INFORMATION AND CONTROL 22, 89–99 (1973)

# Observability of Nonlinear Systems*

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The purpose of this paper is to investigate the problem of observability of nonlinear systems. Two sufficient conditions of global observability of nonlinear systems are presented: (1) the ratio condition which is the generalization of Fujisawa and Kuh's (1971) ratio condition of circuit theory, (2) the strongly positive semidefinite condition. The relationships between these two conditions as well as the condition of positive definiteness of Fitts (1970) are given.

## 1. INTRODUCTION

In estimating the state of a dynamic system, the question of observability of the system becomes important. The observability problem for a system is to use the output information to determine uniquely the state of the system. Many authors have developed criteria of observability of linear systems. For nonlinear systems Kostyukovskii (1968a, 1968b) first gave conditions of global observability. But it was pointed out by Fitts (1970) and Griffith and Kumar (1971) that the sufficient condition of Kostyukovskii is incorrect.

In this paper we show that a theorem of circuit theory (Fujisawa and Kuh, 1971) and a theorem given by Berger and Berger (1968) can be generalized as sufficient conditions of global observability of nonlinear systems. The relation between the observability problem of a system and the univalence of the corresponding observability mapping is explained in Section 2. Section 3 contains the main results of sufficient conditions of observability and the mutual implications. Examples are included in Section 4 and the conclusions are in Section 5.

* This research was supported in part by the National Science Foundation's Grants GK-5570, GK-27873, GK-22905, and by the Air Force Office of Scientific Research under Grant No. F44620-69-C-0116.

## 2. Observability and Univalence

Before discussing the observability problem we give a description of the system as follows:

$$\dot{x} = f(t, x) \qquad f : [t_0, t_1] \times \Omega \subset E^1 \times E^n \to E^n \tag{1}$$

and the measurement equation

$$y = h(t, x) \qquad h : [t_0, t_1] \times \Omega \subset E^1 \times E^n \to E^m \tag{2}$$

where the state $x$ is not available for direct measurement. The initial state $x(t_0)$ is unknown and belongs to the set $\Omega_0 \subseteq \Omega$. Suppose that the $k$th order derivatives of $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ exist for every $x \in \Omega$ and for every $t \in [t_0, t_1]$ where $km \geqslant n$.

By the smoothness assumptions for $y(t)$ we can take Taylor series expansion of $y(t)$ as

$$\begin{aligned} y(t) &= y(t_0) + \dot{y}(t_0)(t - t_0) + (\ddot{y}(t_0)/2!)(t - t_0)^2 \\ &\quad + \cdots + (y^{(k-1)}(t_0)/(k-1)!)(t - t_0)^{k-1} + (y^{(k)}(t^*)/k!)(t - t_0)^k \end{aligned}$$

where $t^*$ is a certain instant between $t_0$ and $t$, and

$$\begin{aligned} y(t_0) &= h(x(t_0), t_0) \triangleq h_0(x(t_0), t_0) \\ \dot{y}(t_0) &= \frac{\partial h_0}{\partial t}(x(t_0), t_0) + \left(\frac{\partial h_0}{\partial x(t_0)}(x(t_0), t_0)\right) f(x(t_0), t_0) \triangleq h_1(x(t_0), t_0) \\ &\vdots \\ y^{(k-1)}(t_0) &= \frac{\partial h_{k-2}}{\partial t}(x(t_0), t_0) + \left(\frac{\partial h_{k-2}}{\partial x(t_0)}(x(t_0), t_0)\right) f(x(t_0), t_0) \triangleq h_{k-1}(x(t_0), t_0) \end{aligned}$$

These equations can be represented as a nonlinear map

$$z = H(x(t_0))$$

where

$$z = \begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \vdots \\ y^{(k-1)}(t_0) \end{bmatrix}, \qquad H(x(t_0)) = \begin{bmatrix} h_0(x(t_0), t_0) \\ h_1((t_0), t_0) \\ \vdots \\ h_{k-1}(x(t_0), t_0) \end{bmatrix}.$$

Let us call this nonlinear map $H$ the "observability mapping" of the system.

We will show that the observability problem of the system is closely related to the univalence of the observability mapping $H$. (A one-to-one mapping is said to be univalent.) At first we give the definition of global observability of nonlinear systems.

DEFINITION. The system described by (1) and (2) is said to be completely observable in $\Omega_0$ on the time interval $[t_0, t_1]$ if there exists a one-to-one correspondence between the set $\Omega_0$ of initial states and the set of trajectories of the observed output $y(t)$ for $t \in [t_0, t_1]$.

Now if the observability map $H$ is one-to-one from $\Omega_0$ to $H(\Omega_0)$, then by the data $z$, the initial state $x(t_0)$ of the system can be uniquely determined. Hence, according to the above definition of observability, the system is completely observable.

*Remark.* The univalence of map $H$ from $\Omega_0$ to $H(\Omega_0)$ is only a sufficient condition of observability for nonlinear continuous-time systems. The reason is that the vector $z$ shown before does not represent the whole trajectory $y(t)$, $t \in [t_0, t_1]$.

## 3. Main Results

DEFINITION. The $m$th leading principal minor of a square matrix $A$, denoted by $\Delta_m$, is defined to be the determinant of the matrix obtained by deleting the last $n-m$ columns and rows of $A$.

THEOREM 1. *Given* $H : E^n \to E^n$,

$$H \in D \text{ (Differentiable) with Jacobian matrix } J(x).$$

*If there exists a constant* $\epsilon > 0$ *such that the absolute values of the leading principal minors* $\Delta_1, \Delta_2, \ldots, \Delta_n$ *of* $J(x)$ *satisfy*

$$|\Delta_1| \geqslant \epsilon, \frac{|\Delta_2|}{|\Delta_1|} \geqslant \epsilon, \ldots, \frac{|\Delta_n|}{|\Delta_{n-1}|} \geqslant \epsilon$$

*for all* $x \in E^n$, *then* $H$ *is one-to-one from* $E^n$ *onto* $E^n$.

*Remark.* The main structure of the proof of Theorem 1 is similar to Fujisawa and Kuh (1971). A complete proof is given in the Appendix. The difference between Theorem 1 and the results of Fujisawa and Kuh (1971) is that in the assumption of this theorem, $H$ is only differentiable instead of assuming $H \in C'$ (continuously differentiable). It is shown that $H \in C'$ is not necessary in the proof of this theorem. The conditions on leading principal minors of Theorem 1 are called "ratio condition" by Fujisawa and Kuh (1971).

COROLLARY 1. *The system described by* (1) *and* (2) *is completely observable in the set* $\Omega_0$ *of initial states on the time interval* $[t_0, t_1]$ *if*:

(1) $km = n$ *where* $n$: *number of elements of the state*
$m$: *number of outputs*
$k$: $k$th *derivatives of* $f(\cdot, \cdot)$ *and* $h(\cdot, \cdot)$ *exist*;

(2) *the observability mapping of this system is differentiable*;

(3) *the Jacobian matrix of the observability mapping satisfies the ratio condition uniformly*.

*Remark.* This Corollary is actually the same as Theorem 1 but with a different statement.

Next we are going to extend the results of Theorem 1 to the case where the mapping $H$ is from $E^n$ space to $E^{km}$ space with $km \geqslant n$.

THEOREM 2. *Given* $H: E^n \to E^{km}$, $km \geqslant n$

$$H \in D \text{ with Jacobian matrix } J(x).$$

*If there exists a constant* $n \times km$ *matrix* $A$ *such that the* $n \times n$ *matrix* $AJ(x)$ *satisfies the ratio condition for all* $x \in E^n$, *then* $H$ *is one-to-one from* $E^n$ *onto* $H(E^n)$.

*Proof.* Since the $km \times n$ matrix $J(x)$ is the Jacobian matrix of $H(x)$, then $AJ(x)$, where $A$ is the constant $n \times km$ matrix satisfying the condition of the theorem, will be the Jacobian matrix of the mapping $AH(x)$ which consists of $n$ components. Now by the given assumptions and Theorem 1 we see that $AH(x)$ is a one-to-one mapping from $E^n$ onto $E^n$. We claim that this result implies that $H(x)$ is one-to-one from $E^n$ onto $H(E^n)$. Suppose $H$ is not one-to-one from $E^n$ onto $H(E^n)$. Then there exists at least two points $x_1, x_2$ of $E^n$ where $x_1 \neq x_2$ such that $H(x_1) = H(x_2)$. Now we multiply both sides by matrix $A$ and we have

$$AH(x_1) = AH(x_2).$$

Hence $AH$ is not one-to-one. So we proved that if $AH(x)$ is one-to-one from

$E^n$ onto $E^n$ then $H(x)$ is one-to-one from $E^n$ onto $H(E^n)$. We complete the proof.

*Remark* 1. In Theorem 2 if $H(x)$ is from $E^n \to E^n$ then it reduces to the special case of Theorem 1. Since $A$ equals the identity matrix in Theorem 2 then $AJ = J$ and Theorem 2 reduces to Theorem 1.

*Remark* 2. Let us restate the results of Fitts (1970) as follows

Given $H: E^n \to E^{km}$, $km \geqslant n$

$H \in D$ with Jacobian matrix $J(x)$

$\Omega$ is a convex subset of $E^n$.

If there exists a constant $n \times km$ matrix $A$ such that $AJ(x)$ *is positive definite* for all $x \in \Omega$, then $H(x)$ is one-to-one from $\Omega$ onto $H(\Omega)$.

Now this condition of positive definiteness guarantees that $AH(x)$ is univalent but not necessarily onto $E^n$. However, the ratio condition of Theorem 2 shows that $AH(x)$ is a univalent mapping of $E^n$ onto $E^n$. Furthermore, there is no implication between these conditions. That a matrix satisfies the ratio condition doesn't imply that it is positive definite (as illustrated in Example 1 of Section 4). The converse is not true either (see Example 2).

Now we restate Theorem 2 in terms of the observability property explicitly.

COROLLARY 2. *The system described by* (1) *and* (2) *is completely observable in the set* $\Omega_0$ *of the initial states on the time interval* $[t_0, t_1]$ *if*:

(1) $km \geqslant n$ *where* $k, m, n$ *are the same as that of Corollary* 1;

(2) *the observability mapping* $H(x)$ *is differentiable*;

(3) *there exists a constant* $n \times km$ *matrix* $A$ *such that* $AJ(x)$ *satisfies the ratio condition uniformly where* $J(x)$ *is the Jacobian matrix of* $H(x)$.

THEOREM 3. *Given* $H: \Omega \subset E^n \to E^{km}$, $km \geqslant n$,

$\Omega$ *is an open convex bounded subset of* $E^n$,

$H \in C'$ *with Jacobian matrix* $J(x)$.

*If there exists a constant* $n \times km$ *matrix* $A$ *such that the square matrix* $AJ(x)$ *satisfies* (1) $\det AJ(x) > 0$, *for all* $x \in \Omega$, and (2) $AJ(x) + (AJ(x))^T$ *has nonnegative principal minors for all* $x \in \Omega$ *where* $T$ *denotes transpose of a matrix, then* $H$ *is one-to-one from* $\Omega$ *onto* $H(\Omega)$.

*Proof.* $AJ(x)$ is the Jacobian matrix of $AH(x)$. By the application of Theorem 4-4 given by Berger and Berger (1968), $AH(x)$ is one-to-one from $\Omega$ onto $AH(\Omega)$. Similar to the proof of Theorem 2 we conclude that $H(x)$ is one-to-one from $\Omega$ onto $H(\Omega)$.

*Remark.* The hypotheses (1) and (2) will be called "*strongly positive semidefinite*" conditions.

From Theorem 3, we have the following sufficient condition for observability.

COROLLARY 3. *The system described by* (1) *and* (2) *is completely observable in the set* $\Omega_0$ *of initial states on the time interval* $[t_0, t_1]$ *if*:

(1) $km \geqslant n$ *where* $k$, $m$, $n$ *are the same as that of Corollary* 1;

(2) *the observability mapping* $H(x)$ *is continuously differentiable*;

(3) $\Omega$ *is an open bounded convex subset of* $E^n$;

(4) *there exists a constant* $n \times km$ *matrix* $A$ *such that* $AJ(x)$ *satisfies the strongly positive semidefinite condition for all* $x \in \Omega$ *where* $J(x)$ *is the Jacobian matrix of* $H(x)$.

*Remark* 1. If $A = B + C$, where $B$ is positive definite and $C$ is skew-symmetric, then $\det A \geqslant \det B$. (Bellman, 1970, p. 137). This can be easily proved by induction on $n$. It follows that Fitts' results imply strongly positive semidefinite. Nevertheless the converse is not true (see example 3 of Section 4).

*Remark* 2. From Examples (4) and (5) of Section 4 we see that the ratio condition does not imply the strongly positive semidefinite, and the strongly positive semidefinite does not imply the ratio condition.

## 4. EXAMPLES

To illustrate the relationships between the sufficient conditions which were obtained in Section 3 and the condition of positive definiteness given by Fitts (1970), we will show some examples in this section.

EXAMPLE 1. Consider the system

$$\dot{x}_1 = \tfrac{1}{2}x_1^2 + e^{x_2} + x_2$$

$$\dot{x}_2 = x_1^2,$$

with measurement equation

$$y = x_1 .$$

So the derivative of $y$ will be

$$\dot{y} = \dot{x}_1 = \tfrac{1}{2}x_1^2 + e^{x_2} + x_2 ,$$

the observability mapping $H(x)$ is

$$H(x) = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{2}x_1^2 + e^{x_2} + x_2 \end{bmatrix},$$

the Jacobian matrix of $H(x)$ is

$$J(x) = \begin{bmatrix} 1 & 0 \\ x_1 & e^{x_2} + 1 \end{bmatrix}$$

$$| \Delta_1 | = 1 \geqslant 1$$

$$| \Delta_2 |/| \Delta_1 | = | e^{x_2} + 1 | \geqslant 1 \qquad \text{for all} \quad x \in E^2.$$

So $J(x)$ satisfies ratio condition with $\epsilon = 1$ and let $A$ equal the identity matrix then the sufficient condition of Theorem 2 is satisfied.

*Claim.* For this $J(x)$, there is no constant $2 \times 2$ matrix $A$ such that $AJ(x)$ will be positive definite for all $x \in E^2$.

*Proof.* Let $A$ be a constant $2 \times 2$ matrix with entries undetermined:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

So

$$AJ(x) = \begin{bmatrix} a_{11} + a_{12}x_1 & a_{12}(e^{x_2} + 1) \\ a_{21} + a_{22}x_1 & a_{22}(e^{x_2} + 1) \end{bmatrix}.$$

The symmetric part of $AJ(x)$ is

$$AJ(x)_{\text{sym}} = \begin{bmatrix} a_{11} + a_{12}x_1 & \frac{1}{2}[(a_{21} + a_{22}x_1) + a_{12}(e^{x_2} + 1)] \\ \frac{1}{2}[(a_{21} + a_{22}x_1) + a_{12}(e^{x_2} + 1)] & a_{22}(e^{x_2} + 1) \end{bmatrix}$$

For $AJ(x)$ to be positive definite, it is necessary that $a_{11} + a_{12}x_1 > 0$ for all $x$. This requires that $a_{12} = 0$, $a_{11} > 0$. Also it is required that $\det AJ(x)_{\text{sym}}$ be greater than zero.

Since

$$\det(AJ(x))_{\text{sym}} = a_{11}a_{22}(e^{x_2} + 1) - \tfrac{1}{4}(a_{21} + a_{22}x_1)^2,$$

obviously $a_{22}$ has to be greater than zero. Now with these restrictions on $a_{11}$, $a_{22}$, and $a_{12}$, no matter what matrix $A$ will be, we can always find some $x_1$ such that the second term of det $AJ(x)_{\text{sym}}$ is greater than the first term. So for this $J(x)$ we can never find a constant $2 \times 2$ matrix $A$ such that $AJ(x)$ is positive definite for all $x \in E^2$. We complete the proof.

By this example we have shown that the sufficient condition of Theorem 2 does not necessarily imply the sufficient condition of Fitts (1970).

EXAMPLE 2. Consider the system

$$\dot{x}_1 = x_2 e^{-x_1}$$
$$\dot{x}_2 = x_1,$$

and the measurement equation is

$$y = e^{x_1}.$$

The observability mapping and the Jacobian matrix are as follows

$$H(x) = \begin{bmatrix} e^{x_1} \\ x_2 \end{bmatrix}, \qquad J(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that $J(x)$ satisfies the positive definite condition for all $x \in E^2$.

*Claim.* For this $J(x)$ there does not exist any $2 \times 2$ constant matrix $A$ such that $AJ(x)$ satisfies the ratio condition.

*Proof.* Let $A$ be a $2 \times 2$ constant matrix with entries to be determined, then

$$AJ(x) = \begin{bmatrix} a_{11}e^{x_1} & a_{12} \\ a_{21}e^{x_1} & a_{22} \end{bmatrix}.$$

We cannot find a proper $a_{11}$ such that there exists an $\epsilon > 0$ with $|a_{11}e^{x_1}| \geqslant \epsilon$ for all $x_1 \in E^1$. So there does not exist any $2 \times 2$ constant matrix $A$ such that $AJ(x)$ satisfies the ratio condition.

EXAMPLE 3. The system is given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + e^{x_2},$$

and the measurement equation

$$y = x_2.$$

The observability mapping $H(x)$ and the Jacobian matrix $J(x)$ are as follows

$$H(x) = \begin{bmatrix} x_2 \\ -x_1 + e^{x_2} \end{bmatrix}, \qquad J(x) = \begin{bmatrix} 0 & 1 \\ -1 & e^{x_2} \end{bmatrix}.$$

Let $A = I$ (identity matrix), it is easy to see that $J(x)$ satisfies the strongly positive semidefinite condition. So the system is observable.

It is trivial to check that no constant $2 \times 2$ matrix $A$ exists such that $AJ(x)$ will be positive definite for all $x \in \Omega$. Hence this system does not satisfy the Fitts' condition.

EXAMPLE 4. In fact, Example 1 may also be used as an example to show that the ratio condition does not imply the strongly positive semidefiniteness.

EXAMPLE 5. The Jacobian matrix of Example 2 is also strongly positive semidefinite. So it can be used as an example to show that the strongly positive semidefiniteness does not imply the ratio condition.

## 5. CONCLUSIONS

Two sufficient conditions of global observability of nonlinear systems have been presented. The generalization of Fujisawa and Kuh's ratio condition of circuit theory to the observability problem of control theory is shown in Theorem 1 and Corollary 1. Then the results are extended to more general nonlinear systems as in Theorem 2 and Corollary 2. The strongly positive semidefinite condition is obtained in Theorem 3 and Corollary 3 by the application of a theorem given by Berger and Berger. The relationships between ratio condition, strongly positive semidefinite and positive definite are given, which indicates that the positive definiteness is a weaker sufficient condition for observability of nonlinear systems than the strongly positive semidefiniteness of Theorem 3. Examples in Section 4 document the mutual relationships.

## APPENDIX

*Proof of Theorem* 1. By mathematical induction, $H: E^1 \to E^1$, $H \in D$ with Jacobian $dH/dx$ for $n = 1$, and by ratio condition $|\Delta_1| = |dH/dx| \geqslant \epsilon > 0$.

By the theorem of Zaring (1967) we have that $H$ is a strictly monotone mapping from $E^1$ onto $E^1$, so $H$ is one-to-one from $E^1$ onto $E^1$.

For $n = k - 1$, assume that the theorem is true.

For $n = k$, we are going to prove that $H$ is one-to-one from $E^k$ onto $E^k$ if $H: E^k \to E^k$, $H \in D$ with Jacobian $J_k(x)$ and ratio condition is satisfied uniformly.

In other words, under these conditions, we try to prove that for any given $y \in E^k$ there exists a unique point $x \in E^k$ such that $y = H(x)$ or

$$y_i = H_i(x), \qquad i = 1, 2, \ldots, k. \tag{4}$$

Let us introduce the following notations.

$$x_k^- = [x_1, \ldots, x_{k-1}]^T,$$

$$y_k^- = [y_1, \ldots, y_{k-1}]^T,$$

$$H_k^- = [H_1, \ldots, H_{k-1}]^T,$$

where $T$ denotes the transpose. Now the first $k - 1$ equations of (4) can be expressed as

$$y_k^- = H_k^-(x_k^-, x_k).$$

If the value of $x_k$ is kept fixed, the mapping $H_k^-$ of $E^{k-1}$ into itself is differentiable. Furthermore, the Jacobian matrix $J_{k-1}$ of $H_k^-$ satisfies the ratio condition uniformly. So $H_k^-$ is one-to-one from $E^{k-1}$ onto $E^{k-1}$. Thus, by global inverse function theory, $x_k^-$ can be represented as a function of $y_k^-$ and $x_k$ as follows:

$$x_k^- = g_k^-(y_k^-, x_k). \tag{5}$$

Substituting (5) into the $k$th equation of (4), $y_k$ is represented as a function of $y_k^-$ and $x_k$. The dependence of $y_k$ on $x_k$ can be determined in the following way provided that $y_k^-$ is kept fixed (Kuh and Hajj, 1971). The differentiation of (4) yields

$$dy = J_k \, dx,$$

or equivalently

$$\begin{bmatrix} dy_k^- \\ dy_k \end{bmatrix} = J_k \begin{bmatrix} dx_k^- \\ dx_k \end{bmatrix}.$$

Since $dy_k^- = O$ and $\det J_k \neq 0$, Cramer's rule can be used to obtain

$$dy_k/dx_k = \det J_k / \det J_{k-1}.$$

Again by ratio condition, and by the theorem of Zaring (1967), if $y_k^-$ is kept fixed, $y_k$ is a strictly monotone function of $x_k$ for $-\infty < x_k < \infty$. Furthermore, the range of $y_k$ covers the whole real line $-\infty < y_k < \infty$.

Now let $y^* = [y_1^*, y_2^*,..., y_k^*]^T$ be an arbitrary given point of $E^k$. For any value $x_k = s$ there is one and only one point

$$x(s) = [x_1(s), x_2(s),..., x_{k-1}(s), s]^T$$

such that

$$[y_1^*, y_2^*,..., y_{k-1}^*, y_k]^T = H(x(s)).$$

Now by the property of the dependence of $y_k$ on $x_k = s$, there exists one and only one value $s^*$ of $s$ for which the following relation holds:

$$y^* = H(x(s^*)).$$

Thus, for given arbitrary $y^* \in E^k$, there exists a unique $x(s^*) \in E^k$ such that $y^* = H(x(s^*))$. So $H$ is one-to-one from $E^k$ onto $E^k$. This completes the proof of Theorem 1.

RECEIVED: January 13, 1972

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