HW2

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1

Determine those values of λ for which the following set of equations may possess a nontrivial solution

$$\begin{cases} 3x_1 + x_2 - \lambda x_3 = 0 \\ 4x_1 - 2x_2 - 3x_3 = 0 \\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0 \end{cases}$$

and for each permisssible value of λ, determine the most general solution.

Write this in the form $Ax = 0$ we know

$$\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}$$

and we can use Carmer's rule. To have nontrivial solution, we want $\det A = 0$ at least, otherwise according to the formula, we'll get 0 on the numerator and thus all solution would be trivial. So we want

$$-6\lambda - 6\lambda - 16\lambda - 4\lambda^2 - 4\lambda + 36 = -4(\lambda^2 + 8\lambda - 9) = -4(\lambda + 9)(\lambda - 1) = 0$$

this gives $\lambda = 1, -9$. Nest the general solution. For $\lambda = 1$, after some basic gaussian elimination, we obtain

$$\begin{cases} 2x_1 - x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

So letting $x_1 = k$, we obtain the solution $x = [k, -k, 2k]$ for $k \in \mathbb{R}$.
For $\lambda = -9$, similarly, we obtain

$$\begin{cases} -2x_1 - 3x_3 = 0 \\ -3x_1 + x_2 = 0 \end{cases}$$

So letting $x_1 = 3k$, we obtain the solution $x = [3k, 9k, -2k]$ for $k \in \mathbb{R}$.

2

a)

By investigating ranks of relevant matrices, show that the following set of equations possess a one-parameter family of solutions:

$$\begin{cases} 2x_1 - x_2 - x_3 = 2 \\ x_1 + 2x_2 + x_3 = 2 \\ 4x_1 - 7x_2 - 5x_3 = 2 \end{cases}$$

b)

Determine the general solution

a)

Write in matrix form $Ax = b$ where $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$ and the augumented matrix $A_{aug} = \begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$. Then, to find ranks,

$$rank(A) = rank(\begin{bmatrix} 0 & -5 & -3 \\ 1 & 2 & 1 \\ 0 & -15 & -9 \end{bmatrix}) = 2$$

$$rank(A_{aug}) = rank(\begin{bmatrix} 0 & -5 & -3 & -2 \\ 1 & 2 & 1 & 2 \\ 0 & -15 & -9 & -6 \end{bmatrix}) = 2$$

Thus it's a consistent system and since $2 < 3$ thus there should be infinite solutions with the number of independent variables $3 - 2 = 1$, i.e., one-parameter family of solutions.

b)

Start by letting $x_3 = k$, then from $-5x_2 - 3x_3 = -2$, we have $x_2 = \dfrac{2 - 3k}{5}$; next from $x_1 + 2x_2 + x_3 = 2$, we have $x_1 = \dfrac{k + 6}{5}$ so finally the general solution is $x = [\frac{k+6}{5}, \frac{2-3k}{5}, k]$

3

Determine whether the vector $\{6, 1, -6, 2\}$ is in the vector space generated by the basis vectors $\{1, 1, -1, 1\}$, $\{-1, 0, 1, 1\}$, $\{1, -1, -1, 0\}$

Write this question in matrix form, we are to solve x for

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} 6 \\ 1 \\ -6 \\ 2 \end{bmatrix}$$

And the solution is $x = [3, -1, 2]$, thus the vector can be represented by the 3 basis vectors.

4

Find eigen pairs of matrix $\begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

First find the characteristic polynomial, denote A to be the matrix then

$$\det(\lambda I - A) = \lambda(\lambda-1)(\lambda-3) + 2(\lambda-1) = (\lambda-1)^2(\lambda-2)$$

So the eigenvalues are $1, 2$. For $\lambda = 1$, we solve

$$(\lambda I - A)x = 0 = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix} x$$

And this gives the eigenvectors of the form $x = [2m, k, m]$ for $m, k \in \mathbb{R}$. Especially, we have two orthogonal choice, $x = [2, 0, 1]$ and $x = [0, 1, 0]$.
For $\lambda = 2$, we solve

$$(\lambda I - A)x = 0 = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & -2 \\ 1 & 0 & =1 \end{bmatrix} x$$

And this gives the eigenvectors $x = [n, n, n]$ for $n \in \mathbb{R}$.

5

Gram-Schmidt orthogonalization for following set of vectors. $[-1, 2, 0]^\top, [1, 1, -1]^\top, [1, -1, 1]^\top$.

Let the vectors be x_is. First we find orthogonal set: $\hat{x}_1 = x_1$, $\hat{x}_2 = x_2 - (\hat{x}_1, x_2)x_1/(\hat{x}_1, \hat{x}_1) = [1.2, 0.6, -1]^\top$, then $\hat{x}_3 = x_3 - (\hat{x}_1, x_3)\hat{x}_1/(\hat{x}_1, \hat{x}_1) - (\hat{x}_2, x_3)\hat{x}_2/(\hat{x}_2, \hat{x}_2) = [4/7, 2/7, 6/7]^\top$, then after normalization, we obtain the orthonormal set of vectors: $u_1 = [-1/\sqrt{5}, 2/\sqrt{5}, 0]^\top$, $u_2 = [6/\sqrt{70}, 3/\sqrt{70}, -5/\sqrt{70}]^\top$, $u_3 = [4/\sqrt{56}, 2/\sqrt{56}, 6/\sqrt{56}]^\top$

6

Matrix for the following quadratic form. And determine if it's positive definite, and the shape it makes.

$$1.5x_1^2 - x_1x_3 + x_2^2 + 1.5x_3^2$$

As of form $x^\top Ax$, we have $A = \begin{bmatrix} 1.5 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 1.5 \end{bmatrix}$. And its characteristic polynomial is $(\lambda-1)^2(\lambda-2)$ thus all eigenvalues are positive, so positive definite. So it's a Ellipsoid.

7

Given $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 4 & 0 \\ 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & 5 \\ 3 & 1 & 2 \end{bmatrix}$, verify $(AB)^{-1} = B^{-1}A^{-1}$.

After some calculations we get, $A^{-1} = \frac{1}{46}\begin{bmatrix} 12 & 2 & 4 \\ -3 & 11 & -1 \\ 7 & 5 & -13 \end{bmatrix}$; $B^{-1} = \frac{1}{36}\begin{bmatrix} 5 & -7 & 15 \\ -11 & 1 & 3 \\ -2 & 10 & -6 \end{bmatrix}$; $AB = \begin{bmatrix} 4 & -8 & 0 \\ 9 & -3 & 21 \\ -5 & -9 & 1 \end{bmatrix}$ and its inverse $(AB)^{-1} = \frac{1}{828}\begin{bmatrix} 93 & 4 & -84 \\ -57 & 2 & -42 \\ -48 & 38 & 30 \end{bmatrix}$ and $B^{-1}A^{-1} = \frac{1}{1656}\begin{bmatrix} 186 & 8 & -168 \\ -114 & 4 & -84 \\ -96 & 76 & 60 \end{bmatrix} = (AB)^{-1}$