Machine Learning in Robotics
Gaussian Mixture Model and EM algorithm

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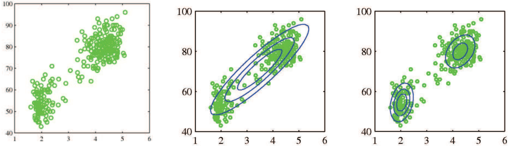
Today Lecture Outline

- Gaussian Mixture Model
- GMM Learning
- General Expectation Maximization Algorithm

Mixture models

Consider the problem of modeling a pdf given a dataset of examples $\boldsymbol{X} = \{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(n)}\}$

- If the form of the underlying pdf is known (e.g. Single Gaussian distribution), the problem could be solved using the Maximum Likelihood Estimation method



Old Faithful data from Bishop2006

- Now we will consider an alternative density estimation method which is modeling the pdf with a mixture of parametric densities. In particular, we will focus on mixture models of Gaussian densities

$$p(\boldsymbol{x}|\boldsymbol{\theta}) = \sum_{j=1}^{K} p(\boldsymbol{x}|\boldsymbol{\theta}_j)p(\omega_j) = \sum_{j=1}^{K} \pi_j \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$$

Gaussian Mixture Model (GMM)

- Mixture of Gaussians
  - A superposition of K Gaussian densities $p(\boldsymbol{x}) = \sum_{j=1}^{K} \pi_j \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$
  - Parameters $\pi_j, \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j$
  - Properties: Asymmetry, multi-modality $0 \leq \pi_j \leq 1, \quad \sum_{j=1}^{K} \pi_j = 1$
- Previously, we estimated parameters for a single Gaussian distribution by MLE
- Log-likelihood function

$$l(\boldsymbol{\theta}) = \ln p(\boldsymbol{X}|\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{i=1}^{n} \left[ln \left[\sum_{k=1}^{K} \pi_k \mathcal{N}(\boldsymbol{x}^{(i)}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right] \right]$$

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Find the maximum of this function by differentiation
for $\boldsymbol{\Sigma}_k = \sigma_k^2 \boldsymbol{I}$

$$\frac{\partial l}{\partial \boldsymbol{\mu}_j} = 0 \rightarrow \hat{\boldsymbol{\mu}}_j = \frac{\sum_{i=1}^{n} p(\omega_j|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})\boldsymbol{x}^{(i)}}{\sum_{i=1}^{n} p(\omega_j|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})}$$

$$\frac{\partial l}{\partial \sigma_j} = 0 \rightarrow \hat{\sigma}_j^2 = \frac{\sum_{i=1}^{n} p(\omega_j|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})(\boldsymbol{x}^{(\boldsymbol{i})} - \boldsymbol{\mu}_j)^2}{\sum_{i=1}^{n} p(\omega_j|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})}$$

$$\frac{\partial l}{\partial \pi_j} = 0 \rightarrow \hat{\pi}_j = \frac{\sum_{i=1}^{n} p(\omega_j|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})}{\sum_{k=1}^{K} \sum_{i=1}^{n} p(\omega_k|\boldsymbol{x}^{(i)}, \boldsymbol{\theta})}$$

Gaussian Mixture Model (GMM)

- NOT a closed form analytical solution for GMM parameters
- Due to responsibility depends on the GMM parameters
- Highly non-linear coupled system of equations

$\Rightarrow$ Iterative Numerical Optimization Technique is necessary. EM algorithm

EM for GMM

Given a Gaussian Mixture Model, the goal is to maximize the likelihood function w.r.t the parameters

1. Initialize $\pi_j, \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j$
2. E-step: Evaluate the responsibilities using the current parameters

$$p(\omega_k|\mathbf{x}^{(i)}, \theta) = \frac{\pi_k \mathcal{N}(\mathbf{x}^{(i)}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\mathbf{x}^{(i)}|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)}$$

3. M-step: Re-estimate the parameters using the current responsibilities

$$\hat{\boldsymbol{\mu}}_k = \frac{1}{n_k} \sum_{i=1}^{n} p(\omega_k|\mathbf{x}^{(i)}, \theta)\mathbf{x}^{(i)}$$

$$\hat{\boldsymbol{\Sigma}}_k = \frac{1}{n_k} \sum_{i=1}^{n} p(\omega_k|\mathbf{x}^{(i)}, \theta)(\mathbf{x}^{(i)} - \hat{\boldsymbol{\mu}}_k)(\mathbf{x}^{(i)} - \hat{\boldsymbol{\mu}}_k)^T$$

$$\hat{\pi}_k = \frac{n_k}{n} \quad \text{where} \quad n_k = \sum_{i=1}^{n} p(\omega_k|\mathbf{x}^{(i)}, \theta)$$

4. Evaluate the log-likelihood and check for convergence of either the parameters or the log-likelihood. If not converged, go to step 2.

$$l(\boldsymbol{\theta}) = \ln p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \ln \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}^{(i)}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

Example

Probability to get a credit (A,B,C,D) from a lecture depends on the mean. Assume that the number of students for each credit (A,B,C,D) is a, b, c, d. Estimate the mean.

$$\begin{aligned} \omega_1 &= A & P(A) &= 1/2 \\ \omega_2 &= B & P(B) &= \mu \\ \omega_3 &= C & P(C) &= 2\mu \\ \omega_4 &= D & P(D) &= 1/2 - 3\mu \\ &\text{where} & 0 &\leq \mu \leq 1/6 \end{aligned}$$

$$P(A) + P(B) + P(C) + P(D) = 1$$

$$P(a, b, c, d|\mu) =$$

$$\frac{\partial lnP(a, b, c, d|\mu)}{\partial \mu} =$$

$$\Rightarrow \mu = \frac{b + c}{6(b + c + d)}$$

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\end{array}
$$

$$
\begin{aligned}
& P(A) + P(B) + P(C) + P(D) = 1 \\
& P(a,b,c,d|\mu) = K(1/2)^a(\mu)^b(2\mu)^c(1/2-3\mu)^d \\
& \frac{\partial lnP(a,b,c,d|\mu)}{\partial \mu} = \frac{b}{\mu} + \frac{2c}{2\mu} - \frac{3d}{1/2-3\mu} = 0 \\
& \Rightarrow \mu = \frac{b+c}{6(b+c+d)}
\end{aligned}
$$

Example

If it is known that $a = 14, b = 6, c = 9, d = 10$, then $\mu = 1/10$. However, now assume that c and d are known, but a and b are unknown.

$$a = \frac{0.5}{0.5 + \mu} h \ , \ b = \frac{\mu}{0.5 + \mu} h \ \Leftrightarrow \ \mu = \frac{b + c}{6(b + c + d)}$$

EM algorithm : Start with an initial parameter. Iterate E-step and M-step.

- Initialization : $\mu(0)$
- E-step : $b = \frac{\mu(t)}{1/2 + \mu(t)} h = \mathbb{E}[b | \mu(t)]$
- M-step : $\mu(t + 1) = \frac{b + c}{6(b + c + d)}$

Example

Given $h = 20, c = 9, d = 10$, estimate μ with an initial guess $\mu(0) = 0$ by the EM algorithm

t	$\mu(t)$	$b(t)$
0	0	0
1	0.0789	1. 3636
2	0.0848	1.4504
3	0.0852	1.4555
4	0.0852	1.4557
5	0.0852	1.4558
6	0.0852	1.4558

The Expectation-Maximization algorithm

- The EM is a general method for finding the ML estimate of the parameters of a pdf when the data has missing values

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- Assume a dataset containing two types of features
    - A set of features $\boldsymbol{X}$ whose value is known. We call these the *incomplete* data
    - A set of features $\boldsymbol{Z}$ whose value is unknown. We call these the *missing* data
    - $\boldsymbol{\theta}$: model parameters

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- We now define a joint pdf $p(\boldsymbol{X}, \boldsymbol{Z}|\boldsymbol{\theta})$ called the complete-data likelihood
- As suggested by its name, the EM algorithm operates by performing two basic operations over and over:
    - An Expectation step
    - A Maximization step

The Expectation-Maximization algorithm

- EXPECTATION : Find the expected value of the log-likelihood $\ln p(\boldsymbol{X}, \boldsymbol{Z}|\boldsymbol{\theta})$ with respect to the unknown data $\boldsymbol{Z}$, given the data $\boldsymbol{X}$ and the current parameter estimates $\boldsymbol{\theta}$

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{old}) = \sum_{\boldsymbol{Z}} p(\boldsymbol{Z}|\boldsymbol{X}, \boldsymbol{\theta}^{old}) \ln p(\boldsymbol{X}, \boldsymbol{Z}|\boldsymbol{\theta})$$

- MAXIMIZATION : Find the argument $\boldsymbol{\theta}$ that maximizes the expected value defined by $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{old})$

$$\boldsymbol{\theta}^{new} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{old})$$

- Convergence properties
  - Each iteration (E+M) is guaranteed to increase the log-likelihood.
  - EM algorithm is guaranteed to converge to a local maximum of the likelihood function.

EM algorithm to find MAP solution

- EM can be used to find MAP (maximum a posterior) solutions for models in which a prior $p(\boldsymbol{\theta})$ is defined over parameters.
- E-step : same as EM for ML
- M-step : Find the argument $\boldsymbol{\theta}$ that maximizes the expected value defined by $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{old}) + lnp(\boldsymbol{\theta})$ instead of $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{old})$

GMM revisited

- Consider the problem of maximizing the likelihood for the complete data set $\{\boldsymbol{X}, \boldsymbol{Z}\}$
- If complete data set $\{\boldsymbol{X}, \boldsymbol{Z}\}$ is given, the complete-data log likelihood function can be maximized trivially in closed form.
- In practice, the latent variables are not given. Therefore, we consider the expectation of the complete-data log-likelihood, wrt the posterior distribution of the latent variables.

$$\mathbb{E}[z_k^{(i)}] = \frac{\pi_k \mathcal{N}(\boldsymbol{x}^{(i)} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\boldsymbol{x}^{(i)} | \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)} = \gamma(z_k^{(i)})$$

$$\mathbb{E}_z \left[\ln P(\boldsymbol{X}, \boldsymbol{Z} | \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\pi})\right] = \sum_{i=1}^{n} \sum_{k=1}^{K} \gamma(z_k^{(i)}) \left\{ \ln \pi_k + \ln \mathcal{N}(\boldsymbol{x}^{(i)} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right\}$$

Relation to k-means

- EM for GMM
  - Soft assignment of data points to clusters
- K-means
  - Hard assignment of data points to clusters
  - A special case of "EM for GMM"

$$\mathbb{E}[z_k^{(i)}] = \gamma(z_k^{(i)}) = \begin{cases} 1 & \text{if } \left|\boldsymbol{x}^{(i)} - \boldsymbol{\mu}_k\right| < \left|\boldsymbol{x}^{(i)} - \boldsymbol{\mu}_j\right| \quad , \forall j \\ 0 & \text{else} \end{cases}$$

EM algorithm

- For learning from partly unobserved data
- Maximum Likelihood estimate (MLE) vs. EM estimate
    - ML estimate
    - EM estimate

EM algorithm

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$$\boldsymbol{\theta} = argmax_{\boldsymbol{\theta}} p(\boldsymbol{X}|\boldsymbol{\theta})$$

  - EM estimate

$$\boldsymbol{\theta} = argmax_{\boldsymbol{\theta}} \mathbb{E}_Z[p(\boldsymbol{X}, \boldsymbol{Z}|\boldsymbol{\theta})]$$

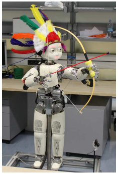
Detection of target and arrow using GMM

Task: To perform archery by a humanoid robot iCub.
Proposed approach: reinforcement learning algorithms for learning the skill of archery

- EM based Reinforcement Learning (PoWER)
- chained vector regression (ARCHER)

Subproblem: Image processing

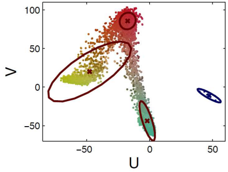
- To detect where the target is
- To get the relative position of arrow from the target



Kormushev, Petar, et al. *Learning the skill of archery by a humanoid robot iCub*. IEEE-RAS International Conference on Humanoid Robots, 2010.

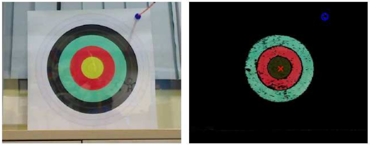
Detection of target and arrow using GMM

- The color detection is done in YUV color space.
- Only U and V components are used to ensure robustness against luminosity.
- A three component GMM for target, a single component GMM for arrow tip
- Bayesian Information Criterion (BIC) is used for optimizing the number of components in each GMM



Detection of target and arrow using GMM

After learning the likelihood value of each pixel in a new image can be used for classification of pixels. The classified image can be used to detect the center of target (red cross) and arrow (blue circle) in below figure.

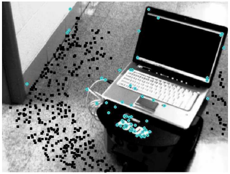


Ground Plane Detection

Task : In mobile navigation, detection of ground and non-ground is useful in various application such as: object recognition, obstacle avoidance during autonomous navigation. This paper uses it for object tracking and following.

Conrad D. and DeSouza G. N. *Homography-based Ground Plane Detection for Mobile Robot Navigation Using a Modified EM Algorithm*. IEEE International Conference on Robotics and Automation. 2010.

Ground Plane Detection



- From two images, a large number of pixel correspondences are found by SIFT algorithm.
- EM used to classify pixel correspondences ($\boldsymbol{x}$) from two images into 2 classes: *Ground plane* and *Non-Ground plane* in order to segment out the ground.
- Robot control: The robot uses pixels on the target object to follow. obstacle avoidance during autonomous navigation. It keeps the target object in the center of image view.

Ground Plane Detection

- Homography : a transformation matrix that relates the pixel coordinates of planar points as seen from two different viewing angles.

$$s\hat{p}_i = Hp_i$$

- Homography H is defined as $H = \hat{A}(R + \frac{t}{d}n^T)A^{-1}$ with $\hat{A}$ and A containing the intrinsic parameters of the cameras. The parameter Homography H is $\theta = \{R, t, n, d\}$, which is rotation matrix, translation vector, normal vector of the plane, distance between the camera and plane.
- These parameters will be updated via EM.
- The pair of corresponding pixels $\hat{p}_i$, p_i is referred to as pixel correspondence x_i.

Ground Plane Detection

- Expectation Maximization Algorithm

$$P(\boldsymbol{X}|\boldsymbol{Z},\boldsymbol{\theta}) = \frac{exp(-\frac{err_i^2}{2\sigma^2})}{\sum_i exp(-\frac{err_i^2}{2\sigma^2})} \quad with \quad err_i = \| \hat{p}_i - \frac{H_{ground} p_i}{s} \|$$

- where $\sigma = 3$, $H = A_1(R + \frac{\boldsymbol{t}}{d}\boldsymbol{n}^T)A_2^{-1}$, $\boldsymbol{\theta} = (R, \boldsymbol{t}, d, \boldsymbol{n})$, $\boldsymbol{X} = \boldsymbol{x}$ and $\boldsymbol{Z} = ground\ plane$.
- $P(\boldsymbol{X}|C_{non-ground},\boldsymbol{\theta}) = \frac{1-exp(-\frac{err_i^2}{2\sigma^2})}{\sum_i 1-exp(-\frac{err_i^2}{2\sigma^2})}$.
- After computing all posterior probabilities (E-step), the new model parameters are updated (M-step).
- In the M-step, an optimization algorithm (Simplex method) is used because it does not require an explicit gradient and shows faster convergence.
- Ground detection rate : 99.6$\%$.

Announcements

- Further Reading
  - MLE: Duda Chap 3.1, 3.2, Bishop Chap. 1.2.4
  - GMM: Duda Chap. 3.4, Bishop Chap. 9.2
  - EM: Mitchell Chap. 6.12, Bishop Chap. 9.2-9.3
- Next Lecture
  - Nonparametric density estimation
  - Kernel Density Estimation, k-NNR, Parzen window