ANALYSIS OF 2D STEADY-STATE HEAT CONDUCTION IN A SQUARE PLATE

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Introduction

The work aims at numerical analysis of a two-dimensional steady-state heat conduction problem in a square steel plate.

The plate has dimensions of 1 m × 1 m × 1 cm and is heated at its middle with a concentrated heat source of 1 kW. The analysis involves the finite element method using linear triangular elements to determine the temperature distribution under different boundary conditions and mesh refinements.

Problem Description

- **Plate dimensions**: 1 m × 1 m × 1 cm
- **Heat source**: 1 kW concentrated at the centre of the plate
- **Heat flux model**:

  $q = 1000\ \delta(x - x_c\ , y - y_c)$ W/m2

  where (x_c, y_c) are the coordinates of the plate centre
- **Analysis approach**: 2D steady-state heat conduction using linear triangular elements

Two cases are analysed:

1. Three sides insulated with the fourth side maintained at 25°C
2. Three sides insulated with the fourth side subjected to convective heat transfer:

   $Q_n = h(T - T_0)$

   where ,

   T_0 = 25°C

   h = 10 W/(m²-K)

Theoretical Background

Governing Equation

The steady-state heat conduction equation in two dimensions is:

$$\boxed{-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = Q}$$

where:

- T is the temperature (°C)
- k is the thermal conductivity (W/m-K)
- Q is the internal heat generation per unit volume (W/m³)

FEM Formulation

To analyse the plate under the given thermal loading we use Galerkin FEM. Under Galerkin FEM, the governing differential equation is used to derive its weak form and appropriate weights and boundary conditions are applied.

The weak form of the governing differential equation is given by,

$$\int_\Omega w\left[-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - f\right] dA = 0$$

$$\int_\Omega \left(\frac{\partial w}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}k\frac{\partial T}{\partial y}\right) dA - \int_{\Gamma_q} w q_n \, dS - \int_\Omega w f \, dA = 0$$

$$\boxed{\int_\Omega \nabla w \cdot (k\nabla T)\, dA = \int_{\Gamma_q} w q_n \, dS + \int_\Omega w f \, dA}$$

The plate is discritised using triangular elements with nodal points at each of its vertices. The temperature function for each triangular element is approximated using linear polynomial functions in x and y which are later mapped to barycentric coordinates.

Element wise the weights functions can be approximated as,

$$w = N_i,\, T \approx \sum_j N_j T_j$$

$$\int_{\Omega^e} \nabla N_i \cdot (k\nabla T^h)\, dA = \int_{\Gamma_q^e} N_i q_n \, dS + \int_{\Omega^e} N_i f \, dA$$

Element Formulation for 3-Noded Linear Triangular Element :

$$T(x, y) = N_1(x, y)T_1 + N_2(x, y)T_2 + N_3(x, y)T_3$$

$$N_1(x, y) = \frac{1}{\Delta}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y)$$

$$N_2(x, y) = \frac{1}{\Delta}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y)$$

$$N_3(x, y) = \frac{1}{\Delta}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y)$$

To make the analyses simpler we move from (x,y) coordinated to (ξ, **η)** coordinates. Here these barycentric coordinates represent area ratios associated with a single triangular element.

Jacobian:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{3} N_i(\xi, \eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

For each triangular element the Jacobian J can be estimated by finding the difference between the coordinates of the nodes at each of its vertices, helping in mapping the coordinates to the master coordinates.

$$K_e = \int_{-1}^{1} \int_{-1}^{1} Q^T(\xi, \eta)\, D\, Q(\xi, \eta)\, |J(\xi, \eta)|\, d\xi\, d\eta$$

$$K_e = \int_{-1}^{1}\int_{-1}^{1} Q^T D Q \, |J| \, d\xi \, d\eta$$

$$Q = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix} \qquad D = k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Where Q represents the derivates of the shape functions with respect to the (x,y) coordinates. Through this expression we can evaluate the unknown temperature at the node points and thereby understand the temperature distribution along the plate.

The load vector is given by

$$F_e = \int_{-1}^{1}\int_{-1}^{1} N^T \mathbf{f} \, |J| \, d\xi \, d\eta$$

Using the Ke formula, the local stiffness matrix for each triangular element is formed and later on assembled to obtain the global stiffness matrix K. When the weight function associated with the nodes are considered arbitrary, it can be established that the Final Equation would be :

$$\mathbf{K} \cdot \mathbf{T} = \mathbf{F}$$

where,

K is the Global Stiffness Matrix

T is the Temperature vector

F is the Load vector

Procedure

The following cases and boundary conditions were considered to arrive at the result

Discretization

Case1a:

- Two triangular elements were used to discritize the plate
- A source with magnitude $q = 1000\ \delta(x - x_c\ ,\ y - y_c)$ was applied at the centre of the plate (0.5,0.5)
- Drichlet boundary condtion :Top side (chosen as fourth side) of the plate was held at 25 °C

Case1b:

- Four triangular elements were used to discritize the plate
- At the centre node (0.5, 0.5) the source with magnitude q=1000 was applied
- Drichlet boundary condtion :Top side of the plate was held at 25 °C

Case1c:

- Eight elements were used to discritize the plate.
- The centre node was identified at location (0.5,0.5) and the source with q=1000 was applied
- Drichlet boundary condtion :Top side of the plate was held at 25 °C

Case1d:

- 100 elements were used to discritize the plate with the area of the elements at the centre different compared to the others (an adaptive mesh approach)
- The centre node was identified at location (0.5,0.5) and the source with q=1000 was applied
- Drichlet boundary condtion:Top side of the plate was held at 25 degree C

Case2

- 100 elements were used to discritize the plate with the area of the elements at the centre, different compared to the others (an adaptive mesh approach)
- Neumann boundary conditions:Three sides were insulated (q = 0) and fourth side was subjected to convection $q_n = h\ (T-T_0)$ where $T_0 = 25$ °C and h = 10W/m2K

Analysis

Case1a

After discritization, the local stiffness matrix for both triangular elements were found. Once the local stiffnesses were estimated, there were assembled to find the global stiffness K. To form the F matrix, the source term given in the form of a Dirac-delta function has to be evaluated. The source term effects are to be distributed equally onto the nodes which are the end points of the element division line. For this the shape functions were evaluated at the centre point of the plate (0.5,0.5) and then multiplied with the given magnitude of heating i.e. 1000. The resultant term was then added to the F matrix. The nodes on the top side were assigned 25 °C. Once both these matrices were formulated, using KT=F, T was found and thereby the temperature distribution was obtained.

Case 1b 1c 1d

Once the plate was discrtised with triangular elements, (similar to Case1a), the global stiffness martrix K was found. The source term was applied at centre node and F matrix was formulated. The nodes at the top side of the plate (held at 25 °C) were identified by their "y" coordinate ("y" coordinate at top side is 1). Using KT = F, the temperature distribution was found.

Case 2:

Similar to procedures in Case1, the global stiffness matrix K was obtained. Unlike in Case1, there was no middle source term but a convective term that was to be added in the F matrix. To do this the top side (chosen as the fourth side) nodes were identified by using the fact that they lie on y = 1 and corresponding nodal F values were stored in the matrix F. Using KT = F, the temperature distribution was estimated.

The F and K are added with the convective terms.

$$\mathbf{F}_i^{(\mathrm{conv})} = \int_{\Omega_e} N_i \cdot h \cdot (T_\infty - T)\, d\Omega$$

$$\mathbf{K}_e^{(\mathrm{conv})} = \int_{\Omega_e} \nabla N_i \cdot h \cdot N_j \, d\Omega$$

Observations

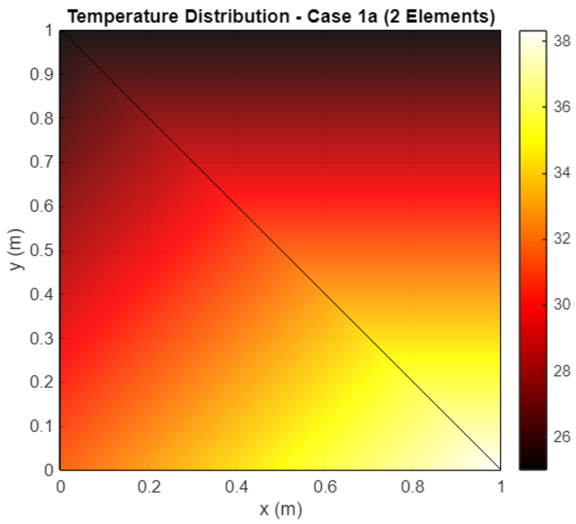
Case1

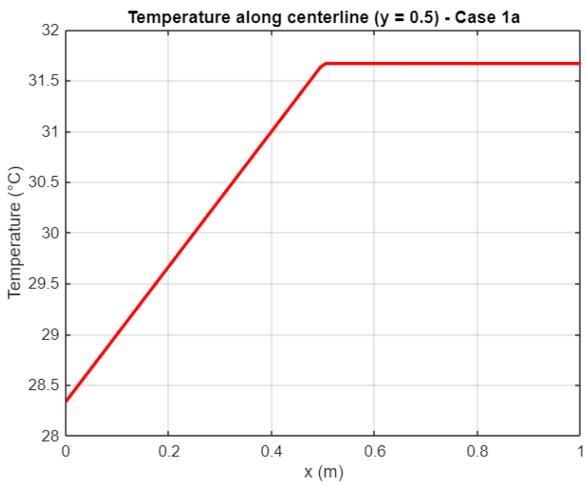
- The results are not very accurate because the discritization is minimal and the size of the element is too big. This reduces the complexity of the problem, but induces a lot of error in the solution.

  $$\|e\|_{L^2_h} = h^{1/2} \cdot \|T - T_h\|_{L^2}$$

  h is the element size
  From the L2 norm error it is understood that the error is higher if the element size is large. There this kind of discritization is not recommended.
- From the given problem statement, the heat source in the physical system should be in the centre of the plate. But the discritization followed does not provide a node at the centre which restricts the heat source to be applied at that point. This forces the use of Direc delta function, which approximates the heat source to be distributed among the element.
  This kind of approximation also induces significant error in the analysis.

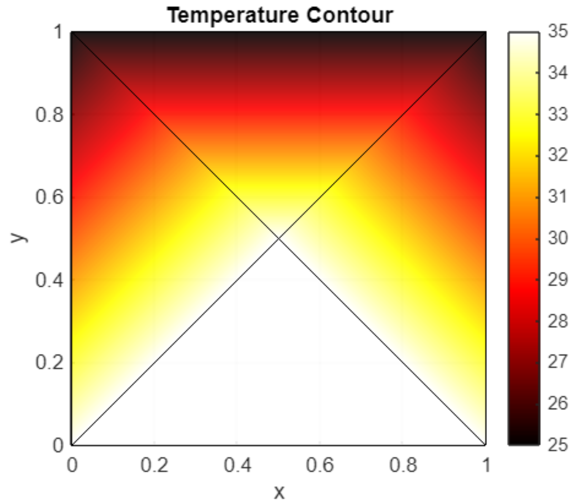
  Dirac delta function : $\overline{\mathbf{F} = \delta(x - x_c, y - y_c) \cdot q}$

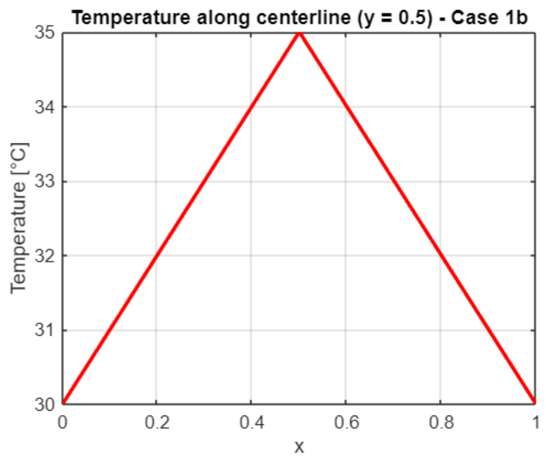




Case 1b

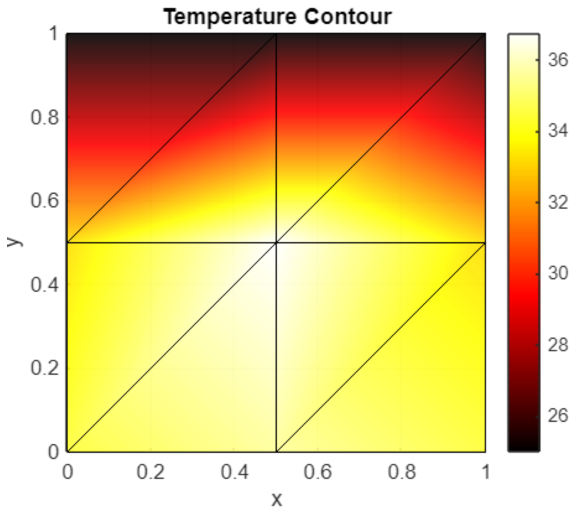
- The results are comparatively better but not accurate because the element size is still bigger.
- The problem of placement of the heat source is rectified as there exists a central node.
- The heat diffusion could not be observed properly as there are not enough nodes to refine the data.

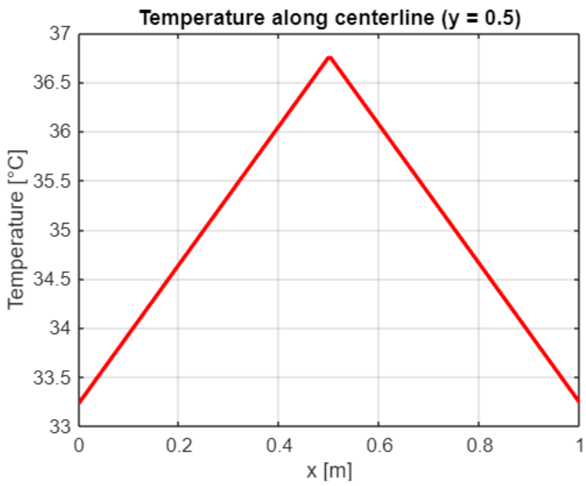




Case 1c

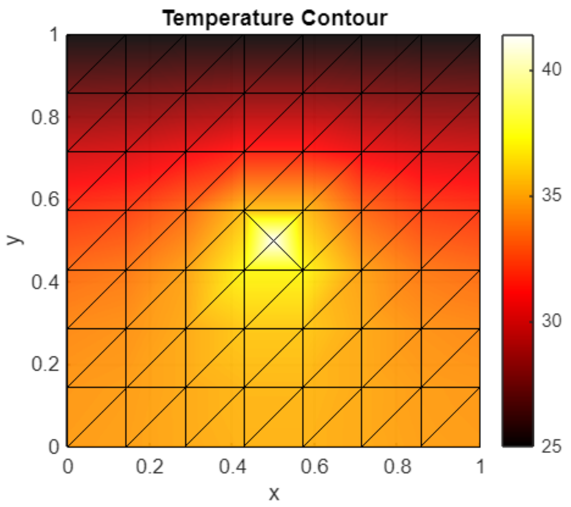
- The results are much reasonable than the before cases because of the refinement in the mesh.
- In this case also, the problem of placement of the heat source is rectified as there exists a central node.
- The heat diffusion pattern observed is reasonable as there more nodes to capture the data.

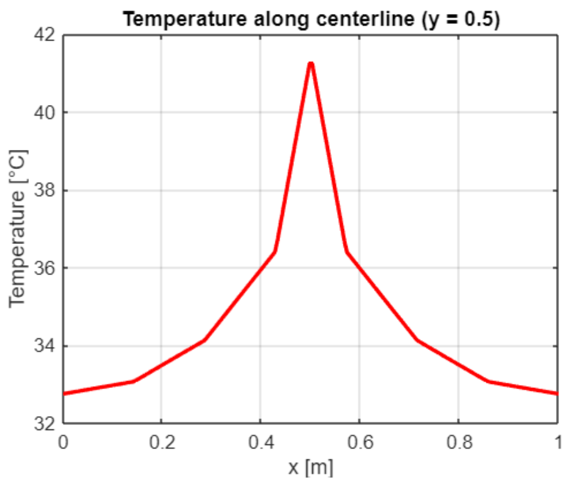




Case 1d

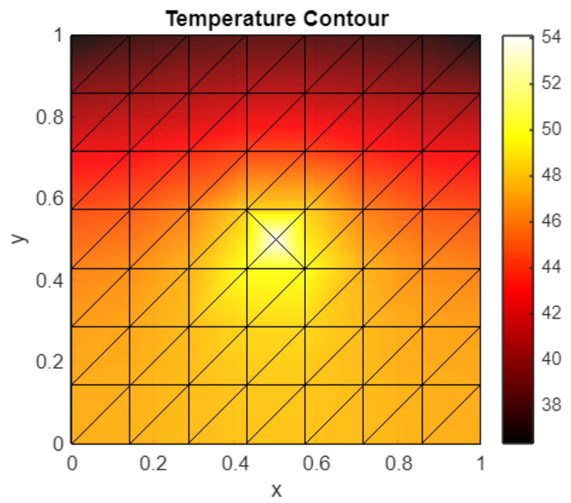
- Temperature contour observed in through this case is more precise and highly comparable to the physical system
- The addition of heat source exactly in the central node with more refined mesh elements surrounded around it paves way to obtain more accurate diffusion pattern.

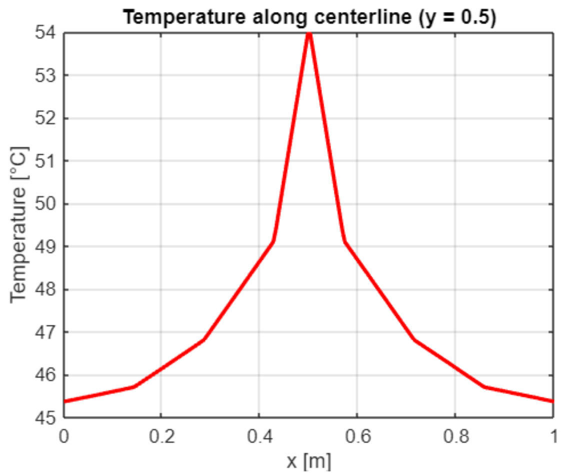




Case 2

- The discritization is same as that of Case 1d and therefore is considerable to best predict the temperature distribution.
- Here, along with the heat source, the 4th side is added with a convective term which induces gradual change in the temperature throughout the plate.





Comparison Between Case 1d and Case 2

In case 1d the temperature of the 4th side is forced to be 25°C regardless of the heat diffusion pattern, which is not true in the physical world.

This problem is sorted out in case 2 by adding a convective term which ensures a gradual gradient in the heat diffusion and hence is more relatable to the physical system.

CONCLUSION

The numerical analysis provides a great understanding on how mesh refinement and boundary conditions affect temperature distribution.

In Case 1, various levels of discretization were discussed. The accuracy of results significantly improved with mesh refinement. Specifically, Case 1d showed that using a finer mesh around the heat source location allowed for a more accurate and realistic temperature distribution.

However, the use of Dirichlet boundary conditions in Case 1, particularly fixing one side at 25°C, introduces a discontinuity in the thermal profile which may not be physically comparable. This issue was addressed in Case 2 by implementing convective boundary conditions, which more naturally simulate heat loss through the boundary. The results from Case 2 demonstrated a smoother and more realistic temperature gradient, validating the appropriateness of using convection models in thermal simulations.