

Welcome to



Matrices & Determinants

$$\begin{aligned}2x + 5y + 3z &= -3 \\4x + 0y + 8z &= 0 \\1x + 3y + 0z &= 2\end{aligned}$$

$$\underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}}$$

$$A^{-1}A = I$$

# Table of contents

<b>Session 01</b>	03	<b>Session 04</b>	64	<b>Session 08</b>	141
<u>Introduction</u>	04	<u>Co-factor of an Element</u>	65	<u>Properties of Inverse of Matrix</u>	143
<u>Order of Matrix</u>	08	<u>Value of <math>3 \times 3</math> Matrix Determinant</u>	67		
<u>Types of Matrices</u>	09	<u>Value of Determinant in terms of Minor and Cofactor</u>	69		
<u>Principal Diagonal of Matrix</u>	15	<u>Properties of Determinant</u>	74		
<u>Trace of Matrix</u>	16			<b>Session 09</b>	163
<u>Types of Matrices</u>	18			<u>Inverse of a Matrix by elementary transformations</u>	164
<b>Session 02</b>	27	<b>Session 05</b>	81	<u>System of Linear Equations</u>	175
<u>Algebra of Matrices</u>	28	<u>Properties of Determinant</u>	82	<u>Cramer's Rule</u>	179
<u>Properties of Addition/Subtraction of Matrices</u>	32	<u>Properties of Determinant</u>	87		
<u>Matrix Multiplication</u>	34	<u>Some important Formulae</u>	97		
<u>Properties of Matrix Multiplication</u>	37			<b>Session 10</b>	190
<u>Power of a Square Matrix</u>	41	<b>Session 06</b>	98	<u>Cramer's Rule</u>	195
<b>Session 03</b>	48	<u>Some important Determinants</u>	99	<u>System of Linear Equations( Matrix Inversion)</u>	203
<u>Polynomial Equation in Matrix</u>	45	<u>Product of Two Determinants</u>	103	<u>Homogeneous System of Linear Equations( Matrix Inversion)</u>	206
<u>Transpose of a Matrix</u>	47	<u>Application of Determinants</u>	107		
<u>Symmetric and Skew Symmetric Matrices</u>	51	<u>Differentiation of Determinant</u>	112		
<u>Properties of Trace of a Matrix</u>	60	<u>Integration/ Summation of Determinant</u>	114		
<u>Determinants</u>	62			<b>Session 11</b>	208
<u>Minor of an element</u>	63	<b>Session 07</b>	119	<u>Characteristic Polynomial and Characteristic Equation</u>	209
		<u>Singular/Non-Singular Matrix</u>	120	<u>Cayley-Hamilton Theorem</u>	210
		<u>Cofactor Matrix &amp; Adjoint Matrix</u>	121	<u>Special Types of Matrices</u>	216
		<u>Properties of Adjoint Matrix</u>	124		
		<u>Inverse of a Matrix</u>	134		
		<u>Matrix Properties</u>	138		



# Session 01

## Introduction to Matrices



## Key Takeaways



B

- A rectangular arrangement of  $m \cdot n$  numbers (real or complex) or expressions (real or complex valued), having  $m$  rows and  $n$  columns is called a matrix. ( $m, n \in N$ )

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix}$$

The diagram illustrates a matrix  $A$  as a rectangular grid of elements  $a_{ij}$ . The vertical dimension is labeled 'Rows' with four arrows pointing upwards from the bottom. The horizontal dimension is labeled 'Columns' with four arrows pointing downwards from the left.

- An element of a matrix is denoted by

$a_{ij}$ : Element of  $i^{th}$  row &  $j^{th}$  column.



## Key Takeaways



B

- A rectangular arrangement of  $m \cdot n$  numbers (real or complex) or expressions (real or complex valued), having  $m$  rows and  $n$  columns is called a matrix. ( $m, n \in N$ )

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix}$$

The diagram illustrates a matrix  $A$  as a rectangular grid of elements  $a_{ij}$ . The vertical dimension is labeled 'Rows' with three arrows pointing right from the first three columns to the fourth column. The horizontal dimension is labeled 'Columns' with four arrows pointing up from the bottom row to the top row.

- Number of elements in a matrix  
= Number of rows  $\times$  Number of columns  
=  $m \times n$



Write  $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$  for the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 3 & -8 \end{bmatrix}$$

Solution :

$$a_{11} = 1$$

$$a_{12} = 0$$

$$a_{13} = 5$$

$$a_{21} = -2$$

$$a_{22} = 3$$

$$a_{23} = -8$$



Find the value  $a_{23}$  in the following matrix

$$A = \begin{pmatrix} 3 & -4 & 0 \\ -2 & 7 & 10 \\ 5 & -6 & 9 \end{pmatrix}$$

A

-6

B

0

C

10

D

5



### Order of a matrix

Order or dimension of a matrix denotes the arrangement of elements as number of rows and number of columns.

- Order = Number of rows  $\times$  Number of columns  $= m \times n$

The diagram shows a matrix  $A_{m \times n}$  represented as a grid of elements  $a_{ij}$ . The matrix has  $m$  rows and  $n$  columns. The first row contains elements  $a_{11}, a_{12}, a_{13}, \dots, a_{1n}$ . The second row contains  $a_{21}, a_{22}, a_{23}, \dots, a_{2n}$ . The  $i$ -th row contains  $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$ , indicated by a vertical ellipsis between the second and third columns. The first column contains  $a_{11}, a_{21}, \dots, a_{m1}$ . The second column contains  $a_{12}, a_{22}, \dots, a_{m2}$ . The  $j$ -th column contains  $a_{1j}, a_{2j}, \dots, a_{mj}$ , indicated by a horizontal ellipsis between the first and second rows. Brackets on the left side group the matrix into a single entity labeled "Name of a matrix". Brackets at the bottom group the matrix into a single entity labeled "Order of a matrix". A brace on the right side, labeled "Rows", groups the  $m$  rows. A brace at the bottom, labeled "Columns", groups the  $n$  columns.

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- Thus, a matrix can also be represented as  $A = [a_{ij}]_{m \times n}$  or  $(a_{ij})_{m \times n}$



## Types of Matrix:

- Row Matrix (row vector) : A matrix having a single row is called a row matrix.

$$A = [a_{ij}]_{1 \times n} = [a_{11} \quad a_{12} \quad a_{13} \cdots a_{1n}]_{1 \times n}$$

Example:  $B = [a \quad b \quad c]_{1 \times 3}$

- Column Matrix (column vector) : A matrix having a single column is called a column matrix.

$$\text{Example: } B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4 \times 1}$$

$$A = [a_{ij}]_{m \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1}$$

- Matrices consisting of one row or one column are called vectors.



## Types of Matrix:

- Zero Matrix (null matrix) : If all the elements of a matrix are zero, then it is called zero or null matrix

$A = [a_{ij}]_{m \times n}$  is called a zero matrix, if  $a_{ij} = 0, \forall i & j.$

Examples:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



## Vertical Matrix

A matrix of order  $m \times n$  is known as vertical matrix if  $m > n$ , where  $m$  is equal to the number of rows and  $n$  is equal to the number of columns.

Example:

$$\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$$

- In the matrix example given the number of rows ( $m$ ) = 4, whereas the number of columns ( $n$ ) = 2.

Therefore, this makes the matrix a vertical matrix.



## Horizontal Matrix

A matrix of order  $m \times n$  is known as vertical matrix if  $n > m$ , where  $m$  is equal to the number of rows and  $n$  is equal to the number of columns.

Example:  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

- In the matrix example given the number of rows ( $m$ ) = 2, whereas the number of columns ( $n$ ) = 4.

Therefore, this makes the matrix a horizontal matrix.

If a matrix has 12 elements, then what are the possible orders it can have?

Solution :

Number of elements = Number of rows × Number of columns

$$12 = m \times n \quad (m, n \in N)$$

Possible Order =  $1 \times 12, 2 \times 6, 3 \times 4, 4 \times 3, 6 \times 2, 12 \times 1$



Construct a  $2 \times 3$  matrix, whose elements are given by  $a_{ij} = \frac{(i+2j)}{3}$ .

Solution :

$$a_{ij} = \frac{(i+2j)}{3}$$

$$a_{11} = 1$$

$$a_{12} = \frac{5}{3}$$

$$a_{13} = \frac{7}{3}$$

$$a_{21} = \frac{4}{3}$$

$$a_{22} = 2$$

$$a_{23} = \frac{8}{3}$$

$$A = \begin{pmatrix} 1 & \frac{5}{3} & \frac{7}{3} \\ \frac{4}{3} & 2 & \frac{8}{3} \end{pmatrix}$$



## Key Takeaways



- Principal Diagonal of a Matrix: Diagonal containing the elements  $a_{ij}$ , where  $i = j$  is called principal diagonal of a matrix
- Examples:

$$A = \begin{bmatrix} 2 & -6 & 10 \\ 5 & 0 & 7 \\ 19 & -3 & -8 \end{bmatrix}_{3 \times 3}$$

$$B = \begin{pmatrix} 2 & 3 & 4 & -5 \\ 1 & 4 & 0 & 6 \\ -3 & 7 & 8 & 9 \end{pmatrix}_{3 \times 4}$$

### Types of Matrix:

- Square Matrix: A matrix where number of rows = number of columns is called square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix}_{n \times n}$$

Example:

$$A = \begin{bmatrix} -4 & 5 & 0 \\ 8 & -1 & 3 \\ 9 & 7 & 2 \end{bmatrix}_{3 \times 3}$$