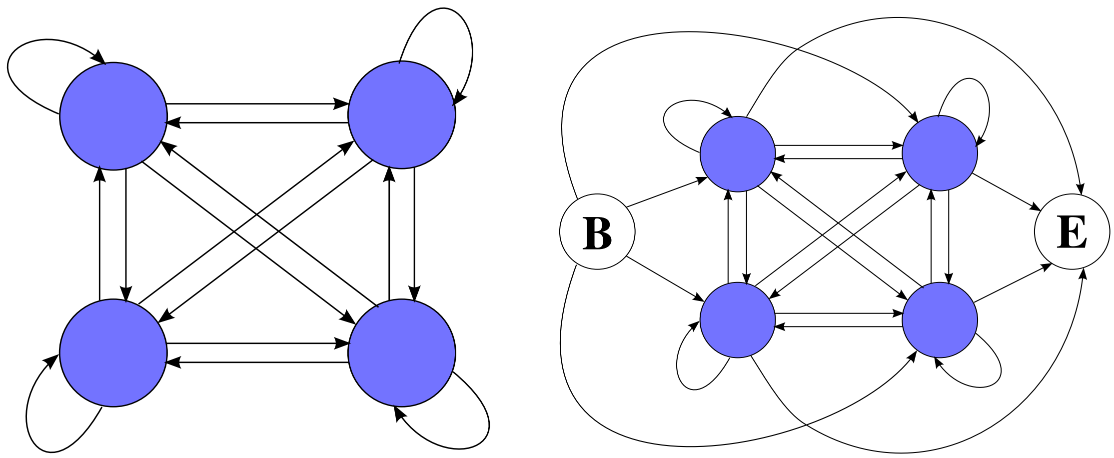
Lehrstuhl für STEUERUNGS- UND REGELUNGSTECHNIK Technische Universität München Prof. Dongheui Lee	**MACHINE LEARNING IN ROBOTICS** Exercises: HMM

Exercise 1

Two HMMs with different structure are shown in Fig. 1.



(a) (b)

Figure 1: HMMs with different structures.

a) Consider the HMM in Fig. 1(a). Show that the sum of the probability of all possible state sequences $\mathcal{Q} = [q_1, \ldots, q_L]$ of length L is equal to 1.

b) Assume that the HMM has a begin (B) and an end (E) state, as in Fig. 1(b). The end state has probability ε. Show that the sum of the probability over all state sequences $\mathcal{Q} = [q_1, \ldots, q_L]$ of length L (and properly terminating by making a transition to the end state) is $p = \varepsilon(1-\varepsilon)^{L-1}$. Use this result to show that the sum of the probability over all possible state sequences of any length is 1.

Hint: Use the result $\sum_{i=0}^{\infty} x^i = 1/(1-x)$ for $0 < x < 1$.

Solution Exercise 1

a) The probability for a sequence $\mathcal{Q} = [q_1, \ldots, q_L]$ with length L is

$$p(\mathcal{Q}) = p(q_L|q_{L-1}, \ldots, q_2|q_1, q_1) = p(q_L|q_{L-1})p(q_{L-1}|q_{L-2}) \ldots p(q_2|q_1)p(q_1)$$

The sum of the probability of all possible state sequences of length L is

$$\sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) = \sum_{q_1} \sum_{q_2} \cdots \sum_{q_L} p(q_L|q_{L-1})p(q_{L-1}|q_{L-2}) \ldots p(q_2|q_1)p(q_1)$$

where $\forall \mathcal{Q}, L$ indicates we are considering all the sequences of length L.

Note that, for a particular q_i, the sum of transition probability for all possible q_{i+1} is 1, i.e.

$$\sum_{q_L} p(q_L|q_{L-1}) = 1, \sum_{q_{L-1}} p(q_{L-1}|q_{L-2}) = 1, \cdots, \sum_{q_2} p(q_2|q_1) = 1$$

Then

$$\sum_{\forall \mathcal{Q},L} p(\mathcal{Q}) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_2|q_1)p(q_1)$$
$$= \sum_{q_1}\sum_{q_2} p(q_2|q_1)p(q_1) = \sum_{q_1} p(q_1) = 1$$

b) The sum of the probability of all possible state sequences of length L and terminating in E is

$$\sum_{\forall \mathcal{Q},L} p(\mathcal{Q}) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_E|q_L)p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_1)$$

where $\forall \mathcal{Q}, L$ indicates we are considering all the sequences of length L.

Note that

$$\sum_{q_1} p(q_1) = 1 \qquad \text{and} \qquad p(q_E|q_L) = \varepsilon$$

and that, for a particular q_i, the sum of transition probability for all possible q_{i+1} (except for $i+1 = E$) is $1-\varepsilon$, i.e.

$$\sum_{q_L} p(q_L|q_{L-1}) = 1-\varepsilon, \sum_{q_{L-1}} p(q_{L-1}|q_{L-2}) = 1-\varepsilon, \cdots, \sum_{q_2} p(q_2|q_1) = 1-\varepsilon$$

Then

$$\sum_{\forall \mathcal{Q},L} p(\mathcal{Q}) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_E|q_L)p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_1)$$
$$= \varepsilon\sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_1) = \varepsilon(1-\varepsilon)^{L-1}$$

The sum of the probability over all possible state sequences of any length is

$$\sum_{\forall \mathcal{Q}} p(\mathcal{Q}) = \sum_{L=1}^{\infty}\sum_{\forall \mathcal{Q},L} p(\mathcal{Q}) = \sum_{L=1}^{\infty}\varepsilon(1-\varepsilon)^{L-1} = \varepsilon\sum_{L-1=0}^{\infty}(1-\varepsilon)^{L-1} = \varepsilon\frac{1}{\varepsilon} = 1$$

Exercise 2

A Hidden Markov Model with 2 states $\{s_1 = H, s_2 = C\}$ and 3 possible observations based on the number of observed sizes $\{small, medium, large\}$ is given with transition probability matrix:

$$A = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

and observation matrix:

$$B = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$$

The prior probabilities of states are $\pi_1 = 0.6, \pi_2 = 0.4$.

a) Compute the probability of the state sequence $\{q_1 = H \ q_2 = H \ q_3 = C \ q_4 = C\}$.

b) Compute the probability $P(o_1 = S \ o_2 = M \ o_3 = S \ o_4 = L \mid q_1 = H \ q_2 = H \ q_3 = C \ q_4 = C)$.

Solution Exercise 2

a) $P(q_1)P(q_2|q_1)P(q_3|q_2)P(q_4|q_3) = 0.6 * 0.7 * 0.3 * 0.6 = 0.0756.$

b) Compute the probability $P(o_1|q_1)P(o_2|q_2)P(o_3|q_3)P(o_4|q_4) = 0.1 * 0.4 * 0.7 * 0.1 = 0.0028.$