HW0

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1

Prove $|AB| = |A|\,|B|$ when both are 2×2 matrices.

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then

$$\begin{aligned}
\text{LHS} &= \begin{vmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{vmatrix} \\
&= a_1a_3b_1b_2 + a_1a_4b_1b_4 + a_2a_3b_2b_3 + a_2a_4b_3b_4 - a_1a_3b_1b_2 - a_2a_3b_1b_4 - a_2a_4b_3b_4 \\
&= a_1a_4b_1b_4 + a_2a_3b_2b_3 - a_2a_3b_1b_4 - a_1a_4b_2b_3 \\
\text{RHS} &= (a_1a_4 - a_2a_3)\,(b_1b_4 - b_2b_3) \\
&= a_1a_4b_1b_4 + a_2a_3b_2b_3 - a_2a_3b_1b_4 - a_1a_4b_2b_3 \\
&= \text{LHS}
\end{aligned}$$

2

Determine $A^\top$, $\operatorname{Adj}(A)$, A^{-1} when $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$A^\top = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\operatorname{Adj}(A) = \begin{bmatrix} 1\cdot 1 - 0 & -(2\cdot 1 - 0) & 2\cdot 0 - 1\cdot 1 \\ -2\cdot 1 + 0 & -1\cdot 1 + 1\cdot(-1) & -0 + (-1)\cdot 2 \\ 0 - (-1)\cdot 1 & 0 - 2\cdot 1 & 1 - 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\operatorname{Adj}(A) = -\frac{1}{2}\operatorname{Adj}(A) = \begin{bmatrix} -1/2 & 1 & 1/2 \\ 1 & -1 & -1 \\ -1/2 & -1 & 3/2 \end{bmatrix}$$

3

Prove $|A| = \left|A^\top\right|$ using cofactor matrix expansion

Denote $A^\top = (b_{ij})_{3\times 3}$ and corresponding cofactor matrix D_{ij}. Then we expand $A^\top$ along the first column and obtain

$$\left|A^\top\right| = b_{11}D_{11} + b_{21}D_{21} + b_{31}D_{31}$$

Notice $b_{ij} = a_{ji}$,$D_{ij} = (-1)^{i+j}B(i\,|\,j) = (-1)^{i+j}A(j\,|\,i) = C_{ji}$, where $B(i\,|\,j)$ denote the new matrix obtained by removing row i and column j from matrix $B = A^\top$, then

$$\left|A^\top\right| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{31}$$

Corresponding to the cofactor expansion of A along first row, and thus $|A| = \left|A^\top\right|$.

4

Gaussian elimination to solve

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

where $b_1^2 + b_2^2 > 0$. Show it's same as using Cramer's Rule and then solve

$$\begin{cases} x_1 + 5x_2 = 3 \\ 7x_1 - 3x_2 = -1 \end{cases}$$

For the general form, multiply a_{21} to first eqn and minus a_{11} multiply second eqn, then we obtain

$$(a_{21}a_{12} - a_{11}a_{22})x_2 = a_{21}b_1 - a_{11}b_2 \implies x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{D_2}{|A|}$$

Similarly, multiply a_{22} to first eqn and minus a_{12} multiply second eqn, then we obtain

$$(a_{11}a_{22} - a_{21}a_{12})x_1 = a_{22}b_1 - a_{12}b_2 \implies x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{D_1}{|A|}$$

where D_i is obtained by replacing A's i-th column to $[b_1, b_2]^\top$.
For the case with actual numbers, using cramer's rule, the solution is

$$x_1 = \frac{-4}{-38} = \frac{2}{19}, x_2 = \frac{-22}{-38} = \frac{11}{19}$$

5

$$P(\lambda) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

Expand it and solve it.

$$P(\lambda) = (4-\lambda)(-\lambda)(2-\lambda) + 0 + (-2) - \lambda - 0 - (4-\lambda)(-2) = -(\lambda-2)(\lambda+1)(\lambda+3)$$

Thus when λ equals $2, -1, -3$, $P(\lambda)$ vanishes.

6

solve

$$Ax = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 5 & 6 & 7 & 2 \\ 2 & 1 & 0 & 1 \end{bmatrix} x = 0$$

Using Gaussian elimination, above is reduced to

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 7 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} x = 0$$

Then start with letting $x_2 = k$, we solve it as $x = [-4k, k, 0, 7k]^\top$ for $k \in \mathbb{R}$.

7

Given augumented matrix, write the equations and see if consistent.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 4 \\ 0 & 3 & 5 & 1 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + x_3 = 4 \\ 3x_2 + 5x_3 = 1 \end{cases}$$

Using gaussian elimination, we can obtain

$$\begin{cases} -4x_3 = 8 \\ 8x_3 = -11 \end{cases}$$

Thus inconsistent.