Gleanings on Engineering Analysis I

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1 On Diagonalization of Matrices

Q diagonalizes A is $Q^\top A Q = D$ is a diagonal matrix. Q is the called *modal matrix*. A general procedure for real symmetric matrix $A_{N \times N}$.

1. notice for real symmetric matrix, it has N different eigenvalues, here we obtain them and name them as λ_i

2. get normalized eigenvectors q_i

3. construct orthogonal modal matrix $Q = [q_1, \ldots, q_N]$

4. then $D = \text{diag}(\lambda_1, \ldots, \lambda_N)$

Then, we write $x = Qy$ and then write the *canonical form* of the quadratic form.

$$x^\top A x = y^\top Q^\top A Q y = \sum_N \lambda_i y_i^2$$

Then clearly, Q and A share the same eigenvalues (*spectral theorem*), and then we call $D = Q^\top A Q$ the similarity transformation.

Theorem 1.1 (Spectral theorem in finite space). *for normal matrix, it's diagonalizable using its orthogonal model matrix. The result share the same eigenvalues and they are similar.*

Remark 1.2. For real and symmetric matrix $\boldsymbol{A}$ with distinct eigenvalues, the eigenvectors are mutually orthogonal and we have decomposition: $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^\top$ where $\boldsymbol{Q}$ is the modal matrix comprised of the orthonormal eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{D}$ is a diagonal matrix having the eigenvalues of $\boldsymbol{A}$ along its diagonal; for nonsymmetric matrix with distinct eigenvalues, the eigenvectors are mutually linearly independent, but not necessarily orthogonal. So we can have the decomposition: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$ and $\boldsymbol{U}$ is the modal matrix containing the linearly-independent eigenvectors of $\boldsymbol{A}$. For nonsymmetric matrix with repeated eigenvalues, it ends up *Jordan canonical form* and we write $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{J}\boldsymbol{U}^{-1}$ where $\boldsymbol{J}$ is the nearly-diagonal Jordan canonical form.

2 On solving DEs

2.1 System of N coupled first-order linear ordinary differential equations

Consider

$$\dot{u}_1(t) = A_{11}u_1(t) + A_{12}u_2(t) + \cdots + A_{1N}u_N(t) + f_1(t)$$
$$\dot{u}_2(t) = A_{21}u_1(t) + A_{22}u_2(t) + \cdots + A_{2N}u_N(t) + f_2(t)$$
$$\vdots$$
$$\dot{u}_N(t) = A_{N1}u_1(t) + A_{N2}u_2(t) + \cdots + A_{NN}u_N(t) + f_N(t)$$

In matrix form, we write $\dot{\boldsymbol{u}}(t) = \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}(t)$. If $\boldsymbol{f} \equiv 0$ then the system is homogeneous.
To solve this, we first diagonalize $\boldsymbol{A}$.

1. Diagonalize $\boldsymbol{A}$ and obtain $D = U^{-1}\boldsymbol{A}U$

2. Consider $\dot{\boldsymbol{u}}(t) = U\dot{\boldsymbol{v}}(t)$

3. then we have $\dot{\boldsymbol{v}}(t) = U^{-1}AU\boldsymbol{v}(t) + U^{-1}\boldsymbol{f}(t) = D\boldsymbol{v}(t) + U^{-1}\boldsymbol{f}(t)$

4. we then obtain the uncoupled system and the homogeneous system will have an obvious soution: $v_i(t) = c_i e^{\lambda_i t}$ where λ_i is the i-th element of the diagonal of matrix D.

5. For nonhomogeneous system, the particular solution can be found using method of undetermined coefficients.

6. Finally, $\boldsymbol{u}(t) = U\boldsymbol{v}(t)$ is the solution we want.

2.2 N-th order Linear ODE

Consider

$$u^{(N)} = F(t, u, \dot{u}, \ldots, u^{(N-1)})$$

We write it into a system of N first-order differential equations by substitutions $u_i(t) = u^{(i-1)}(t)$.

Remark 2.1. For linear equations, both the real and imaginary parts of complex solutions are by themselves solutions of the differential equations, and a linear combination of the real and imaginary parts, is also a solution of the linear equations. The real and imagainary parts is obtained using *Euler's formula*: $e^{\theta i t} = \cos(\theta t) + i\sin(\theta t)$

Remark 2.2. Above we are only discussing *autonomous* system meaning $\boldsymbol{A}$ does not depend on time.

3 Decompositions

3.1 SVD

For matrix A of size $m \times n$ we want to decomposite it to $A = U\Sigma V^\top$ where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix. In the following, u_i and v_i are column vectors of u and V.

1. $AA^\top u_i = \sigma_i^2 u_i = \sigma_i A v_i$, solving left singular vectors of A, and singular values σ_i. We also want $\sigma_1 \geq \sigma_2 \cdots \sigma_p \geq 0$ where $p = \min(m, n)$.

2. Alternatively, we can solve right singular vectors from $A^\top A v_i = \sigma_i^2 v_i = \sigma_i A^\top u_i$

3. obtain full U and V and orthonormalizing them.

4. fill Σ left columns in 0.

5. The final result can be also expressed in the form: $A = \sum_{i=1}^{p} \sigma_i u_i v_i^\top$.

Remark 3.1. in image compression, we might use $k < p$.

3.2 Polar decomposition

Following SVD, we might futher write $A = U\Sigma V^\top = UV^\top V\Sigma V^\top$ and break this up into $A = WR$ where $W = UV^\top$ and $R = V\Sigma V^\top$.

Notice U and V are both orthogonal so that W is also orthogonal; and R is symmetric since Σ is symmetric.

1. Notice $A^\top A = R^\top R = R^2$, we first solve R.

2. Notice we can diagonalize R^2 and R: $Q^\top R^2 Q = \text{diag}(\lambda_1^2, \cdots, \lambda_N^2)$, $Q^\top R Q = \text{diag}(\lambda_1, \cdots, \lambda_N)$

3. then $R = Q \text{diag}(\lambda_1, \cdots, \lambda_N) Q^\top$

4. $W = AR^{-1}$

3.3 QR

Write $A = [u_1, u_2, \ldots, u_S]$ and we want $A = QR$ where Q is orthonormal matrix $[q_1, q_2, \ldots, q_S]$ and R is right triangular matrix. Following Gram-schmidt, we are transforming u_i to q_i. For $i = 2, 3, \ldots, S$,

$$\hat{u}_i = u_i - \sum_{k=1}^{i-1} \langle u_i, q_k \rangle q_k$$

then we normalize them $q_i = \frac{\hat{u}_i}{\|\hat{u}_i\|}$. And this GS orthogonalization is actually a matrix decomposition for A because we actually have $\hat{u}_i = \|\hat{u}_i\| q_i$, $\|\hat{u}_i\| = \langle \hat{u}_i, q_i \rangle$, and

$$u_i = \|\hat{u}_i\| q_i + \sum_{k=1}^{i-1} \langle u_i, q_k \rangle q_k$$

Above in a nicer way:

$$i = 1 : u_1 = \langle u_1, q_1 \rangle q_1,$$
$$i = 2 : u_2 = \langle u_2, q_1 \rangle q_1, + \langle u_2, q_2 \rangle q_2,$$
$$\vdots$$

or in matrix form: $A = QR$ where $Q = [q_1, q_2, \ldots, q_S]$ and

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle & \cdots & \langle u_S, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle & \cdots & \langle u_S, q_2 \rangle \\ \vdots & & & & \langle u_S, q_S \rangle \end{bmatrix}$$

3.4 LU decomposition

3.5 CHolesky decomposition

4 Differential Eigenproblems

4.1 Function Spaces, bases, and orthogonalization

Definition 4.1. A set of functions $u_i(x)$ for $i = 1, 2, \ldots, M$ are *linearly independent* over an interval $a \le x \le b$ is their linear combination is zero is all coefficients are zero $\forall x \in [a, b]$.

Remark 4.2. For any piecewise continuous function $u(x)$ we have expression:

$$u(x) = \sum_{i=1}^{\infty} a_n u_n(x)$$

where $a_n \in \mathbb{R}$ and a_n are not all zeros, $u_n(x)$ are linear-independent *basis functions*. When there are actually infinite sum, we say $u(x)$ is *infinite dimensional*.

Definition 4.3. We also want basis functions mutually orthogonal, so here we define the inner product of two functions.

$$\langle u_m(x), u_n(x) \rangle = \int_a^b w(x) u_m(x) u_n(x) \, \mathrm{d}x$$

orthogonality means $\langle u_m(x), u_n(x) \rangle \equiv 0 \forall x \in [a, b]$, and $w(x)$ is a weight function. $w(x) \equiv 1$ unless stated otherwise.

Definition 4.4 (Legendre polynomials)**.** Obtained from Gram-schmidt process on $u_i(x) = x^i$ for $i = 0, 1, \ldots$ and $-1 \leq x \leq 1$, the first several terms are $\tilde{u}_i(x) = 1, x, \frac{1}{2}(3x^2 - 1), \ldots$. Then any piecewise continuous function $f(x)$ in the interval $-1 \leq x \leq 1$ can be written as

$$f(x) = \sum_{i=1}^{\infty} \langle f(x), \tilde{u}_i(x) \rangle \, \tilde{u}_i(x)$$

Remark 4.5. When considering bases $\cos(ix), \sin(ix)$ for $i = 0, 1, \ldots$ and $0 \leq x \leq 2\pi$, we end up with *Fourier series*,

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{i=1}^{\infty} a_i \frac{\cos(ix)}{\sqrt{\pi}} + \sum_{i=1}^{\infty} b_i \frac{\sin(ix)}{\sqrt{\pi}}$$

$$a_0 = \left\langle f(x), \frac{1}{\sqrt{2\pi}} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \, \mathrm{d}x$$

$$a_i = \left\langle f(x), \frac{\cos(ix)}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos(ix) \, \mathrm{d}x$$

$$b_i = \left\langle f(x), \frac{\sin(ix)}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin(ix) \, \mathrm{d}x$$

Definition 4.6 (Eigenproblems)**.** Consider n-th order differential equation $\mathcal{L}u(x) = f(x)$ where

$$\mathcal{L} = a_0 + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \cdots + a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}$$

We then tend to find λ_i so that $\mathcal{L}u_i(x) = \lambda_i u_i(x)$. We call the nontrivial solution $u_i(x)$ the eigenvalue of $\mathcal{L}$ and $u_i(x)$ the *eigenfunction* of $\mathcal{L}$.

Given that the differential operator $\mathcal{L}$ is self-adjoint (defined below) and boundary conditions,

1. Solve $\mathcal{L}u_i(x) = \lambda_i u_i(x)$ as a N-th order Linear order ODE, excluding the trivial solution.

2. Usually we end up in a characteristic equation with roots being the eigenvalues

3. And when solving for coefficients of general solutions, make sure the norm of eigenfunctions being 1, resulting in normalized eigenfunctions.

4. write $f(x) = \sum_{i=1}^{\infty} \hat{a}_i u_i(x)$ where $\hat{a}_i = \langle f(x), u_i(x) \rangle$

5. write $u(x) = \sum_{i=1}^{\infty} a_i u_i(x)$ where $a_i = \frac{\hat{a}_i}{\lambda_i}$, so that $\mathcal{L}u(x) = f(x)$

6. The final solution can be written as

$$u(x) = \sum_{i=1}^{\infty} \frac{\langle f(x), u_i(x) \rangle}{\lambda_i} u_i(x)$$

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5   Adjointness

Definition 5.1 (from matrix transpose to adjoint operator)**.** Consider matrix A of size $n \times n$, arbitrary vectors u and v or size $n \times 1$. We define $A^\top$ the matrix such that $\langle v, Au \rangle = \langle A^\top v, u \rangle$. And in real case, $\langle v, Au \rangle = \langle u, A^\top v \rangle$. Likewise, for operator $\mathcal{L}$ we define its adjoint $\mathcal{L}^*$ such that

$$\langle v, \mathcal{L}u \rangle = \langle u, \mathcal{L}^* v \rangle$$

where $u(x)$ and $v(x)$ are arbitrary functions with homogeneous boundary conditions.

Remark 5.2. An easy way to find an operator's adjoint is to use test function, then do integration by parts.

Example 5.3 (Second order linear differential equation with variable coefficients)**.**

$$\mathcal{L}u = \frac{1}{w(x)} \left(a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) \right) = 0$$

where $x \in [a, b]$ and $w(x)$ is a weight function.

1. Consider test function $v(x)$ and find

$$\langle v, \mathcal{L}u \rangle = \int_a^b w(x)v(x) \left(\frac{1}{w(x)} \left(a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) \right) \right) \, \mathrm{d}x$$

$$= \int_a^b \left(a_0 vu'' + a_1 vu' + a_2 vu \right) \, \mathrm{d}x$$

2. using integration by parts to down-order u: $\int p \, \mathrm{d}q = pq - \int q \, \mathrm{d}p$, we obtain the following:

   - $\int_a^b a_0 vu'' \, \mathrm{d}x = \int_a^b a_0 v \, \mathrm{d}u' = a_0 vu' \big|_a^b - \int_a^b u'(a_0 v)' \, \mathrm{d}x = a_0 vu' \big|_a^b - (a_0 v)' u \big|_a^b + \int_a^b u(a_0 v)'' \, \mathrm{d}x$
   - $\int_a^b a_1 vu' \, \mathrm{d}x = \int_a^b a_1 v \, \mathrm{d}u = a_1 vu \big|_a^b - \int_A^b u(a_1 v)' \, \mathrm{d}x$
   - $a_2(x)$ term with $u(x)$ so no need to do integration by parts

3. combine these we end up with

$$\langle v, \mathcal{L}u \rangle = \left(a_0 vu' - (a_0 v)' u + a_1 vu \right)_a^b + \int_a^b w(x)u(x) \left(\frac{1}{w(x)} \left((a_0 v)'' - (a_1 v)' + a_2 v \right) \right) \, \mathrm{d}x$$

4. above should equal to $\langle u, \mathcal{L}^* u \rangle$ so given homogeneous BC, $\mathcal{L}^* v = \frac{1}{w(x)} \left((a_0 v)'' - (a_1 v)' + a_2 v \right)$

Remark 5.4. variable coefficients move inside the derivatives, and the odd-order derivatives change sign as compared to $\mathcal{L}$.

Definition 5.5. A differential operator $\mathcal{L}$ is *self-adjoint*, or Hermitian, if $\mathcal{L} = \mathcal{L}^*$.

Remark 5.6. Similar to symmetric matrix, its eigenvectors are mutually orthogonal if its eigenvalues are distinct, self-adjoint operator's eigenfunctions are mutually orthogonal if its eigenvalues are distinct.

5.1   Sturm-Liouville differential operator

For differential equations in the previous example, use product rule we have

$$\mathcal{L}^* v = \frac{1}{w(x)} \left((a_0 v)'' - (a_1 v)' + a_2 v \right) = \frac{1}{w(x)} \left(a_0 v'' + (2a_0' - a_1)v' + (a_0'' - a_1' + a_2)v \right)$$

Letting $\mathcal{L} = \mathcal{L}^*$, we want

$$a_1 = 2a_0' - a_1 \implies a_1 = a_0'$$
$$a_2 = a_0'' - a_1' + a_2$$

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So a self-adjoint differential operator may have the form:

$$\mathcal{L}u = \frac{1}{w(x)}\left(a_0 u'' + a_0' u' + a_2 u\right)$$

$$\mathcal{L} = \frac{1}{w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right)$$

where $p(x) > 0$ and $w(x) > 0$ in $a \leq x \leq b$ and the boundary conditions are homogeneous.

Remark 5.7. Similarly, we have the forth-order Sturm-Liouville differential operator

$$\mathcal{L} = \frac{1}{w(x)}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(s(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right)$$

Remark 5.8 (Eigenfunctions of Sturm-Liouville Operators). Still we assume homogeneous boundary conditions, and consider the eigenproblrm $w(x)\mathcal{L}u_i(x) = -\lambda_i w(x)u_i(x)$. We have

- the eigenvalues are distinct and nonnegative

- the eigenfunctions $u_i(x)$ are orthogonal wrt the weight function $w(x)$.

Remark 5.9. some special cases

- taking $p \equiv 1$ and $q \equiv 0$ on $x \in [0,1]$ produces eigenfunctions. Either fourier sine series or fourier cosine series, for different set of boundary conditions.

- taking $p(x) = x$, $q(x) = \frac{-\nu^2}{x}$, $w(x) = x$ on $x \in [0,1]$ produces Bessel functions, with $\lambda_i = \mu_i^2$

- taking $p(x) = 1 - x^2$, $q(x) = 0$, $w(x) = 1$ on $x \in [-1,1]$ produces Legendre Polynomials, with $\lambda_i = \nu(\nu+1)$

- Legendre polynomials have recursion relation:

$$(\nu+1)P_{\nu+1}(x) - (2\nu+1)xP_\nu(x) + \nu P_{\nu-1}(x) = 0$$

- taking $p(x) = \sqrt{1-x^2}$, $q(x) = 0$, $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $x \in [-1,1]$ produces Chebyshev polynomials, with $\lambda_i = \nu^2$

- Chebyshev polynomials have recursion relation:

$$T_{\nu+1}(x) - 2xT_\nu(x) + T_{\nu-1}(x) = 0$$

Remark 5.10. for nonhomogeneous boundary conditions, we can apply certain transformation.

6 Separation of Variables

A new way to solve PDEs, by assume solution $u(x,t) = \phi(x)\psi(t)$.

6.1 General hyperbolic partial differential equation

Consider $\mathcal{M}\ddot{u} + \mathcal{K}u = 0$ where $\mathcal{M}$ and $\mathcal{K}$ are linear differential operators in space, and dots denote differential in time.

Substituting into the governing equation gives

$$\ddot{\psi}\mathcal{M}\phi + \psi\mathcal{K}\phi = 0 \implies \frac{\mathcal{K}\phi}{\mathcal{M}\phi} = -\frac{\ddot{\psi}}{\psi} = \lambda$$

it's a constant since LHS only on space and RHS only on time.

Then we solve uncoupled differential equations

$$\mathcal{K}\phi(x) = \lambda\mathcal{M}\phi(x)$$

$$\ddot{\psi}(t) = -\lambda\psi(t)$$

6.2 Electrmagnetics

6.3 Schrödinger Equation

7 Complex variables

7.1 Complex functions

On trigonometric functions, using taylor series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{iz} = \cos z + i \sin z$$

Theorem 7.1 (De Moivre's Theorem).

$$z^n = (re^{i\theta})^n r^n (\cos(n\theta) + i \sin(n\theta))$$

On hyperbolic functions, similarly we have

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{e^z - e^{-z}}{2i}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \frac{e^z + e^{-z}}{2}$$

$$\sinh(iz) = i \sin z, \sin(iz) = i \sinh(z)$$

$$\cosh(iz) = \cos z, \cos(iz) = \cosh(z)$$

On expenentials, for $z = x + iy$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^z = e^x (\cos y + i \sin y)$$

On Logarithms, for $z = re^{i\theta} \neq 0$

$$\log z = \log r + i\theta = \log |z| + i(\theta_0 + 2k\pi)$$

a multi-valued function. We define the pricinple argument where $-\pi < \theta_0 \leq \pi$, with $k \in \mathbb{Z}$.
on fractional powers

$$z^{n/m} = \exp\left\{\log z^{n/m}\right\}$$

$$= \exp\left\{(n/m)\log(r + i(\theta_0 + 2k\pi))\right\}$$

$$= r^{n/m} \exp\left\{i(n/m)(\theta_0 + 2k\pi)\right\}, k = 0, 1, 2, \ldots, m - 1$$

on inverse trigonometric functions, these values are solved from quadratic equation. For example with $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$ we have $(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$ and thus $w = \sin^{-1} z = \frac{1}{i} \log(iz + (1 - z^2)^{1/2})$. Similarly we have the following results

$$\sin^{-1} z = \frac{1}{i} \log(iz + (1 - z^2)^{1/2})$$

$$\cos^{-1} z = \frac{1}{i} \log(z + (z^2 - 1)^{1/2})$$

$$\tan^{-1} z = \frac{1}{2i} \log \frac{i - z}{i + z}$$

$$\sinh^{-1} z = \log(z + (1 + z^2)^{1/2})$$

$$\cosh^{-1} z = \log(z + (z^2 - 1)^{1/2})$$

$$\tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}$$

7.2 Analytic functions

Definition 7.2 (Analyticity). A complex function $f(z)$ is anlytic at a point $z = z_0$ if $f(z)$ is *differentiable* at z_0 and in some neighbourhood of z_0.

- and is *entire* if analytic in the entire complex plane.

- and points where a function is not analytic, are *singular points*.

Theorem 7.3 (Goursat's theorem). *If $f(z)$ is analytic at a point z_0, then $f'(z)$ is continuous at z_o.*

Theorem 7.4 (Cauchy-Riemann equations). *In order for a function $w = f(z) = u(x, y) + iv(x, y)$ to be analytic in some region in $\mathbb{R}^2$, the following should be satisfied:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Notice with this we further have the continuity of these partial derivatives. Thus we might conclude: Cauchy-Riemann equations $\implies$ $f(z)$ being analytic $\implies$ continuity of partial derivatives.

We can also derive its expression in polar coordinates. Using change of variables we end up with

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Corollary 7.5. *Then the derivatives have the forms*

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}z} = f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\
&= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \\
&= \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) (\cos \theta - i \sin \theta) \\
&= \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) \frac{-i}{r} (\cos \theta - i \sin \theta)
\end{aligned}$$

On the slopes of u and v. Consider curve $u(x, y) = U_1$ then

$$\mathrm{d}u = 0 = \frac{\partial u}{\partial x} \mathrm{d}x + \frac{\partial u}{\partial y} \mathrm{d}y \implies \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)_{u = U_1} = -\frac{\partial u/\partial x}{\partial u/\partial y}$$

Similarly we have

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)_{v = V_1} = \frac{\partial u/\partial y}{\partial u/\partial x} = -\frac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)_{u = U_1}}$$

.

Also notice that we have $\nabla^2 u = \nabla^2 v = 0$ and we call the solutions of laplace's equation the harmonic functions. And call $u(x, y)$ and $v(x, y)$ the harmonic conjugates of one another.

7.3 Branch points and branch cuts

Consider the multi-valued function

$$\log z = \log |z| + i(\theta_0 + 2k\pi)$$

where the principle argument is in range $-\pi < \theta_0 \leq \pi$ and $k \in \mathbb{Z}$.

Definition 7.6 (branch). • branch is a single-valued and analytic portion of a multi-valued function, say we can fix $k = 0$, which is the principle branch.

- The branch point is a point that must be inside any patch that is necessary to change branches of a function. In the example, $z = 0$ is a branch point of $\log z$.

- The branch cut is a curve along which the function is discontinuous if it is to remain single-valued. For example $y = 0, x \leq 0$ the negative real axis ia a branch cut of $\log z$.

7.4 Conformal mapping and boundary value problems

First, we view complex function $w = f(z) = u(x, y) + iv(x, y)$ as a mapping from x, y-plane to u, v-plane. We want the mapping to be one-to-one and conformal.

Theorem 7.7 (one-to-one mapping). *A mapping $w = f(z)$ is one-to-one at z_0 if*

- *$f(z)$ is analytic at z_0*

- *$f'(z_0) \neq 0$ (points where $f'(z) = 0$ are called the critical points)*

And then a unique inverse mapping function $z = F(w)$ exists at z_0.

Sketch of the proof. Notice we have

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

Above linear system has unique solution if *Jacobian* is not zero.

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \neq 0$$

Thus if further given the condition that the function is analytic, we can use the Cauchy-Riemann equation and obtain that $J = |f'(z)|^2 \neq 0$.

Theorem 7.8 (conformal mapping). *conformal means locally angle-preserving. We can prove that the condition for being conformal is same for being one-to-one.*

Sketch of the proof. For mapping $w = f(z)$ and two curves C, T where C is in x, y-plane and is mapped to T in u, v-plane. We consider points z_0 and $z_o + \mathrm{d}z$ on curve C and their image w_0 and $w_0 + \mathrm{d}w$. We also define θ and η to be the angles of the tangents to the curves C and T at z_0 and w_0.

We have the following relations

$$\mathrm{d}z = |\mathrm{d}z| e^{i\theta}$$

$$\mathrm{d}w = |\mathrm{d}w| e^{i\eta}$$

$$f'(z_0) = |f'(z_0)| e^{i \, \mathrm{Arg}(f'(z_0))}$$

Then we plug in above into $\mathrm{d}w = f'(z_0) \, \mathrm{d}z$, we have

$$|\mathrm{d}w| = |f'(z_0)| \, |\mathrm{d}z| \exp i \, \mathrm{Arg}(f'(z_0) + \theta)$$
$$\eta = \mathrm{Arg}(f'(z_0)) + \theta$$

This means that mapping $f(z)$ stretches the infinitesimal curve segment $\mathrm{d}z$ by a factor $|f'(z_0)|$ and rotates it by an angle $\mathrm{Arg}(f'(z_0))$. So to preserve angles

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Remark 7.9. conformality $\iff$ analyticity and non-zero derivatives

Theorem 7.10. *Subject to conformal mapping, Laplace's equation in the x, y-plane transforms to Laplace's equation in the u, v-plane.*

With this theorem, we can simplify some problem by using conformal mapping first and solve the problem in the new plane where the conditions are in a simpler form. Here's a useful transformation: Linear Fractional (Bilinear or Möbius) Transformation

$$w = f(z) = \frac{az + b}{cz + d} \implies z = F(w) = \frac{-dw + b}{cw - a}$$

We can use this to map three specified points z_1, z_2, z_3 in the x, y-plane into three specified image points w_1, w_2, w_3 in the u, v-plane, the constants are determined by solving:

$$\frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$$

And when a points is speficied to be at infinity, we set the fators that includes that point equal to 1. For example, if $w_1 = \infty$, then $(w - w_1)/(w_3 - w_1) = 1$.

This map will transform lines and circles to lines or circles.

Other conformal mappings are elementary mappings: translation, stretching, rotation, inversion. If only working in a restricted region, exponentials, trigonometrics can also be choices of conformal mapping.

Example 7.11 (temperature distribution, dirichilet type)**.** Consider Laplace's equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ subject to the boundary condition in the complex plane $z = x + iy$

$$T(x, y) = 0, \left| z - \frac{1}{2} \right| = \frac{1}{2}$$

$$T(x, y) = 1, \left| z - \frac{1}{4} \right| = \frac{1}{4}$$

It's an area between two isothermal circles that pass through the origin.

- -

We consider transform the circle boundary to line boundary. Consider mapping:

$$z_1 = 0 \to w_1 = \infty$$

$$z_2 = \frac{1}{2} \to w_2 = 0$$

$$z_3 = 1 \to w_3 = 1$$

Plug in above to the formula we end up with

$$w = f(z) = \frac{1 - z}{z}$$

And the boundaies are mapped as follows

$$T(x, y) = 0, \left| z - \frac{1}{2} \right| = \frac{1}{2} \to u = 1$$

$$T(x, y) = 1, \left| z - \frac{1}{4} \right| = \frac{1}{4} \to u = 0$$

In the new plane, we have

$$\frac{\partial^2 \hat{T}^2}{\partial u^2} + \frac{\partial^2 \hat{T}}{\partial v^2} = 0$$

with the above boundary conditions.

Notice that there's no change in the v-direction. So we assume $\hat{T} = \hat{T}(u)$. Then we have $\frac{d^2\hat{T}}{du^2} = 0 \implies \hat{T}(u,v) = Au + B$. With the boundary condition, we obtain $\hat{T}(u,v) = u$.

Next we find the inverse map to get the solution in the original plane. From $w = \frac{1-z}{z}$ we end up with $u(x,y) = \frac{x}{x^2+y^2} - 1$, thus

$$T(x,y) = \hat{T}(u(x,y), v(x,y)) = \frac{x}{x^2+y^2} - 1$$

7.5 Diricilet Problem in a Circular Disk

The Laplace's equation in acylindrical coordinates is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

with the boundary conditions on $r = R$:

$$\phi(r,\theta) = \phi(R,\sigma) = f(\sigma)$$

method 1: seperation of variables

We assume that $\phi(r,\theta) = P(r)Q(\theta)$, then substute it into the equation and obtain

$$\frac{r^2}{p}\frac{d^2P}{dr^2} + \frac{r}{P}\frac{dP}{dr} = \frac{-1}{Q}\frac{d^2Q}{d\theta^2} = \lambda^2$$

It's now an eigenproblem.

$$r^2\frac{d^2P}{dr^2} + r\frac{dP}{dr} - \lambda^2 P = 0$$
$$d^2Q\,d\theta^2 + \lambda^2 Q = 0$$

Solving these we have $P(r) = C_1 r^\lambda + C_2 r^{-\lambda}$, $\lambda \neq 0$; $P(\lambda) = C_3 + C_4\log r$, $\lambda = 0$, and $Q(\theta) = C_5\cos(\lambda\theta) + C_6\sin(\lambda\theta)$, $\lambda \neq 0$; $Q(\theta) = C_7 + C_8\theta$, $\lambda = 0$.

So the general solution is

$$\phi(r,\theta) = (C_1 r^\lambda + C_2 r^{-\lambda})(C_5\cos(\lambda\theta) + C_6\sin(\lambda\theta)) + (C_3 + C_4\log r)(C_7 + C_8\theta)$$

Notice the function must be bounded thus $C_2 = C_4 = 0$ and the function must be 2π-periodic to be single-valued, thus $C_8 = 0$. So we end up with

$$\phi(r,\theta) = c_1 + r^\lambda(c_3\cos(\lambda\theta) + c_4\sin(\lambda\theta))$$

Next to make sure the cos and sin terms have period 2π, we plug in $\theta + 2\pi$ and end up with $\cos(2\pi\lambda) = 1$ and $\sin(2\pi\lambda) = 0$. So the eigenvalues must be $\lambda = 1, 2, 3, \ldots$. And thus the eigenfunction solution is

$$\phi(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

Then apply the boundary condition at $r = R$ we have

$$f(\sigma) = \phi(R,\sigma) = A_0 + \sum_{n=1}^{\infty} R^n\left(A_n\cos(n\sigma) + B_n\sin(n\sigma)\right)$$

And this is just the fourier series expansion of $f(\sigma)$ where we let

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\sigma)\, d\sigma$$

$$A_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\sigma) \cos(n\sigma)\, d\sigma$$

$$B_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\sigma) \sin(n\sigma)\, d\sigma$$

Finally the simplified solution is:

$$\phi(r,\theta) = \frac{1}{\pi} \int_0^{2\pi} f(\sigma) \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(\sigma - \theta)) \right) d\sigma$$

Next we rewrite this using the following relations:

$$\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(\sigma - \theta)) = \Re\left(\sum_{n=1}^{\infty} \left(\frac{r}{R} e^{i(\sigma-\theta)}\right)^n \right)$$

$$\sum_{n=1}^{\infty} x^n = -1 + \frac{1}{1-x}$$

We end up with the following:

Theorem 7.12 (Poisson integral formula for Dirichlet problem in a circular disk). *Suppose the boundary condition to be $\phi(R,\sigma) = f(\sigma)$ then the Laplace's equation have solution:*

$$\phi(r,\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \phi(R,\sigma) \frac{1}{R^2 + r^2 - 2Rr\cos(\sigma - \theta)}\, d\sigma$$

where $0 \le r \le R$ and $\theta \in \mathbb{R}$. And $\phi(R,\sigma) = f(\sigma)$ are the values of $\phi(r,\theta)$ on the circular boundary at $r = R$.

With above formula we obtain following two theorem:

Theorem 7.13 (Gauss' mean value theorem). *If C is a circle that is completely within a region R in which $\phi(z)$ is analytic, then $\phi(z)$ at the center of the circle is the average of the value of $\phi(z)$ along C.*

Theorem 7.14 (Maximum and minimum modulus theorem). *Let $\phi(z)$ be continuous and nonconstant throughout a closed bounded region R and analytic within the interior of R. Then the maximum value of $|\phi(z)|$ occurs on the boundary of R. IF $\phi(z)$ is nowhere zero in R, then the minimum value of $|\phi(z)|$ must also occur on the boundary.*

method 2: conformal map

we will build a conformal map and solve the problem in upper half plane. Consider the mapping:

$$z_1 = -1 \to w_1 = 0$$
$$z_2 = -i \to w_2 = 1$$
$$z_3 = 1 \to w_3 = \infty$$

We solve it in similar method: $w = f(z) = i\frac{1+z}{1-z}$ and $z = e^{i\sigma} = F(w) = \frac{w-i}{w+i}$. We can check that it indeed map the unit circle to upper half plane.

Consider some point on the unit circle $e^{i\sigma}$ whose image in the upper half plane is ξ. To obtain the formula in the new plane, we first rewrite the formula.

$$\phi(z) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \phi(R, \sigma) \frac{1}{R^2 + r^2 - 2Rr\cos(\sigma - \theta)}\, d\sigma$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \phi(Re^{1\sigma}) \frac{R^2 - r^2}{|Re^{i\sigma} - re^{i\theta}|^2}\, d\sigma$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\sigma}) \frac{1 - |z|^2}{|e^{i\sigma} - z|^2}\, d\sigma, \text{if } R = 1$$

Notice the unit circle is mapped to the real axis and thus: $\hat{\phi}(e^{i\sigma}) = \hat{\phi}(F(\xi)) = \hat{\phi}(\xi, 0)$. Following this we can also derive the following.

$$1 - |z|^2 = 1 - \left|\frac{(u + iv) - i}{(u + iv) + i}\right|^2 = \frac{4v}{u^2 + (a + v)^2}$$

$$\left|e^{i\sigma} - z\right|^2 = \left|\frac{\xi - i}{\xi + i} - \frac{w - i}{w + i}\right|^2 = 4\frac{(u - \xi)^2 + v^2}{(1 + \xi^2)(u^2 + (1 + v)^2)}$$

$$ie^{i\sigma}\, d\sigma = \left(\frac{1}{\xi + i} - \frac{\xi - i}{(\xi + i)^2}\right) d\xi$$

$$d\sigma = \frac{2}{\xi^2 + 1}\, d\xi$$

Substituting these into the formula gives

$$\hat{\phi}(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\xi, 0) \frac{4v}{u^2 + (1 + v)^2} \frac{1}{4} \frac{(1 + \xi^2)(u^2 + (1 + v)^2)}{(\xi - u)^2 + v^2} \frac{2}{1 + \xi^2}\, d\xi$$

$$= \frac{v}{\pi} \int_{-\infty}^{\infty} \hat{\phi}(\xi, 0) \frac{1}{(\xi - u)^2 + v^2}\, d\xi, v > 0$$

where $\hat{\phi}(\xi, 0)$ being the values on the boundary with $v = 0$.

Remark 7.15. Fourier transform may also be used to derive this.

7.6   Application to Fluid Flows

Definition 7.16 (2D incompressible flow condition). Continuity equation:

$$\nabla \cdot \boldsymbol{V} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

where $\boldsymbol{V} = v_x(x, y)i + v_y(x, y)j$ is the velocity vector, $v_x(x, y)$ here denotes the velocity components and i denotes the unit vector in x-direction.

Definition 7.17 (Streamfunction). Consider streamfunction $\psi(x, y)$ defined by

$$v_x = \frac{\partial \psi}{\partial y}, v_y = -\frac{\partial \psi}{\partial x}$$

so that the flow is automatically incompressible. We also call the contours of constant streamfunction streamlines. We can also see that the slop of the streamline is:

$$\frac{dy}{dx} = \frac{-\partial \psi/\partial x}{\partial \psi/\partial y} = \frac{v_y}{v_x}$$

it's the direction of the velocity vector thus streamlines can indicate the local flow direction (velocity direction).

Definition 7.18 (vorticity)**.** Vorticity is the curl of the velocity field, in 2D cartesian coordinates, we write

$$\omega = \nabla \times \boldsymbol{V} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

And in 3D cartesian coordinates, we write

$$\nabla \times \boldsymbol{V} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

If the flow is irrotational, we have $\omega = 0$ in which case there exists a scalar function $\phi(x, y)$ for which $\nabla \times (\nabla \phi) = 0$.

Remark 7.19. The curl of the gradient of any scalar function is zero. In 2D case we have

$$\nabla \times (\nabla \phi) = \frac{\partial(\partial \phi/\partial y)}{\partial x} - \frac{\partial(\partial \phi/\partial x)}{\partial y} = 0$$

And in 3d case we have

$$\nabla \times (\nabla \phi) = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}, \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) = 0$$

Definition 7.20 (velocity potential)**.** We write the velocity vector as a gradient of a scalar function $\phi(x, y)$ according to $\boldsymbol{V} = \nabla \phi$ where the components are $v_x = \frac{\partial \phi}{\partial x}$. We call this scalar function ϕ the velocity potential.

Corollary 7.21. *From the incompressible condition, we have $\nabla \cdot \boldsymbol{V} = 0$ and thus*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Also the streamfunction has the property (2D case):

$$w = \nabla \times \boldsymbol{V} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

Definition 7.22 (potential flow)**.** Ideal flow: two-dimensional, incompressible, inviscid, irrotational.

Corollary 7.23. *For the ideal flow, the streamfunction, velocity potential are homonic functions, and are harmonic conjugates of one another, i.e.:*

$$\nabla^2 \phi = 0, \nabla^2 \psi = -\omega = 0, v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

And thus here we define the complex potential function:

$$\Phi(z) = \phi(x, y) + i\psi(x, y)$$

where $\Phi(z)$ is an analytic function. Correspondingly we have the complex velocity by taking derivatives.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{\partial \phi}{\partial x} + i\frac{\partial psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y} = v_x - iv_y = \bar{W}(z)$$

The velocity potential drives the flow along the streamlines from higher to lower potential. We also define the Stagnation points in the potential flow where the velocity $\boldsymbol{V} = 0$, or $\bar{W} = \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0$.

In polar coordinates, we have the following results:

$$\boldsymbol{V} = v_r(r, \theta)\boldsymbol{e}_r + v_\theta(r, \theta)\boldsymbol{e}_\theta$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\bar{W}(z) = \frac{d\Phi}{dz} = (v_r - iv_\theta)e^{-i\theta}$$

Next, we define basic flows by specifying their complex potentials.

Uniform flow: of speed U at an angle α with positive x-axis.

$$\Phi(z) = Ue^{-i\alpha}z$$

$$\bar{W}(z) = Ue^{-i\alpha} = U(\cos\alpha - i\sin\alpha)$$

$$v_x = U\cos\alpha$$

$$v_y = U\sin\alpha$$

Sources and sinks: Q being the volumetric flow rate. If $Q > 0$, it's a source and sink if negative.

$$\Phi(z) = \frac{Q}{2\pi}\log z$$

$$\Phi(z) = \frac{Q}{2\pi}(\log r + i\theta)$$

$$\phi(x, y) = \Re(\Phi) = \frac{Q}{2\pi}\log r$$

$$\psi(x, y) = \Im(\Phi) = \frac{Q}{2\pi}\theta$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

Point Vortex: Γ being the circulation strength. If $\Gamma > 0$, it's counterclockwise rotation.

$$\Phi(z) = -i\frac{\Gamma}{2\pi}\log z$$

Doublet: obtained by placing a source and a sink a distance ϵ apart and taking $\epsilon \to 0$ in a particular manner. The resulting velocity potential is $\Phi = \frac{S}{z}$ where S is the strength of the doublet.

Corner and Wedge flows:

$$\Phi(z) = \frac{A}{n}z^n, n \geq \frac{1}{2}$$

$$\phi(r, \theta) = \frac{A}{n}r^n \cos(n\theta)$$

$$\psi(r, \theta) = \frac{A}{n}r^n \sin(n\theta)$$

$$\bar{W}(z) = Az^{n-1} = Ar^{n-1}e^{in\theta}e^{-i\theta}$$

$$v_r(r, \theta) = Ar^{n-1}\cos(n\theta)$$

$$v_\theta(r, \theta) = -Ar^{n-1}\sin(n\theta)$$

Next we can consider using conformal mapping on the flows.

$$\bar{W}(z) = \frac{\mathrm{d}\Phi}{\mathrm{d}z}$$

$$\hat{\bar{W}}(w) = \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}w}$$

$$\Phi = \hat{\Phi}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}w}f'(z)$$

$$\hat{W}(z) = f'(z)\hat{\bar{W}}(w)$$

The magnitudes of the velocities differ by a factor of $f'(z)$ in the z- and w-planes, while the values of ϕ and ψ are the same at image points, i.e. $\Phi(z) = \hat{\Phi}(w)$.

The critical points of the mapping, i.e. where $f'(z) = 0$ are stagnation points in the physical domain i.e. $\bar{W}(z) = 0$.

Two useful general purpose mappings:

Schwarz-Christoffel Transformation: it maps the interior of a polygon with n vertices in the z-plane to the upper half of the w-plane. The inverse transformation is

$$z = F(w) = A\int (w - u_1)^{k_1}(w - u_2)^{k_2}\cdots(w - u_n)^{k_1 n}\,\mathrm{d}w + B$$

where $k_i = \frac{\alpha_i}{\pi} - 1$

Joukowski Transformation: the mapped image is a cylinder. We have

$$z = F(w) = w + \frac{c^2}{w}$$

$$w = f(z) = \frac{1}{2}z \pm \left(\frac{z}{2}\right)^2 - c^{2\,1/2}$$

8 Integration in the complex plane and residue theory

9 Variational Methods

We use differential calculus to find extrema of functions, and use variatinoal calculus to find extrema of functionals, say a definite integral involving an unknown function and its derivatives The result should be a function.

9.1 Integration by parts

starting from product rule of two functions $p(x)$ and $q(x)$, $\frac{\mathrm{d}}{\mathrm{d}}(pq) = p\frac{\mathrm{d}q}{\mathrm{d}x} + q\frac{\mathrm{d}p}{\mathrm{d}x}$ and thus integration

$$pq = \int p\frac{\mathrm{d}q}{\mathrm{d}x}\,\mathrm{d}x + \int q\frac{\mathrm{d}q}{\mathrm{d}x}\,\mathrm{d}x \implies \int p\,\mathrm{d}q = pq - \int q\,\mathrm{d}p$$

Alternatively, for a definite integral over $x_0 \le x \le x_1$, we have

$$\int_{x_0}^{x_1} p\,\mathrm{d}q = pq|_{x_0}^{x_1} - \int_{x_0}^{x_1} q\,\mathrm{d}p$$

In the two-dimensional context, integration by parts is replaced by the divergence theorem.

Theorem 9.1 (2D divergence theorem)**.**

$$\iint_A \nabla \cdot \boldsymbol{v}\,\mathrm{d}A = \oint_C \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s$$

where $\boldsymbol{v}$ is any vector, $\boldsymbol{n}$ is the outward facing unit normal to the curve C bounding the area A. Notice that $\mathrm{d}s$ proceeds in the counterclockwise direction along the boundary.

Corollary 9.2. *Consider apply divergence theorem to $\boldsymbol{v} = \psi \nabla \phi$ where $\phi(x, y)$ and $\phi(x, y)$ are arbitrary 2D scalar functions. Then*

$$\nabla \cdot \boldsymbol{v} = \nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi$$
$$\boldsymbol{v} \cdot \boldsymbol{n} = \psi (\nabla \phi \cdot \boldsymbol{n})$$

Then in the Green's first theorem, we have

$$\iint_A \nabla \psi \cdot \nabla \phi \, \mathrm{d}A = \oint_C \psi (\nabla \phi \cdot \boldsymbol{n}) \, \mathrm{d}s - \iint_A \psi \nabla^2 \phi \, \mathrm{d}A$$

In one dimension, with $\psi = \psi(x)$ and $\phi = \phi(x)$ we have $\nabla = \frac{\mathrm{d}}{\mathrm{d}x} i$ and $\boldsymbol{n}_0 = -i$, $\boldsymbol{n}_1 = i$ and $\mathrm{d}A = \mathrm{d}x$ thus

$$\int_{x_0}^{x_1} \frac{\mathrm{d}\psi}{\mathrm{d}x} \frac{\mathrm{d}\phi}{\mathrm{d}x} \, \mathrm{d}x = \psi \frac{\mathrm{d}\phi}{\mathrm{d}x} \Big|_{x=x_1} - \psi \frac{\mathrm{d}\phi}{\mathrm{d}x} \Big|_{x=x_0} - \int_{x_0}^{x_1} \psi \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} \, \mathrm{d}x$$

Consider $\psi(x) = q(x)$ and $\phi(x) = \int p(x) \, \mathrm{d}x$ we have the integration by parts formula in 1D. For 2D case, we have $\boldsymbol{v} = P(x, y) i + Q(x, y) j$, $\boldsymbol{n} = \frac{\mathrm{d}y}{\mathrm{d}s} i - \frac{\mathrm{d}x}{\mathrm{d}s} j$ and $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j$. The formula goes to:

$$\iint_A \left(\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\mathrm{d}Q}{\mathrm{d}y} \right) \mathrm{d}x \, \mathrm{d}y = \oint_C \left(P \frac{\mathrm{d}y}{\mathrm{d}x} - Q \frac{\mathrm{d}x}{\mathrm{d}x} \right) \mathrm{d}s = \oint_C (P \, \mathrm{d}y - Q \, \mathrm{d}x)$$

Lemma 9.3 (fundamental lemma of the calculus of variations). *If $g(x)$ is a continuous function in $x_0 \le x \le x_1$ and if*

$$\int_{x_0}^{x_1} g(x) h(x) \, \mathrm{d}x = 0$$

where $h(x)$ is an arbitrary function in the same interval with $h(x_0) = h(x_1) = 0$ then $g(x) = 0$ at every point in the inverval $x_0 \le x \le x_1$.

can be proved by contradiction.

9.2 Calculus of variations

Definition 9.4 (Varation of a function). consider the functional $I[u(x)]$ of the function $u(x)$ of the form

$$I[u(x)] = \int_{x_0}^{x_1} F(x, u(x), u'(x)) \, \mathrm{d}x$$

where $u(x)$ is smooth and is the family of curves passing through the speficied end points $u(x_0) = u_0$, $u(x_1) = u_1$.

Then consider $\hat{u}(x) = u(x) + \left(\frac{\partial u}{\partial \epsilon} \big|_{\epsilon=0} \right) \epsilon + O(\epsilon^2)$. And the variation of u is

$$O(\epsilon) = \delta u = \left(\frac{\partial u}{\partial \epsilon} \Big|_{\epsilon=0} \right) \epsilon := \epsilon \eta(x)$$

where $\eta(x_0) = \eta(x_1) = 0$.

Next we derive the definition of functional derivative, starting from

$$\frac{\mathrm{d}I[u(x)]}{\mathrm{d}\epsilon} \Big|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{I[\hat{u}(x)] - I[u(x)]}{\epsilon}$$
$$= \lim_{\epsilon \to 0} \frac{I[u + \epsilon \eta] - I[u]}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{x_0}^{x_1} F(x, u + \epsilon\eta, u' + \epsilon\eta') \, \mathrm{d}x - \int_{x_0}^{x_1} F(x, u, u') \, \mathrm{d}x \right)$$

Then using Talor series expansion for the first integral, we have

$$\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{x_0}^{x_1} F(x, u, u') + \epsilon\eta \frac{\partial F}{\partial u} + \epsilon \frac{\mathrm{d}\eta}{\mathrm{d}x} \frac{\partial F}{\partial u'} + O(\epsilon^2) \, \mathrm{d}x - \int_{x_0}^{x_1} F(x, u, u') \, \mathrm{d}x \right)$$

$$= \int_{x_0}^{x_1} \eta \frac{\partial F}{\partial u} + \frac{\mathrm{d}\eta}{\mathrm{d}x} \frac{\partial F}{\partial u'} \, \mathrm{d}x$$

Applying the integration by parts to the second part by letting $p = \frac{\partial F}{\partial u'}$ and $q = \eta$, we end up with

$$\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} = \int_{x_0}^{x_1} \eta \frac{\partial F}{\partial u} \, \mathrm{d}x + \left(\frac{\partial F}{\partial u'} \eta \right)\bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) \, \mathrm{d}x = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) \right) \eta \, \mathrm{d}x$$

Thus we define the functional derivative

Definition 9.5 (The functional derivative)**.**

$$\frac{\delta I}{\delta u} = \frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right)$$

so that we have $\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\big|_{\epsilon=0} = \int_{x_0}^{x_1} \frac{\delta I}{\delta u} \eta \, \mathrm{d}x$

At the extrema of the functional $I[u(x)]$, we would expect that $\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\big|_{\epsilon=0} = 0$. Nest we find the stationary function of the functional I. We write

$$\frac{\mathrm{d}I}{\mathrm{d}\epsilon} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_{x_0}^{x_1} F \, \mathrm{d}x = \int_{x_0}^{x_1} \frac{\mathrm{d}F}{\mathrm{d}\epsilon} \, \mathrm{d}x$$

$$\frac{\mathrm{d}F}{\mathrm{d}\epsilon} = \frac{\partial F}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}\epsilon} + \frac{\partial F}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}\epsilon} + \frac{\partial F}{\partial u'} \frac{\mathrm{d}u'}{\mathrm{d}\epsilon}$$

Notice that x is independent of ϵ so the first term vanishes. Next using the definition of $u(x) = \hat{u}(x) - \epsilon\eta(x)$ we have

$$\frac{\mathrm{d}u}{\mathrm{d}\epsilon} = -\eta(x), \frac{\mathrm{d}u'}{\mathrm{d}\epsilon} = -\eta'(x)$$

and thus to make sure $\frac{\mathrm{d}I}{\mathrm{d}\epsilon} = 0$, we want

$$\int_{x_0}^{x_1} \left(\frac{\partial F}{\partial u} \eta + \frac{\partial F}{\partial u'} \eta' \right) \, \mathrm{d}x = 0$$

Using integrating by parts again, the second term goes to

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial u'} \eta' \, \mathrm{d}x = \left(\frac{\partial F}{\partial u'} \eta \right)\bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) \eta \, \mathrm{d}x$$

Definition 9.6 (Natural boundary condition)**.** To continue, here we add the natural boundary condition for integrand $F(x, u, u')$. It's

$$\frac{\partial F}{\partial u'}\bigg|_{x_0} = \frac{\partial F}{\partial u'}\bigg|_{x_1} = 0$$

Remark 9.7. If $u(x)$ are fixed at the end points, then $\delta u = 0$ at the end points so we don't need this natural boundary condition.

Now the first term vanishes, we can plug it in and end up with

$$\int_{x_0}^{x_1} \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) \right) \eta \, \mathrm{d}x = 0$$

notice η is an arbitrary function thus we end up with the following equation:

Definition 9.8 (Euler-Lagrange equation)**.** For an integrand of the form $F = F(x, u, u')$, the Euler equation is:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) = 0$$

It's necessary, but not sufficient condition for an extremum of $I[u(x)] = \int F \, \mathrm{d}x$.

Proposition 9.9. *Another form obtained by expanding the terms using total derivative:*

$$\frac{\partial F}{\partial u} - \left(\frac{\partial^2 F}{\partial u' \partial x} + \frac{\partial^2 F}{\partial u' \partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial^2 F}{\partial u'^2} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right) = 0$$

Corollary 9.10. *A special case when $F = F(u, u')$ that does not explicitly depend on the independent variable x. We can do the following: Supposing $x(u)$ exists.*

$$
\begin{aligned}
I[u(x)] &= \int_{x_0}^{x_1} F(u, u') \, \mathrm{d}x \\
&= \int_{u_0}^{u_1} F\left(u, \left(\frac{\mathrm{d}x}{\mathrm{d}u} \right)^{-1} \right) \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \\
&= \int_{u_0}^{u_1} \bar{F}(u, x') \, \mathrm{d}u \\
\bar{F}(u, x') &= x' F(u, (x')^{-1})
\end{aligned}
$$

Then since here $\partial \bar{F} / \partial x = 0$ the Euler equation is reduced to

$$-\frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{\partial \bar{F}}{\partial x'} \right) = 0 \implies \frac{\partial \bar{F}}{\partial x'} = c$$

It can be alternatively derived in another way and obtained in a different form, with a different name: Beltrami identity:

$$F - \frac{\partial F}{\partial u'} \frac{\mathrm{d}u}{\mathrm{d}x} = c$$

Corollary 9.11. *This form can also be obtained when encountering end points on different curves. Supposing $u(0) = f(x_0)$ and $u(1) = g(x_1)$. We can actually show that the Euler equation is of the usual form except at the ends, where we have the transversality conditions:*

$$F + (f' - u') \frac{\partial F}{\partial u'} = 0, x = x_0$$

$$F + (g' - u') \frac{\partial F}{\partial u'} = 0, x = x_1$$

9.3   Summary on notation and definitions

- the functional of interest: $I[u(x)] = \int_{x_0}^{x_1} F(x, u, u') \, \mathrm{d}x$

- variation of function u: $\delta u = O(\epsilon) = \epsilon \eta(x)$

- especially here we have $\delta u' = (\delta u)'$ meaning the variational operator δ and the differential operator $\mathrm{d}/\mathrm{d}x$ are commutative if the differentiation is wrt an independent variable.

- variation of functional $I[u]$:

$$\delta I = \delta \int_{x_0}^{x_1} F(x, u, u') \, \mathrm{d}x = \int_{x_0}^{x_1} \delta F(x, u, u') \, \mathrm{d}x = 0$$

- total differential of $F(x, u, u')$:

$$\mathrm{d}F = \frac{\partial F}{\partial x} \, \mathrm{d}x + \frac{\partial F}{\partial u} \, \mathrm{d}u + \frac{\partial F}{\partial u'} \, \mathrm{d}u'$$

- (total) variation of $F(x, u, u')$:

$$\delta F = \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

- functional derivative: As $\left. \dfrac{\mathrm{d}I}{\mathrm{d}\epsilon} \right|_{\epsilon=0} = \lim_{\epsilon \to 0} \dfrac{I[u + \delta u] - I[u]}{\epsilon} = \displaystyle\int_{x_0}^{x_1} \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) \right) \eta \, \mathrm{d}x$, we define

$$\frac{\delta I}{\delta u} = \frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right)$$

- Euler equation for integrand $F(x, u, u')$: $\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) = 0$

Some fun fact about variational principles

- straight line: shortest distance between two points

- circle: shortest curve enclosing a given area (Dido's problem)

- sphere: smallest area enclosing a given volume

- isosceles triangle: a triangle having the shortest perimeter with a given base line and area

In higher dimension, the results are:

- Euler equation for $F(x, u, u', u'')$

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\partial F}{\partial u''} \right) = 0$$

- expanding results in a 4-th order ode, which gives rise to the following

- two possible boundary condition sets: first is to specify u and u' at the end points so that δu and $\delta u'$ at x_0, x_1 are both zero. Or use natural boundary conditions, that at x_0, x_1 we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u''} \right) - \frac{\partial F}{\partial u'} = 0, \frac{\partial F}{\partial u''} = 0$$

- for $F(x, u, u', \ldots, u^{(n)})$, the Euler equation is

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u'} \right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\partial F}{\partial u''} \right) - \cdots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\frac{\partial F}{\partial u^{(n)}} \right) = 0$$

We'll move to function of more independent variables.

9.4 Functional of Two Independent Variables

Generally, we consider the following functional where $u(x, y)$ depends on two independent variables x and y.

$$I[u(x, y)] = \iint_A F(x, y, u, u_x, u_y) \, \mathrm{d}x \, \mathrm{d}y$$

Here subscripts denote partial differential instead of components. By setting the variation of the functional equal to 0, the Euler Equation is obtained via the following steps:

$$\delta I[u(x, y)] = 0$$
$$= \iint_A \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_x} \delta(u_x) + \frac{\partial F}{\partial u_y} \delta(u_y) \, \mathrm{d}x \, \mathrm{d}y$$
$$= \iint_A \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_x} (\delta u)_x + \frac{\partial F}{\partial u_y} (\delta u)_y \, \mathrm{d}x \, \mathrm{d}y$$
$$= \iint_A \frac{\partial F}{\partial u} \delta u + \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u \right] + \left[\frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] \mathrm{d}x \, \mathrm{d}y$$
$$= \iint_A \frac{\partial F}{\partial u} \delta u + \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) \right] - \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] \mathrm{d}x \, \mathrm{d}y$$

Next we use the divergence theorem on the second term above. Let $P = \dfrac{\partial F}{\partial u_x} \delta u$ and $Q = \dfrac{\partial F}{\partial u_y} \delta u$, then we have

$$\iint_A \left(\frac{\partial F}{\partial u} - \left(\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right) \right) \delta u \, \mathrm{d}x \, \mathrm{d}y + \oint_C \left(P \frac{\mathrm{d}y}{\mathrm{d}x} - Q \frac{\mathrm{d}x}{\mathrm{d}x} \right) \mathrm{d}s = 0$$

where the second term can be written explicitly as

$$\oint_C \left(\frac{\partial F}{\partial u_x} \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial F}{\partial u_y} \frac{\mathrm{d}x}{\mathrm{d}s} \right) \delta u \, \mathrm{d}s$$

This results in the following Euler equation.

Theorem 9.12. *Euler's equation for integrand* $F(x, y, u, u_x, u_y)$ *is*

$$\frac{\partial F}{\partial u} - \left(\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right) = 0$$

So that the first integration above goes to 0. For the second integration, either we speficy $u(x, y)$ at the boundary so that $\delta u = 0$, or we apply the natural boundary condition:

$$\frac{\partial F}{\partial u_x} \frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial F}{\partial u_y} \frac{\mathrm{d}x}{\mathrm{d}s} = 0$$

In vector form, we write the Euler equation as:

$$\frac{\partial F}{\partial u} - \nabla \cdot \frac{\partial F}{\partial \nabla u} = 0$$

The natural boundary condiiton is:

$$\boldsymbol{n} \cdot \frac{\partial F}{\partial \nabla u} = 0$$

Remark 9.13 (arc length formula in 2D). smooth function $u(x, y)$ enclosing an area A have area:

$$\iint_A \sqrt{1 + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2} \, \mathrm{d}x \, \mathrm{d}y$$

A generalization to the length formula in 1D

$$\int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2} \, \mathrm{d}x$$

9.5 Inverse problem

Convert known differential equations into their equivalent variational form. Here's the general steps we can follow.

1. Write the differential equation in operator form: $\mathcal{L}u = 0$

2. times δu then take integral and let it be zero: $\int_\Omega \mathcal{L}u\delta u\, \mathrm{d}\boldsymbol{x} = 0$

3. using integration by parts and other formula to finally obtain $\delta I[u] = 0$

Here's an example of the 2D rectangle Dirichlet problem. We begin with the differential equation:

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$$

along with the specified boundary condition on the boundary C: $u(x, y) = u_0(x, y)$.

$$
\begin{aligned}
0 &= \iint_\Omega \Delta u \delta u\, \mathrm{d}x\, \mathrm{d}y \\
&= \iint_\Omega u_{xx}\delta u\, \mathrm{d}x\, \mathrm{d}y + \iint_\Omega u_{yy}\delta u\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} \delta u\, \mathrm{d}u_x \right) \mathrm{d}y + \int_{x_0}^{x_1} \left(\int_{y_0}^{y_1} \delta u\, \mathrm{d}u_y \right) \mathrm{d}x \\
&= \int_{y_0}^{y_1} \left(0 - \int_{x_0}^{x_1} u_x\, \mathrm{d}\delta u \right) \mathrm{d}y + \int_{x_0}^{x_1} \left(0 - \int_{y_0}^{y_1} u_y\, \mathrm{d}\delta u \right) \mathrm{d}x \\
&= -\int_{y_0}^{y_1} \int_{x_0}^{x_1} u_x\delta u_x\, \mathrm{d}x\, \mathrm{d}y - \int_{x_0}^{x_1} \int_{y_0}^{y_1} u_y\delta u_y\, \mathrm{d}y\, \mathrm{d}x \\
&= -\frac{1}{2} \int_\Omega \delta\left(u_x^2 + u_y^2 \right) \mathrm{d}x\, \mathrm{d}y \\
&= \delta \int_\Omega \nabla u \cdot \nabla u\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}
$$

Remark 9.14. This process can always be done on linear, self-adjoint differential operators.

Remark 9.15. For poisson equation: $\mathcal{L}u = f$ similarly we have the inverse problem solution:

$$I[u] = \iint_\Omega \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + 2fu \right) \mathrm{d}x\, \mathrm{d}y$$

9.6 Functionals of Two Dependent Variables

Theorem 9.16. *Euler's Equation for* $F(x, y, u, v, u_x, v_x, u_y, v_y)$ *are*

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} \right) = 0$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y} \right) = 0$$

Either we use fixed boundary conditions or the following natural boundary conditions:

$$\frac{\partial F}{\partial u_x}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial F}{\partial u_y}\frac{\mathrm{d}x}{\mathrm{d}s} = 0$$

$$\frac{\partial F}{\partial v_x}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial F}{\partial v_y}\frac{\mathrm{d}x}{\mathrm{d}s} = 0$$

9.7 Constrained Functionals

Integral constraints

Use Lagrange multiplier method, notice the multiplier hereis treated as a number not a function.

Consider the functional $I[u] = \int_{x_0}^{x_1} F(x, u, u') \, \mathrm{d}x$ subject to $\int_{x_0}^{x_1} G(x, u, u') \, \mathrm{d}x = C$. We build $J[u] = \int_{x_0}^{x_1} \tilde{F}(\lambda, x, u, u') \, \mathrm{d}x$ where $\tilde{F}(\lambda, x, u, u') = F + \lambda G$ and then solve $\delta J[u] = 0$. We call $\tilde{F}$ the augmented integrand. Multiple constraints are treated similarly. We write $\tilde{F} = F + \sum_i \lambda_i G_i$. The Euler's equation is simimlar:

$$\frac{\partial \tilde{F}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial u'}\right) = 0$$

One specific type of problems: Sturm-Liouville problems, where the integrand has the form: $F(x, u, u') = q(x)u^2 - p(x)(u')^2$, and the constraint has the form $G(x, u, u') = r(x)u^2$ and $C = 1$. Three functions r, p, q are known. The simplified Euler's equation is:

$$q(x)u + \lambda r(x)u - \frac{\mathrm{d}}{\mathrm{d}x}\left(-p(x)u'\right) = 0$$

That's how this type of problem got its name for this is a Sturm-Liouville equation. Notice here λ is an eigenvalue of the Sturm-Liouville differential operator.

Algebraic and differential constraints

Consider the functional $I[u, v] = \int_{x_0}^{x_1} F(x, u, v, u', v') \, \mathrm{d}x$ subject to $\phi(u, v, u', v') = 0$.

Notice in this case to use the multiplier, it's no longer a number but a continuous function of the independent variable(s). We write the augmented integrand as

$$\tilde{F} = F + \lambda(x)\phi$$

The Euler's equation is the same: $\frac{\partial \tilde{F}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial u'}\right) = 0$ and $\frac{\partial \tilde{F}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial v'}\right) = 0$.

10 Hamilton's Principle

10.1 Hamilton's Principle for Discrete Systems

Consider the system to be a single $O(1)$-sized rigid particle, a point mass m acted upon by a force $\boldsymbol{f}(t)$ and $\boldsymbol{r}(t)$ is the position vector of the system and t is independent variable.

Theorem 10.1 (Hamilton's Variational Principle for a Single Particle). *In order to determine the actual trajectory $\boldsymbol{r}(t)$ of the system over the finite time $t_0 \leq t \leq t_1$, Hamilton's principle asserts that the trajectory must be such that $\boldsymbol{r}(t)$ is a stationary funciton of the functional*

$$\int_{t_0}^{t_1} \delta T + \boldsymbol{f} \cdot \delta \boldsymbol{r} \, \mathrm{d}t = 0$$

- *For the first term, T is the kinetic energy of the system and it equals $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2 := \frac{1}{2}m\boldsymbol{v}^2$. It's a scalar.*

- *For the second term, $\boldsymbol{f} \cdot \delta \boldsymbol{r}$ is the virtual work owing to the application of the actual resultant external force $\boldsymbol{f}$ acting on the system through the infinitesimal virtual displacement $\delta \boldsymbol{r}$. Some examples including the gravitational potential, electrostatic potential, and deformation potential.*

Remark 10.2. Notice the principle only asserts that $\boldsymbol{r}(t)$ is the stationary solution, thus not necessaryily a minimum.

By variation, the Euler's equation for this system is the newoton's second law.

$$m\ddot{\bm{r}} - \bm{f} = 0$$

Remark 10.3. There's a subtle difference betweent the differential form (local law) and the variatonal form (global law). Newton's second law requires two initial conditions at the initial time, position and velocity, thus a initial-value problem. And the Hamilton's principle requires the positions of initial and ending time, thus a boundary-value problem.

Next we consider the case when nonconservative forces is involved.

Definition 10.4 (Conservative forces). $\bm{f}_c$, forces that the virtual work done W_c is independent of the path and only depends on the initial and final positions. Such work is recoverable, and the process is reversible.

If the work done depend on the path, then the force is nonconservative.

Definition 10.5 (Mechanical energy). The capacity to do work.

- kinetic energy: from its motion

- potential energy: from its position in a conservative force field.

A system is conservative when its mechanical energy is conserved.

Thus far, we write

$$\bm{f} \cdot \delta\bm{r} = \bm{f}_c \cdot \delta\bm{r} + \bm{f}_{nc} \cdot \delta\bm{r}$$
$$W_c = \int_{\bm{r}_1}^{\bm{r}_2} \bm{f}_c \cdot \delta\bm{r} = \int_{\bm{r}_1}^{\bm{r}_2} \delta\Gamma = \Gamma(\bm{r}_2) - \Gamma(\bm{r}_1)$$
$$\Gamma = -V$$
$$\bm{f}_c \cdot \delta\bm{r} = -\delta V$$

In cartisian coordinates, we also have

$$\delta V = \nabla V \cdot \delta\bm{r} \implies \bm{f}_c = -\nabla V$$

So the Conservative force maybe expressed as the gradient of the scalar potential energy function.

Theorem 10.6 (Hamilton's Variational Principle for Conservative and Nonconservative Forces).

$$\int_{t_0}^{t_1} \delta(T - V) + \bm{f}_{nc} \cdot \delta\bm{r}\, \mathrm{d}t = 0$$

And if there's only conservative force, we may write

$$\delta \int_{t_0}^{t_1} L\, \mathrm{d}t := \delta\mathcal{A} = 0$$

where $L = T - V$ is called the Lagrangian. And $\mathcal{A}$ is the action integral. This formula is also referred as the lease action principle (LAP).

10.2 Hamilton's Principle for Continuous Systems

In the previouis subsection, the particles are nondeformable and thus the virtual work is only that from the external forces acting on the system. Now we turn to continuous systems comprised of deformable bodies in which the mass is distributed throughout the body.

We can apply the discrete version of the principle to differential elements of the continuous system $\bar{V}$ for volumn, $\bar{A}$ for area and $\bar{S}$ for length, and we sum over the entire volumn through integration. Some other changes including

- Lagrangian: $L = \int \mathcal{L}\, \mathrm{d}\bar{V}$ where $\mathcal{L}$ is the Lagrangian density, and it equals the kinetic energy per unit minus the potential energy per unit: $\mathcal{L} = \mathcal{T} - \mathcal{V}$

- Other than external forces, also put internal forces that deform the continuous system

10.3 Euler-Lagrange Equations

For now we only consider the discrete cases. We begin with following series of definitions and formulae. For a dynamical system with n degrees of freedom,

- Degree of freedom: the minimum number of scalar dependent variables required to fully describe the motion or state of the system.

- Generalized coordinates (dependent): $q_i(t)$, $i = 1, 2, \ldots, n$;

- Generalized velocities: $\dot{q}_i(t)$, where dots denote differentiation wrt time;

- Virtual displacements: δq_i, the infinitesimally small variations.

- Kinetic energy: $T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$,

- Potential energy: $V(q_1, \ldots, q_n)$,

- Lagrangian: $L(t, q_i, \dot{q}_i) = T(t, q_i, \dot{q}_i) - V(t, q_i)$,

- Euler-Lagrange equations for an n-DOF with conservative forces: $\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$, $i = 1, 2, \ldots, n$;

- and the equivalent form: $\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = 0$ since $\frac{\partial V}{\partial \dot{q}_i} = 0$

- The virtual work: $\boldsymbol{f} \cdot \delta \boldsymbol{r} = \sum_i \boldsymbol{f} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} \delta q_i = \sum_i Q_i \delta q_i$, and we call $Q_i = \boldsymbol{f} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i}$ the generalized force.

- The conservative force: $Q_i = \boldsymbol{f} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} = -\frac{\partial V}{\partial q_i}$

- The Euler-Lagrangian equations of motion for an n-DOF system with nonconservative forces: $\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -Q_i$ where Q_i are the generalized forces owing to nonconservative forces acting on the system, and the potential owing to conservative forces are included in the Lagrangian $L = T - V$.

- Generalized momenta: $p_i = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}$ and we can use $\dot{p}_i$ to simplify the E-L equations: $\frac{\partial L}{\partial q_i} = \dot{p}_i$.

Proposition 10.7 (Invariance of Euler-Lagrangian Equations)**.** *E-L equations are independent of the coordinate system, say q_i and q_i^*, $i = 1, 2, \ldots, n$, under the condition that matrix $\dfrac{\partial q_j^*}{\partial q_i}$ is non-singular. Then for $L(t, q_i, \dot{q}_i) = L^*(t, q_i^*, \dot{q}_i^*)$ we have*

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L^*}{\partial q_i^*} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dot{q}_i^*}\right) = 0$$

Corollary 10.8. *During the derivation, a byproduct is that*

$$\frac{\partial \dot{q}_j^*}{\partial \dot{q}_i} = \sum_i \frac{\partial q_j^*}{\partial q_i}$$

10.4 From the First Law of Thermodynamics to the Hamilton's Principle

Theorem 10.9 (Conservation of Mass)**.** *For a closed system, where no matter is moving across the system boundary, then the mass of the system is a constant: $\dot{m} = 0$.*

Theorem 10.10 (The first law of thermodynamics)**.** *The conservation of energy: time rate of change of total energy of the system = time rate of energy transfer across the system boundary. And we call a system adiabatic if no hear transfer across the boundary. By assuming no work entering or exiting the system across the boundary, it requires that $\dot{E} = 0$.*

To derive the Hamilton's principle, we neglect heat transfer, work crossing the system boundary, and changes in thermodynamic internal energy, thus $E = T + V - W_{nc} = 2T - L - W_{nc}$ is a constant. We also assume the system is autonomous. The derivation is skipped. Here's a byproduct.

Corollary 10.11. *During the derivation, a byproduct is that*

$$E = 2T - L - W_{nc} = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L - W_{nc}$$

Also performing the inverse problem from the E-L equations can also give Hamilton's principle. We consider

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + Q_i \right) \delta q_i \, \mathrm{d}t \\
&= - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) \Bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + Q_i \delta q_i \right) \, \mathrm{d}t \\
&= 0 + \int_{t_0}^{t_1} \left(\delta L + \sum_i Q_i \delta q_i \right) \, \mathrm{d}t \\
&= \int_{t_0}^{t_1} (\delta L + \boldsymbol{f}_{nc} \cdot \delta \boldsymbol{r}) \, \mathrm{d}t \\
&= \int_{t_0}^{t_1} (\delta T + \boldsymbol{f} \cdot \delta \boldsymbol{r}) \, \mathrm{d}t
\end{aligned}
$$

10.5 Conservation of Mechanical Energy and the Hamiltonian

Consider the system with only conservative forces acting on, $Q_i = 0$, $W_{nc} = 0$. The first law of thermodynamics reads $E = H = T + V$ being a constant. Here H is the total mechanical energy of a conservative system. And similarly we have

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i p_i - L$$

Proposition 10.12 (Hamilton's canonical equations)**.** *We have following two equivalent expressions for Euler-Lagrangian Equations of motion using Hamiltonian $i = 1, 2, \ldots, n$, notice p_i is generalized momenta and L is assumed to have the form $L(q_i, \dot{q}_i)$*

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \frac{\partial H}{\partial p_i} = \dot{q}_i$$

Remark 10.13. With above two relations, we can find that H and L are the Legendre transform of each other. In ideal condition, for function $f(x)$, we write its Legendre transform $f^*(p) = (px - f(x)) \mid_{x=(f')^{-1}(p)}$

Theorem 10.14 (Noether's Theorem)**.** *This theorem will build connection between Hamilton's principle and Conservation laws. It assets that if the Lagrangian exhibits some mathematical symmetry, for which it is invariant under a transformation in one of the variables on which it depends, denoted by s, then an associated property C exists that is conserved such that $\dot{C} = 0$.*

First example starting from Hamilton's principle for conservative autonomous systems, L is independent of $s = t$ and thus invariant, then

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}t}(2T - L - W_{nc}) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = 0$$

so here $C = H = T + V$.

Then consider the E-L equations, as we have proved the invariance of E-L equations under $L = L^*$, so the variable here is $s = q_i$. And we can also consider the case where L is independent of q_i so that $\frac{\partial}{\partial q_i} = 0$, then L

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} = 0$$

So we can let $C = \frac{\partial L}{\partial \dot{q}_i} = p_i$.

11 Classical Machanics

Other than the examples, here're some important definites and formulae.

- in three-dimensional cylindrical coordinates we have the kinetic energy: $T = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{q}_1^2 + (q_1\dot{q}_2)^2 + \dot{q}_3^2\right)$

  - The generalized force $m\ddot{q}_1 - mq_1\dot{q}_2^2 = Q_1 = -\frac{\partial V}{\partial q_1}$ where $mq_1\dot{q}_2^2$ is the centrifugal acceleration
  - The generalized force $mq_1^2\ddot{q}_2 + 2mq_1\dot{q}_1\dot{q}_2 = Q_2 = -\frac{\partial Q}{\partial q_2}$ where $2mq_1\dot{q}_1\dot{q}_2$ is the Coriolis acceleration.
  - And $m\ddot{q}_3 = Q_3 = -\frac{\partial V}{\partial q_3}$

- in the dynamics problems with constraints, we may use augmented Lagrangian $\tilde{L} = T - V + \lambda(t)\phi(q_i)$. And we call holonomic constraints for those constrain the coordinates of the system.

- Principle of virtual work: at the static status, all forces are balanced, system do not move with time, and thus the virtual work must be zero.

- In statics problem, the potential energy is the total virtual work (sum or integration of $\boldsymbol{f} \cdot \delta\boldsymbol{q}$) that would be required to move the system through a virtual displacement $\delta\boldsymbol{q}$ from its unloaded condition to its fully-loaded condition, where it is in static equilibrium.