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Robust Kalman Filtering based on Multiple Hypothesis Techniques

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Abstract: In this paper, a new robust state estimator for linear systems with parametric uncertainties is proposed. The uncertainties affecting the system are regarded as unknown sequences of quantized parametric uncertainties. And then the exact robust estimator is derived by handling the uncertainty parameter sequence hypotheses by means of multiple hypotheses testing(MHT) techniques. However, since the exact filter has to treat ever expanding hypotheses, a suboptimal estimator based on zero scan back concept is proposed. A benchmark example for robust Kalman filtering is demonstrated to compare the performance of the proposed filter with those of an existing robust Kalman filter.

Keywords: Robust Kalman Filter, Uncertainty, Estimation, Multiple Hypotheses Test

1. INTRODUCTION

Recently, the robust Kalman filtering problem has been received much interest because of its robust behavior against parametric uncertainties which are often found in many real control and signal processing applications [1–3]. As a result, there have been many approaches to solve this robust Kalman filter design problem from various viewpoints - game theoretic approach, convex optimization approach, set-valued estimation approach and so on [4–7]. The design purpose of the robust Kalman filter is to minimize the mean square estimation errors and to guarantee robustness in worst-cases. This filter design criterion is successfully achieved by the previous methods for linear time-invariant (LTI) filtering problem. However, in linear time-varying (LTV) filtering problem, the selection of proper design parameters mitigating the performance degradation due to the parameter uncertainties remains an open problem up to date [7, 8]. For this limitation, the existing robust filter design methods entail inherited conservatism on the estimation performance.

In this paper, aside from the previous methods, the robust Kalman filter(RKF) problem is formulated using the quantized random uncertainties with Markovian transition properties. And an exact(optimal) solution for the RKF problem is introduced. However, the optimal solution is not practically implementable, since it has to handle ever expanding hypotheses. In order to cope with this problem, a suboptimal RKF is proposed based on the zero-scan-back(ZSB) concept introduced in [9] and [10]. At each time step, every quantized uncertainties are considered and evaluated through hypothesis generation and evaluation process. And at the end of each time step, by making the Bayesian summation of all hypotheses, the uncertainty hypotheses are combined to produce one state estimate.

Interestingly, the proposed RKF has the same structure of the conventional RKF - the Kalman filter sturucture with a scale factor compensation term. Moreover, the compensation term in our approach is evaluated at each sampling time in consideration of the measurements and assumed probabilistic properties of the parameter uncertainties. Consequently, it posesses self-adaptation property for time-varying parameter uncertainties. It means that the new approach may give a solution for the aforementioned open problem of the filter design parameter selection. The performance of the proposed ZSB filter is compared with that of Krein space RKF in [8] through the benchmark problem therein.

2. EXACT SOLUTION FOR ROBUST ESTIMATION

In this section, a new robust filter is derived based on the multiple hypothesis testing(MHT) technique. Let's consider the following system with parametric uncertainties in the system matrix.

$$x_{k+1} = (F_k + \Delta F_k)x_k + G_k u_k \quad (1)$$
$$z_k = H_k x_k + v_k \quad (2)$$

In the above equations, u_k and v_k are zero-mean Gaussian process with covariance Q_k and R_k, respectively. And we accept the following assumption of the structural parameter uncertainty introduced in [8].

$$\Delta F_k = L_k \Delta_k M_k \quad (3)$$

where Δ_k is a scalar uncertainty parameter. In addition, we assume that at each time the uncertainty Δ_k takes its values in the quantized uncertainty set of $\left\{\Delta_k^1, \cdots, \Delta_k^{N_k}\right\}$. Then the robust filter problem can be regarded as the problem to estimate a parameter uncertainty sequence in the set of the possible sequences of $\left\{\left\{\Delta_0^{j(0)}, \cdots, \Delta_k^{j(k)}\right\}, j(i) \in \{1, 2, \cdots, N_i\}\right\}$.

Let every sequence of system parameter uncertainty obey Markovian transition properties such as the following assumption **A1**.

A1. Each uncertainty $\Delta_i^{j(i)}$ is an element of the set $D_i \triangleq \left\{\Delta^j,\ j = 1, 2, \cdots, N_i\right\}$. And the transition probability from Δ_i^l to Δ_{i+1}^m is given with π_i^{lm}.

In order to apply MHT techniques, an uncertainty hypothesis is defined as a sequence of system parameter uncertainties. And let η^k be the cumulative uncertainty hypothesis such as

$$\eta^k \triangleq \left\{ \eta_k^i,\ i = 1, 2, \cdots, \prod_{\tau=1}^{k} N_\tau \right\}$$

where η_k^i is the i-th uncertainty hypothesis from initial time to k, so it can be represented by the corresponding uncertainty sequence of $\left\{ \Delta_0^{j(0)}, \cdots, \Delta_{k-1}^{j(k-1)} \right\}$. Since η^k represents all possible histories of uncertainties for the system (1), the robust filtering problem can be treated as multiple hypotheses testing and handling problem of the hypotheses included in η^k.

MHT techniques typically consist of three parts: hypothesis generation, hypothesis evaluation, and hypothesis reduction. In the *hypothesis generation*, a new hypothesis η_{k+1}^l can be easily made by just adding a new possible uncertainty Δ_k^j for the next time to each of the previous hypothesis η_k^i.

Hypothesis evaluation is the process evaluating the *a posteriori* probability of each hypothesis. Assume that η_k^i was made by attaching the new possible uncertainty Δ_{k-1}^i to the previous hypothesis η_{k-1}^j. Then the *a posteriori* probability of η_k^i can be evaluated recursively through Bayesian rule as follows:

$$\begin{aligned} p\left(\eta_k^i | Z^k\right) &= p\left(\eta_{k-1}^l, \Delta_{k-1}^i | z_k, Z^{k-1}\right) \\ &= \frac{1}{c} p\left(z_k | \eta_k^i, Z^{k-1}\right) p\left(\Delta_{k-1}^i | \eta_{k-1}^j, Z^{k-1}\right) \\ &\quad \times p\left(\eta_{k-1}^j | Z^{k-1}\right) \end{aligned} \tag{4}$$

where z_k is the measurement at k while Z^k is the cumulative measurements of $\{z_i,\ i = 1, \cdots, k\}$. In addition, c is the normalizing constant making the total probability be unity. In the above equations, $p\left(z_k | \eta_k^i, Z^{k-1}\right)$ is evaluated conventionally by the residual of the Kalman filter conditioned by $\{\eta_k^i, Z^{k-1}\}$, and $p\left(\Delta_{k-1}^i | \eta_{k-1}^j, Z^{k-1}\right)$ is the Markov-transition probability given by **A1**.

Since every hypothesis in η^k has its own probability evaluated by (4), the exact probability density function of the state can be obtained as follows:

$$p(x_k | Z^k) = \sum_{i=1}^{N_{\eta^k}} N\left(x_k : \hat{x}_{k|k}^i, P_{k|k}^i\right) p\left(\eta_k^i | Z^k\right) \tag{5}$$

where N_{η^k} is the number of hypotheses in η^k, and $N(x : \bar{x}, P)$ is the Gaussian pdf with mean $\bar{x}$ and covariance P. $\hat{x}_{k|k}^i$ and $P_{k|k}^i$ are the *a posteriori* estimate and its covariance of the Kalman filter conditioned by η_k^i. In fact, (5) will be the optimal exact solution for our robust filtering problem if the above pdf is practically implementable. However, since the number of hypotheses η^k is increasing exponentially as time elapses, (5) can not be evaluated and accordingly we need hypothesis reduction techniques to maintain the hypotheses within an allowable size. Hereafter we will derive a robust filter by applying the zero-scan-back(ZSB) hypothesis reduction technique.

3. ZERO SCAN BACK SOLUTION FOR ROBUST ESTIMATION

The zero scan back method is one of the most widely accepted concepts for hypothesis reduction in MHT filtering area [9, 10]. The basic idea of the zero scan back is to make only one hypothesis remain at the end of each scan (or at the end of each sampling interval) by utilizing the hypothesis reduction or combine techniques. As a result of this concept, the zero scan back filter has the structure in which hypothesis branching process alternates with hypotheses combining process.

The *Hypothesis branching* process at time k generates new hypotheses representing which uncertain parameter actually works at the k-th time interval. Since we assume that the uncertainty Δ_k should take its value from the set of $\left\{ \Delta_k^1, \cdots, \Delta_k^{N_k} \right\}$, the number of hypotheses generated at time k will be N_k. Once the hypotheses are generated, the Kalman filters conditioned by each hypothesis are carried out. Speaking in detail, if η_k^i is the hypothesis generated at time k for taking Δ_{k-1}^i into account, the Kalman filter conditioned by η_k^i will be the Kalman filter for the model of (1) with $\Delta F_{k-1} = \Delta F_{k-1}^i = L_{k-1} \Delta_{k-1}^i M_{k-1}$.

After the Kalman filtering conditioned by the hypotheses, *Hypothesis combining* process is performed. In this process, the Bayesian sum of the filtering results are evaluated to produce the unique resultant state estimate and its covariance. This process can be thought as the approximation of the exact pdf of (5) into one Gaussian pdf.

In order to derive a simple result of ZSB suboptimal filtering, let's make some more assumptions.

A2. The set of uncertainty matrices $\{\Delta_i^{j(i)}\}$ consists of symmetric elements. In other words,

$$\sum_{j=1}^{N_i} \Delta_i^{j(i)} = 0 \ \text{ for all } \ i\,. \tag{6}$$

A3. The transition probability from Δ_i^l to Δ_{i+1}^m or π_i^{lm} obeys uniform distribution.

As overviewed above, in the ZSB filtering, the new hypotheses are generated from the previous (representative) hypothesis and merged again to produce the representative hypothesis for the next time. Hereafter we explain the ZSB filtering process by pursuing the procedures carried out from $t = k$ to $t = k + 1$.

Let's begin with the ZSB state estimate $\hat{x}_{k|k}$ and its error covariance $P_{k|k}$ which come out from the representative hypothesis at k. The first step of branching is very simple and the resultant hypotheses will be $\left\{ \eta_{k+1}^i, i = 1, \cdots, N_k \right\}$ where η_{k+1}^i denotes the hypothesis assuming the uncertainty Δ_k^i.

The next step is Kalman filtering in consideration of the newly generated hypotheses. The Kalman filter constrained by η_{k+1}^i can be summarized as follows:

Kalman filter conditioned by η^i_{k+1}

Time Update (System Propagation):

$$\begin{aligned} P^i_{k+1|k} &= [F_k + \Delta F^i_k] P_{k|k} [F_k + \Delta F^i_k]^T + G_k Q_k G^T_k \\ &\approx F_k P_{k|k} F^T_k + G_k Q_k G^T_k \qquad (7) \\ \hat{x}^i_{k+1|k} &= F_k \hat{x}_{k|k} + \Delta F^i_k \hat{x}_{k|k} \\ &= F_k \hat{x}_{k|k} + L_k \Delta^i_k M_k \hat{x}_{k|k} \qquad (8) \end{aligned}$$

Measurement Update:

$$\begin{aligned} K^i_{k+1} &= P^i_{k+1|k} H^T_{k+1} \\ &\quad \times \left[H_{k+1} P^i_{k+1|k} H^T_{k+1} + R_{k+1} \right]^{-1} \qquad (9) \\ P^i_{k+1|k+1} &= (I - K^i_{k+1} H_{k+1}) P^i_{k+1|k} \qquad (10) \\ \hat{x}^i_{k+1|k+1} &= \hat{x}^i_{k+1|k} + K^i_{k+1} r^i_{k+1} \qquad (11) \\ r^i_{k+1} &= (z_{k+1} - H_{k+1} \hat{x}^i_{k+1|k}) \qquad (12) \end{aligned}$$

Note that (7), (8), (9), and (10) produces the same value for all i-th Kalman filter. It means that K^i_{k+1}, $P^i_{k+1|k}$ and $P^i_{k+1|k+1}$ are independent of η_i.

The final step is merging all the hypotheses of $\{\eta^i_{k+1}, i = 1, \cdots, N_k\}$ to produce ZSB representative hypothesis. Since the resultant *a posteriori* pdf of x_{k+1} is expressed by

$$\begin{aligned} p(x_{k+1}|Z^{k+1}) &= \sum_{i=1}^{N_k} p\left(x_{k+1}|\eta^i_{k+1}, Z^{k+1}\right) \\ &\quad \times p\left(\eta^i_{k+1}|Z^{k+1}\right), \qquad (13) \end{aligned}$$

the mean and covariance of the ZSB representative hypothesis can be computed using the following equations [9]. And they actually will be the ZSB state estimate and its covariance for ZSB-RKF(zero scan back robust Kalman filter).

Equations for ZSB-RKF State Estimator

$$\begin{aligned} \hat{x}_{k+1|k+1} &= \sum_{i=1}^{N_k} \left[\hat{x}^i_{k+1|k+1} p^i_{k+1} \right] \\ &= \hat{x}^n_{k+1|k} + K_{k+1} r^n_{k+1} \\ &\quad + [I - K_{k+1} H_{k+1}] L_k \bar{\Delta}_k M_k \hat{x}_{k|k} \qquad (14) \end{aligned}$$

$$\begin{aligned} P_{k+1|k+1} &= \sum_{i=1}^{N_k} P^i_{k+1|k+1} p^i_{k+1} \\ &\quad + \sum_{i=1}^{N_{k+1}} (\hat{x}^i_{k+1|k+1} - \hat{x}_{k+1|k+1})(\leftarrow)^T \\ &= P^n_{k+1|k+1} \\ &\quad + \sum_{j=1}^{N_k} \{ [I - K_{k+1} H_{k+1}] L_k d^{\Delta}_k(j) M_k \hat{x}_{k|k} \} \\ &\qquad \{\leftarrow\}^T p^i_{k+1} \qquad (15) \end{aligned}$$

where $p^i_{k+1} = p\left(\eta^i_{k+1}|Z^{k+1}\right)$, $\hat{x}^n_{k+1|k} = F_k \hat{x}_{k|k}$, $r^n_{k+1} = z_{k+1} - H_{k+1} \hat{x}^n_{k+1|k}$, $K_{k+1} = K^i_{k+1}$, $P_{k+1|k} = P^i_{k+1|k}$, $P^n_{k+1|k+1} = P^i_{k+1|k+1}$. Remind that K^i_{k+1}, $P^i_{k+1|k}$ and $P^i_{k+1|k+1}$ are independent of η_i in our case. In addition, in the above equations, the average $\bar{\Delta}_k$ and deviation of Δ^j_k are defined as follows:

$$\bar{\Delta}_k = \left[\sum_{i=1}^{N_k} \Delta^i_k p^i_{k+1} \right], \qquad (16)$$

$$d^{\Delta}_k(j) = \Delta^j_k - \sum_{i=1}^{N_k} \Delta^i_k p^i_{k+1} = \Delta^j_k - \bar{\Delta}_k. \qquad (17)$$

With the ZSB concept and the assumption of **A3**, (4) is reduced to

$$p\left(\eta^i_k|Z^k\right) = \frac{1}{c} p\left(z_k|\Delta^i_{k-1}, Z^{k-1}\right). \qquad (18)$$

For evaluating the above probability, the following assumption **A4** is conventionally introduced as in [9].

A4. The likelihood $p\left(z_k|\Delta^j_{k-1}, Z^{k-1}\right)$ can be approximated to Gaussian distribution as follows.

$$\begin{aligned} & p(z_k|\Delta^i_{k-1}, Z^{k-1}) \sim \\ & \qquad N(z_k : H_k \hat{x}^i_{k|k-1}, H_k P^i_{k|k-1} H^T_k + R_k) \qquad (19) \end{aligned}$$

On the ZSB-RFK expressed by (14) and (15), some comments are made. First of all, the state estimate of the proposed ZSB-RFK consists of the nominal Kalman filter terms and the scale factor compensation terms evaluated by the uncertainty estimate of $\bar{\Delta}_k$ in (16). Especially, in the righthand side of (14), the first two terms form a conventional Kalman filter equation while the last term is the compensation term. From the same viewpoint, the last term of (15) will be the uncertainty increment due to the scale factor compensation by the last term of (14). In fact, this kind of filter structure including scale factor compensation can be frequently found in many conventional RKFs [7,8].

Next, notice that the proposed ZSB-RKF no more requires tuning procedure. As stated above, in our ZSB-RKF, the compensation term is automatically evaluated based on the estimation of uncertainties while in conventional RKFs it is achieved by additional tuning. However, this kind of tuning is very difficult, especially in time varying filter design case.

4. EXAMPLE

In this section, a simple benchmark problem stated in [8] is demonstrated to compare the proposed ZSB-RFK with the previous RKF in [8]. The benchmark system with uncertainty δ is as follows:

$$x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix} x_k + \begin{bmatrix} -6 \\ 1 \end{bmatrix} u_k \qquad (20)$$

$$z_k = \begin{bmatrix} -100 & 10 \end{bmatrix} x_k + v_k \qquad (21)$$

where the δ is bounded as $-0.3 \leq \delta \leq 0.3$.

For ZSB-RFK design, the parametric uncertainty Δ is quantized to five equally spaced quantities from -0.3 to 0.3. And the equations in the previous section are applied. Fig. 1 and Fig. 2 show the error variances of the proposed ZSB-RFK, Krein space RKF [8], and nominal($\delta = 0$)

Kalman filter with respect to the variation of δ. As shown in the figure, the proposed filter shows the best average performance in the presence of the uncertainties. Consequently, it can be said that the proposed ZSB-RKF guarantees robustness against the variation of Δ without extra tuning efforts.

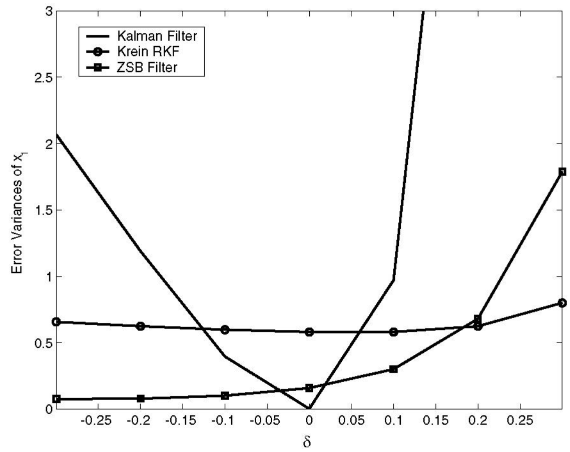

Fig. 1 The Estimation Error Variances for x_1

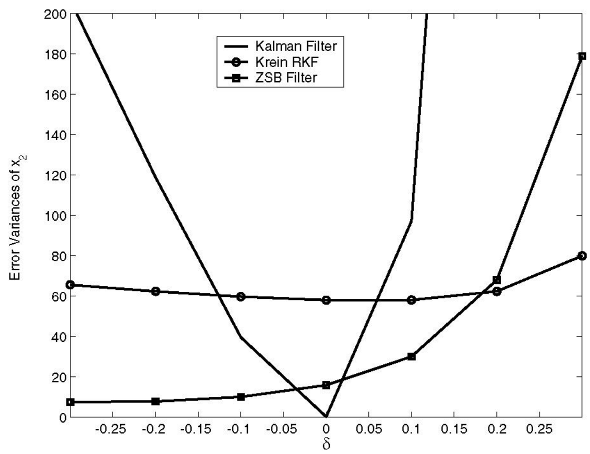

Fig. 2 The Estimation Error Variances for x_2

5. CONCLUSION

In this paper, a new robust estimator for the systems with uncertain parameters is proposed based on the framework of zero scan back multiple hypothesis filtering method. The resultant filter has the structure of Kalman filter with scale factor compensation. Since the compensation term is automatically evaluated using the uncertainty estimate, the proposed filter does not require the effort of additional tuning which has been essential to most conventional robust filters. Through the simple representative benchmark example, it is shown that our filter outperforms conventional ones even without the tuning efforts.

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