HW0

Yuanxing Cheng, A20453410, CS577-f22

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A

let $a = [1,2,3]^\top$, $b = [4,5,6]^\top$, $c = [-1,1,3]^\top$.

1

Find $2a - b$

$$2a - b = [2 \cdot 1 - 4, 2 \cdot 2 - 5, 2 \cdot 3 - 3]^\top = [-2, -1, 0]^\top$$

2

Find $\hat{a}$ the unit vector along a

$$\hat{a} = \frac{a}{|a|} = \frac{[1,2,3]^\top}{\sqrt{1^2+2^2+3^2}} = \left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]$$

3

Find $\|a\|$ and teh angle of a relative to positive x axis

$$\|a\| = \|a\|_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\theta = \arccos\left(\frac{a \cdot i}{|a|\,|i|}\right) = \arccos\left(\frac{1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0}{\sqrt{14} \cdot 1}\right) = \arccos \frac{1}{\sqrt{14}}$$

4

Find direction cosines of a

$$\theta_1 = \theta = \arccos\left(1/\sqrt{14}\right)$$

$$\theta_2 = \arccos\left(\frac{a \cdot j}{|a|\,|i|}\right) = \arccos\left(\frac{1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0}{\sqrt{14} \cdot 1}\right) = \arccos\left(\frac{2}{\sqrt{14}}\right)$$

$$\theta_3 = \arccos\left(\frac{a \cdot k}{|a|\,|i|}\right) = \arccos\left(\frac{1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1}{\sqrt{14} \cdot 1}\right) = \arccos\left(\frac{3}{\sqrt{14}}\right)$$

5

Find angle between a and b

$$\theta_{a,b} = \arccos\left(\frac{a \cdot b}{|a| \cdot |b|}\right) = \arccos\left(\frac{1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6}{\sqrt{14} \cdot \sqrt{4^2+5^2+6^2}}\right) = \left(\frac{32}{7\sqrt{22}}\right)$$

6

Find $a \cdot b$ and $b \cdot a$

$$a \cdot b = b \cdot a = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

7

Find $a \cdot b$ by using the angle between a and b

$$a \cdot b = |a|\,|b| \cos\theta_{a,b} = 7\sqrt{22} \cdot \frac{32}{7\sqrt{22}} = 32$$

8

The scalar projection of b onto $\hat{a}$

$$|\mathrm{Proj}\,(b; \hat{a})| = b \cdot \hat{a} = \frac{32}{\sqrt{14}}$$

checking,

$$\mathrm{Proj}\,(b; \hat{a}) = \frac{aa^\top}{a^\top a} b = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} b / |a|^2 = [32, 64, 96]^\top / 14 = [16/7, 32/7, 48/7]^\top$$

9

Find a vector which is perpendicular to both a

consider $x = [0, -3, x_3]^\top$ and then since x is perpendicular to a, we have

$$0 = x \cdot a = -3 \cdot 2 + 3x_3 \implies x_3 = 2$$

so $x = [0, -3, 2]^\top \perp a$

10

Find $a \times b$ and $b \times a$

using matrix notation

$$a \times b = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = -3i - (-6)j + (-3)k = [-3, 6, -3]^\top$$

$$b \times a = -a \times b = [3, -6, 3]^\top$$

11

Find a vector which is perpendicular to both a and b

$a \times b$ is perpendicular to both vectors

12

Find the linear dependency between these vecs

notice $b - 3a + c = 0$, thus they are linearly dependent.

13

Find $a^\top b$ and $ab^\top$

$$a^\top b = a \cdot b = 32, ab^\top = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

B

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & -5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$, $d = [1, 2, 3]^\top$

1

Find $2A - B$

using elementwise operation

$$2A - B = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

2

Find AB and BA

$$AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}, BA = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

3

Find $(AB)^\top$ and $B^\top A^\top$

$$(AB)^\top = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}, B^\top A^\top = (AB)^\top$$

4

Find $|A|$ and $|C|$

$$a \cdot b = b \cdot a = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

Notice C is from last problem whose row vectors are linearly dependent, thus has 0 determinant.

$$|A| = 2 + 0 + 60 - 0 - (-8) - 15 = 55, |C| = 0$$

5

Find whose matrix row vecs form an orthogonal set

$$a \cdot b = b \cdot a = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

for A, $a_2^{row} \cdot a_3^{row} = -13$ thus A is not; C not full rank, thus not. Now check B,

$$b_2^{row} \cdot b_3^{row} = 6 - 2 - 4 = 0$$

$$b_1^{row} \cdot b_3^{row} = 3 - 4 + 1 = 0$$
$$b_2^{row} \cdot b_1^{row} = 2 + 2 - 4 = 0$$

6

Find A^{-1} and B^{-1}

Using adjoint matrices.

$$A^{-1} = \frac{1}{55}\begin{bmatrix} 2-15 & -(-2)+15 & 6-(-6) \\ -(-4)+0 & -1-0 & -3+12 \\ 20-0 & -5+0 & -2-8 \end{bmatrix} = \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & 9/55 \\ 4/11 & -1/11 & -2/11 \end{bmatrix},$$
$$B^{-1} = \frac{1}{-42}\begin{bmatrix} 1-8 & -2+(-2) & -8-1 \\ -2+(-12) & 1-3 & -(-4)+2 \\ -4-3 & -(-2)+6 & 1-4 \end{bmatrix} = \begin{bmatrix} 1/6 & 2/21 & 3/14 \\ 1/3 & 1/21 & -1/7 \\ 1/6 & -4/21 & 1/14 \end{bmatrix}$$

7

Find C^{-1}

As row's are linearly dependent, inverse didn't exist.

8

Find Ad

$$Ad = [14, 9, 7]^\top$$

9

Find the scalar projection of the rows of A onto the vector d with noralizing d

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$$A\hat{d} = [\sqrt{14}, 9/\sqrt{14}, 7/\sqrt{14}]^\top$$

10

Find the vector projection of the rows of A onto the vector d with noralizing d

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$$\text{Proj}\left(A; \hat{d}\right)^\top = \left(\frac{dd^\top}{d^\top d}A^\top\right)^\top = \frac{1}{d^\top d}Add^\top = A\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} / \, |d|^2 = \begin{bmatrix} \sqrt{14} & 2\sqrt{14} & 3\sqrt{14} \\ 9/\sqrt{14} & 18/\sqrt{14} & 27/\sqrt{14} \\ 7/\sqrt{14} & \sqrt{14} & 21/\sqrt{14} \end{bmatrix}$$

11

Find linear combination of the columns of A using the elements of d

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$$Ad = [14, 9, 7]^\top$$

12

Find x for $Bx = d$

Notice $B = [d, e, f]$ so if $x = [x_1, x_2, x_3]^\top$,

$$x_1 d + x_2 e + x_3 f = d \implies x = [1, 0, 0]^\top$$

Since B nonsingular, so that's the only solution.

13

Find x for $Cx = d$ and reason

Using Gaussian elimination we end up with

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 3 & 6 \end{bmatrix} [x_1, x_2, x_3]^\top = [1, -2, 4]^\top$$

Seeing last rows we know there's no such x satisfies the equation.

C

$D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $E = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

1

Find eigenvalues and eigenvectors of D

$$\det(\lambda I - D) = (\lambda - 1)(\lambda - 2) - 6 = (\lambda - 4)(\lambda + 1)$$

so the eigenvalues are -1 and 4, then solve eigenvectors,

$$(-I - D)\, x = 0 \implies x = [1, -1]^\top$$

$$(4I - D)\, x = 0 \implies x = [2, 3]^\top$$

2

Find the dot product between the eigenvalues of D

$$1 \cdot 2 - 1 \cdot 3 = -1$$

3

Find the dot product between the eigenvalues of E

$$\det(\lambda I - E) = (\lambda - 2)(\lambda - 5) - 4 = (\lambda - 6)(\lambda - 1) = 0 \implies \lambda_1 = 1, \lambda_2 = 6$$

$$(I - E)\, x = 0 \implies x = [2, 1]^\top$$

$$(6I - E)\, x = 0 \implies x = [-1, 2]^\top$$

$$-2 + 2 = 0$$

4

Find the property of the eigenvalues of E and reason

the eigenvectors of E are orthogonal due to the fact that E is symmetric. Reason
Let $Ex = \lambda_1 x$ and $Ey = \lambda_2 y$ where $\lambda_1 \neq \lambda_2$ we have

$$\begin{aligned} \lambda_1 x \cdot y &= Ex \cdot y \\ &= (Ex)^\top y = x^\top Ey \\ &= x \cdot (Ey) = x \cdot \lambda_2 y \\ \implies (\lambda_1 - \lambda_2) x \cdot y &= 0 \end{aligned} \tag{0.1}$$

Since we've assumed $\lambda_1 \neq \lambda_2$, $x \cdot y = 0$ thus orthogonal.

5

Find trivial x for $Fx = 0$

Trivial solution for $Fx = 0$ is just zero vector $[0, 0]^\top$.

6

Find non-trivial x for $Fx = 0$

non trivial solution, we can have $x = [2, -1]^\top$ or $x = [4, -2]^\top$

7

the only solution x to $Dx = 0$ and reason why single solution

it only has trivial solution $[0, 0]^\top$. Reason is by gaussian elimination or null space dimention equals to $n - rank(D) = 2 - 2 = 0$ thus null space contains only zero vector.

D

$f(x) = x^2 + 3$, $g(x) = x^2$, $q(x, y) = x^2 + y^2$

1

first and second derivative of f wrt x

$$f'(x) = 2x, f''(x) = 2$$

2

partial derivatives of q

$$\frac{\partial q}{\partial x} = 2x, \frac{\partial q}{\partial y} = 2y$$

3

gradient of q

$$\nabla q = [\frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}]^\top = [2x, 2y]^\top$$

4

the derivative of $f \circ g$ wrt x, with and without using chain rule

$$f \circ g(x) = x^4 + 3 \implies (f \circ g)'(x) = 4x^3 \; (f \circ g)'(x) = \left.\frac{\partial f(y)}{\partial y}\right|_{y=g(x)} \frac{\partial g(x)}{\partial x} = 2(x^2) * 2x = 4x^3$$

E

Doing A, B, C with python, see following notebook pdf.

A Appendix