Final Exam

Linear Algebra

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Instructions

1. All solutions must be presented with rigor and clarity. Justify your answers thoroughly.
2. When referencing any results from the course, provide proper citations or references.
3. Maintain a concise and legible writing style throughout your answers.
4. Best of luck!

Exercise 1

Let V be a K-vector space of dimension n, and let $C = \{v_1, \dots, v_r\} \subset V$. Prove that:

(i) There exists a basis B of V such that $B \subseteq C$ if and only if C is a generating system of V.

(ii) There exists a basis B of V such that $B \supseteq C$ if and only if C is linearly independent.

Exercise 2

Let K be a field, and let $B = \{v_1, \dots, v_n\}$ be a basis of K^n, where $v_i = (v_{i,1}, \dots, v_{i,n})$ for $i = 1, \dots, n$. Let $\Delta = \det((v_{i,j})_{1 \le i,j \le n})$. For $j = 1, \dots, n$, the map $\varphi_j : K^n \to K$ is defined as follows:

$$\varphi_j(x_1, \dots, x_n) = \Delta^{-1} \cdot \det \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{j-1,1} & v_{j-1,2} & \cdots & v_{j-1,n} \\ x_1 & x_2 & \cdots & x_n \\ v_{j+1,1} & v_{j+1,2} & \cdots & v_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,n} \end{pmatrix}$$

i) Verify that $\varphi_j \in (K^n)^*$ for all $1 \le j \le n$.

ii) Prove that $B^* = \{\varphi_1, \dots, \varphi_n\} \subset (K^n)^*$ is the dual basis of B.

iii) Let $S = \langle v_1 - v_n, \dots, v_{n-1} - v_n \rangle$. Find a basis for S^o.