DHANAMANJURI UNIVERSITY

JUNE 2022

Name of Programme	**:**	**B.Sc Mathematics (Honours)**
Semester	**:**	**II**
Paper Type	**:**	**Core III**
Paper Code	**:**	**CMA-103**
Paper Title	**:**	**Real Analysis**
Full Marks	**:**	**100**
Duration	**:**	**3 Hours**

The figures in the margin indicate full marks for the questions.
Answer all the questions:

1. State the completeness property of $\mathbb{R}$. Prove that if A and B are nonempty subsets of $\mathbb{R}$ that satisfy the property: $a \leq b$ for all $a \in A$ and all $b \in B$, then $\sup A \leq \inf B$. 4

2. Show that the series $\dfrac{1 \cdot 2}{3^2 \cdot 4^2} + \dfrac{3 \cdot 4}{5^2 \cdot 6^2} + \dfrac{5 \cdot 6}{7^2 \cdot 8^2} + \cdots$ is convergent. 4

3. Prove that a sequence cannot converge to more than one limit. 5

4. What is an absolutely convergent series? Prove that every absolutely convergent series is convergent. 5

5. Prove that an upper bound u of a nonempty set $S \subset \mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$. 6

6. Prove the following:

   i) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

   ii) $1 > 0$.

   iii) If $n \in \mathbb{N}$, then $n > 0$.

   6

7. Prove that the following statements are equivalent:

   i) S is a countable set.
   ii) There exists a surjection from $\mathbb{N}$ to S.
   iii) There exists an injection from S to $\mathbb{N}$.

   6

8. State and prove the Archimedean Property. 6

9. State and prove the Sandwich Theorem for limits. 6

10. Prove that a necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded. 6

11. State and prove Cantor's Theorem. 7

12. If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, prove that there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. 7

13. If $S \subset \mathbb{R}$ contains at least two points and has the property that if $x, y \in S$ and $x < y$, then $[x, y] \subseteq S$, prove that S is an interval. 8

    Or

    State and prove the Density Theorem for rational numbers.

14. State and prove the Bolzano-Weierstrass Theorem for sequences. 8

    Or

    State and prove Cauchy's General Principle of Convergence.

15. State and prove Leibnitz's Test for the convergence of an alternating series. 8

16. State and prove Cauchy's Root Test for the convergence of positive term series. 8