

# Trigonometric Functions Review

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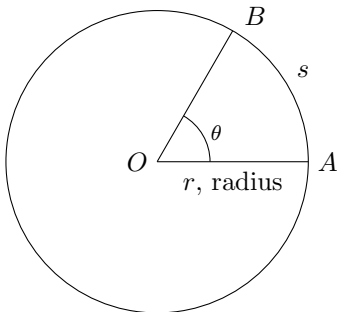
## 1 Basics

radian and degree

They are the unit of angle. And the standard unit is radian.

- 1 radian is defined as the angle subtended from the center of a circle which intercepts an arc equal in length to the radius of the circle.
- 360 degree is one complete revolution.

Using radian measure, we have the relation,  $s = \theta r$ .



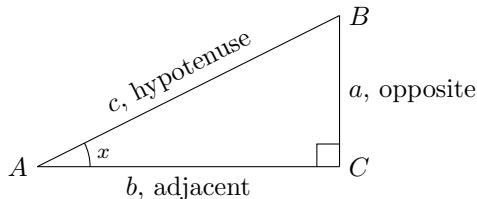
$\pi$

$\pi$  is defined as the ratio of a circle's circumference  $C$  to its diameter  $d$ . So  $C = \pi d = 2\pi r$ . A right angle is  $\pi/2 = 90$  degree.

trigonometric function

Given a right triangle, an angle  $x$  in radian measure. For this angle  $x$ , define the opposite side to be the side opposite to  $x$ , with length  $a$ , define the adjacent side to be the side between  $x$  and the right angle, with length  $b$ , and the hypotenuse side to be the side opposite to the right angle, with length  $c$ , then

$\sin(x) = a/c$ ,  $\cos(x) = b/c$ , and  $\tan(x) = a/b$ ; then  $\csc(x) = 1/\sin(x)$ ,  $\sec(x) = 1/\cos(x)$  and  $\cot(x) = 1/\tan(x)$ .

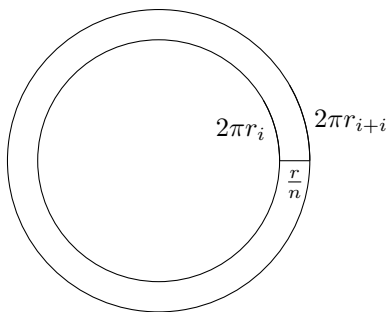


## 2 Conclusions

area of circle

Using limit and squeeze theorem, we separate the area of a circle into  $n$  number of annulus. Now the area of circle with radius  $r$ ,  $A$  satisfies the following:

$$\sum_{i=0}^{n-1} \frac{r}{n} \left( 2\pi \frac{r(i+1)}{n} \right) > A > \sum_{i=0}^{n-1} \frac{r}{n} \left( 2\pi \frac{ri}{n} \right)$$



Above inequality is reduced to

$$\pi r^2 \frac{n+1}{n} > A > \pi r^2 \frac{n-1}{n}$$

Then by squeeze theorem (for series not functions),  $\lim_{n \rightarrow \infty} A = A = \pi r^2$ .

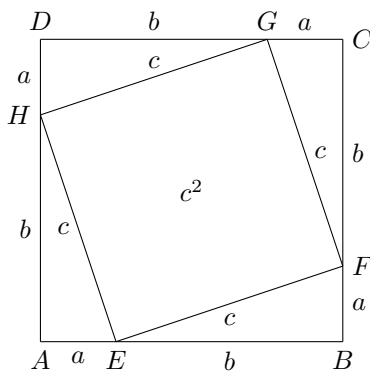
Similar idea can be used to find the area of a sector. For a sector of radius  $r$  and angle  $\theta$ , its area is  $\frac{1}{2}\theta r^2$ . A linear function of  $\theta$ .

We can also plug in the formula for arc length  $s = \theta r$  and above becomes  $\frac{1}{2}\theta s$ , which is quite similar to the one for triangle.

Pythagorean Theorem

**Theorem 2.1.** In any right triangle with sides of lengths  $a$  and  $b$ , and hypotenuse of length  $c$ , we have  $a^2 + b^2 = c^2$ .

*Proof.* We can put four triangles of the same size into the following square.

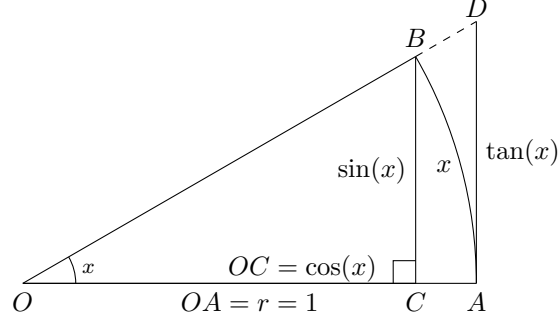


□

**Proposition 2.2.**  $\sin^2(x) + \cos^2(x) = 1$

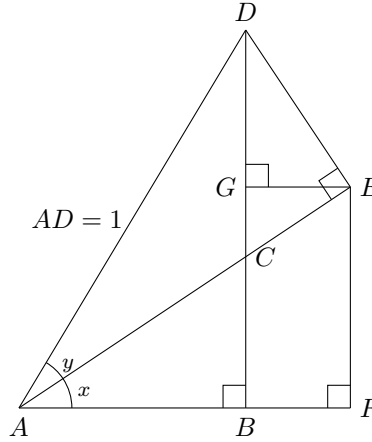
some simple relations

- for any triangle with sides of length  $a$ ,  $b$ , and  $c$ :  $a + b > c$ ,  $b + c > a$ , and  $c + a > b$
- when  $x$  is close to 0, we have  $\sin(x) < x < \tan(x)$



- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\sin(x + 2\pi) = \sin(x)$ ,  $\cos(x + 2\pi) = \cos(x)$
- $\sin(-x) = -\sin(x)$ ,  $\cos(-x) = \cos(x)$ ,  $\tan(-x) = -\tan(x)$
- $\sin(x + \pi) = -\sin(x)$ ,  $\cos(x + \pi) = -\cos(x)$
- $\sin(x + \pi/2) = \cos(x)$ ,  $\cos(x + \pi/2) = -\sin(x)$

$\sin(x + y)$  and  $\cos(x + y)$



From above figure, we can write the following:

$$DB = \sin(x + y) = DG + GB = DG + EF$$

$$AB = \cos(x + y) = AF - AB = AF - EG$$

$$\angle CDE = \angle CAB = x$$

$$DE = AD \sin(y) = \sin(y), DG = DE \cos(x) = \sin(y) \cos(x), EG = DE \sin(x) = \sin(y) \sin(x)$$

$$AE = AD \cos(y) = \cos(y), EF = AE \sin(x) = \cos(y) \sin(x), AF = AE \cos(x) = \cos(y) \cos(x)$$

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

### other formulas

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)} \\ &= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}\end{aligned}$$

$$\sin(x-y) = \sin(x+(-y)) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x-y) = \cos(x+(-y)) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x-y) = \tan(x+(-y)) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

$$\sin(2x) = \sin(x+x) = 2\sin(x)\cos(x) = \frac{2\sin(x)\cos(x)}{\sin^2(x) + \cos^2(x)} = \frac{2\tan(x)}{1 + \tan^2(x)}$$

$$\cos(2x) = \cos(x+x) = \cos^2(x) - \sin^2(x) = \frac{\cos^2(x) - \sin^2(x)}{\sin^2(x) + \cos^2(x)} = \frac{1 - \tan^2(x)}{1 + \tan^2(x)}$$

$$\tan(2x) = \tan(x+x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

### half-angle formulas

use the identity:

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\cos(x) = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2)$$

then solve this quadratic equation we have

$$\sin(x/2) = \sqrt{\frac{\cos(x) + 1}{2}} \text{ or } -\sqrt{\frac{\cos(x) + 1}{2}}$$

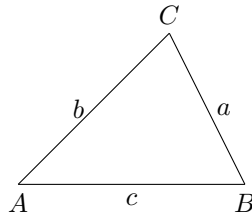
$$\cos(x/2) = \sqrt{\frac{1 - \cos(x)}{2}} \text{ or } -\sqrt{\frac{1 - \cos(x)}{2}}$$

where the sign can be determined using the quadrant of the half-angle.

### Law of sines

**Theorem 2.3.** For any triangle  $\triangle ABC$  with sides of length  $a, b, c$ , we have the following:

$$\frac{a}{\sin(\angle A)} = \frac{c}{\sin(\angle C)} = \frac{b}{\sin(\angle B)}$$



The proof uses the fact that the area of this triangle  $S = \frac{1}{2}c \cdot b \sin \angle A = \frac{1}{2}c \cdot a \sin \angle B$

Law of cosines

**Theorem 2.4.** For any triangle  $\triangle ABC$  with sides of length  $a, b, c$ , we have the following:

$$c^2 = a^2 + b^2 - 2ab \cos \angle C$$

$$a^2 = b^2 + c^2 - 2bc \cos \angle A$$

$$b^2 = c^2 + a^2 - 2ca \cos \angle B$$

*Proof.* Notice that  $c = a \cos \angle B + b \cos \angle A$ , we multiply this by  $c$  and obtain

$$c^2 = ac \cos \angle B + bc \cos \angle A$$

Do this for all three sides and we have

$$a^2 = ac \cos \angle B + ab \cos \angle C$$

$$b^2 = bc \cos \angle A + ba \cos \angle C$$

$$c^2 = ca \cos \angle B + cb \cos \angle A$$

And the rest part of the proof is obvious.  $\square$

### 3 Extensions

inverse trigonometric functions, not the reciprocal trigonometric functions

With  $y = \sin(x)$ , where  $x \in \mathbb{R}$ , we define  $x = \arcsin(y)$  where  $y \in [-1, 1]$  and range  $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Similarly we can define the following:

- $y = \arcsin(x)$  where  $x \in [-1, 1]$  and  $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $y = \arccos(x)$  where  $x \in [-1, 1]$  and  $y \in [0, \pi]$
- $y = \arctan(x)$  where  $x \in \mathbb{R}$  and  $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

composition of inverse trigonometric functions

We have the following result:

$$\begin{array}{lll} \sin(\arcsin(x)) = x & \cos(\arcsin(x)) = \sqrt{1-x^2} & \sin(\arccos(x)) = \sqrt{1-x^2} \\ \sin(\arccos(x)) = \sqrt{1-x^2} & \cos(\arccos(x)) = x & \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x} \\ \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}} & \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} & \tan(\arctan(x)) = x \end{array}$$

relation between inverse trigonometric functions

We have the following result:

$$\begin{array}{l} \arcsin(-x) = -\arcsin(x) \\ \arccos(-x) = \pi - \arccos(x) \\ \arctan(-x) = -\arctan(x) \\ \arccos(x) = \frac{\pi}{2} - \arcsin(x) \\ \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan(x), x > 0 \\ \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2} - \arctan(x), x < 0 \end{array}$$

hyperbolic functions

Using exponential function  $e^x$  we define:

$$\begin{aligned}\sinh(x) &= \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}} \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}\end{aligned}$$

relation between hyperbolic functions

Using exponential function  $e^x$  we define:

$$\begin{aligned}\sinh(-x) &= -\sinh(x) \\ \cosh(-x) &= \cosh(x) \\ \cosh^2(x) - \sinh^2(x) &= 1 \\ \cosh(x) + \sinh(x) &= e^x \\ \cosh(x) - \sinh(x) &= e^{-x} \\ \sinh(x + y) &= \sinh(x) \cosh(y) + \cosh(x) \sinh(y) \\ \cosh(x + y) &= \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \\ \operatorname{arcsinh}(x) &= \ln(\sqrt{x + \sqrt{x^2 + 1}}) \\ \operatorname{arccosh}(x) &= \ln(\sqrt{x + \sqrt{x^2 - 1}})\end{aligned}$$

## 4 Exercises

show that  $\frac{\sin(x)}{x} \rightarrow 1$  as  $x \rightarrow 0$ .

show that  $\frac{1 - \cos(x)}{0.5x^2} \rightarrow 1$  as  $x \rightarrow 0$ .

show that  $\frac{\tan(x)}{x} \rightarrow 1$  as  $x \rightarrow 0$ .

show that  $a \sin(x) + b \cos(x) = k \sin(x + \theta)$ , where  $k = \sqrt{a^2 + b^2}$  and  $\theta = \arctan(\frac{b}{a})$