Time-Varying Systems and Computations

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Homework 1

Task Description

The homework provides problems to perform some type of '*experimental Linear Algebra*'. The goal is to re-activate some facts on linear time invariant systems and signals. Another goal is to get students to use Matlab as a tool to do numerical experiments. It makes most sense to solve the given problems using Matlab. You can use Matlab's 'Diary Files' functionality (see Matlab Online Help) to perform the computations and then insert your comments into the diary file using a standard text editor. It is ok to deliver the print out from the diary file. It is also ok to deliver hand-written documents which have been scanned.

Date of delivery: **November 7, 2011, 10:00 o'clock**
Send the deliverable via email to **tvsc@ldv.ei.tum.de.**

1 Toeplitz Matrix and Discrete Fourier Transformation

Creating Data

Create test vectors u, t using the following Matlab commands

```
u = randn(5,1);
t = randn(4,1);
Q = 1/sqrt(8)*dftmtx(8);
```

Perform Computations

1. Compute $\boldsymbol{y} = \boldsymbol{t} \star \boldsymbol{u}$ using the convolution sum (conv command in Matlab), and check for the length of $\boldsymbol{y}$.
2. Pad the vectors $\boldsymbol{u}$ and $\boldsymbol{t}$ with zeros such that both vectors have the length of $\boldsymbol{y}$. The padded vectors are denoted by $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{t}}$.
3. Compute the Fourier transforms of the vectors $\boldsymbol{U} = \mathcal{F}(\hat{\boldsymbol{u}})$ and $\boldsymbol{\Lambda} = \mathcal{F}(\hat{\boldsymbol{t}})$.
4. Compute $\boldsymbol{y} = \mathcal{F}^{-1}(\boldsymbol{\Lambda} \cdot \boldsymbol{U})$.
5. Construct the Toeplitz matrix $\boldsymbol{T} \in \mathbb{R}^{8\times 5}$ using the vector $\boldsymbol{t}$ and compute the convolution as $\boldsymbol{y} = \boldsymbol{T} \cdot \boldsymbol{u}$.
6. Extend $\boldsymbol{T}$ to construct the cyclic Toeplitz matrix $\boldsymbol{T_c} \in \mathbb{R}^{8\times 8}$ using the vector $\boldsymbol{t}$.
7. Compute the Eigenvalue decomposition ($\Lambda = eig(\boldsymbol{T_c})$) and compare the values of $\boldsymbol{\Lambda}$ with the vector $\boldsymbol{T} = \mathcal{F}(\hat{\boldsymbol{t}})$.

8. Compute the matrix vector product $\boldsymbol{Q} \cdot \hat{\boldsymbol{t}}$ and compare the result with $\mathcal{F}(\hat{\boldsymbol{t}})$.
9. Determine and compare the Euclidean length of the vectors $\boldsymbol{Q} \cdot \hat{\boldsymbol{t}}$ and $\mathcal{F}(\hat{\boldsymbol{t}})$.

Discuss Results

Record the results of the computations and inspect the data. Write up what you observe.

2 Systems of Equations

Creating Data

Using Matlab create the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ by means of the following command

```
A_1 = randn(5,3)*randn(3,4);
A_2 = randn(5,4);
```

Create the two linear systems of equations of the form $\boldsymbol{A}_1 \cdot \boldsymbol{x} = \boldsymbol{b}_1$ and $\boldsymbol{A}_2 \cdot \boldsymbol{x} = \boldsymbol{b}_2$, e.g. using the following Matlab commands:

```
x = randn(4,1);
b_1 = A_1*x;
b_2 = A_2*x + randn(5,1);
```

Perform Computations

1. Solve the previously generated linear systems of equations $\boldsymbol{A}_i \cdot \boldsymbol{x} = \boldsymbol{b}_i$, $i = 1, 2$ by computing the Pseudo-Inverse $\boldsymbol{A}_i^{\dagger} = (\boldsymbol{A}_i^T \boldsymbol{A}_i)^{-1} \boldsymbol{A}_i^T$ in the form $\tilde{\boldsymbol{x}}_i = \boldsymbol{A}_i^{\dagger} \cdot \boldsymbol{b}_i$. What phenomenon can you observe?
2. Solve the linear systems of equations using a pseudo-inverse computed in Matlab by the command "pinv(A)". What do you observe ?
3. Compute the LU factorizations for the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Check result.
4. Compute the QR factorizations for the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Check result.

Discuss Results

Record the results of the computations and inspect the data. Write up what you observe.

3 Singular Value Decomposition

Creating Data

Using Matlab create matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ by means of the following commands

```
A = randn(5,3)*randn(3,4);
B = A'*A;
C = A*A';
```

Perform Computations

1. Compute the Singular Value Decomposition (SVD) of the matrix $\boldsymbol{A}$ in the form

   $$\boldsymbol{A} = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^T.$$

2. For this particular example check the validity of the following expressions:

   $$\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^T = \mathbf{1}_5, \quad \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{V}\boldsymbol{V}^T = \mathbf{1}_4.$$

3. Compute the Eigenvalue Decomposition (EVD) of the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ in the form

   $$\boldsymbol{B} = \boldsymbol{Q}_1 \cdot \boldsymbol{\Lambda}_1 \cdot \boldsymbol{Q}_1^T \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{Q}_2 \cdot \boldsymbol{\Lambda}_2 \cdot \boldsymbol{Q}_2^T$$

4. For this particular example check the validity of the following expressions:

   $$\boldsymbol{Q}_1^T\boldsymbol{Q}_1 = \mathbf{1}_4, \quad \boldsymbol{Q}_2^T\boldsymbol{Q}_2 = \mathbf{1}_5.$$

5. Compare the matrices $\boldsymbol{U}, \boldsymbol{V}$ with the matrices $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ along with $\boldsymbol{\Sigma}, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2$. What can you observe?

6. Check the determinants of the matrices $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. Based on the determinants, what can be said about the properties of $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$?

7. Compute the QR-Decomposition of the matrix $\boldsymbol{A}$ in the form $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, with $\boldsymbol{Q}^T\boldsymbol{Q} = \mathbf{1}_5$, and $\boldsymbol{R}_{ij} = 0$, for $i > j$ (upper triangular matrix). Compare the column spaces of the matrices $\boldsymbol{Q}$ determined by means of the QR-decomposition and the matrix $\boldsymbol{U}$ determined via SVD. What can you observe ?

8. Determine one vector $\boldsymbol{x} \in \ker \boldsymbol{A}$ for the specific matrix $\boldsymbol{A}$.

9. Compute the matrix $\hat{\boldsymbol{A}}$, which is the closest to $\boldsymbol{A}$ and which satisfies $rank\,(\boldsymbol{A}) = 2$.

4 Projections and Orthogonal Transformations

1. Create a projection matrix $\boldsymbol{P}_u = \boldsymbol{u} \cdot (\boldsymbol{u}^T\boldsymbol{u})^{-1}\boldsymbol{u}^T$ using a random vector $\boldsymbol{u} \in \mathbb{R}^5$

2. Using this matrix $\boldsymbol{P}_u$, check the validity of the following statements

   - $\boldsymbol{P}_u^2 = \boldsymbol{P}_u$.
   - $\boldsymbol{P}_u = \boldsymbol{P}_u^T$.
   - $rank\,(\boldsymbol{P}_u) = 1$.
   - the eigenvalues of $\boldsymbol{P}_u$ have the values 0 und 1.

3. Compute the matrix $\boldsymbol{H} = \mathbf{1}_5 - 2\boldsymbol{P}_u$ and check $\boldsymbol{H}$ for orthogonality (general expression as well as numerically).

4. Compute the determinant of $\boldsymbol{H}$. What can you say about the properties of $\boldsymbol{H}$.

5. Consider the orthogonal matrix $\boldsymbol{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$, which is a Givens-Rotation. The parameters satisfy the condition $c^2 + s^2 = 1$. Derive a formula for computing the parameters c und s, such that

   $$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}, \quad z^2 = x^2 + y^2$$

   holds.

5 State-Space Description of linear time-invariant systems

Manual Work

Consider the linear time-invariant filter examples shown in Figures 1a) and 1b).

1. From Figure 1a) read off the equations for $\boldsymbol{x}^1_{k+1}$, $\boldsymbol{x}^2_{k+1}$ and $\boldsymbol{x}^3_{k+1}$ and $\boldsymbol{y}_k$ in terms of the variable $\boldsymbol{x}^1_k$, $\boldsymbol{x}^2_k$ and $\boldsymbol{x}^3_k$ and u_k.
2. Rearrange the equations into a matrix notation of the state-space equations in the form
$$\begin{bmatrix} \boldsymbol{x}_{k+1} \\ y_k \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right] \cdot \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix}.$$
3. repeat step 1 for the system shown in Figure 1b).
4. repeat step 2 for the result of step 3.
5. repeat step 1 for the system shown in Figure 1c).
6. repeat step 2 for the result of step 5.

Data Processing

1. Determine the state-space description for the system shown in Figure 2.
2. For the parameters take the values $a_0 = 1$, $a_1 = 2$, $a_2 = 1$ and $a_3 = 0.5$. Find the output signal of the system to the input signal
$$u = \begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}$$
starting at $k = 0$ by propagating the state-space equations.

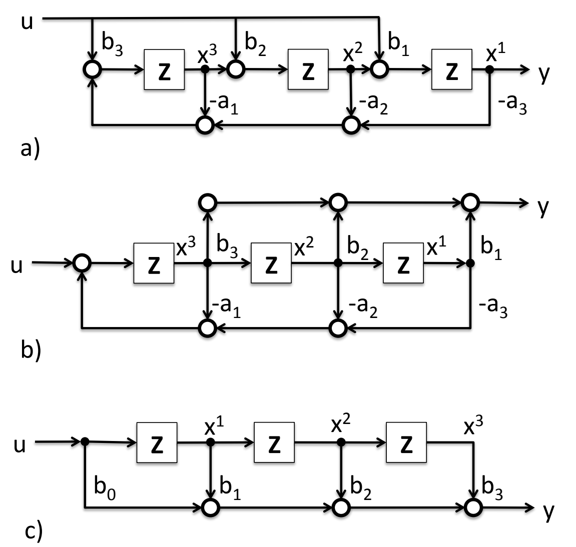

Figure 1: a), b) and c): 3 Examples of Linear Time Invariant Filters

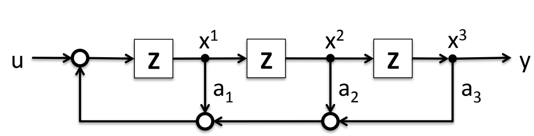

Figure 2: Another Example of a Linear Time Invariant Filter