Calculus I Recitation

Yuanxing Cheng

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The recitation is based on *Calculus, 9th edition, Stewart.*

1 Functions and Limits

1.1 Four Ways to Represent a Function

function, it's domain and range

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function $y = f(x)$ and X, Y are two sets. $x \in X$ and $y \in Y$. A function is a rule that assigns each $x \in X$ to exactly one element $y = f(x) \in Y$. Domain of a function X is the set of inputs accepted by the function so that the function makes sense.

- age, height are positive by default
- 1/0 doesn't make sense

range can be obtained after you have the domain $Y = \{y = f(x) \,|\, x \in X\}$.

piecewise defined function

Divide domain of f into subsets $X_1, X_2, \ldots$ without intersection, so $X_i \cap X_j = \emptyset$. Then,

$$f(x) = \begin{cases} f_1(x), & x \in X_1 \\ f_2(x), & x \in X_2 \\ \vdots \end{cases}$$

even function and odd function

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function $f(x)$ is even if $f(x) = f(-x)$ and is odd if $f(x) = -f(x)$, for any x in its domain.

increasing and decreasing

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function $f(x)$ is increasing on an interval I if $f(x_1) \geq f(x_2)$, for any $x_1, x_2 \in I$, $x_1 > x_2$.
function $f(x)$ is decreasing on an interval I if $f(x_1) \leq f(x_2)$, for any $x_1, x_2 \in I$, $x_1 > x_2$.

1.2 Mathematical Models

mathematical model

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math description of a phenomenon, usually by equations. For example someone's age vs year.

year (at Jan. 1st)	age
2020	1
2021	2
2022	3

linear function: y is a linear function of x.

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when changes of y is proportion to the changes of x. For an arbitrary point belong to this relationship, (x, y) and a fixed point (x_1, y_1) also belong to this relationship,

$$y - y_1 = m(x - x_1), k \neq 0$$
$$y = mx - mx_1 + y_1 = mx + b$$

we call m the slope and b the y-intercept.

polynomial function: $P(x)$

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, n is nonnegative integer, $a_n \neq 0$ and its degree is n. A linear function is a polynomial of degree 1, and a quadratic function is a polynomial of degree 2, cubic for 3.

power function

$f(x) = x^a$. If a is an integer n,

- if n is even, then $f(x)$ is even for $f(x) = f(-x)$
- if n is odd, then $f(x)$ is odd for $f(x) = -f(-x)$

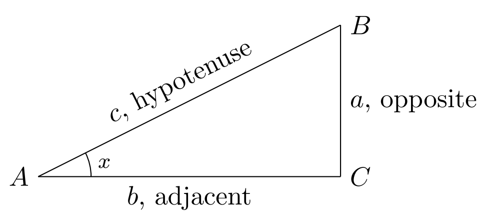
If $a = 1/n$ with n a positive integer, $f(x) = x^a$ is a root function. If $a = -1$, $f(x) = x^{-1}$ is a reciprocal function.

rational function

$f(x) = P(x)/Q(x)$, where P and Q are two polynomial functions.

Trigonometric function

Given a right triangle, an angle x in radian measure. For this angle x, define the opposite side to be the side opposite to x, with length a, define the adjacent side to be the side between x and the right angle, with length b, and the hypotenuse side to be the side opposite to the right angle, with length c, then
$\sin(x) = a/c$, $\cos(x) = b/c$, and $\tan(x) = a/b$; then $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ and $\cot(x) = 1/\tan(x)$.



period function

if $f(x) = f(x + T)$, where T is a constant.

algebraic function, and transcendental if not

functions constructed using addition, subtraction, multiplication, division, raising to a whole number power, and taking roots. And some transcendental functions: $\sin(x)$, $\log(x)$, e^x.

Exponential and Logarithmic function

Exponential function has the form $y = f(x) = b^x$, and logarithmic function has the form $y = f(x) = \log_b x = \log x / \log b$

1.3 New Functions from Old Functions

shifting a function $f(x)$

new function $g(x) = f(x-h) + v$, has the plot same as shift the plot of $f(x)$ v units to the right vertically and h units upward horizontally. Notice when $h < 0$, shifting h units to the right equals to shifting $|h|$ units to the left.

stretching and reflecting a function $f(x)$

new function $g(x) = v \times f(x/h)$, has the plot same as stretch the plot of $f(x)$ by a factor of v vertically and h units horizontally.
If $v < 0$, we do a reflection about the line $y = 0$ (x-axis) first then stretch by a factor of $|v|$ vertically, and if $h < 0$, we do a reflection about the line $x = 0$ (y-axis) first then stretch by a factor of $|h|$ horizontally.
if $|v| < 1$ or $|h| < 1$, it's a shrinking operation, not stretching.

combination of functions $f(x)$ and $g(x)$

$f + g$ sum, $f - g$ difference, fg product, and f/g quotient. The domain of the new function is the intersection of domains of f and g.

composition of functions $f(x)$ and $g(x)$

$(f \circ g)(x) = f(g(x))$. To make sure x is accepted, first $g(x)$ need to make sense, then exclude those x so that if letting $z = g(x)$, $f(z)$ make sense.

1. Let $z = g(x)$ so that $f \circ g(x) = f(z)$
2. find domain of $f(z)$, say set/interval Z
3. simplify $z = g(x) \in Z$ and obtain $x \in X_1$
4. find domain of $g(x)$, X_2
5. the domain of $f \circ g(x)$ is then $X_1 \cap X_2$

1.4 The Tangent and Velocity Problems

secant line and tangent line

With a given curve C, a secant line is a line passing through two points of a curve. In most cases, as one point is brought towards the other, the secant line tends to be the tangent line at the other point.
The slope of the tangent line is the limit of the slopes of the secant lines.

difference quotient

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

1.5 The Limit of a Function

the limit of $f(x)$ as x approaches a

Suppose $f(x)$ is defined when x is near the number a, Then we write

$$\lim_{x \to a} f(x) = L$$

or $f(x) \to L$ as $x \to a$, if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on either side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, 0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$.

left-side limits

Suppose $f(x)$ is defined when x is near the number a, and also $x < a$. Then we write

$$\lim_{x \to a^-} f(x) = L$$

or $f(x) \to L$ as $x \to a^-$, if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on left side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, 0 < a - x < \delta$ we have $|f(x) - L| < \epsilon$.
Right-side limit is defined in a similar way.
And we have $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

infinite limits

In this case the limit doesn't exist.
Suppose $f(x)$ is defined when x is near the number a, Then we write

$$\lim_{x \to a} f(x) = \infty$$

or $f(x) \to \infty$ as $x \to a$, if we can make the values of $f(x)$ arbitrarily large by restricting x to be sufficiently close to a (on either side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall M > 0, \exists \delta > 0$ s.t. $\forall x, 0 < |x - a| < \delta$ we have $f(x) > M$.
Negative infinite limits, one-sided infinite limits can be defined similarly.
And then we call $x = a$ the vertical asymptote of the curve $f(x)$.

1.6 Calculating Limits using the Limit Laws

limit laws

Following operation is interchangeable with finding the limits.

- summation
- difference
- scalar multiplication
- product
- quotient (excluding the case where the denominator has limit 0)
- power to n or $1/n$, n is any positive integer

direct substitution property

if the function limit at a is equal to the function value at a. And we call the function is continuous at a if this property hold.

the Squeeze theorem

Lemma 1.1. *if $f(x) \leq g(x)$ when x is near a, and the limits of f and g exist at a, then* $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Theorem 1.2. *With this, we can show that if $f \leq g \leq h$ when x is near a, and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, *then* $\lim_{x\to a} g(x) = L$.

1.7 The Precise Definition of a Limit

see the $\epsilon - \delta$ language part in subsection 1.5.

1.8 Continuity

point continuity

A function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$

type of discontinuity at a point $x = a$

- removable: when we can define a new value for $f(a)$ so that $f(x)$ can regain continuity at $x = a$. (a pothole)
- jump: when the left limit and the right limit exist but they are not equal (stairs)
- infinite: when the left limit or the right limit doesn't exist. ($1/x$ and $\sin(1/x)$)

left continuity and interval continuity

A function f is continuous from the right at a point $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.
A function f is continuous on an interval if it is continuous at every number in the interval.

Properties of continuous function

Theorem 1.3. *If f and g are both continuous at a, then their combination $f \pm g$, fg, f/g where $g(a) \neq 0$, are continuous at a.*
If g is continuous at a and f is continuous at $g(a)$, then the composition $f(g(x))$ is continuous at a.

Intermediate Value Theorem

Theorem 1.4. *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

meaning a continuous function takes on every intermediate value between the function values at two ends of the interval.

1.9 Exercises

Define $y(x) = \frac{x^2-x}{2(x-2)}$ and $g(x) = \frac{x}{2}$. State the difference between them and plot them.

Given that $y = x^2$, is y a function of x? is x a function of y? If not, add a restriction to make it/them function(s).

are the following functions odd, even, or neither. x^2 where $x > 0$, $\tan(x + \pi/4)$, x^{20}, x^{-20}, e^{x^3}, x^x where x is a integer.

transform the function $f(x) = \frac{1}{2x} + 2$ to $g(x) = -\frac{1}{x-2}$

find difference quotient of $x^2 + 2x + 3$ and $\frac{2}{x}$

plot function $y = \operatorname{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1. & x < 0 \end{cases}$ and function $z = g(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$. And then find the limit, left limit and right limit of $f(x)$ and $g(x)$ at $x = 0$.

An object is moving straightforward. The location x and time t has an approximated relation $x = 2t^3 + t^2 + 5t$. What's the average speed from $t = 1$ to $t = 3$?

For the previous question, first simplify the difference quotient of the distance function $\frac{x(t+\Delta t) - x(t)}{\Delta t}$ at $t = 1$. What's the limit when Δt goes to 0? Also what's the limit of $x(t)$ at $t = 1$? Use words to explain their difference.

Use $\epsilon - \delta$ language to show that $\lim_{x\to 1} 1/x = 1$ and $\lim_{x\to 2} x^3 = 8$.

Use $\epsilon - \delta$ language to prove $x\sin(1/x)$ has limit 0 at $x = 0$ and $\sin(1/x)$ doesn't have limit at $x = 0$.

Use $\epsilon - \delta$ language to prove that the limit at point $x = a$ doesn't exist when the left limit doesn't equal to the right limit.

Use $\epsilon - \delta$ language to prove that function of the form kx^n is continuous everywhere, where $k \neq 0$ and n is an integer.

Use $\epsilon - \delta$ language to prove the properties of the continuous function Theorem 1.3.

Show that there's a solution of the equation $x^4 - 5 = 0$ between 0 and 2.
