Final Exam

Linear Algebra

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Instructions

1. All solutions must be presented with rigor and clarity. Justify your answers thoroughly.
2. When referencing any results from the course, provide proper citations or references.
3. Maintain a concise and legible writing style throughout your answers.
4. Best of luck!

Question 1

Consider three complex vector spaces U, V, and W, each of finite dimension, and equipped with inner products. Let us examine two linear functions:

$$U \xrightarrow{f} V \xrightarrow{g} W$$

Prove that if the image of f ($\mathrm{Im}(f)$) is equal to the null space of g ($\mathrm{Nu}(g)$), then the linear function $h = f \circ f^* + g^* \circ g : V \to V$ represents an isomorphism.

Hint: Consider v, an element in the kernel of h. What can you deduce about $\langle g(v), g(v) \rangle$?

Question 2

Let V be a finite-dimensional vector space over a field K, with n as its dimension. Additionally, let f and $g : V \to V$ be two endomorphisms of V such that $f \circ g = g \circ f$.

1. Prove that if f is diagonalizable, and its characteristic polynomial has simple roots, then g is also diagonalizable.
2. Show that if $f^n = 0$ and $f^{n-1} \neq 0$, then there exists a polynomial $P \in K[X]$ such that $g = P(f)$.

Question 3

Let V be a complex vector space of finite dimension equipped with an inner product, and let $f : V \to V$ be a linear function. If $f^* + f = 0$, then prove that f is diagonalizable, and its eigenvalues have null real parts.

Question 4

Let $n \in \mathbb{N}$ and let $\Bbbk$ be a field.

(a) Let A and B be two matrices of $M_n(\Bbbk)$. Prove that there exists an invertible matrix $C \in M_n(\Bbbk)$ such that $A = CB \iff \{x \in \Bbbk^n : Ax = 0\} = \{x \in \Bbbk^n : Bx = 0\}$.

(b) If a matrix $A \in M_n(\Bbbk)$ has the same rank as its square A^2, then there exists an invertible matrix D such that $A^2 = DA$.

Question 5

Let n be in $\mathbb{N}$ and let $V = \mathbb{R}[X]_{\leq n}$ be the vector space of polynomials with real coefficients that are either zero or have degree at most n.

(a) For each $t \in \mathbb{R}$, we consider the linear function $\phi_t : V \to \mathbb{R}$ defined as follows:

$$\phi_t(p) = p(t) \text{ for every } p \in V.$$

Show that if $t_0, \ldots, t_n$ are $n+1$ *pairwise distinct* elements of $\mathbb{R}$, then the set $\{\phi_{t_0}, \ldots, \phi_{t_n}\}$ forms a basis for the dual space V^*.

(b) Now, suppose $n = 2$, and let $I : V \to \mathbb{R}$ be the linear function defined as $I(p) = \int_0^1 p(x)dx$ for every polynomial $p \in V$. Express I as a linear combination of ϕ_0, $\phi_{1/2}$, and ϕ_1.

(c) Continuing with the assumption that $n = 2$, is it possible to express the function I from the previous part as a linear combination of two functions ϕ_{t_1} and ϕ_{t_2}, by appropriately choosing the numbers t_1 and t_2?