DHANAMANJURI UNIVERSITY
DECEMBER-2022

Name of Programme	:	B.A/B.Sc. Mathematics (Honours)
Semester	:	3rd
Paper Type	:	Core – VI (Theory)
Paper Code	:	CMA-206
Paper Title	:	Group Theory

Full Marks : 100
Pass Marks : 40 **Duration : 3 Hours**

The figures in the margin indicate full marks for the questions
Answer all the questions:

1. Answer the following: **4 x 1 = 4**

a) The number of distinct elements in the symmetric group S_n on n symbols is

i) n ii) $n!$
iii) $n-1$ iv) $(n-1)!$

b) Let G be a group and let H and K be two subgroup of G. The $H \cup K$ is also a subgroup of G if and only if:

i) both H and K are cyclic
ii) both H and K are abelian.
iii) $H \sqsubset K$ or $K \sqsubset H$
iv) H is cyclic or K is cyclic.

c) Let N be a normal subgroup of a finite group G. Then $O\left(\frac{G}{N}\right)$ is equal to

i) $\frac{O(G)}{O(N)}$ ii) $O(G) - O(N)$
iii) $O(G) + O(N)$ iv) $O(G) \times O(N)$

d) Number of Sylow $2-subgroups$ of S_3 is

i) 1 ii) 3
iii) 0 iv) 2

2. Answer the following: **12 x 1 = 12**

a) In the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication. Find the value of the product $(i.j).k$.

b) Define the dihedral group.

c) Write the permutation (3 4 5) in 2 – rowed notation using 6 symbols.

d) Define the centralizer of an arbitrary group G.

e) What is an index of a subgroup H of a group G.

f) For the group $(Z,^*)$ where $*$ is defined by
$a^*b = a + b + 1$
$\forall\, a, b\ \in Z$, find the identity element

g) Define a transposition and show that the identity permutation I is an even permutation.

h) Write the statement of the Fermat's Little theorem.

i) Define a quotient group.

j) When is a homomorphism f on groups is an automorphism?

k) Define Sylow's $p - subgroups$ of a group G.

l) Write the correct relationship among $A(G), Aut(G) and\ I(G)$ associated with an arbitrary group G.

3. Answer any 12 (Twelve) of the following questions: **12 x 3 = 36**

a) State and prove the cancellation laws in a group $(G,^*)$.

b) Let $G = \{-1, 1, i, -i\}, i = \sqrt{-1}$. Show that $(G, 0)$ forms a group by forming the composition table and under matrix multiplication.

c) Show that in a group G the left inverse of an element is also the right inverse.

d) Prove that the centre of a group G is a normal subgroup of the group G.

e) If every element of a subgroup H of a subgroup G is its own inverse then prove that G must be an Abelian group.

f) A relation is defined by $ab^{-1} \in H \Leftrightarrow a \equiv b\ (mod\ H)$ where H is a subgroup of G. Write the three relations to be satisfied by '$\equiv$' in G.

g) Express the permutation $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$ as product of transpositions.

h) Prove that any two right cossets of a subgroup H of a group G are either disjoint or identical.

i) Show that every subgroup of a cyclic group is normal.

j) Show that the mapping $g: C \to C$ such that $g(z) = \overleftarrow{z}, \overleftarrow{z}$ is conjugate of $z \in C$ is an isomorphism if C is the set of complex numbers.

k) Prove that a finite group G is a $p - group$ if and only if $O(G) = p^n$ for some positive integer $n \in N$, set of $+ve$ integer.

l) Let G be a group and let $g \in G$. Define $T_g: G \to G$ by $T_g(x) = gxg^{-1} \forall x \in G$. Show that T_g is an isomorphism.

(m) Show that $Aut\ (G)$ is a normal subgroup of $A(G)$.

n) Find a Sylow 2 –subgroups of S_3.

o) Show that arbitrary intersection of subgroups of a group G is again a subgroup of G.

4. Answer any two of the following: **6 x 2 = 12**

a) Show that a non empty set G equipped with a binary composition multiplication $(\bullet)$ forms a group if

i) $(\bullet)$ is associative and

ii) the two equations $ax = b; ya = b\ \forall a, b \in G$ have unique solutions in G.

b) Prove that

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \theta \in IR$$

forms a non abelian group under multiplication of martrices.