HW4

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1

Reduce the differential eigenproblem $(1-x^2)u'' - xu' + \lambda u = 0$ to Sturm-Liouville form.

From the form

$$a_0 \frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}u_i}{\mathrm{d}x} + \left(a_2(x) + \lambda_i a_3(x)\right) u_i = 0$$

$$\begin{aligned} a_0(x) &= 1 - x^2 \\ a_1(x) &= -x \\ a_2(x) &= 0 \\ a_3(x) &= 1 \end{aligned}$$

To obtain the SL form $\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}u_i\right) + \left(q(x) + \lambda_i w(x)\right) u_i = 0$, we let $p(x) = \exp\left(\int \frac{a_1}{a_0}\,\mathrm{d}x\right) = \sqrt{x^2-1}$, $q(x) = \frac{a_2}{a_0}p = 0$ and $w(x) = \frac{a_3}{a_0}p = -\frac{1}{\sqrt{x^2-1}}$

2

Find the eigenvalues and eigenfunctions of the differential eigenproblem

$$u'' + \lambda u = 0, u'(0) = 0, u(1) = 0$$

For case $\lambda = 0$, we end up with solution $u(x) = c_1 x + c_2$. Plug in the boundary condition we have

$$\begin{cases} c_1 &= 0 \\ c_1 + c_2 &= 0 \end{cases} \implies c_i = 0$$

This is trivial, so next we consider $\lambda = -\mu^2$ where $\mu > 0$. The solution is $u = c_1 e^{\mu t} + c_2 e^{-\mu t}$. Plug in the boundary condition we have

$$\begin{cases} (c_1 - c_2)\mu &= 0 \\ c_1 e^{\mu} + c_2 e^{-\mu} &= 0 \end{cases} \implies c_i = 0$$

Still trivial, so we consider $\lambda = \mu^2$. The solution is then $u = c_1 e^{i\mu t} + c_2 e^{-i\mu t} = c_3 \cos(\mu t) + i c_4 \sin(\mu t)$. For $u(x) = c_3 \cos(\mu t)$, plug in the boundary condition we have

$$\begin{cases} -c_3 \mu \sin(0) &= 0 \\ c_3 \cos(\mu) &= 0 \end{cases} \implies \mu = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

For $u(x) = c_4 \sin(\mu t)$, plug in the boundary condition we have

$$\begin{cases} c_4 \mu \cos(0) & = 0 \\ c_4 \sin(\mu) & = 0 \end{cases} \implies c_4 = 0$$

Excluding these trivial solutions, we end up with eigenvalues $\lambda_i = \mu^2 = (k+0.5)^2\pi^2, k \in \mathbb{Z}$, and the corresponding eigenfunctions are $u_k(t) = c_3 \cos((k+0.5)\pi t)$, $k = 1, 2, 3, \ldots$.
Finally, we normalizing the functions, and obtain $c_3 = \sqrt{2}$.

3

Find the eigenvalues and eigenfunctions of the differential eigenproblem $\lambda = \mu^2$

$$x^2 u'' + xu' + \mu^2 u = 0, u(1) = 0, u(4) = 0$$

Plug in the general form $u(x) = x^r$ we obtain $r^2 + \mu^2 = 0$ so $r = \pm i\mu$ and $u(x) = x^{\pm i\mu} = e^{\pm(\ln x)i\mu}$.
Then use Euler's formula and obtain $u(x) = c_1 \cos(\mu \ln x)$, or $u(x) = c_2 \sin(\mu \ln x)$.
For $u(x) = c_1 \cos(\mu \ln x)$, the boundary condition gives $c_1 = 0$ thus trivial, and for $u(x) = c_2 \sin(\mu \ln x)$ the boundary condition gives $c_2 \sin(\mu \ln 4) = 0$ thus $\mu = \frac{k\pi}{\ln 4}$ where $k \in \mathbb{Z}$. Similarly after normalizing it we have the eigenfunctions $\sqrt{2}\sin(\frac{k\pi \ln x}{\ln 4})$ and corresponding eigenvalues are $\mu^2 = \frac{k^2\pi^2}{(\ln 4)^2}$

4

Verify that the set of functions $\phi_n(x) = \sin\left(\frac{n\pi x}{l}\right), n = 1, 2, \ldots$ on $x \le x \le l$ are orthogonal with unit weight, and find their norms.

For distinct n, m, we have the following:

$$\begin{aligned} (\phi_n, \phi_m) &= \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) \mathrm{d}x \\ &= \frac{1}{2} \int_0^l \cos\left(\frac{(n-m)\pi x}{l}\right) - \cos\left(\frac{(n+m)\pi x}{l}\right) \mathrm{d}x \\ &= \frac{1}{2\pi} \left(\frac{l}{n-m} \sin\left(\frac{(n-m)\pi x}{l}\right)\bigg|_0^l - \frac{l}{n+m} \sin\left(\frac{(n+m)\pi x}{l}\right)\bigg|_0^l \right) = 0 \end{aligned}$$

This is the orthogonality; for the norm, we have

$$\begin{aligned} (\phi_n, \phi_n) &= \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\ &= \frac{1}{2} \int_0^l 1 - \cos\left(\frac{2n\pi x}{l}\right) \mathrm{d}x \\ &= l/2 \end{aligned}$$

Thus their norms are $\sqrt{\frac{l}{2}}$.

5

Show that Bessel's equation of order ν: $x^2u'' + xu' + (x^2 - \nu^2)u = 0$ is not self-adjoint. Then find a such that the following equation is self-adjoint.

$$a \sin x u'' + \cos x u' + 2u = 0$$

According to the standard form of a general second order linear differential equation: $\mathcal{L}u = \frac{1}{w(x)}\left(a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x)\right)$, here

$$\begin{aligned} a_0(x) &= x^2, \\ a_1(x) &= x, \\ a_2(x) &= x^2, \\ \lambda &= \nu^2, \\ w(x) &= 1 \end{aligned}$$

As in the requirement: $a_1(x) = a_0'(x)$, apparently $(x^2)' = 2x \neq x$, thus not self-adjoint. This requirement also give rise to the solution to a. We only want $(a \sin x)' = \cos x$ thus $a = 1$.

6

Expand the polynomial $3x^4 - 4x^2 - x$ in terms of a linear combination of the first four Chebyhev polynomials.

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

We can compare the coefficients from higher order.

$$\begin{aligned} a_4 &= 3/8, \\ a_3 &= 0, \\ a_2 &= (-4 - (-8) \times 3/8)/2 = -1/2, \\ a_1 &= -1, \\ a_0 &= -(3/8 + 1/2) = -7/8 \end{aligned}$$

7

A rectangular membrane with its corners at $(0,0), (a,0), (0,b), (a,b)$ has its edges clamped. The governing equation is the two-dimensional wave equation in Cartesian coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where c is the wavespeed, and u is the vibrational amplitude. Show that the natural vibrational frequencies are given by

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

and that the corresponding eigenfunctions determining the modes of the vibration are

$$\phi_{mn} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

Using separation of variable method, we first obtain

$$-\frac{c^2\Delta\phi}{\phi} = -\frac{\ddot{\psi}}{\psi} = \lambda = (k_1 + k_2)\omega^2$$

First we solve spatial equation: $\Delta\phi = -\frac{\omega^2}{c^2}\phi$, where the boundary condition is zero edges due to the clamp. Here we do another separation of variable: $\phi(x, y) = \phi_1(x)\phi_2(y)$ then we have $\phi_2(y)\Delta\phi_1(x) + \phi_1(x)\Delta\phi_2(y) = -\frac{(k_1+k_2)\omega^2}{c^2}\phi_1(x)\phi_2(y)$ and then we could have

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\phi_1(x) + \frac{k_1\omega^2}{c^2}\phi_1(x) = 0 \\
\frac{\partial^2}{\partial y^2}\phi_1(y) + \frac{k_2\omega^2}{c^2}\phi_2(y) = 0 \\
\phi_1(0) = \phi_1(a) = 0 \\
\phi_2(0) = \phi_2(b) = 0
\end{aligned}$$

The general solution is $\phi_1(x) = c_1\cos\left(\sqrt{k_1}\frac{\omega_n}{c}x\right) + c_2\sin\left(\sqrt{k_1}\frac{\omega_n}{c}x\right)$ and from the initial condition, $c_1 = 0$ and $\sin\left(\sqrt{k_1}\frac{\omega_n a}{c}\right) = 0$ so the eigenvalues satisfy $\sqrt{k_1}\frac{\omega_n a}{c} = n\pi, n = 1, 2, \ldots$ and the eigenfunctions are $\phi_1(x) = c_2\sin\left(\frac{n\pi x}{a}\right)$. Similarly, $\phi_2(y) = c_4\sin\left(\frac{m\pi y}{b}\right)$, $m, n = 1, 2, \ldots$. The arbitrary constants here are taken as 1 and thus

$$\phi_{mn}(x, y) = \phi_1(x)\phi_2(y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

Their natural frequencies are $k_1\omega^2 = \frac{n\pi c}{a}$ and $k_2\omega^2 = \frac{m\pi c}{b}$, then

$$\omega_{mn} = \sqrt{(k_1 + k_2)\omega^2} = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$