Linear Control Systems
Lecture # 25
Time-Varying Systems

Change of Variables, Controllability & Observability

Change of Variables

Consider the system

$$\dot{x} = A(t)x$$

and the change of variables

$$x(t) = P(t)z(t)$$

where $P(t)$ is nonsingular and continuously differentiable for all t

$$\dot{x}(t) = P(t)\dot{z}(t) + \dot{P}(t)z(t)$$

$$A(t)x(t) = A(t)P(t)z(t)$$

$$\dot{x} = A(t)x \;\Rightarrow\; P(t)\dot{z}(t) + \dot{P}(t)z(t) = A(t)P(t)z(t)$$

$$\dot{z}(t) = \left[P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)\right] z(t) \stackrel{\text{def}}{=} F(t)z(t)$$

Let $\Phi_A(t, t_0)$ and $\Phi_F(t, t_0)$ be the transition matrices of $A(t)$ and $F(t)$, respectively. What is the relationship between $\Phi_A(t, t_0)$ and $\Phi_F(t, t_0)$?

$$z(t) = \Phi_F(t, t_0)z(t_0) = \Phi_F(t, t_0)P^{-1}(t_0)x(t_0)$$

On the other hand,

$$z(t) = P^{-1}(t)x(t) = P^{-1}(t)\Phi_A(t, t_0)x(t_0)$$

$$\Rightarrow \quad P^{-1}(t)\Phi_A(t, t_0)x(t_0) = \Phi_F(t, t_0)P^{-1}(t_0)x(t_0), \;\; \forall\, x(t_0)$$

$$\Rightarrow \quad P^{-1}(t)\Phi_A(t, t_0) = \Phi_F(t, t_0)P^{-1}(t_0)$$

$$\Rightarrow \quad \boxed{\Phi_F(t, t_0) = P^{-1}(t)\Phi_A(t, t_0)P(t_0)}$$

Periodic Systems

Consider the system $\dot{x} = A(t)x$, where $A(t)$ is a continuous T-periodic function of t; i.e.,

$$A(t+T) = A(t), \quad \forall\, t$$

Let $\Phi(t,\tau)$ be the transition matrix of $A(t)$ and define a constant matrix R by

$$e^{RT} = \Phi(T,0)$$

There is always a matrix R (although it is not unique)

Special Case: If $\Phi(T,0)$ is diagonalizable, then there is a nonsingular matrix Q such that

$$Q^{-1}\Phi(T,0)Q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\text{Take } \; R_1 = \begin{bmatrix} \ln(\lambda_1) & & & \\ & \ln(\lambda_2) & & \\ & & \ddots & \\ & & & \ln(\lambda_n) \end{bmatrix}$$

$$\text{Take } \boldsymbol{R} = \frac{\mathbf{1}}{\boldsymbol{T}} \boldsymbol{Q} \boldsymbol{R}_{\mathbf{1}} \boldsymbol{Q}^{-\mathbf{1}}$$

Then

$$\begin{aligned}
\exp(\boldsymbol{RT}) &= \exp(\boldsymbol{Q}\boldsymbol{R}_1\boldsymbol{Q}^{-1}) = \boldsymbol{Q}\exp(\boldsymbol{R}_1)\boldsymbol{Q}^{-1} \\
&= \boldsymbol{Q}\begin{bmatrix} \boldsymbol{\lambda}_1 & & & \\ & \boldsymbol{\lambda}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{\lambda}_n \end{bmatrix}\boldsymbol{Q}^{-1} = \Phi(\boldsymbol{T}, \mathbf{0})
\end{aligned}$$

Lemma: Let $P(t) = \Phi(t, 0)e^{-Rt}$

- $P(t)$ is nonsingular for all t
- $P(t+T) = P(t)$ for all t

Proof: $P(t)$ is nonsingular because it is the product of two nonsingular matrices

$$\begin{aligned} P(t+T) &= \Phi(t+T, 0)e^{-R(t+T)} \\ &= \Phi(t+T, T)\Phi(T, 0)e^{-RT}e^{-Rt} \\ &= \Phi(t+T, T)e^{RT}e^{-RT}e^{-Rt} \\ &= \Phi(t+T, T)e^{-Rt} \end{aligned}$$

Because $A(t)$ is T-periodic, $\Phi(t+T, T) = \Phi(t, 0)$

$$P(t+T) = \Phi(t, 0)e^{-Rt} = P(t)$$

$$
\begin{aligned}
\dot{P}(t) &= \frac{\partial}{\partial t}\left[\Phi(t,0)\right] e^{-Rt} + \Phi(t,0)\frac{d}{dt}\left[e^{-Rt}\right] \\
&= A(t)\Phi(t,0)e^{-Rt} - \Phi(t,0)e^{-Rt}R \\
&= A(t)P(t) - P(t)R
\end{aligned}
$$

Apply the change of variables $x(t) = P(t)z(t)$

$$
\begin{aligned}
\dot{z} &= \left[P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)\right] z(t) \\
&= P^{-1}(t)\left[A(t)P(t) - \dot{P}(t)\right] z(t) \\
&= P^{-1}(t)P(t)R\, z(t) \;=\; R\, z(t)
\end{aligned}
$$

Floquet decomposition: A T-periodic system $\dot{x} = A(t)x$ can be transformed by a T-periodic change of variables $x(t) = P(t)z(t)$ into an equivalent time-invariant system.

Example:

$$A(t) = \begin{bmatrix} (-1+\cos t) & 0 \\ 0 & (-2+\cos t) \end{bmatrix}, \quad T = 2\pi$$

$$\begin{aligned} \Phi(t,0) &= \begin{bmatrix} e^{\int_0^t (-1+\cos\tau)\ d\tau} & 0 \\ 0 & e^{\int_0^t (-2+\cos\tau)\ d\tau} \end{bmatrix} \\ &= \begin{bmatrix} e^{(-t+\sin t)} & 0 \\ 0 & e^{(-2t+\sin t)} \end{bmatrix} \end{aligned}$$

Find $\boldsymbol{R}$ such that

$$\begin{aligned} e^{\boldsymbol{R}T} &= \Phi(T,0) = \begin{bmatrix} e^{(-T+\sin T)} & 0 \\ 0 & e^{(-2T+\sin T)} \end{bmatrix} \\ &= \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \end{aligned}$$

$$\boldsymbol{R} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$
\begin{aligned}
P(t) &= \Phi(t,0)e^{-Rt} \\
&= \begin{bmatrix} e^{(-t+\sin t)} & 0 \\ 0 & e^{(-2t+\sin t)} \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{bmatrix} \\
&= \begin{bmatrix} e^{\sin t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}
\end{aligned}
$$

What is the effect of the change of variables on the input-output response?

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Zero-state response:

$$y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\, d\tau$$

$$x(t) = P(t)z(t)$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$P(t)\dot{z}(t) + \dot{P}(t)z(t) = A(t)P(t)z(t) + B(t)u(t)$$

$$
\begin{aligned}
\dot{z} &= \left[P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)\right] z(t) \\
&\quad + P^{-1}(t)B(t)u(t) \\
&= F(t)z(t) + P^{-1}(t)B(t)u(t)
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= C(t)x(t) + D(t)u(t) \\
&= C(t)P(t)z(t) + D(t)u(t)
\end{aligned}
$$

$$\{A, B, C, D\} \longrightarrow \boxed{x = Pz} \longrightarrow \{F, P^{-1}B, CP, D\}$$

$$F = P^{-1}AP - P^{-1}\dot{P}$$

$$
\begin{aligned}
&C(t)P(t)\Phi_F(t,\tau)P^{-1}(\tau)B(\tau) \\
&\quad = \; C(t)P(t)P^{-1}(t)\Phi_A(t,\tau)P(\tau)P^{-1}(\tau)B(\tau) \\
&\quad = \; C(t)\Phi_A(t,\tau)B(\tau)
\end{aligned}
$$

The input-output response is invariant to state transformations

What is the effect of the change of variables on internal stability?

$$A(t) \;\rightarrow\; F(t) = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)$$

$$\Phi_F(t, t_0) = P^{-1}(t)\Phi_A(t, t_0)P(t_0)$$

Lyapunov Transformation: A change of variables $x(t) = P(t)z(t)$ is a Lyapunov transformation if both $P(t)$ and $P^{-1}(t)$ are bounded and continuously differentiable for all t

Uniform stability and uniform asymptotic stability are invariant to Lyapunov transformations

Example:

$$A(t) = \begin{bmatrix} (-1 + \cos t) & 0 \\ 0 & (-2 + \cos t) \end{bmatrix}$$

$x(t) = P(t)z(t)$ with

$$P(t) = \begin{bmatrix} e^{\sin t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$$

transforms the system into

$$\dot{z} = R\, z = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z$$

$\boldsymbol{x}(t) = \boldsymbol{P}(t)\boldsymbol{z}(t)$ is a Lyapunov transformation because both $\boldsymbol{P}(t)$ and $\boldsymbol{P}^{-1}(t)$ are bounded and continuously differentiable for all t

The system $\dot{\boldsymbol{z}} = \boldsymbol{R}\boldsymbol{z}$ is asymptotically stable

Hence, the system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ is uniformly asymptotically stable

Controllability

Definition: The system $\dot{x} = A(t)x + B(t)u$, or the pair $(A(t), B(t))$, is said to be controllable on $[t_0, t_f]$ if given any initial state x_0, there is a continuous control $u(t)$ that steers the state of the system from $x(t_0) = x_0$ to $x(t_f) = 0$. It is said to be reachable on $[t_0, t_f]$ if given any final state x_f, there is a continuous control $u(t)$ that steers the state of the system from $x(t_0) = 0$ to $x(t_f) = x_f$

The Controllability Gramian of $(A(t), B(t))$ is defined by

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)\, dt$$

In the time-invariant case we took $t_0 = 0$ & $\Phi(t_0, t) = e^{-At}$

Lemma: $W_c(t_0, t_f)$ is positive definite if and only if there is no vector $x_a \neq 0$ such that

$$x_a^T \Phi(t_0, t) B(t) \equiv 0, \quad \forall\, t \in [t_0, t_f]$$

Theorem: The pair $(A(t), B(t))$ is controllable (reachable) on $[t_0, t_f]$ if and only if the controllability Gramian $W_c(t_0, t_f)$ is positive definite

Proofs are the same as in the time-invariant case

Example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ e^{-2t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad \Phi(t_0, t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t_0 - t)} \end{bmatrix}$$

$$\Phi(t_0, t)B(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t_0 - t)} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-(t + t_0)} \end{bmatrix}$$

Is there $x_a \neq 0$ such that $x_a^T \Phi(t_0, t)B(t) \equiv 0$ for all $t \in [t_0, t_f]$?

$$x_a = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad x_a^T \Phi(t_0, t) B(t) = \alpha + \beta e^{-(t+t_0)}$$

Can we find α and β (not both zero) such that $\alpha + \beta e^{-(t+t_0)} \equiv 0$ for all $t \in [t_0, t_f]$?

The answer is NO. The system is controllable on any interval $[t_0, t_f]$

Alternative method:

$$\begin{aligned} W_c(t_0, t_f) &= \int_{t_0}^{t_f} \Phi(t_0, t) B(t) B^T(t) \Phi^T(t_0, t)\, dt \\ &= \int_{t_0}^{t_f} \begin{bmatrix} 1 \\ e^{-(t+t_0)} \end{bmatrix} \begin{bmatrix} 1 & e^{-(t+t_0)} \end{bmatrix} dt \\ &= \begin{bmatrix} (t_f - t_0) & W_{c12} \\ W_{c12} & W_{c22} \end{bmatrix} \end{aligned}$$

$$W_{c12} = -\left(e^{-(t_f+t_0)} - e^{-2t_0}\right)$$

$$W_{c22} = -\frac{1}{2}\left(e^{-2(t_f+t_0)} - e^{-4t_0}\right)$$

Verify that $\det[W_c(t_0, t_f)] \neq 0$

Observability

Definition: The system

$$\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t)$$

or the pair $(A(t), C(t))$, is said to be observable on $[t_0, t_f]$ if any initial state $x(t_0) = x_0$ can be uniquely determined from $y(t)$ on $[t_0, t_f]$. It is said to be (re)constructible on $[t_0, t_f]$ if any final state $x(t_f) = x_f$ can be uniquely determined from $y(t)$ on $[t_0, t_f]$

The observability Gramian of $(A(t), C(t))$ is defined by

$$W_o(t_0, t_f) = \int_{t_0}^{t_f} \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0)\, dt$$

Lemma: $\boldsymbol{W_o(t_0, t_f)}$ is positive definite if and only if there is no vector $\boldsymbol{x_a \neq 0}$ such that

$$\boldsymbol{C(t)\Phi(t, t_0)x_a \equiv 0, \quad \forall\, t \in [t_0, t_f]}$$

Theorem: The pair $\boldsymbol{(A(t), C(t))}$ is observable (constructible) on $\boldsymbol{[t_0, t_f]}$ if and only if the observability Gramian $\boldsymbol{W_o(t_0, t_f)}$ is positive definite