Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$, Prove that : $\int_0^{\pi/2} f(x)\, dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$

Solution:

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operate $R_1 \to R_1 - \sec x\ R_3$

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$$= (\sec^2 x + \cot x \operatorname{cosec} x)\,(\cos^4 x - \cos^2 x)$$

$$f(x) = \left(1 + \frac{\cos^3 x}{\sin^2 x} - \cos^3 x\right)(\cos^2 x - 1) = -\sin^2 x \frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\sin^2 x}$$