DHANAMANJURI UNIVERSITY
JUNE - 2024

Name of Programme	:	**B.A/B.Sc. Mathematics (Honours)**
Semester	:	**6^{th}**
Paper Type	:	**Core XIV (Theory)**
Paper Code	:	**CMA-314**
Paper Title	:	**Ring Theory and Linear Algebra**
Full Marks	:	**100**
Pass Marks	:	**40**
Duration	:	**3 Hours**

The figures in the margin indicate full marks for the questions

Answer all the questions:

1. Choose and rewrite the correct answer for each of the following: $1 \times 5 = 5$

a) The polynomial $f(x)=2x^2-2 \in Q[x]$ is

i) not primitive but reducible
ii) Primitive as well as reducible
iii) Primitive as well as irreducible
iv) not primitive as well as irreducible

b) If V(F) be a finite dimensional vector space over the field F of scalars and let W(F) be a subspace of V(F), then A(A(W)) is isomorphic to

i) $\widehat{W}$
ii) A(W)
iii) W
iv) $\widehat{A(W)}$

c) Let V(F) be a finite dimensional inner product space. The vector in V(F) which is orthogonal to each vector $x \in V(F)$ is

i) 1
ii) 0
iii) $\frac{x}{||x||}, x \neq 0$
iv) does not exist

d) Let R be an integral domain. Let $f(x), g(x) \in R[x]$ be such that deg $(f(x)) = m, deg(g(x)) = n$. Then deg $(f(x).g(x))$ is

i) less than $(m + n)$
ii) less than min (m, n)
iii) less than max $(m.n)$
iv) Equal to $(m + n)$

e) Let T be a linear operator on an inner product space V(F). Then T is a normal operator defined on V(F) if

i) $TT^* = I$ ii) $T = T^*$

iii) $T = T^{**}$ iv) $TT^* = T^*T$

If T^* is the adjoint operator of T and I, the identity operator

2. Answer the following questions: $1 \times 5 = 5$

a) Define a prime element in a commutative ring with unity.

b) Let a and b be two no zero elements in a Euclidean domain R. Write the condition for a,b to be relatively prime.

c) Define an eigen value λ of a linear operator T defined on a vector space V(F) and corresponding eigen vector.

d) When is a square matrix A of order nxn said to be diagonalizable?

e) State the Bessel's inequality in a inner product space V(F).

3. Answer any three of the following questions: $3 \times 10 = 30$

a) i) Show that every ideal of a Euclidean domain is a Principal ideal.

ii) Show that a Euclidean domain possesses unity and is also a Principal Ideal Domain (PID) 5+5=10

b) Show that any two non-zero elements a and b in a Euclidean domain R have a greatest common divisor D and it is possible to write D=λa+μb, λ, μϵ R. 10

c) Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Z[x]$. Suppose that for some prime number $p\frac{p}{a_0}, \frac{p}{a_1}, \ldots \frac{p}{a_{n-1}}, p \nmid a_n, p \nmid a_0$ then $f(x)$ is irreducible polynomial over Q, the ring of rationals.

d) i) If F is a Unique Factorisation Domain (UFO) and if $f(x), g(x) \epsilon F[x]$, then show that $c(fg) = c(f).c(g)$

ii) If $f(x)g(x)$ is primitive polynomial then show that $f(x)$ and $g(x)$ are primitives separately.

e) For any prime p, Show that the polynomial $x^{p-1} + x^{p-2} + \cdots x^2 + x + 1$ is irreducible over Q, the field of rational numbers.

4. Answer any three of the following questions: $3 \times 10 = 30$

a) Let {$V_1, V_2, \ldots. V_n$} be a basis of a vector space. Define $\hat{V}_i$: V(F)→F $\hat{V}_i\ ({}_{i=1}\Sigma^n a_i V_i) = a_i$ ∀ i=1,2,....,n Then show that {$\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n$} is a basis of $\hat{V}$. Hence dim V(F)= dim $\hat{V}$(F)

b) Let T be a linear operator on a finite dimensional vector space V(F). Show that c ϵ F is an eigen value of T if and only if T-cI is singular. Also, show that similar matrices have the same characteristic polynomials.

c) Show that every square matrix satisfies its characteristic equation.

d) Let V(F) be an inner product space. Show that $|(u,v)| \leq ||u|| \, ||v|| \; \forall$ u,vϵ V(F).

e) Obtain an orthonormal basis for V(IR), the space of all real polynomials of degree utmost 2, the inner product defined by $(f,g) = \int_0^1 f(x).g(x)dx$.

5. Answer any three of the following questions: **$3 \times 10 = 30$**

a) Let V(F) be an inner product. Show that

$(x, x) = ||x||^2$

i) $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in V(F)$

ii) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

b) Let V(F) be a finite dimensional inner product space and T be a linear operator on V(F). Show that there exists a unique linear operator T* such that (TV,V')= (V,T*V') ∀ V'ϵV(F).

c) Let V(F) be a finite dimensional inner product space and W is a subspace of V(F). Show that V(F)=W⊕ $W^{\perp}$, $W^{\perp}$ is the orthogonal complement of W.

d) i) Let W be the subspace of $IR^5(IR)$ spanned by the vectors

$\alpha = (2,-2,3,4,-1)$ and

$\beta = (0,0,-1,-2,3)$

Describe $A(W)$

ii) Let W_1 and W_2 be two subsets of a vector space $V(F)$. Show that $A(W_1) \sqsupset A(W_2)$ provided $W_1 \sqsubset W_2$ in $V(F)$.

e) Define:

$T: R^3 \rightarrow IR^2$ by

$T(x,y,z) = (x + y, 2z - x)$

If $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\beta' = \{(1,0), (0,1)\}$ are standard bases for IR^3 and IR^2, find $[T]_{\beta\beta'}$

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