Final Project, due December 7

In this project, the QR algorithm is used to find all eigenvalues of a symmetric matrix.

1. Write a MATLAB function $A = tridiag(S)$, to reduce a symmetric matrix S to a tridiagonal matrix A by similarity transformations using Householder reflectors.

Run your code on the matrix $S = hilb(4)$, and make sure that the returned matrix A is symmetric and tridiagonal.

2. Write a MATLAB function $Anew = qralg(A)$ to implement the unshifted QR algorithm on an $m \times m$ tridiagonal matrix A. The iteration should stop when $|A(m, m-1)| < 10^{-12}$, and return the result matrix to $Anew$. For the QR factorization in each step of the iteration, you can use MATLAB built-in function qr.

Test the function on tridiagonal matrix $A = tridiag(hilb(4))$, obtained from Question **1**. You should be able to see that $Anew(m, m)$ equals the smallest eigenvalue of $hilb(4)$.

3. Write a MATLAB function to implement the following steps for a given symmetric matrix S: 1) call $A = tridiag(S)$ to obtain a tridiagonal matrix A; 2) implement $Anew = qralg(A)$ on A to obtain $Anew$ and the smallest eigenvalue of S; 3) repeat applying $qralg$ until all eigenvalues of S are obtained.

Implement this on $S = hilb(4)$. Are the eigenvalues obtained from your function the same as those obtained from MATLAB eigensolver eig?

Adjust the functions to save all the values $|A(m, m-1)|$ (here the dimension m changes depending on different stages in the algorithm), after each QR factorization in the algorithm, into a vector. At the end, after obtaining all eigenvalues of S, draw this vector using $semilogy$. Explain what you observed on the plot.

4. Modify the function $qralg$ in Question **2**, such that it uses the Wilkinson shift at each iteration. The Wilkinson shift value μ is defined by

$$\mu = A(m, m) - sign(\delta) A(m, m-1)^2 \Big/ \left(|\delta| + \sqrt{\delta^2 + A(m, m-1)^2} \right),$$

where

$$\delta = \left(A(m-1, m-1) - A(m, m) \right) / 2.$$

Redo Question **3**, using this shifted QR algorithm, on the matrix $hilb(4)$. Are the obtained eigenvalues the same? Explain the difference in the plot of the vector containing the $|A(m, m-1)|$ values.

5. Implement your program on the matrix $S = diag(15 : -1 : 1) + ones(15, 15)$, with and without the Wilkinson shift in the QR algorithm, and plot the vector containing the values $|A(m, m-1)|$, respectively. Which QR algorithm, with or without shift, converges faster from the picture?