A Student's t Filter for Heavy Tailed Process and Measurement Noise

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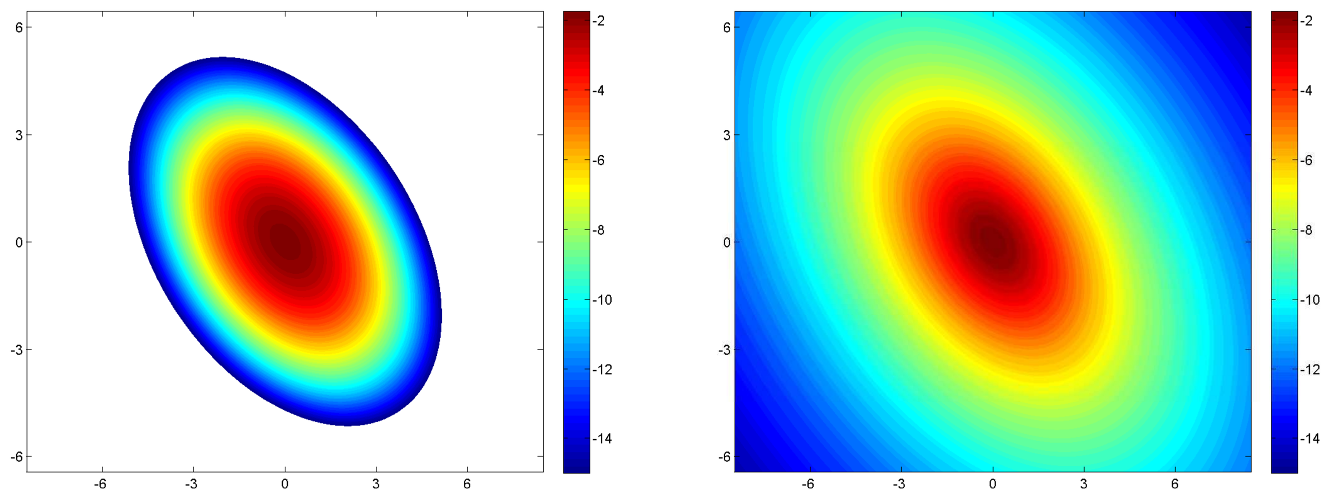
Summary

- Filtering based on Student's t distribution.
- Covers heavy tailed process and measurement noise.
- New filter is a generalization of the ubiquitous Kalman filter.
- Simple to implement, computationally cheap.
- Performance shown on challenging tracking example.

Student's t distribution

Can model many **real world phenomena**: measurement outliers, target maneuvers in tracking, linearization errors in linearized systems. **Heavier tails** than the Gaussian.



(a) $\ln \mathcal{N}(x; \hat{x}, P)$. (b) $\ln \mathrm{St}(x; \hat{x}, P, \nu)$.

Figure 1: Heat maps of density logarithms of Gaussian and t distribution with same parameters $\hat{x}$ and P, and $\nu = 5$ for the t distribution. White areas in (a) correspond to values below -15.

Almost as convenient as the Gaussian with formulas for affine transformations, marginalization, conditioning. The **conditional distribution** of t random vector is **again t** but with increased degrees of freedom.

Filtering problem with Student's t noise

We consider the linear state space model

$$x_{k+1} = F_k x_k + v_k,$$
$$y_k = H_k x_k + e_k.$$

Initial state $x_0 \sim \mathrm{St}(\hat{x}_0, P_0, \eta_0)$, process noise $v_k \sim \mathrm{St}(0, Q_k, \gamma_k)$, measurement noise $e_k \sim \mathrm{St}(0, R_k, \delta_k)$.

No closed form posterior is available in general.

Suggested algorithm

Idea: Assume $x_k \sim \mathrm{St}(\hat{x}_{k|k}, P_{k|k}, \eta_k)$ and employ the rules for the t distribution recursively. Include careful approximations to obtain closed form expressions. Collapses to the Kalman filter for Gaussian noise.

The **time update** consists of an approximation of the joint density $p(x_k, v_k | y_{1:k})$ and a Kalman filter like prediction:

Input $\hat{x}_{k|k}$, $P_{k|k}$, η_k
$P_{k|k}, Q_k, \eta_k, \gamma_k \to \widetilde{P}_{k|k}, \widetilde{Q}_k, \widetilde{\eta}_k$ ▷ joint parameters

$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k}$
$P_{k+1|k} = \widetilde{Q}_k + F_k \widetilde{P}_{k|k} F_k^{\mathrm{T}}$ ▷ prediction similar to KF
Return $\hat{x}_{k+1|k}$, $P_{k+1|k}$

The **measurement update** can be summarized as follows:

Input $\hat{x}_{k|k-1}$, $P_{k|k-1}$, η_{k-1}
$P_{k|k-1}, R_k, \eta_{k-1}, \delta_k \to \widetilde{P}_{k|k-1}, \widetilde{R}_k, \overline{\eta}_{k-1}$ ▷ joint parameters

$S_k = H_k \widetilde{P}_{k|k-1} H_k^{\mathrm{T}} + \widetilde{R}_k$
$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \widetilde{P}_{k|k-1} H_k^{\mathrm{T}} S_k^{-1} (y_k - H_k \hat{x}_{k|k-1})$
$\overline{P}_{k|k} = \widetilde{P}_{k|k-1} - \widetilde{P}_{k|k-1} H_k^{\mathrm{T}} S_k^{-1} H_k \widetilde{P}_{k|k-1}$ ▷ similar to KF

$\Delta^2_{y,k} = (y_k - H_k \hat{x}_{k|k-1})^{\mathrm{T}} S_k^{-1} (y_k - H_k \hat{x}_{k|k-1})$
$P_{k|k} = \frac{\overline{\eta}_{k-1} + \Delta^2_{y,k}}{\overline{\eta}_{k-1} + d_y} \overline{P}_{k|k}$ ▷ matrix depends on y_k
$\eta_k = \overline{\eta}_{k-1} + d_y$ ▷ increase degrees of freedom
Return $\hat{x}_{k|k}$, $P_{k|k}$, η_k

An approximation of $p(x_k, e_k | y_{1:k-1})$ is required. The Kalman filter equations are contained but both $\hat{x}_{k|k}$ and $P_{k|k}$ are **observation dependent**.

The required **density approximations** amount to finding few scalar parameters (common degrees of freedom, matrix scale factors). (Numerical) minimization of the Kullback-Leibler divergence is possible.

Example: maneuvering target in clutter

Constant velocity model with position measurements. Process and measurement noise are corrupted by large outliers. The nominal (outlier free) and true noise statistics are available.

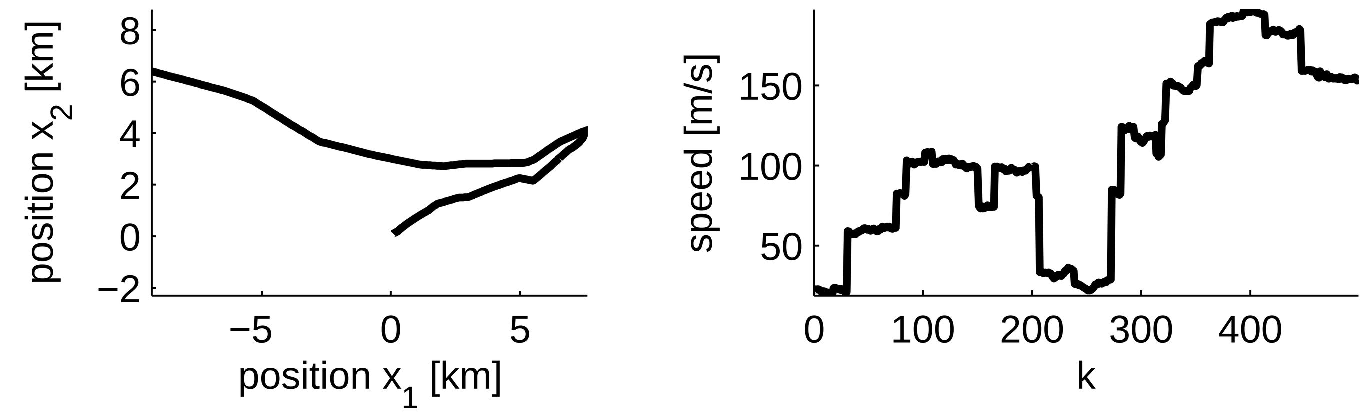


Figure 2: One realization of the example. Position trajectory and speed profile.

Compared are three filters: a Kalman filter with nominal noise parameters KF, a Kalman filter with true parameters KF2 (the best linear unbiased estimator), and the suggested filter TF with nominal parameters and three degrees of freedom for all signals.

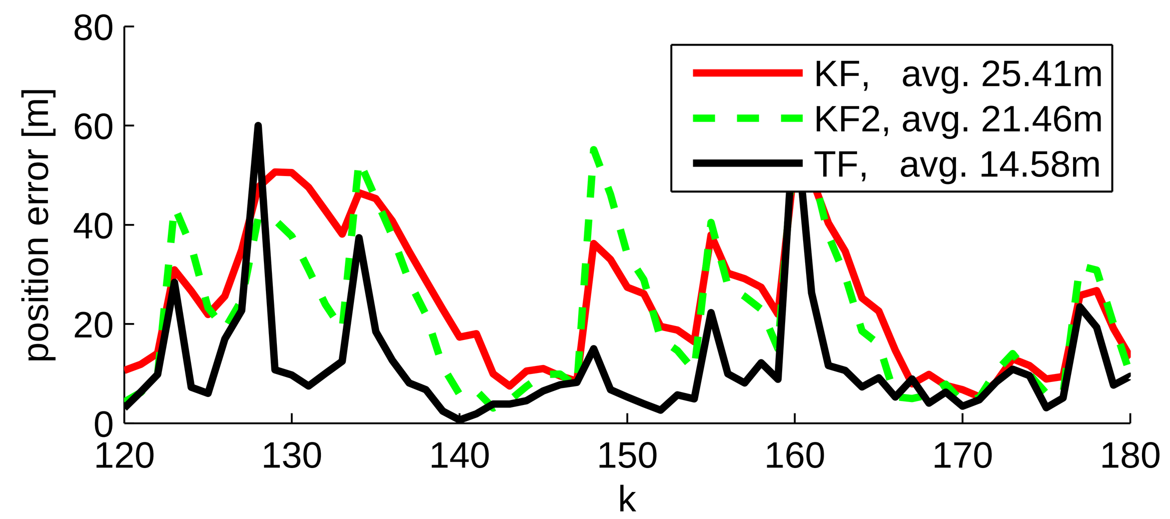


Figure 3: Position error over time for one realization.

All filters show spikes in the position error that stem from either a target maneuver or a measurement outlier. TF is the fastest to decrease the spikes, and outperforms its competitors with mere knowledge of the nominal (outlier free) noise parameters. The results are confirmed over MC runs with randomized noise parameters.

Further information, paper, and technical report on the t distribution available at
`http://users.isy.liu.se/en/rt/roth/`