HW5

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1

Given that $z = 1$ is a root, find the other two roots of $2z^3 - z^2 + 3z - 4 = 0$

Use polynomial dvision, we have $2z^3 - z^2 + 3z - 4 = (z-1)(2z^2 + z + 4)$ thus the other two roots are the roots or $2z^2 + z + 4 = 0$. By the formula, they are $\frac{-1 \pm \sqrt{-31}}{4} = -\frac{1}{4} - \frac{i}{4}\sqrt{31}$

2

Given that $u = 2 + 3i, v = 1 - 2i, w = -3 - 6i$, find $|u + v|, u + 2v, u - 3v + 2w, uv, uvw, |u/v|, v/w$

- $|u + v| = |3 + i| = \sqrt{9 + 1} = \sqrt{10}$
- $u + 2v = 4 - i$
- $u - 3v + 2w = -7 - 3i$
- $uv = 2 + 3i - 4i - 6i^2 = 8 - i$
- $uvw = -24 + 3i - 48i + 6i^2 = -30 - 45i$
- $|u/v| = \left|\frac{u\bar{v}}{|v|^2}\right| = \left|-\frac{4}{5} + \frac{7}{5}i\right| = \sqrt{\frac{13}{5}}$
- $v/w = \frac{v\bar{w}}{|w|^2} = \frac{3}{15} + \frac{4}{15}i$

3

Express the function $f(z) = \frac{2z + i}{z + i}$ in both cartesian and polar form, and determine the forms taken by u and v in each case, i.e. $f(z) = u(x, y) + iv(x, y)$ and $f(z) = u(r, \theta) + iv(r, \theta)$

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first we plug in $z = x + iy$ to get the cartesian form.

$$\begin{aligned} f(z) &= \frac{(2x + i(2y + 1))(x - i(y + 1))}{(x + i(y + 1))(x - i(y + 1))} \\ &= \frac{2x^2 + (2y + 1)(y + 1)}{x^2 + (y + 1)^2} + i\frac{-x}{x^2 + (y + 1)^2} \end{aligned} \tag{0.1}$$

using above result, we can easily get the polar form: $f(z) = u + iv = r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{u^2 + v^2}$$
$$\theta = \arctan\left(\frac{v}{u}\right)$$

4

Use the Cauchy-Riemann equations to show that the function $f(z) = \frac{1}{z^2+1}$ is differentiable. Use the result to find $f'(z)$ both in its cartisian form and as a function of z, and locate any points where the derivative is not defined.

To use Cauchy-Riemann equation, first write original function in cartisian form. Plug in $z = x + iy$ we have

$$f(z) = \frac{1}{z^2+1} = u + iv$$

where $u(x, y) = \dfrac{x^2 - y^2 + 1}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$ and $v(x, y) = \dfrac{-2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$. Next we find their derivatives, letting $d = (x^2 - y^2 + 1)^2 + 4x^2y^2$

$$\begin{aligned}
d_x &= \frac{\partial d}{\partial x} = 2(x^2 - y^2 + 1)2x + 8xy^2 \\
d_y &= \frac{\partial d}{\partial y} = -2(x^2 - y^2 + 1)2y + 8yx^2 \\
\frac{\partial u}{\partial x} &= \frac{2xd - d_x(x^2 - y^2 + 1)}{d^2} \\
\frac{\partial v}{\partial y} &= \frac{-2xd - d_y(-2xy)}{d^2} \\
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= \frac{4xd - d_x(x^2 - y^2 + 1) + d_y(-2xy)}{d^2} \\
&= \frac{16x^3y^2 - 8xy^2(x^2 - y^2 + 1) - 2xy(4yx^2 + 4y^3 - 4y)}{d^2} = 0 \\
\frac{\partial u}{\partial y} &= \frac{-2yd - d_y(x^2 - y^2 + 1)}{d^2} \\
\frac{\partial v}{\partial x} &= \frac{-2yd - d_x(-2xy)}{d^2} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{-4yd - d_y(x^2 - y^2 + 1) - d_x(-2xy)}{d^2} \\
&= \frac{-16x^2y^3 - 8yx^2(x^2 - y^2 + 1) + 8x^2y(x^2 - y^1 + 1) + 16x^2y^3}{d^2} = 0
\end{aligned}$$

So $f(z)$ is analytic in $\mathbb{C}$ thus differentiable. Next to find its derivative, using quotient rule,

$$\begin{aligned}
f'(z) = \frac{-2z}{(z^2+1)^2} &= \frac{-2x^5 + 4x^3y^2 - 4x^3 + 6xy^4 - 4xy^2 - 2x}{((x^2 - y^2 + 1)^2 - 4x^2y^2)^2 + (4xy(x^2 - y^2 + 1))^2} \\
&+ \frac{6x^4y + 4x^2y^3 + 4x^2y - 2y^5 + 4y^3 - 2y}{((x^2 - y^2 + 1)^2 - 4x^2y^2)^2 + (4xy(x^2 - y^2 + 1))^2} i
\end{aligned}$$

If $z^2 + 1 = 0$ then the derivative is not defined, which results in $z = \pm i$.

5

Use change of variable from the cartesian coordinates (x, y) to the polar coordinates r, θ given by $x = r\cos\theta$ and $y = r\sin\theta$ to show that the derivative of a single-valued analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ is given by

$$f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\theta - i\sin\theta)$$

or

$$f'(z) = \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\left(\frac{-i}{r}\right)(\cos\theta - i\sin\theta)$$

The second expression is direct result of Cauchy-Riemann equation in polar form. Here I only derive the first one. Notice by chain rule,

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\end{aligned}$$

This solves for the following

$$\begin{aligned}\frac{\partial u}{\partial x} &= \cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial \theta} \\ \frac{\partial u}{\partial y} &= \sin\theta\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial \theta}\end{aligned}$$

then

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ &= \cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial \theta} + i\left(\sin\theta\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial \theta}\right) \\ &= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\theta - i\sin\theta)\end{aligned}$$

7

Show that $u(x, y) = xy$ and $v(x, y) = x^3 - 3xy^2$ are both harmonic functions, but they are not harmonic conjugates.

First to show harmonic, we use laplace operator.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0, \Delta v = 6x - 6x = 0$$

Next to show they are not harmonic conjugates, we check if Cauchy-Riemann equation satisfied.

$$\frac{\partial u}{\partial x} = y, \frac{\partial v}{\partial y} = -6x \neq y, \frac{\partial u}{\partial y} = x, \frac{\partial v}{\partial x} = 3x^2 - 6xy \neq -x$$