Trigonometric Functions Review

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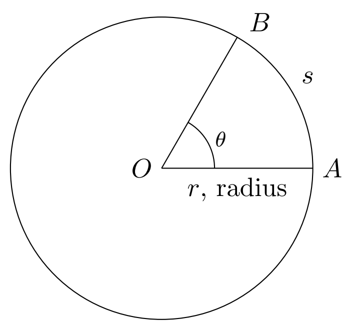
1 Basics

radian and degree

They are the unit of angle. And the standard unit is radian.

- 1 radian is defined as the angle subtended from the center of a circle which intercepts an arc equal in length to the radius of the circle.
- 360 degree is one complete revolution.

Using radian measure, we have the relation, $s = \theta r$.

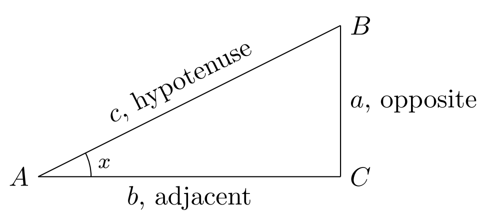


π

π is defined as the ratio of a circle's circumference C to its diameter d. So $C = \pi d = 2\pi r$. A right angle is $\pi/2 = 90$ degree.

trigonometric function

Given a right triangle, an angle x in radian measure. For this angle x, define the opposite side to be the side opposite to x, with length a, define the adjacent side to be the side between x and the right angle, with length b, and the hypotenuse side to be the side opposite to the right angle, with length c, then
$\sin(x) = a/c$, $\cos(x) = b/c$, and $\tan(x) = a/b$; then $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ and $\cot(x) = 1/\tan(x)$.

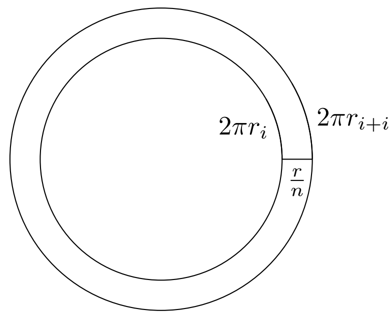


2 Conclusions

area of circle

Using limit and squeeze theorem, we separate the area of a circle into n number of annulus. Now the area of circle with radius r, A satisfies the following:

$$\sum_{i=0}^{n-1} \frac{r}{n}\left(2\pi\frac{r(i+1)}{n}\right) > A > \sum_{i=0}^{n-1} \frac{r}{n}\left(2\pi\frac{ri}{n}\right)$$



Above inequality is reduced to

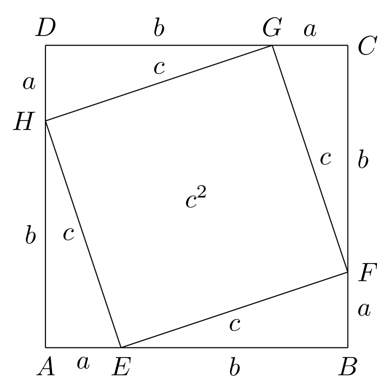
$$\pi r^2\frac{n+1}{n} > A > \pi r^2\frac{n-1}{n}$$

Then by squeeze theorem (for series not functions), $\lim_{n\to\infty} A = A = \pi r^2$.
Similar idea can be used to find the area of a sector. For a sector of radius r and angle θ, its area is $\frac{1}{2}\theta r^2$. A linear function of θ.
We can also plug in the formula for arc length $s = \theta r$ and above becomes $\frac{1}{2}\theta s$, which is quite similar to the one for triangle.

Pythagorean Theorem

Theorem 2.1. *In any right triangle with sides of lengths a and b, and hypotenuse of length c, we have $a^2 + b^2 = c^2$.*

Proof. We can put four triangles of the same size into the following shape.

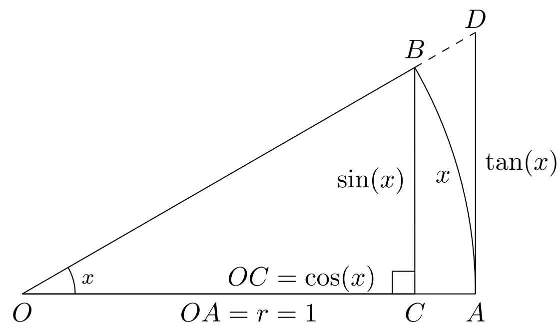


□

Proposition 2.2. $\sin^2(x) + \cos^2(x) = 1$

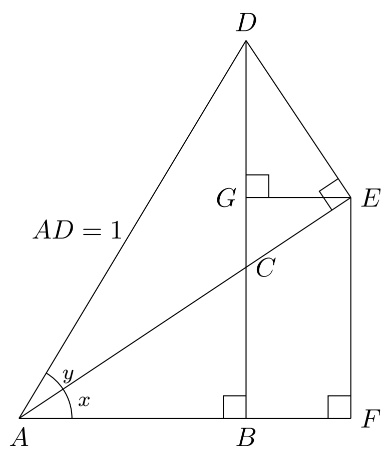
some simple relations

- for any triangle with sides of length a, b, and c: $a+b>c$, $b+c>a$, and $c+a>b$
- when x is close to 0, we have $\sin(x) < x < \tan(x)$



- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\sin(x+2\pi) = \sin(x)$, $\cos(x+2\pi) = \cos(x)$
- $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$, $\tan(-x) = -\tan(x)$
- $\sin(x+\pi) = -\sin(x)$, $\cos(x+\pi) = -\cos(x)$
- $\sin(x+\pi/2) = \cos(x)$, $\cos(x+\pi/2) = -\sin(x)$

$\sin(x+y)$ and $\cos(x+y)$



From above figure, we can write the following:

$$
\begin{aligned}
&DB = \sin(x+y) = DG + GB = DG + EF \\
&AB = \cos(x+y) = AF - AB = AF - EG \\
&\angle CDE = \angle CAB = x \\
&DE = AD\sin(y) = \sin(y), DG = DE\cos(x) = \sin(y)\cos(x), EG = DE\sin(x) = \sin(y)\sin(x) \\
&AE = AD\cos(y) = \cos(y), EF = AE\sin(x) = \cos(y)\sin(x), AF = AE\cos(x) = \cos(y)\cos(x) \\
&\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \\
&\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)
\end{aligned}
$$

other formulas

$$
\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)} \\
&= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\
\sin(x-y) &= \sin(x+(-y)) = \sin(x)\cos(y)-\cos(x)\sin(y) \\
\cos(x-y) &= \cos(x+(-y)) = \cos(x)\cos(y)+\sin(x)\sin(y) \\
\tan(x-y) &= \tan(x+(-y)) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \\
\sin(2x) &= \sin(x+x) = 2\sin(x)\cos(x) = \frac{2\sin(x)\cos(x)}{\sin^2(x)+\cos^2(x)} = \frac{2\tan(x)}{1+\tan^2(x)} \\
\cos(2x) &= \cos(x+x) = \cos^2(x)-\sin^2(x) = \frac{\cos^2(x)-\sin^2(x)}{\sin^2(x)+\cos^2(x)} = \frac{1-\tan^2(x)}{1+\tan^2(x)} \\
\tan(2x) &= \tan(x+x) = \frac{2\tan(x)}{1-\tan^2(x)}
\end{aligned}
$$

half-angle formulas

use the identity:

$$
\begin{aligned}
\cos(2x) &= 2\cos^2(x)-1 = 1-2\sin^2(x) \\
\cos(x) &= 2\cos^2(x/2)-1 = 1-2\sin^2(x/2)
\end{aligned}
$$

then solve this quadratic equation we have

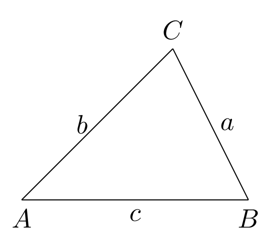
$$
\begin{aligned}
\sin(x/2) &= \sqrt{\frac{\cos(x)+1}{2}} \text{ or } -\sqrt{\frac{\cos(x)+1}{2}} \\
\cos(x/2) &= \sqrt{\frac{1-\cos(x)}{2}} \text{ or } -\sqrt{\frac{1-\cos(x)}{2}}
\end{aligned}
$$

where the sign can be determined using the quadrant of the half-angle.

Law of sines

Theorem 2.3. *For any triangle $\triangle ABC$ with sides of length a, b, c, we have the following:*

$$
\frac{a}{\sin(\angle A)} = \frac{c}{\sin(\angle C)} = \frac{b}{\sin(\angle B)}
$$



The proof uses the fact that the area of this triangle $S = \frac{1}{2}c \cdot b \sin\angle A = \frac{1}{2}c \cdot a \sin\angle B$

Law of cosines

Theorem 2.4. *For any triangle $\triangle ABC$ with sides of length a, b, c, we have the following:*

$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$
$$a^2 = b^2 + c^2 - 2bc\cos\angle A$$
$$b^2 = c^2 + a^2 - 2ca\cos\angle B$$

Proof. Notice that $c = a\cos\angle B + b\cos\angle A$, we multiply this by c and obtain

$$c^2 = ac\cos\angle B + bc\cos\angle A$$

Do this for all three sides and we have

$$a^2 = ac\cos\angle B + ab\cos\angle C$$
$$b^2 = bc\cos\angle A + ba\cos\angle C$$
$$c^2 = ca\cos\angle B + cb\cos\angle A$$

And the rest part of the proof is obvious. □

3 Extensions

inverse trigonometric functions, not the reciprocal trigonometric functions

With $y = \sin(x)$, where $x \in \mathbb{R}$, we define $x = \arcsin(y)$ where $y \in [-1, 1]$ and range $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Similarly we can define the following:

- $y = \arcsin(x)$ where $x \in [-1, 1]$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $y = \arccos(x)$ where $x \in [-1, 1]$ and $y \in [0, \pi]$
- $y = \arctan(x)$ where $x \in \mathbb{R}$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

composition of inverse trigonometric functions

We have the following result:

$$\begin{array}{lll} \sin(\arcsin(x)) = x & \cos(\arcsin(x)) = \sqrt{1-x^2} & \sin(\arccos(x)) = \sqrt{1-x^2} \\ \sin(\arccos(x)) = \sqrt{1-x^2} & \cos(\arccos(x)) = x & \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x} \\ \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}} & \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} & \tan(\arctan(x)) = x \end{array}$$

relation between inverse trigonometric functions

We have the following result:

$$\arcsin(-x) = -\arcsin(x)$$
$$\arccos(-x) = \pi - \arccos(x)$$
$$\arctan(-x) = -\arctan(x)$$
$$\arccos(x) = \frac{\pi}{2} - \arcsin(x)$$
$$\arctan(\frac{1}{x}) = \frac{\pi}{2} - \arctan(x), x > 0$$
$$\arctan(\frac{1}{x}) = -\frac{\pi}{2} - \arctan(x), x < 0$$

hyperbolic functions

Using exponential function e^x we define:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

relation between hyperbolic functions

Using exponential function e^x we define:

$$\begin{aligned}
\sinh(-x) &= -\sinh(x) \\
\cosh(-x) &= \cosh(x) \\
\cosh^2(x) - \sinh^2(x) &= 1 \\
\cosh(x) + \sinh(x) &= e^x \\
\cosh(x) - \sinh(x) &= e^{-x} \\
\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
\operatorname{arcsinh}(x) &= \ln(\sqrt{x + \sqrt{x^2+1}}) \\
\operatorname{arcCosh}(x) &= \ln(\sqrt{x + \sqrt{x^2-1}})
\end{aligned}$$

4 Exercises

show that $\frac{\sin(x)}{x} \to 1$ as $x \to 0$.

show that $\frac{1-\cos(x)}{0.5x^2} \to 1$ as $x \to 0$.

show that $\frac{\tan(x)}{x} \to 1$ as $x \to 0$.