Communication

Tabulation of the Symbolic Midframe Jacobian of a Robot Manipulator

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Abstract

The symbolic midframe Jacobian for a general 6-DOF manipulator, a 12-DOF, 6-cylindric-jointed manipulator (6C) and a 24-DOF manipulator in which all four kinematic parameters of each link transform vary, admits to direct, manageable tabulation. For 6-link manipulators, the midframe Jacobians ${}^3\mathbf{J}_a$ and ${}^3\mathbf{J}_\alpha$ associated with rates of change in the link lengths a and twists α do not depend upon the kinematic parameters of the first link frame (d_1, θ_1, a_1, α_1). Similarly, the Jacobians ${}^3\mathbf{J}_d$ and ${}^3\mathbf{J}_\theta$ associated with the rates of change in link offsets d and rotations θ do not depend on the kinematic parameters of the sixth link frame (d_6, θ_6, a_6, α_6). A general property of a manipulator Jacobian, expressed in any frame, is that the column associated with the rate of change of a kinematic parameter is independent of that kinematic parameter. Furthermore, the determinant of the manipulator Jacobian is invariant with respect to its frame of representation, hence, the manipulator singularities may be calculated with relative ease in midframe. Finally, the complexity of the general form of ${}^3\mathbf{J}_a$ versus ${}^3\mathbf{J}_d$ and ${}^3\mathbf{J}_\alpha$ versus ${}^3\mathbf{J}_\theta$, as measured in types of symbols (+, −, ·, sines, cosines, parameters) and quantity of symbols of each type, is the same.

1. Introduction

Computation of the Jacobian is a critical requirement in many robot applications (Whitney 1972; Raibert and Craig 1981; Salisbury and Craig 1982; Orin and Schrader 1984). The manipulator Jacobian relates the joint rates to the end-effector linear and angular velocities. In the static case, the transpose of the Jacobian relates end-effector forces to joint torques. The singularities of the Jacobian indicate where the manipulator loses some "dexterity," velocity, and force control, in configuration space.

As part of his doctoral dissertation, Renaud (1980) demonstrated that the midframe Jacobian of a robot manipulator has a simpler symbolic form and requires fewer calculations to evaluate numerically than the Jacobian expressed in any other link frame. Renaud apparently developed a computer program to also generate the symbolic form of a midframe Jacobian, given the manipulator's kinematic parameters. Other researchers appear to have duplicated his effort and have their own symbolic Jacobian generators (Orin and Schrader 1984; Waldrin, Wang, and Bolin 1985).

Unfortunately, computer programs to handle symbolic equations, although available commercially, may not be easily accessible to some researchers. This paper makes available for the first time the complete symbolic expressions for the midframe Jacobian of an arbitrary 6-DOF manipulator (refer to the Appendix). Substitution of specific kinematic parameter values into the table expressions allows the user to easily compute the midframe Jacobian for any particular instance of a 6-DOF arm. The tables in the Appendix do much more than this, however, since they also provide the symbolic midframe Jacobian for six cylindric jointed manipulators (6C) and those manipulators in which all four kinematic parameters of a link transform vary. The latter situation is of interest in link parameter estimation and robot calibration (Hayati 1983; Wu and Lee 1985; Whitney, Lozinski, and Rourke 1986).

Typically, the joints of a serial kinematic chain are assumed to be prismatic or revolute. A cylindric joint supports prismatic and revolute action simultaneously. Tables A7–A12 in the Appendix depict the contributions to the midframe Jacobian due to prismatic action in the link offsets $\boldsymbol{d}$. Tables A21–A29 account for the contributions to the midframe Jacobian due to revolute action of the rotation angles θ. Together, the tables provide the symbolic midframe Jacobian for manipulators having as many as six cylindric joints (12 DOF).

The formulation presented here also permits the link

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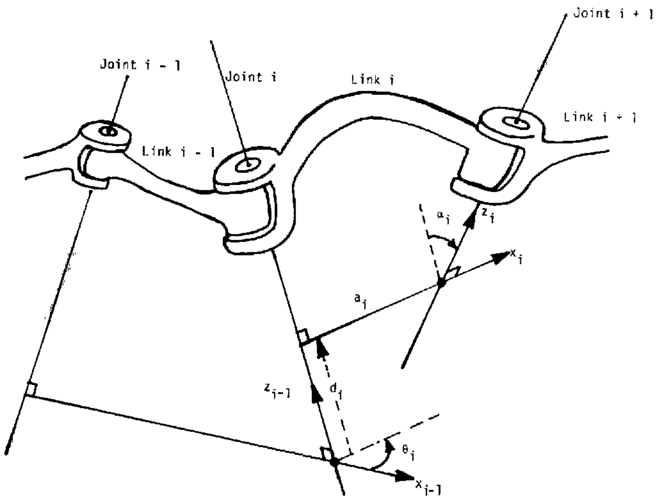
Fig. 1. Definitions of the DH kinematic parameters used in this paper.



length a and twist α to vary. Since a and α usually remain fixed in the standard Denavit–Hartenberg representation of a kinematic serial chain (Denavit and Hartenberg 1955; Paul 1981), allowing them to vary assumes either separate actuation control for those parameters or uncertainty in their values.

Appendix Tables A1–A6 and A13–A20 take care of the additional degrees of freedom induced by variations in a and α. The tabulation in the Appendix, therefore, provides the symbolic midframe Jacobian for manipulators with no more than six joints, whose structure may vary due to actuation or changes in any of the link kinematic parameters (24 DOF).

2. Link Frame Description

The homogeneous matrix ${}^{i-1}\mathbf{A}_i$ relates link i coordinate frame F_i to link $i-1$ coordinate frame F_{i-1} in a robot manipulator (Fig. 1) consisting of an open, serial kinematic chain (Denavit and Hartenberg 1955; Paul 1981),

$$ {}^{i-1}\mathbf{A}_i = \left[\begin{array}{c|c} {}^{i-1}\mathbf{R}_i & {}^{i-1}\mathbf{p}_i \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right]. \tag{1}$$

For link twist α_i let $\tau_i := \cos\alpha_i$ and $\sigma_i := \sin\alpha_i$, and for joint angle θ_i let $c_i := \cos\theta_i$, $s_i := \sin\theta_i$. Then

$$ {}^{i-1}\mathbf{R}_i = \begin{bmatrix} c_i & -\tau_i \cdot s_i & \sigma_i \cdot s_i \\ s_i & \tau_i \cdot c_i & -\sigma_i \cdot c_i \\ 0 & \sigma_i & \tau_i \end{bmatrix} \tag{2}$$

is the rotation matrix describing the relative orientation of F_i to F_{i-1}. Recall that the transpose $({}^{i-1}\mathbf{R}_i)^t = {}^i\mathbf{R}_{i-1}$ of a rotation matrix equals the inverse $({}^{i-1}\mathbf{R}_i)^{-1}$ and that ${}^i\mathbf{R}_k \cdot {}^k\mathbf{R}_m = {}^i\mathbf{R}_m$ and ${}^i\mathbf{R}_i = \mathbf{I}$, the identity transform.

Let a_i be the normal distance between the z-axes of the two frames F_{i-1} and F_i and d_i a displacement along $\mathbf{z}_{i-1}$, the z-axis of frame F_{i-1}. Then, in general,

$$ {}^{i-1}\mathbf{p}_i = [a_i \cdot c_i \quad a_i \cdot s_i \quad d_i]^t \tag{3}$$

describes the position of the F_i origin with respect to the F_{i-1} origin.

The z-axis, $\mathbf{z}_{i-1}$, of frame F_{i-1} coincides with joint i axis along which prismatic motion (d_i variable) or around which revolute motion (θ_i variable) takes place. The x-axis of frame F_i defines the direction of the linear motion generated by variations in a_i and the axis of rotation for α_i. In frame F_{i-1} this direction is $[c_i \quad s_i \quad 0]^t$.

2.1. Vector Notation

The representation of a vector (position, velocity, or acceleration) depends upon two coordinate frames: its frame of definition and its frame of expression. Two superscripts can describe this state of affairs. The vector ${}^{i,j}\mathbf{u}_k$ is expressed in frame F_i, designated by the first superscript, and defined in frame F_j, specified by the second superscript (Craig 1986). The superscript of definition is not affected by any coordinate transformation. Finally, the subscript k is used as an index, typically, to identify the head of the vector with a specific point, often the origin, in frame F_k.

The vector ${}^{i,j}\mathbf{v}_k$, for example, might designate the velocity of frame F_k origin as observed from frame F_j and expressed in frame F_i coordinates. If the frame of expression F_i and definition F_j are the same, the superscript j is often elided to simplify the notation: ${}^i\mathbf{v}_k := {}^{i,i}\mathbf{v}_k$. As a further illustration, the z-axis, $\mathbf{z}_i$, and x-axis, $\mathbf{x}_i$, vectors of the link frame F_i have representation ${}^i\mathbf{z}_i = [0 \quad 0 \quad 1]^t$ and ${}^i\mathbf{x}_i = [1 \quad 0 \quad 0]^t$.

2.2. Frame Velocity

The general velocity of a coordinate frame triad may be described by the six-component *frame velocity vector*

$$ {}^{i,0}\mathbf{V}_k := \begin{bmatrix} {}^{i,0}\mathbf{v}_k \\ {}^{i,0}\mathbf{w}_k \end{bmatrix}. \tag{4}$$

Featherstone (1984) calls this the "spatial velocity" of a rigid body. The first three components of a frame velocity vector describe the linear velocity of frame F_k origin with respect to the base frame, expressed in F_i coordinates. The remaining three components describe the angular velocity of frame F_k with respect to F_0, again expressed in F_i coordinates. When a frame is attached to a rigid body, the frame velocity may be interpreted as the body's linear and angular velocity components. The velocity of any point in the rigid body is the vector sum of the frame linear velocity with the cross product of the frame angular velocity and the position vector of the point with respect to the body frame.

3. Derivation of the Complete Manipulator Jacobian

This section relates the end-effector frame velocity to each of the link parameter velocities via the Jacobian.

3.1. Link Frame Velocity Vector

Perhaps one of the more direct ways of obtaining an expression for a manipulator Jacobian is to derive an expression for the link frame velocity and determine how that velocity transforms to other coordinate frames. The velocity ${}^{i,0}\mathbf{v}_i$ of the frame F_i origin with respect to the base frame F_0, expressed in frame F_i itself, is the sum of the linear velocity of the origin of frame F_{i-1}, ${}^{i,0}\mathbf{v}_{i-1}$, plus the translational velocity $\mathbf{v}_{ti}$ of frame F_i:

$$\mathbf{v}_{ti} = \dot{d}_i \cdot {}^{i,i-1}\mathbf{z}_{i-1} + \dot{a}_i \cdot {}^{i,i}\mathbf{x}_i, \tag{5}$$

due to the variation in link offset d_i and length a_i, plus the velocity ${}^{i,0}\mathbf{w}_i \times {}^{i,i-1}\mathbf{p}_i$ produced by the angular rotation of frame F_i with respect to the base frame:

$$\begin{aligned} {}^{i,0}\mathbf{v}_i &= {}^{i,0}\mathbf{v}_{i-1} + \dot{d}_i \cdot {}^{i,i-1}\mathbf{z}_{i-1} + \dot{a}_i \cdot {}^{i,i}\mathbf{x}_i + {}^{i,0}\mathbf{w}_i \times {}^{i,i-1}\mathbf{p}_i \\ &= {}^{i}\mathbf{R}_{i-1} \cdot ({}^{i-1,0}\mathbf{v}_{i-1} + \dot{d}_i \cdot {}^{i-1}\mathbf{z}_{i-1} \\ &\quad + \dot{a}_i \cdot {}^{i-1,i}\mathbf{x}_i) + {}^{i,0}\mathbf{w}_i \times {}^{i,i-1}\mathbf{p}_i, \end{aligned} \tag{6}$$

where

$${}^{i-1,i}\mathbf{x}_i = [c_i \quad s_i \quad 0]^t \tag{7}$$

and

$${}^{i,i-1}\mathbf{p}_i = {}^{i}\mathbf{R}_{i-1} \cdot {}^{i-1}\mathbf{p}_i = [a_i \quad \sigma_i \cdot d_i \quad \tau_i \cdot d_i]^t. \tag{8}$$

Equation (6) portrays the simultaneous velocity effects on frame F_i when all the kinematic parameters vary.

The angular velocity ${}^{i,0}\mathbf{w}_i$ of frame F_i with respect to the base frame F_0 and expressed in F_i coordinates may be written as the sum of the angular velocity of frame F_{i-1} with respect to the base frame and the angular velocity induced by $\dot{\theta}_i$ about the z-axis of frame F_{i-1} and the rate of twist $\dot{\alpha}_{i-1}$ about the x-axis of frame F_{i-1}:

$$\begin{aligned} {}^{i,0}\mathbf{w}_i &= {}^{i,0}\mathbf{w}_{i-1} + \dot{\theta}_i \cdot {}^{i,i-1}\mathbf{z}_{i-1} + \dot{\alpha}_{i-1} \cdot {}^{i,i-1}\mathbf{x}_{i-1} \\ &= {}^{i}\mathbf{R}_{i-1} \cdot ({}^{i-1,0}\mathbf{w}_{i-1} + \dot{\theta}_i \cdot {}^{i-1}\mathbf{z}_{i-1} + \dot{\alpha}_{i-1} \cdot {}^{i-1}\mathbf{x}_{i-1}). \end{aligned} \tag{9}$$

The cross product in Eq. (6), after a substitution from Eq. (9), may be expressed in the matrix form

$$\begin{aligned} {}^{i,0}\mathbf{w}_i \times {}^{i,i-1}\mathbf{p}_i &= -{}^{i,i-1}\mathbf{p}_i \times {}^{i,0}\mathbf{w}_i \\ &= -{}^{i,i-1}\mathbf{P}_i \cdot {}^{i}\mathbf{R}_{i-1} \cdot ({}^{i-1,0}\mathbf{w}_{i-1} \\ &\quad + \dot{\theta}_i \cdot {}^{i-1}\mathbf{z}_{i-1} + \dot{\alpha}_{i-1} \cdot {}^{i-1}\mathbf{x}_{i-1}), \end{aligned} \tag{10}$$

where

$${}^{i,i-1}\mathbf{P}_i := \begin{bmatrix} 0 & -\tau_i \cdot d_i & \sigma_i \cdot d_i \\ \tau_i \cdot d_i & 0 & -a_i \\ -\sigma_i \cdot d_i & a_i & 0 \end{bmatrix} \tag{11}$$

is the skew-symmetric matrix, $({}^{i,i-1}\mathbf{P}_i)^t = -{}^{i,i-1}\mathbf{P}_i$, whose operation is equivalent to the vector cross product operation $({}^{i,i-1}\mathbf{p}_i \times \cdots)$.

Equations (6), (9), and (10) allow us to express the velocity of the origin of frame F_i as a linear transformation of the linear and angular velocities of frame F_{i-1}. This recursive formulation constitutes the primary computational step in determining the symbolic form of the manipulator Jacobian. Numerical computations of the Jacobian employ these equations as well. This approach extends similar developments described in Orin and Schrader (1984), Renaud (1981), and Waldron et al. (1985). The following paragraphs elaborate further.

3.2. Frame Velocity Transformation

If $\mathbf{\Phi}$ is defined as a 3×3 matrix of zero elements, then the *frame velocity transformation*

$${}^{i}\mathbf{G}_{i-1} := \begin{bmatrix} {}^{i}\mathbf{R}_{i-1} & -{}^{i,i-1}\mathbf{P}_i \cdot {}^{i}\mathbf{R}_{i-1} \\ \mathbf{\Phi} & {}^{i}\mathbf{R}_{i-1} \end{bmatrix} \tag{12}$$

helps to succinctly combine the velocity transformations in Eqs. (6) and (9) as the frame (spatial) velocity transformation

$${}^{i,0}\mathbf{V}_i = {}^{i}\mathbf{G}_{i-1} \cdot \left({}^{i-1,0}\mathbf{V}_{i-1} + \begin{bmatrix} \dot{a}_i \cdot {}^{i-1,i}\mathbf{x}_i + \dot{d}_i \cdot {}^{i-1}\mathbf{z}_{i-1} \\ \dot{\alpha}_{i-1} \cdot {}^{i-1}\mathbf{x}_{i-1} + \dot{\theta}_i \cdot {}^{i-1}\mathbf{z}_{i-1} \end{bmatrix} \right). \tag{13}$$

The inverse of Eq. (13),

$$^{i-1}\mathbf{G}_i := ({}^i\mathbf{G}_{i-1})^{-1} = \begin{bmatrix} {}^{i-1}\mathbf{R}_i & {}^{i-1}\mathbf{R}_i \cdot {}^{i,i-1}\mathbf{P}_i \\ \mathbf{\Phi} & {}^{i-1}\mathbf{R}_i \end{bmatrix}, \quad (14)$$

may be verified by showing that the product of Eqs. (12) and (14) produces the identity transform **I**. Since ${}^{i,i}\mathbf{P}_i = \mathbf{\Phi}$ follows directly from its definition and ${}^i\mathbf{R}_i = \mathbf{I}$, then ${}^i\mathbf{G}_i = \mathbf{I}$.

Successive application of frame velocity transformations suggest the definitions

$${}^k\mathbf{G}_i := {}^k\mathbf{G}_{k-1} \cdot {}^{k-1}\mathbf{G}_{k-2} \cdots {}^{i-2}\mathbf{G}_{i-1} \cdot {}^{i-1}\mathbf{G}_i, \qquad k > i, \quad (15)$$

and

$${}^k\mathbf{G}_i := {}^k\mathbf{G}_{k+1} \cdot {}^{k+1}\mathbf{G}_{k+2} \cdots {}^{i-2}\mathbf{G}_{i-1} \cdot {}^{i-1}\mathbf{G}_i, \qquad k < i. \quad (16)$$

One can show that the determinant or ${}^k\mathbf{G}_i$, $\det[{}^k\mathbf{G}_i]$, is 1 for any k and i. From Eqs. (12), (14)–(16), and direct computation, the frame velocity transforms satisfy the property

$${}^m\mathbf{G}_k \cdot {}^k\mathbf{G}_i := {}^m\mathbf{G}_i, \qquad m,k,i \geq 0. \quad (17)$$

Since the base frame of the manipulator is assumed fixed, ${}^{i,0}\mathbf{v}_0 = \mathbf{0}$. Recursive application of Eq. (13), starting at the base frame, $i - 1 = 0$, and working out to the end-effector frame, $i = n$, and repeated utilization of Eq. (17) yields the frame velocity vector at the end-effector

$$\begin{aligned} {}^{n,0}\mathbf{V}_n = \sum_{i=1}^{n} {}^n\mathbf{G}_{i-1} \cdot \begin{bmatrix} \dot{d}_i \cdot {}^{i-1}\mathbf{z}_{i-1} \\ \dot{\theta}_i \cdot {}^{i-1}\mathbf{z}_{i-1} \end{bmatrix} \\ + \sum_{i=1}^{n} {}^n\mathbf{G}_i \cdot \begin{bmatrix} \dot{a}_i \cdot {}^i\mathbf{x}_i \\ \dot{\alpha}_i \cdot {}^i\mathbf{x}_i \end{bmatrix}. \end{aligned} \quad (18)$$

The subscript i on the frame velocity transform in the last term of Eq. (18) reflects the fact that the variations in the link length a_i and twist α_i act along and about the x-axis of frame F_i directly and thus may be expressed more simply in that frame than in the previous frame F_{i-1}.

3.3. The Manipulator Jacobian When All Link Parameters Vary

Define the generalized coordinate displacement vectors associated with the link parameters as

$$\boldsymbol{a} := [a_1 \quad a_2 \quad \cdots \quad a_n]^t, \quad (19a)$$

$$\boldsymbol{d} := [d_1 \quad d_2 \quad \cdots \quad d_n]^t, \quad (19b)$$

$$\boldsymbol{\alpha} := [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^t, \quad (19c)$$

$$\boldsymbol{\theta} := [\theta_1 \quad \theta_2 \quad \cdots \quad \theta_n]^t. \quad (19d)$$

Define unit vectors

$$\boldsymbol{u}_1 := [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^t, \quad (20a)$$

$$\boldsymbol{u}_3 := [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^t, \quad (20b)$$

$$\boldsymbol{u}_4 := [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]^t, \quad (20c)$$

$$\boldsymbol{u}_6 := [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^t, \quad (20d)$$

associated with the parameter displacements, and the vectors

$${}^n\boldsymbol{a}_i := {}^n\mathbf{G}_i \cdot \mathbf{u}_1, \qquad i = 1, \ldots, n, \quad (21a)$$

$${}^n\boldsymbol{d}_i := {}^n\mathbf{G}_{i-1} \cdot \mathbf{u}_3, \qquad i = 1, \ldots, n, \quad (21b)$$

$${}^n\boldsymbol{\alpha}_i := {}^n\mathbf{G}_i \cdot \mathbf{u}_4, \qquad i = 1, \ldots, n, \quad (21c)$$

$${}^n\boldsymbol{\theta}_i := {}^n\mathbf{G}_{i-1} \cdot \mathbf{u}_6, \qquad i = 1, \ldots, n. \quad (21d)$$

Then Eq. (18) may be cast into the form

$$\begin{aligned} {}^{n,0}\mathbf{V}_n &= [{}^n\mathbf{a}_1 \quad {}^n\mathbf{a}_2 \quad \cdots \quad {}^n\mathbf{a}_n]\, \dot{\boldsymbol{a}} \\ &\quad + [{}^n\mathbf{d}_1 \quad {}^n\mathbf{d}_2 \quad \cdots \quad {}^n\mathbf{d}_n]\, \dot{\boldsymbol{d}} \\ &\quad + [{}^n\boldsymbol{\alpha}_1 \quad {}^n\boldsymbol{\alpha}_2 \quad \cdots \quad {}^n\boldsymbol{\alpha}_n]\, \dot{\boldsymbol{\alpha}} \\ &\quad + [{}^n\boldsymbol{\theta}_1 \quad {}^n\boldsymbol{\theta}_2 \quad \cdots \quad {}^n\boldsymbol{\theta}_n]\, \dot{\boldsymbol{\theta}} \\ &= {}^n\mathbf{J}_a \cdot \dot{\boldsymbol{a}} + {}^n\mathbf{J}_d \cdot \dot{\boldsymbol{d}} + {}^n\mathbf{J}_\alpha \cdot \dot{\boldsymbol{\alpha}} + {}^n\mathbf{J}_\theta \cdot \dot{\boldsymbol{\theta}}, \end{aligned} \quad (22)$$

where

$${}^n\mathbf{J}_a := [{}^n\boldsymbol{a}_1 \quad {}^n\boldsymbol{a}_2 \quad \cdots \quad {}^n\mathbf{a}_n], \quad (23)$$

$${}^n\mathbf{J}_d := [{}^n\boldsymbol{d}_1 \quad {}^n\boldsymbol{d}_2 \quad \cdots \quad {}^n\mathbf{d}_n] \quad (24)$$

are the manipulator Jacobian transformations that relate the prismatic motion $\dot{\boldsymbol{a}}$ and $\dot{\boldsymbol{d}}$ to the frame velocity vector, and

$${}^n\mathbf{J}_{\boldsymbol{\alpha}} := [{}^n\boldsymbol{\alpha}_1 \quad {}^n\boldsymbol{\alpha}_2 \quad \cdots \quad {}^n\boldsymbol{\alpha}_n], \quad (25)$$

$${}^n\mathbf{J}_{\boldsymbol{\theta}} := [{}^n\boldsymbol{\theta}_1 \quad {}^n\boldsymbol{\theta}_2 \quad \cdots \quad {}^n\boldsymbol{\theta}_n] \quad (26)$$

are manipulator Jacobian transformations relating the contribution of the revolute angular velocities $\dot{\boldsymbol{\alpha}}$ and $\dot{\boldsymbol{\theta}}$ to the frame velocity vector.

If we choose to express the joint coordinates as a $4n$ component vector, $2n$ prismatic and $2n$ revolute coordinates, and write the manipulator Jacobian as

$$ {}^{n}\mathbf{J} := [{}^{n}\mathbf{J}_a \quad {}^{n}\mathbf{J}_d \quad {}^{n}\mathbf{J}_\alpha \quad {}^{n}\mathbf{J}_\theta], \tag{27} $$

we obtain the form

$$ {}^{n,0}\mathbf{V}_n = {}^{n}\mathbf{J} \begin{bmatrix} \dot{a} \\ \dot{d} \\ \dot{\alpha} \\ \dot{\theta} \end{bmatrix}. \tag{28} $$

The columns of ${}^{n}\mathbf{J}$ may be computed symbolically or numerically through the use of Eqs. (21).

4. Coordinate Transformation of the Manipulator Jacobian

To express the frame velocity vector, Eq. (18), in another frame, simply multiply by the appropriate **G** matrix:

$$ \begin{aligned} {}^{m,0}\mathbf{V}_n &= {}^{m}\mathbf{G}_n \cdot {}^{n,0}\mathbf{V}_n \\ &= {}^{m}\mathbf{G}_n \cdot {}^{n}\mathbf{J}_a \cdot \dot{a} + {}^{m}\mathbf{G}_n \cdot {}^{n}\mathbf{J}_d \cdot \dot{d} \\ &\quad + {}^{m}\mathbf{G}_n \cdot {}^{n}\mathbf{J}_\alpha \cdot \dot{\alpha} + {}^{m}\mathbf{G}_n \cdot {}^{n}\mathbf{J}_\theta \cdot \dot{\theta}. \end{aligned} \tag{29} $$

Equation (29) indicates that the manipulator Jacobian transforms according to

$$ {}^{m}\mathbf{J} = {}^{m}\mathbf{G}_n \cdot {}^{n}\mathbf{J} \tag{30} $$

or, in general,

$$ {}^{m}\mathbf{J} = {}^{m}\mathbf{G}_k \cdot {}^{k}\mathbf{J}, \qquad m, k \in (0, \ldots, n). \tag{31} $$

The Jacobian transformation law expressed in Eq. (31) allows us to compute the Jacobian in any convenient reference frame of the manipulator. For example, to compute the Jacobian in frame k, substitute k for n in Eqs. (21).

Renaud (1980) provides an argument for the minimal, symbolic complexity of the midframe Jacobian. His argument may be extended to demonstrate the minimal complexity of the midframe Jacobian of a manipulator in which all the kinematic parameters of each link are allowed to vary. Direct generation of the symbolic midframe Jacobian for such manipulators is, indeed, quite manageable.

5. Symbolic Computation of the Jacobian Column Vectors

An explicit computation of ${}^{i}\mathbf{G}_{i-1}$ requires both

$$ {}^{i}\mathbf{R}_{i-1} = \begin{bmatrix} c_i & s_i & 0 \\ -\tau_i \cdot s_i & \tau_i \cdot c_i & \sigma_i \\ \sigma_i \cdot s_i & -\sigma_i \cdot c_i & \tau_i \end{bmatrix} \tag{32} $$

and

$$ -{}^{i}\mathbf{P}_i \cdot {}^{i}\mathbf{R}_{i-1} = \begin{bmatrix} -d_i \cdot s_i & d_i \cdot c_i & 0 \\ a_i \cdot \sigma_i \cdot s_i - d_i \cdot \tau_i \cdot c_i & -d_i \cdot \tau_i \cdot s_i - a_i \cdot \sigma_i \cdot c_i & a_i \cdot \tau_i \\ a_i \cdot \tau_i \cdot s_i + d_i \cdot \sigma_i \cdot c_i & d_i \cdot \sigma_i \cdot s_i - a_i \cdot \tau_i \cdot c_i & -a_i \cdot \sigma_i \end{bmatrix}. \tag{33} $$

From Eqs. (21), (31), and the relation ${}^{k}\mathbf{G}_k = \mathbf{I}$ for any k,

$$ {}^{i}\boldsymbol{a}_i = \boldsymbol{u}_1, \tag{34a} $$

$$ {}^{i-1}\boldsymbol{d}_i = \boldsymbol{u}_3, \tag{34b} $$

$$ {}^{i}\boldsymbol{\alpha}_i = \boldsymbol{u}_4, \tag{34c} $$

$$ {}^{i-1}\boldsymbol{\theta}_i = \boldsymbol{u}_6 \tag{34d} $$

for $n \geqslant i \geqslant 1$. Applying the frame velocity transform ${}^{i+1}\mathbf{G}_i$ to ${}^{i}\mathbf{a}_i$ *and* ${}^{i}\boldsymbol{\alpha}_i$ transforms them from frame F_i coordinates to frame F_{i+1} coordinates. Applying ${}^{i}\mathbf{G}_{i-1}$ to ${}^{i-1}\mathbf{d}_i$ and ${}^{i-1}\boldsymbol{\theta}_i$ transforms these Jacobian column vectors from frame F_{i-1} coordinates to frame F_i coordinates:

$$ \begin{aligned} {}^{i+1}\mathbf{a}_i &= {}^{i+1}\boldsymbol{G}_i \cdot \boldsymbol{u}_1 \\ &= [c_{i+1} \quad -\tau_{i+1} \cdot s_{i+1} \quad \sigma_{i+1} \cdot s_{i+1} \quad 0 \quad 0 \quad 0]^t, \end{aligned} \tag{35a} $$

$$ {}^{i}\mathbf{d}_i = {}^{i}\boldsymbol{G}_{i-1} \cdot \boldsymbol{u}_3 = [0 \quad \sigma_i \quad \tau_i \quad 0 \quad 0 \quad 0]^t, \tag{35b} $$

$$ {}^{i+1}\boldsymbol{\alpha}_i = {}^{i+1}\boldsymbol{G}_i \cdot \boldsymbol{u}_4 = \begin{bmatrix} -d_{i+1} \cdot s_{i+1} \\ a_{i+1} \cdot \sigma_{i+1} \cdot s_{i+1} - d_{i+1} \cdot \tau_{i+1} \cdot c_{i+1} \\ a_{i+1} \cdot \tau_{i+1} \cdot s_{i+1} + d_{i+1} \cdot \sigma_{i+1} \cdot c_{i+1} \\ c_{i+1} \\ -\tau_{i+1} \cdot s_{i+1} \\ \sigma_{i+1} \cdot s_{i+1} \end{bmatrix}, \tag{35c} $$

$$ {}^{i}\theta_i = {}^{i}\boldsymbol{G}_{i-1} \cdot \boldsymbol{u}_6 = [0 \quad a_i \cdot \tau_i \quad -a_i \cdot \sigma_i \quad 0 \quad \sigma_i \quad \tau_i]^t. \tag{35d} $$

Observe that ${}^{i+1}\mathbf{a}_i$, ${}^{i}\mathbf{d}_i$, ${}^{i+1}\boldsymbol{\alpha}_i$, ${}^{i}\theta_i$ do not depend upon their associated parameter, a_i, d_i, α_i, and θ_i, respectively. This means that a coordinate transformation on the unit vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_3$, $\boldsymbol{u}_4$, and $\boldsymbol{u}_6$ to an even more distal frame produces Jacobian column vectors which are not a function of the parameter variables a_i, d_i, α_i, or θ_i associated with the unit vector.

Applying the frame velocity transform ${}^{i-1}\boldsymbol{G}_i$ to ${}^{i}\boldsymbol{a}_i$ and ${}^{i}\boldsymbol{\alpha}_i$

transforms them from frame F_i coordinates to frame F_{i-1} coordinates. Applying ${}^{i-2}\boldsymbol{G}_{i-1}$ to ${}^{i-1}\boldsymbol{d}_i$ and ${}^{i-1}\theta_i$ transforms these Jacobian column vectors from frame F_{i-1} coordinates to frame F_{i-2} coordinates:

$$ {}^{i-1}\boldsymbol{a}_i = [c_i \quad s_i \quad 0 \quad 0 \quad 0 \quad 0]^t, \tag{36a}$$

$$ {}^{i-2}\boldsymbol{d}_i = [\sigma_{i-1} \cdot s_{i-1} \quad -\sigma_{i-1} \cdot c_{i-1} \quad \tau_{i-1} \quad 0 \quad 0 \quad 0]^t, \tag{36b}$$

$$ {}^{i-1}\boldsymbol{\alpha}_i = [-d_i \cdot s_i \quad d_i \cdot c_i \quad 0 \quad c_i \quad s_i \quad 0]^t, \tag{36c}$$

$$ {}^{i-2}\boldsymbol{\theta}_i = \begin{bmatrix} a_{i-1} \cdot \tau_{i-1} \cdot s_{i-1} + d_{i-1} \cdot \sigma_{i-1} \cdot c_{i-1} \\ d_{i-1} \cdot \sigma_{i-1} \cdot s_{i-1} - a_{i-1} \cdot \tau_{i-1} \cdot c_{i-1} \\ -a_{i-1} \cdot \sigma_{i-1} \\ \sigma_{i-1} \cdot s_{i-1} \\ -\sigma_{i-1} \cdot c_{i-1} \\ \tau_{i-1} \end{bmatrix}. \tag{36d}$$

Observe in Eqs. (36) that a coordinate transformation on the unit vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_3$, $\boldsymbol{u}_4$, and $\boldsymbol{u}_6$ to a more proximal frame produces Jacobian column vectors which are not a function of the parameter variables a_i, d_i, α_i, or θ_i associated with the unit vector. This observation and the previous one established for Eqs. (35) are, in effect, algebraic simplification theorems which reduce the complexity of the symbolic expressions for the Jacobian when transformed to other frames. In summary, column ${}^k\boldsymbol{s}_j \in \{{}^k\boldsymbol{d}_j, {}^k\boldsymbol{\theta}_j, {}^k\boldsymbol{a}_j, {}^k\boldsymbol{\alpha}_j\}$ in the Jacobian ${}^k\boldsymbol{J}$ cannot be a function of the associated kinematic parameter $s_j \in \{d_j, \theta_j, a_j, \alpha_j\}$ for any k or j.

Compare Eqs. (35) and (36). For each set of four columns the overall complexity, as measured in the number of symbols or numerical operators, is the same. Should only d_i and θ_i vary, however, then the adjacent distal frame columns are simpler. Similarly, should only a_i and α_i vary, the adjacent proximal frame expressions are simpler.

6. The Symbolic Midframe Jacobian

The midframe Jacobian ${}^3\mathbf{J}$ for a completely general arm with six link frames, wherein the four kinematic parameters associated with each frame may vary, may be symbolically computed from the coordinate transformations

$$ {}^3\boldsymbol{a}_i = {}^3\boldsymbol{G}_i \cdot {}^i\boldsymbol{a}_i, \qquad i = 1, \ldots, 6, \tag{37a}$$

$$ {}^3\boldsymbol{d}_i = {}^3\boldsymbol{G}_i \cdot {}^i\boldsymbol{d}_i, \qquad i = 1, \ldots, 6, \tag{37b}$$

$$ {}^3\boldsymbol{\alpha}_i = {}^3\boldsymbol{G}_i \cdot {}^i\boldsymbol{\alpha}_i, \qquad i = 1, \ldots, 6, \tag{37c}$$

$$ {}^3\boldsymbol{\theta}_i = {}^3\boldsymbol{G}_i \cdot {}^i\boldsymbol{\theta}_i, \qquad i = 1, \ldots, 6. \tag{37d}$$

Twelve of the 24 columns have already been computed, namely, for $i = 2$, 3, and 4 in Eqs. (37a) and (37c) and for $i = 3$, 4, and 5 in Eqs. (37b) to (37d). The recursive nature of the Jacobian column generation allows the remaining terms to be computed as follows.

Let ${}^k\boldsymbol{s}_j$ be representative of any of the column vectors ${}^k\boldsymbol{a}_j$ or ${}^k\boldsymbol{\alpha}_j$. From Eqs. (35) the vector ${}^2\boldsymbol{s}_1$ is known; hence,

$$ {}^3\boldsymbol{s}_1 = {}^3\boldsymbol{G}_2 \cdot {}^2\boldsymbol{s}_1 \tag{38}$$

is a simple calculation.

Similarly, calculate

$$ {}^3\boldsymbol{s}_5 = {}^3\boldsymbol{G}_4 \cdot {}^4\boldsymbol{s}_5, \tag{39}$$

where ${}^4\boldsymbol{s}_5$ is obtained directly from Eqs. (36). Adding 1 to each index in the resulting expression converts ${}^3\boldsymbol{s}_5$ to ${}^4\boldsymbol{s}_6$, which then can be transformed to

$$ {}^3\boldsymbol{s}_6 = {}^3\boldsymbol{G}_4 \cdot {}^4\boldsymbol{s}_6. \tag{40}$$

Equations (38) to (40) yield six additional Jacobian columns. A similar development for ${}^k\boldsymbol{d}_j$ and ${}^k\theta_j$ produces another six columns, bringing the total to 24.

The computations indicated by Eqs. (34) to (40) and those suggested in the previous sentence yield the columns of the Jacobians ${}^3\boldsymbol{J}_a$, ${}^3\boldsymbol{J}_d$, ${}^3\boldsymbol{J}_\alpha$, *and* ${}^3\boldsymbol{J}_\theta$. These symbolic computations were performed both manually and by a PASCAL program written specifically for this purpose. The midframe Jacobian columns for an arbitrary 6-link manipulator are displayed in the Appendix (Tables A1–A6, Tables A7–A12, Tables A13–A20, Tables A21–A29, respectively). Notice that ${}^3\boldsymbol{J}_a$ and ${}^3\boldsymbol{J}_\alpha$ do not depend upon any of the kinematic parameters in ${}^0\boldsymbol{A}_1$, whereas ${}^3\boldsymbol{J}_d$ and ${}^3\boldsymbol{J}_\theta$ do not depend on any of the kinematic parameters in ${}^5\boldsymbol{A}_6$.

7. Application of the Jacobian Tables

The symbolic equations listed in the tables allow the user to directly generate the symbolic midframe Jacobian for any general manipulator with $n \leq 6$ degrees of freedom and the end-effector frame Jacobian for $n \leq 3$. The symbolic midframe Jacobian for manipulators describable by $n \leq 6$ link frames and 24 or less degrees of freedom can also be obtained from the tables.

To compute the midframe Jacobian

$$ {}^3\boldsymbol{J} = [{}^3\boldsymbol{J}_a \quad {}^3\boldsymbol{J}_d \quad {}^3\boldsymbol{J}_\alpha \quad {}^3\boldsymbol{J}_\theta] \tag{41}$$

of a robot manipulator, the user selects the appropriate tables associated with those parameters that vary and substitutes the robot's kinematic parameters.

7.1. Example: PUMA 260 Robot

Kinematic parameters for the PUMA 260 series robot in symbolic form are given in Table 1.

Table 1. PUMA 260 Kinematic Parameters

Joint	d	θ	a	α
1	0	θ_1	0	90°
2	0	θ_2	a_2	0°
3	d_3	θ_3	0	−90°
4	d_4	θ_4	0	90°
5	0	θ_5	0	−90°
6	0	θ_6	0	0°

All the joints of the PUMA 260 are revolute, and the kinematic parameters $\boldsymbol{d}$, $\boldsymbol{a}$, and $\boldsymbol{\alpha}$, are assumed constant. Only the Appendix Tables A21–A29 apply. Substituting the PUMA parameter values of Table 1 into these symbolic equations yields and PUMA 260 midframe Jacobian ${}^3\boldsymbol{J} = [{}^3\boldsymbol{\theta}_1\ {}^3\boldsymbol{\theta}_2\ {}^3\boldsymbol{\theta}_3\ {}^3\boldsymbol{\theta}_4\ {}^3\boldsymbol{\theta}_5\ {}^3\boldsymbol{\theta}_6]$:

$$
{}^3\boldsymbol{J} = \begin{bmatrix} d_3 \cdot c_{2+3} & a_2 \cdot s_3 & 0 & 0 & d_4 \cdot c_4 & d_4 \cdot s_4 \cdot s_5 \\ a_2 \cdot c_2 & 0 & 0 & 0 & d_4 \cdot s_4 & -d_4 \cdot c_4 \cdot s_5 \\ -d_3 \cdot s_{2+3} & a_2 \cdot c_3 & 0 & 0 & 0 & 0 \\ s_{2+3} & 0 & 0 & 0 & s_4 & -c_4 \cdot s_5 \\ 0 & -1 & -1 & 0 & -c_4 & -s_4 \cdot s_5 \\ c_{2+3} & 0 & 0 & 1 & 0 & c_5 \end{bmatrix} \tag{42}
$$

where $c_{2+3} := \cos(\theta_2 + \theta_3)$ and $s_{2+3} := \sin(\theta_2 + \theta_3)$.

If the offset d_3 and twist α_2 tend to vary significantly under load conditions, the Jacobian in Eq. (42) would be augmented by the columns ${}^3\boldsymbol{d}_3 = [0\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0]^t$ (Table A9) and ${}^3\boldsymbol{\alpha}_2 = [-d_3 \cdot s_3\ \ 0\ \ -d_3 \cdot c_3\ \ c_3\ \ 0\ \ -s_3]^t$ (Table A14) to account for the increased degrees of freedom. Columns 1 and 2 must also be modified to account for the fact that α_2 is no longer fixed at zero in the expressions found in Tables A21–A24.

$$
{}^3\boldsymbol{\theta}_1 = \begin{bmatrix} -a_2 \cdot \sigma_2 \cdot c_2 + d_3 \cdot (\tau_2 \cdot c_2 \cdot c_3 - s_2 \cdot s_3) \\ a_2 \cdot \tau_2 \cdot c_2 \\ a_2 \cdot \sigma_2 \cdot c_2 - d_3 \cdot (\tau_2 \cdot c_2 \cdot s_3 + s_2 \cdot c_3) \\ \tau_2 \cdot c_2 \cdot s_3 + s_2 \cdot c_3 \\ \sigma_2 \cdot c_2 \\ \tau_2 \cdot c_2 \cdot c_3 - s_2 \cdot s_3 \end{bmatrix}, \tag{43}
$$

$$
{}^3\boldsymbol{\theta}_2 = \begin{bmatrix} a_2 \cdot \tau_2 \cdot s_3 + \sigma_2 \cdot d_3 \cdot c_3 \\ a_2 \cdot \sigma_2 \\ a_2 \cdot \tau_2 \cdot c_3 - \sigma_2 \cdot d_3 \cdot s_3 \\ \sigma_2 \cdot s_3 \\ -\tau_2 \\ \sigma_2 \cdot c_3 \end{bmatrix}. \tag{44}
$$

7.2. Numeric Computation of the Jacobian

Once the symbolic midframe Jacobian for a particular manipulator has been determined from the tables, a subroutine can be written to numerically evaluate the Jacobian for any particular values of the kinematic parameters. Such specialized numeric computations will often increase the computational speed. For example, Orin and Schrader (1984) indicate that the numerical evaluation of the midframe Jacobian for a general 6-DOF arm requires about 93 floating-point multiplications. In contrast, the symbolic midframe for the PUMA 260 only requires 11 multiplications. This result is typical of commercial manipulators.

7.3. Manipulator Singularities

The solutions to the equation $\det[{}^n\boldsymbol{J}] = 0$ furnish the manipulator singularities. Since

$$
\det[{}^n\boldsymbol{J}] = \det[{}^n\boldsymbol{G}_k \cdot {}^k\boldsymbol{J}] = \det[{}^n\boldsymbol{G}_k] \cdot \det[{}^k\boldsymbol{J}] = \det[{}^k\boldsymbol{J}], \tag{45}
$$

$0 \leq k \leq n$, the manipulator singularities may be calculated in any convenient frame. Frame velocity transformations applied to a manipulator Jacobian leave the Jacobian determinant invariant. Although reassuring, this result should not come as a surprise, since the physical structure of the manipulator determines its singularities and, as such, should be independent of how the Jacobian is represented. Gorla and Renaud (1984) have tabulated the singularities of many industrial robots.

The minimal complexity of the midframe Jacobian suggests solving det $[{}^3\boldsymbol{J}] = 0$ to find the manipulator singularities for a 6-DOF arm. Tabulation of det $[{}^3\boldsymbol{J}]$ for the general case does not appear manageable, or even desirable, and therefore should be left on a case-by-case basis. For most commercial manipulators the midframe Jacobian matrix is relatively sparse, and its determinant may be computed symbolically with relative ease.

In any case, if the link length a_6 and twist α_6 are fixed, the kinematic parameters of frame 6 cannot influence the manipulator singular points. Similarly, if the offset d_1 and rotation θ_1 are fixed, the kinematic parameters of frame 1 cannot have any bearing on the manipulator singularities. These statements also hold for the pseudo-inverse of the midframe Jacobian generated when all 24 parameters of a 6-link frame manipulator vary.

8. Conclusion

The symbolic midframe Jacobian for a general 6-DOF manipulator or a 24-DOF, 6-link manipulator admits to a manageable tabulation. The Appendix furnishes such a tabulation. For 6-link manipulators, the midframe Jacobians ${}^3\boldsymbol{J}_a$ and ${}^3\boldsymbol{J}_\alpha$ do not depend upon d_1, θ_1, a_1, α_1; and ${}^3\boldsymbol{J}_d$ and ${}^3\boldsymbol{J}_\theta$ do not depend on d_6, θ_6, a_6, and α_6.

An interesting, general property of manipulator Jacobians is that column ${}^k\boldsymbol{s}_j \in \{{}^k\boldsymbol{d}_j, {}^k\theta_j, {}^k\boldsymbol{a}_j, {}^k\alpha_j\}$ in the Jacobian ${}^k\boldsymbol{J}$ cannot be a function of the associated kinematic parameter $s_j \in \{d_j, \theta_j, a_j, \alpha_j\}$ for any k or j. Furthermore, the determinant of a manipulator Jacobian remains invariant under a frame velocity transformation on the Jacobian.

The availability of these tables should assist application engineers, teachers, students, and researchers in the robotics community who do not have easy access to a symbolic processor and would profit from the insight provided by a symbolically rendered manipulator Jacobian or the improved computational speed resulting from direct calculation of the Jacobian from its symbolic form. Since the symbolic midframe Jacobian for most commercial manipulators is quite sparse, the Jacobian determinant and the midframe inverse are often easily attainable in symbolic form as well. Analysis of manipulator singularities then becomes tractable for most commercial manipulators.

Acknowledgment

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Appendix: Symbolic Midframe Jacobian Tables

Table A1. Column 1 of ${}^3J_a$

j	${}^3\boldsymbol{a}_1[j]$
1	$c_2 \cdot c_3 - \tau_2 \cdot s_2 \cdot s_3$
2	$-c_2 \cdot \tau_3 \cdot s_3 - \tau_2 \cdot s_2 \cdot \tau_3 \cdot c_3 + \sigma_2 \cdot s_2 \cdot \sigma_3$
3	$c_2 \cdot \sigma_3 \cdot s_3 + \tau_2 \cdot s_2 \cdot \sigma_3 \cdot c_3 + \sigma_2 \cdot s_2 \cdot \tau_3$
4	0
5	0
6	0

Table A2. Column 2 of ${}^3J_a$

j	${}^3\boldsymbol{a}_2[j]$
1	c_3
2	$-\tau_3 \cdot s_3$
3	$\sigma_3 \cdot s_3$
4	0
5	0
6	0

Table A3. Column 3 of ^3J_a

j	$^3a_3[j]$
1	1
2	0
3	0
4	0
5	0
6	0

Table A4. Column 4 of ^3J_a

j	$^3a_4[j]$
1	c_4
2	s_4
3	0
4	0
5	0
6	0

Table A5. Column 5 of ^3J_a

j	$^3a_5[j]$
1	$c_4 \cdot c_5 - \tau_4 \cdot s_4 \cdot s_5$
2	$s_4 \cdot c_5 + \tau_4 \cdot c_4 \cdot s_5$
3	$\sigma_4 \cdot s_5$
4	0
5	0
6	0

Table A6. Column 6 of ^3J_a

j	$^3a_6[j]$
1	$c_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) - \tau_4 \cdot s_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) + \sigma_4 \cdot s_4 \cdot \sigma_5 \cdot s_6$
2	$s_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) + \tau_4 \cdot c_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) - \sigma_4 \cdot c_4 \cdot \sigma_5 \cdot s_6$
3	$\sigma_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) + \tau_4 \cdot \sigma_5 \cdot s_6$
4	0
5	0
6	0

Table A7. Column 1 of ^3J_d

j	$^3d_1[j]$
1	$\sigma_1 \cdot s_2 \cdot c_3 + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot s_3$
2	$-\sigma_1 \cdot s_2 \cdot \tau_3 \cdot s_3 + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot \tau_3 \cdot c_3 + (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot \sigma_3$
3	$\sigma_1 \cdot s_2 \cdot \sigma_3 \cdot s_3 - (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot \sigma_3 \cdot c_3 + (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot \tau_3$
4	0
5	0
6	0

Table A8. Column 2 of ^3J_d

j	$^3d_2[j]$
1	$\sigma_2 \cdot s_3$
2	$\sigma_2 \cdot \tau_3 \cdot c_3 + \tau_2 \cdot \sigma_3$
3	$-\sigma_2 \cdot \sigma_3 \cdot c_3 + \tau_2 \cdot \tau_3$
4	0
5	0
6	0

Table A9. Column 3 of ^3J_d

j	$^3d_3[j]$
1	0
2	σ_3
3	τ_3
4	0
5	0
6	0

Table A10. Column 4 of ^3J_d

j	$^3d_4[j]$
1	0
2	0
3	1
4	0
5	0
6	0

Table A11. Column 5 of ^3J_d

j	$^3d_5[j]$
1	$\sigma_4 \cdot s_4$
2	$-\sigma_4 \cdot c_4$
3	τ_4
4	0
5	0
6	0

Table A12. Column 6 of ${}^3J_d$

j	${}^3d_6[j]$
1	$c_4 \cdot \sigma_5 \cdot s_5 + \tau_4 \cdot s_4 \cdot \sigma_5 \cdot c_5 + \sigma_4 \cdot s_4 \cdot \tau_5$
2	$s_4 \cdot \sigma_5 \cdot s_5 - \tau_4 \cdot c_4 \cdot \sigma_5 \cdot c_5 - \sigma_4 \cdot c_4 \cdot \tau_5$
3	$-\sigma_4 \cdot \sigma_5 \cdot c_5 + \tau_4 \cdot \tau_5$
4	0
5	0
6	0

Table A13. Column 1 of ${}^3J_\alpha$

j	${}^3\alpha_1[j]$
1	$-d_2 \cdot s_2 \cdot c_3 + (a_2 \cdot \sigma_2 \cdot s_2 - d_2 \cdot \tau_2 \cdot c_2) \cdot s_3 - c_2 \cdot d_3 \cdot s_3$ $-\tau_2 \cdot s_2 \cdot d_3 \cdot c_3$
2	$d_2 \cdot s_2 \cdot \tau_3 \cdot s_3 + (a_2 \cdot \sigma_2 \cdot s_2 - d_2 \cdot \tau_2 \cdot c_2) \cdot \tau_3 \cdot c_3$ $+(a_2 \cdot \tau_2 \cdot s_2 + d_2 \cdot \sigma_2 \cdot c_2) \cdot \sigma_3 + c_2 \cdot (a_3 \cdot \sigma_3 \cdot s_3 - d_3 \cdot \tau_3 \cdot c_3)$ $+\tau_2 \cdot s_2 \cdot (d_3 \cdot \tau_3 \cdot s_3 + a_3 \cdot \sigma_3 \cdot c_3) + \sigma_2 \cdot s_2 \cdot a_3 \cdot \tau_3$
3	$-d_2 \cdot s_2 \cdot \sigma_3 \cdot s_3 - (a_2 \cdot \sigma_2 \cdot s_2 - d_2 \cdot \tau_2 \cdot c_2) \cdot \sigma_3 \cdot c_3$ $+(a_2 \cdot \tau_2 \cdot s_2 + d_2 \cdot \sigma_2 \cdot c_2) \cdot \tau_3 + c_2 \cdot (a_3 \cdot \tau_3 \cdot s_3 + d_3 \cdot \sigma_3 \cdot c_3)$ $-\tau_2 \cdot s_2 \cdot (d_3 \cdot \sigma_3 \cdot s_3 - a_3 \cdot \tau_3 \cdot c_3) - \sigma_2 \cdot s_2 \cdot a_3 \cdot \sigma_3$
4	$c_2 \cdot c_3 - \tau_2 \cdot s_2 \cdot s_3$
5	$-c_2 \cdot \tau_3 \cdot s_3 - \tau_2 \cdot s_2 \cdot \tau_3 \cdot c_3 + \sigma_2 \cdot s_2 \cdot \sigma_3$
6	$c_2 \cdot \sigma_3 \cdot s_3 + \tau_2 \cdot s_2 \cdot \sigma_3 \cdot c_3 + \sigma_2 \cdot s_2 \cdot \tau_3$

Table A14. Column 2 of ${}^3J_\alpha$

j	${}^3\alpha_2[j]$
1	$-d_3 \cdot s_3$
2	$a_3 \cdot \sigma_3 \cdot s_3 - d_3 \cdot \tau_3 \cdot c_3$
3	$a_3 \cdot \tau_3 \cdot s_3 + d_3 \cdot \sigma_3 \cdot c_3$
4	c_3
5	$-\tau_3 \cdot s_3$
6	$\sigma_3 \cdot s_3$

Table A15. Column 3 of ${}^3J_\alpha$

j	${}^3\alpha_3[j]$
1	0
2	0
3	0
4	0
5	0
6	1

Table A16. Column 4 of ${}^3J_\alpha$

j	${}^3\alpha_4[j]$
1	$-d_4 \cdot s_4$
2	$d_4 \cdot c_4$
3	0
4	c_4
5	s_4
6	0

Table A17. Column 5 of $^3J_\alpha$

j	$^3\alpha_5[j]$
1	$-c_4 \cdot d_5 \cdot s_5 - \tau_4 \cdot s_4 \cdot d_5 \cdot c_5 - d_4 \cdot s_4 \cdot c_5 + (a_4 \cdot \sigma_4 \cdot s_4 - d_4 \cdot \tau_4 \cdot c_4) \cdot s_5$
2	$-s_4 \cdot d_5 \cdot s_5 + \tau_4 \cdot c_4 \cdot d_5 \cdot c_5 + d_4 \cdot c_4 \cdot c_5 - (d_4 \cdot \tau_4 \cdot s_4 + a_4 \cdot \sigma_4 \cdot c_4) \cdot s_5$
3	$\sigma_4 \cdot d_5 \cdot c_5 + a_4 \cdot \tau_4 \cdot s_5$
4	$c_4 \cdot c_5 - \tau_4 \cdot s_4 \cdot s_5$
5	$s_4 \cdot c_5 + \tau_4 \cdot c_4 \cdot s_5$
6	$\sigma_4 \cdot s_5$

Table A18. First Two Elements of Column 6 of $^3J_\alpha$

$$\begin{aligned}
{}^3\alpha_6[1] = {} & \\
& c_4 \cdot [-c_5 \cdot d_6 \cdot s_6 - \tau_5 \cdot s_5 \cdot d_6 \cdot c_6 - d_5 \cdot s_5 \cdot c_6 \\
& + (a_5 \cdot \sigma_5 \cdot s_5 - d_5 \cdot \tau_5 \cdot c_5) \cdot s_6] \\
& - \tau_4 \cdot s_4 \cdot [-s_5 \cdot d_6 \cdot s_6 + \tau_5 \cdot c_5 \cdot d_6 \cdot c_6 + d_5 \cdot c_5 \cdot c_6 \\
& - (d_5 \cdot \tau_5 \cdot s_5 + a_5 \cdot \sigma_5 \cdot c_5) \cdot s_6] + \sigma_4 \cdot s_4 \cdot (\sigma_5 \cdot d_6 \cdot c_6 \\
& + a_5 \cdot \tau_5 \cdot s_6) - d_4 \cdot s_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) \\
& + (a_4 \cdot \sigma_4 \cdot s_4 - d_4 \cdot \tau_4 \cdot c_4) \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) \\
& + (a_4 \cdot \tau_4 \cdot s_4 + d_4 \cdot \sigma_4 \cdot c_4) \cdot \sigma_5 \cdot s_6
\end{aligned}$$

$$\begin{aligned}
{}^3\alpha_6[2] = {} & \\
& s_4 \cdot [-c_5 \cdot d_6 \cdot s_6 - \tau_5 \cdot s_5 \cdot d_6 \cdot c_6 - d_5 \cdot s_5 \cdot c_6 \\
& + (a_5 \cdot \sigma_5 \cdot s_5 - d_5 \cdot \tau_5 \cdot c_5) \cdot s_6] \\
& + \tau_4 \cdot c_4 \cdot [-s_5 \cdot d_6 \cdot s_6 + \tau_5 \cdot c_5 \cdot d_6 \cdot c_6 + d_5 \cdot c_5 \cdot c_6 \\
& - (d_5 \cdot \tau_5 \cdot s_5 + a_5 \cdot \sigma_5 \cdot c_5) \cdot s_6] \\
& - \sigma_4 \cdot c_4 \cdot (\sigma_5 \cdot d_6 \cdot c_6 + a_5 \cdot \tau_5 \cdot s_6) \\
& + d_4 \cdot c_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) \\
& - (d_4 \cdot \tau_4 \cdot s_4 + a_4 \cdot \sigma_4 \cdot c_4) \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) \\
& + (d_4 \cdot \sigma_4 \cdot s_4 - a_4 \cdot \tau_4 \cdot c_4) \cdot \sigma_5 \cdot s_6
\end{aligned}$$

Table A19. Third Element of Column 6 of $^3J_\alpha$

$$\begin{aligned}
{}^3\alpha_6[3] = {} & \\
& \sigma_4 \cdot [-s_5 \cdot d_6 \cdot s_6 + \tau_5 \cdot c_5 \cdot d_6 \cdot c_6 + d_5 \cdot c_5 \cdot c_6 \\
& - (d_5 \cdot \tau_5 \cdot s_5 + a_5 \cdot \sigma_5 \cdot c_5) \cdot s_6] \\
& + \tau_4 \cdot [(\sigma_5 \cdot d_6 \cdot c_6 + a_5 \cdot \tau_5 \cdot s_6) \\
& + a_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6)] \\
& - a_4 \cdot \sigma_4 \cdot \sigma_5 \cdot s_6
\end{aligned}$$

Table A20. Last Three Elements of Column 6 of $^3J_\alpha$

j	$^3\alpha_6[j]$
4	$c_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) - \tau_4 \cdot s_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) + \sigma_4 \cdot s_4 \cdot \sigma_5 \cdot s_6$
5	$s_4 \cdot (c_5 \cdot c_6 - \tau_5 \cdot s_5 \cdot s_6) + \tau_4 \cdot c_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) - \sigma_4 \cdot c_4 \cdot \sigma_5 \cdot s_6$
6	$\sigma_4 \cdot (s_5 \cdot c_6 + \tau_5 \cdot c_5 \cdot s_6) + \tau_4 \cdot \sigma_5 \cdot s_6$

Table A21. First Two Elements of Column 1 of $^3J_\theta$

$$
\begin{aligned}
{}^3\theta_1[1] = {} & (a_1 \cdot \tau_1 \cdot s_2 + \sigma_1 \cdot d_2 \cdot c_2) \cdot c_3 \\
& + \{a_1 \cdot \tau_1 \cdot \tau_2 \cdot c_2 - a_1 \cdot \sigma_1 \cdot \sigma_2 \\
& - \sigma_1 \cdot (d_2 \cdot \tau_2 \cdot s_2 + a_2 \cdot \sigma_2 \cdot c_2) + \tau_1 \cdot a_2 \cdot \tau_2\} \cdot s_3 \\
& - \sigma_1 \cdot s_2 \cdot d_3 \cdot s_3 + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot d_3 \cdot c_3
\end{aligned}
$$

$$
\begin{aligned}
{}^3\theta_1[2] = {} & -(a_1 \cdot \tau_1 \cdot s_2 + \sigma_1 \cdot d_2 \cdot c_2) \cdot \tau_3 \cdot s_3 \\
& + \{a_1 \cdot \tau_1 \cdot \tau_2 \cdot c_2 - a_1 \cdot \sigma_1 \cdot \sigma_2 \\
& - \sigma_1 \cdot (d_2 \cdot \tau_2 \cdot s_2 + a_2 \cdot \sigma_2 \cdot c_2) + \tau_1 \cdot a_2 \cdot \tau_2\} \cdot \tau_3 \cdot c_3 \\
& + \{-a_1 \cdot \tau_1 \cdot \sigma_2 \cdot c_2 - a_1 \cdot \sigma_1 \cdot \tau_2 \\
& + \sigma_1 \cdot (d_2 \cdot \sigma_2 \cdot s_2 - a_2 \cdot \tau_2 \cdot c_2) - \tau_1 \cdot a_2 \cdot \sigma_2\} \cdot \sigma_3 \\
& + \sigma_1 \cdot s_2 \cdot (a_3 \cdot \sigma_3 \cdot s_3 - d_3 \cdot \tau_3 \cdot c_3) \\
& - (\sigma_1 \cdot \tau_2 \cdot c_2 + (\tau_1 \cdot \sigma_2) \cdot (d_3 \cdot \tau_3 \cdot s_3 + a_3 \cdot \sigma_3 \cdot c_3) \\
& + (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot a_3 \cdot \tau_3
\end{aligned}
$$

Table A22. Third Element of Column 1 of $^3J_\theta$

$$
\begin{aligned}
{}^3\theta_1[3] = {} & (a_1 \cdot \tau_1 \cdot s_2 + \sigma_1 \cdot d_2 \cdot c_2) \cdot \sigma_3 \cdot s_3 \\
& - \{a_1 \cdot \tau_1 \cdot \tau_2 \cdot c_2 - a_1 \cdot \sigma_1 \cdot \sigma_2 \\
& - \sigma_1 \cdot (d_2 \cdot \tau_2 \cdot s_2 + a_2 \cdot \sigma_2 \cdot c_2) + \tau_1 \cdot a_2 \cdot \tau_2\} \cdot \sigma_3 \cdot c_3 \\
& + \{-a_1 \cdot \tau_1 \cdot \sigma_2 \cdot c_2 - a_1 \cdot \sigma_1 \cdot \tau_2 \\
& + \sigma_1 \cdot (d_2 \cdot \sigma_2 \cdot s_2 - a_2 \cdot \tau_2 \cdot c_2) - \tau_1 \cdot a_2 \cdot \sigma_2\} \cdot \tau_3 \\
& + \sigma_1 \cdot s_2 \cdot (a_3 \cdot \tau_3 \cdot s_3 + d_3 \cdot \sigma_3 \cdot c_3) \\
& + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot (d_3 \cdot \sigma_3 \cdot s_3 - a_3 \cdot \tau_3 \cdot c_3) \\
& - (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot a_3 \cdot \sigma_3
\end{aligned}
$$

Table A23. Last Three Elements of Column 1 of $^3J_\theta$

j	$^3\theta_1[j]$
4	$\sigma_1 \cdot s_2 \cdot c_3 + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot s_3$
5	$-\sigma_1 \cdot s_2 \cdot \tau_3 \cdot s_3 + (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot \tau_3 \cdot c_3$ $+ (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot \sigma_3$
6	$\sigma_1 \cdot s_2 \cdot \sigma_3 \cdot s_3 - (\sigma_1 \cdot \tau_2 \cdot c_2 + \tau_1 \cdot \sigma_2) \cdot \sigma_3 \cdot c_3$ $+ (-\sigma_1 \cdot \sigma_2 \cdot c_2 + \tau_1 \cdot \tau_2) \cdot \tau_3$

Table A24. Column 2 of $^3J_\theta$

j	$^3\theta_2[j]$
1	$a_2 \cdot \tau_2 \cdot s_3 + \sigma_2 \cdot d_3 \cdot c_3$
2	$a_2 \cdot \tau_2 \cdot \tau_3 \cdot c_3 - a_2 \cdot \sigma_2 \cdot \sigma_3 - \sigma_2 \cdot (d_3 \cdot \tau_3 \cdot s_3 + a_3 \cdot \sigma_3 \cdot c_3) + \tau_2 \cdot a_3 \cdot \tau_3$
3	$-a_2 \cdot \tau_2 \cdot \sigma_3 \cdot c_3 - a_2 \cdot \sigma_2 \cdot \tau_3 + \sigma_2 \cdot (d_3 \cdot \sigma_3 \cdot s_3 - a_3 \cdot \tau_3 \cdot c_3)$ $- \tau_2 \cdot a_3 \cdot \sigma_3$
4	$\sigma_2 \cdot s_3$
5	$\sigma_2 \cdot \tau_3 \cdot c_3 + \tau_2 \cdot \sigma_3$
6	$-\sigma_2 \cdot \sigma_3 \cdot c_3 + \tau_2 \cdot \tau_3$

Table A25. Column 3 of $^3J_\theta$

j	$^3\theta_3[j]$
1	0
2	$a_3 \cdot \tau_3$
3	$-a_3 \cdot \sigma_3$
4	0
5	σ_3
6	τ_3

Table A26. Column 4 of $^3J_\theta$

j	$^3\theta_4[j]$
1	0
2	0
3	0
4	0
5	0
6	1

Table A27. Column 5 of $^3J_\theta$

j	$^3\theta_5[j]$
1	$a_4 \cdot \tau_4 \cdot s_4 + d_4 \cdot \sigma_4 \cdot c_4$
2	$d_4 \cdot \sigma_4 \cdot s_4 - a_4 \cdot \tau_4 \cdot c_4$
3	$-a_4 \cdot \sigma_4$
4	$\sigma_4 \cdot s_4$
5	$-\sigma_4 \cdot c_4$
6	τ_4

Table A28. First Two Elements of Column 6 of $^3J_\theta$

$$
\begin{aligned}
{}^3\theta_6[1] &= c_4 \cdot (a_5 \cdot \tau_5 \cdot s_5 + d_5 \cdot \sigma_5 \cdot c_5) \\
&\quad - \tau_4 \cdot s_4 \cdot (d_5 \cdot \sigma_5 \cdot s_5 - a_5 \cdot \tau_5 \cdot c_5) \\
&\quad - \sigma_4 \cdot s_4 \cdot a_5 \cdot \sigma_5 - d_4 \cdot s_4 \cdot \sigma_5 \cdot s_5 \\
&\quad - (a_4 \cdot \sigma_4 \cdot s_4 - d_4 \cdot \tau_4 \cdot c_4) \cdot \sigma_5 \cdot c_5 \\
&\quad + (a_4 \cdot \tau_4 \cdot s_4 + d_4 \cdot \sigma_4 \cdot c_4) \cdot \tau_5
\end{aligned}
$$

$$
\begin{aligned}
{}^3\theta_6[2] &= \\
&\quad s_4 \cdot (a_5 \cdot \tau_5 \cdot s_5 + d_5 \cdot \sigma_5 \cdot c_5) \\
&\quad + \tau_4 \cdot c_4 \cdot (d_5 \cdot \sigma_5 \cdot s_5 - a_5 \cdot \tau_5 \cdot c_5) \\
&\quad + \sigma_4 \cdot c_4 \cdot a_5 \cdot \sigma_5 + d_4 \cdot c_4 \cdot \sigma_5 \cdot s_5 \\
&\quad + (d_4 \cdot \tau_4 \cdot s_4 + a_4 \cdot \sigma_4 \cdot c_4) \cdot \sigma_5 \cdot c_5 \\
&\quad + (d_4 \cdot \sigma_4 \cdot s_4 - a_4 \cdot \tau_4 \cdot c_4) \cdot \tau_5
\end{aligned}
$$

Table A29. Last Four Elements of Column 6 of $^3J_\theta$

j	$^3\theta_6[j]$
3	$\sigma_4 \cdot (d_5 \cdot \sigma_5 \cdot s_5 - a_5 \cdot \tau_5 \cdot c_5) - \tau_4 \cdot a_5 \cdot \sigma_5 - a_4 \cdot \tau_4 \cdot \sigma_5 \cdot c_5 - a_4 \cdot \sigma_4 \cdot \tau_5$
4	$c_4 \cdot \sigma_5 \cdot s_5 + \tau_4 \cdot s_4 \cdot \sigma_5 \cdot c_5 + \sigma_4 \cdot s_4 \cdot \tau_5$
5	$s_4 \cdot \sigma_5 \cdot s_5 - \tau_4 \cdot c_4 \cdot \sigma_5 \cdot c_5 - \sigma_4 \cdot c_4 \cdot \tau_5$
6	$-\sigma_4 \cdot \sigma_5 \cdot c_5 + \tau_4 \cdot \tau_5$

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