

Welcome to



## Matrices & Determinants

$$\begin{aligned}2x + 5y + 3z &= -3 \\4x + 0y + 8z &= 0 \\1x + 3y + 0z &= 2\end{aligned}$$

$$\begin{matrix} & \overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^A & \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} & = & \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}} \end{matrix}$$

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# **Session 01**

## **Introduction to Matrices**

## Key Takeaways

- A rectangular arrangement of  $m \cdot n$  numbers (real or complex) or expressions (real or complex valued), having  $m$  rows and  $n$  columns is called a matrix. ( $m, n \in N$ )

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Diagram illustrating the structure of a matrix  $A$  with  $m$  rows and  $n$  columns. The elements are arranged in a grid. The first row is labeled  $a_{11}, a_{12}, a_{13}, \dots, a_{1n}$ . The second row is labeled  $a_{21}, a_{22}, a_{23}, \dots, a_{2n}$ . The third row is labeled  $\vdots, \vdots, \vdots, \vdots$ . The last row is labeled  $a_{m1}, a_{m2}, a_{m3}, \dots, a_{mn}$ . Arrows point from the labels "Rows" and "Columns" to the corresponding dimensions of the matrix.

- An element of a matrix is denoted by  $a_{ij}$ : Element of  $i^{th}$  row &  $j^{th}$  column.



## Key Takeaways

- A rectangular arrangement of  $m \cdot n$  numbers (real or complex) or expressions (real or complex valued), having  $m$  rows and  $n$  columns is called a matrix. ( $m, n \in N$ )

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Diagram illustrating the structure of a matrix  $A$  with  $m$  rows and  $n$  columns. The elements are arranged in a grid. The first row is labeled  $a_{11}, a_{12}, a_{13}, \dots, a_{1n}$ . The second row is labeled  $a_{21}, a_{22}, a_{23}, \dots, a_{2n}$ . The third row is labeled  $\vdots, \vdots, \vdots, \vdots$ . The last row is labeled  $a_{m1}, a_{m2}, a_{m3}, \dots, a_{mn}$ . Arrows point from the right to each row, labeled "Rows". Arrows point from below to each column, labeled "Columns".

- Number of elements in a matrix  
= Number of rows  $\times$  Number of columns  
=  $m \times n$



Write  $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$  for the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 3 & -8 \end{bmatrix}$$

Solution :

$$a_{11} = 1$$

$$a_{12} = 0$$

$$a_{13} = 5$$

$$a_{21} = -2$$

$$a_{22} = 3$$

$$a_{23} = -8$$



Find the value  $a_{23}$  in the following matrix

$$A = \begin{pmatrix} 3 & -4 & 0 \\ -2 & 7 & 10 \\ 5 & -6 & 9 \end{pmatrix}$$

A

-6

B

0

C

10

D

5

## Order of a matrix

Order or dimension of a matrix denotes the arrangement of elements as number of rows and number of columns.

- Order = Number of rows  $\times$  Number of columns =  $m \times n$

Name of a matrix

Order of a matrix

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Rows

Columns

- Thus, a matrix can also be represented as  $A = [a_{ij}]_{m \times n}$  or  $(a_{ij})_{m \times n}$





## Types of Matrix:

- Row Matrix (row vector) : A matrix having a single row is called a row matrix.

$$A = [a_{ij}]_{1 \times n} = [a_{11} \quad a_{12} \quad a_{13} \cdots a_{1n}]_{1 \times n}$$

Example:  $B = [a \quad b \quad c]_{1 \times 3}$

- Column Matrix (column vector) : A matrix having a single column is called a column matrix.

Example:  $B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4 \times 1}$

$$A = [a_{ij}]_{m \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1}$$

- Matrices consisting of one row or one column are called vectors.



## Types of Matrix:



- Zero Matrix (null matrix) : If all the elements of a matrix are zero, then it is called zero or null matrix

$A = [a_{ij}]_{m \times n}$  is called a zero matrix, if  $a_{ij} = 0, \forall i \text{ \& } j$ .

Examples:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



## Vertical Matrix

A matrix of order  $m \times n$  is known as vertical matrix if  $m > n$ , where  $m$  is equal to the number of rows and  $n$  is equal to the number of columns.

Example: 
$$\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$$

- In the matrix example given the number of rows ( $m$ ) = 4, whereas the number of columns ( $n$ ) = 2.

Therefore, this makes the matrix a vertical matrix.



## Horizontal Matrix

A matrix of order  $m \times n$  is known as vertical matrix if  $n > m$ , where  $m$  is equal to the number of rows and  $n$  is equal to the number of columns.

Example: 
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

- In the matrix example given the number of rows ( $m$ ) = 2, whereas the number of columns ( $n$ ) = 4.

Therefore, this makes the matrix a horizontal matrix.



If a matrix has 12 elements, then what are the possible orders it can have?

Solution :

Number of elements = Number of rows  $\times$  Number of columns

$$12 = m \times n \quad (m, n \in N)$$

Possible Order =  $1 \times 12, 2 \times 6, 3 \times 4, 4 \times 3, 6 \times 2, 12 \times 1$





Construct a  $2 \times 3$  matrix, whose elements are given by  $a_{ij} = \frac{(i+2j)}{3}$ .

Solution :

$$a_{ij} = \frac{(i+2j)}{3}$$

$$a_{11} = 1$$

$$a_{12} = \frac{5}{3}$$

$$a_{13} = \frac{7}{3}$$

$$a_{21} = \frac{4}{3}$$

$$a_{22} = 2$$

$$a_{23} = \frac{8}{3}$$

$$A = \begin{pmatrix} 1 & \frac{5}{3} & \frac{7}{3} \\ \frac{4}{3} & 2 & \frac{8}{3} \end{pmatrix}$$

- Principal Diagonal of a Matrix: Diagonal containing the elements  $a_{ij}$ , where  $i = j$  is called principal diagonal of a matrix
- Examples:

$$A = \begin{bmatrix} 2 & -6 & 10 \\ 5 & 0 & 7 \\ 19 & -3 & -8 \end{bmatrix}_{3 \times 3}$$

$$B = \begin{pmatrix} 2 & 3 & 4 & -5 \\ 1 & 4 & 0 & 6 \\ -3 & 7 & 8 & 9 \end{pmatrix}_{3 \times 4}$$

## Types of Matrix:

- Square Matrix: A matrix where number of rows = number of columns is called square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \quad \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix}_{n \times n}$$

Example:

$$A = \begin{bmatrix} -4 & 5 & 0 \\ 8 & -1 & 3 \\ 9 & 7 & 2 \end{bmatrix}_{3 \times 3}$$