained in Section 4.2. This can be achieved by solving either of the *superconformal Ward identities* (WI)

$$\left(\partial_{w_1}+\frac{1}{2}\partial_z\right)\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}\Big|_{w_1=z}\overset{!}{=}0\quad\text{and}\quad\left(\partial_{w_2}+\frac{1}{2}\partial_z\right)\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}\Big|_{w_2=z}\overset{!}{=}0\tag{4.9}$$

both in the bulk and in the boundary. Above we isolated the full superconformal blocks

$$\begin{aligned}\mathfrak{F}^{\text{blk}}_{\Delta_1\Delta_2}(z,w_1,w_2)\equiv\sum_{\chi_{\text{blk}}}\sum_{(\Delta,[r_1,r_2,r_3])}&c_{(\Delta,[r_1,r_2,r_3])}a_{(\Delta,[r_1,r_2,r_3])}\\&\times\Omega^{\frac{\Delta_1+\Delta_2}{2}}\mathfrak{f}^{\text{blk}}_{\Delta}(z)\mathfrak{h}^{\text{blk}}_{r_1,r_2,r_3]}(w_1,w_2)\end{aligned}\tag{4.10a}$$

out of (3.25) in the bulk, with $\Omega\equiv\sigma/\xi$, (3.19) and

$$\begin{aligned}\mathfrak{F}^{\text{bdy}}_{\Delta_1\Delta_2}(z,w_1,w_2)\equiv\sum_{\chi_{\text{bdy}}}\sum_{(\delta,[r_+,r_-])}&c_{(\delta,[r_+,r_-])}\mu_{\Delta_1(\delta,[r_+,r_-])}\mu_{\Delta_2(\delta,[r_+,r_-])}\\&\times\mathfrak{f}^{\text{bdy}}_{\delta}(z)\mathfrak{h}^{\text{bdy}}_{r_+}(w_1,w_2)\mathfrak{h}^{\text{bdy}}_{r_-}(w_1,w_2)\end{aligned}\tag{4.10b}$$

out of (3.27) in the boundary channel. Like (3.3), these do not depend on any of the kinematic variables x_1,x_2 nor R-symmetry variables u_1,u_2 but rather only on the set of conformal and R-symmetry invariants, which is here $\{z,w_1,w_2\}$ as introduced in Section 3.3.

Derivation The conditions in (4.9) are derived by demanding that (4.10) remain well-behaved under superconformal symmetry. Specifically, their supersymmetrization should not introduce any poles when w_1 or w_2 approaches z. We now outline this derivation, following and adapting the approach of [54] for four-point Ward identities in $d=4$, $\mathcal{N}=4$ theories. In Section 3.3, we argued that the conformal invariant for a two-point correlation function in the presence of a boundary simplifies to $z_1=z_2\equiv z$. Building on this, we observe that the superconformal completions (3.18) become singular when either $w_1\to z$ or $w_2\to z$. Without loss of generality, we focus on the case $w_1\neq w_2$ and take $w_1\to z$ in the following discussion. Supersymmetrizing the full superconformal blocks (4.10) amounts to ensuring that they satisfy the linearized SUSY transformation

$$\begin{aligned}\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(\hat z,\hat w_1,\hat w_2)=&\,\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(z,w_1,w_2)\\&+(\Delta z\partial_z+\Delta w_1\partial_{w_1}+\Delta w_2\partial_{w_2})\,\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(z,w_1,w_2)\end{aligned}\tag{4.11}$$

where $\Delta z\equiv\hat z-z$, $\Delta w_i\equiv\hat w_i-w_i$, and from (3.18) we read off

$$\Delta z\equiv-\frac{\theta^{11}\bar\theta^{11}}{w_1-z}-\frac{\theta^{12}\bar\theta^{21}}{w_2-z}+\mathcal{O}((\theta)^4),\qquad\Delta w_1\equiv-\frac{\theta^{11}\bar\theta^{11}}{w_1-z}-\frac{\theta^{21}\bar\theta^{12}}{w_1-z}+\mathcal{O}((\theta)^4).$$

Picking up the singular terms in (4.11) only, we have as $w_1\to z$

$$\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(\hat z,\hat w_1,\hat w_2)\sim\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}-\left[\frac{\theta^{11}\bar\theta^{11}}{w_1-z}\partial_z+\left(\frac{\theta^{11}\bar\theta^{11}}{w_1-z}+\frac{\theta^{21}\bar\theta^{12}}{w_1-z}\right)\partial_{w_1}\right]\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}.\tag{4.12}$$

The final simplification comes from noting that the defect introduces parity-symmetry $\mathcal{P}$ of $\theta, \bar{\theta}$ by reflection along $x_\perp = 0$ in Figure 2. Since $\mathcal{P}\theta^{1b} \mapsto \theta^{2b}$, $\mathcal{P}\bar{\theta}^{\dot{a}1} \mapsto \bar{\theta}^{\dot{a}2}$ and $\mathcal{P}^2 = 1$, we have $\theta^{11}\bar{\theta}^{11} \overset{\mathcal{P}}{\sim} \theta^{12}\bar{\theta}^{21}$ and (4.12) factorizes,[10]

$$\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(\hat{z}, \hat{w}_1, \hat{w}_2) \sim \mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(z, w_1, w_2) - \frac{\theta^{11}\bar{\theta}^{11}}{w_1 - z}\left(\partial_z + 2\partial_{w_1}\right)\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(z, w_1, w_2).$$

To ensure that the superconformal block $\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(\hat{z}, \hat{w}_1, \hat{w}_2)$ is well-defined and free of any harmonic singularities, we thus require

$$\left(\partial_{w_1} + \frac{1}{2}\partial_z\right)\mathfrak{F}^{\text{blk/bdy}}_{\Delta_1\Delta_2}(z, w_1, w_2)\bigg|_{w_1 = z} \overset{!}{=} 0.$$

This is the first of (4.9). The superconformal WI for $w_2 \to z$ is obtained similarly.

One may now ultimately use (4.9) to solve for the coefficients $c_{(\Delta,[r_1,r_2,r_3])}$ and $c_{(\delta,[r_+,r_-])}$ and thereby complete the bootstrap analysis. The steps are carried out explicitly in Appendix D for the boundary channel, where we proceed perturbatively using `Mathematica`.

5 Conclusion

In this work, we reviewed the bootstrap program for superconformal field theories (SCFTs) in the presence of defects, with a particular focus on $\frac{1}{2}$-BPS boundary defects in $\mathcal{N} = 4$ SYM. Utilizing superspace techniques, we fully leveraged superconformal symmetry to derive the complete superblock expansion for two-point correlation functions. This framework not only enhances our understanding of supersymmetric gauge theories but also provides a robust foundation for exploring holographic dualities and advancing theoretical applications.

Outlook A compelling direction for future research is the extension of these techniques to other SCFTs, particularly Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [65]. As a three-dimensional SCFT with $\mathcal{N} = 6$ supersymmetry, ABJM is dual to M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$, serving as a lower-dimensional counterpart to $\mathcal{N} = 4$ SYM. Defects in ABJM, such as those involving a Nahm pole (codimension-1), have been explored from the perspective of integrability [66–68], while recent advances have examined supersymmetric boundary conditions [69] and the operator spectrum in $\frac{1}{2}$-BPS domain walls [70]. These studies lay the groundwork for extending defect bootstrap techniques to ABJM, enabling the exploration of its defect CFT in a manner analogous to the work presented here.

A promising next step would thus be to derive the superblock expansion for the complete set of two-point functions involving chiral primary operators in ABJM theory. This would deepen our understanding of operator dynamics, structure constants, and the intricate interplay between supersymmetry and holography in lower dimensions. Such an extension would provide new insights into the rich structure of SCFTs and their dualities, building on the methodology established for $\mathcal{N} = 4$ SYM in [1] and reviewed in this article.

[10] Credit for this observation goes to Menglei Tian [64].

Acknowledgements

I sincerely thank Menglei Tian for our insightful discussions, her thorough review of an earlier draft of this article, as well as for the enjoyable company in our office at NBB. I am also grateful to Prof. Charlotte Kristjansen for initiating this project and to Dr. Adam Chalabi for his time, guidance, and invaluable feedback. Special thanks also go to Manuel Artime Alvarez, Ian Le Meur and Michael Sacks for proofreading this document. Finally, I extend my heartfelt gratitude to Milla for her support over the past few months.

A Elements of Conformal Field Theory

General quantum field theories (QFTs) admit the Poincaré group as the symmetry group of relativistic field theory in flat space, whose general transformation is of the form

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$$

with Λ^μ_ν describing Lorentz transformations and a^μ a spacetime translation. A special class of QFTs are *conformal field theories* (CFTs), which enjoy additional spacetime symmetries under *conformal transformations*. The latter are the angle-preserving transformations and allow to place strong additional constraints on the theory. Given that

$$\theta_{uv} = \arccos \frac{g_{\mu\nu}(x) u^\mu v^\nu}{|g_{\mu\nu}(x) u^\mu u^\nu||g_{\mu\nu}(x) v^\mu v^\nu|}$$

is the angle between two spacetime vectors u^μ and v^μ at a point x^μ in spacetime, conformal transformations must act on the metric $g_{\mu\nu}$ as

$$g_{\mu\nu}(x) \mapsto \Omega(x)^2 g_{\mu\nu}(x) \quad \text{(Weyl transformation)} \tag{A.1}$$

with Ω a position-dependent scale factor. For an infinitesimal transformation parametrized by ϵ^μ and $\Omega^2(x) \approx 1 + \omega(x)$ we find the corresponding Killing vectors as

$$\mathcal{L}_\epsilon \eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \overset{!}{=} \omega(x) \eta_{\mu\nu}, \tag{A.2}$$

where we exploited Weyl invariance to rescale the metric to the flat Minkowski metric $\eta_{\mu\nu}$. After taking a trace to solve for ω and performing some algebra, one arrives at the equation

$$\partial^\mu \partial_\mu \epsilon_\nu = \frac{1}{d}(2-d)\partial_\nu \partial_\sigma \epsilon^\sigma$$

with d the spacetime dimension. The case $d = 2$ appears as an obvious special case, which is discussed in detail eg. in [6] but will not concern us here. Assuming $d > 2$ in what follows, the most general solution to this equation writes

$$\epsilon^\mu(x) = \underbrace{a^\mu + \omega^\mu{}_\nu x^\nu}_{\text{Poincaré}} + \underbrace{\lambda x^\mu}_{\text{dilatations}} + \underbrace{b^\mu x^2 - 2x^\mu b\cdot x}_{\text{SCT}}.$$

We see that they contain the Poincaré transformations,[11] enhanced by infinitesimal *dilatations* parametrized by λ, as well as *special conformal transformations* (SCT) parametrized by b^μ.[12] The conformal generators are

$$P_\mu \text{ (translations)}, \; M_{\mu\nu} \text{ (Lorentz transformations)}, \; D \text{ (dilatations)}, \; K_\mu \text{ (SCTs)}$$

and their algebra is found to be isomorphic to $\mathfrak{so}(d, 2)$ in Lorentzian signature or $\mathfrak{so}(d+1, 1)$ in Euclidean signature.[13] In Lorentzian signature, we have schematically

$$[M_{\mu\nu}, M_{\rho\sigma}],\ [M_{\mu\nu}, P_\rho] \sim \text{Poincaré algebra},\ [M_{\mu\nu}, K_\rho] \sim \text{vector},\ [M_{\mu\nu}, D] \sim \text{scalar}, \tag{A.3a}$$

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu \tag{A.3b}$$

$$[P_\mu, K_\nu] = 2(g_{\mu\nu} D - M_{\mu\nu}) \quad \text{(closure)}. \tag{A.3c}$$

[11] After all, these are the isometries of flat space satisfying the conformal Killing equation (A.2) for $\omega \equiv 0$.
[12] These can also be thought of as translations conjugated by an *inversion* of the form $x^\mu \to x^\mu / x^2$.
[13] For flat space of signature (m, n), $m + n = d$, the conformal group is isomorphic to $\mathfrak{so}(m + 1, n + 1)$.

We see that P and K act as raising and lowering operators for the dilatation operator respectively. This will play an important role in what follows.

States in A CFT To think about the different states of a quantum theory, one needs to choose a foliation of spacetime and define the evolution between different leaves [33]. In Poincaré invariant theories, the foliation is usually done along surfaces of equal time, while the exponentiation of the Hamiltonian P^0 gives the unitary evolution operator

$$U = e^{iP^0 \Delta t}$$

and the states living on these surfaces can be characterized by their momenta $P_\mu|k\rangle = k_\mu|k\rangle$. There is however another convenient choice that one can perform in a CFT, where we want to describe states created by insertions of local operators: we foliate the spacetime along spheres of various radii and the evolution operator now involves the dilatation operator

$$U = e^{iD\Delta t}.$$

This is called *radial quantization*, and will always be assumed in the following. In this scheme, operators in correlation functions are ordered so that those inserted at larger radial distance are moved to the left. This change of foliation is akin to a Wick rotation from a Lorentzian signature to an Euclidean theory, where time is now identified with the radial coordinate, the states living on the spheres are classified according to their $\mathfrak{so}(2)$ *scaling dimension* $\Delta \in \mathbb{R}$ and $\mathfrak{so}(d)$ spin $J \in \mathbb{Z}_{\geq 0}/2$. The conformal generators now satisfy

$$D^\dagger = D, \quad P_\mu^\dagger = K_\mu, \quad K_\mu^\dagger = P_\mu, \quad M_{\mu\nu}^\dagger = M_{\nu\mu}.$$

Physical representations should have an energy spectrum bounded from below. From (A.3b) we see that there must therefore exist a highest-weight state $|\Delta, J\rangle$ such that

$$D|\Delta, J\rangle = \Delta|\Delta, J\rangle, \quad M_{\mu\nu}|\Delta, J\rangle = \Sigma_{\mu\nu}|\Delta, J\rangle, \quad K_\mu|\Delta, J\rangle = 0, \tag{A.4}$$

with the matrices $\Sigma_{\mu\nu}$ corresponding to the generators of the spin J representation of $\mathfrak{so}(d)$. This state is called a *conformal primary* (CP), while states $P_{\mu_1}...P_{\mu_n}|\Delta, J\rangle$ obtained by successive application of the momentum operator are called *conformal descendants* (CDs). The set of all CDs descending from a CP (including the CP itself) forms a *conformal multiplet*. Lastly, *unitarity bounds* on Δ of the form $\Delta \geq \Delta_{\text{min}}(J)$ within a given multiplet can be derived by requiring that the norm[14] of the states be non-negative across the entire multiplet. *Long* multiplets have $\Delta > \Delta_{\text{min}}$ such that all states have positive norm, while for $\Delta = \Delta_{\text{min}}$ there might *null states* whose norm vanishes. These must be removed from the Hilbert space, giving rise to so-called *short* multiplets.

Operators in a CFT Now, note that fields of a QFT may be dimensionful, while requiring invariance under e.g. constant rescalings $x \mapsto \lambda x$ would force all dimensionful quantities

[14] There is a natural inner product on the Hilbert space of states in radial quantization, which is inherited from the 2-point functions of local operators through the state-operator map, see next paragraph.

to vanish. It is thus natural to expect local operators $\mathcal{O}_\Delta(x)$ built out of fields and their derivatives to also transform under conformal transformations like

$$\mathcal{O}_\Delta(x) \mapsto \Omega(x)^{-\Delta}(R(x)\cdot\mathcal{O}_\Delta)(x) \tag{A.5}$$

with Ω as in (A.1), Δ the scaling dimension[15] of the operator and R the rotational part of the conformal transformation, i.e. a spin J rotation matrix. Infinitesimally, (A.5) means

$$\begin{aligned}
[P_\mu, \mathcal{O}_\Delta(x)] &= \partial_\mu \mathcal{O}_\Delta(x), \\
[M_{\mu\nu}, \mathcal{O}_\Delta(x)] &= (x_\nu\partial_\mu - x_\mu\partial_\nu + \Sigma_{\mu\nu})\mathcal{O}_\Delta(x), \\
[D, \mathcal{O}_\Delta(x)] &= (x^\mu\partial_\mu + \Delta)\mathcal{O}_\Delta(x), \\
[K_\mu, \mathcal{O}_\Delta(x)] &= (2x_\mu(x\cdot\partial) - x^2\partial_\mu + 2\Delta x_\mu)\mathcal{O}_\Delta(x).
\end{aligned}$$

Then defining $|\mathcal{O}_\Delta\rangle \equiv \mathcal{O}_\Delta(0)|0\rangle$ we find that this state satisfies (A.4). In light of radial quantization, this hints that states on the sphere are generated by operator insertions inside of the sphere. The vacuum state $|0\rangle$ corresponds to doing nothing, and thus to inserting the identity operator $\mathbb{1}$, while primary states stem from operators inserted at the origin. We thus identify local operators $\mathcal{O}_\Delta(x)$ in a CFT with primary states $|\mathcal{O}_\Delta\rangle$, called *primary operators* of *weight* Δ. Descendant operators are obtained by taking derivatives on $\mathcal{O}_\Delta$ in analogy to acting with the momentum operator P_μ on $|\mathcal{O}_\Delta\rangle$; this is the *state-operator map*. All local operators of a unitary theory must reside in unitary representations, or *multiplets* of the conformal algebra. In this work we mostly focus on scalar operators ($J = 0$).

B Differential Equations for Superblocks

B.1 Bulk Conformal Block

We start from (4.1),

$$\begin{aligned}
C^{(1,2)}_{SO(5,1)}\mathcal{F}(P_1,P_2) &= \frac{1}{2}\left(L^{(1)MN} + L^{(2)MN}\right)\left(L^{(1)}_{MN} + L^{(2)}_{MN}\right)\mathcal{F}(P_1,P_2) \\
&= \frac{1}{2}\left(P_1^M\frac{\partial}{\partial P_{1,N}} - P_1^N\frac{\partial}{\partial P_{1,M}} + P_2^M\frac{\partial}{\partial P_{2,N}} - P_2^N\frac{\partial}{\partial P_{2,M}}\right) \\
&\quad\times\left(P_{1,M}\frac{\partial\mathcal{F}}{\partial P_1^N} - P_{1,N}\frac{\partial\mathcal{F}}{\partial P_1^M} + P_{2,M}\frac{\partial\mathcal{F}}{\partial P_2^N} - P_{2,N}\frac{\partial\mathcal{F}}{\partial P_2^M}\right)
\end{aligned}$$

where we abbreviated $\mathcal{F}^{\text{blk},\Delta}_{\Delta_1\Delta_2}(P_1,P_2) \equiv \mathcal{F}(P_1,P_2)$ as in (3.26a) for notational convenience. Expanding,

$$\begin{aligned}
C^{(1,2)}_{SO(5,1)}\mathcal{F} = \frac{1}{2}\Bigg[&P_1^M\frac{\partial}{\partial P_{1,N}}\left(P_{1,M}\frac{\partial\mathcal{F}}{\partial P_1^N} - P_{1,N}\frac{\partial\mathcal{F}}{\partial P_1^M} + P_{2,M}\frac{\partial\mathcal{F}}{\partial P_2^N} - P_{2,N}\frac{\partial\mathcal{F}}{\partial P_2^M}\right) \\
&-P_1^N\frac{\partial}{\partial P_{1,M}}\left(P_{1,M}\frac{\partial\mathcal{F}}{\partial P_1^N} - P_{1,N}\frac{\partial\mathcal{F}}{\partial P_1^M} + P_{2,M}\frac{\partial\mathcal{F}}{\partial P_2^N} - P_{2,N}\frac{\partial\mathcal{F}}{\partial P_2^M}\right)\Bigg] + (1\leftrightarrow 2).
\end{aligned}$$

[15] This coincides with their mass dimension classically but may receives corrections at the quantum level.

Distributing, using the Leibniz rule while keeping in mind that $P_i \cdot P_i = 0$ and $\partial_{P_i^M} P_i^M = 6$ for $i = 1, 2$, we arrive at

$$C^{(1,2)}_{SO(5,d)}\mathcal{F}(P_1,P_2) = (P_1 \cdot P_2)\frac{\partial^2 \mathcal{F}}{\partial P_1 \cdot \partial P_2} - P_1^M P_2^N \frac{\partial^2 \mathcal{F}}{\partial P_1^N \partial P_2^M} \\ - 5P_1 \cdot \frac{\partial \mathcal{F}}{\partial P_1} - P_1^M P_1^N \frac{\partial^2 \mathcal{F}}{\partial P_1^N \partial P_1^M} + (1 \leftrightarrow 2)$$

and finally the quadratic Casimir of $SO(5,1)$ in embedding space acts on $\mathcal{F}(P_1, P_2)$ as

$$C^{(1,2)}_{SO(5,1)}\mathcal{F}(P_1,P_2) = 2(P_1 \cdot P_2)\frac{\partial^2 \mathcal{F}}{\partial P_1 \cdot \partial P_2} - 2P_1^M P_2^N \frac{\partial^2 \mathcal{F}}{\partial P_1^N \partial P_2^M} \\ - 5P_1 \cdot \frac{\partial \mathcal{F}}{\partial P_1} - P_1^M P_1^N \frac{\partial^2 \mathcal{F}}{\partial P_1^N \partial P_1^M} - 5P_2 \cdot \frac{\partial \mathcal{F}}{\partial P_2} - P_2^M P_2^N \frac{\partial^2 \mathcal{F}}{\partial P_2^N \partial P_2^M}.$$

In embedding space, (3.26a) writes

$$\mathcal{F}(P_1,P_2) \equiv \xi^{-\frac{\Delta_1+\Delta_2}{2}} \frac{\mathfrak{f}^{\mathrm{blk}}_\Delta(\xi)}{(2V \cdot P_1)^{\Delta_1}(2V \cdot P_2)^{\Delta_2}}$$

with ξ and V defined around (3.6). We also know that the eigenvalue of of the $SO(5,1)$-Casimir in a scalar conformal multiplet with label Δ is $\tilde{C}_{SO(5,1)} \equiv -\Delta(\Delta-4)$. Our equation will then be $C^{(1,2)}_{SO(5,1)}\mathcal{F}(P_1,P_2) = \tilde{C}_{SO(5,1)}\mathcal{F}(P_1,P_2)$. With the help of `Mathematica` we find

$$4\xi^2(1+\xi)\partial^2 f(\xi) + 2\xi\left[2(1+\xi)(1+\Delta_1+\Delta_2) - 4)\right]\partial f(\xi) \\ + \left[(\Delta - \Delta_1 - \Delta_2)(\Delta + \Delta_1 + \Delta_2 - 4) - 4\Delta_1\Delta_2\xi\right] f(\xi) = 0$$

for $f(\xi) \equiv \xi^{-\frac{\Delta_1+\Delta_2}{2}}\mathfrak{f}^{\mathrm{blk}}_\Delta(\xi)$. This ODE admits two solutions in terms of hypergeometric functions

$$\mathfrak{f}^{\mathrm{blk}}_\Delta(\xi) = \xi^{\frac{4-\Delta}{2}}\,{}_2F_1\left(\frac{4-\Delta+\Delta_1-\Delta_2}{2}, \frac{4-\Delta-\Delta_1+\Delta_2}{2}, 3-\Delta; -\xi\right), \tag{B.1}$$

$$\mathfrak{f}^{\mathrm{blk}}_\Delta(\xi) = \xi^{\frac{\Delta}{2}}\,{}_2F_1\left(\frac{\Delta+\Delta_1-\Delta_2}{2}, \frac{\Delta-\Delta_1+\Delta_2}{2}, \Delta-1; -\xi\right). \tag{B.2}$$

To pick the right solution, we look at their asymptotics as the two bulk operators are brought together. From (3.6) we see that $\lim_{x_1 \to x_2} \xi = 0$ while ${}_2F_1 \to 1$ in this limit. Recall that we expect the OPE (2.8) to scale as

$$\mathcal{W}_{\Delta_1}(x_1)\mathcal{W}_{\Delta_2}(x_2) \sim \sum_\Delta \frac{\mathcal{O}_\Delta(x_2)}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta}} + \underbrace{\cdots}_{\text{descendants}}$$

as $x_1 \to x_2$, so the function f with the correct asymptotics is that of (B.2). This is precisely (4.2), where we have inserted the constant factor $4^{\Delta/2}$ to match the normalisation of [1].

B.2 Bulk R-symmetry Block

We start from (4.3),

$$C^{(1,2)}_{SO(6)}\mathcal{H}(u_1,u_2) = \frac{1}{2}\left(L^{(1)MN} + L^{(2)MN}\right)\left(L^{(1)}_{MN} + L^{(2)}_{MN}\right)\mathcal{H}(u_1,u_2)$$

for $\mathcal{H}(u_1,u_2) \equiv \mathcal{H}^{\text{blk},[r_1,r_2,r_3]}_{\Delta_1\Delta_2}(u_1,u_2)$ as in (3.26b) and $M,N = 1,...,6$. Expanding,

$$\begin{aligned}C^{(1,2)}_{SO(6)}\mathcal{H} = \frac{1}{2}\Bigg[&u_1^M\frac{\partial}{\partial u_{1,N}}\left(u_{1,M}\frac{\partial\mathcal{H}}{\partial u_1^N} - u_{1,N}\frac{\partial\mathcal{H}}{\partial u_1^M} + u_{2,M}\frac{\partial\mathcal{H}}{\partial u_2^N} - u_{2,N}\frac{\partial\mathcal{H}}{\partial u_2^M}\right)\\ &-u_1^N\frac{\partial}{\partial u_{1,M}}\left(u_{1,M}\frac{\partial\mathcal{H}}{\partial u_1^N} - u_{1,N}\frac{\partial\mathcal{H}}{\partial u_1^M} + u_{2,M}\frac{\partial\mathcal{H}}{\partial u_2^N} - u_{2,N}\frac{\partial\mathcal{H}}{\partial u_2^M}\right)\Bigg] + (1\leftrightarrow 2).\end{aligned}$$

Distributing, using the Leibniz rule while keeping in mind that $u_i\cdot u_i = 0$ and $\partial_{u_i^M}u_i^M = 6$ for $i = 1,2$, we arrive at

$$\begin{aligned}C^{(1,2)}_{SO(6)}\mathcal{H}(u_1,u_2) = 2(u_1\cdot u_2)&\frac{\partial^2\mathcal{H}}{\partial u_1\cdot\partial u_2} - 2u_1^M u_2^N\frac{\partial^2\mathcal{H}}{\partial u_1^N\partial u_2^M}\\ &- 5u_1\cdot\frac{\partial\mathcal{H}}{\partial u_1} - u_1^M u_1^N\frac{\partial^2\mathcal{H}}{\partial u_1^N\partial u_1^M} - 5u_2\cdot\frac{\partial\mathcal{H}}{\partial u_2} - u_2^M u_2^N\frac{\partial^2\mathcal{H}}{\partial u_2^N\partial u_2^M}.\end{aligned}$$

Meanwhile, the eigenvalue of the quadratic $SO(6)$-Casimir operator acting on $\mathcal{H}$ in the R-symmetry representation with Dynkin labels $[2m,2n,2m]$ is [1, 71]

$$\tilde{C}_{[2m,2n,2m]} \equiv 2[n^2 + 2n(m+1) + m(2m+3)].$$

Ultimately we thus want to solve

$$C^{(1,2)}_{SO(6)}\mathcal{H}^{\text{blk},[2m,2n,2m]}_{\Delta_1,\Delta_2}(u_1,u_2) = \tilde{C}_{[2m,2n,2m]}\mathcal{H}^{\text{blk},[2m,2n,2m]}_{\Delta_1,\Delta_2}(u_1,u_2)$$

Carrying through the Leibniz rule on the LHS, one gets a differential equation for $\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]}(\sigma,\bar{\sigma})$. Equivalently, swapping the R-symmetry invariants $(\sigma,\bar{\sigma})$ for those of Section 3.3 this writes

$$\begin{aligned}\left[\left(\sum_{i=1}^{2} w_i(w_i-1)^2\partial^2_{w_i}\right) + k(w_1,w_2)\partial_{w_1} + k(w_2,w_1)\partial_{w_2}\right]&\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]}(w_1,w_2)\\ &= \tilde{C}_{[r_1,r_2,r_3]}\mathfrak{h}^{\text{blk}}_{[r_1,r_2,r_3]}(w_1,w_2)\end{aligned}$$

where

$$k(w_1,w_2) = \left(\frac{w_1(w_1-1)}{w_1-w_2} + \frac{w_1-1}{w_1w_2-1} - 2\right)(w_1-1).$$

Unlike the bulk conformal blocks, this does not admit a closed-form solution in general, though specific cases such as $[0,2n,0]$ or $[2m,0,2m]$ allow for such a solution [1].

B.3 Boundary Conformal Block

We start from (4.5),

$$C^{(P_2)}_{SO(4,1)}\mathcal{F}(P_2)=\frac{1}{2}\left(P_2^A\frac{\partial}{\partial P_{2,B}}-P_2^B\frac{\partial}{\partial P_{2,A}}\right)\left(P_{2,A}\frac{\partial\mathcal{F}(P_2)}{\partial P_2^B}-P_{2,B}\frac{\partial\mathcal{F}(P_2)}{\partial P_2^A}\right)$$

where $\mathcal{F}(P_2)\equiv\mathcal{F}^{\mathrm{bdy},R}_{\delta_1\delta_2}(P_1,P_2)$ as in (3.28a), seen as a function of P_2 only. This is time the Casimir operator has eigenvalue $\tilde{C}_{S0(4,1)}=-\delta(\delta-2)$ when acting on a scalar conformal multiplet of weight δ. The differential equation in embedding space writes

$$C^{(P_2)}_{SO(4,1)}\mathcal{F}(P_2)=-\delta(\delta-2)F_{\Delta_1,\Delta_2}(P_2). \tag{B.3}$$

With the help of `Mathematica` we find the LHS

$$\begin{aligned}C^{(P_2)}_{SO(4,1)}\mathcal{F}(P_2)=\frac{1}{2^{2+\Delta_1+\Delta_2}(P_1^d)^{2+\Delta_1}(P_2^d)^{2+\Delta_2}}&\Big[-6(P_1^dP_2^d)^2(1+2\xi)\partial\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)\\&-\left((1+2\xi)^2(P_1^dP_2^d)^2-(P_1^dP_2^d)^2\right)\partial^2\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)\Big]\end{aligned}$$

where we denote $P_i^d\equiv V\cdot P_i$. Cancelling $(P_1^dP_2^d)^2$ and rewriting

$$(1+2\xi)^2-1=4\xi^2+4\xi=4\xi(\xi+1)$$

this is

$$\begin{aligned}C^{(P_2)}_{SO(4,1)}\mathcal{F}(P_2)&=\frac{-6(1+2\xi)\partial\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)-4\xi(\xi+1)\partial^2\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)}{2^{2+\Delta_1+\Delta_2}(P_1^d)^{\Delta_1}(P_2^d)^{\Delta_2}}\\&=\frac{1}{2^{\Delta_1+\Delta_2}(P_1^d)^{\Delta_1}(P_2^d)^{\Delta_2}}\left(-\frac{3}{2}(1+2\xi)\partial\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)-\xi(\xi+1)\partial^2\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)\right).\end{aligned}$$

Meanwhile, the RHS is

$$-\delta(\delta-2)\mathcal{F}(P_2)=-\delta(\delta-2)\frac{\mathfrak{f}^{\mathrm{bdy}}_\delta(\xi)}{2^{\Delta_1+\Delta_2}(P_1^d)^{\Delta_1}(P_2^d)^{\Delta_2}}.$$

Equation (B.3) thus writes

$$\xi(\xi+1)\mathfrak{f}''(\xi)+\frac{3}{2}(1+2\xi)\mathfrak{f}'(\xi)-\delta(\delta-2)\mathfrak{f}(\xi)=0. \tag{B.4}$$

The Casimir is acting on P_2 so we are implicitly using the BOE for $\mathcal{W}_{\Delta_2}(x_2,u_2)$ here, assuming $x_{2,\perp}\ll 1$. From (3.6) we see that small $x_{2,\perp}$ means large ξ. It is thus convenient to define $\chi=1/\xi$ and introduce $g(\chi)=f(1/\chi)$ to carry out our expansion. The ODE (B.4) becomes

$$\chi^2(1+\chi)g''(\chi)-\frac{1}{2}\chi[-4(1+\chi)+3(2+\chi)]g'(\chi)-\delta(\delta-2)g(\chi)=0$$

in terms of these new variables, which is solved by

$$g(\chi)=\chi^{3-\delta}{}_2F_1\left(2-\delta,3-\delta,4-2\delta;-\chi\right), \tag{B.5}$$

$$g(\chi)=\chi^{\delta}{}_2F_1\left(\delta,\delta-1,2\delta-4;-\chi\right). \tag{B.6}$$

As for the bulk case, we now look at the expected scaling when $x_{2,\perp} \to 0$ to pick up the correct solution, only this time using the BOE (3.10),

$$\mathcal{W}_{\Delta_2}(\vec{x}_2, x_{2,\perp}) \sim \sum_{\hat{\mathcal{O}}} \mu_{\mathcal{O}\hat{\mathcal{O}}} \frac{\hat{\mathcal{O}}(\vec{x}_2)}{(2x_{2,\perp})^{\Delta-\delta}} + ..., \quad \hat{\mathcal{O}} \in [0]_\delta^{[2r_+,2r_-]}.$$

The correct solution is thus (B.6). Translating it back to $\mathfrak{f}_\delta^{\text{bdy}}(\xi)$, this is precisely (4.6), where we have inserted the constant factor $4^{-\delta}$ to match the normalisation of [1].

B.4 Boundary R-symmetry Block

The boundary R-symmetry is $SO(4) \cong SO(3)_+ \oplus SO(3)_-$. In this section, we first argue that we may thus solve for either of $\mathfrak{h}_{r\pm}^{\text{bdy}}(\sigma^\pm)$ by making the change of variables (4.7) and using the corresponding $SO(3)_\pm$-Casimir on $\mathcal{H}(u_2) \equiv \mathcal{H}_{\Delta_1\Delta_2}^{\text{bdy},[r_+,r_-]}(u_1, u_2)$ as in (3.28b), but now viewed as a function of u_2 only. As a matter of fact, let us redefine

$$u = v + w, \quad \bar{u} = v - w, \quad v = (\vec{v}, 0), \quad w = (0, \vec{w}),$$

to reflect the breaking of R-symmetry into $SO(3)_\pm$. Clearly, v parametrizes $SO(3)_+$ and w $SO(3)_-$. We deduce

$$\begin{gathered} u^2 = u \cdot u = |\vec{v}|^2 + |\vec{w}|^2 \overset{!}{=} 0 \quad \Rightarrow \quad |\vec{v}|^2 = -|\vec{w}|^2 \\ u_i \cdot \bar{u}_i = |\vec{v_i}|^2 - |\vec{w_i}|^2 = 2|\vec{v_i}|^2 = -2|\vec{w_i}|^2 \end{gathered}$$

and the R-symmetry invariants (3.7) may be rewritten as

$$\begin{aligned} \sigma &= \frac{u_1 \cdot u_2}{\sqrt{u_1 \cdot \bar{u}_1}\sqrt{u_2 \cdot \bar{u}_2}} = \frac{\vec{v_1} \cdot \vec{v_2} + \vec{w_1} \cdot \vec{w_2}}{2|\vec{v_1}||\vec{v_2}|} = \frac{\vec{v_1} \cdot \vec{v_2} + \vec{w_1} \cdot \vec{w_2}}{2|\vec{w_1}||\vec{w_2}|}, \\ \bar{\sigma} &= \frac{u_1 \cdot \bar{u}_2}{\sqrt{u_1 \cdot \bar{u}_1}\sqrt{u_2 \cdot \bar{u}_2}} = \frac{\vec{v_1} \cdot \vec{v_2} - \vec{w_1} \cdot \vec{w_2}}{2|\vec{v_1}||\vec{v_2}|} = \frac{\vec{v_1} \cdot \vec{v_2} - \vec{w_1} \cdot \vec{w_2}}{2|\vec{w_1}||\vec{w_2}|}. \end{aligned}$$

We can now substitute $\sigma, \bar{\sigma}$ for $\sigma^\pm$ to get

$$\sigma^+ \equiv \sigma + \bar{\sigma} = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}||\vec{v_2}|}, \quad \sigma^- \equiv \sigma - \bar{\sigma} = \frac{\vec{w_1} \cdot \vec{w_2}}{|\vec{w_1}||\vec{w_2}|}.$$

With these redefinitions, $\mathfrak{h}_{r\pm}^{\text{bdy}}(\sigma^\pm)$, we find that the R-symmetry block for $SO(3)_\pm$ has to be a function of $\sigma^\pm$ only, $\mathfrak{h}_{r\pm}^{\text{bdy}}(\sigma, \bar{\sigma}) \equiv \mathfrak{h}_{r\pm}^{\text{bdy}}(\sigma^\pm)$. When acting with, say, the Casimir of $SO(3)_+$ on $\mathcal{H}(u_2)$ we actually only care about the $\vec{v_2}$-dependence, so effectively in this case

$$\mathcal{H}(v_2) \sim |\vec{v_2}|^{\Delta_2} \mathfrak{h}_{r+}^{\text{bdy}}(\sigma^+)$$

with all the remaining terms factorizing (and eventually dropping out of the differential equation). The Casimir operator of $SO(3)_+$ is

$$C_{SO(3)_+}^{(v)} = \frac{1}{2} L^{ij} L_{ij}, \quad i, j = 1, ..., 3$$

with $L^{ij} = v^i \frac{\partial}{\partial v_j} - v^j \frac{\partial}{\partial v_i}$. Explicitly

$$
\begin{aligned}
C^{(v)}_{SO(3)_+}\mathcal{H} &= \frac{1}{2}\left(v^i \frac{\partial}{\partial v_j} - v^j \frac{\partial}{\partial v_i}\right)\left(v_i \frac{\partial \mathcal{H}}{\partial v^j} - v_j \frac{\partial \mathcal{H}}{\partial v^i}\right) \\
&= \frac{1}{2} v^i \frac{\partial}{\partial v_j}\left(v_i \frac{\partial \mathcal{H}}{\partial v^j} - v_j \frac{\partial \mathcal{H}}{\partial v^i}\right) - \frac{1}{2} v^j \frac{\partial}{\partial v_i}\left(v_i \frac{\partial \mathcal{H}}{\partial v^j} - v_j \frac{\partial \mathcal{H}}{\partial v^i}\right) \\
&= \frac{1}{2}\left(v^i \delta^j_i \frac{\partial \mathcal{H}}{\partial v^j} + v^i v_i \frac{\partial^2 \mathcal{H}}{\partial v_j \partial v^j} - 3 v^i \frac{\partial \mathcal{H}}{\partial v^i} - v^i v_j \frac{\partial^2 \mathcal{H}}{\partial v_j \partial v^i}\right) \\
&\quad - \frac{1}{2}\left(3 v^j \frac{\partial \mathcal{H}}{\partial v^j} + v^j v_i \frac{\partial^2 \mathcal{H}}{\partial v_i \partial v^j} - v^j \delta^i_j \frac{\partial \mathcal{H}}{\partial v^i} - v^j v_j \frac{\partial^2 \mathcal{H}}{\partial v_i \partial v^i}\right) \\
&= -2 v^i \frac{\partial \mathcal{H}}{\partial v^i} + v^i v_i \frac{\partial^2 \mathcal{H}}{\partial v_j \partial v^j} - v^i v_j \frac{\partial^2 \mathcal{H}}{\partial v_j \partial v^i}
\end{aligned}
$$

and its eigenvalue in the $SO(3)_\pm$ representation $[k_\pm]$ is $k_\pm(k_\pm + 1)$ [1, 71]. For convenience we use the original $SO(6)$ symmetry to pick a frame where

$$
u_1 = (1, 0, 0, 0, 0, i) \quad \Rightarrow \quad \bar{u}_1 = (1, 0, 0, 0, 0, -i).
$$

Then

$$
v_1 = \frac{u_1 + \bar{u}_1}{2} = (1, 0, 0, 0, 0, 0) \quad \Rightarrow \quad \vec{v_1} = (1, 0, 0) \in SO(3)_+
$$

and

$$
w_1 = \frac{u_1 - \bar{u}_1}{2} = (0, 0, 0, 0, 0, i) \quad \Rightarrow \quad \vec{w_1} = (0, 0, i) \in SO(3)_-.
$$

In particular,

$$
\begin{aligned}
&\sigma^+ = \frac{v_2^1}{|\vec{v_2}|} \quad \Leftrightarrow \quad (v_2^1)^2 = (\sigma^+)^2\left((v_2^1)^2 + (v_2^2)^2 + (v_2^3)^2\right) \\
&\Leftrightarrow \quad (v_2^1)^2(1 - (\sigma^+)^2) = (\sigma^+)^2\left((v_2^2)^2 + (v_2^3)^2\right) \quad \Leftrightarrow \quad v_2^1 = \sigma^+ \sqrt{\frac{(v_2^2)^2 + (v_2^3)^2}{1 - (\sigma^+)^2}}
\end{aligned}
$$

and our frame is

$$
\vec{v_1} = (1, 0, 0) \quad \text{and} \quad \vec{v_2} = \left(\sigma^+ \sqrt{\frac{(v_2^2)^2 + (v_2^3)^2}{1 - (\sigma^+)^2}}, v_2^2, v_2^3\right) \tag{B.7}
$$

In this frame,

$$
\mathcal{H}(v_2) \sim |\vec{v_2}|^{\Delta_2} \mathfrak{h}^{\text{bdy}}_{r^+}(\sigma^+)\Big|_{(\text{B.7})} = \left(\frac{(v_2^2)^2 + (v_2^3)^2}{1 - (\sigma^+)^2}\right)^{\frac{\Delta_2}{2}} \mathfrak{h}^{\text{bdy}}_{r^+}(\sigma^+)
$$

while

$$
C^{(v_2)}_{SO(3)_+}\mathcal{H}\Big|_{(\text{B.7})} = \left(\frac{(v_2^2)^2 + (v_2^3)^2}{1 - (\sigma^+)^2}\right)^{\frac{\Delta_2}{2}} \left(-2\sigma^+ \partial_{\sigma^+} \mathfrak{h}^{\text{bdy}}_{r^+} - ((\sigma^+)^2 - 1)\partial^2_{\sigma^+} \mathfrak{h}^{\text{bdy}}_{r^+}\right).
$$

Combining the two sides and inserting the eigenvalue of the Casimir in the representation $r_+ = [k]$, we end up with the differential equation

$$
k(k+1)\mathfrak{h}^{\text{bdy}}_k + 2\sigma^+ \partial_{\sigma^+} \mathfrak{h}^{\text{bdy}}_k + ((\sigma^+)^2 - 1)\partial^2_{\sigma^+} \mathfrak{h}^{\text{bdy}}_k = 0,
$$

whose solution is found to be a sum of Legendre polynomials. In the variables (w_1, w_2) of Section 3.3, this is

$$\left(w^2\partial_w^2 + \frac{2w^2}{w-w^{-1}}\partial_w\right)\mathfrak{h}_k^{\text{bdy}}(w) = k(k+1)\mathfrak{h}_k^{\text{bdy}}(w)$$

whose solution is

$$\mathfrak{h}_k^{\text{bdy}}(w) = w^{-k}{}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; w^2\right).$$

C Finding the Contributing Representations

C.1 Bulk Channel Representations

In this Section we look at the supermultiplets descending from the superprimaries of (3.23).

- Let us start with $\chi_{\text{blk}} = B_1\bar{B}_1[0;0]_{2n}^{[0,2n,0]}$. We generate the full $\frac{1}{2}$-BPS supermultiplet by acting with all the Q's and $\bar{Q}$'s on the SP $[0;0]_{2n}^{[0,2n,0]}$ and imposing the shortening conditions, keeping in mind that both Q and $\bar{Q}$ are Grassmann variables. We also argued in Section 4.2.1 that only diagonal representations should be retained. These can be found in Table 1 of Dolan and Osborn [60] and are

$$\begin{aligned}
&\text{level 0:} && [0;0]_{2n}^{[0,2n,0]}, \quad \text{(superprimary)}\\
&\text{level 1:} && [1;1]_{2n+1}^{[1,2n-2,1]},\\
&\text{level 2:} && \left\{[2;2]_{2n+2}^{[0,2n-2,0]}, \quad [0;0]_{2n+2}^{[2,2n-4,2]}\right\},\\
&\text{level 3:} && [1;1]_{2n+3}^{[1,2n-4,1]},\\
&\text{level 4:} && [0;0]_{2n+4}^{[0,2n,0]}.
\end{aligned} \tag{C.1}$$

  Further restricting our attention to scalar representations with $j, \bar{j} = 0$, we get precisely (4.8). The $\frac{1}{2}$-BPS bulk superblock of (3.25) will thus include the coefficients

$$c_{(2n,[0,2n,0])}, \quad c_{(2n+2,[2n,2n-4,2])}, \quad c_{(2n+4,[0,2n-4,0])}.$$

  Finally, we are free to rescale all coefficients w.l.o.g. by $c_{(2n,[0,2n,0])}$, leaving only two coefficients to effectively solve for later on using the Ward identities, see Section 4.3.

- Moving on to $\chi_{\text{blk}} = B_1\bar{B}_1[0;0]_{2n+4m}^{[2m,2n,2m]}$, the $\frac{1}{4}$-BPS supermultiplet is generated in the exact same way, only with different shortening conditions entering the procedure.

The scalar diagonal representations are listed in Table 2 of [60] and read

$$
\begin{aligned}
&\text{level 0:} \quad [0;0]^{[2m,2n,2m]}_{4m+2n}, \qquad \text{(superprimary)}\\
&\text{level 2:} \quad \left\{\begin{array}{c}
[0;0]^{[2m-2,2n,2m-2]}_{4m+2n+2}, [0;0]^{[2m-2,2n+2,2m-2]}_{4m+2n+2}, [0;0]^{[2m-2,2n+4,2m-2]}_{4m+2n+2},\\
2[0;0]^{[2m-1,2n,2m-1]}_{4m+2n+2}, 2[0;0]^{[2m-1,2n+2,2m-1]}_{4m+2n+2}, [0;0]^{[2m,2n-2,2m]}_{4m+2n+2},\\
3[0;0]^{[2m,2n,2m]}_{4m+2n+2}, 2[0;0]^{[2m+1,2n-2,2m+1]}_{4m+2n+2}, [0;0]^{[2m+2,2n-4,2m+2]}_{4m+2n+2}
\end{array}\right\},\\
&\text{level 4:} \quad \left\{\begin{array}{c}
[0;0]^{[2m-4,2n+4,2m-4]}_{4m+2n+4}, 2[0;0]^{[2m-3,2n+2,2m-3]}_{4m+2n+4}, 3[0;0]^{[2m-2,2n,2m-2]}_{4m+2n+4},\\
[0;0]^{[2m-2,2n+2,2m-2]}_{4m+2n+4}, 2[0;0]^{[2m-1,2n-2,2m-1]}_{4m+2n+4}, 2[0;0]^{[2m-1,2n,2m-1]}_{4m+2n+4},\\
[0;0]^{[2m,2n-4,2m]}_{4m+2n+4}, [0;0]^{[2m,2n-2,2m]}_{4m+2n+4}, [0;0]^{[2m,2n,2m]}_{4m+2n+4}
\end{array}\right\},\\
&\text{level 6:} \quad [0;0]^{[2m-2,2n,2m-2]}_{4m+2n+6}.
\end{aligned}
$$

The $\frac{1}{4}$-BPS bulk superblock of (3.25) will thus include a coefficients for each of the representations above, up to rescaling e.g. $c_{(4m+2n,[2m,2n,2m])} \equiv 1$. One must then solve for the remaining coefficients using the Ward identities, see Section 4.3.

- Finally, the long supermultiplet for $\chi_{\text{blk}} = L\bar{L}[0;0]^{[2m,2n,2\bar{m}]}_{\delta}$ is obtained in an similar fashion, though the procedure is more tedious. They can be found in [1], where a limiting relation between $\frac{1}{4}$-BPS and long blocks is also exhibited.

C.2 Working out the $B_1\bar{B}_1[0;0]^{[0,2,0]}_2$ supermultiplet with LieART

In this section, we show how to use LieART [63] to generate the full $\frac{1}{2}$-BPS supermultiplet (C.1) with $n = 1$, $B_1\bar{B}_1[0;0]^{[0,2,0]}_2$, We start by definining the superprimary $[0;0]^{[0,2,0]}_2$, which we call `PrimaryRep`.

```
In[12]:= (*Auxiliary function*)
PrintState[Rep_] := Print["[", DynkinLabel[Rep[[1]]], ";", DynkinLabel[Rep[[2]]], "]^", DynkinLabel[Rep[[3]]]]

In[13]:= ClearAll;
p = 2; Δ = p; NSUSY = 4;
(* Construct the primary state as an irrep of SU(2)⊗SU(2)⊗SU(4) (0,p,0), [0;0]_p^(0,p,0) *)
LorentzPrimary = ProductIrrep[Irrep[SU2][1], Irrep[SU2][1]]; (*Lorentz SU(2) spin labels*)
RPrimary = Irrep[A][0, p, 0]; (*R-symmetry Dynkin labels*)
PrimaryRep = ProductIrrep[LorentzPrimary, RPrimary];
Print["Level 0: Δ =", Δ]
PrintState[PrimaryRep]

Level 0: Δ =2
[(0);(0)]^(020)
```

Next, we define the supercharge $Q \in [1;0]^{[1,0,0]}_{\frac{1}{2}}$ and similarly $\bar{Q} \in [0;1]^{[0,0,1]}_{\frac{1}{2}}$.

```
In[20]:= (*Now the supercharges Q,Q̄, cf. Dolan http://arxiv.org/abs/hep-th/0209056 p.21*)
Q = ProductIrrep[Irrep[SU2][2], Irrep[SU2][1], Irrep[A][1, 0, 0]];
Qbar = ProductIrrep[Irrep[SU2][1], Irrep[SU2][2], Irrep[A][0, 0, 1]];
PrintState[Q];
PrintState[Qbar];

[(1);(0)]^(100)
[(0);(1)]^(001)
```

Then acting at level 1 with Q and $\bar{Q}$ one gets

```
In[28]:= Q ⋆ PrimaryRep
         Q ⋆ PrimaryRep // InputForm
         Print["Level 1: Δ =", Δ + 1/2]

Out[28]= (2, 1, 20) + (2, 1, 60)

         Level 1: Δ = 5/2

Out[29]//InputForm=
         IrrepPlus[ProductIrrep[Irrep[A][1], Irrep[A][0], Irrep[A][0, 1, 1]],
          ProductIrrep[Irrep[A][1], Irrep[A][0], Irrep[A][1, 2, 0]]]

In[31]:= Qbar ⋆ PrimaryRep
         Qbar ⋆ PrimaryRep // InputForm
         Print["Level 1: Δ =", Δ + 1/2]

Out[31]= (1, 2, 20bar) + (1, 2, 60bar)

         Level 1: Δ = 5/2

Out[32]//InputForm=
         IrrepPlus[ProductIrrep[Irrep[A][0], Irrep[A][1], Irrep[A][1, 1, 0]],
          ProductIrrep[Irrep[A][0], Irrep[A][1], Irrep[A][0, 2, 1]]]
```

From this we learn

$$\begin{aligned} Q: &\quad [1;0]_{\frac{1}{2}}^{[1,0,0]} \otimes [0;0]_2^{[0,2,0]} = [1;0]_{\frac{5}{2}}^{[0,1,1]} \oplus {\color{red}[1;0]_{\frac{5}{2}}^{[1,2,0]}}, \\ \bar{Q}: &\quad [0;1]_{\frac{1}{2}}^{[0,0,1]} \otimes [0;0]_2^{[0,2,0]} = [0;1]_{\frac{5}{2}}^{[1,1,0]} \oplus {\color{red}[0;1]_{\frac{5}{2}}^{[0,2,1]}}. \end{aligned}$$

The superconformal representations in red above are precisely the null states that should be removed from the shortening condition $B_1\bar{B}_1$ on the superprimary [41]. The procedure continues at level 2 and higher until we have acted with all of the Q's and $\bar{Q}$'s on $[0;0]_2^{[0,2,0]}$.

C.3 Boundary Channel Representations

In this Section we look at the boundary SMs of (3.24).

- Starting with $\chi_{\text{bdy}} = B_1[0]_k^{[2k,0]}$, we generate the full supermultiplet by acting with the Q's on the corresponding SP and imposing the relevant shortening conditions. The result is displayed in Section 4.4 of [41]. We further restrict our attention to scalar multiplets and apply the Racah-Speiser algorithm to find

$$\begin{aligned} &\text{level 0:} \quad [0]_k^{[2k,0]} \quad \text{(superprimary)} \\ &\text{level 1:} \quad [0]_{k+1}^{[2k-2,2]} \\ &\text{level 2:} \quad [0]_{k+2}^{[2k-4,0]} \end{aligned}$$

  The first $\frac{1}{2}$-BPS superblock of (3.27) will thus feature the coefficients

$$c_{(k,[k,0])}, \quad c_{(k+1,[k-1,1])}, \quad c_{(k+2,[k-2,0])}.$$

  Rescaling $c_{(k,[k,0])} \equiv 1$, there are then only two remaining coefficients to solve for using the superconformal Ward identities, see Section 4.3.

- The exact same steps can be applied to generate the second $\frac{1}{2}$-BPS supermultiplet of (3.23), when $\chi_{\text{bdy}} = B_1[0]_k^{[0,2k]}$. We find

$$\begin{aligned} &\text{level 0:} \quad [0]_k^{[0,2k]} \quad \text{(superprimary)} \\ &\text{level 1:} \quad [0]_{k+1}^{[2,2k-2]} \\ &\text{level 2:} \quad [0]_{k+2}^{[0,2k-4]} \end{aligned}$$

The second $\frac{1}{2}$-BPS superblock of (3.27) will thus feature the coefficients

$$c_{(k,[0,k])}, \quad c_{(k+1,[1,k-1])}, \quad c_{(k+2,[0,k-2])}.$$

Rescaling $c_{(k,[0,k])} \equiv 1$, there are then only two remaining coefficients to solve for using the superconformal Ward identities, see Section 4.3.

- Next, we move on to the $\frac{1}{4}$-BPS supermultiplet $\chi_{\text{bdy}} = B_1[0]_{k_++k_-}^{[2k_+,2k_-]}$. Looking again at Section 4.4 of [41] we find the following scalar representations

$$\begin{aligned} &\text{level 0:} \quad [0]_{k_++k_-}^{[2k_+,2k_-]}, \quad \text{(superprimary)} \\ &\text{level 1:} \quad [0]_{k_++k_-+1}^{\{[2k_++2,2k_--2]\oplus[2k_+,2k_--2]\}\oplus T\{...\}\oplus[2k_+,2k_-]\oplus[2k_+-2,2k_--2]}, \\ &\text{level 2:} \quad [0]_{k_++k_-+2}^{\{[2k_+,2k_--4]\oplus[2k_+,2k_--2]\}\oplus T\{...\}\oplus[2k_+,2k_-]\oplus[2k_+-2,2k_--2]}, \\ &\text{level 3:} \quad [0]_{k_++k_-+3}^{[2k_+-2,2k_--2]}. \end{aligned} \tag{C.2}$$

Applying RS, representations at level 1 are all the $[0]_{k_++k_-+1}^{[2r_+,2r_-]}$ ones with the pairs $[2r_+, 2r_-]$ belonging to

$$\begin{aligned} &\{[2k_+ + 2, 2k_- - 2] \oplus [2k_+, 2k_- - 2]\} \oplus T\{...\} \oplus [2k_+, 2k_-] \oplus [2k_+ - 2, 2k_- - 2] \\ &\quad = [2k_+ + 2, 2k_- - 2] \oplus [2k_+, 2k_- - 2] \oplus [2k_+ - 2, 2k_- + 2] \oplus [2k_+ - 2, 2k_-] \\ &\qquad \oplus [2k_+, 2k_-] \oplus [2k_+ - 2, 2k_- - 2], \end{aligned} \tag{C.3a}$$

while at level 2 these are all the $[0]_{k_++k_-+2}^{[2r_+,2r_-]}$ ones with the pairs $[2r_+, 2r_-]$ belonging to

$$\begin{aligned} &\{[2k_+, 2k_- - 4] \oplus [2k_+, 2k_- - 2]\} \oplus T\{...\} \oplus [2k_+, 2k_-] \oplus [2k_+ - 2, 2k_- - 2] \\ &\quad = [2k_+, 2k_- - 4] \oplus [2k_+, 2k_- - 2] \oplus [2k_+ - 4, 2k_-] \oplus [2k_+ - 2, 2k_-] \\ &\qquad \oplus [2k_+, 2k_-] \oplus [2k_+ - 2, 2k_- - 2]. \end{aligned} \tag{C.3b}$$

The $\frac{1}{4}$-BPS superblock of (3.27) will thus feature a coefficient $c_{(\delta,[r_+,r_-])}$ for all the representations listed in (C.2), up to rescaling the superprimary coefficient

$$c_{(k_++k_-,[k_+,k_-])} \equiv 1.$$

All the remaining coefficients can be solved for using the superconformal Ward identities, see Section 4.3. An important remark is in order: above, we highlighted in green the representations arising from the Racah-Speiser algorithm which that match those in [1], where the authors derive the contributing representations using the superconformal characters of Dolan [62]. There is no contradiction however, as one finds solving the superconformal Ward identities (see Section 4.3) that the coefficients for the black representations must all vanish.

- Finally, we look at the supermultiplet $\chi_{\text{bdy}} = L[0]_\delta^{[2k_+,2k_-]}$. We generate the the full supermultiplet by acting with the supercharges Q's on the SP $[0]_\delta^{[2k_+,2k_-]}$. The result can be found in Section 4.4 of [41]. We find

$$\begin{aligned}
&\text{level 0:} \quad [0]_\delta^{[2k_+,2k_-]}, \quad \text{(superprimary)}\\
&\text{level 1:} \quad [0]_{\delta+1}^{[2k_+\pm2,2k_-\pm2]\oplus4[2k_+,2k_-]\oplus\{2[2k_+\pm2,2k_-]\}\oplus T\{\dots\}},\\
&\text{level 2:} \quad [0]_{\delta+2}^{\{[2k_+\pm4,2k_-]\oplus4[2k_+\pm2,2k_-]\}\oplus T\{\dots\}\oplus2[2k_+\pm2,2k_-\pm2]\oplus8[2k_+,2k_-]},\\
&\text{level 3:} \quad [0]_{\delta+3}^{[2k_+\pm2,2k_-\pm2]\oplus4[2k_+,2k_-]\oplus\{2[2k_+\pm2,2k_-]\}\oplus T\{\dots\}},\\
&\text{level 4:} \quad [0]_{\delta+4}^{[2k_+,2k_-]}.
\end{aligned} \tag{C.4}$$

Applying RS at level 1, we have the representations $[0]_{\delta+1}^{[2r_+,2r_-]}$ for the pair $[2r_+, 2r_-]$ being an element of

$$\begin{aligned}
&[2k_+ \pm 2, 2k_- \pm 2] \oplus 4[2k_+, 2k_-] \oplus \{2[2k_+ \pm 2, 2k_-]\} \oplus T\{\dots\}\\
&= [2k_+ + 2, 2k_- + 2] \oplus [2k_+ - 2, 2k_- - 2] \oplus [2k_+ + 2, 2k_- - 2] \oplus [2k_+ - 2, 2k_- + 2]\\
&\qquad \oplus 4[2k_+, 2k_-] \oplus 2[2k_+ + 2, 2k_-] \oplus 2[2k_+ - 2, 2k_-]\\
&\qquad\qquad \oplus 2[2k_+, 2k_- + 2] \oplus 2[2k_+, 2k_- - 2].
\end{aligned}$$

At level 2, these are $[0]_{\delta+2}^{[2r_+,2r_-]}$ with $[2r_+, 2r_-]$ one of

$$\begin{aligned}
&\{[2k_+ \pm 4, 2k_-] \oplus 4[2k_+ \pm 2, 2k_-]\} \oplus T\{\dots\} \oplus 2[2k_+ \pm 2, 2k_- \pm 2] \oplus 8[2k_+, 2k_-]\\
&= [2k_+ \pm 4, 2k_-] \oplus 4[2k_+ \pm 2, 2k_-] \oplus [2k_+, 2k_- \pm 4] \oplus 4[2k_+, 2k_- \pm 2]\\
&\qquad \oplus 2[2k_+ \pm 2, 2k_- \pm 2] \oplus 8[2k_+, 2k_-]\\
&= [2k_+ + 4, 2k_-] \oplus [2k_+ - 4, 2k_-] \oplus 4[2k_+ + 2, 2k_-] \oplus 4[2k_+ - 2, 2k_-]\\
&\qquad \oplus [2k_+, 2k_- + 4] \oplus [2k_+, 2k_- - 4]\\
&\qquad \oplus 4[2k_+, 2k_- + 2] \oplus 4[2k_+, 2k_- - 2]\\
&\qquad \oplus 2[2k_+ + 2, 2k_- + 2] \oplus 2[2k_+ - 2, 2k_- - 2]\\
&\qquad \oplus 2[2k_+ + 2, 2k_- - 2] \oplus 2[2k_+ - 2, 2k_- + 2] \oplus 8[2k_+, 2k_-],
\end{aligned}$$

while at level 3 one gets $[0]_{\delta+3}^{[2r_+,2r_-]}$ with $[2r_+, 2r_-]$ one of

$$\begin{aligned}
&[2k_+ \pm 2, 2k_- \pm 2] \oplus 4[2k_+, 2k_-] \oplus \{2[2k_+ \pm 2, 2k_-]\} \oplus T\{\dots\}\\
&\quad = [2k_+ \pm 2, 2k_- \pm 2] \oplus 4[2k_+, 2k_-] \oplus 2[2k_+ \pm 2, 2k_-] \oplus 2[2k_+, 2k_- \pm 2]\\
&= [2k_+ + 2, 2k_- + 2] \oplus [2k_+ - 2, 2k_- - 2] \oplus [2k_+ + 2, 2k_- - 2] \oplus [2k_+ - 2, 2k_- + 2]\\
&\qquad \oplus 4[2k_+, 2k_-] \oplus 2[2k_+ + 2, 2k_-] \oplus 2[2k_+ - 2, 2k_-]\\
&\qquad\qquad \oplus 2[2k_+, 2k_- + 2] \oplus 2[2k_+, 2k_- - 2].
\end{aligned} \tag{C.5a}$$

The long superblock of (3.27) will thus feature a coefficient $c_{(\delta,[r_+,r_-])}$ for all the representations listed in (C.4), up to rescaling the superprimary coefficient $c_{(\delta,[k_+,k_-])} \equiv 1$. All the remaining coefficients can be solved for using the superconformal Ward identities, see Section 4.3. We highlighted again in green the representations arising from the Racah-Speiser algorithm which that match those in [1], where the authors derive the contributing representations using the superconformal characters of Dolan [62]. One finds again that the coefficients for the black representations must all vanish when solving the Ward identities. We show explicitly in Appendix D how to determine the boundary channel coefficients perturbatively using `Mathematica`.

D Solving the Superconformal Ward Identities

In this Section we show how to determine the boundary channel coefficients solving the Ward identities (4.9) perturbatively in `Mathematica`. We focus here on the $\frac{1}{4}$-BPS supermultiplet, but the same procedure can be straightforwardly generalized to the other SMs of (3.24) by adapting the Ansatz for the superblock and the order of truncation. We start by defining the superblocks $\mathfrak{f}_\delta^{\text{bdy}}$ and $\mathfrak{h}_k^{\text{bdy}}$ as found in Section 4.1.2. The task is computationally challenging

```
In[ ]:= Clear["Global`*"]

orderz = 4;

(*building blocks, as derived*)
ξ[z_] := (z - 1)^2/(4 z);
HyperApprox[a_, b_, c_, z_, order_] := Sum[Pochhammer[a, n] Pochhammer[b, n]/Pochhammer[c, n] * z^n/n!, {n, 0, order}];
f[z_, δ_] := (4 ξ[z])^-δ * HyperApprox[δ, δ - 1, 2 * δ - 2, -ξ[z]^-1, orderz];
h[w_, k_] := w^-k * HyperApprox[1/2, -k, 1/2 - k, w^2, 4];
```

so we also approximate the hypergeometrics ${}_2F_1$ by their series

$$
{}_2F_1(a,b,c;z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1
$$

where $(q)_n$ is the (rising) Pochhamer symbol. Recall that we are implicitly using the BOE for $\mathcal{W}_{\Delta_2}(x_2, u_2)$ in the boundary channel, hence assuming $x_{2,\perp} \ll 1$. From (3.6) we see that small $x_{2,\perp}$ means large ξ, so $\xi^{-1} \ll 1$ and the series above converges. To further simplify our task, we truncate the series to $n = 4$ from now on. Next we set up the Ansatz for the superblock `F`, adding all coefficients found in Section C.3 for the $\frac{1}{4}$-BPS supermultiplet. We also change variables from (w_+, w_-) to (w_1, w_2) using the replacement rule `subswPM`. The coefficients are all gathered in a vector `Coefs`, numbered according to their order in which their appear in (C.3a), (C.3b) and finally at level 3. The superprimary coefficient is rescaled to 1 as can be seen in the definition of `Flevel0`.
Next, we compute the Ward identity `Ward1` (4.9) for `F`. We here also define the `Assumptions` to be taken into account when simplifying expressions later on. The RHS of (4.9) being 0, we are free to multiply this expression to get rid of the factor of powers of parameters, which only render the handling of expressions `Mathematica` more difficult. We can now simplify

```
In[•]:= subswPM = {wP → (w1 * w2)^(1/2), wM → (w1/w2)^(1/2)};
Coefs = Table[Subscript[c, i][kP, kM], {i, 1, 13}];

(*F(z,w1,w2), from my notes*)
δ = kP + kM;

Flevel0 = h[wP, kP] × h[wM, kM] × f[z, δ];

Flevel1 = (Coefs[[1]] × h[wP, kP + 1] × h[wM, kM - 1] +
     Coefs[[2]] × h[wP, kP] × h[wM, kM - 1] +
     Coefs[[3]] × h[wP, kP - 1] × h[wM, kM + 1] +
     Coefs[[4]] × h[wP, kP - 1] × h[wM, kM] +
     Coefs[[5]] × h[wP, kP] × h[wM, kM] +
     Coefs[[6]] × h[wP, kP - 1] × h[wM, kM - 1]) f[z, δ + 1];

Flevel2 = (Coefs[[7]] × h[wP, kP] × h[wM, kM - 2] +
     Coefs[[8]] × h[wP, kP] × h[wM, kM - 1] +
     Coefs[[9]] × h[wP, kP - 2] × h[wM, kM] +
     Coefs[[10]] × h[wP, kP - 1] × h[wM, kM] +
     Coefs[[11]] × h[wP, kP] × h[wM, kM] +
     Coefs[[12]] × h[wP, kP - 1] × h[wM, kM - 1]) f[z, δ + 2];

Flevel3 = Coefs[[13]] × h[wP, kP - 1] × h[wM, kM - 1] × f[z, δ + 3];

F = Flevel0 + Flevel1 + Flevel2 + Flevel3 /. subswPM;
```

```
In[•]:= $Assumptions = {z > 1 && w1 > 0 && w2 > 0 && wP > 0 && wM > 0 && kP > 2 && kM > 2};
Ward1 = (D[F, w1] + 1/2 D[F, z]) /. {w1 → z};
(*Ward2 =( D[F,w2]+ 1/2 D[F,z])/.{w2→z};*)
Clear[F]

In[•]:= Ward1
```

Out[•]=

$$\left(\frac{(-1+z)^2}{z}\right)^{-kM-kP}\left(\frac{z}{w2}\right)^{-kM/2}(w2\,z)^{-kP/2}\left(-\frac{kP\,w2}{2\left(\frac{1}{2}-kP\right)}-\frac{3\,(1-kP)\,kP\,w2^2\,z}{4\left(\frac{1}{2}-kP\right)\left(\frac{3}{2}-kP\right)}-\frac{15\,(1-kP)\,(2-kP)\,kP\,w2^3\,z^2}{16\left(\frac{1}{2}-kP\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)}-\frac{35\,(1-kP)\,(2-kP)\,(3-kP)\,kP\,w2^4\,z^3}{32\left(\frac{1}{2}-kP\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)}\right)$$

$$\left(1-\frac{kM\,z}{2\left(\frac{1}{2}-kM\right)w2}-\frac{3\,(1-kM)\,kM\,z^2}{8\left(\frac{1}{2}-kM\right)\left(\frac{3}{2}-kM\right)w2^2}-\frac{5\,(1-kM)\,(2-kM)\,kM\,z^3}{16\left(\frac{1}{2}-kM\right)\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)w2^3}-\frac{35\,(1-kM)\,(2-kM)\,(3-kM)\,kM\,z^4}{128\left(\frac{1}{2}-kM\right)\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)w2^4}\right)$$

$$\left(1-\frac{4\,(-1+kM+kP)\,(kM+kP)\,z}{(-2+2\,(kM+kP))\,(-1+z)^2}+\frac{8\,(-1+kM+kP)\,(kM+kP)^2\,(1+kM+kP)\,z^2}{(-2+2\,(kM+kP))\,(-1+2\,(kM+kP))\,(-1+z)^4}-\frac{16\,(-1+kM+kP)\,(kM+kP)\,(1+kM+kP)^2\,(2+kM+kP)\,z^3}{3\,(-2+2\,(kM+kP))\,(-1+2\,(kM+kP))\,(-1+z)^6}+\frac{16\,(-1+kM+kP)\,(kM+kP)\,(1+kM+kP)^2\,(2+kM+kP)^2\,(3+kM+kP)\,z^4}{3\,(-2+2\,(kM+kP))\,(-1+2\,(kM+kP))\,(1+2\,(kM+kP))\,(-1+z)^8}\right)+$$

$$\cdots 11 \cdots + \frac{1}{2}\left(\cdots 1 \cdots\right)$$

Full expression not available (original memory size: 1 MB)

```
In[•]:= (*RHS of Ward is zero so can multiply to get rid of powers of parameters*)
newWard1 = (Ward1 * ((-1 + z)^2/z)^(+kM+kP) (z/w2)^(+kM/2) (w2 z)^(+kP/2)) // PowerExpand // Expand
Clear[Ward1]
```

Out[•]=

$$\frac{2\,kM^2\,(-1+z)^{-2+2\,(-1-kM-kP)+2\,(kM+kP)}}{-2+2\,(kM+kP)}-\frac{2\,kM^3\,(-1+z)^{-2+2\,(-1-kM-kP)+2\,(kM+kP)}}{-2+2\,(kM+kP)}+\cdots 406734\cdots+$$

$$\frac{6125\,kM^3\,kP^{12}\,w2^{\frac{1}{2}(-1+kM)-\frac{kM}{2}+\frac{1-kP}{2}+\frac{kP}{2}}\,(-1+z)^{-8+2\,(-4-kM-kP)+2\,(kM+kP)}\,z^{16+\frac{1-kM}{2}+\frac{kM}{2}+\frac{1-kP}{2}+\frac{kP}{2}}\,c_{13}[kP,kM]}{3072\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2\,(3+kM+kP))\,(-1+2\,(3+kM+kP))\,(1+2\,(3+kM+kP))}-$$

$$\frac{1225\,kM^4\,kP^{12}\,w2^{\frac{1}{2}(-1+kM)-\frac{kM}{2}+\frac{1-kP}{2}+\frac{kP}{2}}\,(-1+z)^{-8+2\,(-4-kM-kP)+2\,(kM+kP)}\,z^{16+\frac{1-kM}{2}+\frac{kM}{2}+\frac{1-kP}{2}+\frac{kP}{2}}\,c_{13}[kP,kM]}{6144\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2\,(3+kM+kP))\,(-1+2\,(3+kM+kP))\,(1+2\,(3+kM+kP))}$$

Full expression not available (original memory size: 1.1 GB)

the powers of parameters appearing in `newWard1` to obtain `simplifiedWard1`. This is done using parallel programming.

Moving on, we further simplify our expressions by taking a series expansion in z: this yields `simplifiedWard1Series`. One should keep in mind that $\xi \gg 1$ corresponds to $z \ll 1$ so this approach is consistent with the fact that we are in the boundary channel. One need to be careful with the order of truncation of this series expansion here, which should match the order of truncation of the series expansion of the hypergeometric functions above.

```
In[•]:= (*COMPUTE: simplify the powers*)
simplifiedWard1 = Total[ParallelMap[# /. Power[z_, exp_] :> z^Simplify[exp] &, List @@ newWard1]]
Export["C:/Users/kervy/Desktop/TMP/Erasmus/Charlotte Kristjansen/exprQuarterExtraSimplified.m", simplifiedWard1];
Clear[newWard1]
```

Out[•]=

$$-\frac{kM}{2\left(\frac{1}{2}-kM\right)w2}+\frac{kM^2}{4\left(\frac{1}{2}-kM\right)w2}+\frac{kM\,kP}{4\left(\frac{1}{2}-kM\right)w2}-\frac{kP\,w2}{2\left(\frac{1}{2}-kP\right)}+\frac{kM\,kP\,w2}{4\left(\frac{1}{2}-kP\right)}+\cdots 406727 \cdots+$$

$$\frac{30625\,kM\,kP^{12}\,z^{17}\,c_{13}[kP,kM]}{3072\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2(3+kM+kP))(-1+2(3+kM+kP))(1+2(3+kM+kP))(-1+z)^{16}}-$$

$$\frac{42875\,kM^2\,kP^{12}\,z^{17}\,c_{13}[kP,kM]}{6144\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2(3+kM+kP))(-1+2(3+kM+kP))(1+2(3+kM+kP))(-1+z)^{16}}+$$

$$\frac{6125\,kM^3\,kP^{12}\,z^{17}\,c_{13}[kP,kM]}{3072\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2(3+kM+kP))(-1+2(3+kM+kP))(1+2(3+kM+kP))(-1+z)^{16}}-$$

$$\frac{1225\,kM^4\,kP^{12}\,z^{17}\,c_{13}[kP,kM]}{6144\left(\frac{3}{2}-kM\right)\left(\frac{5}{2}-kM\right)\left(\frac{7}{2}-kM\right)\left(\frac{9}{2}-kM\right)\left(\frac{3}{2}-kP\right)\left(\frac{5}{2}-kP\right)\left(\frac{7}{2}-kP\right)\left(\frac{9}{2}-kP\right)(-2+2(3+kM+kP))(-1+2(3+kM+kP))(1+2(3+kM+kP))(-1+z)^{16}}$$

Full expression not available (original memory size: 0.7 GB)

```
In[•]:= (*list of powers of w2*)
powersw2 = Union[Exponent[#, w2] & /@ List @@ simplifiedWard1]
```

Out[•]= $\left\{-5,-\frac{9}{2},-4,-\frac{7}{2},-3,-\frac{5}{2},-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4,\frac{9}{2},5\right\}$

```
In[•]:= (*series in z, careful with the order*)
simplifiedWard1Series = Total[ParallelMap[Normal[Series[#, {z, 0, orderz}]] &, List @@ simplifiedWard1]]
Export["C:/Users/kervy/Desktop/TMP/Erasmus/Charlotte Kristjansen/exprQuarterExtraSimplifiedSeries.m", simplifiedWard1Series];
```

Out[•]=

$$kM+kP+\frac{kM}{-1+kM+kP}-\frac{kM^2}{-1+kM+kP}+\frac{kP}{-1+kM+kP}-\frac{2\,kM\,kP}{-1+kM+kP}-\frac{kP^2}{-1+kM+kP}-\frac{kM}{2\left(\frac{1}{2}-kM\right)w2}+\frac{kM^2}{4\left(\frac{1}{2}-kM\right)w2}+\frac{kM^2}{2(-1+2\,kM)\,w2}+\frac{kM\,kP}{4\left(\frac{1}{2}-kM\right)w2}+\frac{kM\,kP}{2(-1+2\,kM)\,w2}-\frac{kP\,w2}{2\left(\frac{1}{2}-kP\right)}+$$

$$\frac{kM\,kP\,w2}{4\left(\frac{1}{2}-kP\right)}+\frac{kP^2\,w2}{4\left(\frac{1}{2}-kP\right)}+\frac{kM\,kP\,w2}{2(-1+2\,kP)}+\frac{kP^2\,w2}{2(-1+2\,kP)}+kM\,z+kP\,z+\cdots 6037 \cdots+\frac{12\,w2^2\,z^4\,c_{12}[kP,kM]}{(-5+2\,kP)(-3+2\,kP)}-\frac{18\,kP\,w2^2\,z^4\,c_{12}[kP,kM]}{(-5+2\,kP)(-3+2\,kP)}+\frac{6\,kP^2\,w2^2\,z^4\,c_{12}[kP,kM]}{(-5+2\,kP)(-3+2\,kP)}+$$

$$\frac{5}{2}z^3\,c_{13}[kP,kM]+18\,z^4\,c_{13}[kP,kM]+kM\,z^4\,c_{13}[kP,kM]+kP\,z^4\,c_{13}[kP,kM]-\frac{36\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-\frac{30\,kM\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-\frac{6\,kM^2\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-$$

$$\frac{30\,kP\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-\frac{12\,kM\,kP\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-\frac{6\,kP^2\,z^4\,c_{13}[kP,kM]}{2+kM+kP}-\frac{7\,z^4\,c_{13}[kP,kM]}{2(-3+2\,kM)\,w2}+\frac{7\,kM\,z^4\,c_{13}[kP,kM]}{2(-3+2\,kM)\,w2}-\frac{7\,w2\,z^4\,c_{13}[kP,kM]}{2(-3+2\,kP)}+\frac{7\,kP\,w2\,z^4\,c_{13}[kP,kM]}{2(-3+2\,kP)}$$

Full expression not available (original memory size: 3.9 MB)

Intuitively, this is because

$$\xi=\frac{(z-1)^2}{4z}\sim\frac{1}{z}\quad\text{as } z\to 0.$$

One is now finally in a position to solve the WI order by order in z and w_2.

```
In[•]:= (*list of coefs powers of z*)
powersz = Union[Exponent[#, z] & /@ List @@ simplifiedWard1Series]
```

Out[•]= $\left\{0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4\right\}$

```
In[•]:= (*then look at each power of w2 in each power of z*)
Clear[finalSol]
finalSol = {};
SolCoefs = Coefs
```

Out[•]= $\{c_1[kP,kM],c_2[kP,kM],c_3[kP,kM],c_4[kP,kM],c_5[kP,kM],c_6[kP,kM],c_7[kP,kM],c_8[kP,kM],c_9[kP,kM],c_{10}[kP,kM],c_{11}[kP,kM],c_{12}[kP,kM],c_{13}[kP,kM]\}$

Starting at z^0, we pick up the relevant coefficient `eqcoef` in `simplifiedWard1Series`. We then pick up the coefficients of all powers of w_2 in `eqcoef`. Requiring that they all vanish yields a set of equations, `eqsz`. It only remains to solve these for the `Coefs`. Here we find

$$c_1\equiv c_{(k_++k_-+1,[k_++1,k_--1])}=-\frac{2k_-}{2k_--1},$$

$$c_3\equiv c_{(k_++k_-+1,[k_+-1,k_-+1])}=-\frac{2k_+}{2k_+-1},$$

$$c_5\equiv c_{(k_++k_-+1,[k_+,k_-])}=0.$$

```
In[•]:= (*start with z^0*)
        eqcoef = Coefficient[simplifiedWard1Series, z, 0];
        eqsz = {};
        eqsz = Append[eqsz, # == 0] & /@ (Coefficient[eqcoef, w2, #] & /@ powersw2) // FullSimplify // Flatten;
        sol = Solve[eqsz, Coefs] // Flatten
        (*finalSol=Union[finalSol,sol]//FullSimplify*)
        Clear[eqcoef, eqsz]

        ••• Solve: Equations may not give solutions for all "solve" variables.
```

$$\text{Out[•]=}\ \left\{c_1[\text{kP}, \text{kM}] \to -\frac{2\,\text{kM}}{-1+2\,\text{kM}},\ c_3[\text{kP}, \text{kM}] \to -\frac{2\,\text{kP}}{-1+2\,\text{kP}},\ c_5[\text{kP}, \text{kM}] \to 0\right\}$$

```
In[•]:= SolCoefs = SolCoefs /. sol
        Clear[sol]
```

$$\text{Out[•]=}\ \left\{-\frac{2\,\text{kM}}{-1+2\,\text{kM}},\ c_2[\text{kP}, \text{kM}],\ -\frac{2\,\text{kP}}{-1+2\,\text{kP}},\ c_4[\text{kP}, \text{kM}],\ 0,\ c_6[\text{kP}, \text{kM}],\ c_7[\text{kP}, \text{kM}],\ c_8[\text{kP}, \text{kM}],\ c_9[\text{kP}, \text{kM}],\ c_{10}[\text{kP}, \text{kM}],\ c_{11}[\text{kP}, \text{kM}],\ c_{12}[\text{kP}, \text{kM}],\ c_{13}[\text{kP}, \text{kM}]\right\}$$

Note that c_5 corresponds to one of the black representations in (C.3a) which arise from RS but not from using superconformal characters (see discussion of Section 4.2) and it is thus expected that it vanishes.

Repeating the same steps at the next order in z, we find that $c_2 = c_4 = 0$, which was also expected. The procedure continues until all coefficients are fully fixed.

```
In[•]:= (*continue with z^(1/2)*)
        eqcoef = Coefficient[simplifiedWard1Series, z, 1/2];
        eqsz = {};
        eqsz = Append[eqsz, # == 0] & /@ (Coefficient[eqcoef, w2, #] & /@ powersw2) // FullSimplify // Flatten;
        sol = Solve[eqsz, Coefs] // Flatten
        (*finalSol=Union[finalSol,sol]//FullSimplify*)
        Clear[eqcoef, eqsz]

        ••• Solve: Equations may not give solutions for all "solve" variables.

Out[•]= {c2[kP, kM] -> 0, c4[kP, kM] -> 0}

In[•]:= SolCoefs = SolCoefs /. sol
        Clear[sol]
```

$$\text{Out[•]=}\ \left\{-\frac{2\,\text{kM}}{-1+2\,\text{kM}},\ 0,\ -\frac{2\,\text{kP}}{-1+2\,\text{kP}},\ 0,\ 0,\ c_6[\text{kP}, \text{kM}],\ c_7[\text{kP}, \text{kM}],\ c_8[\text{kP}, \text{kM}],\ c_9[\text{kP}, \text{kM}],\ c_{10}[\text{kP}, \text{kM}],\ c_{11}[\text{kP}, \text{kM}],\ c_{12}[\text{kP}, \text{kM}],\ c_{13}[\text{kP}, \text{kM}]\right\}$$

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