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The extended Luenberger observer for nonlinear systems

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Abstract: For nonlinear single-input single-output systems $\dot{x} = f(x, u)$, $y = h(x, u)$, the relationships for a state transformation into the nonlinear observer canonical form are developed. It is possible to dimension a nonlinear observer by an eigenvalue assignment without solving the nonlinear partial differential equations for the transformation, if the transformed nonlinearities are linearized about the reconstructed state. With reference to the extended Kalman filter algorithm, this nonlinear observer design is called the extended Luenberger observer.

Keywords: Nonlinear systems, Observer, Canonical form, State transformation, Output injection, Extended linearization, Extended Luenberger observer.

1. Introduction

Consider nonlinear systems

$$\dot{x} = f(x, u), \quad x(0) = x_0, \tag{1.1}$$

$$y = h(x, u), \tag{1.2}$$

where the state x is an n-vector and the input u and the output y are scalars. The nonlinear functions $f(x, u)$ and $h(x, u)$, which in general also depend on u, are assumed to be real analytic and sufficiently smooth just as the time function $u(t)$. Furthermore, it is assumed that the system (1.1), (1.2) is locally observable, i.e. the observability matrix

$$Q(x, \bar{u}) = \begin{bmatrix} \mathcal{N}_f^0 \\ \mathcal{N}_f \\ \vdots \\ \mathcal{N}_f^{n-1} \end{bmatrix} \frac{\partial h}{\partial x} \tag{1.3}$$

has full rank in the considered domains of x and $\bar{u}$ [11]. In (1.3), the linear differential operator $\mathcal{N}_f$,

$$\mathcal{N}_f \frac{\partial h}{\partial x} := \frac{\partial h}{\partial x} \frac{\partial f}{\partial x} + f^{\mathrm{T}} \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}} + \dot{\bar{u}}^{\mathrm{T}} \frac{\partial}{\partial \bar{u}}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}} \tag{1.4}$$

with

$$\mathcal{N}_f^0 \frac{\partial h}{\partial x} = \frac{\partial h}{\partial x} \quad \text{and} \quad \bar{u} = [u, \dot{u}, \ldots, u^{(n-1)}]^{\mathrm{T}},$$

is repeatedly applied to the row vector $\partial h / \partial x$.

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In [4], the nonlinear observer canonical form

$$\dot{x}^* = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} x^* - \begin{bmatrix} a_1(y, u, \dot{u}, \dots, u^{(n)}) \\ \vdots \\ \vdots \\ a_n(y, u, \dot{u}) \end{bmatrix}$$

$$= E_n x^* - a(y, u^*) = f^*(x^*, u^*), \quad x^*(0) = x_0^*, \tag{1.5}$$

$$y = h^*(x_n^*, u), \quad x_n^* = h^{*-1}(y, u), \tag{1.6}$$

is introduced for the system (1.1), (1.2). The canonical nonlinear functions $a(y, u^*)$ and $f^*(x^*, u^*)$ also depend on the time derivatives

$$u^* = [u, \dot{u}, \dots, u^{(n)}]^{\mathrm{T}}. \tag{1.7}$$

The name *observer form* refers to the property that a nonlinear observer can be designed in canonical coordinates as in the linear case by an eigenvalue assignment. Considering this, the observer is set up in the following form:

$$\dot{\hat{x}}^* = E_n \hat{x}^* - a(y, u^*) + g^*[h^{*-1}(y, u) - \hat{x}_n^*], \quad \hat{x}^*(0) = \hat{x}_0^*, \tag{1.8}$$

where the functions $a(y, u^*)$ are realized by output and input injection. The transient behavior of the observer can be predetermined by assignment of the vector elements g_i^*, $i = 1(1)n$, which are equal to the coefficients of the characteristic polynomial of the linear part of (1.8).

This canonical observer design assumes that the nonlinear functions $a(y, u^*)$ and $h^{*-1}(y, u)$ can be calculated. But a transformation of nonlinear systems into observer canonical form is practicable only for few exceptional cases, due to the required integration of partial differential equations and inversion of nonlinear equations [1,4,6,7].

In this paper, a new method for the nonlinear observer design is developed which does not require an explicit solution of the defining equations of the state transformation.

2. Transformation into observer canonical form

In a first step, the transformation relationships derived in [1] will be extended to input-dependent functions $f(x, u)$ and $h(x, u)$, and to the general canonical output equation (1.6). The required state transformation is set up as a smooth one-to-one vector function

$$x = w(x^*, \bar{u}), \tag{2.1}$$

$$x^* = w^{-1}(x, \bar{u}), \tag{2.2}$$

which depends also on the input vector $\bar{u}$, defined in (1.4). Then (2.1) is differentiated with respect to time whereby $\dot{x}^*$ is substituted from (1.5), yielding

$$\dot{x} = \frac{\partial w}{\partial x^*} f^*(x^*, u^*) + \frac{\partial w}{\partial \bar{u}} \dot{\bar{u}}.$$

A comparison with the original differential equation (1.1) yields

$$f(x, u) = \frac{\partial w}{\partial x^*} f^*(x^*, u^*) + \frac{\partial w}{\partial \bar{u}} \dot{\bar{u}}. \tag{2.3}$$

In this equation, the Jacobian matrix

$$\frac{\partial w}{\partial x^*} = \left[\frac{\partial w}{\partial x_1^*} \; \frac{\partial w}{\partial x_2^*} \; \cdots \; \frac{\partial w}{\partial x_n^*} \right] \tag{2.4}$$

occurs. Later on, it will be seen that the column vectors of the Jacobian can be calculated more easily as a function of x than as a function of x^*. Therefore, it is necessary to introduce new symbols

$$\overline{\frac{\partial w}{\partial x_i^*}}(x, \bar{u}) := \frac{\partial w}{\partial x_i^*}\left[w^{-1}(x, \bar{u}), \bar{u}\right], \quad i=1(1)n, \tag{2.5}$$

for this functional dependence. Throughout the paper, a bar over a symbol will mean dependence on x instead of x^*.

In order to obtain the defining equations for the vectors (2.5), equation (2.3) is differentiated partially with respect to x_i^*, $i=1(1)n$. This leads, as shown in Appendix A, to the recursive formulas

$$\overline{\frac{\partial w}{\partial x_{i+1}^*}} = \mathscr{L}_f \overline{\frac{\partial w}{\partial x_i^*}} = \mathscr{L}_f^i \overline{\frac{\partial w}{\partial x_1^*}}, \quad i=1(1)n-1, \tag{2.6}$$

$$-\overline{\frac{\partial w}{\partial x^*}}\,\overline{\frac{\partial a}{\partial y}}\,\overline{\frac{\partial h^*}{\partial x_n^*}} = \mathscr{L}_f^n \overline{\frac{\partial w}{\partial x_1^*}}. \tag{2.7}$$

Here, the linear differential operator $\mathscr{L}_f$,

$$\mathscr{L}_f \overline{\frac{\partial w}{\partial x_i^*}} := \frac{\partial f}{\partial x}\overline{\frac{\partial w}{\partial x_i^*}} - \frac{\partial}{\partial x}\left(\overline{\frac{\partial w}{\partial x_i^*}}\right) f - \frac{\partial}{\partial \bar{u}}\left(\overline{\frac{\partial w}{\partial x_i^*}}\right)\dot{\bar{u}} \quad \text{with} \quad \mathscr{L}_f^0 \overline{\frac{\partial w}{\partial x_i^*}} = \overline{\frac{\partial w}{\partial x_i^*}}, \tag{2.8}$$

is applied to column vectors. By means of this operator notation, the Jacobian (2.4) reads

$$\overline{\frac{\partial w}{\partial x^*}} = \left[\mathscr{L}_f^0, \mathscr{L}_f, \ldots, \mathscr{L}_f^{n-1}\right]\overline{\frac{\partial w}{\partial x_1^*}}. \tag{2.9}$$

For the determination of the starting vector $\overline{\partial w/\partial x_1^*}$ of the recursive formulas (2.6), (2.7) and (2.9), the equivalence of the outputs (1.2) and (1.6) is used

$$h(x, u) = h^*(x_n^*, u). \tag{2.10}$$

The technique outlined in Appendix B is applied for the partial differentiation of (2.10) with respect to x_i^*, $i=1(1)n$. This yields for the starting vector

$$\overline{\frac{\partial w}{\partial x_1^*}}(x, \bar{u}) = \overline{\frac{\partial h^*}{\partial x_n^*}} Q^{-1}(x, \bar{u})[0 \quad \cdots \quad 0 \quad 1]^{\mathrm{T}}, \tag{2.11}$$

i.e., the last column of the inverse observability matrix (1.3) multiplied by $\overline{\partial h^*/\partial x_n^*}$. Thereby it must be postulated that the nonlinear system is locally observable.

With that, the partial derivatives of the transformation function $w(x^*, \bar{u})$ with respect to x_i^*, $i=1(1)n$, are defined by (2.9) and (2.11) as functions of x and $\bar{u}$. A corresponding result follows from (2.7) and (2.11) for the derivatives of the nonlinearities, $a(y, u^*)$ and $h^*(x_n^*, u)$. A problem which still remains unsolved is the actual computation of the nonlinear functions $w(x^*, \bar{u})$, $a(y, u^*)$ and $h^*(x_n^*, u)$ from these relationships. This requires an integration of nonlinear partial differential equations that result from the defining equations (2.7), (2.9) and (2.11) formulated in dependence on x^*.

In the next section, it is shown in which way a nonlinear observer can even be designed on the basis of the unsolved partial differential equations.

3. Extended linearization and observer design

According to [5,9,10], the nonlinear observer

$$\dot{\hat{x}} = f(\hat{x}, u) + g(\hat{x}, u^*)[y - h(\hat{x}, u)], \quad \hat{x}(0) = \hat{x}_0, \quad \hat{y} = h(\hat{x}, u), \tag{3.1}$$

belongs to the system (1.1), (1.2). In general, it must be assumed that the gain vector $g(\hat{x}, u^*)$ is a function of the reconstructed state $\hat{x}$ and, as proved in the following, of u^* which is defined by (1.7). In order to determine the gains, the observer (3.1) is transformed into the canonical coordinates $\hat{x}^*$,

$$\begin{aligned}\dot{\hat{x}}^* &= E_n\hat{x}^* - a(\hat{y}, u^*) + \left[\frac{\partial w}{\partial \hat{x}^*}\right]^{-1} g(\hat{x}, u^*)[h^*(x_n^*, u) - h^*(\hat{x}_n^*, u)] \\ &= f^*(\hat{x}^*, u^*) + g^*(\hat{x}^*, u^*)(y - \hat{y}),\end{aligned} \tag{3.2}$$

as shown in Section 2, using the transformations

$$\hat{x} = w(\hat{x}^*, \bar{u}), \qquad \hat{x}^* = w^{-1}(\hat{x}, \bar{u}).$$

This representation differs from the canonical observer design (1.8) by the expression $a(\hat{y}, u^*)$ instead of $a(y, u^*)$, and by the correction term which depends on $y - \hat{y}$ instead of $x_n^* - \hat{x}_n^*$. Based on the results of Section 2, it is seen that the design with equation (1.8) is not possible. Only the partial derivatives of $a(y, u^*)$ and $h^*(x_n^*, u)$ are known whereas the functions themselves remain unknown.

The dimensioning of the vector $g^*(\hat{x}^*, u^*)$ is based on the differential equation of the observer error $\tilde{x}^*$ in the canonical coordinates

$$\dot{\tilde{x}}^* = E_n\tilde{x}^* + a(y, u^*) - a(\hat{y}, u^*) + g^*(\hat{x}^*, u^*)[h^*(x_n^*, u) - h^*(\hat{x}_n^*, u)], \quad \tilde{x}^*(0) = \hat{x}_0^* - x_0^*. \tag{3.3}$$

Through an *extended linearization* of this equation about the reconstructed trajectory $\hat{x}_n^*(t)$,

$$\dot{\tilde{x}}^* = E_n\tilde{x}^* - \left(\frac{\partial a}{\partial \hat{y}}\frac{\partial h^*}{\partial \hat{x}_n^*} + g^*\frac{\partial h^*}{\partial \hat{x}_n^*}\right)\tilde{x}_n^* + \mathrm{O}(\tilde{x}_n^{*2}), \tag{3.4}$$

the following two advantages are achieved:

• The partial derivatives of $a(\hat{y}, u^*)$ and $h^*(x_n^*, u)$, known through (2.7) and (2.11) as functions of $\hat{x}$ and u^*, can be substituted.

• The error differential equation is obtained in the time-variable linear observer canonical form [1] and can be dimensioned by an eigenvalue assignment.

The choice of

$$g^*(\hat{x}^*, u^*) = \left(p - \frac{\partial a}{\partial \hat{y}}\frac{\partial h^*}{\partial \hat{x}_n^*}\right)\Big/\frac{\partial h^*}{\partial \hat{x}_n^*} \quad \text{with} \quad \frac{\partial h^*}{\partial \hat{x}_n^*} \neq 0, \tag{3.5}$$

leads to a linear time-invariant differential equation for the approximated observer error $\tilde{x}^*$ with the coefficients of the characteristic polynomial,

$$p = [p_1, p_2, \ldots, p_n]^{\mathrm{T}},$$

which must be specified. Consequently, the gain vector of the observer (3.1) reads

$$g(\hat{x}, u^*) = \overline{\frac{\partial w}{\partial \hat{x}^*}}\left(p - \overline{\frac{\partial a}{\partial \hat{y}}}\,\overline{\frac{\partial h^*}{\partial \hat{x}_n^*}}\right)\Big/\overline{\frac{\partial h^*}{\partial \hat{x}_n^*}}. \tag{3.6}$$

Here, the notation (2.5) has been transferred to the functional dependence on $\hat{x}$ or $\hat{x}^*$, respectively. By substitution of the transformation relationships (2.7), (2.9) and (2.11) into (3.6), the dimensioning rule

$$g(\hat{x}, u^*) = \left\{\left[p_1\mathscr{L}_f^0 + p_2\mathscr{L}_f + \cdots + p_n\mathscr{L}_f^{n-1} + \mathscr{L}_f^n\right]\left(\overline{\frac{\partial h^*}{\partial \hat{x}_n^*}}Q^{-1}(\hat{x}, \bar{u})\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\right)\right\}\Big/\overline{\frac{\partial h^*}{\partial \hat{x}_n^*}} \tag{3.7}$$

is obtained for the nonlinear observer (3.1). As opposed to all known design methods for nonlinear systems based on canonical forms, the application of this dimensioning rule requires neither an integration of differential equations nor an inversion of nonlinear equations. The only assumption is that the nonlinear system (1.1), (1.2) is locally observable, i.e., that the inverse $Q^{-1}(\hat{x}, \bar{u})$ of the observability matrix (1.3) exists. The dimensioning rule (3.7) is also consistent with *Ackermann's formula* for linear time-invariant observers [3].

For the computation of (3.7), the function $\overline{\partial h^*}/\partial \hat{x}_n^*$, which can be chosen arbitrarily, is of particular importance. By an appropriate choice of $\overline{\partial h^*}/\partial \hat{x}_n^* \neq 0$, the first column of the transformation Jacobian (2.11) can be simplified such that the application of the operators $\mathscr{L}_f^i$, $i = 1(1)n$, becomes as simple as possible. Finally, the obtained result is divided by the previously chosen functions $\overline{\partial h^*}/\partial \hat{x}_n^* \neq 0$.

The developed design method for nonlinear observers is essentially based on the extended linearization of the observer error system in the canonical coordinates. With reference to the *extended Kalman filter*, which is also based on a linearization about the current state estimate [2], this nonlinear observer design is called the *extended Luenberger observer*. For the admissibility of the linearization, a suitably chosen initial value $\hat{x}(0) = \hat{x}_0$ for the extended observer algorithm is an important factor.

4. Example

The extended Luenberger observer design is demonstrated for a second order polynomial system

$$\dot{x} = \begin{bmatrix} ax_1 - bx_1x_2 \\ cx_1x_2 - dx_2 - ex_2u \end{bmatrix}, \quad y = x_2. \tag{4.1}$$

For this system, the observability matrix (1.3),

$$Q(x, \bar{u}) = \begin{bmatrix} \mathscr{N}_f^0 \\ \mathscr{N}_f \end{bmatrix} \frac{\partial h}{\partial x} = \begin{bmatrix} 0 & 1 \\ cx_2 & cx_1 - d - eu \end{bmatrix}, \tag{4.2}$$

has full rank for $cx_2 \neq 0$. According to (2.11), the starting vector

$$\frac{\overline{\partial w}}{\partial x_1^*} = \begin{bmatrix} 1/cx_2 \\ 0 \end{bmatrix} \frac{\overline{\partial h^*}}{\partial x_2^*} \tag{4.3}$$

is calculated. This vector can be simplified by a corresponding choice of

$$\frac{\overline{\partial h^*}}{\partial x_2^*} = cx_2 \neq 0. \tag{4.4}$$

Now, the computation of the observer gains (3.7),

$$\begin{aligned} g(\hat{x}, u^*) &= \left\{ \left(p_1 \mathscr{L}_f^0 + p_2 \mathscr{L}_f + \mathscr{L}_f^2 \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \Big/ \frac{\overline{\partial h^*}}{\partial \hat{x}_2^*} \\ &= \left\{ p_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} a - b\hat{x}_2 \\ c\hat{x}_2 \end{bmatrix} + \begin{bmatrix} (a - b\hat{x}_2)^2 - bd\hat{x}_2 - be\hat{x}_2 u \\ c\hat{x}_2(a - b\hat{x}_2) \end{bmatrix} \right\} / c\hat{x}_2, \end{aligned} \tag{4.5}$$

requires by a factor of 5 less floating point operations than without a canonical output function $h^*(x_2^*, u)$. Moreover, a dependence on the time derivative of u is avoided.

5. Concluding remarks

In conclusion, some results and consequences for an extended Luenberger observer design are briefly summarized.

- An extended Luenberger observer exists for smooth nonlinear systems which are locally observable in the considered domains of state and input.
- This nonlinear observer design method contains – besides the choice of the function $\overline{\partial h^*}/\partial x_n^*$ – only recursive calculation steps. Therefore, a computer-aided design can be carried out easily by use of a symbolic programming language, e.g. MACSYMA, REDUCE or MUMATH.
- The most limiting restrictions are due to the assumed n-fold differentiability of the nonlinear system functions, and if necessary of the input function $u(t)$. Provided that time derivatives of the input are needed, then these are precisely only realizable if $u(t)$ is generated in a dynamical system. Hence, the differentiability of the system nonlinearities also includes the differentiability of the input function.
- The extended Luenberger observer is a method dual to the pseudolinearization for the nonlinear state feedback design [8].
- The differential operators $\mathscr{L}_f$ and $\mathscr{N}_f$ used in the paper contain the *Lie derivatives* of a column and a row vector, respectively, with respect to a nonlinear vector field $f(x, u)$. But here the input dependence of the vector field is to be taken into consideration.
- An application of the extended Luenberger observer for multiple-output systems is straightforward. In that case, a canonical output function belongs to each output variable and the computation of the observer gains can be simplified even more by an appropriate choice of those functions.

Appendix A: Derivation of the differential operator $\mathscr{L}_f$

The partial differentiations of (2.3) with respect to x_i^*, $i = 1(1)n$, result in

$$\frac{\partial f}{\partial x_i^*} = \frac{\partial}{\partial x_i^*}\left[\frac{\partial w}{\partial x^*}\right] f^* + \frac{\partial w}{\partial x^*}\frac{\partial f^*}{\partial x_i^*} + \frac{\partial}{\partial x_i^*}\left[\frac{\partial w}{\partial \bar{u}}\right]\dot{\bar{u}} \tag{A1}$$

where $f = f[w(x^*, \bar{u}), u]$ is taken into consideration. Assuming the differentiability required for the various functions and using the notation (2.5) and the canonical form (1.5), (1.6) each term of (A1) can be converted as follows:

$$\frac{\partial f}{\partial x_i^*} = \frac{\partial f}{\partial x}\frac{\overline{\partial w}}{\partial x_i^*}, \tag{A2}$$

$$\begin{aligned}\frac{\partial}{\partial x_i^*}\left[\frac{\partial w}{\partial x^*}\right] f^* &= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x_i^*}\right)\frac{\partial w}{\partial x^*} f^* \quad \text{with (2.3)} \\ &= \frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x_i^*}\right)\left[f - \frac{\overline{\partial w}}{\partial \bar{u}}\dot{\bar{u}}\right],\end{aligned} \tag{A3}$$

$$\frac{\partial w}{\partial x^*}\frac{\partial f^*}{\partial x_i^*} = \begin{cases} \dfrac{\overline{\partial w}}{\partial x_{i+1}^*} & \text{for } i = 1(1)n-1, \\ -\dfrac{\overline{\partial w}}{\partial x^*}\dfrac{\overline{\partial a}}{\partial y}\dfrac{\overline{\partial h^*}}{\partial x_n^*} & \text{for } i = n, \end{cases} \tag{A4}$$

$$\frac{\partial}{\partial x_i^*}\left[\frac{\partial w}{\partial \bar{u}}\right]\dot{\bar{u}} = \left[\frac{\partial}{\partial \bar{u}}\left(\frac{\overline{\partial w}}{\partial x_i^*}\right) + \frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x_i^*}\right)\frac{\overline{\partial w}}{\partial \bar{u}}\right]\dot{\bar{u}}. \tag{A5}$$

By substitution of (A2)–(A5) in (A1), the recursive formulas

$$\frac{\overline{\partial w}}{\partial x^*_{i+1}} = \frac{\partial f}{\partial x}\frac{\overline{\partial w}}{\partial x^*_i} - \frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x^*_i}\right) f - \frac{\partial}{\partial \bar{u}}\left(\frac{\overline{\partial w}}{\partial x^*_i}\right)\dot{\bar{u}}, \quad i=1(1)n-1, \tag{A6}$$

$$-\frac{\overline{\partial w}}{\partial x^*}\frac{\overline{\partial a}}{\partial y}\frac{\overline{\partial h^*}}{\partial x^*_n} = \frac{\partial f}{\partial x}\frac{\overline{\partial w}}{\partial x^*_n} - \frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x^*_n}\right) f - \frac{\partial}{\partial \bar{u}}\left(\frac{\overline{\partial w}}{\partial x^*_n}\right)\dot{\bar{u}}, \tag{A7}$$

are obtained. By introducing the operator $\mathscr{L}_f$ – defined in (2.8) – (A6) and (A7) can be written as (2.6) and (2.7), respectively.

Appendix B: Derivation of the starting vector $\overline{\partial w}/\partial x_1^*$

Partial differentiation of (2.10) with respect to x_i^*, $i=1(1)n$, yields

$$\frac{\partial h}{\partial x}\frac{\overline{\partial w}}{\partial x^*_i} = \delta_{in}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad i=1(1)n. \tag{B1}$$

Notation (2.5) and the Kronecker delta δ_{in} have been used here. Substitution of (A6) in the last $n-1$ equations yields

$$\frac{\partial h}{\partial x}\left[\frac{\partial f}{\partial x}\frac{\overline{\partial w}}{\partial x^*_{i-1}} - \frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x^*_{i-1}}\right) f - \frac{\partial}{\partial \bar{u}}\left(\frac{\overline{\partial w}}{\partial x^*_{i-1}}\right)\dot{\bar{u}}\right] = \delta_{in}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad i=2(1)n. \tag{B2}$$

Moreover, the first $n-1$ equations of (B1) are differentiated totally with respect to t, yielding

$$\left[f^{\mathrm{T}}\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}} + \dot{\bar{u}}^{T}\frac{\partial}{\partial \bar{u}}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}}\right]\frac{\overline{\partial w}}{\partial x^*_i} + \frac{\partial h}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\overline{\partial w}}{\partial x^*_i}\right) f + \frac{\partial}{\partial \bar{u}}\left(\frac{\overline{\partial w}}{\partial x^*_i}\right)\dot{\bar{u}}\right] = 0, \quad i=1(1)n-1, \tag{B3}$$

whereby $\dot{x}$ is substituted from (1.1). After an index shift the last two terms of (B3) are introduced into (B2), yielding

$$\left[\frac{\partial h}{\partial x}\frac{\partial f}{\partial x} + f^{\mathrm{T}}\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}} + \dot{\bar{u}}^{\mathrm{T}}\frac{\partial}{\partial \bar{u}}\left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}}\right]\frac{\overline{\partial w}}{\partial x^*_{i-1}} = \delta_{in}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad i=2(1)n. \tag{B4}$$

By using the differential operator $\mathscr{N}_f$ as defined in (1.4), equation (B4) is written more concisely as

$$\left[\mathscr{N}_f\frac{\partial h}{\partial x}\right]\frac{\overline{\partial w}}{\partial x^*_{i-1}} = \delta_{in}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad i=2(1)n. \tag{B5}$$

These equations are of the same type as (B1), if $\partial h/\partial x$ is replaced by $\mathscr{N}_f\partial h/\partial x$ there. Through repeated application of the operations (B2) and (B3) to the equations that are formed in each case, a total number of n systems of equations with a decreasing number of equations is obtained:

$$\left[\mathscr{N}_f^{k-1}\frac{\partial h}{\partial x}\right]\frac{\overline{\partial w}}{\partial x^*_{i-(k-1)}} = \delta_{in}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad i=k(1)n;\ k=1(1)n, \quad \text{with } \mathscr{N}_f^0\frac{\partial h}{\partial x} = \frac{\partial h}{\partial x}. \tag{B6}$$

Every first equation in this system, i.e. $i=k$, is used for the determination of $\overline{\partial w}/\partial x_1^*$:

$$\left[\mathscr{N}_f^{k-1}\frac{\partial h}{\partial x}\right]\frac{\overline{\partial w}}{\partial x^*_1} = \delta_{kn}\frac{\overline{\partial h^*}}{\partial x^*_n}, \quad k=1(1)n. \tag{B7}$$

These equations can be summarized by use of definition (1.3) for $Q(x, \bar{u})$ to

$$Q(x, \bar{u})\frac{\overline{\partial w}}{\partial x^*_1} = \frac{\overline{\partial h^*}}{\partial x^*_n}[0 \quad \cdots \quad 0 \quad 1]^{\mathrm{T}} \tag{B8}$$

from which (2.11) follows directly.

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