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On the Observability of Nonlinear Systems: I

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I. INTRODUCTION

Recently, much interest has been shown [1, 2, 3] in the problem of estimating the states and parameters of a nonlinear dynamic system from a knowledge of its input and observed output on a time interval $[t_0, t_1]$. The nature of this problem motivates the study of observability of such systems. The observability problem is concerned with determining conditions under which a knowledge of the input-output data uniquely determines the state of the system.

For linear dynamic systems, the observability conditions are well known [4]. For nonlinear systems, local observability conditions are reported in [5]. It was Kostyukovskii [6, 7] who first reported general observability conditions for nonlinear systems. The present paper derives its inspiration from the work of Kostyukovskii and obtains results different from those of Kostyukovskii. The results of the present paper cover much more ground than those of [6, 7] and only part I is reported. Part II will be published shortly.

The paper is organized as follows. Definitions and the problem statement are discussed in Section II. Sections III, IV and V contain results on functional dependence and related topics. Necessary conditions and a sufficient condition are stated in Sections VI and VII. Several illustrative examples are included in Section VIII and the conclusions are in Section IX.

II. PROBLEM STATEMENT

Consider a nonlinear process described by the ordinary differential equation

$$\dot{x} = f(t, x, u), \qquad x(t_0) = x_0, \tag{1}$$

and the observation process described by

$$y = h(t, x) \tag{2}$$

where x is a n vector, y is a m vector, u is a r vector of inputs, $t \in [t_0, t_1]$, x_0 belongs to a set K of admissible initial conditions and u belongs to a set U of admissible inputs.

It is assumed that $f(\cdot)$ has continuous partial derivatives with respect to t, x and u up to and including order $n - 1$ and continuous mixed partial derivatives of the same order. Further, $u \in C^{n-1}$, $h(\cdot)$ has continuous partial derivatives with respect to t and x up to and including order n and continuous mixed partial derivatives of the same order.

A solution of (1) is denoted by $\varphi(t; x_0, u)$ and the corresponding observation $h(t, \varphi(t; x_0, u))$ by $y(t; x_0, u)$.

Given $u \in U$, denote by the set X the totality of all solution points $\varphi(t; x_0, u)$ for $t \in [t_0, t_1]$ $x_0 \in K$.

For $x_0 \in K$, $u \in U$, the solution $\varphi(t; x_0, u)$ is said to be observable with respect to K on $[t_0, t_1]$ if there does not exist another $x_0' \in K$ such that $y(t; x_0, u) \equiv y(t; x_0', u)$ on $[t_0, t_1]$.

The problem is to find necessary and sufficient conditions for a solution to be observable.

The nature of the problem is such that the details are quite involved. The reader is urged to consult the examples at the end of the paper as he reads through the theoretical development.

III. Functional Dependence

The discussion will be limited to a scalar observation process for the sake of simplicity. Extension to a vector observation process is straight-forward. Define

$$\begin{aligned}
F_0(t, x) &= h(t, x)\\
F_1(t, x, u) &= \frac{\partial F_0}{\partial t} + \left(\frac{\partial F_0}{\partial x}\right)' f(t, x, u)\\
F_2(t, x, \dot{u}) &= \frac{\partial F_1}{\partial t} + \left(\frac{\partial F_1}{\partial x}\right)' f(t, x, u) + \left(\frac{\partial F_1}{\partial u}\right)' \dot{u} \qquad (3)\\
&\vdots\\
F_n(t, x, u, \dot{u}, \ldots, u^{(n-1)}) &= \frac{\partial F_{n-1}}{\partial t} + \cdots + \left(\frac{\partial F_{n-1}}{\partial u^{(n-2)}}\right)' u^{(n-1)}
\end{aligned}$$

where $'$ denotes the transpose.

Let $v(t) = \mathrm{col}(t, u, \dot{u}, u^{(n-1)})$. This is a known function of time. Then, the functions $F_i(\cdot)$ in (3) correspond to $F_i(v(t), x)$, $i = 0, 1,\ldots, n$.

The problem is to find conditions at a point (t_0, x_0) under which $F_n(v(t), x)$ is functionally dependent on $F_0, F_1, \ldots, F_{n-1}$. To this end, the following $n \times n$ matrix is defined.

$$I_n(v(t), x) = \left\{\frac{\partial F_i(v(t), x)}{\partial x_j}\right\} \qquad \begin{matrix} i = 0, 1,\ldots, n-1 \\ j = 1, 2,\ldots, n \end{matrix}. \tag{4}$$

LEMMA 1. *$F_n(v(t), x)$ depends functionally on $F_0(v(t), x),\ldots, F_{n-1}(v(t), x))$ according to*

$$F_n(v(t), x) = \Phi(v(t), F_0(v(t), x),\ldots, F_{n-1}(v(t), x)) \qquad \text{for} \quad (t, x)$$

in a neighborhood of (t_0, x_0) which satisfies $|I_n(v(t_0), x_0)| \neq 0$.

Proof. Define the vector $z = \mathrm{col}(z_0, z_1, \ldots, z_{n-1})$ and the functions

$$p_i(v(t), x, z) = F_i(v(t), x) - z_i\,; \qquad i = 0, 1,\ldots, n-1 \tag{5}$$

Let (t_0, x_0) be fixed such that $|I_n(v(t_0), x_0)| \neq 0$. Let ω be $n \times 1$ constant vector with components

$$\omega_i = F_i(v(t_0), x_0); \qquad i = 0, 1,\ldots, n-1 \tag{6}$$

$$p_i(v(t_0), x_0, \omega) = F_i(v(t_0), x_0) - \omega_i = 0, \qquad i = 0, 1,\ldots, n-1 \tag{7}$$

The Jacobian determinant

$$\frac{\partial(p_0, \ldots, p_{n-1})}{\partial(x_1, \ldots, x_n)} = |I_n(v(t_0), x_0)| \neq 0 \tag{8}$$

by hypothesis. The above relation implies that x_0 is an isolated root of the algebraic system (6). Equations (7) and (8) imply, by the implicit function theorem, that there exists a continuously differentiable $n \times 1$ vector function $\psi(v(t), z)$ uniquely defined in a neighborhood $N(t_0, \omega)$ such that

$$x_0 = \psi(v(t_0), \omega)$$

and

$$p_i(v(t), \psi(v(t), z), z) \equiv 0 \tag{9}$$

on $N(t_0, \omega)$, $i = 1, 0,\ldots, n-1$.

Using (9) and

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial F_i}{\partial x_j}, \qquad \frac{\partial p_i}{\partial z_{j-1}} = \delta_{i,j-1} \qquad \begin{matrix} i = 0, 1,\ldots, n-1, \\ j = 1,\ldots, n \end{matrix}$$

we get

$$\left(\frac{\partial F_i}{\partial x}\right)'\left(\frac{\partial \psi}{\partial z_j}\right)=\delta_{i,j}\qquad \begin{matrix} i=0,1,\dots,n-1\\ j=0,1,\dots,n-1\end{matrix}$$

which implies

$$\frac{\partial\psi(v(t_0),\omega)}{\partial z}=I_n^{-1}(v(t_0),x_0). \tag{10}$$

The function $G(t,z)=\binom{t}{\psi(v(t),z)}$ where $G(t_0,\omega)=\binom{t_0}{x_0}$ maps $N(t_0,\omega)$ onto $N(t_0,x_0)$. This mapping is one to one, since

$$\frac{\partial(G_1,\dots,G_{n+1})}{\partial(t,z)}(t_0,\omega)=\left|\frac{\partial\psi(v(t_0),\omega)}{\partial z}\right|\neq 0$$

by (10). For each point $(t,z)\in N(t_0,\omega)$, Eq. (9) holds and the corresponding unique $(t,x)\in N(t_0,x_0)$ yields via (5)

$z_i=F_i(v(t),x);\qquad i=0,1,\dots,n-1\qquad$ which holds for $(t,x)\in N(t_0,x_0)$.

Thus, $F_n(v(t),x)=F_n(v(t),\psi(v(t),z))=\Phi(v(t),z)$. Finally,

$$F_n(v(t),x)=\Phi(v(t),F_0,F_1,\dots,F_{n-1})\qquad\text{for}\quad (t,x)\in N(t_0,x_0)$$

where $|I_n(v(t_0),x_0)|\neq 0$. The function Φ is continuously differentiable with respect to each argument.

Remarks. 1. Defining $D_n(v(t))=\{x\colon |I_n(v(t),x)|\neq 0\}$, it is not hard to show that the above functional dependence holds for $x\in D_n(v(t))$.

2. If $|I_n(v(t),x)|=0$, we define matrices, corresponding to (4) as follows:

$$I_{n-\alpha}^k(v(t),x)=\left\{\frac{\partial F_i(v(t),x)}{\partial x_j}\right\};\qquad \begin{matrix} i=0,1,\dots,n-\alpha-1\\ j=j_1,j_2,\dots,j_{n-\alpha}\end{matrix}$$

is a $(n-\alpha)\times(n-\alpha)$ matrix formed by using the top $(n-\alpha)$ rows and $(n-\alpha)$ columns of I_n. There are $\binom{n}{\alpha}$ such matrices $I_{n-\alpha}^k$; hence, $k=1,2,\dots\binom{n}{\alpha}$ and $\alpha=1,2,\dots,n-1$.

Correspondingly, we define

$$D_{n-\alpha}^k(v(t))=\left\{x: |I_{n-\alpha}^k|\neq 0 \text{ and } |I_{n-\alpha+1}^j|=0 \text{ for } j=1,2,\dots\binom{n}{\alpha-1}\right\}$$

and

$$D_{n-\alpha}=\bigcup_{k=1}^{M}D_{n-\alpha}^k\qquad\text{where}\qquad M=\binom{n}{\alpha}.$$

A lemma corresponding to Lemma 1 may now be stated.

LEMMA 2. *For each* $x \in D_{n-\alpha}(v(t))$, $F_{n-\alpha}(v(t), x)$ *depends functionally on* $F_0, F_1, \ldots, F_{n-\alpha-1}$ *according to*

$$F_{n-\alpha}(v(t), x) = \Phi(v(t), F_0(v(t), x), \ldots, F_{n-\alpha-1}(v(t), x)) \quad \text{for } \alpha = 1, 2, \ldots, n-1.$$

The proof is similar to that of Lemma 1 and hence is omitted.

IV. THE MULTIVALUED PROPERTY OF Φ

It was shown in section III that on $D_n(v(t)) = \{x: | I_n(v(t), x | \neq 0\}$ the following functional dependence holds.

$$F_n(v(t), x) = \Phi(v(t), F_0(v(t), x), \ldots, F_{n-1}(v(t), x)) \tag{11}$$

$\Phi(\cdot)$, considered as a function of x is uniquely defined for each $x \in D_n(v(t))$. However, for fixed t, $\Phi(\cdot)$ considered as a function of $F_0, \ldots, F_{n-1}$ is in general multivalued.

To see this, fix $t = t_0$ and consider the system

$$\omega = \mathrm{col}(F_0(v(t_0), x), \ldots, F_{n-1}(v(t_0), x)).$$

The set of roots of the above algebraic system in the n unknowns $x_1, x_2, \ldots, x_n$ is denoted by

$$P_n(v(t_0), \omega) = \{x : \omega = \mathrm{col}(F_0(v(t_0), x))\cdots\} \cap X \tag{12}$$

Points $x_0 \in P_n$ which satisfy $| I_n(v(t_0), x_0)| \neq 0$ are isolated. Hence, there can be at most a countable number of points $x_0{}^i$ in $P_n(v(t_0), \omega) \cap D_n(v(t_0))$.

Applying the functional dependence representation to each point $x_0{}^i \in P_n \cap D_n$, we get

$$F_n(v(t_0), x_0{}^i) = \Phi(v(t_0), \omega_0, \omega_1, \ldots, \omega_{n-1})$$

where, in general, the numbers $F_n(v(t_0), x_0{}^i)$ are not all equal. Thus, Φ evaluated at (t_0, ω) is multivalued.

At each (t_0, ω), define the single valued functions

$$\Phi^i(v(t_0), \omega_0, \ldots, \omega_{n-1})$$

which have the property

$$F_n(v(t_0), x_0{}^i) = \Phi^i(v(t_0), \omega_0, \ldots, \omega_{n-1}).$$

For fixed t, the functions Φ^i can be interpreted geometrically as continuously varying surfaces defined over the ω plane. Two surface Φ^j and Φ^{j+1} may meet for some points in the ω plane. Points of $D_n(v(t))$ for which these surfaces meet comprise the set $E_n{}^j(v(t))$ defined in the next section.

A simple argument shows that two of the Φ^i functions, $\Phi^j(v(t), \omega)$ and $\Phi^{j+1}(v(t), \omega)$ are equal at a point ω if and only if there exist two points $x_0{}^1$ and $x_0{}^2 \in D_n(v(t))$ such that

$$\operatorname{col}(F_0(v(t), x_0{}^1), \ldots, F_{n-1}(v(t), x_0{}^1)) = \operatorname{col}(F_0(v(t), x_0{}^2), \ldots, F_{n-1}(v(t), x_0{}^2))$$

and

$$F_n(v(t), x_0{}^1) = F_n(v(t), x_0{}^2). \tag{13}$$

V. The $E_n{}^j$ Sets

The previous section shows that for each (t, ω), there are functions $\Phi^i(v(t), \omega)$ some of which may have the same value.

A function Φ^k is isolated at (t, ω) if there are no other functions Φ^i at (t, ω) such that $\Phi^k(v(t), \omega) = \Phi^i(v(t), \omega)$. If $\Phi^k(v(t), \omega) = \Phi^i(v(t), \omega)$ at (t, ω), the function Φ^k is called a multiple function (nonisolated). A function Φ^k is a multiple function at (t, ω) if and only if the conditions (13) hold.

Let subsets $E_n{}^j(v(t))$ of $D_n(v(t))$ be defined as follows:

$$E_n{}^j(v(t)) = \bigcup_{\omega} E_n{}^j(v(t), \omega) \tag{14}$$

where

$$\begin{aligned} E_n{}^j(v(t), \omega) &= \{x^1 : \exists x_2 \ni \omega = \operatorname{col}(F_0(v, x^1), \ldots, F_{n-1}(v, x^1)) \\ &= \operatorname{col}(F_0(v, x^2), \ldots, F_{n-1}(v, x^2)) \end{aligned}$$

and

$$(\Phi^j(v(t), \omega) = F_n(v(t), x^1) = F_n(v(t), x^2)\} \cap D_n(v(t)).$$

Suppose a solution $\varphi(t; x_0, u)$ of (1) is in $E_n{}^j(v(t))$ for all $t \in [t_0, t_1]$. Then, at each t, there exists a $n \times 1$ vector $c(t)$ such that

$$\begin{aligned} \omega(t) &= \operatorname{col}(F_0(v(t), \varphi(t)), \ldots, F_{n-1}(v(t), \varphi(t))) \\ &= \operatorname{col}(F_0(v(t), c(t)), \ldots, F_{n-1}(v(t), c(t))) \end{aligned}$$

and

$$F_n(v(t), \varphi(t)) = F_n(v(t), c(t)) = \Phi^j(v(t), \omega(t)) \qquad \text{for all } t \in [t_0, t_1].$$

Clearly, $c(t) \in E_n{}^j(v(t)) \subset D_n(v(t))$ for all $t \in [t_0, t_1]$.

The vector function $c(t)$ is called a companion function to $\varphi(t; x_0, u)$. For a given $\varphi(t; x_0, u)$, there may be more than one companion function. The companion function need not be in $C'[t_0, t_1]$ even though Φ is.

Remark. This remark pertains to the case where we work with the sets $D_{n-\alpha}$ of section III instead of D_n. Corresponding to (14), we define

$$E^j_{n-\alpha}(v(t), \omega) = \{x^1 : \exists x^2 \ni \omega = \mathrm{col}(F_0(v, x^1), \ldots, F_{n-\alpha-1}(v, x^1)) \\ = \mathrm{col}(F_0(v, x^2), \ldots, F_{n-\alpha-1}(v, x^2))$$

and

$$\Phi^j(v(t), \omega) = F_{n-\alpha}(v(t)\, x^1) = F_{n-\alpha}(v(t), x^2)\} \cap D_{n-\alpha}(v(t)).$$

Further,

$$E^j_{n-\alpha}(v(t)) = \bigcup_{\omega} E^j_{n-\alpha}(v(t), \omega).$$

VI. Necessary Conditions

Theorem 1. *Let x_0^1 and x_0^2 be any points in $P_n(\omega, v(t_0)) \cap E_n^j(v(t))$. If $\varphi(t; x_0^1, u)$ is observable with respect to K on $[t_0, t_1]$, then both the solutions $\varphi(t; x_0^1, u)$* and *$\varphi(t; x_0^2, u)$ cannot be in $E_n^j(v(t))$ for all $t \in [t_0, t_1]$.*

Proof. Suppose $\varphi(t; x_0^i, u) \in E_n^j(v(t))$, $i = 1, 2$ for all $t \in [t_0, t_1]$. $\varphi(t; x_0^i, u) \in E_n^j(v(t))$ implies

$$F_n(v(t), \varphi(t; x_0^i)) = \Phi^j(v(t), F_0(v(t), \varphi(t; x_0^i)), \ldots, F_{n-1}(v(t), \varphi(t; x_0^i))); \qquad i = 1, 2.$$

By (3),

$$F_k(v(t), \varphi(t; x_0^i)) \triangleq y^{(k)}(t; x_0^i, u), \qquad k = 0, 1, \ldots, n.$$

Thus,

$$y^{(n)}(t; x_0^i) = \Phi^j(v, y, \dot{y}, \ldots, y^{(n-1)}), \qquad i = 1, 2 \qquad \text{for all } t \in [t_0, t_1]. \tag{15}$$

Thus, the two observations satisfy the same differential equation. The initial conditions for both are the same for, by hypothesis, $y^{(k)}(t_0; x_0^1) \triangleq F_k(v(t_0), x_0^1) = F_k(v(t_0), x_0^2) \triangleq y^{(k)}(t_0; x_0^2)$, $k = 0, 1, \ldots, n-1$. Also Φ^j is differentiable which implies uniqueness of the solution. Hence, any two solutions of (15) which have the same initial conditions must be identically equal. Thus, $\varphi(t; x_0^1, u)$ and $\varphi(t; x_0^2, u)$ yield the same observation. This means that $\varphi(t; x_0^1, u)$ is unobservable with respect to K on $[t_0, t_1]$. This contradiction of the statement of the theorem implies that the assumption $\varphi(t; x_0^i) \in E_n^j$ is false. The theorem is thus proven.

The following Corollary is a direct result of Theorem 1.

COROLLARY 1. *If x_0^i are points in $P_n(v(t_0), \omega) \cap E_n^j(v(t_0))$, and if the solutions $\varphi(t; x_0^i, u)$ are in $E_n^j(v(t))$ for $t \in [t_0, t_1]$, then the solutions $\varphi(t; x_0^i, u)$ are unobservable with respect to K on $[t_0, t_1]$, where $i = 1, 2$.*

The set of roots of the algebraic system

$$\omega = \mathrm{Col}(F_0(v(t), x),..., F_{n-\alpha-1}(v(t), x))$$

in the n unknowns $x_1, ..., x_n$ is denoted by

$$P_{n-\alpha}(v(t), \omega) = \{x : \omega = \mathrm{Col}(F_0(v(t), x),..., F_{n-\alpha-1}(v(t), x))\} \cap X \quad \text{for} \quad \alpha = 1, 2,..., n-1.$$

THEOREM 2. *Let x_0^1 qnd x_0^2 be any points in $P_{n-\alpha}(v(t_0), \omega) \cap E_{n-\alpha}^j(v(t_0))$, for $\alpha = 1, 2,..., n-1$. If $\varphi(t; x_0^1)$ is observable with respect to K on $[t_0, t_1]$, then both solutions $\varphi(t; x_0^1)$ and $\varphi(t; x_0^2)$ cannot be in $E_{n-\alpha}^j(v(t))$ for all $t \in [t_0, t_1]$.*

The proof is similar to that of Theorem 1 and hence is omitted.
The following Corollary is a direct result of Theorem 2.

COROLLARY 2. *If x_0^i are points*

$$P_{n-\alpha}(v(t_0), \omega) \cap E_{n-\alpha}^j(v(t_0)), \qquad \textit{for } \alpha = 1, 2,..., n-1,$$

and if the solutions $\varphi(t; x_0^i, u)$ are in $E_{n-\alpha}^j(v(t))$ for all $t \in [t_0, t_1]$, then the solutions $\varphi(t; x_0^i; u)$ are unobservable with respect to K on $[t_0, t_1]$ where $i = 1, 2$.

THEOREM 3. *Let $c(t)$ be a companion function of a solution*

$$\varphi(t; x_0^1, u) \in E_n^j(v(t)).$$

If $c(t)$ is differentiable on $[t_0, t_1]$, then it is a solution of $\dot{x} = f(t; x, u)$, and thus $\varphi(t; x_0^1, u)$ is unobservable with respect to K on $[t_0, t_1]$.

Proof. To show that $\dot{c}(t) = f(t, c(t), u)$, consider

$$F_k(v(t), \varphi(t)) = F_k(v(t), c(t)), \qquad k = 0, 1,..., n.$$

Differentiating both sides with respect to t, there results:

$$\left(\frac{\partial F_k(v, c)}{\partial x}\right)' (\dot{c} - f(t, c, u)) \equiv 0, \qquad k = 0, 1,..., n-1$$

which can be written in the form

$$I_n(v(t), c(t))(\dot{c} - f(t, c, u)) \equiv 0 \qquad \text{on} \quad [t_0, t_1].$$

But $c(t) \in D_n$ implies I_n is nonsingular. This implies $\dot{c} = f(t, c, u)$ and $[t_0, t_1]$ which implies, in turn, c is a solution of (1) and hence $\varphi(t, x_0^1, u)$ is unobservable.

VII. Sufficient Conditions

Define

$$P_{n+\alpha}(v(t), \omega) = \{x : \omega = \mathrm{Col}(F_0(v(t), x), \dots, F_{n+\alpha-1}(v(t), x))\} \cap X, \qquad \alpha = 0, 1, 2, \dots.$$

$\bar{P}_{n+\alpha}(v(t)) = \bigcup_\omega \{P_{n+\alpha}(v(t), \omega) : P_{n+\alpha}(v, \omega)$ contains more than one point$\}$.
Note that

$$\bar{P}_n(v(t)) \supset \bar{P}_{n+1} \supset \cdots \supset \bar{P}_{n+\alpha} \cdots$$

So that

$$\tilde{\bar{P}}_n \subset \tilde{\bar{P}}_{n+1} \subset \cdots \subset \tilde{\bar{P}}_{n+\alpha} \cdots$$

where $\tilde{\bar{P}}$ is the complement of $\bar{P}$.

Theorem 4. *For some α, $\alpha = 0, 1, 2, \dots$, if there exists a $t^* \in [t_0, t_1]$ such that the solution $\varphi(t^*) \in \tilde{\bar{P}}_{n+\alpha}(v(t^*))$, then φ is observable with respect to K on $[t_0, t_1]$.*

Proof. Assume $\varphi(t^*) \in \tilde{\bar{P}}_{n+\alpha}(v(t^*))$ and defined $x_0^1 = \varphi(t^*; x_0^1, u)$. It is now required to show that any $x_0^2 \in X$, $y(t; x_0^1) \not\equiv y(t; x_0^2)$. Choose any $x_0^2 \in X$ and denote the solution passing through x_0^2 at t^* by $\varphi(t; x_0^2, u)$. Note that $x_0^1 \in \tilde{\bar{P}}_{n+\alpha}(v(t^*))$. It cannot be true that

$$\mathrm{Col}(F_0(v(t^*), x_0^1), \dots, F_{n-1+\alpha}(\cdot)) = \mathrm{Col}(F_0(v(t^*), x_0^2), \dots, F_{n-1+\alpha}(\cdot))$$

for, if this did hold, then $x_0^1 \in \bar{P}_{n+\alpha}(v(t^*))$ which is contrary to the hypothesis. Thus, there is some k, $0 \leqslant k \leqslant n - 1 + \alpha$ such that $F_k(v(t^*), x_0^1) \neq F_k(v(t^*), x_0^2)$ or $y^{(k)}(t^*; x_0^1) \neq y^{(k)}(t^*; x_0^2)$. Assuming that $n - 1 + \alpha$ derivatives of y are continuous, there exists an interval about t^* on which $y^{(k)}(t; x_0^1) \neq y^{(k)}(t; x_0^2)$. It thus follows that $y(t; x_0^1) \not\equiv y(t; x_0^2)$ which implies $\varphi(t; x_0^1, u)$ is observable with respect to K on $[t_0, t_1]$.

VIII. Examples

Example 1. This is a counter-example to Kostyukovskii's sufficient condition.

Consider $\dot{x}_1 = x_1$, $\dot{x}_2 = 2x_2$, $y = x_1^2 + x_2^2$. Thus, $F_0(x) = x_1^2 + x_2^2$, $F_1(x) = 2x_1^2 + 4x_2^2$ and $F_2(x) = 4x_1^2 + 16x_2^2$.

$$I_2(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ 4x_1 & 8x_2 \end{bmatrix}$$

and $|I_2(x)| = 8x_1x_2$. Further, $I_1^1(x) = 2x_1$ and $I_1^2(x) = 2x_2$.

$$D_2 = \{x : |I_2(x)| \neq 0\}$$

which is the (x_1, x_2) plane excluding the x_1 and x_2 axes.

$$D_1^1 = \{x : |I_1^1(x)| \neq 0, |I_2(x)| = 0\} = \{x : x_1 \neq 0, x_2 \neq 0\}$$

Similarly,

$$D_1^2 = x_2 \text{ axis} - \{0\}.$$

Hence, $D_1 = D_1^1 \cup D_1^2$ which is the x_1 and x_2 axes excluding the origin. According to Kostyukovskii, all solutions in D_2 should be observable which is shown to be false. We solve $\omega_0 = x_1^2 + x_2^2$ and $\omega_1 = 2x_1^2 + 4x_2^2$ for x_1 and x_2. Thus,

$$x_1 = \pm(2\omega_0 - \tfrac{1}{2}\omega_1)^{1/2} \triangleq \pm A \qquad \text{and} \qquad x_2 = \pm(\tfrac{1}{2}\omega_1 - \omega_0)^{1/2} \triangleq \pm B.$$

The roots x_0^i are,

$$x_0^1 = \begin{pmatrix} A \\ B \end{pmatrix}, \qquad x_0^2 = \begin{pmatrix} -A \\ B \end{pmatrix},$$
$$x_0^3 = \begin{pmatrix} -A \\ -B \end{pmatrix} \qquad \text{and} \qquad x_0^4 = \begin{pmatrix} A \\ -B \end{pmatrix}. \tag{16}$$

$F_2(x_0^1) = F_2(x_0^2) = F_2(x_0^3) = F_2(x_0^4)$ implying that Φ is not a multivalued function of ω_0 and ω_1 for $x \in D_1$. A similar argument shows that Φ is a multivalued function of ω_0 for $x \in D_1$, taking on a value of $2\omega_0$ and $4\omega_0$. Let $\Phi^1(\omega_0) = 2\omega_0$ and $\Phi^2(\omega_0) = 4\omega_0$.

The set $E_2(\omega) = \{x^1 : \omega_0 = F_0(x)$ and $\omega_1 = F_1(x)$ has more than one solution x^1 and x^2 in D_2, and $F_2(x^1) = F_2(x^2)\}$. The set $E_2 = \bigcup_\omega E_2(\omega) = D_2$.

The sets $E_1^j(\omega^0) = \{x^1 : \omega_0 = F_0(x)$ has more than one solution x^1 and x^2 in D_1, and $F_1(x^1) = F_1(x^2) = \Phi^j(\omega_0)\}$, $j = 1, 2$. It follows that $E_1^1 = \bigcup_{\omega_0} E_1^1(\omega_0) = D_1^1$, and $E_1^2 = \bigcup_{\omega_0} E_1^2(\omega_0) = D_1^2$.

Solutions starting at any point in $E_1^j \cap P_1(\omega_0)$ remain in E_1^j for all t, $j = 1, 2$. By Corollary 2 solutions φ starting at these points yield the same output $y(t)$. Solutions in E_1^j are thus not observable, $j = 1, 2$.

By Corollary I, all solutions φ starting at the roots given by (16) yield the same output since these $\varphi \in E_2$ for all t. Thus, $\varphi \in E_2$ are not observable.

EXAMPLE 2. This is similar to the one considered by Detchmendy and

Sridhar [2] in investigating their nonlinear filter. The appropriate equations are, $\dot{x}_1 = x_2$, $\dot{x}_2 = -2x_1 - 3x_2 - x_1^3x_3$, $\dot{x}_3 = -x_3x_4$, $\dot{x}_4 = 0$ and $y = h(x) = x_1$. The problem is to determine $x_1(0)$, $x_2(0)$, $x_3(0)$ and $x_4(0)$ from a knowledge of $x_1(t)$ on the interval $[0, t_1]$.

In accordance with the theoretical development, the equations corresponding to (3) are,

$$\begin{aligned}
F_0(x) &= x_1 \\
F_1(x) &= x_2 \\
F_2(x) &= 2x_1 - 3x_2 - x_1^3x_3 \\
F_3(x) &= 6x_1 + 7x_2 + 3x_1^3x_3 + x_1^3x_3x_4 - 3x_1^2x_2x_3 \\
F_4(x) &= -14x_1 - 15x_2 - x_1^3x_3 + x_1^2x_3(3 + x_4)(3x_2 - x_1x_4) \\
&\quad + 3x_1^5x_3^2 + 3x_1x_2x_3(3x_1 - 2x_2 + x_1x_4)
\end{aligned}$$

The matrix corresponding to (4) is given by

$$I_4(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 - 3x_1^2x_3 & -3 & -x_1^3 & 0 \\ i_{41} & 7 - 3x_1^2x_3 & i_{43} & x_1^3x_3 \end{bmatrix}$$

where $i_{41} = 6(1 - x_1x_2x_3) + 3x_1^2x_3(3 + x_4)$ and $i_{43} = x_1^3(3 + x_4) - 3x_1^2x_2$. $|I_4(x)| = -x_1^6x_3$.

$$I_3^1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 - 3x_1^2x_3 & -3 & -x_1^3 \end{bmatrix} \quad \text{and} \quad |I_3^1(x)| = -x_1^3.$$

Note that $|I_3^2| = |I_3^3| = |I_3^4| = 0$, since the right hand column of each corresponding matrix contains all zeros.

$$I_2^1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; |I_2^1(x)| = 1 \quad \text{and} \quad |I_2^i(x)| = 0, \quad i \neq 1.$$

$$I_1^1(x) = 1 \quad \text{and} \quad I_1^i(x) = 0 \quad \text{for } i = 2, 3, 4.$$

$$D_4 = \{x : |I_4(x)| \neq 0\} = \{x : x_1 \neq 0 \text{ and } x_3 \neq 0\}$$

$$D_3^1 = \{x : |I_3^1(x)| = 0 \text{ and } |I_4(x)| = 0\} = \{x : x_1 \neq 0 \text{ and } x_3 = 0\}$$

$$D_3^i = \{x : |I_3^i(x)| \neq 0 \text{ and } |I_4(x)| = 0\} = \text{empty set} \quad \text{for } i = 1, 2, 3, 4.$$

$$D_3 = \bigcup_i D_3^i = D_3^1$$

$$D_2^1 = \{x : |I_2^1(x)| \neq 0 \text{ and } |I_3^i(x)| = 0 \text{ for } i = 1, 2, 3, 4\} = \{x : x_1 = 0\}$$

D_2^i are empty sets for $i \neq 1$.

$$D_2 = \cup D_2^i = D_2^1$$

D_1^1, D_1^2, D_1^3 and D_1^4 are empty sets and so is $D_1 = \bigcup D_1^i$.

It is easy to verify that Φ is not multivalued for $x \in D_4$, and $x \in D_3$ and $x \in D_2$. To see this examine the roots of

$$\begin{aligned}\omega_0 &= x_1\\ \omega_1 &= x_2\\ \omega_2 &= -2x_1 - 3x_2 - x_1{}^3x_3\\ \omega_3 &= 6x_1 + 7x_2 + x_1{}^3x_3(3 + x_4) - 3x_1{}^2x_2x_3\end{aligned} \tag{17}$$

with the constraints $x_1 \neq 0$ and $x_3 \neq 0$. The first two of the above equations determine x_1 and x_2. The third equation determines

$$x_3 = \tfrac{1}{2}(\omega_2 + 2\omega_0 + 3\omega_1) \neq 0$$

and the last one determines x_4. It is easy to see that for each $\omega = \mathrm{col}(\omega_0, \omega_1, \omega_2, \omega_3)$, there exists only one root x^0. Thus, Φ cannot be multivalued for $x \in D_4$.

For $x \in D_3$, a similar argument shows that the root x^0 for the algebraic system

$$\begin{aligned}\omega_0 &= x_1\\ \omega_1 &= x_2\\ \omega_2 &= -2x_1 - 3x_2 - x_1{}^3x_3\end{aligned} \tag{18}$$

under the constraint $x_1 \neq 0$ and $x_3 = 0$ is given by $x^0 = \mathrm{col}(\omega_0 \neq 0, \omega_1, 0, x_4 = \text{free})$. Thus, for $x \in D_3$, $\Phi(\omega)$ has a value of $6\omega_0 + 7\omega_1$ and is not multivalued. For $x \in D_2$, the root x^0 of

$$\omega_0 = x_1, \qquad \omega_1 = x_2 \tag{19}$$

under the constraint $x_1 = 0$ is given by $\mathrm{col}(0, \omega_1, x_3 = \text{free}, x_4 = \text{free})$.

The set E_4 is empty since (17) has only one root. $E_3(\omega) = \{x^1 : (18)$ has more than one root x^1 and x^2, and $F_3(x^1) = F_3(x^2)\}$ $E_3 = \bigcup E_3(\omega) = D_3$. $E_2 = D_2$ and E_1 is the empty set.

To determine which solutions are not observable, note that points of the form $\mathrm{col}(\omega_0 \neq 0, \omega_1, 0, x_4 = \text{free})$ are in $P_3(\omega) \cap E_3$ and solutions with these points as initial conditions remain in E_3 for all t. Thus, by Corollary 2, these solutions all result in the same output $y = x_1$.

Points of the form $\mathrm{col}(0, \omega_1, x_3 = \text{free}, x_4 = \text{free})$ are in $P_2(\omega) \cap E_2$. However, solutions with these initial conditions do not remain in E_2 for all t. To cause the solution to remain in E_2 for all t, choose initial conditions of the form $\mathrm{col}(0, 0, x_3 = \text{free}, x_4 = \text{free})$ and apply Corollary 2.

To summarize, initial conditions of the form $\mathrm{col}(x_{10} \neq 0, x_{20}, 0, x_{40})$ or $\mathrm{col}(0, 0, x_{30}, x_{40})$ result in unobservable solutions.

To apply the sufficient conditions, note that $P_4(\omega)$ is the set of points x such that (17) has more than one solution. Points in $\bar{P}_4(\omega)$ are of the form col $(0, \omega_1, x_3, x_4)$ or col$(\omega_0, \omega_1, 0, x_4)$ and thus $\bar{P}_4 = \bigcup P_4(\omega) = \{x : x_1 = 0$ or $x_3 = 0\}$ and $\bar{P}_4 = \{x : x_1 \neq 0$ and $x_3 \neq 0\} = D_4$. Any solution φ such that $\varphi(t^*) \in \bar{P}_4$ is observable with respect to R^4 on $[0, t_1]$.

IX. Conclusions

This paper has considered the important problems of observability of nonlinear systems. The nature of the problem is such that the details are quite involved. Necessary conditions and a sufficient condition for observability have been proven. The problem considered in this paper raises several questions. How meaningfully can one select an appropriate test input in order to make the system observable? This has implications in process identification methods. Is it possible to select inputs which make the unobservable part of the state space small in some sense? In otherwords, can we talk of degree of observability?

These and other problems are currently under investigation and will be reported soon.

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