Lehrstuhl für STEUERUNGS- UND REGELUNGSTECHNIK Technische Universität München Prof. Dongheui Lee	**MACHINE LEARNING IN ROBOTICS** Exercises : Markov processes

Exercise 1

A protocol for data transmission shall be analysed using a Markov chain with 3 states. The probability for the transition from $state1$ (*check interface for incoming data*) to $state2$ (*check address*) is 0.1. The address is correct with probability 0.4. In this case, there is a transition to $state3$ (*message received*). Otherwise, the system returns to $state1$. If a message was received and there is no further message (probability 0.7), the system leaves $state3$ and enters in the $state1$. If there is a further message, it enters in the $state2$.

a) Specify the matrix of transition probabilities.

b) Draw the corresponding Markov chain.

c) What is the probability for the system to be in $state1$?

Solution Exercise 1

a)

$$\mathbf{A} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{pmatrix}$$

remember: $\sum_j a_{ij} = 1 \forall i$

b) The resulting Markov chain is shown in Fig. 1

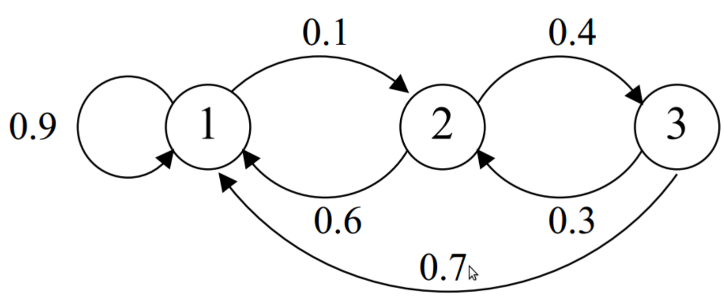

Figure 1: Markov chain

c) We are looking for the probability that the system is in $state1$. Since we are not specifying any time, this means we are looking for the steady state probability. The steady state probability vector $\mathbf{p}$ is defined as $\mathbf{p} = \mathbf{A}^T\mathbf{p}$, i.e.

$$p_1 = 0.9p_1 + 0.6p_2 + 0.7p_3 \tag{1}$$

$$p_2 = 0.1p_1 + 0.3p_3 \tag{2}$$

$$p_3 = 0.4p_2 \tag{3}$$

where it hold that

$$\sum_i p_i = p_1 + p_2 + p_3 = 1 \tag{4}$$

Solving 4 for p_2 and inserting it in 2 and 3, respectively, gives:

$$p_3 = \frac{2}{7} - \frac{2}{7}p_1 \tag{5}$$

$$p_1 = \frac{10}{11} - \frac{13}{10}p_3 \tag{6}$$

Inserting 5 in 6 gives the stationary probability to be in $state1$:

$$p_1 = \frac{44}{51} = 0.8627.$$

Exercise 2

An urn contains N balls, consisting of some white and some black balls. At each stage, a coin is flipped with a probability $p,\ 0 < p < 1$, of landing heads. If head appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tail appears, then a ball is chosen from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the n-th stage.

a) Is $X_n,\ n \geq 0$ a Markov chain? If so, explain why.

b) Compute the probabilities $P(X_{n+1} = X_n + 1|X_n)$, $P(X_{n+1} = X_n|X_n)$ and $P(X_{n+1} = X_n - 1|X_n)$ that define the described system.

Solution Exercise 2

a) $X_n,\ n \geq 0$ is a Markov Chain since the number of white balls at time $n+1$ only depends on X_n and the coin flip result.

b) Let i be the number of white balls on the urn at time n. Then, $P_{ij} = 0$ for $j < i-1$ and $j > i+1$.
X_n : the number of white balls in the urn after the n_{th} stage.
$N - X_n$: the number of black balls in the urn after the n_{th} stage.
p : the probability that a coin flip is head.
$1-p$: the probability that a coin flip is tail.

$P(X_{n+1} = X_n + 1|X_n) = P(draw\ a\ black\ ball\ AND\ coin\ flip\ is\ head) = P(draw\ a\ black\ ball) \cdot P(coin\ flip\ is\ head) = \frac{N-X_n}{N} \cdot p.$

$P(X_{n+1} = X_n - 1|X_n) = P(draw\ a\ white\ ball\ AND\ coin\ flip\ is\ tail) = P(draw\ a\ white\ ball) \cdot P(coin\ flip\ is\ tail) = \frac{X_n}{N} \cdot (1-p)$

$P(X_{n+1} = X_n|X_n) = 1 - P(X_{n+1} = X_n + 1|X_n) - P(X_{n+1} = X_n - 1|X_n) = 1 - \frac{N-X_n}{N} \cdot p - \frac{X_n}{N} \cdot (1-p).$