DHANAMANJURI UNIVERSITY
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Name of Programme:	B.A/B.Sc. Mathematics (Honours)
Semester:	6^{th}
Paper Type:	Core XIV (Theory)
Paper Code:	CMA-314
Paper Title:	Ring Theory and Linear Algebra
Full Marks:	100
Pass Marks:	40
Duration:	3 Hours

The figures in the margin indicate full marks for the questions
Answer all the questions:

1. Choose and rewrite the correct answer for each of the following: ($1 \times 5 = 5$)

(a) The polynomial $f(x) = 2x^2 - 2 \in Q[x]$ is

- (i) not primitive but reducible
- (ii) Primitive as well as reducible
- (iii) Primitive as well as irreducible
- (iv) not primitive as well as irreducible

(b) If $V(F)$ be a finite dimensional vector space over the field F of scalars and let $W(F)$ be a subspace of $V(F)$, then $A(A(W))$ is isomorphic to

- (i) $\hat{W}$
- (ii) $A(W)$
- (iii) W
- (iv) $\overline{A(W)}$

(c) Let $V(F)$ be a finite dimensional inner product space. The vector in $V(F)$ which is orthogonal to each vector $x \in V(F)$ is

(i) 1

(ii) 0

(iii) $\frac{x}{\|x\|}, x \neq 0$

(iv) does not exist

(d) Let R be an integral domain. Let $f(x), g(x) \in R[x]$ be such that $\deg(f(x)) = m, \deg(g(x)) = n$. Then $\deg(f(x) \cdot g(x))$ is

(i) less than $m+n$

(ii) less than $\min(m,n)$

(iii) less than $\max(m,n)$

(iv) equal to $m+n$

(e) Let T be a linear operator on an inner product space $V(F)$. Then T is a normal operator defined on $V(F)$ if

(i) $TT^* = I$

(ii) $T = T^*$

(iii) $T = T^{**}$

(iv) $T^*T = TT^*$

2. Answer the following questions: (1 $\times$ 5 = 5)

(a) Define a prime element in a commutative ring with unity.

(b) Let a and b be two non-zero elements in a Euclidean domain R. Write the condition for a, b to be relatively prime.

(c) Define an eigenvalue λ of a linear operator T defined on a vector space $V(F)$ and corresponding eigenvector.

(d) When is a square matrix A of order $n \times n$ said to be diagonalizable?

(e) State the Bessel's inequality in an inner product space $V(F)$.

3. Answer any three of the following questions: $(3 \times 10 = 30)$

(a) (i) Show that every ideal of a Euclidean domain is a Principal ideal.

(ii) Show that a Euclidean domain possesses unity and is also a Principal Ideal Domain (PID).

(b) Show that any two non-zero elements a and b in a Euclidean domain R have a greatest common divisor D and it is possible to write $D = \lambda a + \mu b$, $\lambda, \mu \in R$.

(c) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in Z[x]$. Suppose that for some prime number p, $pa_0, pa_1, \ldots, pa_n$, then $f(x)$ is irreducible polynomial over Q, the ring of rationals.

(d) (i) If F is a Unique Factorisation Domain (UFO) and if $f(x), g(x) \in F[x]$, then show that $c(fg) = c(f) \cdot c(g)$.

(ii) If $f(x)g(x)$ is primitive polynomial then show that $f(x)$ and $g(x)$ are primitives separately.

(e) For any prime p, Show that the polynomial $x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible over Q, the field of rational numbers.

4. Answer any three of the following questions: $(3 \times 10 = 30)$

(a) Let $\{V_1, V_2, \ldots, V_n\}$ be a basis of a vector space. Define

$$\hat{v}_i : V(F) \to F \quad \hat{v}(\sum_{i=1}^{n} \alpha_i V_i) = \alpha_i \quad \forall i = 1, 2, \ldots, n$$

Then show that $\{\hat{v}_1, \hat{v}_2, \ldots, \hat{V}_n\}$ is a basis of $\hat{V}$. Hence dim $V(F) = \dim \hat{v}(F)$.

(b) Let T be a linear operator on a finite dimensional vector space $V(F)$. Show that c $\in$ F is an eigen value of T if and only if T $-$ cI is singular. Also, show that similar matrices have the same characteristic polynomials

(c) Show that every square matrix satisfies its characteristic equation.

(d) Let $V(F)$ be an inner product space. Show that $|(u, v)| \leq ||u| \; ||v|| \;\; \forall u, v \in V(F)$.

(e) Obtain an orthonormal basis for V(IR), the space of all real polynomials of degree utmost 2, the inner product defined by $(f, g) = \int_0^1 f(x).g(x)dx$.

5. Answer any three of the following questions: $3 \times 10 = 30$

(a) Let $V(F)$ be an inner product space. Show that:

(i) $\|x+y\| \leq \|x\| + \|y\|$ and $\|y\| \| \langle x, y \rangle$, $x, y \in V(F)$

(ii) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

(b) Let $V(F)$ be a finite-dimensional inner product space and T be a linear operator on $V(F)$. Show that there exists a unique linear operator T^* such that $\langle TV, V' \rangle = \langle V, T^* V' \rangle$ for all $V' \in V(F)$.

(c) Let $V(F)$ be a finite-dimensional inner product space and W is a subspace of $V(F)$. Show that $V(F) = W \oplus W^{\perp}$, $W^{\perp}$, where $W^{\perp}$ is the orthogonal complement of W.

(d) (i) Let W be the subspace of $IR^5(IR)$ spanned by the vectors

$$\alpha = (2, -2, 3, 4, -1) \quad \text{and} \quad \beta = (0, 0, -1, -2, 3).$$

Describe $A(W)$.

(ii) Let W_1 and W_2 be two subsets of a vector space $V(F)$. Show that $A(W_1) \supset A(W_2)$ provided $W_1 \subset W_2$ in $V(F)$.

(e) Define:

$$T : R^3 \to R^2 \text{ by}$$

$$T(x, y, z) = (x+y, 2z-x).$$

If $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $\beta' = \{(1,0), (0,1)\}$ are standard bases for IR^3 and IR^2, find $[T]_{\beta\beta'}$.