Calculus I Recitation

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The recitation is based on *Calculus, 9th edition, Stewart.*

1 Functions and Limits

1.1 Four Ways to Represent a Function

function, it's domain and range

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function $y = f(x)$ and X, Y are two sets. $x \in X$ and $y \in Y$. A function is a rule that assigns each $x \in X$ to exactly one element $y = f(x) \in Y$. Domain of a function X is the set of inputs accepted by the function so that the function makes sense.

- age, height are positive by default
- $1/0$ doesn't make sense

range can be obtained after you have the domain $Y = \{y = f(x) \,|\, x \in X\}$.

piecewise defined function

Divide domain of f into subsets $X_1, X_2, \ldots$ without intersection, so $X_i \cap X_j = \emptyset$. Then,

$$f(x) = \begin{cases} f_1(x), & x \in X_1 \\ f_2(x), & x \in X_2 \\ \vdots \end{cases}$$

even function and odd function

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function $f(x)$ is even if $f(x) = f(-x)$ and is odd if $f(x) = -f(x)$, for any x in its domain.

increasing and decreasing

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function $f(x)$ is increasing on an interval I if $f(x_1) \geq f(x_2)$, for any $x_1, x_2 \in I$, $x_1 > x_2$.
function $f(x)$ is decreasing on an interval I if $f(x_1) \leq f(x_2)$, for any $x_1, x_2 \in I$, $x_1 > x_2$.

1.2 Mathematical Models

mathematical model

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math description of a phenomenon, usually by equations. For example someone's age vs year.

year (at Jan. 1st)	age
2020	1
2021	2
2022	3

linear function: y is a linear function of x.

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when changes of y is proportion to the changes of x. For an arbitrary point belong to this relationship, (x, y) and a fixed point (x_1, y_1) also belong to this relationship,

$$y - y_1 = m(x - x_1), k \neq 0$$
$$y = mx - mx_1 + y_1 = mx + b$$

we call m the slope and b the y-intercept.

polynomial function: $P(x)$

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, n is nonnegative integer, $a_n \neq 0$ and its degree is n. A linear function is a polynomial of degree 1, and a quadratic function is a polynomial of degree 2, cubic for 3.

power function

$f(x) = x^a$. If a is an integer n,

- if n is even, then $f(x)$ is even for $f(x) = f(-x)$
- if n is odd, then $f(x)$ is odd for $f(x) = -f(-x)$

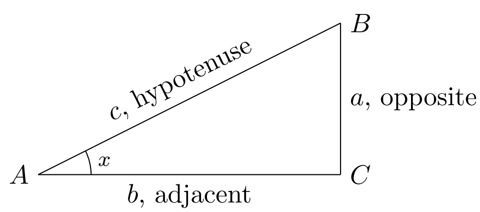
If $a = 1/n$ with n a positive integer, $f(x) = x^a$ is a root function. If $a = -1$, $f(x) = x^{-1}$ is a reciprocal function.

rational function

$f(x) = P(x)/Q(x)$, where P and Q are two polynomial functions.

Trigonometric function

Given a right triangle, an angle x in radian measure. For this angle x, define the opposite side to be the side opposite to x, with length a, define the adjacent side to be the side between x and the right angle, with length b, and the hypotenuse side to be the side opposite to the right angle, with length c, then
$\sin(x) = a/c$, $\cos(x) = b/c$, and $\tan(x) = a/b$; then $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ and $\cot(x) = 1/\tan(x)$.



period function

if $f(x) = f(x+T)$, where T is a constant.

algebraic function, and transcendental if not

functions constructed using addition, subtraction, multiplication, division, raising to a whole number power, and taking roots. And some transcendental functions: $\sin(x)$, $\log(x)$, e^x.

Exponential and Logarithmic function

Exponential function has the form $y = f(x) = b^x$, and logarithmic function has the form $y = f(x) = \log_b x = \log x / \log b$

1.3 New Functions from Old Functions

shifting a function $f(x)$

new function $g(x) = f(x-h) + v$, has the plot same as shift the plot of $f(x)$ v units to the right vertically and h units upward horizontally. Notice when $h < 0$, shifting h units to the right equals to shifting $|h|$ units to the left.

stretching and reflecting a function $f(x)$

new function $g(x) = v \times f(x/h)$, has the plot same as stretch the plot of $f(x)$ by a factor of v vertically and h units horizontally.
If $v < 0$, we do a reflection about the line $y = 0$ (x-axis) first then stretch by a factor of $|v|$ vertically, and if $h < 0$, we do a reflection about the line $x = 0$ (y-axis) first then stretch by a factor of $|h|$ horizontally.
if $|v| < 1$ or $|h| < 1$, it's a shrinking operation, not stretching.

combination of functions $f(x)$ and $g(x)$

$f + g$ sum, $f - g$ difference, fg product, and f/g quotient. The domain of the new function is the intersection of domains of f and g.

composition of functions $f(x)$ and $g(x)$

$(f \circ g)(x) = f(g(x))$. To make sure x is accepted, first $g(x)$ need to make sense, then exclude those x so that if letting $z = g(x)$, $f(z)$ make sense.

1. Let $z = g(x)$ so that $f \circ g(x) = f(z)$
2. find domain of $f(z)$, say set/interval Z
3. simplify $z = g(x) \in Z$ and obtain $x \in X_1$
4. find domain of $g(x)$, X_2
5. the domain of $f \circ g(x)$ is then $X_1 \cap X_2$

1.4 The Tangent and Velocity Problems

secant line and tangent line

With a given curve C, a secant line is a line passing through two points of a curve. In most cases, as one point is brought towards the other, the secant line tends to be the tangent line at the other point.
The slope of the tangent line is the limit of the slopes of the secant lines.

difference quotient

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

1.5 The Limit of a Function

the limit of $f(x)$ as x approaches a

Suppose $f(x)$ is defined when x is near the number a, Then we write

$$\lim_{x \to a} f(x) = L$$

or $f(x) \to L$ as $x \to a$, if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on either side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, 0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$.
If $f(x)$ is only defined on one side of a, check the next definition.

left-side limits

Suppose $f(x)$ is defined when x is near the number a, and also $x < a$. Then we write

$$\lim_{x \to a^-} f(x) = L$$

or $f(x) \to L$ as $x \to a^-$, if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on left side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, 0 < a - x < \delta$ we have $|f(x) - L| < \epsilon$.
Remarks:

- Right-side limit is defined in a similar way.
- $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$
- If the domain of f stays in one side of a, for example $f(x) = \frac{x}{\sqrt{x}}$, the limit at a can still be defined properly to be the side limit. Thus $\lim_{x \to 0} \frac{x}{\sqrt{x}} = 0$.

infinite limits

In this case the limit doesn't exist.
Suppose $f(x)$ is defined when x is near the number a, Then we write

$$\lim_{x \to a} f(x) = \infty$$

or $f(x) \to \infty$ as $x \to a$, if we can make the values of $f(x)$ arbitrarily large by restricting x to be sufficiently close to a (on either side of a) but not equal to a.
In $\epsilon - \delta$ language, the condition is: $\forall M > 0, \exists \delta > 0$ s.t. $\forall x, 0 < |x - a| < \delta$ we have $f(x) > M$.
Negative infinite limits, one-sided infinite limits can be defined similarly.
And then we call $x = a$ the vertical asymptote of the curve $f(x)$.

1.6 Calculating Limits using the Limit Laws

limit laws

Following operation is interchangeable with finding the limits.

- summation
- difference
- scalar multiplication
- product
- quotient (excluding the case where the denominator has limit 0)
- power to n or $1/n$, n is any positive integer

direct substitution property

if the function limit at a is equal to the function value at a. And we call the function is continuous at a if this property hold.

the Squeeze theorem

Lemma 1.1. *if $f(x) \leq g(x)$ when x is near a, and the limits of f and g exist at a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.*

Theorem 1.2. *With this, we can show that if $f \leq g \leq h$ when x is near a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

1.7 The Precise Definition of a Limit

see the $\epsilon - \delta$ language part in subsection 1.5. For $\lim_{x\to a} f(x) = L$, we mean that $f(x)$ will approach L, as long as we move x to a point close enough to a, but not a itself.

- a, the target point, where we want to know about the function limit
- ϵ measures how close the function value $f(x)$ at the moving point x to the limit L, with $|f(x) - L| < \epsilon$
- we want ϵ arbitrarily small, so ϵ can be any positive number
- δ measures how close the moving point x to the target point a, with $0 < |x - a| < \delta$
- In many simple cases, δ is a function of ϵ. So if we can prove that $0 < |x - a| < \delta(\epsilon)$ leads to $|f(x) - L| < \epsilon$, then the limit L is found.

The easiest case is when $f(x) = x, x \neq a$, we can consider $\delta(\epsilon) = \epsilon$ so that we have $\lim_{x\to a} f(x) = a$.

1.8 Continuity

point continuity

A function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$

type of discontinuity at a point $x = a$

- removable: when we can define a new value for $f(a)$ so that $f(x)$ can regain continuity at $x = a$. (a pothole)
- jump: when the left limit and the right limit exist but they are not equal (stairs)
- infinite: when the left limit or the right limit doesn't exist. ($1/x$ and $\sin(1/x)$)

left continuity and interval continuity

A function f is continuous from the right at a point $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.
A function f is continuous on an interval if it is continuous at every number in the interval.

Properties of continuous function

Theorem 1.3. *If f and g are both continuous at a, then their combination $f \pm g$, fg, f/g where $g(a) \neq 0$, are continuous at a.*
If g is continuous at a and f is continuous at $g(a)$, then the composition $f(g(x))$ is continuous at a.

Intermediate Value Theorem

Theorem 1.4. *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

meaning a continuous function takes on every intermediate value between the function values at two ends of the interval.

1.9 Exercises

Define $y(x) = \frac{x^2-x}{2(x-2)}$ and $g(x) = \frac{x}{2}$. State the difference between them and plot them.

Given that $y = x^2$, is y a function of x? is x a function of y? If not, add a restriction to make it/them function(s) and plot them.

are the following functions odd, even, or neither. x^2 where $x > 0$, $\tan(x + \pi/4)$, x^{20}, x^{-20}, e^{x^3}, x^x where x is a integer.

transform the function $f(x) = \frac{1}{2x} + 2$ to $g(x) = -\frac{1}{x-2}$, and plot what you gain at each step.

find difference quotient of $x^2 + 2x + 3$ and $\frac{2}{x}$.

plot function $y = \operatorname{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1. & x < 0 \end{cases}$ and function $z = g(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$. And then find the limit, left limit and right limit of $f(x)$ and $g(x)$ at $x = 0$.

An object is moving straightforward. The location x and time t has an approximated relation $x = 2t^3 + t^2 + 5t$. What's the average speed from $t = 1$ to $t = 3$?

For the previous question, first simplify the difference quotient of the distance function $\frac{x(t+\Delta t)-x(t)}{\Delta t}$ at $t = 1$. What's the limit when Δt goes to 0? Also what's the limit of $x(t)$ at $t = 1$? Use words to explain their difference.

Use $\epsilon - \delta$ language to show that $\lim_{x\to 1} 1/x = 1$ and $\lim_{x\to 2} x^3 = 8$.

Use $\epsilon - \delta$ language to prove $x\sin(1/x)$ has limit 0 at $x = 0$ and $\sin(1/x)$ doesn't have limit at $x = 0$. Then use Squeeze theorem to prove the first part.

Use $\epsilon - \delta$ language to prove that the limit at point $x = a$ doesn't exist when the left limit doesn't equal to the right limit.

Use $\epsilon - \delta$ language to prove that function of the form kx^n is continuous everywhere, where $k \neq 0$ and n is an integer.

Use $\epsilon - \delta$ language to prove the properties of the continuous function Theorem 1.3.

Show that there's a solution of the equation $x^4 - 5 = 0$ between 0 and 2.

Show function $f(x) = x$ and function $g(x) = \sin(x)$ intersect with each other at only one point. Could this number of intersection be an even number?

In the film *The Curious Case of Benjamin Button*, Benjamin was born at the age of 85. His age decreases as time passes until the end of his life at age 0. Now consider somewhere on earth a normal child Xavier was born at the same time when Benjamin was born. Assume Xavier is gonna live for at least 85 years, prove there exists a time when Benjamin and Xavier will be at the same age. Make a story of this kind on your own. Maybe transferring water from cup A to cup B.

2 Derivatives

2.1 Derivatives and Rates of Change

tangent line to the curve $f(x)$ at the point $P(a, f(a))$

The line through P with slope:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.
Notice that we can define $g(x, a) = \frac{f(x)-f(a)}{x-a}$ and it will be the slope of the secant line passing points $(a, f(a))$ and $(x, f(x))$.
Another form can be obtained by a change of variable. Letting $x = a + h$, we have

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

average velocity and instantaneous velocity

With a displacement function, or a position function $x(t)$, we define the following:

- Average velocity: $\bar{v}(t) = \frac{x(t+\Delta t)-x(t)}{\Delta t}$
- Instantaneous velocity (or just call it velocity): $v(t) = \lim\limits_{\Delta t \to 0} \bar{v}(t)$
- Speed: the absolute value of velocity $|v(t)|$

the derivative of a function f at a point a, denoted by $f'(a)$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if the limit exists.

rate of change of y with respect to x

- Average rate of change of y with respect to x: $\frac{\Delta y}{\Delta x} = \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$
- Instantaneous rate of change of y with respect to x: $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

2.2 The Derivative as a Function

the derivative of a function f

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

and it is defined where $f'(x)$ exists. Other notations

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$$

$$f'(a) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=a}$$

differentiable function

If $f'(a)$ exists, we say f is differentiable at point $x = a$.

differentiability implies continuity

Theorem 2.1. *If f is differentiable at a, then f is continuous at a.*

Proof.

$$\begin{aligned}
\lim_{h\to 0} f(x+h) - f(x) &= \lim_{h\to 0} (f(x+h) - f(x)) \left(\frac{h}{h}\right) \\
&= \lim_{h\to 0} h \frac{f(x+h) - f(x)}{h} \\
&= 0 f'(x) = 0
\end{aligned}$$

□

And the converse is not true.

cases of function that is not differentiable at point $x = a$.

- $f(a)$ doesn't exist
- $\lim_{x\to a} f(x)$ doesn't exist
- $\lim_{x\to a} f(x) \neq f(a)$
- $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ doesn't exist
  - limit is infinity: notice in this case we say the function has a vertical tangent line and it's not the vertical asymptote.
  - left limit does not equal to right limit

The first three cases are for discontinuity.

higher derivatives

The nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$ and is obtained from f by differentiating n times. Other notations:

$$f^{(n)}(x) = y^{(n)} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

$$f''(x) = (f'(x))'$$

In the case of position function $x(t)$, we call

- position $x(t)$
- velocity $x'(t) = v(t) = \dot{x}(t)$
- acceleration $x''(t) = a(t) = \dot{v}(t) = \ddot{x}(t)$
- jerk $x'''(t) = j(t) = \dot{a}(t)$

2.3 Differentiation Formulas

derivatives of constant function and power function

- Constant function $f(x) = c$, $f' = 0$
- Power function $f(x) = x^n$, $f' = nx^{n-1}$, where n is a positive integer. And this is also true for any real number n.

rules

- $(f \pm g)' = f' \pm g'$
- $(fg)' = f'g + fg'$
- $(f/g)' = \frac{f'g-fg'}{g^2}$

2.4 Derivatives of Trigonometric Functions

derivatives of trigonometric functions

- $\cos'(x) = -\sin(x)$
- $\sin'(x) = \cos(x)$

two special limits

- $\sin(x)/x \to 1$ as $x \to 0$
- $(\cos(x) - 1)/x \to 0$ as $x \to 0$

2.5 The Chain Rule

the chain rule

if g is differentiable at x and f is differentiable at $g(x)$, the the composite function $F = f \circ g$ is differentiable at x and F' is given by

$$F'(x) = f'(g(x))g'(x)$$

Or in Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

2.6 Implicit Differentiation

implicit function

In the case of 2 variables x and y, we define a function like $y = f(x)$ before. The implicit function has the form $R(x, y) = 0$, like the one for the unit circle, $x^2 + y^2 - 1 = 0$.
And the trick here is to use chain rule. Take $h(x)g(y) = 0$ for example, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(h(x)g(y)) = h'(x)g(y) + h(x)\frac{\mathrm{d}}{\mathrm{d}x}g(y) = h'(x)g(y) + h(x)\left(\frac{\mathrm{d}}{\mathrm{d}y}g(y)\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

The last step is to check if there's any substitution with $h(x)g(y) = 0$.

2.7 Related Rates

related rates problems

Suppose we have relation $y = f(z)$ and $z = g(x)$, then with $\frac{\mathrm{d}z}{\mathrm{d}x} = g'$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

Usually, one of the relation is given and the other can be obtained through geometry and physical laws.

2.8 Linear Approximations and Differentials

linearization

Linearization is the tangent line at point $(a, f(a))$ and is used to approximate curve $f(x)$ when x is near a.

$$L(x) = f(a) + f'(a)(x - a) \approx f(x), x \text{ near } a$$

Here L is the linearization of f at a and $f \approx L$ when x near a is called the linear approximation or tangent line approximation of f at a.

differentials

If $y = f(x)$ where f is differentiable, the differential $\mathrm{d}x$ is an independent variable and we define differential $\mathrm{d}y$ in terms of $\mathrm{d}x$: $\mathrm{d}y = f'(x)\,\mathrm{d}x$.

about the errors

when using L to approximate f, we define the following errors

- absolute error $\epsilon(x) = |f(x) - L(x)|$
- relative error $\eta(x) = \left|\frac{f(x)-L(x)}{f(x)}\right|$
- percentage error $\delta(x) = 100\% \times \eta(x)$

2.9 About Euler's number

e

We have the following characterizations of number e.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

and

$$\lim_{x\to 1} \frac{e^x - 1}{x} = 1$$

2.10 Exercises

use the definition of limit to prove the sum, difference, product and quotient rule for derivatives

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find the derivatives of the functions of general types: linear, polynomial, power, rational, trigonometric. Exponential and logarithmic functions are in chapter 6 but you can try them out now.

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find the tangent line at point $x = 1$ for the following functions: $y = \frac{3}{x^2}$, $y = \sqrt{5x-3}$, $y = \frac{x+1}{x+2}$, $y = 6$

If f is odd, what about f'? And what if f is even?

Use the chain rule to find the derivatives of the following functions: $\sin(\cos(2x))$, $\sqrt[4]{x^3+1}$, $\cos\left(\frac{x}{\sqrt{x^3+6}}\right)$

Use the chain rule to prove that the derivative of the inverse function is the reciprocal of the derivative of the original function, i.e., letting $g = f^{<-1>}$ to be the inverse of f, then $g' = \frac{1}{f'}$ under suitable conditions.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x's and y's of the curve $x^2 \sin(y^2) = 1$.

Suppose initially you are 0.5 meter tall and weight 3 kilogram. Let h be your height, w be your weight, B is your BMI and $B = w/h^2$. Given that $h = 0.5 + 0.02t$, and $w = 3 + t$ for some period, find $\mathrm{d}B/\mathrm{d}t$.

Consider a ball with radius r, with volume $V = \frac{4}{3}\pi r^3$. Due to thermal expansion, $r(t) = 5 + \sin(2t)$, what is $\frac{\mathrm{d}V}{\mathrm{d}t}$ when $r = 5$ cm and 6 cm?

2.11 Practice Exam

A

Consider $f(x) = (x+1)/(x-1)^2$

Rewrite f in rational form. What's the domain of $f(x)$ and $f(\sin(x))$?

What's $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to 1} f(x)$?

Show that by shifting and stretching $f(x)$, you cannot get an odd or even function.

Find f'.

Now let $y = f(x)$. Write the implicit definition of this relation.

Bonus

Consider $y^2 = x\sin(x)$

Suppose $y \geq 0$, then write $y = f(x)$. What's the domain of $f(x)$ and $f(x^2+1)$?

What's $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to 1} f(x)$?

Find f'.

Use implicit differentiation to find f'.

3 Applications of Differentiation

3.1 Maximum and Minimum Values

global maximum/minimum, extreme values

Let $c \in D$, the domain of f. Then $f(c)$ is the

- global maximum, or absolute maximum, value of f on D, if $f(c) \geq f(x)$, $\forall x \in D$.
- global minimum, or absolute minimum, value of f on D, if $f(c) \leq f(x)$, $\forall x \in D$.

They are the extreme values of f.

local maximum/minimum

Let the neighborhood of c is in D, the domain of f. Then $f(c)$ is the

- local maximum value of f, if $f(c) \geq f(x)$, when x is near c
- local minimum value of f, if $f(c) \leq f(x)$, when x is near c

Here being true "near" c means being true on some open interval containing c.

The Extreme Value Theorem

Theorem 3.1. *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value at $f(c)$ and an absolute minimum value $f(d)$ at some $c, d \in [a, b]$.*

critical number

A critical number of f is a number c in the domain D such that either $f'(c) = 0$ or $f'(c)$ doesn't exist.

The Fermat's Theorem

Theorem 3.2. *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. In terms of critical numbers, c is a critical number of f.*

closed interval method

To find the absolute maximum/minimum value of a function f on a closed interval $[a, b]$,

1. find the values of f at the critical numbers of f in (a, b)
2. find the values of f at the endpoints of the interval
3. the largest of the values from step 1 and 2 is the absolute maximum value, and the smallest of these values is the absolute minimum value.

3.2 The Mean Value Theorem

Rolle's Theorem

Theorem 3.3. *Let f be a function that satisfies the following three conditions:*

- *f is continuous on $[a, b]$*
- *f is differentiable on (a, b)*
- *$f(a) = f(b)$*

then $\exists c \in (a, b)$ *such that* $f'(c) = 0$.

The Mean Value Theorem

Theorem 3.4. *Let f be a function that satisfies the following two conditions:*

- *f is continuous on $[a, b]$*
- *f is differentiable on (a, b)*

then $\exists c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

some facts

Proposition 3.5. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant on (a, b). And if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f = g + c$ on (a, b) where c is a constant.*

3.3 What Derivatives Tell Us about the Shape of a Graph

increasing, decreasing test

- if $f'(x) > 0$ on an interval, then f is increasing on that interval
- if $f'(x) < 0$ on an interval, then f is decreasing on that interval

first derivative test

- if $f'(x)$ changes from positive to negative at a, then f has a local maximum at a
- if $f'(x)$ changes from negative to positive at a, then f has a local minimum at a
- if $f'(x)$ is positive to the left and right of a, or negative to the left and right of a, then f has no local maximum or minimum at a

concave upward and downward

If the graph of f lies above all its tangents on an interval I, then f is concave upward on I. If the graph of f lies below all its tangents on an interval I, then f is concave downward on I.

concavity test

- if $f''(x) > 0$ on an interval, then f is concave upward on that interval
- if $f''(x) < 0$ on an interval, then f is concave downward on that interval

inflection point

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward.

first derivative test

- if $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a
- if $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a

3.4 Limits at Infinity; Horizontal Asymptotes

limit at infinity

We use the notation $\lim_{x\to\infty} f(x) = L$. Let f be a function defined on some interval (a, ∞), then limit of f at positive infinity is L if values of f can be made arbitrarily close to L by requiring x to be sufficiently large.
In $\epsilon - \delta$ language, $\forall \epsilon > 0$, $\exists M > 0$ such that $\forall x > M$, $|f(x) - L| < \epsilon$.
Limit at negative infinity can be defined similarly.

horizontal asymptote

Line $y = L$ is a horizontal asymptote of function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ of $\lim_{x\to-\infty} f(x) = L$

limit of negative rational power function at infinity

Theorem 3.6. *if r is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$. Further if x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.*

infinity limits at infinity

Notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicates that the values of $f(x)$ become infinitely large as x goes to infinity.
In $\epsilon - \delta$ language, $\forall M > 0$, $\exists x(M) > 0$ such that $\forall x > x(M)$, $f(x) > M$.

3.5 Summary of Curve Sketching

Guidelines for Plotting $y = f(x)$

1. domain and plot range: select where to plot the function so that the interesting parts are included
2. intercepts: mark $f(0)$ on y-axis and roots for $f(x) = 0$ on x-axis if possible
3. symmetry: check if f is odd or even, or can be shifted or stretched to an odd or even function g, and also check if the function is periodic or not
4. asymptotes: use dashed line to plot horizontal asymptotes and vertical asymptotes. Slant asymptotes and other higher order asymptotes will be discussed later
5. intervals of increase and decrease: calculate $f'(x)$ and check its positivity
6. local maximum or minimum: solve $f'(x) = 0$ and check if $f''(x) \neq 0$
7. concavity and points of inflection: calculate $f''(x)$ and check its positivity

slant asymptote

If $\lim_{x\to\infty} f(x) - (kx + b) = 0$ or $\lim_{x\to-\infty} f(x) - (kx + b) = 0$, then line $y = kx + b$ is a slant asymptote for $f(x)$.

3.6 Exercises

find the global/local maximum/minimum of function: $x^3 - 3x^2 + 6$ on interval $[-1, 2]$.

find the critical numbers of functions: $x + 1/x$, $\tan(x)$, $x\sin(6x^2)$.

Assume $0 < \alpha < \beta < \pi/2$, prove

$$\frac{\beta - \alpha}{\cos^2 \alpha} < \tan\beta - \tan\alpha < \frac{\beta - \alpha}{\cos^2 \beta}$$

Prove $|\sin x - \sin y| \leq |x - y|$, $x, y \in \mathbb{R}$

Prove Cauchy's Mean Value Theorem.

Theorem 3.7. *Let f, g be two functions that satisfy the following three conditions:*

- *f, g is continuous on $[a, b]$*
- *f, g is differentiable on (a, b)*
- *$g' \neq 0$ when $x \in (a, b)$*

then $\exists c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Assuming f is differentiable at $[0, \infty)$ and $0 \leq f(x) \leq \frac{x}{1+x^2}$, prove there exists a $c > 0$ such that

$$f'(c) = \frac{1 - c^2}{(1 + c^2)^2}$$

Let $f(x) = x + \frac{1}{x}$. Find the intervals where it's increasing, where it's decreasing, where it's concave upward and where it's concave downward.

Let $f(x) = \sin(x)$. Find the intervals where it's increasing, where it's decreasing, where it's concave upward and where it's concave downward. Then find all its inflection points.

Use $\epsilon - \delta$ language to say that $\sin(x)$ doesn't have limit at infinity.

Calculate the following limit where $a_n, b_n \neq 0$.

$$\lim_{x\to\infty} \frac{a_n x^n}{b_n x^n}, \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1}}{b_n x^n + b_{n-1}x^{n-1}}$$

Calculate the asymptote for the following function.

$$\frac{a_3x^3 + a_2x^2 + a_1x + a_0}{b_1x + b_0}$$

where $a_3, b_1 \neq 0$

Consider a semi-ellipse above the x-axis with expression: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y \geq 0$. Find the area of the largest rectangle that can be inscribed inside.

Find the point on the parabola $y = 6x^2$ that is closest to the point $(3, 3)$.