Notes on Lectures on Logarithmic Sobolev Inequalities

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1 Markov Semi-groups

1.1 Markov Semi-groups and Generators

Definition 1.1. A familiy $(P_t)_{t\geq 0}$ of linear operators on a Banach space $(\mathcal{B}, \|\cdot\|)$ is a *semi-group* iff it satisfies the following:

- $P_0 = I$, the identity on $\mathcal{B}$
- the map $t \to P_t$ is continuous in the sense that for all $f \in \mathcal{B}$, $t \to P_t f$ is a continuous map from $\mathbb{R}^+$ into $\mathcal{B}$
- $\forall f \in \mathcal{B}$ and $(t, s) \in (\mathbb{R}^+)^2$
$$P_{t+s} f = P_t P_s f$$

Under most cases, $\mathcal{B}$ will be the set $\mathcal{C}(\Omega)$ of real-valued functions bounded continuous functions on a Polish space Ω equipped with the uniform norm. Other cases, we might need uniform boundedness.

Definition 1.2. A semi-group $(P_t)_{t\geq 0}$ is *Markov* iff

- for any $t \in \mathbb{R}^+$, $P_t \mathbb{1} = \mathbb{1}$
- for any $t \in \mathbb{R}^+$, P_t preserves positivity, i.e., for any $f \in \mathcal{B}, t \in \mathbb{R}^+$, $f \geq 0 \implies P_t f \geq 0$

Definition 1.3. P_t is *contractive* iff for any $f \in \mathcal{B}$,

$$\|P_t f\| \leq \|f\| \tag{1.1}$$

where $\|\cdot\|$ denotes the norm on $\mathcal{B}$.

Definition 1.4. The *infinitesimal generator* $\mathcal{L}$ of a semi-group P_t is defined by

$$\mathcal{L} f := \lim_{t\to 0^+} \frac{(P_t - I) f}{t} \tag{1.2}$$

for any function f for which the limit makes sense. The domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is the set of functions of $\mathcal{C}(\Omega)$ for which above limit exists.

Theorem 1.5 (Hille-Yoshida theorem for Markov semi-groups)**.** *A Linear operator $\mathcal{L}$ is the infinitesimal generator of a Markov semi-group $(P_t, t \in \mathbb{R}^+)$ on $\mathcal{B}$ iff*

1. *$\mathbb{1} \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}\mathbb{1} = 0$*
2. *$\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{B}$*
3. *$\mathcal{L}$ is closed: iff any sequence f_n of $\mathcal{D}(\mathcal{L})$ converging to f and such that $\mathcal{L} f_n$ converge, then $\lim_{n\to\infty} \mathcal{L} f_n = \mathcal{L} f$*
4. *for anyy $\lambda > 0$, $(\lambda I - \mathcal{L})$ is invertible. And its inverse is bounded*
$$\sup_{\|f\|\leq 1} \|(\lambda I - \mathcal{L})^{-1} f\| \leq 1/\lambda$$
and preserve positivity, i.e., for all $f \geq 0$, $(\lambda I - \mathcal{L})^{-1} f \geq 0$

1.2 Invariant Measures of a semi-group

Definition 1.6. Let $(P_t)_{t\geq 0}$ be a Markov semi-group. A probability measure μ on (Ω, Σ) is invariant with respect to the semi-group $(P_t)_{t\geq 0}$ iff for any $f \in \mathcal{C}(\Omega)$ and any $t \geq 0$

$$\mu\left(P_t f\right) = \mu(f) \tag{1.3}$$

The set of invariant measures for a semi-group $(P_t)_{t\geq 0}$ will be denoted hereafter $\mathcal{J} = \mathcal{J}(P)$

This invariant probability is also characterized by

Proposition 1.7. *μ on (Ω, Σ) is invariant with respect to the semi-group $(P_t)_{t\geq 0}$ iff for any $f \in \mathcal{D}(\mathcal{L})$,*

$$\mu(\mathcal{L}f) = 0 \tag{1.4}$$

Extension

Proposition 1.8. *Let $\mu \in \mathcal{J}(P)$ with respect to a Markov semi-group $(P_t)_{t\geq 0}$ (i.e., μ is invariant to P_t). $(P_t)_{t\geq 0}$ can be extended to any $L^p(\mu)$ for $p \geq 1$.*

Prove using Jensen's inequality and Hahn-Banach theorem.

Definition 1.9. A Markov semi-group $\{P_t, t \geq 0\}$ is $L^q(\mu)$ *ergodic* for $q \in (1, \infty)$ and $\mu \in \mathcal{J}(P_t)$ iff for any function $f \in L^q(\mu)$,

$$\lim_{t\to\infty} \int (P_t f - \mu f)^q \, \mathrm{d}\mu = 0$$

A Markov semi-group $\{P_t, t \geq 0\}$ is $L^q(\mu)$ is *uniformly ergodic* iff $\mathcal{J}(P_t)$ is reduced to a unique probability measure and

$$\lim_{t\to\infty} \|P_t f - \mu f\|_\infty = 0$$

A stronger property than invariance

Definition 1.10. A probability measure μ on (Ω, Σ) is *reversible* for a Markov semi-group $\{P_t, t \geq 0\}$ iff for any $(f, g) \in \mathcal{B}$ and time $t \geq 0$,

$$\mu(gP_t f) = \mu(f P_t g) \tag{1.5}$$

Equivalently, we say that $(P_t)_{t\geq 0}$ satisfies the detailed balance condition for the probability μ.

And the set of reversible measures of a Markov semi-group $\{P_t, t \geq 0\}$ will be denoted as $\mathcal{J}_0(P_t)$.

1.3 Markov Processes