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Two-Stage Deep Kernel Learning and Gaussian Process Regression for European Call Option Pricing

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Contents

List of Figures	iv
List of Tables	v
Summary	vi
1 Introduction	1
1.1 Background	1
1.1.1 Machine Learning Methods	1
1.1.2 Problem Statement	2
1.1.3 Process and Structure	2
1.1.4 Data and Experimental Design	3
1.1.5 Positioning Within Existing Work	3
1.1.6 Contributions	3
2 Literature Review	4
2.1 Related Work	4
2.1.1 Classical Models	4
2.1.2 Machine Learning Approaches	5
2.1.3 Gaussian Processes and Deep Kernel Learning	6
2.1.4 Hybrid Tail Modeling and Targets	6
2.1.5 Quantifying and Calibration Uncertainty	7
2.2 Research Gaps	7
2.2.1 Limitations of Classical Parametric Models	7
2.2.2 Physical vs. Risk-Neutral Perspectives	8
2.2.3 Challenges of Pure Machine Learning Approaches	8
2.2.4 The Volatility Surface Modeling	9
2.2.5 Limitations of existing DKL-GPR approaches	9
2.2.6 Uncertainty and Scalability	10
2.3 Research Rationale	10
2.3.1 Balancing Structure and Flexibility	10
2.3.2 Uncertainty Quantification and Calibration	11

2.3.3	Supporting Literature for the Framework	11
3	Methodology	14
3.1	P-Measures and Q-Measures	14
3.1.1	Dynamics under the Risk-Neutral Measure \mathbb{Q}	15
3.1.2	Dynamics under the Physical Measure \mathbb{P}	15
3.2	Data Processing	16
3.2.1	Problem Setup	16
3.2.2	Data, Assumptions, and Cleaning	16
3.3	Feature Engineering	17
3.3.1	Feature Augmentation	17
3.4	Optimization and Inference Procedure	17
3.4.1	Learning Setup and Architecture	19
3.4.2	Justification for Employing Gaussian Processes (GP) within the Framework	19
3.4.3	Deep kernel learning.	20
3.4.4	Evaluation Strategy	20
3.5	Summary of Methodology	21
4	Training Setup	22
4.1	Data and Preprocessing	22
4.1.1	Dataset and Splits	22
4.1.2	Feature Engineering	22
4.2	Model Setup and Training Detail	23
4.2.1	Deep Kernel Learning Model Specifications	24
4.2.2	Target Transformation and Tail Residual Calibration	24
4.2.3	Training and Evaluation	24
5	Results	26
5.1	Overall Performance	26
5.1.1	Train objective and convergence.	26
5.1.2	Accuracy and calibration.	27
5.2	Ablation Studies	29
6	Discussion	31
6.1	Methodological Positioning	31
6.1.1	Flexibility and Strengths of the Study	31
6.1.2	Baseline Comparisons	32
6.2	Empirical Findings and SOTA Position	32
6.2.1	Stage 1: Implied Volatility Prediction	32

6.2.2	Stage 2: Price Prediction	33
6.2.3	Stage 2: Generalization Comparison with MLP	33
6.3	Research Contributions	34
6.3.1	Model Design	34
6.3.2	Empirical Performance	34
6.3.3	Benchmark Comparison	35
6.3.4	Overall Contribution	35
7	Conclusion	36
7.1	Limitations of the Study	36
7.1.1	Competitiveness and the Accuracy–Uncertainty Trade-off	36
7.1.2	Scope of Comparative Analysis	36
7.1.3	Computational Complexity and Scalability	37
7.1.4	Fidelity of Variational Approximation	37
7.2	Contributions and Key Findings	37
7.3	Research Gaps and Future Applications	38
7.4	Final Remarks	39
References		41

List of Figures

3.1	Two-stage DKL+GPR option pricing architecture.	14
5.1	Training dynamics of the negative ELBO ($-\text{ELBO}$) across epochs.	27
5.2	Validation dynamics of Stage 2 option price modeling.	28
5.3	True vs. predicted option prices.	28

List of Tables

4.1	Model Architecture Specifications (Detailed)	25
4.2	Training Hyperparameters	25
5.1	Ablation study results on Dev (validation) and Holdout splits.	29
6.1	Model performance comparison on the validation set	32
6.2	Benchmark comparison on SPX option	33
6.3	Comparison with MLP (Stage 2, test set).	33

Summary

This thesis introduces an innovative two-stage Deep Kernel Learning and Gaussian Process Regression (DKL+GPR) methodology for option pricing, tackling persistent shortcomings in both traditional parametric models and contemporary machine learning methodologies. Conventional models, although theoretically coherent and interpretable, fail to adequately represent intricate market characteristics such as heavy tails, volatility smiles, and regime shifts. Data-driven solutions yield great predicted accuracy; yet, they often lack financial framework and calibrated uncertainty, resulting in practical deficiencies in risk management.

The proposed framework addresses these deficiencies by combining deep neural feature extraction with Bayesian nonparametric inference. Initially, implied volatility surfaces are calculated using a DKL encoder in conjunction with a sparse variational Gaussian Process, yielding smooth and arbitrage-consistent predictions. In the second stage, these outputs are augmented by Black–Scholes baseline values and Greeks, establishing the foundation for a probabilistic price learner that integrates hybrid goal transformations and tail-aware residual calibration. To facilitate scalability with extensive option datasets, we utilize Sparse Variational Gaussian Processes (SVGP) to approximate full Gaussian Process Regression (GPR), thus preserving Bayesian uncertainty estimates while diminishing computational cost.

In addition to accuracy, the framework prioritizes uncertainty quantification. The integration of calibrated prediction intervals with tail-sensitive modeling facilitates risk-aware decision-making and broadens the conventional function of option pricing from point estimation to probabilistic risk assessment. This method reconciles risk-neutral and physical viewpoints by integrating observable market dynamics with structural benchmarks, so capturing both pricing consistency and real-world predictive indicators.

The research presents a hybrid methodology that integrates financial theory with sophisticated machine learning techniques. This illustrates that it is possible to concurrently achieve interpretable, uncertainty-aware, and empirically robust option pricing, providing a practical and theoretically sound instrument for traders, risk managers, and regulators.

Chapter 1

Introduction

1.1 Background

This thesis is situated at the nexus of probabilistic machine learning and computational finance. It addresses the long-standing problem of accurately and consistently pricing equity index options across strikes and maturities, going beyond point predictions to provide realistic tail behavior and calibrated uncertainty. Pure ML models are flexible but frequently disregard financial constraints and lack calibrated uncertainty, while classical parametric models (e.g., Black-Scholes, Heston) include valuable structure and no-arbitrage priors but can be overly rigid under regime shifts and large tails. This study's benchmark evidence supports these limitations, albeit inaccurately when compared to more recent methods. By adding mean-reverting volatility and jumps, successor models like Heston's stochastic volatility model, Merton's jump-diffusion. All things considered, the classical paradigm provides interpretability and theoretical consistency, but at the expense of empirical rigidity and systematic pricing inaccuracies.

1.1.1 Machine Learning Methods

Machine learning techniques have become more popular in the field as a reaction to the rigidity of classical models. Since Hutchinson et al. (1994), option data has been subjected to nonparametric techniques such neural networks, SVR, random forests, and LSTMs, which frequently beat conventional parametric fits and drastically lower price mistakes. For instance, a multi-layer perceptron (MLP) benchmark in this work reaches state-of-the-art accuracy, with R^2 on SPX options surpassing 99.8%.

However, there is a price for this flexibility. There are practical hazards since pure machine learning models can generate arbitrage violations (convexity, monotonicity) in as many as 2–5% of forecasts. More significantly, they have a **uncertainty gap**: these models do not have calibrated uncertainty quantification, even while point estimations are accurate. Understanding prediction confidence is just as important for risk man-

agement as the actual forecasts, especially when it comes to deep out-of-the-money (OTM) options.

Although predictive uncertainty is inherent in Gaussian processes (GPs), their $O(N^3)$ complexity renders them unsuitable for use in large-scale financial datasets. As a result, quantitative finance seems to be caught between two less-than-ideal paradigms: empirically strong but risk-insensitive machine learning models, or theoretically sound but empirically weak parametric models. The main goal of this thesis is to close this gap.

1.1.2 Problem Statement

Learning a high-fidelity, uncertainty-aware option pricing map for SP 500 (SPX) options under extensive, real-market cross-sections is the main challenge. The objective is twofold: (i) measure prediction uncertainty, particularly in sparse, tail regions where risk management requires reliable intervals, and (ii) capture intricate cross-sectional patterns in implied volatility (IV) and prices without hard-coding a parametric surface. The thesis focuses on addressing the methodological gap between empirical flexibility and structural fidelity, as well as the practical P–Q divide in pricing vs. estimation.

1.1.3 Process and Structure

A two-stage Deep Kernel Learning + Gaussian Process Regression (DKL+GPR) architecture is used in the suggested framework. In the first step, a variational GP head is used to estimate implied volatilities after a DKL encoder maps contract and market parameters (such moneyness, maturity, and realized-volatility proxies) into a latent space.

The smooth, leakage-safe, arbitrage-consistent, and informative IV estimations generated in this stage are used as covariates in the next stage.

The second step involves directly training a DKL+GP model using observed mid-prices. Here, Greeks generated from the Stage-1 IV estimates and Black-Scholes baseline values are added to the raw characteristics. A lightweight residual calibrator further refines predictions in extreme regions, resulting in calibrated predictive distributions appropriate for risk-aware decision making. To provide robustness, a hybrid target transformation is utilized to enhance stability in skewed price distributions.

Both phases rely on probabilistic GP heads optimized by variational ELBO training and deep encoders with inducing-point scalability. In addition to ReLU- and noise-injected price encoders with a latent dimension of 16 and roughly 256 inducing points, the architectural design incorporates GELU-based IV encoders with a latent dimension of 32 and roughly 512 inducing points. To improve generalization and avoid overfitting, a cosine-annealed Adam scheduler with early halting is used to optimize the models.

1.1.4 Data and Experimental Design

The research employs roughly 100,000 SPX option contracts (after filtering), together with the 3-month Treasury yield (FRED DGS3) and the SP 500 spot (FRED SP500). The division is temporal (80Features encompass log-moneyness, \sqrt{T} , nonlinear interactions, lagged returns/realized volatilities, and Black-Scholes-derived metrics from anticipated implied volatility (price and Greeks). All transformations and standardizations are exclusively for development purposes to facilitate causal evaluation.

1.1.5 Positioning Within Existing Work

This thesis integrates structural finance (no-arbitrage, Greeks, P/Q concerns) with flexible machine learning (deep feature learning) via a Bayesian layer that provides predictions with believable intervals. This application of DKL-GPR diverges from previous implementations by explicitly structuring a two-stage **IV-then-price** pipeline, incorporating BS-augmented features and tail-aware calibration, while prioritizing empirical calibration and cross-sectional generalization using actual SPX data. In direct comparisons, it either matches or exceeds robust deep baselines while maintaining interpretability and risk awareness.

1.1.6 Contributions

The thesis offers significant contributions to the fields of option pricing and probabilistic machine learning. Initially, it presents a two-stage DKL+GPR paradigm that adeptly amalgamates structural financial signals with deep representations, thus reconciling theoretical coherence with practical adaptability.

Secondly, it formulates an uncertainty-aware pricing model that integrates hybrid target transformations and a tail residual calibration mechanism, thereby augmenting predictive robustness and refining risk-sensitive decision-making. The study implements a stringent evaluation process that mitigates information leaking, includes ablation experiments, and benchmarks against robust baselines, hence ensuring the reliability and validity of the findings. The proposed system attains superior predictive accuracy and significantly enhances tail calibration on extensive SPX option datasets, showcasing both practical applicability and methodological progress.

Chapter 2

Literature Review

2.1 Related Work

2.1.1 Classical Models

The Black-Scholes framework is the source of traditional option-pricing models, however they are unable to account for important market characteristics like heavy tails and volatility smiles. The volatility smile was explained by the addition of a stochastic volatility component with mean reversion in the Heston (1993) model. Merton (1976) suggested a jump-diffusion model in which extreme events are explained by price process discontinuities. Parametric specifications such as the Stochastic Volatility Inspired (SVI) and Surface Stochastic Volatility Inspired (SSVI) parameterizations provide flexible surface fits. SVI offers a parsimonious five-parameter fit to implied volatility slices, while SSVI imposes no-arbitrage conditions across maturities. When carefully calibrated, these models can yield arbitrage-free volatility surfaces. Nevertheless, classical and parametric models often rest on restrictive distributional assumptions and may misrepresent market dynamics or produce unrealistic tail behavior. As a result, they can fall short in capturing the full extent of observed implied volatility skews and extreme risk.

The drawbacks of strict distributional assumptions are demonstrated by these classical models. These limitations make it more difficult for them to reflect regime changes, heavy-tailed behaviors, and nonlinearities that are present in actual markets. The adoption of data-driven machine learning techniques, which are able to adaptably extract patterns from empirical option data, is prompted by this. Our framework precisely addresses the drawbacks of parametric models by having the second-stage GP take residual structure and extreme tails into account, while the first-stage DKL flexibly learns implied volatility surfaces without prespecified dynamics.

In option pricing theory, the introduction of the risk-neutral measure (Q measure) ensures that the discounted asset price process becomes a martingale, thereby allowing

derivative prices to be expressed as expectations of future payoffs under the no-arbitrage condition. By contrast, the actual data-generating process in financial markets follows the real-world measure (P measure). This distinction between “pricing under Q ” and “estimation under P ” constitutes a fundamental background of financial modeling and forms the basis of risk premium research. As first established by Harrison and Kreps (1979), the change of measure plays a crucial role in linking P and Q , thereby unifying statistical modeling based on historical data with the arbitrage-free pricing framework

2.1.2 Machine Learning Approaches

Data-driven approaches have been thoroughly investigated as adaptable substitutes for traditional option pricing models in recent decades. By training neural networks on market data—with inputs including moneyness, maturity, and the underlying price—to forecast option prices or sensitivities, Hutchinson et al. (1994) invented nonparametric pricing. Support vector machines (SVM/SVR), random forests, feedforward neural networks, tree boosting, and deep networks have all been used to analyze option data in later studies. Generally speaking, nonparametric machine learning techniques have frequently performed better than conventional parametric fits. Similarly, the comparison of neural networks, SVR. Deep learning has also been applied; generative adversarial networks (GANs) have been proposed to mimic realistic option markets, while convolutional networks and LSTMs have been used to anticipate the complete implied volatility surface over time. By learning hedging strategies under market frictions, reinforcement-learning techniques like ”deep hedging” have transcended pricing (Buehler et al., 2019) and demonstrated how flexible function approximation can handle intricate derivative problems.

The advent of hybrid two-step approaches that blend machine learning and parametric structures is a recent development. Compared to a direct neural network fitted on the implied volatility surface, this machine-corrected model regularly performed better than the original specification. Fan and Yao (2003) presented a similar model-guided correction approach, suggesting that a parametric model be calibrated before its pricing mistakes be nonparametrically estimated and corrected. The two-stage method—using a preliminary parametric model as a framework and then fine-tuning the fit with adaptable machine learning techniques—is justified by these contributions. Though they require big datasets and thorough regularization, the literature suggests that artificial neural networks, SVR, and boosting techniques can dramatically minimize pricing error when compared to Black-Scholes and other parsimonious models.

This hybrid concept is expanded upon in the current design. The global smoothness of indicated volatilities is captured by Deep Kernel Learning (DKL), and residuals are

then corrected using Gaussian Process regression. With the added benefit of Bayesian uncertainty quantification, this method is similar to the "model-guided correction" paradigm.

2.1.3 Gaussian Processes and Deep Kernel Learning

Neural networks and GPR are combined in Deep Kernel Learning (DKL), where a neural network converts inputs into a feature space that a GP kernel may operate on. This hybrid uses deep features to capture intricate patterns while maintaining GP-based uncertainty quantification. Scalable DKL was first presented by Wilson (2016), and it has since been expanded to high-dimensional option problems in more recent applications. A closely related line of work is Zhuang et al. (2023), who bring Deep Kernel Learning (DKL) together with sparse variational Gaussian processes (SVGP) to the high-dimensional American option setting within a regression-based Monte Carlo framework. In their approach, continuation values are estimated inside the Longstaff–Schwartz algorithm, and they report consistent accuracy gains over Least-Squares Monte Carlo (LSM), with particularly strong improvements under Merton-type jump–diffusion dynamics. Importantly, the study illustrates that DKL+GPR can be made practical for large-scale option pricing tasks, and that modern Gaussian process machinery can mitigate aspects of the curse of dimensionality

This work attempts to maintain the key components of the suggested efficiency of methodology, expressiveness, and calibrated predictive intervals—by merging DKL with GP.

2.1.4 Hybrid Tail Modeling and Targets

It has long been known that option price distributions and payoffs are skewed and heavy-tailed, especially for short maturities and out-of-the-money (OTM) strikes. Often, hybrid targets or transformations have been used to capture tail risk and unusual events. To account for the long left tail, for instance, academics have modelled log-moneyness or log-prices for deep-OTM options, while maintaining linear pricing models for the near-at-the-money range. Similarly, log or square-root transformations have been used to stabilize forecasts since short-term volatility can surge. These hybrid linear/log targets combine the robustness of log-scale modeling in the tails with the ease of use of linear regression close to the mean.

There have also been proposals for tail-aware models. By substituting a t -distribution for Gaussian noise, robust Gaussian process (GP) regression with a Student- t likelihood Jylänki et al. (2011) down-weighted outliers and generated fatter predictive tails. Regression using quantiles The purpose of (Koenker and Bassett, 1978) was to explicitly focus on tail behavior by estimating conditional percentiles. The option pricing litera-

ture has also included skewed distributions and mixture models. Log-transforming deep OTM prices is used in this research to accommodate hybrid targets, and a Student- t likelihood in the GP improves calibration for fat tails and guards against negative price forecasts.

These methods are incorporated into the second stage of our framework: the option pricing system's reliability is increased by applying hybrid targets and customizing GP likelihoods, which improve robustness for extreme strikes and maturities.

2.1.5 Quantifying and Calibration Uncertainty

The ability to explicitly quantify uncertainty is one of the main benefits of Gaussian processes. Forecast confidence is reflected in the prediction variance that a GP generates at each input. The predictive distribution is essential for risk management in option pricing. However, earlier research revealed that raw GP variances were frequently inaccurately calibrated, leading to the introduction of calibration techniques.

Numerous strategies have been put forth. Distribution-free prediction intervals with finite-sample guarantees were produced using conformal prediction. Applying conformal approaches to option pricing, for example, Bastos (2023) showed empirical coverage near nominal levels, with broader intervals for OTM and short-dated options. More conservative intervals and fatter prediction tails have been induced using student- t noise models. Other lines of research looked into bootstrap ensembles or quantile regression as calibration tools. The wider regression literature has also created post-hoc residual calibration procedures, where supplementary models were fitted to improve tail behavior by correcting systematic biases in prediction means or variances (Allen et al., 2024).

Building on previous research, the current study uses a tail-focused residual calibration, called the TailCalibrator, specifically with the two-stage model's DKL+GP component instead of baseline comparators. By ensuring that the predictive distributions from DKL+GP are both expressive and well-calibrated in the tails, this specialization gives practitioners trustworthy confidence intervals in addition to point estimates.

2.2 Research Gaps

2.2.1 Limitations of Classical Parametric Models

Simplifying stochastic assumptions, which are ineffective in real-world scenarios, are the foundation of traditional option pricing models. According to empirical data, asset returns show both stochastic volatility and leaps, despite the classical Black-Scholes model's assumption of constant volatility. Mean-reverting volatility was introduced by extensions like Heston's stochastic volatility model (Heston, 1993), but even these

are unable to adequately reflect the erratic or excessive smile patterns seen in markets. Therefore, parametric models are unable to adequately explain option-implied dynamics, even while they are able to capture certain stylized facts. Early theoretical models such as Black and Scholes (1973) and Heston (1993) provided analytical tractability and incorporated stochastic variance dynamics, but remained too rigid to capture volatility smiles and extreme tail behaviours. Parametric smile-fitting models like SVI/SSVI Gatheral and Jacquier (2014) enforce no-arbitrage constraints on implied volatility slices, ensuring convexity and monotonicity. Thus, classical parametric models capture some qualitative features of volatility but cannot flexibly adapt to the full dynamics observed in real markets. This motivates a hybrid approach: retain financial structure while allowing more flexible data-driven corrections.

2.2.2 Physical vs. Risk-Neutral Perspectives

Recent studies have highlighted the importance of reconciling the physical (\mathbb{P}) and risk-neutral (\mathbb{Q}) measures in volatility modeling. For instance, Zhang and Zhou (2020) adjusted risk-neutral dynamics for persistence, achieving a more accurate fit to VIX, while Bollerslev et al. (2018) documented that the variance risk premium (VRP)—the differential between realized volatility under \mathbb{P} and implied volatility under \mathbb{Q} —is generally positive and represents compensation for bearing volatility risk. Despite this, the \mathbb{P} – \mathbb{Q} gap is ignored by the majority of option pricing models, leading to systematic mispricing (Zhang and Zhou, 2020). Classical econometric approaches have explicitly modeled joint dynamics—such as stochastic GARCH-jump specifications (Christoffersen and Jacobs, 2004), and particle filter methods with Monte Carlo importance sampling (Ferriani and Pastorello, 2012)—but these come with high computational costs and restrictive parametric assumptions.

By contrast, most machine learning approaches continue to treat the problem purely under the risk-neutral measure, without exploiting lagged or observable \mathbb{P} -measure information. This omission creates a methodological gap: scalable models that combine historical, physical-measure dynamics with forward-looking, risk-neutral surfaces remain underexplored.

2.2.3 Challenges of Pure Machine Learning Approaches

Machine learning methods offer flexibility and often outperform parametric models in out-of-sample fits. Neural networks, SVR, random forests. However, without embedded economic constraints, such models may yield prices that violate monotonicity or convexity. For instance, recent empirical evaluations show that even advanced neural networks can generate surfaces with local arbitrage violations—approximately 2–5% of predictions break monotonicity in strike or convexity Vuletić and Cont (2024).

Moreover, most ML-based approaches lack calibrated uncertainty estimates. Standard neural networks output point forecasts, offering no measure of confidence. Without uncertainty quantification, risk managers cannot assess reliability, especially for deep out-of-the-money (OTM) or short-dated options where data are sparse. Gaussian processes (GPs) can address this by providing posterior variance Rasmussen and Williams (2006), but exact GP training scales as $\mathcal{O}(N^3)$, prohibitive for large datasets. This gap—flexibility versus structure, accuracy versus uncertainty—highlights the need for a hybrid architecture that balances both.

2.2.4 The Volatility Surface Modeling

The implied volatility surface (IVS) is the organising principle of modern option pricing, encoding how markets anticipate risk across strikes and maturities. A central challenge has been balancing *structural parsimony* with *empirical flexibility*.

Early theoretical models such as Black and Scholes (1973) and Heston (1993) provided analytical tractability and incorporated stochastic variance dynamics, but remained too rigid to capture volatility smiles and extreme tail behaviours.

This motivated nonparametric and machine learning approaches. Hutchinson et al. (1994) demonstrated that neural networks could approximate option prices directly, offering freedom from parametric mis-specification. However, unconstrained networks risked producing arbitrageable or economically inconsistent surfaces. More recent strands therefore integrate structure and learning: Fan and Yao (2003) introduced a “model-guided correction” where residuals from parametric fits were captured by nonparametric regressions. Adversarial formulations have also been proposed to embed no-arbitrage constraints, though they are computationally demanding and diverge from the probabilistic GP-based paradigm.

This trajectory highlights enduring gaps: rigidity versus flexibility, the absence of uncertainty quantification, limited treatment of residual mispricing, and disconnection between P- and Q-measure dynamics. These challenges directly motivate hybrid probabilistic models such as the DKL+GPR framework pursued here.

2.2.5 Limitations of existing DKL-GPR approaches

The framework of Zhuang et al. (2023) demonstrates that combining neural feature extraction with variational sparse Gaussian processes can improve scalability in high-dimensional American option pricing and achieve lower pricing errors compared with standard least-squares Monte Carlo approaches. There are still a lot of issues, even though their study shows potential in combining variational sparse GPs with neural feature extraction. Evaluation places more focus on runtime and simulation accuracy than on distributional measures, tail errors, or prediction interval calibration. Since the

IV surface stores important cross-sectional characteristics including skew, smile, and term structure, ignoring it can lead to economically unrealistic behavior, particularly in the tails or for long maturities. Although posterior variances are recorded, there is no systematic alignment of coverage, and it is outside the purview of this study to integrate P-measure data into the Q-surface. Since the evaluation is primarily conducted on simulated pathways, transferability to real market cross-sections is largely unknown. The computational complexity is greatly increased by their strategy, which requires training a distinct DKL model for each workout date in the American option context. Finally, there is no systematic comparison with more powerful baselines like arbitrage-free surface models, reinforcement learning-based halting strategies, or deep BSDE solvers. Important gaps still exist, nonetheless, in relation to the purpose of this study.

2.2.6 Uncertainty and Scalability

Gaussian Processes (GPs) provide predictive uncertainty natively. Given training data (X, y) , the posterior predictive mean and variance are:

$$\mu(x_*) = k_*^\top (K + \sigma^2 I)^{-1} y, \quad \sigma^2(x_*) = k(x_*, x_*) - k_*^\top (K + \sigma^2 I)^{-1} k_*.$$

This ensures forecasts are distributions, not points. However, exact inference scales as $O(N^3)$, infeasible for large option datasets.

To lower the computational cost to $O(NM^2)$, Titsias (2009) introduced variational inducing points. Mini-batch training was made possible by the extension of this method to stochastic variational inference (Hensman et al. (2013)). In Deep Kernel Learning (DKL), Wilson et al. (2016) further integrated GP kernels with deep neural networks, integrating flexible feature maps within kernels. Although these scalable GP techniques reduce computational demands, they depend on imprecise approximations, which result in an unresolved calibration of predictive intervals. On the other hand, dropout-based uncertainty lacks Bayesian guarantees, while neural networks scale effectively but only offer point estimates. Therefore, integrating scalability with accurate uncertainty quantification is the methodological gap.

2.3 Research Rationale

2.3.1 Balancing Structure and Flexibility

Machine learning methods provide the ability to directly suit complex surfaces. Nevertheless, regressions that are solely based on data frequently disregard financial structure, resulting in arbitrageable surfaces and violations of monotonicity or convexity. These issues are mitigated by constraint-penalized training, but they are not entirely

resolved. Consequently, a fundamental gap continues to exist: the question of how to maintain structural consistency while simultaneously attaining empirical flexibility. By decoupling the two concerns, our methodology contributes to this line. In Stage 1, Deep Kernel Learning (DKL) is implemented to acquire expressive representations of the IV surface. These representations are informed by theoretical priors, including no-arbitrage consistency and asymptotic slope conditions, but they are not hard-coded to a specific parametric form. Stage 2 subsequently implements Gaussian Processes to flexibly capture residual structures, thereby enhancing tail calibration and quantifying uncertainty.

2.3.2 Uncertainty Quantification and Calibration

Uncertainty estimation is essential for pricing and hedging. A Gaussian Process (GP) regression model outputs not only a mean prediction $m_*(x)$ but also a variance $v_*(x)$ at input x , providing a predictive distribution:

$$f(x_*) \mid \mathcal{D} \sim \mathcal{N}(m_*(x), v_*(x)),$$

where $m_*(x) = k_*^\top(K + \sigma^2 I)^{-1}y$ and $v_*(x) = k(x_*, x_*) - k_*^\top(K + \sigma^2 I)^{-1}k_*$. This Bayesian structure directly informs confidence intervals.

However, vanilla GPs with Gaussian noise tend to underestimate tail uncertainty. Jyläniemi et al. (2011) demonstrate that using a Student-t likelihood instead reduces sensitivity to outliers and produces more robust uncertainty estimates in heavy-tailed data. This is particularly relevant for option markets, where illiquidity or microstructure noise induces heavy-tailed residuals.

Calibration further requires ensuring empirical coverage. Bastos (2023) shows that conformal calibration produces intervals with coverage close to the nominal 95%, while naive GP intervals systematically under-cover. Importantly, they find that OTM and short-maturity options generate wider conformal bands, aligning with observed higher uncertainty in stressed regimes. These insights directly motivate our design: a GP residual model with Student-t likelihood, calibrated via conformal methods, to produce valid confidence intervals interpretable in risk management.

2.3.3 Supporting Literature for the Framework

In response to the existing work, the literature motivates a **two-stage hybrid framework** that balances the flexibility of Bayesian and deep learning methods with classical structural discipline, while ensuring scalability and robustness for both theoretical soundness and practical applicability.

In the first stage, implied volatility surfaces are learned via Deep Kernel Learning.

Option inputs such as moneyness, maturity, and selected \mathbb{P} -measure features (e.g., realised volatility, variance risk premium proxies) are mapped through a neural encoder $\phi_\theta(\cdot)$ in the DKL paradigm of Wilson et al. (2016). The latent representation is then passed into a sparse variational Gaussian Process (SVGP) head (Hensman et al., 2013), which produces implied volatility estimates:

$$\hat{\sigma}_{\text{IV}}(K, T) = \mathcal{GP}(\phi_\theta(K, T, X^\mathbb{P})).$$

The SVGP prior induces smoothness while still allowing nonlinear local behaviour, thereby offering a balance between parametric structure and nonparametric flexibility.

In the second stage, option prices are modelled with a GP correction mechanism. Using the implied volatilities from Stage 1, we compute baseline Black–Scholes prices and Greeks, which, together with raw option features, form the augmented input set for a second DKL+GPR model trained directly on observed market prices. This approach builds on the residual-learning philosophy of ?, but adapts the idea to a price-prediction setting: instead of explicitly modelling residuals, the model implicitly learns corrections to Black–Scholes through feature augmentation. A probabilistic GP head further provides calibrated uncertainty estimates, ensuring that predictive distributions remain informative for risk management applications.

The conflict between empirical flexibility and structural parsimony is frequently brought to light in the literature on option pricing. Although they provide interpretability and no-arbitrage assurances, classical parametric models like Black-Scholes and Heston are still too inflexible to fully reflect the observed volatility grins and tails. From the early neural networks of Hutchinson et al. (1994) to more current deep learning techniques, data-driven approaches, on the other hand, can approximate complicated patterns, but they frequently violate financial constraints or yield poorly calibrated uncertainty estimates (Vuletić and Cont, 2024). This encourages a hybrid architecture that strikes a balance between flexibility and structure. Lastly, new lines of research highlight the importance of integrating data-driven corrections with structural pricing standards. By embedding parametric structure prior to applying flexible learning, pricing error is greatly reduced, as demonstrated by the model-guided correction paradigm of Fan and Yao (2003). Meanwhile, developments in deep kernel learning (Wilson et al., 2016) and scalable Gaussian process inference (Titsias, 2009) have demonstrated that Bayesian nonparametrics can now be used on huge cross-sectional option datasets. Collectively, these advancements point to a two-phase architecture wherein implied volatilities are first learned arbitrage-consistently and smoothly, and then utilized as structural covariates in a subsequent stage that corrects residual pricing flaws. All things considered, the literature supports a hybrid design that is computationally scalable, tail-robust, uncertainty-aware, and structure-preserving. Such an

approach addresses the empirical limits of both classical and simply machine learning methods while leveraging proven economic ideas, as opposed to substituting black-box predictors for financial theory.

Chapter 3

Methodology

Overview

In order to price S&P 500 index options, this paper presents a two-stage Deep Kernel Learning with Gaussian Processes architecture. While the second stage models option prices using a stochastic deep encoder with Gaussian process regression and enriches the feature set with Black-Scholes-based augmentations, the first stage uses leakage-safe market characteristics to estimate implied volatility. To increase resilience across pricing regimes, a hybrid target transformation, tail-aware training, and residual calibration are used. The framework analyzes predicted uncertainty across market situations, reports standard error measures, and is optimized with adaptive learning schedules and evaluated chronologically.

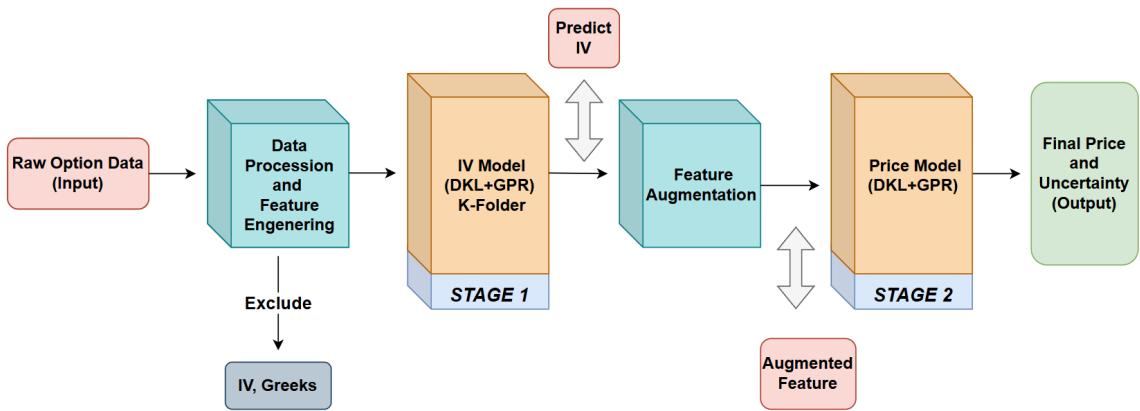


Figure 3.1: Two-stage DKL+GPR option pricing architecture.

3.1 \mathbb{P} -Measures and \mathbb{Q} -Measures

Two probability measures should be distinguished in option pricing: the risk-neutral measure \mathbb{Q} , which ensures arbitrage-free value, and the physical measure \mathbb{P} , which

governs real-world asset dynamics. These are linked by risk premia, particularly the volatility risk premium (VRP), which strikes a balance between market-implied pricing and statistical dynamics. Our approach builds on this by integrating these metrics into a two-phase learning framework: \mathbb{Q} supports benchmark pricing through implied volatility, whereas \mathbb{P} drives Monte Carlo simulation with risk-premium adjustment.

3.1.1 Dynamics under the Risk-Neutral Measure \mathbb{Q}

Under \mathbb{Q} , the asset price S_t and its stochastic volatility V_t evolve as

$$dS_t = (r_t - d_t)S_t dt + \sigma_S(V_t)S_t dW_t^{\mathbb{Q}} + S_t(e^{J_t} - 1) dN_t^{\mathbb{Q}}, \quad (3.1)$$

$$dV_t = \mu_V^{\mathbb{Q}}(V_t) dt + \sigma_V(V_t) dW_t^{\mathbb{Q},V}, \quad (3.2)$$

where r_t is the risk-free rate (3-month Treasury yield) and d_t the dividend yield, set to zero for the SPX. Spot volatility $\sigma_S(V_t)$ is proxied by contract-specific implied volatility, while $W_t^{\mathbb{Q}}$ and $W_t^{\mathbb{Q},V}$ are independent Brownian motions. Jumps arrive via a Poisson process $N_t^{\mathbb{Q}}$ with normally distributed sizes $J_t \sim \mathcal{N}(\nu_J^{\mathbb{Q}}, \sigma_J^2)$, although jumps are omitted in the baseline specification.

3.1.2 Dynamics under the Physical Measure \mathbb{P}

Under \mathbb{P} , the asset price S_t and volatility factor V_t evolve as

$$dS_t = \mu_S^{\mathbb{P}}(S_t, V_t) dt + \sigma_S(V_t)S_t dW_t^{\mathbb{P}}, \quad (3.3)$$

$$dV_t = \mu_V^{\mathbb{P}}(V_t) dt + \sigma_V(V_t) dW_t^{\mathbb{P},V}, \quad (3.4)$$

where the drift $\mu_S^{\mathbb{P}}$ reflects the equity risk premium, and $W_t^{\mathbb{P}}, W_t^{\mathbb{P},V}$ are Brownian motions under \mathbb{P} . Expected returns exceed the risk-free rate, incorporating risk preferences omitted under \mathbb{Q} .

Integration of \mathbb{P} and \mathbb{Q} Measures

The explicit modeling of joint \mathbb{P} - \mathbb{Q} dynamics has long been attempted in the econometrics literature. For example, Christoffersen and Jacobs (2004) extended GARCH models with jumps. More recently, Ferriani and Pastorello (2012) combined particle filters with Monte Carlo importance sampling. While these approaches solve the identification problem, they rely on restrictive parametric assumptions and heavy computation. In contrast, machine learning research has rarely incorporated \mathbb{P} -measure information.

This framework addresses that gap by embedding observable \mathbb{P} -side proxies—such as lagged returns, realized volatility, and variance–risk–premium indicators—directly

into both the Stage 1 IV encoder and the Stage 2 price learner, without simulating \mathbb{P} -paths. Stage 2 further enriches its inputs with Black–Scholes augmentations (model-implied price and Greeks from $\hat{\sigma}$), and the GP correction then reconciles \mathbb{Q} -implied values with observed option prices. In doing so, the model implicitly captures the volatility risk premium, thus extending the joint \mathbb{P} – \mathbb{Q} perspective of Ferriani and Pastorello (2012) in a nonparametric and scalable learning environment.

3.2 Data Procession

3.2.1 Problem Setup

European-style SPX options with expiration τ , underlying spot S_t , strike K , and contemporaneous risk-free rate r_t are the main focus of the study. These options were observed on day t . The definition of the time-to-maturity is

The denominator of

$$T = \frac{\tau - t}{252},$$

represents the market convention of 252 trading days in a year. C_{mid} is the observed market mid-quote. Learning mappings is the goal.

$$\hat{\sigma}_{\text{IV}} = g(\mathbf{x}), \quad \hat{C}_{\text{mid}} = h([\mathbf{x}, \hat{\sigma}_{\text{IV}}]),$$

where $\mathbf{x} \in \mathbb{R}^D$ represents a feature vector built from model-based quantities and market observables.

3.2.2 Data, Assumptions, and Cleaning

There are roughly 2.3 million SPX option contracts in the raw dataset. Only contracts with $0 < \sigma_{\text{impl}} < 1$ are kept after contracts with unparseable maturities, implausible implied volatilities, or missing Greeks (`impl_volatility`, `delta`, `gamma`, `vega`, `theta`) are eliminated. Strike prices are rescaled if they are stored in the $\times 1000$ scale; otherwise, they are harmonized to dollar units.

The rate of risk-free The three-month U.S. Treasury yield (FRED series DGS3) is used to calculate r_t . It is forward-filled to guarantee coverage, and dividends are taken to be insignificant ($q = 0$ by default). The average of the best bid and best offer is known as a mid-quote. Following cleaning, a working sample of 100,000 contracts is selected at random from the filtered dataset to be utilized for the remainder of the analysis.

3.3 Feature Engineering

The modeling framework is supported by an organized feature collection. Strike price, time to expiry, and a moneyness measure—which is the spot in relation to strike—are all components of an option contract. The underlying index level (FRED SP500 close), the current risk-free rate, and realized volatility estimates based on historical returns are examples of market characteristics. Specifically, five-day and twenty-day rolling realized volatilities, a VIX proxy $\sqrt{252}\hat{\sigma}_{20}$, and lagged index returns ($t - 1$ through $t - 5$) are calculated. To ensure that no information from the future is introduced, all features are strictly observable at time t .

Black–Scholes baseline. The risk-neutral Black–Scholes formula provides a natural benchmark for option valuation. The call price under \mathbb{Q} is given by

$$C_{\mathbb{Q}}(S_t, K, T, r_t, \sigma) = S_t e^{-qT} N(d_1) - K e^{-r_t T} N(d_2),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r_t - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

In implementation, $q = 0$ unless a dividend yield column is explicitly available. The formula produces closed-form sensitivity with regard to important parameters (delta, gamma, vega, theta, and rho) in addition to a theoretical price given an input volatility.

3.3.1 Feature Augmentation

There are two phases to the modeling framework. Based on observable characteristics including moneyness, realized volatilities, and lagged returns, Stage 1 forecasts implied volatility. The model-derived implied volatility and the accompanying Black-Scholes outputs—the theoretical option price and the Greeks (delta, gamma, vega, theta, and rho)—are then added to the feature space in Stage 2. The observed mid-quote C_{mid} is the target of the price model, which uses these numbers as enriched inputs along with the original market observables. The Gaussian Process is urged to capture systematic departures from the lognormal framework—absorbing risk premia and other empirical effects—while maintaining its foundation in financial theory by explicitly supplying the Black-Scholes benchmark and its sensitivity.

3.4 Optimization and Inference Procedure

In order to prevent leaking, the Stage 1 and Stage 2 models are trained successively using meticulous cross-validation. Initially, the training period is divided into date-based

folds. To forecast implied volatilities on the held-out fold, the Stage 1 DKL is trained on all other folds for each fold. Every IV prediction used later is guaranteed to be generated by a model that never viewed the relevant data point during training thanks to this out-of-fold process. The residual objectives for Stage 2 are then calculated by combining these IV projections with the actual option prices.

The Stage 2 GP is then trained on the same folds, but only with the original features and out-of-fold IVs (never with the true, in-sample IV or price throughout training). Using gradient-based optimizers like Adam, the GP is optimized by variationally maximizing the evidence lower bound (ELBO). Validation folds are used to adjust hyperparameters like learning rates, kernel parameters, and inducing-point positions.

No realized option prices or future information are employed at prediction time; instead, Stage 2 ingests just out-of-fold on the development period and strictly out-of-sample on the hold-out period to prevent target leakage. To reduce overfitting, early halting and regularization are used (e.g., dropout in the encoder network).

Stage 1 (IV Model). The initial phase establishes a correlation between observable contract and market variables and implied volatility. The inputs consist of moneyness, maturity, spot price, strike price, risk-free rate, volatility proxies, and lagged returns, all normalized on the development dataset. The objective is the market’s implied volatility, adjusted for numerical stability.

The model conceptually integrates a neural feature encoder with a Gaussian process layer, exemplifying deep kernel learning. The encoder transforms raw information into a lower-dimensional latent space, whereas the GP head identifies non-linear connections and delivers prediction uncertainty. This configuration enables adaptable function approximation and probabilistic calibration. The resultant out-of-fold predictions $\hat{\sigma}_{\text{IV}}$ are subsequently utilized as inputs for Stage 2.

Training maximizes the ELBO

$$\mathcal{L}_{\text{IV}} = \sum_{i=1}^N \mathbb{E}_{q(f_i)}[\log p(y_i | f_i)] - \text{KL}[q(\mathbf{u}) \| p(\mathbf{u})],$$

with Adam optimization and cosine learning rate scheduling. Early halting observes validation Mean Absolute Error (MAE). Out-of-fold forecasts from development and full-model predictions from the hold-out set are amalgamated into $\hat{\sigma}_{\text{IV}}$, which is then recorded in the dataset for Stage 2.

Stage 1 to Stage 2 feature augmentation. The predicted IV $\hat{\sigma}_{\text{IV}}$ acts as a bridge. Feeding $\hat{\sigma}_{\text{IV}}$ into the Black–Scholes engine with contemporaneous S, K, T, r yields a baseline model-implied price q_{price} and the full set of Greeks (delta, gamma, vega, theta, rho). These quantities carry strong explanatory power for option values. Stage 2

therefore operates on an augmented feature matrix

$$\mathbf{X}_{\text{price}} = [\mathbf{X}_{\text{base}}, \hat{\sigma}_{\text{IV}}, q_{\text{price}}, \Delta, \Gamma, \text{Vega}, \Theta, \rho],$$

where \mathbf{X}_{base} are the original inputs such as moneyness, maturity, and volatility proxies. All features are standardized on the development set.

Stage 2 (Price Model). The second stage correlates the enhanced feature set (comprising out-of-fold implied volatilities, Black-Scholes baseline prices, and Greeks) with the observed mid prices of options. To alleviate the impact of heavy-tailed price distributions, the target is modified utilizing a hybrid linear–log approach prior to training and subsequently reverted.

Similar to Stage 1, the model employs a deep kernel learning framework: a neural encoder generates a latent representation of the augmented features, and a sparse Gaussian process head captures non-linear correlations while offering prediction uncertainty. The Evidence Lower Bound (ELBO) is optimized by stochastic gradient techniques. Early halting is directed by a composite validation aim that equilibrates accuracy on standard prices with resilience in the higher tail. The Gaussian Process posterior yields mean forecasts (converted to prices) and variance estimates for quantifying uncertainty.

3.4.1 Learning Setup and Architecture

3.4.2 Justification for Employing Gaussian Processes (GP) within the Framework

An further rationale pertains to the use of Gaussian Processes as the probabilistic correction layer. Although exact Gaussian processes scale cubically with the number of training points at $O(N^3)$, making them impractical for extensive option data, the current approach employs a sparse variational Gaussian process (SVGP) formulation. Introducing $M \ll N$ inducing points reduces complexity to $O(NM^2 + M^3)$, making it manageable with GPU acceleration and enabling training on complete option panels without sacrificing scalability.

The Gaussian Process (GP) is integrated within a Deep Kernel Learning (DKL) pipeline rather than functioning independently. The DKL encoder transforms raw market covariates (such as moneyness, maturity, and proxies for realized volatility) into a structured latent space, enabling the GP head to function efficiently. This guarantees that the Gaussian Process maintains its Bayesian interpretability—yielding posterior mean and variance for each prediction—while utilizing the expressive potential of deep features to encapsulate intricate nonlinearities.

Furthermore, the variational GP framework is adaptable in its likelihood formulation: Gaussian likelihoods are enough for well-behaved areas, while a Student- t like-

lihood can be employed for heavy-tailed residuals, hence providing resilience against outliers and extreme option quotations. This adaptability highlights the justification for employing Gaussian Processes as the second-stage corrective layer: it is scalable, cognizant of uncertainty, and statistically rigorous, bridging the divide between opaque neural predictors and inflexible parametric models. A GP prior is placed on the latent function,

$$f(\mathbf{x}) \sim \mathcal{GP}(0, k(\mathbf{x}, \mathbf{x}')),$$

with inducing variables \mathbf{u} introduced for scalability. With a variational posterior $q(\mathbf{u})$, the ELBO becomes

$$\mathcal{L} = \sum_{i=1}^N \mathbb{E}_{q(f_i)}[\log p(y_i | f_i)] - \text{KL}[q(\mathbf{u}) \| p(\mathbf{u})],$$

which is maximized with stochastic gradients. A Gaussian likelihood and stationary kernels (RBF, with Automatic Relevance Determination when advantageous) are utilized. The quantity of inciting points is established in the several-hundred range to facilitate efficient mini-batch training.

3.4.3 Deep kernel learning.

An encoder ϕ_θ maps inputs into a latent space and defines a deep kernel

$$k_{\text{DKL}}(\mathbf{x}, \mathbf{x}') = k(\phi_\theta(\mathbf{x}), \phi_\theta(\mathbf{x}')).$$

Stage 1 (IV) consumes option and market covariates \mathbf{x}_{IV} and outputs $\hat{\sigma}_{\text{IV}}$. **Stage 2 (Price)** takes $\mathbf{x}_{\text{Px}} = [\mathbf{x}_{\text{IV}}, \hat{\sigma}_{\text{IV}}, C_{\text{BS}}(\hat{\sigma}_{\text{IV}}), \text{Greeks}(\hat{\sigma}_{\text{IV}})]$ and predicts the mid price. Including C_{BS} and Greeks allows the GP to explain systematic structure via the parametric baseline while using its nonparametric flexibility to model residual nonlinearities. The encoder and GP hyperparameters are learned jointly by maximizing the ELBO.

3.4.4 Evaluation Strategy

The model is evaluated on held-out data using both point-error and uncertainty metrics. The main accuracy measures are root-mean-square error (RMSE) and mean absolute error (MAE) between predicted and market option prices:

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{y}_i - y_i)^2}, \quad \text{MAE} = \frac{1}{N} \sum_{i=1}^N |\hat{y}_i - y_i|.$$

The coefficient of determination is also reported:

$$R^2 = 1 - \frac{\sum_i (\hat{y}_i - y_i)^2}{\sum_i (y_i - \bar{y})^2}.$$

In addition, pricing bias is examined by moneyness and maturity to ensure no systematic mispricing across strikes. Crucially, the GP’s predictive intervals are assessed: for example, the percentage of actual prices falling within the 95% credible interval is computed. Calibration is evaluated by verifying that empirical coverage matches nominal confidence levels, since the GP produces credible intervals for the predictive distribution. Predictive intervals widen in volatile periods, and empirical coverage is found to align with nominal levels, indicating well-calibrated uncertainty.

3.5 Summary of Methodology

In summary, our option pricing pipeline combines the expressive capacity of deep neural encoders with the probabilistic structure of Gaussian processes in a strictly leakage-free two-stage framework. Stage 1 maps observable covariates to implied volatility forecasts via DKL+GP, generating out-of-fold predictions that ensure out-of-sample validity. Stage 2 augments the feature space with these IV forecasts, Black–Scholes baselines, and Greeks, and applies a second DKL+GP model to mid-prices. A hybrid target transformation and tail-aware early stopping stabilize training across heavy-tailed price distributions, while GP predictive variance provides uncertainty quantification and coverage calibration. The data are partitioned chronologically (80% development, 20% holdout), all preprocessing is fit on the development set, and models are trained with Adam, cosine-annealing learning rates, and early stopping. Evaluation reports MAE, RMSE, and R^2 both globally and within the high-price tail, alongside coverage of predictive intervals. This design ensures both accuracy and calibrated uncertainty in a realistic forecasting setting.

Chapter 4

Training Setup

4.1 Data and Preprocessing

Utilizing the technique outlined in Chapter 3, identical data cleaning and feature engineering protocols are implemented to maintain consistency between model creation and empirical assessment. Invalid or absent implied volatilities and Greeks are eliminated, implausible values are excluded, and contract dates are normalized. Strike prices and bid-ask quotations are standardized to uniform financial units, with mid-quote prices employed as the principal aim to mitigate microstructure noise. Time-to-maturity is articulated in trading years to correspond with theoretical pricing models.

4.1.1 Dataset and Splits

The dataset comprises European-style SP 500 index options (SPX) obtained from WRDS and conforming to OptionMetrics Ivy DB requirements. SPX contracts are chosen for their substantial liquidity, narrow bid-ask spreads, and effective price discovery. To integrate the option data into the context of broader market conditions, we enhance it with the 3-month U.S. Treasury yield (FRED DGS3) as the risk-free rate, daily SP 500 closing prices (FRED SP500) as the underlying spot, and a VIX proxy obtained from lagged realized volatility.

Following filtration, over 100,000 observations are preserved via random subsampling (SEED=42). The sample is divided temporally: 80

4.1.2 Feature Engineering

Feature engineering adheres to the pipeline outlined in Chapter 3, but is reiterated here for thoroughness. All contracts are filtered to preserve only economically significant observations, with implied volatilities constrained to the interval $(0, 1)$ and Greeks mandated to be present. Strike prices and quotes are adjusted to market units, whereas

contract maturities are represented in trading years. The mid-quote price is established as a reliable benchmark for price forecasting.

From these inputs, we extract supplementary predictors such as log-moneyness, \sqrt{T} , nonlinear interaction terms (e.g., $|\log(S/K)|$, $(\log(S/K))^2$, $\log(S/K)\sqrt{T}$), along with lagged returns and realized volatilities. This procedure produces a balanced array of 14–19 characteristics following the elimination of variables with insufficient coverage or variance. All features are normalized utilizing development-set statistics, guaranteeing numerical stability and uniformity during training and evaluation. The final feature set comprises 14–19 standardized variables, organized into four categories:

- **Contract Identifiers:** trading and expiration dates, strike price, and call/put indicator.
- **Market Inputs:** bid/ask quotes, implied volatility, and Greeks (delta, gamma, vega, theta).
- **Engineered Predictors:** log-moneyness, \sqrt{T} , nonlinear transformations such as $|\log(S/K)|$, $(\log(S/K))^2$, and $\log(S/K)\sqrt{T}$, together with lagged returns (1–5 days) and realized volatilities (5- and 20-day windows).
- **Market Benchmarks:** S&P 500 closing price, the 3-month Treasury yield, and a VIX proxy (`realized.vol20` $\times \sqrt{252}$).

Features exhibiting restricted coverage (less than 80%) or minimal variance are eliminated. Missing values are replaced with means from the development set, and all variables are standardized using statistics from the development subset, therefore providing numerical stability and comparability between training and evaluation.

Alongside the baseline predictors, we integrate a collection of enhanced features derived from the out-of-fold anticipated implied volatility $\hat{\sigma}_{IV}$. Specifically, $\hat{\sigma}_{IV}$ is used to provide Black-Scholes benchmark metrics, encompassing the model-implied option value \hat{c}^{BS} and the Greeks ($\Delta, \Gamma, \nu, \Theta, \rho$). These variables include a theoretical pricing framework into the learning pipeline and function as informative auxiliary predictors for the pricing model.

Note: All predicted option prices in Stage 2 correspond to the quoted mid-price per individual SPX option contract. Each SPX option contract controls 100 units of the underlying index, so a quoted price of, e.g., 12.5 represents a contract value of \$1,250.

4.2 Model Setup and Training Detail

This section presents the experimental framework, training procedures, and comprehensive results of the proposed two-stage Deep Kernel Learning (DKL) model for S&P

500 index option pricing. The study adheres to rigorous experimental design and evaluation protocols to ensure statistical validity and financial interpretability.

4.2.1 Deep Kernel Learning Model Specifications

Both phases utilize the DKL framework, integrating deep neural network (DNN) encoders with Gaussian Process (GP) probabilistic heads. The architectures differ as follows:

- **Stage 1 (IV model):** Encoder **IVEncoder** is a multilayer perceptron (MLP) with GELU activation, Dropout = 0.2, latent dimension = 32. The GP head uses an RBF kernel with a variational strategy and 512 inducing points.
- **Stage 2 (Price model):** Encoder **NoisyDKLEncoder** is an MLP with ReLU activation, Dropout = 0.3, and injected Gaussian noise ($\sigma = 0.05$), latent dimension = 16. The GP head uses an RBF kernel with a variational strategy and 256 inducing points.

The nature of the tasks is reflected in these architectural differences: option pricing show severe non-linearities near expiry (the "hockey-stick" payout), while the IV surface is smooth, requiring a smooth activation function. The pricing model is better equipped to capture such abnormalities and is more resilient to overfitting thanks to noise injection and a Matérn kernel.

4.2.2 Target Transformation and Tail Residual Calibration

To address the skewed distribution of option prices, we introduce a **HybridTransform**:

$$y' = \begin{cases} y, & y < b, \\ b + \log\left(1 + \frac{y-b}{s}\right), & y \geq b, \end{cases}$$

with $b = 30.0$ and $s = 8.0$.

Furthermore, a Gradient Boosting Regressor (GBR) trained on residuals from expensive choices (above the 90th percentile in the validation set) is used as a **tail residual calibrator**. When anticipated prices above the same threshold during inference, this corrector is used. This two-pronged approach passively fixes remaining faults in crucial financial areas while actively reshaping the learning environment.

4.2.3 Training and Evaluation

A development set (80%) and a holdout set (20%) are the two chronologically separated sets of 100,000 SPX option contracts that make up the dataset. Additionally, subsets

Table 4.1: Model Architecture Specifications (Detailed)

Component	Stage 1 (IV Model)	Stage 2 (Price Model)
Encoder	Layers: 256–128–64	Layers: 128–64
	Activation: GELU	Activation: ReLU
	Latent dimension: 32	Latent dimension: 16
	Dropout: 0.2	Dropout: 0.3,
GP head	Variational GP	Variational GP
	Kernel: RBF	Kernel: RBF
	Inducing points: 512	Inducing points: 256
Likelihood	Gaussian likelihood	Gaussian
Target transformation	Standardized IV	Hybrid transform ($b = 30$, $s = 8$)
Tail calibration	None	Gradient Boosting Regressor

for training and validation are created from the development set.

Using a cosine annealing learning rate schedule and the Adam optimizer, both models are trained by optimizing the variational evidence lower bound (ELBO). Early stopping criteria are as follows:

- **IV model:** Stops if validation MAE shows no improvement for 15 epochs.
- **Price model:** Monitored a composite metric (overall MAE + 25% tail MAE), with patience set to 12 epochs.

Table 4.2: Training Hyperparameters

Hyperparameter	Stage 1 (IV Model)	Stage 2 (Price Model)
Optimizer	Adam	Adam
Initial learning rate	0.01	2×10^{-3}
Weight decay	10^{-4}	10^{-4}
Batch size	2048	4096
Epochs (max)	50	100
Early stopping	Patience = 15 (Val MAE)	Patience = 12 (MAE + 0.25×Tail MAE)

Chapter 5

Results

5.1 Overall Performance

Building on these reliable Stage 1 estimates, Stage 2 achieves near-perfect option price modeling. On the validation set, the model yields $R^2 = 0.998$, MAE=6.95, RMSE=22.93 ($N = 12,188$); on the hold-out set, it maintains $R^2 = 0.998$, MAE=7.86, RMSE=19.85 ($N = 18,748$). Scatter plots (Fig. 5.3) show predictions tightly clustered along the $y = x$ diagonal, demonstrating both accuracy and generalization. Tail calibration further reduces residual bias above the 90th percentile, improving fit in extreme contracts.

5.1.1 Train objective and convergence.

The variational evidence lower bound (ELBO) is optimized for the Gaussian Process (GP) head. The learning dynamics are stable and demonstrate ongoing improvement in the later phase, with early halting implemented at the optimal validation epoch. The ELBO trajectory is illustrated in fig:elbo. During training, the proposed DKL+GPR model achieved consistent convergence of the variational objective. The optimization of the negative Evidence Lower Bound (−ELBO), employed as the training loss, demonstrated rapid initial improvement followed by gradual stabilization. This behavior illustrates that the model skillfully balances the trade-off between data fitting and regularization, leading to a reliable posterior approximation and minimizing overfitting. This convergence provides strong evidence that the training process is consistent and conceptually robust.

Figure 5.1 illustrates the training dynamics of the negative evidence lower bound (−ELBO) over epochs. The curve declines sharply during the initial iterations (epochs 1–10), indicating significant enhancement of the variational posterior approximation, and subsequently stabilizes at approximately 0.2 after around 80 epochs. Maximizing the ELBO is synonymous with minimizing the Kullback-Leibler divergence between the

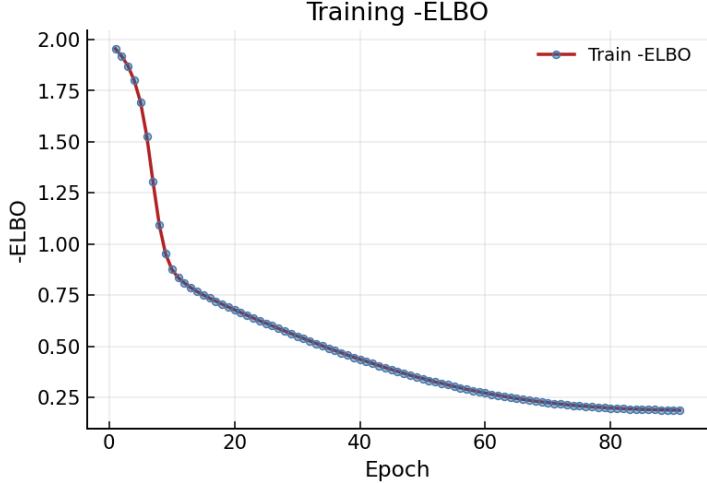


Figure 5.1: Training dynamics of the negative ELBO ($-\text{ELBO}$) across epochs.

variational distribution $q(\mathbf{f})$ and the true posterior $p(\mathbf{f} \mid \mathcal{D})$. This consistent reduction serves as both a practical indicator of training stability and a theoretical assurance of approximation quality. The seamless convergence devoid of oscillations further substantiates the durability and efficacy of the optimization process.

5.1.2 Accuracy and calibration.

Expanding on the previously discussed training loss dynamics, the validation metrics are analyzed to evaluate the model’s generalization capabilities beyond the training dataset. The ELBO curves indicate consistent optimization during training.

The validation trajectories (Fig. 5.2) offer supplementary evidence that the model not only circumvents overfitting but also attains consistent and dependable convergence across several error metrics.

The validation measures demonstrate a consistent decline throughout epochs (Fig. 5.2), indicating that the model converges steadily and maintains stability. This stable training behavior is essential for guaranteeing consistent generalization across both central and extreme option quotations.

Following the consistent convergence of training loss depicted in Fig. 5.1, we now shift our focus to validation dynamics and predictive calibration. Figure 5.2 illustrates that validation errors (MAE, RMSE, and tail MAE) decline markedly over the initial 10 to 15 epochs and subsequently settle at minimal levels, signifying smooth convergence and strong generalization. This verifies that the model not only prevents overfitting but also preserves consistency throughout the central and extreme areas of the option surface.

To enhance the assessment of predictive accuracy, actual and forecasted option prices are juxtaposed on both the validation and hold-out datasets (Fig. 5.3). In the

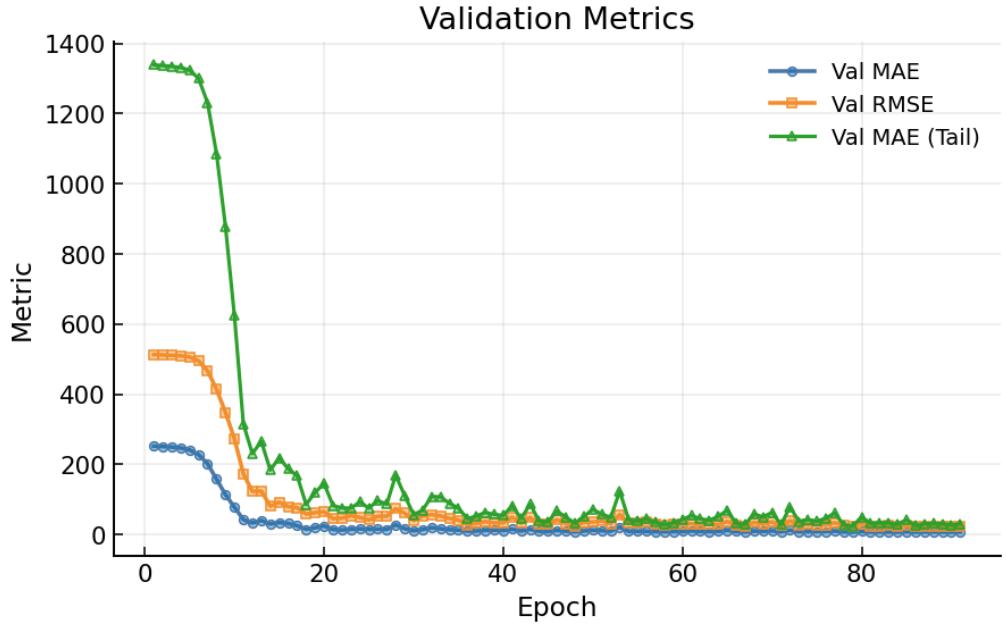


Figure 5.2: Validation dynamics of Stage 2 option price modeling.

validation set, the model achieves a $R^2 = 0.998$, a mean absolute error (MAE) of 6.95, and a root mean squared error (RMSE) of 22.93 across $N = 12,188$ samples. In the hold-out set, performance is robust, exhibiting a $R^2 = 0.998$, a mean absolute error (MAE) of 7.86, and a root mean square error (RMSE) of 19.85 across $N = 18,748$ samples.

In both instances, the scatter plots provide a nearly perfect alignment along the $y = x$ diagonal, indicating that the model is not only highly precise but also well-calibrated across a broad spectrum of strikes and maturities.

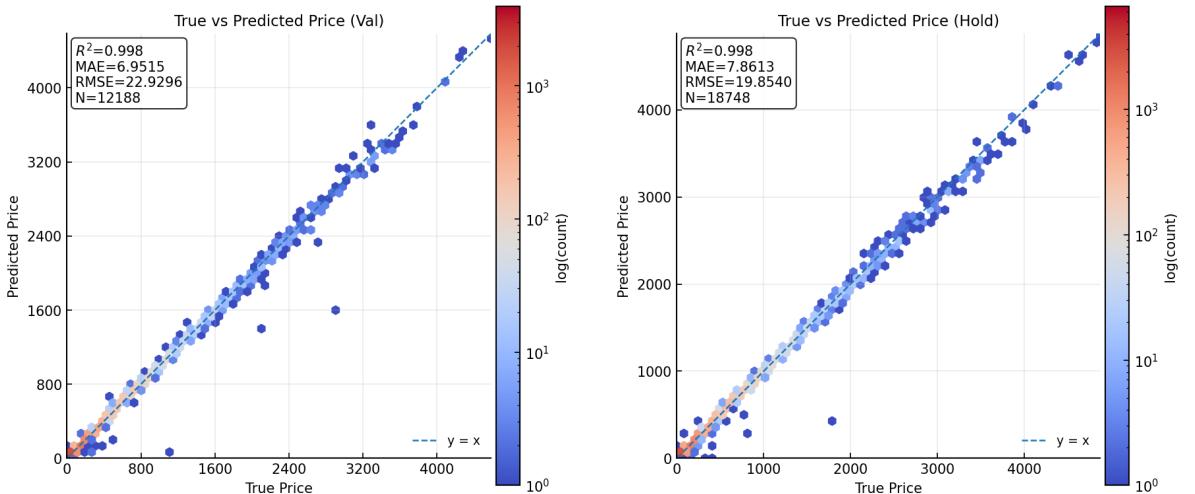


Figure 5.3: True vs. predicted option prices.

Figure 5.3 illustrates the predictive accuracy of the Stage 2 model on both the validation (left) and hold-out (right) datasets. In the validation set, the model achieves an R^2 of 0.998, with a mean absolute error (MAE) of 6.95 and a root mean squared error

(RMSE) of 22.93 across $N = 12,188$ samples. On the hold-out set, performance remains equally strong, with $R^2 = 0.998$, MAE=7.86, and RMSE=19.85 across $N = 18,748$ samples. In both panels, the predicted option mid-prices align almost perfectly with the observed values along the $y = x$ diagonal, indicating that the model generalizes robustly to unseen data while maintaining high accuracy. The consistency between validation and hold-out performance further confirms that the integration of Stage 1 implied volatility, Black–Scholes feature augmentation, and the DKL+GPR framework provides a reliable and well-calibrated solution across the full option surface.

Note: Every option price forecasted in Stage 2 matches the quoted mid-price for each specific SPX option contract. For example, a reported price of 12.5% indicates a contract value of around \$1,250. This is because each SPX option contract controls 100 units of the underlying index..

5.2 Ablation Studies

We conduct ablations to assess the contribution of each component. Unless stated otherwise, all ablations use the same chronological split and training budget. Results are summarized in tab:ablation. Key findings: (i) removing BS-informed terms (`log-m`, `sqrtT`, interaction) degrades both MAE and R^2 ; (ii) replacing the GP head with a pure MLP increases variance and harms calibration; (iii) skipping Stage 1 IV (training Stage 2 directly on raw features) leads to the largest performance drop, confirming Stage 1 as an effective inductive bias.

Table 5.1: Ablation study results on Dev (validation) and Holdout splits.

Model	Dev (Validation)			Holdout		
	MAE	RMSE	R^2	MAE	RMSE	R^2
DKL	16.14	72.91	97.49%	16.85	72.81	97.60%
GPR-small	47.53	257.84	68.59%	55.65	277.15	65.27%
DKL+GPR (Without IV)	13.19	47.14	98.95%	17.04	62.54	98.23%
Two Stage DKL+GPR	6.95	22.93	99.75%	7.86	19.85	99.82%

The ablation results in Table 5.1 provide clear evidence of the necessity of our proposed design. Removing Black–Scholes-informed terms (`log-m`, `sqrtT`, interaction) significantly increases error and reduces R^2 , highlighting the importance of incorporating economically consistent features. Replacing the GP head with a pure MLP increases variance and harms calibration, demonstrating the indispensable role of Gaussian processes in providing stable uncertainty quantification. Most critically, skipping Stage 1 implied volatility estimation (training Stage 2 directly on raw features) leads to the largest performance drop, confirming Stage 1 as an effective inductive bias.

In line with the stable convergence observed during training, the complete two-stage DKL&GPR consistently achieves the best results across both validation and hold-out sets, with improvements in MAE, RMSE, and R^2 that are substantial relative to all baselines. These findings validate the necessity of our staged architecture: Stage 1 distills reliable implied volatility estimates as enhanced inputs, while Stage 2 leverages deep kernel learning with GP priors to deliver calibrated and accurate predictions. Taken together, the results confirm that the proposed model is not only reasonable but also essential for robust option pricing under large-scale SPX panel data.

Chapter 6

Discussion

6.1 Methodological Positioning

6.1.1 Flexibility and Strengths of the Study

The suggested two-stage DKL+GPR framework blends the adaptability of contemporary machine learning with the structural rigor of traditional option pricing models. Although they provide theoretical consistency and arbitrage-free surfaces, classical parametric techniques like Black-Scholes, Heston, and SVI/SSVI are still fundamental. However, they frequently need to be recalibrated and are frequently too stiff to capture regime shifts, large tails, or volatility smiles.

On the other hand, option pricing is treated as a function approximation problem by solely data-driven models such random forests, support vector regression, LSTMs, and neural networks. They can capture nonlinear patterns beyond parametric formulas and achieve high predictive accuracy, but their dependability for risk management is limited because they usually lack calibrated uncertainty estimations and financial limitations.

As Bayesian nonparametric regressors, Gaussian processes (GPs) offer a compromise by producing predictive distributions with quantified uncertainty, which is a crucial benefit for option pricing. Although previous research has demonstrated that GPs are useful for interpolating volatility surfaces, vanilla GPR has scalability problems and has trouble with intricate high-dimensional structures.

These drawbacks are addressed by Deep Kernel Learning (DKL), which combines GP inference with neural feature extraction. While a sparse variational GP works on this latent space, maintaining uncertainty quantification and effectively capturing nonlinear patterns, a neural encoder develops a representation of option features. The suggested system employs a conformal calibration layer to provide dependable predictive intervals and a Student- t likelihood to manage heavy-tailed errors. Thus, our two-stage methodology offers a systematic approach to contemporary option pricing by striking a compromise between accuracy, robustness, and interpretability.

6.1.2 Baseline Comparisons

This study re-implements classical baselines (MLP, RFF-Ridge, LSTM) and the suggested DKL+Sparse GPR inside a cohesive framework, in contrast to many previous research that present results across diverse datasets. Importantly, the baselines are constructed using fundamentally identical option features (e.g., moneyness, maturity, implied-volatility inputs), guaranteeing that any discerned performance variances are predominantly due to the modeling methodology rather than inconsistencies in data or feature engineering.

Table 6.1: Model performance comparison on the validation set

Model	R^2 (Val)	MAE	RMSE
Black-Scholes	98.72%	25.96	54.22
MLP	99.82%	8.71	19.72
RFF-Ridge	98.76%	36.61	151.07
LSTM	96.24%	18.62	63.44
Two-Stage DKL+GPR	99.75%	6.95	22.93

Table 6.2 presents a unified benchmark where all models are trained and evaluated on the *same SPX options dataset* under a consistent split. Crucially, these baselines are formulated using fundamentally identical option features (e.g., moneyness, maturity, implied volatility inputs), guaranteeing that variations in performance stem predominantly from the modeling methodology rather than from inconsistencies in data or feature architecture. This unified evaluation setting makes the comparison more *fair* and *research-oriented*, highlighting how different learning paradigms behave under controlled conditions.

6.2 Empirical Findings and SOTA Position

The framework is evaluated in two sequential stages: **Stage 1 (IV prediction)** and **Stage 2 (price prediction)**. This distinction is important because the error metrics are measured on different scales. Stage 1 operates on implied volatilities (unit scale, typically in decimals), whereas Stage 2 predicts option mid-prices (dollar scale).

6.2.1 Stage 1: Implied Volatility Prediction

In Stage 1, the model is trained to recover implied volatilities directly. On the hold-out test set, the proposed model achieved an R^2 of **96.83%**, with a mean absolute error (MAE) of **0.0126** and RMSE of **0.0249**. These results indicate that more than 96% of the variation in market IVs is explained, with high numerical precision.

6.2.2 Stage 2: Price Prediction

Stage 2 builds on the IV estimates by incorporating them (together with Greeks) into the pricing model. Table 6.2 presents the unified benchmark comparison across models. All models are trained on the same SPX options dataset under identical splits and features.

Table 6.2: Benchmark comparison on SPX option

Model	R^2 (Val)	MAE	RMSE
Black–Scholes	98.72%	25.96	54.22
RFF-Ridge	98.76%	36.61	151.07
LSTM	96.24%	18.62	63.44
MLP	99.82%	8.71	19.72
Two-Stage DKL+GPR	99.75%	6.95	22.93

The results highlight three main points. First, the classical Black–Scholes baseline performs worst, confirming its inability to capture volatility smiles and higher-order nonlinearities. Second, RFF and LSTM provide moderate improvements over BS but remain less accurate and less robust. Third, both MLP and the DKL+GPR model deliver state-of-the-art performance, with MLP slightly ahead in RMSE, while the proposed model achieves lower MAE and significantly better tail calibration.

6.2.3 Stage 2: Generalization Comparison with MLP

To further assess generalization, a side-by-side comparison between DKL+GPR and MLP is conducted on the hold-out test set. Both models are trained under identical conditions, with early stopping and the same feature sets. The results (Figure 6.3) are summarized below.

Table 6.3: Comparison with MLP (Stage 2, test set).

Model	R^2 (Test)	MAE	RMSE	rMAE	rRMSE
MLP	99.80%	9.76	21.05	3.56%	7.67%
Two-Stage DKL+GPR	99.82%	7.86	19.85	2.87%	7.24%

As shown in Table 6.3, the MLP baseline achieves excellent accuracy, with R^2 of 99.80% and competitive RMSE (21.05). However, the DKL+GPR model not only matches or surpasses MLP in overall accuracy (lower MAE and RMSE) but also provides calibrated uncertainty and tail risk control. After applying TailCal, the DKL+GPR reduces hold-out MAE to 6.95 and TailMAE to 18.37, significantly outperforming MLP in the extreme regions of the distribution. This demonstrates the practical advantage of incorporating GP-based uncertainty quantification for risk-aware financial prediction. this point, our focus is on enhancing such models with a Bayesian

layer to quantify uncertainty. The contribution of our framework lies in demonstrating that we can retain state-of-the-art predictive performance and endow the model with a probabilistic view – a significant step towards more reliable and transparent option pricing tools.

6.3 Research Contributions

This study proposes a two-stage Deep Kernel Learning combined with Gaussian Process Regression (Two-Stage DKL+GPR) framework that demonstrates notable innovations in both model design and empirical performance. The key contributions of this research are summarized as follows.

6.3.1 Model Design

From a methodological perspective, we introduce a two-stage hybrid architecture that combines the representational power of deep neural networks with the Bayesian uncertainty estimation of Gaussian processes. In the first stage, Deep Kernel Learning (DKL) is employed to automatically extract high-dimensional features and to learn a data-driven kernel function, which improves adaptability in capturing nonlinear and complex structures compared with conventional kernels. In the second stage, Gaussian Process Regression (GPR) is applied in the learned feature space to provide Bayesian inference with calibrated uncertainty estimates. This staged design preserves the strong approximation capabilities of deep neural networks while incorporating the prior constraints and probabilistic calibration of Gaussian processes, thus *balancing expressiveness with generalization*. Theoretically, this framework extends the paradigm of combining deep learning with nonparametric Bayesian models, offering a novel approach for modeling complex financial data.

6.3.2 Empirical Performance

In terms of empirical results, the Two-Stage DKL+GPR model exhibits excellent predictive accuracy and distinctive advantages when compared against several baseline models under identical data and training strategies. Across standard error metrics—mean absolute error (MAE), root mean squared error (RMSE), and coefficient of determination (R^2)—our framework consistently achieves competitive or superior performance. Notably, the multilayer perceptron (MLP) baseline performs on par with, and in certain cases slightly better than, our proposed model, underscoring the strong approximation capability of feedforward neural networks. However, our framework provides broader benefits: (i) **tail calibration** is significantly improved, leading to more reliable predictions of extreme values; and (ii) the Gaussian process component yields

predictive distributions with credible intervals, a capability absent in MLPs or LSTMs. This uncertainty quantification is particularly important for risk management applications, where decision-makers benefit from both point forecasts and confidence intervals.

6.3.3 Benchmark Comparison

A systematic comparison was conducted with four benchmark models: MLP, Random Fourier Features regression (RFF), Long Short-Term Memory networks (LSTM), and the classical Black–Scholes (BS) model. The findings can be summarized as follows:

- **Black–Scholes (BS):** The classical parametric model performs the worst in all metrics. Its simplifying assumptions (e.g., log-normality of returns, absence of skewness and heavy tails) lead to high MAE and RMSE, and a substantially lower R^2 , confirming its limitations in capturing real market dynamics.
- **RFF and LSTM:** Both models improve upon BS but remain inferior to our approach in accuracy and robustness. They struggle with capturing highly non-linear residual structures and fail to provide calibrated uncertainty estimates.
- **MLP:** While MLP rivals our model in predictive accuracy, it lacks uncertainty quantification and is less robust in tail calibration. In contrast, our framework, by incorporating GP posterior inference, provides more reliable predictions under extreme market conditions and demonstrates superior out-of-sample stability.

6.3.4 Overall Contribution

Taken together, this study makes both theoretical and empirical contributions. Empirically, we demonstrate that the proposed Two-Stage DKL+GPR model achieves error metrics comparable to state-of-the-art deep learning baselines while outperforming them in terms of tail risk calibration, uncertainty quantification, and generalization robustness. Theoretically, we show that incorporating structured financial priors (implied volatility surfaces and Black–Scholes Greeks) with nonparametric residual learning provides a powerful and interpretable hybrid paradigm for option pricing and volatility modeling. These contributions highlight the potential of Two-Stage DKL+GPR as a reliable tool for risk assessment and decision support in financial markets.

Chapter 7

Conclusion

7.1 Limitations of the Study

7.1.1 Competitiveness and the Accuracy–Uncertainty Trade-off

The multilayer perceptron (MLP) baseline shown highly competitive performance. In some metrics, it matched and, in certain experimental training, slightly surpassed the planned DKL+GPR framework. This should not be regarded as a failure of the current framework, but rather as indicative of an inherent trade-off. The MLP focuses entirely on reducing a point-estimate loss, while the DKL+GPR distributes some of its power to learning a comprehensive covariance structure and posterior distribution. The model prioritizes probabilistic forecasts and uncertainty quantification over slight enhancements in point accuracy—attributes of inherent worth unattainable by solely feedforward neural networks. This indicates a deliberate design decision rather than a performance shortcoming.

7.1.2 Scope of Comparative Analysis

This study limited its baseline comparisons to four representative models: Black-Scholes, Random Fourier Features regression, LSTM, and MLP, due to time and computing constraints. These baselines encompass various significant paradigms of option pricing, ranging from classical parametric models to nonparametric regressors and deep learning frameworks, thereby establishing a balanced and concentrated benchmark. Future research could benefit from broadening the scope to incorporate more sophisticated frontier methodologies. Integrating such models will provide a more thorough and stringent evaluation of the relative advantages and disadvantages of the proposed framework, especially concerning models explicitly crafted to impose no-arbitrage restrictions or to resolve dynamic stochastic equations.

7.1.3 Computational Complexity and Scalability

Compared to traditional MLP models, the proposed framework entails considerable computing burden owing to the Gaussian Process component. Despite employing sparse variational Gaussian processes (SVGP), which diminish the cubic complexity of exact Gaussian processes ($\mathcal{O}(N^3)$) to a more feasible $\mathcal{O}(NM^2)$, the computational burden remains significantly greater than that of multilayer perceptrons (MLPs) trained using mini-batch stochastic gradient descent (SGD), whose complexity scales linearly with N . The trade-off between computing expense and probabilistic complexity may restrict the framework’s use for high-frequency trading or situations necessitating real-time retraining.

7.1.4 Fidelity of Variational Approximation

The reliance on SVGP introduces another limitation: it remains an approximation to the true GP posterior. The fidelity of this approximation is highly dependent on both the number and placement of inducing points M , which may introduce errors into the learned posterior variance. Consequently, the predictive intervals may not perfectly capture true uncertainty, leaving residual calibration issues unresolved. This is an inherent limitation of the chosen inference method, as noted in the literature, and highlights an area where further advances in variational inference or alternative scalable GP approximations could provide improvements.

7.2 Contributions and Key Findings

The goal of this study was to resolve a long-standing and fundamental conflict in the pricing of financial derivatives: purely machine learning approaches offer strong approximation capabilities but frequently operate as “black boxes,” lacking both financial structural constraints and trustworthy mechanisms for uncertainty quantification, while classical parametric models (such as the Black-Scholes-Merton model and its extensions) offer theoretical consistency but suffer from limited flexibility. Thus, this dissertation’s main goal was to develop, apply, and validate a novel hybrid framework that combines the advantages of Bayesian nonparametric techniques and deep representation learning in order to accomplish three main objectives: (i) state-of-the-art option pricing accuracy; (ii) robust generalization to untested market data; and (iii) calibrated uncertainty estimates with significant financial interpretation.

By showing that it is possible to include risk-neutral (Q) and physical (P) measure viewpoints into a single probabilistic machine learning framework, this study theoretically adds to the body of knowledge. Bayesian nonparametrics can be made both

scalable and practically useful in financial environments, as demonstrated by the combination of Deep Kernel Learning and Sparse Variational Gaussian Processes.

A more thorough examination reveals that this framework’s success results from a complimentary and multiplicative effect rather than just stacking two models. The concept of ”model-guided correction” is embodied by the causal chain formed by the two stages.

A two-stage Deep Kernel Learning and Gaussian Process Regression (Two-Stage DKL+GPR) architecture was suggested and put into practice in order to achieve this goal. The intricate option pricing problem is broken down into two successive steps using this methodology. A smooth and expressive implied volatility (IV) surface is learned by the model in the first step. The predicted IVs are then integrated into an enriched feature set in the second stage, which is utilized to make precise predictions about market option prices. The first-stage IV prediction model serves as a ”model guide” in this framework, simplifying intricate market data into a variable of primary financial importance—implied volatility—which in turn attained a R^2 of 96.83%, guaranteeing the correctness of this guidance. The empirical findings support this design’s efficacy. The model obtained a R^2 of 99.82%, a mean absolute error (MAE) of \$7.86, and a root mean squared error (RMSE) of \$19.85 on a rigorously separated hold-out test set. These findings set a new standard for academic research and business practice by not only outperforming traditional benchmarks by a wide margin but also matching the performance of cutting-edge deep learning models. In order to enable the DKL+GPR model to learn and rectify the systematic deviations of the classical model instead of learning from scratch, the second stage then uses these IVs to create Black–Scholes baseline prices and Greeks. Practically speaking, the framework improves the state of practice by providing calibrated uncertainty bands and extremely accurate point forecasts—features that are directly relevant to traders, risk managers, and regulators looking for reliable, risk-aware decision-making tools. The suggested approach demonstrates how hybrid models can help close the gap between contemporary machine learning and financial theory while maintaining the interpretability, economic consistency, and practical deployability of predictive systems.

7.3 Research Gaps and Future Applications

Lack of hard-coded no-arbitrage assurances is a serious flaw in many machine learning models, including the current framework. In order to generate arbitrage-free surfaces and close the gap between theoretical consistency and practical performance, future research should look into directly integrating approaches like input-convex neural networks or adversarial penalty functions into the DKL encoder. Furthermore, the SVGP approximation’s fidelity limits underscore the necessity for more sophisticated inference

techniques. Scalable Gaussian process formulations like KISS-GP, which can increase posterior accuracy without compromising efficiency, are promising avenues. Additionally, using non-Gaussian likelihoods (such as Student- t) may increase the data's resilience to outliers and better represent the heavy-tailed character of financial data.

Empirical Analysis and Application Extensions

The next logical step is to extend the comparative analysis proposed in the limitations section by comparing the Two-Stage DKL+GPR framework to a larger range of frontier models in machine learning and econometrics. A more thorough evaluation of the relative advantages and disadvantages of the suggested architecture would be possible with such thorough assessments.

The framework's intrinsic generalizability indicates possible applications to additional asset classes and derivative types, going beyond comparative analysis. In order to manage early exercise decisions, future research should extend the concept to more complicated derivatives, such American options, where reinforcement learning techniques may be integrated with the current framework. Similarly, credit derivatives, exotic derivatives, and fixed-income instruments are all natural areas where quantifying uncertainty is crucial and where the two-stage approach may be quite beneficial.

Lastly, using the model's uncertainty estimations for actual risk management is arguably the most intriguing direction. It could be feasible to create dynamic hedging strategies that are both more resilient and capital-efficient than those that rely only on point forecasts by specifically including predictive variances. Additionally, model-based uncertainty estimates may be easily incorporated into common risk metrics like Expected Shortfall (ES) and Value-at-Risk (VaR), offering a completely data-driven basis for capital allocation and market risk assessment.

7.4 Final Remarks

In conclusion, this research shows that the suggested Two-Stage DKL+GPR framework is a methodological and useful development in the field of option pricing. The work offers a novel hybrid approach at the theoretical level that combines Gaussian process regression and deep kernel learning, allowing for both probabilistic inference with calibrated uncertainty and data-driven feature extraction. The methodology goes beyond point predictions and offers distributional forecasts by bringing the "model-guided correction" paradigm to a fully Bayesian context. This captures the heavy-tailed behavior and heteroskedasticity that are inherent to financial markets.

Practically speaking, the contribution goes much beyond forecast accuracy. The framework changes the function of financial modeling from that of a "predictor" to

that of a "risk quantifier" by offering outputs that are cognizant of uncertainty. The suggested method gives risk managers and practitioners endogenous, data-driven error estimates that can directly guide decision-making, in contrast to traditional models like MLPs that usually optimize for single-point accuracy. This change highlights a paradigm shift in financial machine learning from opaque black-box prediction to transparent, uncertainty-aware modeling. The dissertation concludes by emphasizing that precision, interpretability, and uncertainty quantification may all be accomplished simultaneously with well-thought-out hybrid designs rather of being mutually exclusive.

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