DHANAMANJURI UNIVERSITY

JUNE - 2024

Name of Programme : **B.A/B.Sc. Mathematics (Honours)**
Semester : **6th**
Paper Type : **Core XIII (Theory)**
Paper Code : **CMA-313**
Paper Title : **Metric Spaces and Complex Analysis**
Full Marks : **100**
Pass Marks : **40**
Duration : **3 Hours**

The figures in the margin indicate full marks for the questions

Answer all the questions:

1. Choose and rewrite the correct answer for each of the following: $1 \times 4 = 4$

a) In a metric space (X, d) which of the following is true?

i) every sequence is Cauchy sequence.

ii) every sequence is convergent.

iii) every convergent sequence is Cauchy sequence.

iv) every Cauchy sequence is convergent

b) Let A and B be two subsets of a metric space (X, d). If $B \subset \bar{A}$ then

i) A is dense in X

ii) A is dense in B

iii) A is non dense in B

iv) A is nowhere dense

c) For $f(z) = u(x,y) + iv(x,y)$ the Cauchy – Reimann equations are

i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

ii) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

iii) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

iv) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

d) If $\bar{z}$ *and* $|z|$ are the conjugate and modulus of a complex number z respectively then

i) $z\bar{z} = |z|^2$

ii) $z\bar{z} = -|z|^2$

iii) $z\bar{z} = |\bar{z}|^2$

iv) $z\bar{z} = -|\bar{z}|^2$

2. **Write very short answer for each of the following:** $1 \times 8 = 8$

a) Write down the symmetry property satisfied by a metric "d" on a nonempty set X.

b) Give the reason for the boundedness of discrete metric space.

c) When are two metric spaces said to be homeomorfic?

d) Define a contraction mapping.

e) Are the two sets $A = \{x: \infty < x < 0\}$ and $B = \{x: 0 \leq x < \infty\}$ separated?

f) When is a complex function called harmonic?

g) Whether the limit $\lim_{z \to 0} \frac{\bar{z}}{z}$ exists?

h) Define the term isolated singular point.

3. **Write short answer for each of the following:** $3 \times 10 = 30$

a) Prove that in a metric space every closed sphere is a closed set.

b) Give one example each for an open and a closed set.

c) Define the terms:

i) Neighbourhood

ii) Adharent point in a metric space

d) For any non-empty subset A of a metric space (X, d) show that the function $f: X \to R$ given by $f(x) = d(x, A), for\ x \in X$ is uniformly continuous.

e) Prove that union of connected sets having non empty intersection is connected.

f) Show that the real and imaginary components $u(x, y)$ and $v(x, y)$ of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic.

g) Write down the Laplace equation in polar form. Show that $u(r, \theta) = \left(r + \frac{1}{r}\right) \cos\theta,\ r \neq 0$ is harmonic.

h) Find $f'(z)$ and $f''(z)$ where $f(z) = e^{-x}e^{-iy}$.

i) Evaluate $\int_C \bar{Z}\, dz$ where C is the circle $|z| = 1$.

j) Find all the points at which $f(x + iy) = 2xy + i(x^2 + y^2)$ is differentiable.

4. Answer the following questions: $5 \times 2 = 10$

?

a) Define a metric space.

b) Let $X = R^n$ denotes the set of all n-tuples of real numbers for a fixed $n \in N$. Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in R^n$. A mapping d from $R^n \times R^n$ into R is defined by $(x, y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$. Show that d is a metric on R^n.

Or

Prove that in a metric space every open sphere is an open set.

c) Derive the polar form of Cauchy – Reimann equations $rU_r = V_\theta \text{ and } rV_r = -U_\theta$.

Or

Show that the function defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ is not differentiable at z=0

5. Answer the following questions: $6 \times 2 = 12$

a) State and prove Holder's inequality.

Or

State and prove Minkowski's inquality.

b) Let A and B be any two subsets of metric spaces (X, d). Prove that

i) $A \subseteq B$ implies $int(A) \subseteq int(B)$

ii) $int(A \cap B) = int(A) \cap int(B)$

iii) $int(A \cup B) \supseteq int(A) \cup int(B)$

Or

Let (X, d) be a complete metric space and let $\{F_n\}$ be a decreasing sequence of nonempty subsets of X such that $d(F_n) \to 0 \text{ as } n \to \infty$. Prove that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

6. Answer any two questions from the following: $6 \times 2 = 12$

a) Let Y be a subset of a metric space (X, d). Prove that the following statements are equivalent.

i) Y is connected

ii) Y cannot be expressed as disjoint union of two non-empty closed sets in Y

iii) Φ and Y are the only sets which are both open and closed sets in Y.

b) Discuss the connectedness of the following subsets of the Euclidean space R^2

i) $D = \{(x, y): x \neq 0, and\ y = \sin\frac{1}{x}\}$

ii) $E = \{\ (x, y): x = 0, and - 1 \leq y \leq 1\} \cup D$

c) Define a uniformly continuous function in a metric space. Prove that the image of a Cauchy sequence under a uniform continuous function is again a Cauchy sequence.

7. Answer the following questions: $6 \times 2 = 12$

a) Prove that $u(x, y) = e^x(x\cos y - y\sin y)$ is harmonic. Also find its harmonic conjugate $v(x, y)$ and express the corresponding analytic function $f(z)$ in terms of z.

Or

If $v(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta, r \neq 0$ then find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$. Also find the corresponding analytic function $f(z)$ in terms of z.

b) i) Find the harmonic conjugate of $u(x, y) = y^3 - 3x^2y$.

ii) Find all the points at which $f(x + iy) = 2xy + i(x^2 + y^2)$ is differentiable.

Or

If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$.

8. Answer any two questions: $6 \times 2 = 12$

a) Evaluate the integral $\int_0^{2+i} (\bar{z})^2 dz$ along $y = \frac{x}{2}$

b) Evaluate the integral $\int_C (z^2 + 3z)dz$ counterclockwise from (2,0) to (0,2) along the curve C, where C is the circle $|z| = 2$.

c) Show that the function defined by

$$f(z) = \begin{cases} \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

satisfies C-R Equations at the origin, but f is not differentiable there

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