Dhanamanjuri University

June - 2024

Name of Programme	**:**	**B.A/B.Sc. Mathematics(Honours)**
Semester	**:**	**6^{th}**
Paper Type	**:**	**Core XIII (theory)**
Paper code	**:**	**CMA-313**
Paper Tiltle	**:**	**Metric Spaces and Complex Analysis**
Full Marks	**:**	**100**
Pass Marks	**:**	**40**
Duration	**:**	**3 Hours**

The figures in the margin indicate full marks for the questions
Answer all the questions:

1. Choose and rewrite the correct answer for each of the following: $(1 \times 4 = 4)$

(a) In a metric space (X, d), which of the following is true?

(i) Every sequence is a Cauchy sequence.

(ii) Every sequence is convergent.

(iii) Every convergent sequence is a Cauchy sequence.

(iv) Every Cauchy sequence is convergent.

(b) Let A and B be two subsets of a metric space (X, d). If $B \subset \overline{A}$ then

(i) A is dense in X.

(ii) A is dense in B.

(iii) A is non-dense in B.

(iv) A is nowhere dense.

(c) For $f(z) = u(x, y) + iv(x, y)$, the Cauchy-Riemann equations are

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(ii) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(iii) $\frac{\partial u}{\partial x} = \frac{-\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(iv) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(d) If $\overline{z}$ and $|z|$ are the conjugate and modulus of a complex number z, respectively, then

(i) $z\overline{z} = |z|^2$

(ii) $z\overline{z} = -|z|^2$

(iii) $z\overline{z} = |\overline{z}|^2$

(iv) $z\overline{z} = -|\overline{z}|^2$

2. Write very short answer for each of the following: $(1 \times 8 = 8)$

(a) Write down the symmetry property satisfied by a metric "d" on a nonempty set X.

(b) Give the reason for the boundedness of a discrete metric space.

(c) When are two metric spaces said to be homeomorphic?

(d) Define a contraction mapping.

(e) Are the two sets $A = \{x : \infty < x < 0\}$ and $B = \{x : 0 \leq x < \infty\}$ separated?

(f) When is a complex function called harmonic?

(g) Whether the limit $\lim_{z \to 0} \frac{\overline{z}}{z}$ exists?

(h) Define the term isolated singular point.

3. Write short answer for each of the following: $(3 \times 10 = 30)$

(a) Prove that in a metric space every closed sphere is a closed set.

(b) Give one example each for an open and a closed set.

(c) Define the terms:

   (i) Neighbourhood

   (ii) Adherent point in a metric space

(d) For any non-empty subset A of a metric space (X, d), show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, A)$, for $x \in X$, is uniformly continuous.

(e) Prove that the union of connected sets having non-empty intersection is connected.

(f) Show that the real and imaginary components $u(x, y)$ and $v(x, y)$ of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic.

(g) Write down the Laplace equation in polar form. Show that $u(r, \theta) = \left(r + \frac{1}{r}\right) \cos\theta,\ r \neq 0$ is harmonic.

(h) Find $f'(z)$ and $f''(z)$ where $f(z) = e^{-x}e^{-iy}$.

(i) Evaluate $\int_C \overline{z}\, dz$ where C is the circle $|z| = 1$.

(j) Find all the points at which $f(x + iy) = 2xy + i(x^2 + y^2)$ is differentiable.

4. Answer the following questions:

a) Define a metric space.

b) Let $X = \mathbb{R}^n$ denote the set of all n-tuples of real numbers for a fixed $n \in \mathbb{N}$. Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. A mapping d from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ is defined by:

$$(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

Show that d is a metric on $\mathbb{R}^n$.

Or

Prove that in a metric space every open sphere is an open set.

c) Derive the polar form of the Cauchy-Riemann equations:

$$ru_r = v_\theta \quad \text{and} \quad rv_r = -u_\theta$$

Or

Show that the function defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not differentiable at $z = 0$.

5. Answer the following questions:

a) State and prove Hölder's inequality.

Or

State and prove Minkowski's inequality.

b) Let A and B be any two subsets of metric spaces (X, d). Prove that:

1. $A \subseteq B$ implies $\text{int}(A) \subseteq \text{int}(B)$
2. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$
3. $\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$

Or

Let (X, d) be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty subsets of X such that $d(F_n) \to 0$ as $n \to \infty$. Prove that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

6. Answer any two questions from the following:

a) Let Y be a subset of a metric space (X, d). Prove that the following statements are equivalent:

(i) Y is connected

(ii) Y cannot be expressed as a disjoint union of two non-empty closed sets in Y

(iii) Φ and Y are the only sets which are both open and closed sets in Y.

b) Discuss the connectedness of the following subsets of the Euclidean space $\mathbb{R}^2$:

a) $D = \{(x, y) : x \neq 0, \text{ and } y = \sin \frac{1}{x}\}$

b) $E = \{(x, y) : x = 0, \text{ and } -1 \leq y \leq 1\} \cup D$

c) Define a uniformly continuous function in a metric space. Prove that the image of a Cauchy sequence under a uniform continuous function is again a Cauchy sequence.

7. Answer the following questions:

a) Prove that $u(x, y) = e^x(x \cos y - y \sin y)$ is harmonic. Also find its harmonic conjugate $v(x, y)$ and express the corresponding analytic function $f(z)$ in terms of z.

Or

If $v(r, \theta) = \left(r - \frac{1}{r}\right) \sin \theta$, $r \neq 0$, then find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$. Also find the corresponding analytic function $f(z)$ in terms of z.

b)

1. Find the harmonic conjugate of $u(x, y) = y^3 - 3x^2 y$.
2. Find all the points at which $f(x + iy) = 2xy + i(x^2 + y^2)$ is differentiable.

Or

If $f(z)$ is an analytic function of z, prove that:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 2|f'(z)|^2$$

8. Answer any two questions:

a) Evaluate the integral $\int_0^{2+i}(z^2)dz$ along $y = \frac{x}{2}$.

b) Evaluate the integral $\int_C(z^2 + 3z)dz$, counterclockwise from (2, 0) to (0, 2) along the curve C_1, where C is the circle $|z| = 2$.

c) Show that the function defined by

$$f(z) = \begin{cases} \frac{(1+i)(x^3-(1-i)y^3)}{x^2+y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at the origin, but f is not differentiable there.