# HW3

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## 1

Solve the system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + 2x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 2x_1 + x_2$$

subject to initial condition $x_1(0) = 1$, $x_2(0) = 3$.

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Write the equation as $\frac{\mathrm{d}X}{\mathrm{d}t} = AX$ where $X = [x_1, x_2]^\top$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ For matrix $A$ we find its eigenvalue and eigenvector pairs. We first solve $\det(\lambda I - A) = (\lambda - 1)^2 - (-2)^2 = (\lambda - 3)(\lambda + 1) = 0$ so $\lambda_1 = 3$ and $\lambda_2 = -1$. Their corresponding eigenvectors are obtained by solving $\lambda_i I - A = 0$ which gives $x_1 = [1, 1]^\top$ and $x_2 = [1, -1]^\top$. So we write the solution in the matrix form as

$$X(t) = [x_1, x_2][c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}]^\top = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix}$$

plug in intial conditions we have $X(0) = [1, 3]^\top = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} [c_1, c_2]^\top$, then solve this we have $[c_1, c_2] = [2, -1]$. The final solution is then

$$x_1(t) = 2e^{3t} - e^{-t}$$
$$x_2(t) = 2e^{3t} + e^{-t}$$

## 2

Verify by direct calculation that the matrix $A = \begin{bmatrix} -2 & -3 & -1 \\ 1 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$ satisfies the Cayley–Hamilton theorem.

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$$\begin{aligned} p_A(\lambda) &= (\lambda + 2)(\lambda - 2)(\lambda - 2) + 9 + 3 + 3(\lambda - 2) + 3(\lambda - 2) - 3(\lambda + 2) \\ &= \lambda^3 - 2\lambda^2 - 4\lambda + 8 + 12 + 3\lambda - 18 \\ &= \lambda^3 - 2\lambda^2 - \lambda + 2 \end{aligned}$$

Next we plug in $\lambda = A$, we obtain the following:

$$p_A(A) = \begin{bmatrix} -8 & -9 & -7 \\ 7 & 8 & 7 \\ 9 & 9 & 8 \end{bmatrix} - 2\begin{bmatrix} -2 & -3 & -3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & -3 & -1 \\ 1 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -8+4+2+2 & -9+6+3+0 & -7+6+1+0 \\ 7-6-1+0 & 8-8-2+2 & 7-6-1+0 \\ 9-6-3-0 & 9-6-3-0 & 8-8-2+2 \end{bmatrix} = 0$$

# 3

Determine the general solution of the following system of equations by diagonalization

$$\dot{x}_1 = -10x_1 - 18x_2 + t$$
$$\dot{x}_2 = 6x_1 + 11x_2 + 3$$

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We first rewrite the original system as $\dot{X}(t) = AX(t) + f(t)$ where $A = \begin{bmatrix} -10 & -18 \\ 6 & 11 \end{bmatrix}$ and $f(t) = [t, 3]^\top$. Use the same way, we first get the eigenvalues of matrix $A$, $\lambda_1 = 2$ and $\lambda_2 = -1$, and corresponding eigenvectors are $[3, -2]^\top$ and $[2, -1]^\top$ so we have the diagonalization $A = B\Lambda B^{-1}$ where $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$. We can then let $X = BY$ and left multiply $B^{-1}$ on both side and obtain the following equation.

$$\dot{Y}(t) = \Lambda Y(t) + B^{-1}f(t)$$

We first solve the homogenous equation $y_1(t) = c_1e^{2t}$ and $y_2(t) = c_2e^{-t}$; then guess the particular solution.

$$B^{-1}f(t) = [-t-6, 2t+9]^\top \implies y_1(t) = \frac{1}{2}t + \frac{13}{4}, y_2(t) = 2t + 7$$

We can then combine these two results and obtain the final solution for $Y$ as

$$y_1(t) = c_1e^{2t} + \frac{1}{2}t + \frac{13}{4}, y_2(t) = c_2e^{-t} + 2t + 7$$

So that

$$x_1(t) = 3y_1(t) + 2y_2(t) = 3c_1e^{2t} + 2c_2e^{-t} + \frac{11}{2}t + \frac{95}{4}, x_2(t) = -2y_1(t) - y_2(t) = -2c_1e^{2t} - c_2e^{-t} - 3t - \frac{27}{2}$$

# 4

Determine the general solution of the following system of equations by diagonalization

$$\dot{x}_1 = -2x_1 + 2x_2 + 2x_3 + \sin t$$
$$\dot{x}_2 = -x_2 + 3$$
$$\dot{x}_3 = -2x_1 + 4x_2 + 3x_3$$

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Similarly, we write $\dot{X} = AX + f$ where $A = \begin{bmatrix} -2 & 2 & 2 \\ 0 & -1 & 0 \\ -2 & 4 & 3 \end{bmatrix}$ and $f = [\sin t, 3, 0]^\top$. We first find $A$'s eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and corresponding eigenvectors $[1, 0, 2]^\top$, $[2, 0, 1]^\top$ and $[2, 1, 0]^\top$, so we have the diagonalization $A = B\Lambda B^{-1}$ where $\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. We then let $X = BY$ and left multiply $B^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 0 \\ 2 & -4 & -1 \end{bmatrix}$ and this gives $\dot{Y}(t) = \Lambda Y(t) + B^{-1}f(t)$. So that $y_1(t) = c_1 e^{2t}, y_{2,3}(t) = c_{2,3}e^{-t}$. Then we guess the particular solution using undetermined coefficients.

$$\begin{aligned}
C\cos t - D\sin t + E &= 2C\sin t + 2D\cos t + 2Et + 2F + \frac{1}{3}\left(-\sin t + 6\right) \\
\implies y_1(t) &= \frac{2}{15}\sin t + \frac{1}{15}\cos t - 1 \\
C\cos t - D\sin t + E &= -C\sin t - D\cos t - Et - F + \frac{1}{3}(3 * 3) \\
\implies y_2(t) &= 3 \\
C\cos t - D\sin t + E &= -C\sin t - D\cos t - Et - F + \frac{1}{3}\left(2\sin t - 12\right) \\
\implies y_3(t) &= \frac{1}{3}\sin t - \frac{1}{3}\cos t - 4
\end{aligned}$$

Then the general solution for $Y$ is

$$\begin{aligned}
y_1(t) &= c_1 e^{2t} + \frac{2}{15}\sin t + \frac{1}{15}\cos t - 1 \\
y_2(t) &= c_2 e^{-t} + 3 \\
y_3(t) &= c_3 e^{-t} + \frac{1}{3}\sin t - \frac{1}{3}\cos t - 4
\end{aligned}$$

And finally the general solution for $X$ is

$$\begin{aligned}
x_1(t) &= y_1(t) + 2y_2(t) + 2y_3(t) \\
&= c_1 e^{2t} + 2(c_2 + c_3)e^{-t} + \frac{4}{5}\sin t - \frac{3}{5}\cos t - 3 \\
x_2(t) &= y_2(t) \\
&= c_2 e^{-t} + 3 \\
x_3(t) &= 2y_1(t) + y_3(t) \\
&= 2c_1 e^{2t} + c_3 e^{-t} + \frac{3}{5}\sin t - \frac{1}{5}\cos t - 6
\end{aligned}$$

## 5

Determine the singular value decomposition of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Compare the singular values and singular vectors of $A$ to its eigenvalues and eigenvectors.

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To find the singular values, we find the eigenvalues of $AA^\top = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ which are $8, 2, 0$ and corre-

sponding eigenvectors are $[1,2,1]^\top, [1,-1,1]^\top, [-1,0,1]^\top$. So that $U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$.
Then $v_1 = \frac{1}{\sqrt{8}} A^\top u_1 = [\sqrt{6}/6, \sqrt{3}/2, \sqrt{3}/6]^\top$, similarly, $v_2 = [-\sqrt{3}/3, 0, \sqrt{6}/3]^\top$, and for $v_3$, just make $v$s an orthonormal basis. From $v_2$, we choose $v_3 = [\sqrt{2}, \eta, 1]$, then since $v_1 v_3 = 0$, we see $\eta = -1$, after normalization, we obtain $V = \begin{bmatrix} \sqrt{6}/6 & -\sqrt{3}/3 & \sqrt{2}/2 \\ \sqrt{3}/2 & 0 & -1/2 \\ \sqrt{3}/6 & \sqrt{6}/3 & 1/2 \end{bmatrix}$. So that $A = U\Sigma V^\top$ where $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Next we compare above with its eigenvalues and eigenvectors. We first obtain its characteristic polynomial as $\lambda^3 - 3\lambda^2 + (2-\sqrt{2})\lambda$ so $\lambda_1 = 0$, $\lambda_{2,3} = \frac{3 \pm \sqrt{1+4\sqrt{2}}}{2}$ and corresponding eigenvectors are $[\sqrt{2}, -1, 1]^\top$, $[1, \frac{1 \pm \sqrt{1+4\sqrt{2}}}{2}, 1]^\top$. They aren't very related.

# 6

Consider the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ compute the left singular vectors and singular values of $A$; then using the result find the first two right singular vectors of $A$.

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For the left singular values, we first compute $AA^\top = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ so matrix $A$ has singular value $\sqrt{4} = 2$, $\sqrt{3}$. And the singular vectors are the eigenvectors of $AA^\top$ which are $u_1 = [1,0]^\top$ and $u_2 = [0,1]^\top$. The first 2 right singular values can be found using the formula. $v_1 = \frac{1}{2} A^\top u_1 = \frac{1}{2}[1,0,1,0,1,0,1]^\top$ and $v_2 = \frac{1}{\sqrt{3}} A^\top u_2 = \frac{1}{\sqrt{3}}[0,1,0,1,0,1,0]^\top$.