DHANAMANJURI UNIVERSITY
DECEMBER-2022

Name of Programme	:	B.A/B.Sc. Mathematics (Honours)
Semester	:	3rd
Paper Type	:	Core – VI (Theory)
Paper Code	:	CMA-206
Paper Title	:	Group Theory

Full Marks : 100

Pass Marks : 40 **Duration : 3 Hours**

The figures in the margin indicate full marks for the questions

Answer all the questions:

1. Answer the following: **4 x 1 = 4**

a) The number of distinct elements in the symmetric group S_n on n symbols is

i) n ii) $n!$

iii) $n-1$ iv) $(n-1)!$

b) Let G be a group and let H and K be two subgroup of G. The $H \cup K$ is also a subgroup of G if and only if:

i) both H and K are cyclic

ii) both H and K are abelian.

iii) $H \sqsubset K$ or $K \sqsubset H$

iv) H is cyclic or K is cyclic.

c) Let N be a normal subgroup of a finite group G. Then $O\left(\frac{G}{N}\right)$ is equal to

i) $\frac{O(G)}{O(N)}$ ii) $O(G) - O(N)$

iii) $O(G) + O(N)$ iv) $O(G) \times O(N)$

d) Number of Sylow $2-subgroups$ of S_3 is

i) 1 ii) 3

iii) 0 iv) 2

2. Answer the following: **12 x 1 = 12**

a) In the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication. Find the value of the product $(i.j).k$.

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b) Define the dihedral group.

c) Write the permutation (3 4 5) in 2 – rowed notation using 6 symbols.

d) Define the centralizer of an arbitrary group G.

e) What is an index of a subgroup H of a group G.

f) For the group $(Z,^*)$ where $*$ is defined by
$a^*b = a + b + 1$
$\forall\, a, b \in Z$, find the identity element

g) Define a transposition and show that the identity permutation I is an even permutation.

h) Write the statement of the Fermat's Little theorem.

i) Define a quotient group.

j) When is a homomorphism f on groups is an automorphism?

k) Define Sylow's $p - subgroups$ of a group G.

l) Write the correct relationship among $A(G), Aut(G) and\ I(G)$ associated with an arbitrary group G.

3. Answer any 12 (Twelve) of the following questions: **12 x 3 = 36**

a) State and prove the cancellation laws in a group $(G,^*)$.

b) Let $G = \{-1,1,i,-i\}, i = \sqrt{-1}$. Show that $(G, 0)$ forms a group by forming the composition table and under matrix multiplication.

c) Show that in a group G the left inverse of an element is also the right inverse.

d) Prove that the centre of a group G is a normal subgroup of the group G.

e) If every element of a subgroup H of a subgroup G is its own inverse then prove that G must be an Abelian group.

f) A relation is defined by $ab^{-1} \in H \Leftrightarrow a \equiv b\ (mod\ H)$ where H is a subgroup of G. Write the three relations to be satisfied by '≡' in G.

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g) Express the permutation $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$ as product of transpositions.

h) Prove that any two right cossets of a subgroup H of a group G are either disjoint or identical.

i) Show that every subgroup of a cyclic group is normal.

j) Show that the mapping $g: C \rightarrow C$ such that $g(z) = \overleftarrow{z}, \overleftarrow{z}$ is conjugate of $z \in C$ is an isomorphism if C is the set of complex numbers.

k) Prove that a finite group G is a $p - group$ if and only if $O(G) = p^n$ for some positive integer $n \in N$, set of $+ve$ integer.

l) Let G be a group and let $g \in G$. Define $T_g: G \rightarrow G$ by $T_g(x) = gxg^{-1} \forall x \in G$. Show that T_g is an isomorphism.

(m) Show that $Aut\ (G)$ is a normal subgroup of $A(G)$.

n) Find a Sylow 2 −subgroups of S_3.

o) Show that arbitrary intersection of subgroups of a group G is again a subgroup of G.

4. Answer any two of the following: **6 x 2 = 12**

a) Show that a non empty set G equipped with a binary composition multiplication (•) forms a group if

i) (•) is associative and

ii) the two equations $ax = b; ya = b\ \forall a, b \in G$ have unique solutions in G.

b) Prove that

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \theta \in IR$$

forms a non abelian group under multiplication of martrices.

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e) Show that in a group $(G.^{\bullet})$

i) Identity element in a group is unique

ii) $(x^{-1})^{-1}x\forall x \in G$

iii) $(xy)^{-1} = y^{-1}.x^{-1} \quad \forall x, y \in G$

5. Answer any two of the following: **6 x 2 = 12**

a) Prove that a necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup of G is that $a \in H, b \in h \Rightarrow ab^{-1} \in H$.

b) Let H be a subgroup of a group G and let $N(H)$ be the normalize of H in G. Show that

i) H is normal in N(H)

ii) N(H) is the largest subgroup of G to which H is normal.

c) Let H and K be two subgroups of a group G. Show that the product HK is a subgroup of G iff $HK = KH$.

6. Answer any two of the following questions: **6 x 2 = 12**

a) Let H be a proper subgroup of a finite group G. Let $O(H) = n$ and $O(G) = m$, both m and n are positive integers. Show that $m = kn$ for some $+ve$ integer K.

b) Show that the Number of generators of an infinite cyclic group is precisely 2.

c) Show that every permutation cab be expressed as composite of disjoint cycles, each of length greater than or equal to 2.

7. Answer any two of the following: **6 x 2 = 12**

a) Prove that every group is isomorphic to a permutation group.

b) If $f: G \rightarrow G'$ be onto homomorphism with $K = \ker\ f$ then show that $\frac{G}{K} \cong G'$.

c) Prove that any two Sylow $p - subgroups$ of a finite group are conjugate in G.

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