CME 345: MODEL REDUCTION

Moment Matching

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These slides are based on the recommended textbook: A.C. Antoulas, "Approximation of Large-Scale Dynamical Systems," Advances in Design and Control, SIAM,
ISBN-0-89871-529-6

Outline

$$\begin{aligned}\frac{d}{dt}\mathbf{w}(t) &= \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{w}(t) + \mathbf{D}\mathbf{u}(t)\\ \mathbf{w}(0) &= \mathbf{w}_0\end{aligned}$$

- $\mathbf{w} \in \mathbb{R}^N$: vector state variables
- $\mathbf{u} \in \mathbb{R}^p$: vector of input variables, typically $p \ll N$
- $\mathbf{y} \in \mathbb{R}^q$: vector output variables, typically $q \ll N$

- Goal: construct a Reduced-Order Model (ROM)

$$\begin{aligned} \frac{d}{dt}\mathbf{q}(t) &= \mathbf{A}_r\mathbf{q}(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}_r\mathbf{q}(t) + \mathbf{D}_r\mathbf{u}(t) \end{aligned}$$

  - $\mathbf{q} \in \mathbb{R}^k$: vector of reduced state variables
- ROM resulting from Petrov-Galerkin projection

$$\begin{aligned} \mathbf{A}_r &= (\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V} \in \mathbb{R}^{k\times k} \\ \mathbf{B}_r &= (\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B} \in \mathbb{R}^{k\times p} \\ \mathbf{C}_r &= \mathbf{C}\mathbf{V} \in \mathbb{R}^{q\times k} \\ \mathbf{D}_r &= \mathbf{D} \in \mathbb{R}^{q\times p} \end{aligned}$$

- Let $\mathbf{h}$ denote a general matrix-valued function of time

$$\mathbf{h}: \quad t \in \mathbb{R} \longmapsto \mathbb{R}^{q \times p}$$

Example: impulse response of an LTI system

$$\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

- And let $\mathbf{H}(s) \in \mathbb{R}^{q \times p}$ denote its Laplace transform

$$\mathbf{H}(s) = \int_0^{\infty} \mathbf{h}(t)e^{-st}dt$$

Example: impulse response of an LTI system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- $\mathbf{H}(s)$ is the transfer function associated with the HDM defined by $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ as for each input $\mathbf{U}(s)$, it defines the output

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

- Let $m \in \{0,\ 1,\ \cdots,\ \}$
  The m-th **moment** of $\mathbf{h}:\ t \in \mathbb{R} \longmapsto \mathbb{R}^{q \times p}$ at $s_0 \in \mathbb{C}$ is

$$\eta_m(s_0) = \int_0^\infty t^m e^{-s_0 t} \mathbf{h}(t)\, dt$$

- Hence, the m-th **moment** of $\mathbf{h}$ can be written in terms of the transfer function $\mathbf{H}(s)$ as follows

$$\eta_m(s_0) = (-1)^m \left. \frac{d^m}{ds^m} \mathbf{H}(s) \right|_{s=s_0}$$

  Example: impulse response of an LTI system

$$\begin{aligned} \eta_0(s_0) &= \mathbf{H}(s_0) = \mathbf{C}(s_0 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ \eta_m(s_0) &= m!\ \mathbf{C}(s_0 \mathbf{I}_N - \mathbf{A})^{-(m+1)}\mathbf{B}, \quad \forall m \geq 1 \end{aligned}$$

- Development of $\mathbf{H}(s)$ in Taylor series

$$
\begin{aligned}
\mathbf{H}(s) &= \mathbf{H}(s_0) + \frac{d}{ds}\mathbf{H}(s)\bigg|_{s=s_0} \frac{(s-s_0)}{1!} + \cdots \\
&\quad + \frac{d^m}{ds^m}\mathbf{H}(s)\bigg|_{s=s_0} \frac{(s-s_0)^m}{m!} + \cdots \\
&= \eta_0(s_0) - \eta_1(s_0)\frac{(s-s_0)}{1!} + \cdots + (-1)^m \eta_m(s_0)\frac{(s-s_0)^m}{m!} + \cdots \\
&= \eta_0(s_0) + \eta_1(s_0)\frac{(s_0-s)}{1!} + \cdots + \eta_m(s_0)\frac{(s_0-s)^m}{m!} + \cdots
\end{aligned}
$$

- The **Markov parameters** of the system defined by **h** are defined as the coefficients $\eta_m(\infty)$ of the Laurent series expansion of the transfer function at infinity

$$\mathbf{H}(s) = \eta_0(\infty) + \frac{1}{s}\eta_1(\infty) + \frac{1}{s^2}\eta_2(\infty) + \cdots + \frac{1}{s^m}\eta_m(\infty) + \cdots$$

Example: impulse response of an LTI system

$$\begin{aligned} \eta_0(\infty) &= \mathbf{D} \\ \eta_m(\infty) &= \mathbf{C}\mathbf{A}^{m-1}\mathbf{B},\ \forall m \geq 1 \end{aligned}$$

Proof: Use the property that for $s \to \infty$:

$$(s\mathbf{I}_N - \mathbf{A})^{-1} = \frac{1}{s}\mathbf{I}_N + \frac{1}{s^2}\mathbf{A} + \cdots + \frac{1}{s^{m+1}}\mathbf{A}^m + \cdots$$

- Let $s_0 \in \mathbb{C}$, and let $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ represent the HDM defined by $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$
- *Objective*: construct a ROM $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ such that the first l moments $\{\eta_{r,j}(s_0)\}_{j=0}^{l-1}$ of its transfer function at s_0, $\mathbf{H}_r = \mathbf{C}_r(s_0\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r \in \mathbb{R}^{q \times p}$, match the first l moments $\{\eta_j(s_0)\}_{j=0}^{l-1}$ of the transfer function $\mathbf{H}(s) \in \mathbb{R}^{q \times p}$ of the HDM

$$\eta_{r,j}(s_0) = \eta_j(s_0) \Leftrightarrow \mathbf{H}^{(j)}(s_0) = \mathbf{H}_r^{(j)}(s_0), \ \forall j = 0, \cdots, l-1$$

  - the *direct* matching of the moments is in general a numerically unstable procedure
  - moment matching is performed best today using an equivalent procedure based on Krylov subspaces
- For simplicity, focus is set on the Single Input-Single Output (SISO) $(p = q = 1)$ case throughout the remainder of this chapter

$$\mathbf{B} = \mathbf{b} \in \mathbb{R}^N, \ \mathbf{C}^T = \mathbf{c}^T \in \mathbb{R}^N$$

Theorem

Let $\mathbf{V}$ *be a right Reduced-Order Basis (ROB) such that*

$$range(\mathbf{V}) = \mathcal{K}_k(\mathbf{A}, \mathbf{b}) = span\left\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\right\}$$

and $\mathbf{W}$ *be a left ROB satisfying*

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ *using* $\mathbf{W}$ *and* $\mathbf{V}$ *satisfies*

$$\eta_{r,j}(\infty) = \eta_j(\infty) \Leftrightarrow \mathbf{H}_r^{(j)}(\infty) = \mathbf{H}^{(j)}(\infty), \ \forall j = 0, \cdots, k-1$$

Definition

The order-k Krylov subspace generated by $\mathbf{A} \in \mathbb{R}^{N \times N}$ and $\mathbf{b} \in \mathbb{R}^N$ is

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\}$$

Remark: Constructing $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ requires only the ability to compute the action of the matrix $\mathbf{A}$ onto a vector. In many applications, such a computation can be performed without forming explicitly the matrix $\mathbf{A}$.

The following lemma is introduced to prove the previous theorem

Lemma

The moments of the transfer function of a ROM do not depend on the underlying left and right ROBs, but only on the subspaces associated with these ROBs

Proof of the Theorem.

From the above lemma, it follows that $\mathbf{V}$ can be chosen, for example, as follows

$$\mathbf{V} = [\mathbf{v}_1, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_k] = [\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}]$$

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}_k \Rightarrow \mathbf{A}\mathbf{V}\mathbf{W}^T\mathbf{v}_i = \mathbf{A}\mathbf{V}\mathbf{e}_i = \mathbf{A}\mathbf{v}_i = \mathbf{v}_{i+1} = \mathbf{A}^i\mathbf{b}$$

$$\begin{aligned} \Longrightarrow \quad \eta_{r,0}(\infty) &= \mathbf{D} = \eta_0(\infty) \\ \eta_{r,1}(\infty) &= \mathbf{c}_r\mathbf{b}_r = \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{b} = \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{v}_1 = \mathbf{c}\mathbf{V}\mathbf{e}_1 = \mathbf{c}\mathbf{b} = \eta_1(\infty) \\ \eta_{r,j}(\infty) &= \mathbf{c}_r\mathbf{A}_r^j\mathbf{b}_r = \mathbf{c}\mathbf{V}\mathbf{W}^T(\mathbf{A}\mathbf{V}\mathbf{W}^T)^j\mathbf{b} = \mathbf{c}\mathbf{V}\mathbf{W}^T(\mathbf{A}\mathbf{V}\mathbf{W}^T)^j\mathbf{v}_1 \\ &= \mathbf{c}\mathbf{V}\mathbf{W}^T\mathbf{v}_{j+1} = \mathbf{c}\mathbf{V}\mathbf{e}_{j+1} = \mathbf{c}\mathbf{v}_{j+1} = \mathbf{c}\mathbf{A}^j\mathbf{b} = \eta_j(\infty) \end{aligned}$$

Theorem

Let $s_0 \in \mathbb{C}$, $\mathbf{V}$ be a right ROB satisfying

$$\begin{aligned} range(\mathbf{V}) &= \mathcal{K}_k\left((s_0\mathbf{I}_N - \mathbf{A})^{-1}, (s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right) \\ &= span\left\{(s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_0\mathbf{I}_N - \mathbf{A})^{-k}\mathbf{b}\right\} \end{aligned}$$

and $\mathbf{W}$ be a left ROB satisfying

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,j}(s_0) = \eta_j(s_0) \Leftrightarrow \mathbf{H}_r^{(j)}(s_0) = \mathbf{H}^{(j)}(s_0), \ \forall j = 0, \cdots, k-1$$

- This is a more computationally expensive procedure as the computation of each Krylov basis vector requires the solution of a large-scale system of equations

Theorem

Let $s_i \in \mathbb{C},\ i = 1, \cdots, k$, $\mathbf{V}$ be a right ROB satisfying

$$\mathit{range}(\mathbf{V}) = \mathit{span}\left\{(s_1\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_k\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}$$

and $\mathbf{W}$ be a left ROB satisfying

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,0}(s_i) = \eta_0(s_i) \Leftrightarrow \mathbf{H}_r(s_i) = \mathbf{H}(s_i),\ \forall i = 1, \cdots, k$$

Theorem

Let $s_i \in \mathbb{C}$, $i = 1, \cdots, l$, $\mathbf{V}$ be a right ROB satisfying

$$range(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A})^{-1}, (s_i\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right)$$

and $\mathbf{W}$ be a left ROB satisfying

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}_r^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i),\ \forall i = 1, \cdots, l,\ \forall j = 0, \cdots, k-1$$

Theorem

Let $s_i \in \mathbb{C},\ i = 1, \cdots, 2l$, $\mathbf{V}$ be a right ROB satisfying

$$range(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A})^{-1}, (s_i\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right)$$

and $\mathbf{W}$ be a left ROB satisfying

$$range(\mathbf{W}) = \bigcup_{i=l+1}^{2l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A}^T)^{-1}, (s_i\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T\right)$$

and $\mathbf{W}^T\mathbf{V}$ is nonsingular

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}_r^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i),\ \forall i = 1, \cdots, 2l,\ \forall j = 0, \cdots, k-1$$

- Partial realization requires the construction of $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$, that is the knowledge of the action of $\mathbf{A}$ onto vectors
- Rational interpolation requires the following Krylov space

$$\mathcal{K}_k((s_0\mathbf{I}_N - \mathbf{A})^{-1}, (s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}).$$

  Since the knowledge of the action of $(s_0\mathbf{I}_N - \mathbf{A})^{-1} \in \mathbb{R}^{N \times N}$ is needed, two computationally efficient approaches are possible:
  - if N is small enough, an LU factorization of $s_0\mathbf{I}_N - \mathbf{A}$ can be performed and $(s_0\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{v}$ computed by forward and backward substitution for any vector $\mathbf{v} \in \mathbb{R}^N$
  - if N is too large for an LU factorization to be performed, Krylov subspace recycling techniques allowing the reuse of Krylov subspaces for multiple right-hand sides can be used

- $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ can be efficiently constructed using the Arnoldi factorization method

  Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$
  Output: Orthogonal basis $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

- The following recursion is satisfied:

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k\mathbf{H}_k + \mathbf{f}_k\mathbf{e}_k^T$$

  with $\mathbf{H}_k = \mathbf{V}_k^T\mathbf{A}\mathbf{V}_k$ an upper Hessenberg matrix, $\mathbf{V}_k^T\mathbf{V}_k = \mathbf{I}_k$ and $\mathbf{V}_k^T\mathbf{f}_k = \mathbf{0}$.

Algorithm:

Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$

Output: Orthogonal basis $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

$\mathbf{v}_1 = \mathbf{b}/\|\mathbf{b}\|$;
$\mathbf{w} = \mathbf{A}\mathbf{v}_1$; $\alpha_1 = \mathbf{v}_1^T \mathbf{w}$;
$\mathbf{f}_1 = \mathbf{w} - \alpha_1 \mathbf{v}_1$;
$\mathbf{V}_1 = [\mathbf{v}_1]$; $\mathbf{H} = [\alpha_1]$;
for $j = 1, \cdots, k-1$ **do**
  $\beta_j = \|\mathbf{f}_j\|$; $\mathbf{v}_{j+1} = \mathbf{f}_j / \beta_j$;
  $\mathbf{V}_{j+1} = [\mathbf{V}_j, \ \mathbf{v}_{j+1}]$;
  $\hat{\mathbf{H}}_j = \begin{bmatrix} \mathbf{H}_j \\ \beta_j \mathbf{e}_j^T \end{bmatrix}$
  $\mathbf{w} = \mathbf{A}\mathbf{v}_{j+1}$;
  $\mathbf{h} = \mathbf{V}_{j+1}^T \mathbf{w}$; $\mathbf{f}_{j+1} = \mathbf{w} - \mathbf{V}_{j+1}\mathbf{h}$
  $\mathbf{H}_{j+1} = [\hat{\mathbf{H}}_j, \ \mathbf{h}]$;
end for

- $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$ can be efficiently simultaneously constructed using the Two-sided Lanczos process

  Input: $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{b} \in \mathbb{R}^N$, $\mathbf{c}^T \in \mathbb{R}^N$
  Output: Bi-orthogonal bases $\mathbf{V}_k \in \mathbb{R}^{N \times k}$ and $\mathbf{W}_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$ respectively satisfying $\mathbf{W}_k^T \mathbf{V}_k = \mathbf{I}_k$

- The following recursions are satisfied:

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k \mathbf{T}_k + \mathbf{f}_k \mathbf{e}_k^T,$$

$$\mathbf{A}^T \mathbf{W}_k = \mathbf{W}_k \mathbf{T}_k^T + \mathbf{g}_k \mathbf{e}_k^T,$$

  with $\mathbf{T}_k = \mathbf{W}_k^T \mathbf{A} \mathbf{V}_k$ a tridiagonal matrix, $\mathbf{W}_k^T \mathbf{V}_k = \mathbf{I}_k$, $\mathbf{W}_k^T \mathbf{g}_k = \mathbf{0}$ and $\mathbf{V}^T \mathbf{f}_k = \mathbf{0}$.

Algorithm:

Input: $\mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{b} \in \mathbb{R}^{N}$, $\mathbf{c}^T \in \mathbb{R}^{N}$

Output: Bi-orthogonal bases $\mathbf{V}_k \in \mathbb{R}^{N\times k}$ and $\mathbf{W}_k \in \mathbb{R}^{N\times k}$ for $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ and $\mathcal{K}_k(\mathbf{A}^T, \mathbf{c}^T)$ respectively satisfying $\mathbf{W}_k^T\mathbf{V}_k = \mathbf{I}_k$

$\beta_1 = \sqrt{|\mathbf{b}^T\mathbf{c}^T|}$, $\gamma_1 = \text{sign}(\mathbf{b}^T\mathbf{c}^T)\beta_1$
$\mathbf{v}_1 = \mathbf{b}/\beta_1$, $\mathbf{w}_1 = \mathbf{c}^T/\gamma_1$
for $j = 1, \cdots, k-1$ **do**
    $\alpha_j = \mathbf{w}_j^T\mathbf{A}\mathbf{v}_j$;
    $\mathbf{r}_j = \mathbf{A}\mathbf{v}_j - \alpha_j\mathbf{v}_j - \gamma_j\mathbf{v}_{j-1}$;
    $\mathbf{q}_j = \mathbf{A}^T\mathbf{w}_j - \alpha_j\mathbf{w}_j - \beta_j\mathbf{w}_{j-1}$;
    $\beta_{j+1} = \sqrt{|\mathbf{r}_j^T\mathbf{q}_j|}$, $\gamma_{j+1} = \text{sign}(\mathbf{r}_j^T\mathbf{q}_j)\beta_{j+1}$
    $\mathbf{v}_{j+1} = \mathbf{r}_j/\beta_{j+1}$;
    $\mathbf{w}_{j+1} = \mathbf{q}_j/\gamma_{j+1}$;
end for
$\mathbf{V}_k = [\mathbf{v}_1, \cdots, \mathbf{v}_k]$, $\mathbf{W}_k = [\mathbf{w}_1, \cdots, \mathbf{w}_k]$

Definition

The $\mathcal{H}_2$ norm of a continuous dynamical system $S = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is the $\mathcal{L}_2$ norm of its associated impulse response $\mathbf{h}(\cdot)$.
When $\mathbf{A}$ is stable and $\mathbf{D} = \mathbf{0}$, the norm is bounded and

$$\|S\|_{\mathcal{H}_2} = \left(\int_0^\infty \text{trace}\left(\mathbf{h}^T(t)\mathbf{h}(t) \right) dt \right)^{1/2}$$

- Using Parseval's theorem, one can obtain the expression in the frequency domain using the transfer function $\mathbf{H}(\cdot)$

$$\|S\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left(\mathbf{H}^*(-i\omega)\mathbf{H}(i\omega) \right) d\omega \right)^{1/2}$$

- One can also derive the expression of $\|S\|_{\mathcal{H}_2}$ in terms of the reachability and observability gramians $\mathcal{P}$ and $\mathcal{Q}$.

$$\|S\|_{\mathcal{H}_2} = \sqrt{\text{trace}\left(\mathbf{B}^T \mathcal{Q} \mathbf{B} \right)} = \sqrt{\text{trace}\left(\mathbf{C} \mathcal{P} \mathbf{C}^T \right)}$$

- In the SISO case, the transfer function is a rational function. Assuming for simplicity that it has distinct poles $\lambda_i, \ i = 1, \cdots, N$ associated with the residues h_i, one can write it as

$$\mathbf{H}(s) = \sum_{i=1}^{N} \frac{h_i}{s - \lambda_i}$$

- One can then establish the following theorem:

Theorem

Let $\mathbf{H}_r(\cdot)$ be the transfer function associated with the system $\mathcal{S}_r$ resulting from moment matching using the Lanczos procedure of the underlying system $\mathcal{S}$. Denoting by $h_{r,i}$ and $\lambda_{r,i}, \ i = 1, \cdots, k$ the respective residues and poles of $\mathbf{H}_r(\cdot)$, the following result holds:

$$\|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} h_i \left(\mathbf{H}(-\lambda_i^*) - \mathbf{H}_r(-\lambda_i^*)\right) + \sum_{i=1}^{k} h_{r,i} \left(\mathbf{H}_r(-\lambda_{r,i}) - \mathbf{H}(-\lambda_{r,i})\right)$$

- One would like to build ROBs $(\mathbf{V}, \mathbf{W})$ of a given dimension k such that the corresponding reduced system $\mathcal{S}_r$ is $\mathcal{H}_2$-**optimal**, i.e. minimizes the following problem

$$\min_{\mathcal{S}_r,\ \text{rank}(\mathbf{V})=\text{rank}(\mathbf{W})=k} \|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_2}$$

- One can show that a **necessary condition** is that the ROM matches the first two moments of the HDM at the points $-\lambda_{r,i}$, mirror images of the poles $\lambda_{r,i}$ of the reduced transfer function $\mathbf{H}_r(\cdot)$

$$\mathbf{H}_r(-\lambda_{r,i}) = \mathbf{H}(-\lambda_{r,i}),\ \ \mathbf{H}_r^{(1)}(-\lambda_{r,i}) = \mathbf{H}^{(1)}(-\lambda_{r,i}),\ s = 1, \cdots, k$$

- Unfortunately moment matching ensures that the moments of the transfer function are matched at $\lambda_{r,i}$, not $-\lambda_{r,i}$
- The IRKA (Iterative Rational Krylov Approximation) procedure is an iterative procedure to conciliate these two contradicting goals

- POD in the frequency domain (LTI systems):

$$\begin{aligned}\text{range}(\mathbf{V}) = &= \text{span}\{\mathcal{X}(\omega_1), \cdots, \mathcal{X}(\omega_k)\} \\ &= \text{span}\left\{(j\omega_1 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (j\omega_k \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}\end{aligned}$$

  with $\omega_1, \cdots, \omega_k \in \mathbb{R}^+$

- Rational interpolation with first moment matching at multiple points

$$\text{range}(\mathbf{V}) = \text{span}\left\{(s_1 \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (s_k \mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right\}$$

  with $s_1, \cdots, s_k \in \mathbb{C}$

- Question: would it be possible to extend the two-sided moment matching approach to POD?
- Answer: yes, this is the Balanced POD method

- The Balanced POD method generates snapshots for the dual system in addition to the POD snapshots:

$$\begin{aligned}\mathbf{S} &= \left[(j\omega_1\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}, \cdots, (j\omega_k\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right] \\ \mathbf{S}_{\text{dual}} &= \left[(-j\omega_1\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (-j\omega_k\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T\right]\end{aligned}$$

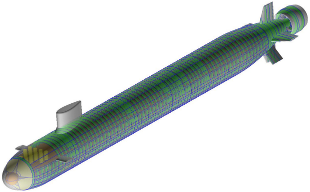
- Right and left reduced-order bases can then be computed as

$$\begin{aligned}\mathbf{S}_{\text{dual}}^T\mathbf{S} &= \mathbf{U}\boldsymbol{\Sigma}\mathbf{Z}^T \quad \text{(SVD)} \\ \mathbf{V} &= \mathbf{S}\mathbf{Z}_k\boldsymbol{\Sigma}_k^{-1/2} \\ \mathbf{W} &= \mathbf{S}_{\text{dual}}\mathbf{U}_k\boldsymbol{\Sigma}_k^{-1/2}\end{aligned}$$

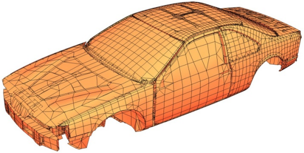
  where a subscript k is relative to the first k components of the singular value decomposition
- If no truncation occurs, this is equivalent to two-sided moment matching at $s_i \in \{\omega_1, \cdots, \omega_k\}$.

- Structural vibrations and interior noise/acoustics

Structural dynamics (Navier)

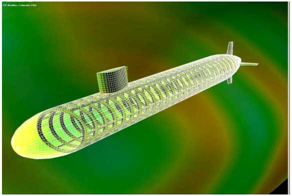


Interior Helmholtz

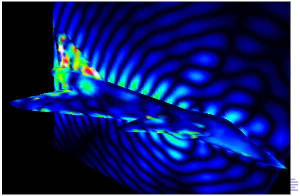


- Scattering (acoustics and electromagnetics)

Exterior Helmholtz



Electromagnetics (Maxwell)
Aeroacoustics (Helmholtz)



- Structural dynamics

$$\mathbf{w}_s(\omega) = \left(\mathbf{K}_s + i\omega\mathbf{D}_s - \omega^2\mathbf{M}_s\right)^{-1}\mathbf{f}_s(\omega)$$

Rayleigh damping $\mathbf{D}_s = \alpha\mathbf{K}_s + \beta\mathbf{M}_s$

- Acoustics

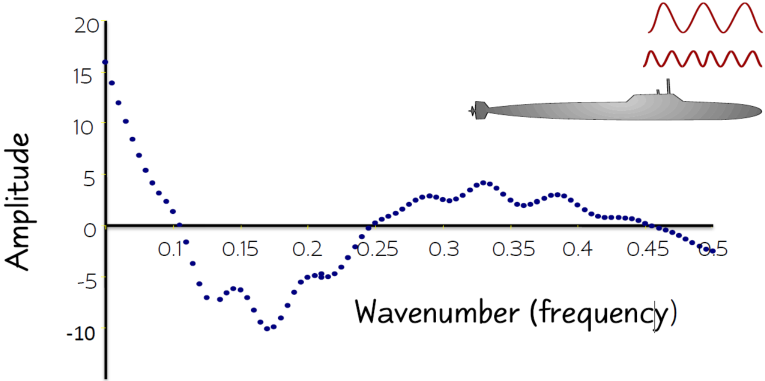
$$\mathbf{w}_f(\omega) = \left(\mathbf{K}_f - \frac{\omega^2}{c_f^2}\mathbf{M}_f + \mathbf{S}_a(\omega)\right)^{-1}\mathbf{f}_f(\omega)$$

- Structural (or vibro)-acoustics

$$\mathbf{w}_v(\omega) = \left(\mathbf{K}_v - \omega^2\mathbf{M}_v + \mathbf{S}_v(\omega)\right)^{-1}\mathbf{f}_v(\omega)$$

$$= \left(\begin{bmatrix} \mathbf{K}_s & \mathbf{C}^T \\ \mathbf{0} & \frac{1}{\rho_f}\mathbf{K}_f \end{bmatrix} - \omega^2\begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ -\mathbf{C} & \frac{1}{\rho_f c_f^2}\mathbf{M}_f \end{bmatrix} + \begin{bmatrix} i\omega\mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{f}_s(\omega) \\ \frac{1}{\rho_f}\mathbf{f}_f(\omega) \end{bmatrix}$$

- Frequency response function $\mathbf{w} = \mathbf{w}(\omega) \Longrightarrow$ problem with multiple left hand sides - very CPU intensive (1,000s of frequencies)



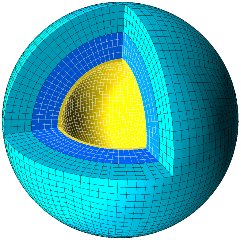
- Approximate $\mathbf{w}(\omega)$ by the Galerkin projection $\mathbf{w} \approx \tilde{\mathbf{w}} = \mathbf{Vq}$

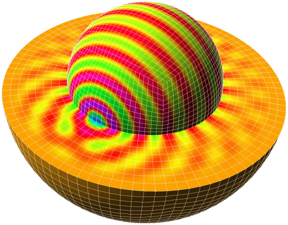
$$\tilde{\mathbf{w}}(\omega) = \mathbf{V} \underbrace{\left(\mathbf{V}^H\mathbf{KV} + i\omega\mathbf{V}^H\mathbf{DV} - \omega^2\mathbf{V}^H\mathbf{MV}\right)^{-1}}_{\textbf{ROM}} \mathbf{V}^H\mathbf{f}$$

- If the columns of $\mathbf{V}$ span the solution and its derivatives at some frequency, the projection is **interpolatory**
- Two ways to compute the vectors in $\mathbf{V}$
  - recursive differentiation with respect to ω at the interpolating frequency
  - construction of a Krylov space that spans the derivatives (special cases)

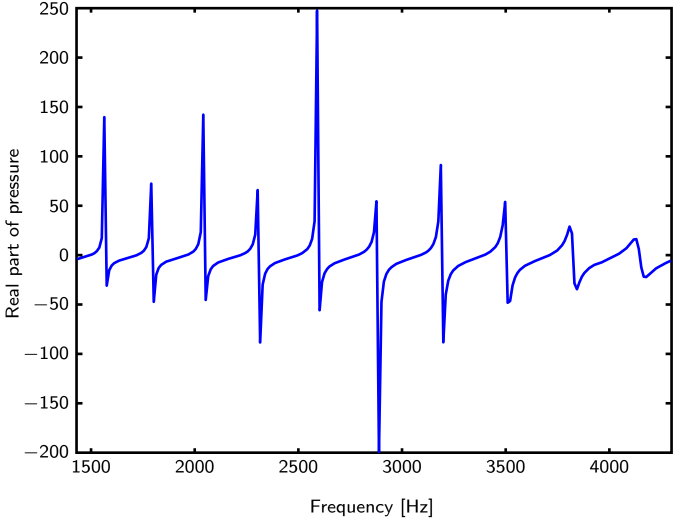
$$\begin{aligned}\text{span}\Big\{&(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f},\\ &(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{M}(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f},\\ &\cdots\\ &[(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{M}]^{n-1}(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})^{-1}\mathbf{f}\Big\}\end{aligned}$$

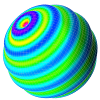
- Frequency sweep analysis of a thick spherical steel shell submerged in water and excited by a point load on its inner surface
- Finite element model using isoparametric cubic elements incorporates with roughly $N = 1,200,000$ dofs





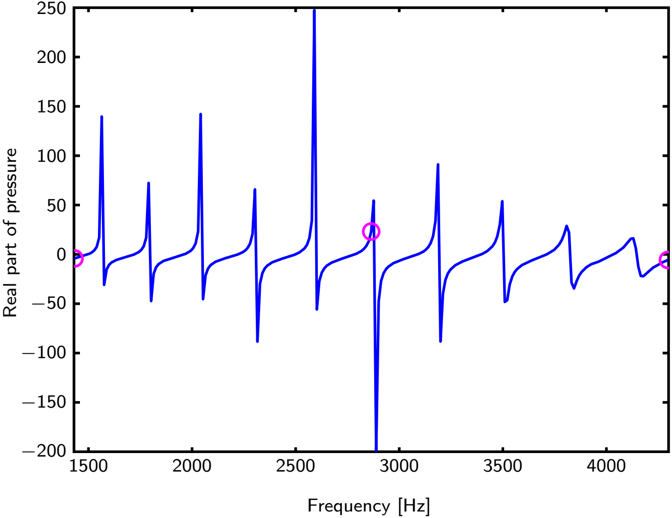
- Frequency sweep analysis of a submerged shell

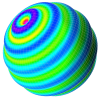




- reference

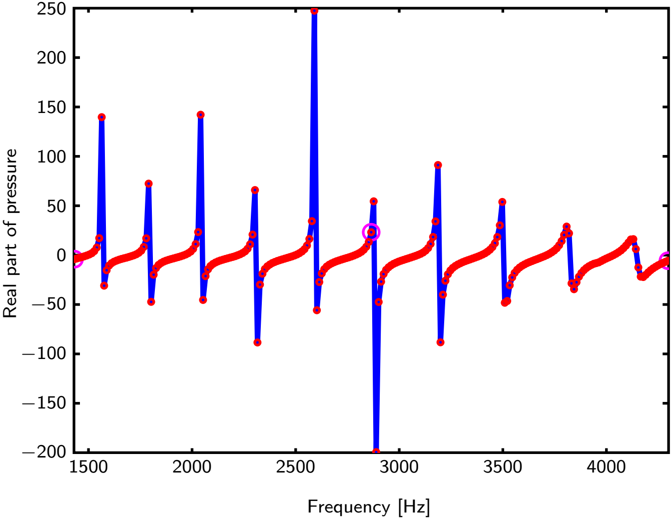
- Frequency sweep analysis of a submerged shell

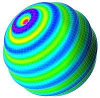




- reference
- interpolating frequencies: 1,430Hz, 2,860Hz, and 4,290Hz

- Frequency sweep analysis of a submerged shell





- reference
- interpolating frequencies: 1,430Hz, 2,860Hz, and 4,290Hz
- 3-points with 32+32+32 vectors
- sampling every 12Hz

- How to choose
  - number of interpolating frequencies
  - location of interpolating frequencies
  - number of derivatives (Krylov vectors)
- Error indicator: relative residual

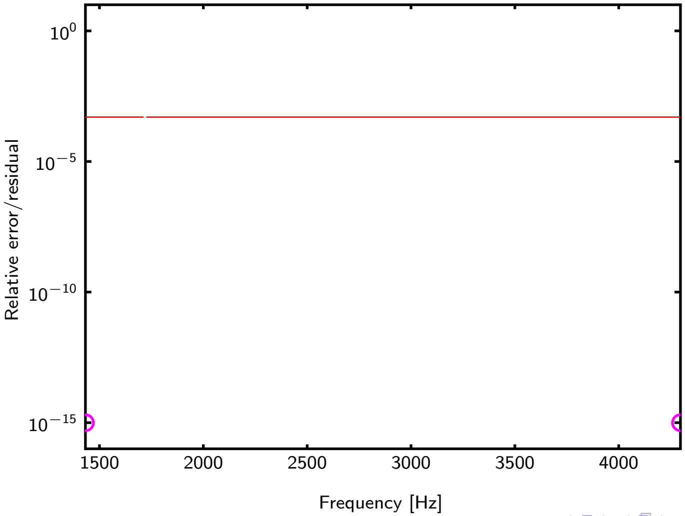
$$\frac{\|(\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{D})\tilde{\mathbf{w}}(\omega) - \mathbf{f}\|}{\|\mathbf{f}\|}$$

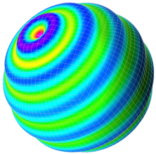
where

$$\tilde{\mathbf{w}}(\omega) = \mathbf{V}\left(\mathbf{V}^H\mathbf{K}\mathbf{V} + i\omega\mathbf{V}^H\mathbf{D}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}$$

1. Specify the number of derivatives per frequency and an accuracy threshold
2. Use two interpolations frequencies at the extremities of the frequency band of interest and construct the ROB
3. Evaluate the residual at some *small* set of the frequencies in between
4. Add a frequency where the residual is largest and update the projection
5. Repeat until the residual is below a threshold at *all sampling points*
6. Check at the end the residual at all sampled (or user-specified) frequencies

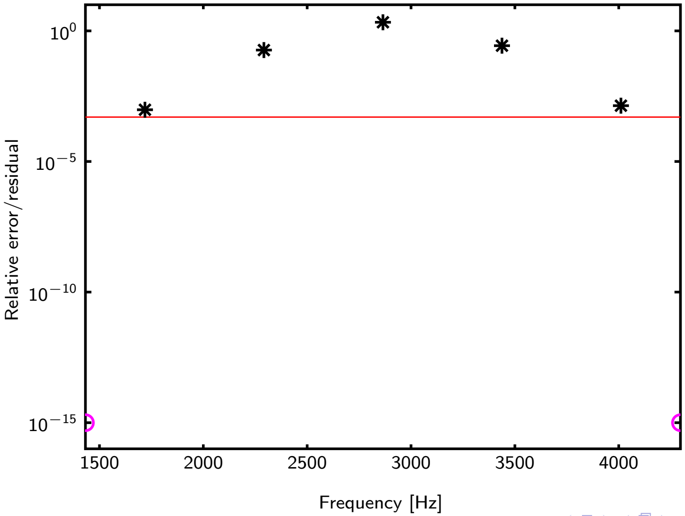
- Frequency sweep analysis of a submerged shell

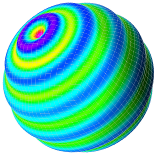




- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers

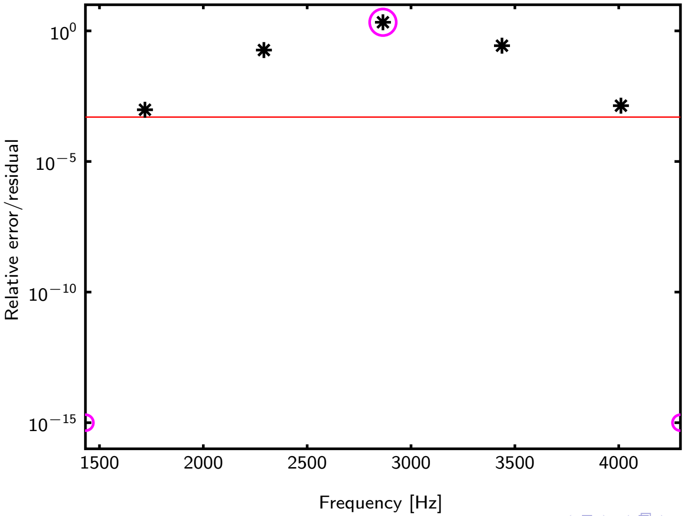
- Frequency sweep analysis of a submerged shell

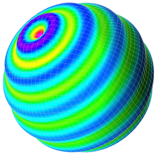




- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *

- Frequency sweep analysis of a submerged shell





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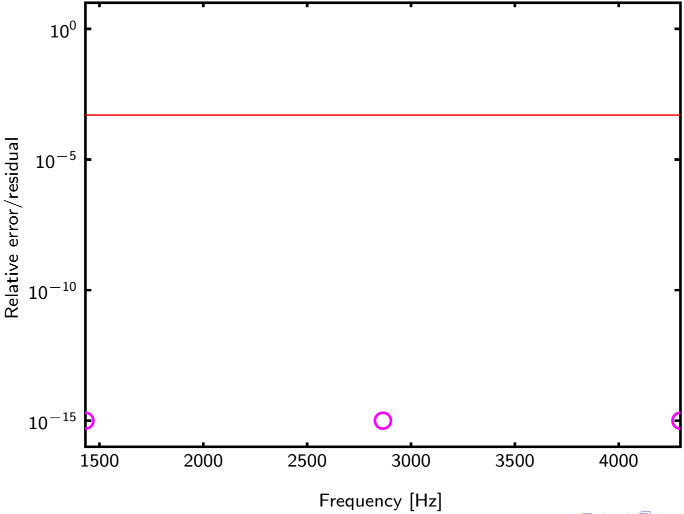

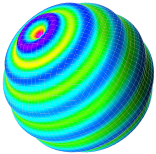

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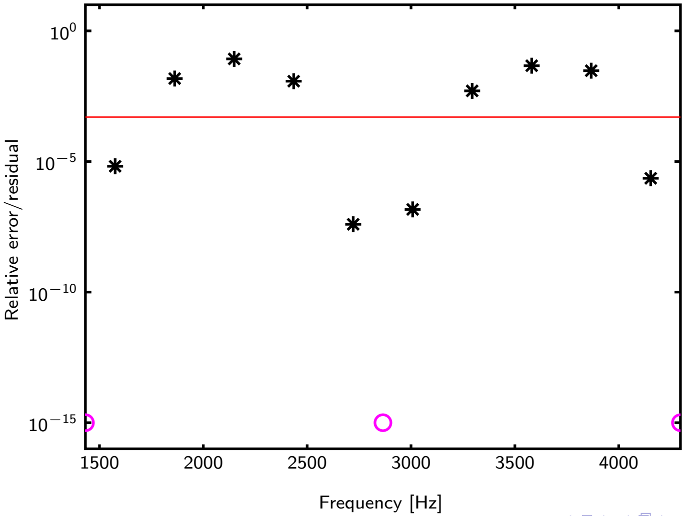

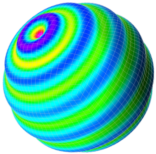

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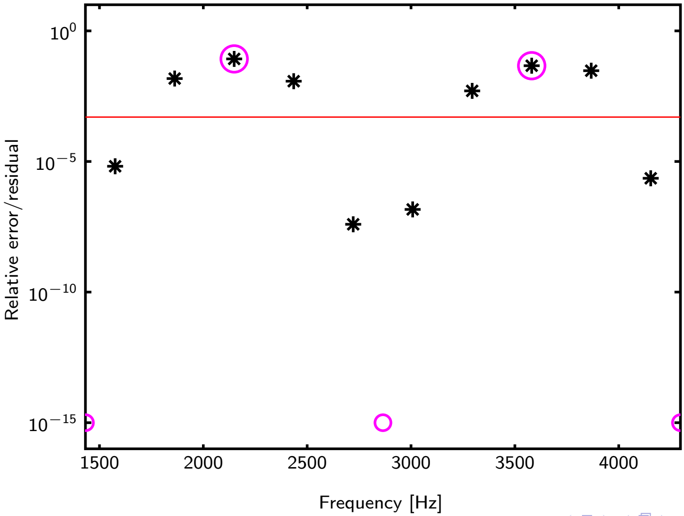

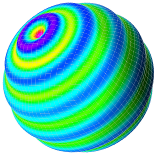

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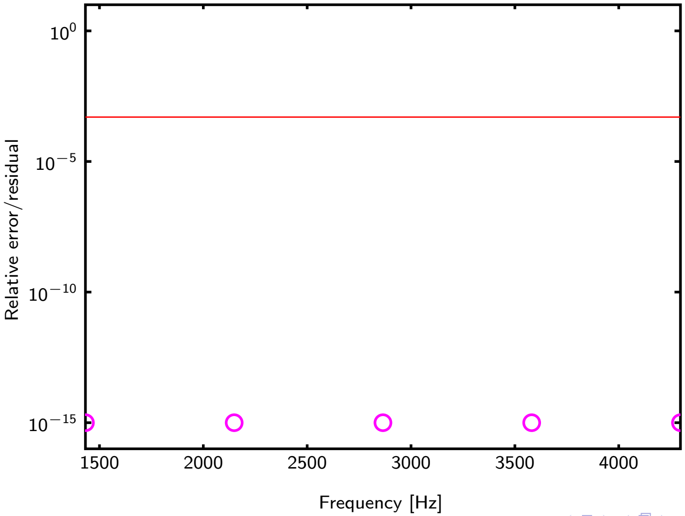

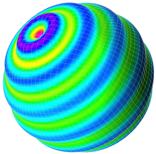

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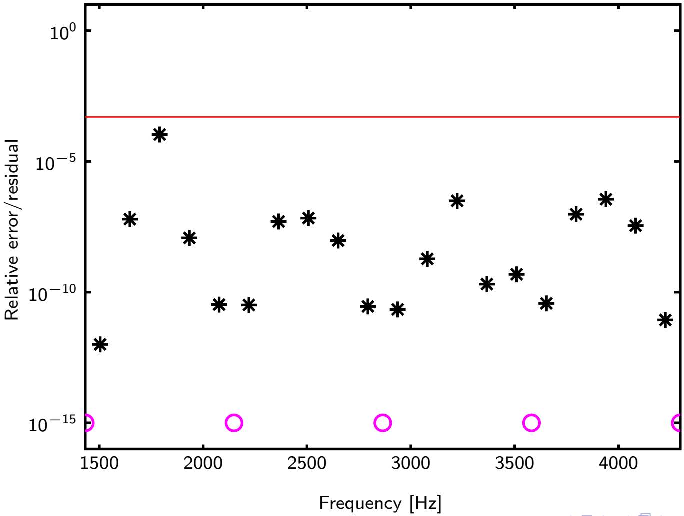

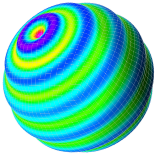

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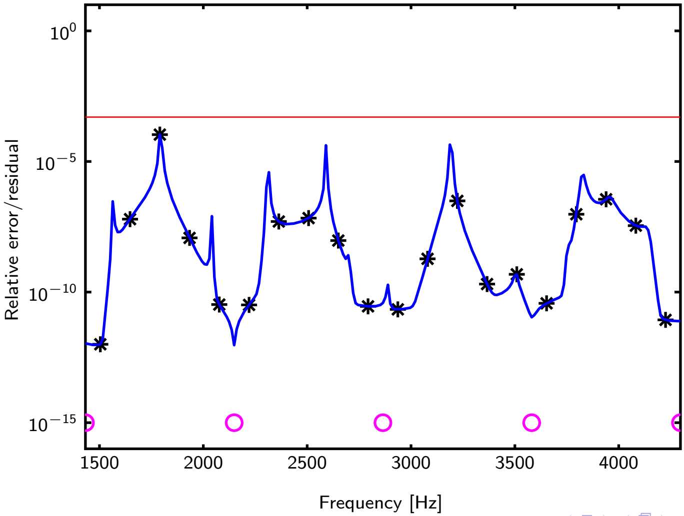
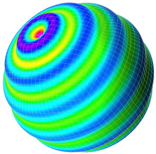
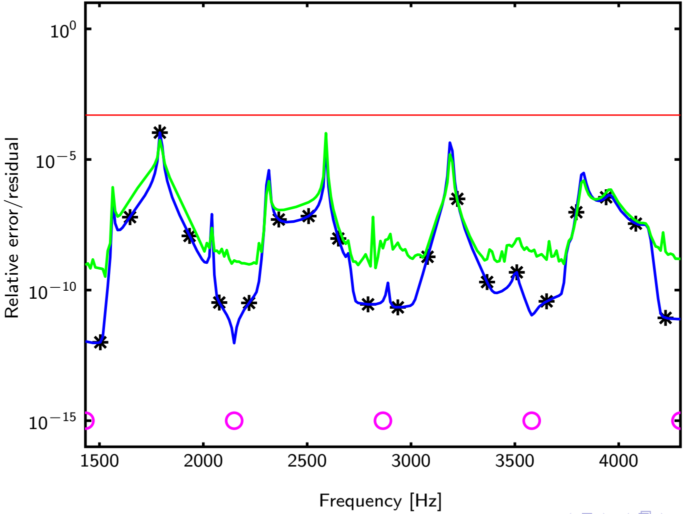
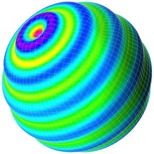




- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual

- Frequency sweep analysis of a submerged shell





- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual
- final error