Decision Theory

Exercise 1

In many pattern classification problems one has the option either to assign the pattern to one of K classes, or to reject it as being unrecognizable. If the cost for rejects is not too high, rejection may be a desirable action. Let

$$C(\alpha_i|\omega_j) = \begin{cases} 0 & if\ i = j,\ for\ i, j = 1, ..., K \\ C_r & if\ i = K + 1 \\ C_s & otherwise \end{cases}$$

where C_r is the loss incurred for choosing the $(K + 1)$th action (rejection), and C_s is the loss incurred for making a substitution error. Show that the minimum risk is obtained if we decide ω_i if $P(\omega_i|\boldsymbol{x}) \geq P(\omega_j|\boldsymbol{x})$ for all j and if $P(\omega_i|\boldsymbol{x}) \geq 1 - \frac{C_r}{C_s}$, and reject otherwise. What happens if $C_r = 0$? What happens if $C_r > C_s$?

Solution Exercise 1

a) In this problem we have K classifiers and a rejection option, so we have in total $K+1$ possible actions. We need to find the optimal policy that minimizes the risk.
The overall risk is given by:

$$R = \int R\big(\alpha(\boldsymbol{x})|\boldsymbol{x}\big)p(\boldsymbol{x})dx \tag{1}$$

A general decision rule is a function $\alpha(\boldsymbol{x})$ which tells us which action to take for every possible observation. Clearly if $\alpha(\boldsymbol{x})$ is choosed such that $R\big(\alpha(\boldsymbol{x})|\boldsymbol{x}\big)$ is as small as possible for every $\boldsymbol{x}$, then the overall risk will be minimized.

The conditional risk is given defined as:

$$R(\alpha_i|\boldsymbol{x}) = \sum_{j=1}^{K} c_{ij} P(\omega_j|\boldsymbol{x}) \tag{2}$$

and it's the risk of taking action α_i(which means choosing the cluster ω_i). We need then to calculate the risk of all actions and find the rule that minimize them.

For the classify actions $i = 1..K$:

$$R(\alpha_i|\boldsymbol{x}) = \sum_{j=1, j\neq i}^{K} C_s P(\omega_j|\boldsymbol{x}) = C_s \sum_{j=1, j\neq i}^{K} P(\omega_j|\boldsymbol{x}) \tag{3}$$

As $\sum_{j=1}^{K} P(\omega_j|\boldsymbol{x}) = 1$, we could write the above equation as:

$$R(\alpha_i|\boldsymbol{x}) = C_s(1 - P(\omega_i|\boldsymbol{x})) \tag{4}$$

To minimize this risk we just have to maximize $P(\omega_i|\boldsymbol{x})$, so we just need to choose the highest $P(\omega_i|\boldsymbol{x})$ of all the classifiers. Written formally:

$$choose\ \omega_i\ \ if P(\omega_i|\boldsymbol{x})\ >\ P(\omega_j|\boldsymbol{x}) \forall j \neq i \tag{5}$$

For the reject action:

$$R(\alpha_{K+1}|\boldsymbol{x}) = C_r \tag{6}$$

Now the only decision missing is to see when are we going to classify and when will we reject. We will just choose the option with the smallest risk:

$$min_{1\leq i\leq K+1} \; [\; C_s(1 - P(\omega_i|\boldsymbol{x})) \;,\; C_r \;] \tag{7}$$

To see when we classify, we just need when the risk of classifying is smaller than rejecting:

$$C_s(1 - P(\omega_i|\boldsymbol{x})) \leq C_r \tag{8}$$

$$P(\omega_i|\boldsymbol{x}) \geq 1 - \frac{C_r}{C_s} \tag{9}$$

b) If $C_r = 0 \;\rightarrow$ always reject.
If $C_r > C_s \;\rightarrow$ always classify.

Exercise 2

A mobile robot is required to navigate towards a goal position while avoiding possible collisions with moving obstacles. The robot can perform only two actions: *maneuvre* (α_1) to avoid the obstacles or *stop* (α_2) at the current position if the obstacles are too close. The *maneuvre* action modifies the robot's desired orientation θ_d according to:

$$\theta_d = \begin{cases} \theta_r + \theta_{av} & \text{if } d < d_{safe} \\ \theta_r & \text{otherwise} \end{cases} \tag{10}$$

where θ_r is the measured orientation of the robot, θ_{av} is the rotation needed to avoid the obstacles and d is the distance between the robot and the obstacles. All the obstacles at a distance $d > d_{safe}$ do not affect the robot's path.

At each time instant, the state of path the robot is executing can be: *obstacle* (ω_1) or *no_obstacle* (ω_2). In a first stage, a human is remotely guiding the robot towards the path. From the collected observations, the robot has learned that $P(\omega_1) = 0.7$ and $P(\omega_2) = 0.3$. Assume as costs $c_{11} = 0$, $c_{12} = c_{21} = 5$ and $c_{22} = 10$, summarized in the following table:

	ω_1	ω_2
α_1	0	5
α_2	5	10

The robot is equipped with a range sensor that generates a 3D point cloud. Points at a distance $d < d_{safe} = 1m$ from the robot (x_1) are considered as obstacles, points at a distance $d \geq d_{safe} = 1m$ (x_2) are considered as free-space (*no_obstacle*). Due to the noisy data d cannot be accurately estimated, suppose: $P(x_1|\omega_1) = 0.8$ and $P(x_2|\omega_2) = 0.7$.

a) Comment the choice of the costs c_{11} and c_{22}.

b) Using the Bayes risk criterion determine which is the best action considering the observations from the range sensor.

Solution Exercise 2

a) $c_{11} = 0 \longrightarrow$ do not penalize the *maneuvre* action when there are obstacles.

$c_{22} = 10 \longrightarrow$ strongly penalize the *stop* action when there are no obstacles.

b) Given $P(x_1|\omega_1) = 0.8$ and $P(x_2|\omega_2) = 0.7$ we can compute $P(x_2|\omega_1) = 1 - P(x_1|\omega_1) = 0.2$ and $P(x_1|\omega_2) = 1 - P(x_2|\omega_2) = 0.3$. The Bayes risk is defined as:

$$R(\alpha_i|\boldsymbol{x}) = \sum_{j=1}^{2} c_{ij} P(\omega_j|\boldsymbol{x}), \quad i = 1, \ldots, 2$$

hence, the only unknowns are the posteriors $P(\omega_j|\boldsymbol{x})$. First, calculate $P(x_1)$ and $P(x_2)$:

$$P(x_1) = P(x_1|\omega_1)P(\omega_1) + P(x_1|\omega_2)P(\omega_2) = 0.8 * 0.7 + 0.3 * 0.3 = 0.65$$
$$P(x_2) = 1 - P(x_1) = 1 - 0.65 = 0.35$$

Then, using the Bayes rule:

$$\begin{aligned}
R(\alpha_1|x_1) &= P(\omega_1|x_1)c_{11} + P(\omega_2|x_1)c_{12} = 0 + P(\omega_2|x_1) * 5 \\
&= \frac{P(x_1|\omega_2)P(\omega_2)}{P(x_1)} * 5 = \frac{0.3 * 0.3}{0.65} * 5 = 0.69 \\
R(\alpha_2|x_1) &= P(\omega_1|x_1)c_{21} + P(\omega_2|x_1)c_{22} = P(\omega_1|x_1) * 5 + P(\omega_2|x_1) * 10 \\
&= \frac{P(x_1|\omega_1)P(\omega_1)}{P(x_1)} * 5 + \frac{P(x_1|\omega_2)P(\omega_2)}{P(x_1)} * 10 = \frac{0.8 * 0.7}{0.65} * 5 + \frac{0.3 * 0.3}{0.65} * 10 = 4.3 + 1.39 = 5.69 \\
R(\alpha_1|x_2) &= P(\omega_1|x_2)c_{11} + P(\omega_2|x_2)c_{12} = 0 + P(\omega_2|x_2) * 5 \\
&= \frac{P(x_2|\omega_2)P(\omega_2)}{P(x_2)} * 5 = \frac{0.7 * 0.3}{0.35} * 5 = 3 \\
R(\alpha_2|x_2) &= P(\omega_1|x_2)c_{21} + P(\omega_2|x_2)c_{22} = P(\omega_1|x_2) * 5 + P(\omega_2|x_2) * 10 \\
&= \frac{P(x_2|\omega_1)P(\omega_1)}{P(x_2)} * 5 + \frac{P(x_2|\omega_2)P(\omega_2)}{P(x_2)} * 10 = \frac{0.2 * 0.7}{0.35} * 5 + \frac{0.7 * 0.3}{0.35} * 10 = 2 + 6 = 8
\end{aligned}$$

From the Bayes risk is obvious that the robot always chooses the *maneuvre* action.

Classifiers

Exercise 3

To play tennis, it is important to consider the weather conditions. Using a *naive* Bayes classifier, classify the days based on whether somebody plays on that day or not using the following Table.

Use the training data from the Table to compute the probability of the following new instance:
(Outlook = sunny, Temperature = cool, Humidity = high, Wind = strong).

Solution Exercise 3

From the Bayesian rule

$$P(\omega|\boldsymbol{x}) = \frac{P(\omega)P(\boldsymbol{x}|\omega)}{P(\boldsymbol{x})} \quad ,$$

The naive Bayes classifier follows the naive assumption (not true in general) that the features are independent of each other i.e. for a m dimensional feature vector $\boldsymbol{x}$:

Day	Outlook	Temperature	Humidity	Wind	Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Overcast	Cool	Normal	Weak	Yes
D7	Rain	Cool	Normal	Strong	No
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

$P(\boldsymbol{x}|\omega) = P(x_1, x_2, ..., x_m|\omega) = P(x_1|\omega)P(x_2|\omega)...P(x_m|\omega)$ Since we will select the class with maximum posterior probability, the normalization term is not important.

$$\begin{aligned}P(\text{tennis}|E) &\propto P(\text{tennis})\prod P(E|\text{tennis})\\ &= P(\text{tennis})P(\text{outlook=sunny}|\text{tennis})P(\text{temperature=cool}|\text{tennis})\\ &P(\text{humidity=high}|\text{tennis})P(\text{wind=strong}|\text{tennis})\end{aligned}$$

In the same way, we calculate the probability NOT to play tennis given the new data.

$$\begin{aligned}P(\text{no tennis}|E) &\propto P(\text{no tennis})\prod P(E|\text{no tennis})\\ &= P(\text{no tennis})P(\text{outlook=sunny}|\text{no tennis})P(\text{temperature=cool}|\text{no tennis})\\ &P(\text{humidity=high}|\text{no tennis})P(\text{wind=strong}|\text{no tennis})\end{aligned}$$

The probabilities are computed as: $P(\text{playtennis=yes}) = \frac{9}{14}$ and $P(\text{playtennis=no}) = \frac{5}{14}$
The conditional probabilities are
$P(\text{humidity=high}|\text{playtennis=yes}) = \frac{3}{9}$
$P(\text{wind=strong}|\text{playtenis=yes}) = \frac{2}{9}$
$P(\text{yes})P(\text{sunny}|\text{yes})P(\text{cool}|\text{yes})P(\text{high}|\text{yes})P(\text{strong}|\text{yes}) = \frac{9}{14} * \frac{2}{9} * \frac{3}{9} * \frac{3}{9} * \frac{2}{9} = 0.0035$
$P(\text{no})P(\text{sunny}|\text{no})P(\text{cool}|\text{no})P(\text{high}|\text{no})P(\text{strong}|\text{no}) = \frac{5}{14} * \frac{3}{5} * \frac{1}{5} * \frac{4}{5} * \frac{3}{5} = 0.0206$
The maximum probability is derived for not playing tennis.