Observability and Observers

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Outline

- Linear Systems
  - Observability
  - Observer design
- Nonlinear Systems
  - Observability
  - Observer design

Observability Definition

- A LTI system is observable if given
  $$\boldsymbol{u}(t)\,,\boldsymbol{y}(t), t \in [0,T], 0 < T < \infty$$
  we can uniquely determine any initial state $\boldsymbol{x}(0)$ from the input and output history over a finite time interval T.
- Unobservable: there are indistinguishable initial states.

Zero-input Response

- $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}\boldsymbol{u}(t)Bd\tau$
- If $\boldsymbol{x}(0)$ can be determined, we can obtain $\boldsymbol{x}(t), 0 < t \leq T$.
- Can consider $\boldsymbol{y}_{ZI}(t)$ only.
  $$\begin{aligned}\boldsymbol{y}_{ZI}(t) &= Ce^{At}\boldsymbol{x}(0)\\ &= \boldsymbol{y}(t) - \boldsymbol{C}\int_0^t e^{A(t-\tau)}\boldsymbol{u}(t)Bd\tau\end{aligned}$$

Unobservable Mode

- System is observable if and only if it has no unobservable modes.
- Measurements may or may not capture all system modes.

$$\mathbf{y}(t) = C\,\mathbf{x}(t)$$
$$\mathbf{x}(t) \in \Re^n, \mathbf{y}(t) \in \Re^l$$
$$\mathbf{y}(t) = C\,e^{At}\mathbf{x}(0)$$
$$= \sum_{i=1}^{n} C\,Z_i \mathbf{x}(0) e^{\lambda_i t}$$
$$C\,Z_i = C\mathbf{v}_i \mathbf{w}_i^T = [0] \Rightarrow \text{unobservable mode } e^{\lambda_i t}$$
$$\mathbf{c}_j^T \mathbf{v}_i = 0,\ j = 1, \cdots, l\ (\text{orthogonal}) \Rightarrow \text{unobservable state } \mathbf{v}_i$$

Rank Test

- A LTI system is observable if and only if the observability matrix has full rank n.
- Rank deficit = number of unobservable modes.

$$rank[\mathcal{O}] = n$$
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (l.n \times n)$$
$$\mathcal{O}\,\mathbf{v}_i = 0 \Leftrightarrow e^{\lambda_i t}\ unobservable, \mathbf{v}_i\ unobservable\ state$$

Cayley-Hamilton Theorem

- Every matrix satisfies its own characteristic equation.
- $A^i, i > n-1$ can be expressed in terms of $A^j,\ j = 0,1,\ \ldots,\ n-1$
- e^{At} can be expressed in terms of $A^j,\ j = 0,1,\ \ldots,\ n-1$

$$\det[\lambda I_n - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$
$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = [0]$$
$$A^n = -\left(a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n\right)$$

Proof of Necessity

- Assume observable with rank deficient matrix gives a contradiction. For rank deficient $\mathcal{O}$

$$\begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{bmatrix} \mathbf{x}_{uo} = \mathbf{0} \Rightarrow C\,A^i \mathbf{x}_{uo} = \mathbf{0}, i = 0,1,\cdots,n-1, \mathbf{x}_{uo} \neq \mathbf{0}$$
$$\Rightarrow \mathbf{y}_{ZI}(t) = C\mathbf{x}(t) = Ce^{At}\mathbf{x}_{uo}$$
$$= C\mathbf{x}_{uo} + CA\mathbf{x}_{uo}t + \cdots CA^i\mathbf{x}_{uo}\frac{t}{i!} + \cdots$$
$$= \sum_{i=0}^{n-1} \alpha_i(t) CA^i \mathbf{x}_{uo} = \mathbf{0}, \forall t$$

- $\mathbf{x}_{uo}$ unobservable state: contradicts observability.

Proof of Sufficiency

- Assume full rank observability matrix then

$$CA^i\boldsymbol{x}(0) = \boldsymbol{0}, i = 0,1,\ldots,n-1 \Rightarrow \boldsymbol{x}(0) = \boldsymbol{0}$$

$$Ce^{At}\boldsymbol{x}(0) = C\left[\sum_{i=0}^{n-1}\alpha_i(t)A^i\right]\boldsymbol{x}(0)$$

$$= [\alpha_0(t)I_n \quad \ldots \quad \alpha_{n-1}(t)I_n]\begin{bmatrix} C \\ \vdots \\ CA^{n-1}\end{bmatrix}\boldsymbol{x}(0)$$

$$\Rightarrow \boldsymbol{x}(0) = \boldsymbol{0}$$

- No unobservable states hence observable.

Example

```
>>A=[zeros(2,1),eye(2);-6,-11,-6] ; C=[1,4/3,1/3];
>> rank(obsv(A,C))
ans =
    1
```

$$rank[\mathcal{O}] = 1$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 4/3 & 1/3 \\ -2 & -8/3 & -2/3 \\ 4 & 16/3 & 4/3 \end{bmatrix} (3\times 3)$$

$\mathcal{O}\,\mathbf{v}_i = 0 \Leftrightarrow e^{\lambda_i t}$ unobservable, $\mathbf{v}_i$ unobservable state

$i = 1, 3$

Example: Rank Test

- Check observability

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

MATLB

```
>> A=[[0;0],eye(2);-6,-11,-6];
>> C=[10,0,2;0,0,1]
>> rank(obsv(A,C)) % Observability rank test
    ans =
        3
```

State Observers

- In practice, the state variables are not all available for measurement (impossible or expensive).
- Need estimate of the state vector for state feedback

  $\hat{\boldsymbol{x}}(t)$ = estimate of the state vector $\boldsymbol{x}(t)$

Full-Order Observer

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

$$\boldsymbol{y} = C\boldsymbol{x}$$

- Estimates all the state variables.

  $$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + L(\boldsymbol{y} - C\hat{\boldsymbol{x}})$$

  $$= A_{obs}\hat{\boldsymbol{x}} + B\boldsymbol{u} + L\boldsymbol{y}$$

  $A_{obs} = A - LC$, $L =$ *observer gain*
- Eigenvalues can be arbitrarily assigned if the system is observable.

Observable Form

- Observable forms are always observable.

  $$A_o = \begin{bmatrix} -\bar{\boldsymbol{a}} & \begin{matrix} I_{n-1} \\ \boldsymbol{0}^T \end{matrix} \end{bmatrix}, C_o = [1 \quad \boldsymbol{0}^T]$$

- Obtain from controller form by transposing A, B, C then interchanging B *and* C.
- Another observable form similarly obtained from phase variable form (same as observer form with reordering the state variables)

Observer: Observer Form

$$\dot{\hat{\boldsymbol{x}}} = \begin{bmatrix} -\bar{\boldsymbol{a}} & \begin{matrix} I_{n-1} \\ \boldsymbol{0}^T \end{matrix} \end{bmatrix} \hat{\boldsymbol{x}} + \boldsymbol{l}_o [1 \quad \boldsymbol{0}^T](\boldsymbol{x} - \hat{\boldsymbol{x}})$$

$$A_{obs} = A_o - \boldsymbol{l}_o C_o = \begin{bmatrix} -(\bar{\boldsymbol{a}} + \boldsymbol{l}_o) & \begin{matrix} I_{n-1} \\ \boldsymbol{0}^T \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} -\bar{\boldsymbol{a}}_d & \begin{matrix} I_{n-1} \\ \boldsymbol{0}^T \end{matrix} \end{bmatrix}$$

$$\boldsymbol{l}_o = \bar{\boldsymbol{a}}_{\boldsymbol{d}} - \bar{\boldsymbol{a}}, \bar{\boldsymbol{a}} = [a_{n-1} \quad a_{n-2} \quad \dots \quad a_0]^T$$

- Observer gain: in terms of coefficients vectors.

Transformation Matrix

- Assume that the system is observable: can be transformed to observable form $A_o = T^{-1}AT, C_o = CT$
- Observability matrix

$$\mathcal{O}_o = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = \mathcal{O}T$$

$$\boldsymbol{l} = T\boldsymbol{l}_o = \mathcal{O}^{-1}\mathcal{O}_o(\overline{\boldsymbol{a}}_{\boldsymbol{d}} - \overline{\boldsymbol{a}})$$

Observers State Feedback

Full order observer estimates the state variables.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

$$\boldsymbol{y} = C\boldsymbol{x}$$

$$\boldsymbol{u} = -K\widehat{\boldsymbol{x}}$$

$$\dot{\widehat{\boldsymbol{x}}} = A\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L(\boldsymbol{y} - C\widehat{\boldsymbol{x}})$$

$$= A_{obs}\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L\boldsymbol{y} = A_{o-c}\widehat{\boldsymbol{x}} + L\boldsymbol{y}$$

$$A_{obs} = A - LC, \qquad A_{o-c} = A - BK - LC$$

L= observer gain, K = controller gain

Plant/Observer Dynamics

Using $\boldsymbol{y} = C\boldsymbol{x}$, $\boldsymbol{u} = -K\widehat{\boldsymbol{x}}$

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} = A\boldsymbol{x} - BK\widehat{\boldsymbol{x}}$$

$$\dot{\widehat{\boldsymbol{x}}} = A_{o-c}\widehat{\boldsymbol{x}} + L\boldsymbol{y} = (A - BK - LC)\widehat{\boldsymbol{x}} + LC\boldsymbol{x}$$

- Write combined state equation.

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\widehat{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \widehat{\boldsymbol{x}} \end{bmatrix}$$

- Estimation error $\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$

Plant/Error Dynamics

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \widehat{\boldsymbol{x}} \Rightarrow \widehat{\boldsymbol{x}} = \boldsymbol{x} - \widetilde{\boldsymbol{x}}$$

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} - BK\widehat{\boldsymbol{x}} = A\boldsymbol{x} - BK(x - \tilde{x})$$
$$= (A - BK)\boldsymbol{x} + BK\widetilde{\boldsymbol{x}}$$

$$\dot{\widehat{\boldsymbol{x}}} = (A - BK - LC)\widehat{\boldsymbol{x}} + LC\boldsymbol{x}$$

$$= (A - BK - LC)(\boldsymbol{x} - \widetilde{\boldsymbol{x}}) + LC\boldsymbol{x}$$

- Subtract $\dot{\widetilde{\boldsymbol{x}}} = (A - LC)\widetilde{\boldsymbol{x}}$
- Combine state and error equations

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\widetilde{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \boldsymbol{0}_{n\times n} & A - LC \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \widetilde{\boldsymbol{x}} \end{bmatrix}$$

Separation Principle

- State and observer-error dynamics:
- $\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\tilde{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \boldsymbol{0}_{n\times n} & A - LC \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix}$
- $\det\{sI_{2n} - A_{c-oe}\} =$
  $\det\{sI_n - A_{con}\}\det\{sI_n - A_{obs}\}$
- $A_{con} = A - BK, A_{obs} = A - LC$
- For observer state feedback, the design of the state feedback and the design of the observer can be carried out separately.

Nonlinear Observability

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u, \boldsymbol{f}, \boldsymbol{g}: \mathcal{R}^n \to \mathcal{R}^n$$
$$y = h(\boldsymbol{x}), h: \mathcal{R}^n \to \mathcal{R}, h(\boldsymbol{0}) = 0$$

- $\boldsymbol{f}, \boldsymbol{g}$ sufficiently smooth
- $\boldsymbol{x}_u(t, \boldsymbol{x}_0)$ solution at time t with input u and IC vector $\boldsymbol{x}_0$
- $y_u(t, \boldsymbol{x}_0) = h(\boldsymbol{x}_u(t, \boldsymbol{x}_0))$ output at time t with input u and IC vector $\boldsymbol{x}_0$

Indistinguishable States

- A pair of states $\boldsymbol{x}_{0i}, i = 1{,}2$ is indistinguishable if $\exists u$ s.t. $y_u(t, \boldsymbol{x}_{01}) = y_u(t, \boldsymbol{x}_{02}), \forall t \geq 0$
- Definition does not require the outputs to be different $\forall u$

Local Observability

- Local Observability at $\boldsymbol{x}_0$:
- There is a neighborhood $U(\boldsymbol{x}_0)$ s.t. $\forall \boldsymbol{x} \neq \boldsymbol{x}_0, \boldsymbol{x} \in U(\boldsymbol{x}_0)$ is distinguishable from $\boldsymbol{x}_0$
- Local Observability: local observability $\forall \boldsymbol{x}_0 \in \mathcal{R}^n$

Autonomous Systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{f}: \mathcal{R}^n \to \mathcal{R}^n$$
$$y = h(\boldsymbol{x}), h: \mathcal{R}^n \to \mathcal{R}, h(\boldsymbol{0}) = 0$$

Autonomous realization is locally observable in a neighborhood $U(\boldsymbol{0})$ if

$$rank \begin{bmatrix} \nabla h \\ \nabla L_{\boldsymbol{f}} h \\ \vdots \\ \nabla L_{\boldsymbol{f}}^{n-1} h \end{bmatrix} = n, \forall \boldsymbol{x} \in U(\boldsymbol{0})$$

Example: Relation to Linear Test

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, y = C\boldsymbol{x}$$
$$\nabla h = C, L_{\boldsymbol{f}} h = CA\boldsymbol{x}, \nabla L_{\boldsymbol{f}} h = CA$$
$$rank \begin{bmatrix} \nabla h \\ \nabla L_{\boldsymbol{f}} h \\ \vdots \\ \nabla L_{\boldsymbol{f}}^{n-1} h \end{bmatrix} = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Necessary and sufficient for LTI observability.

Theorem 11.1: if linearization around $\boldsymbol{0}$ is observable then the system is locally observable around $\boldsymbol{0}$

Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} (1-u^*)x_2 \\ x_1 \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}, u^*)$$
$$y = C\boldsymbol{x} = [1 \quad 0]\boldsymbol{x} = h(\boldsymbol{x})$$
$$\nabla h = C, L_{\boldsymbol{f}} h = C\boldsymbol{f}(\boldsymbol{x}, u^*) = (1-u^*)x_2$$
$$\nabla L_{\boldsymbol{f}} h = [0 \quad 1-u^*]$$
$$rank \begin{bmatrix} \nabla h \\ \nabla L_{\boldsymbol{f}} h \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 0 & 1-u^* \end{bmatrix}$$

Rank depends on the input: rank= $\begin{cases} 2, u^* \neq 1 \\ 1, u^* = 1 \end{cases}$

Observers: Linear Error Dynamics

- Design procedure based on feedback linearization

1. Find an invertible transformation $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x})$ that linearizes $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u$
2. Design an observer for the linear dynamics.
3. Use the inverse transformation $\boldsymbol{T}^{-1}(\boldsymbol{z})$ to recover the state estimate $\hat{\boldsymbol{x}}$.

Weakness: sensitivity to modeling errors.

Transformation

SISO System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u$

$$y = h(\boldsymbol{x})$$

Assume $\exists \boldsymbol{T}(\boldsymbol{x})$ satisfying

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{T}(\boldsymbol{0}) = \boldsymbol{0}, \boldsymbol{z} \in \mathcal{R}^{\boldsymbol{n}}$$

After coordinate transformation

$$\dot{\boldsymbol{z}} = A_0\boldsymbol{z} + \boldsymbol{\gamma}(\boldsymbol{y}, u)$$

$$y = C_0\boldsymbol{z}$$

$$A_o = \begin{bmatrix} -\overline{\boldsymbol{a}} & \begin{matrix} I_{n-1} \\ \boldsymbol{0}^T \end{matrix} \end{bmatrix}, C_0 = \begin{bmatrix} 1 & \boldsymbol{0}^T \end{bmatrix}$$

$$\overline{\boldsymbol{a}} = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix}^T$$

Theorem 11.2

If $\exists$ a coordinate transformation

$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{T}(\boldsymbol{0}) = \boldsymbol{0}$

$$\dot{\boldsymbol{z}} = A_0\boldsymbol{z} + \boldsymbol{\gamma}(\boldsymbol{y}, u)$$

$$y = C_0\boldsymbol{z}$$

Then the observer

$$\dot{\hat{\boldsymbol{z}}} = A_0\hat{\boldsymbol{z}} + \boldsymbol{\gamma}(\boldsymbol{y}, u) + \boldsymbol{l}(y - C_0\hat{\boldsymbol{z}}), \hat{\boldsymbol{z}} \in \mathcal{R}^{\boldsymbol{n}}$$

with the eigenvalues of $A_{obs} = A_o - \boldsymbol{l}C_o$ in the LHP yields the state estimate $\hat{\boldsymbol{x}} = \boldsymbol{T}^{-1}(\hat{\boldsymbol{z}})$,

$\hat{\boldsymbol{x}}(t) \to \boldsymbol{x}(t)$ as $t \to \infty$

Proof

- Let $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}, \tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$

$$\dot{\tilde{\boldsymbol{z}}} = \dot{\boldsymbol{z}} - \dot{\hat{\boldsymbol{z}}} = A_0\boldsymbol{z} + \boldsymbol{\gamma}(\boldsymbol{y}, u) - [A_0\hat{\boldsymbol{z}} + \boldsymbol{\gamma}(\boldsymbol{y}, u) + \boldsymbol{l}(y - C_0\hat{\boldsymbol{z}})]$$

$$\dot{\tilde{\boldsymbol{z}}} = (A_o - \boldsymbol{l}C_o)\tilde{\boldsymbol{z}}$$

$\tilde{\boldsymbol{z}} \to \boldsymbol{0}$ as $t \to \infty$

if $\lambda_i(A_o - \boldsymbol{l}C_o), i = 1, \dots, n$ in LHP

$$\begin{aligned} \tilde{\boldsymbol{x}} &= \boldsymbol{x} - \hat{\boldsymbol{x}} \\ &= \boldsymbol{T}^{-1}(\boldsymbol{z}) - \boldsymbol{T}^{-1}(\boldsymbol{z} - \tilde{\boldsymbol{z}}) \to \boldsymbol{0} \text{ as } t \to \infty, \end{aligned}$$

$\boldsymbol{x} - \hat{\boldsymbol{x}} \to \boldsymbol{0}$ as $t \to \infty$

Example

$$\dot{x}_1 = x_2 + 2x_1^2$$

$$\dot{x}_2 = x_1x_2 + x_1^3u, \qquad y = x_1$$

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ x_2 - x_1^2/2 \end{bmatrix}$$

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 - x_1\dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_1^2 \\ x_1x_2 + x_1^3u - x_1(x_2 + 2x_1^2) \end{bmatrix}$$

$$\dot{\boldsymbol{z}} = \begin{bmatrix} x_2 + 2x_1^2 \\ x_1^3(u - 2) \end{bmatrix} = \begin{bmatrix} x_2 - x_1^2/2 + (5/2)x_1^2 \\ x_1^3(u - 2) \end{bmatrix}$$

Transformed Dynamics

$$\dot{\boldsymbol{z}} = \begin{bmatrix} x_2 - x_1^2/2 + (5/2)x_1^2 \\ x_1^3(u-2) \end{bmatrix} = \begin{bmatrix} z_2 + (5/2)y^2 \\ y^3(u-2) \end{bmatrix}$$

$$y = x_1 = z_1$$

$$\dot{\boldsymbol{z}} = A_0\boldsymbol{z} + \boldsymbol{\gamma}(\boldsymbol{y}, u)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} (5/2)y^2 \\ y^3(u-2) \end{bmatrix}$$

$$y = C_0\boldsymbol{z} = [1 \quad 0]\boldsymbol{z} = z_1$$

Observer

$$\dot{\hat{\boldsymbol{z}}} = A_0\hat{\boldsymbol{z}} + \boldsymbol{\gamma}(\boldsymbol{y}, u) + \boldsymbol{l}(y - C_0\hat{\boldsymbol{z}})$$

$$C_0\hat{\boldsymbol{z}} = [1 \quad 0]\hat{\boldsymbol{z}} = \hat{z}_1$$

$$A_o - \boldsymbol{l}C_o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}[1 \quad 0]$$

$$= \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$$

Stable observer dynamics for $l_i > 0, i = 1,2$

Lipschitz Systems

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, u), y = C\boldsymbol{x}$$

$$A \in \mathcal{R}^{n\times n}, C \in \mathcal{R}^{1\times n}, \boldsymbol{f}: \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$$

Lipschitz in $\boldsymbol{x}$ on an open set $D \in \mathcal{R}^n$

$\|\boldsymbol{f}(\boldsymbol{x}_1, u) - \boldsymbol{f}(\boldsymbol{x}_2, u)\| \le \gamma\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \forall x \in \boldsymbol{D}$

Observer

$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + \boldsymbol{f}(\hat{\boldsymbol{x}}, u) + \boldsymbol{l}(y - C\hat{\boldsymbol{x}}), \boldsymbol{l} \in \mathcal{R}^{n\times 1}$

Theorem 11.3

Given the system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, u), y = C\boldsymbol{x}$ with the observer

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + \boldsymbol{f}(\hat{\boldsymbol{x}}, u) + \boldsymbol{l}(y - C\hat{\boldsymbol{x}}), \boldsymbol{l} \in \mathcal{R}^{n\times 1}$$

If the Lyapunov equation $A_o^T P + PA_o = -Q$ is satisfied with $P = P^T > 0, Q = Q^T > 0$,

$$A_o = A - \boldsymbol{l}C, \qquad \gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$$

then the observer is asymptotically stable.

Proof

$$\dot{\tilde{\boldsymbol{x}}} = \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}}$$
$$= A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, u) - [A\hat{\boldsymbol{x}} + \boldsymbol{f}(\hat{\boldsymbol{x}}, u) + \boldsymbol{l}C(\boldsymbol{x} - \hat{\boldsymbol{x}})]$$
$$= (A - \boldsymbol{l}C)\tilde{\boldsymbol{x}} + \boldsymbol{f}(\boldsymbol{x}, u) - \boldsymbol{f}(\hat{\boldsymbol{x}}, u)$$
$$A_o = A - \boldsymbol{l}C$$
$$\Delta\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, u) - \boldsymbol{f}(\hat{\boldsymbol{x}}, u)$$
$$\dot{\tilde{\boldsymbol{x}}} = A_o\tilde{\boldsymbol{x}} + \Delta\boldsymbol{f}$$

Lypunov Stability Test

Lyapunov function candidate: $V(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T P \tilde{\boldsymbol{x}}$

$$\dot{V}(\tilde{\boldsymbol{x}}) = \dot{\tilde{\boldsymbol{x}}}^T P \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^T P \dot{\tilde{\boldsymbol{x}}} = -\tilde{\boldsymbol{x}}^T Q \tilde{\boldsymbol{x}} + 2\tilde{\boldsymbol{x}}^T P \Delta\boldsymbol{f}$$
$$\tilde{\boldsymbol{x}}^T Q \tilde{\boldsymbol{x}} \geq \lambda_{min}(Q)\|\tilde{\boldsymbol{x}}\|^2$$
$$\|\Delta\boldsymbol{f}\| \leq \gamma\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = \gamma\|\tilde{\boldsymbol{x}}\|$$
$$\dot{V}(\tilde{\boldsymbol{x}}) \leq -\lambda_{min}(Q)\|\tilde{\boldsymbol{x}}\|^2 + 2\gamma\|P\|\|\tilde{\boldsymbol{x}}\|$$
$$\|P\| = \lambda_{max}^{1/2}(P^T P) = \lambda_{max}^{1/2}(P^2) = \lambda_{max}(P)$$
$$\dot{V}(\tilde{\boldsymbol{x}}) \leq -[\lambda_{min}(Q) - 2\gamma\lambda_{max}(P)]\|\tilde{\boldsymbol{x}}\|^2 < 0$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Observer design

$$\boldsymbol{l} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow A_o = A - \boldsymbol{l}C = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

- Determine the Lipschitz constant γ
- Solve the Lyapunov equation with $Q = I_2$

Lipschitz Condition

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$
$$\|\boldsymbol{f}(\boldsymbol{x}_a) - \boldsymbol{f}(\boldsymbol{x}_b)\| \leq \gamma\|\boldsymbol{x}_a - \boldsymbol{x}_b\|$$
$$LHS \leq \sqrt{(x_{2a}^2 - x_{2b}^2)^2}$$
$$= |x_{2a}^2 - x_{2b}^2| = |(x_{2a} - x_{2b})(x_{2a} + x_{2b})|$$

For $|x_{2a}|, |x_{2b}| \leq k$

$$LHS \leq 2k|x_{2a} - x_{2b}| \leq 2k\|\boldsymbol{x}_a - \boldsymbol{x}_b\|$$
$$\gamma = 2k$$

MATLAB Solution

```
>> lyap(A0',eye(2)) % Q=I_2
ans =
   1.5000   0.5000
   0.5000   0.5000
>> eig(P)
ans =
   0.2929
   1.7071
```


Validity Condition

$$\gamma = 2k < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} = \frac{1}{2 \times 1.7071}$$
$$k < \frac{1}{6.8284}$$

- Another choice of $\boldsymbol{l}$ can yield a larger validity region for the observer.
- The choice of $\boldsymbol{l}$ that maximizes the region is not easy to determine.

Separation Principle

- Separate controller and observer design.
- Guaranteed for linear systems.
- In general, not guaranteed to work for nonlinear systems.
- Using a stable state feedback with the state replaced by its estimate from a stable observer can result in an unstable closed-loop system.

Example (Kristic et al.)

$$\dot{x}_1 = -x_1 + x_1^4 + x_1^2 x_2$$
$$\dot{x}_2 = -kx_2 + u, k > 0$$

Design control law with backstepping

$$\phi_1(x_1) = -x_1^2$$
$$\dot{x}_1 = -x_1 + x_1^4 + x_1^2 \phi_1(x_1) = -x_1$$

Error state variable $z = x_2 - \phi_1(x_1) = x_2 + x_1^2$

New State Equations

$$\dot{x}_1 = -x_1 + x_1^2(x_1^2 + x_2) = -x_1 + x_1^2 z$$

$$\dot{z} = \dot{x}_2 - \dot{\phi}_1(x_1) = -kx_2 + u + 2x_1\dot{x}_1$$

$$= -kx_2 + u + 2x_1(-x_1 + x_1^2 z)$$

$$V(x_1, z) = \frac{1}{2}(x_1^2 + z^2)$$

$$\dot{V}(x_1, z) = x_1\dot{x}_1 + z\dot{z}$$

$$= -x_1^2 + x_1^3 z$$

$$+ z[-kx_2 + u + 2x_1(-x_1 + x_1^2 z)]$$

Controller

$$\dot{V}(x_1, z) = x_1\dot{x}_1 + z\dot{z}$$

$$= -x_1^2 + x_1^3 z$$

$$+ z[-kx_2 + u + 2x_1(-x_1 + x_1^2 z)]$$

$$u = -cz - x_1^3 + kx_2 - 2x_1(-x_1 + x_1^2 z), c > 0$$

$$\dot{V}(x_1, z) = -x_1^2 - cz^2 < 0$$

$(x_1, z) = (0,0)$ is a globally asymptotically stable equilibrium point of the system with control u

Stable Dynamics

$$\dot{x}_1 = -x_1 + x_1^2 z$$

$$\dot{z} = -x_1^3 - cz$$

- Control law stabilizes the system if the state vector is measurable.
- In practice, we typically need state estimation.

Observer

- Assume x_1 measured and estimate x_2

$$\dot{\hat{x}}_2 = -k\hat{x}_2 + u$$

$$\tilde{x}_2 = x_2 - \hat{x}_2$$

$$\dot{\tilde{x}}_2 = \dot{x}_2 - \dot{\hat{x}}_2 = -kx_2 + u - (-k\hat{x}_2 + u)$$

$$= -k\tilde{x}_2$$

- $\tilde{x}_2(t) = e^{-kt}\tilde{x}_2(0)$ converges to zero exponentially.
- Substituting $\hat{x}_2$ in place of x_2 leads to an unstable system (finite escape time)

Error Variable

$$x_2 = \hat{x}_2 + \tilde{x}_2$$

$$\dot{x}_1 = -x_1 + x_1^4 + x_1^2(\hat{x}_2 + \tilde{x}_2)$$

$$= -x_1 + x_1^2(\hat{x}_2 + x_1^2 + \tilde{x}_2)$$

$$\dot{x}_2 = -kx_2 + u, k > 0$$

$$\dot{\hat{x}}_2 = -k\hat{x}_2 + u$$

$$\dot{\tilde{x}}_2 = \dot{x}_2 - \dot{\hat{x}}_2 = -k\tilde{x}_2, k > 0$$

Error Variable $z = \hat{x}_2 - \phi_1(x_1) = \hat{x}_2 + x_1^2$

Error Dynamics

Error Variable $z = \hat{x}_2 - \phi_1(x_1) = \hat{x}_2 + x_1^2$

$$\dot{z} = \dot{\hat{x}}_2 + 2x_1\dot{x}_1$$

$$= -k\hat{x}_2 + u + 2x_1[-x_1 + x_1^4 + x_1^2(\hat{x}_2 + \tilde{x}_2)]$$

$$u = -cz - x_1^3 + k\hat{x}_2 - 2x_1(-x_1 + x_1^2 z), c > 0$$

$$\dot{z} = -cz - x_1^3 - 2x_1^3(\hat{x}_2 + x_1^2) + 2x_1[x_1^4 + x_1^2(\hat{x}_2 + \tilde{x}_2)]$$

$$\dot{z} = -cz - x_1^3 + 2x_1^3\tilde{x}_2$$

Closed-loop System

$$z = \hat{x}_2 - \phi_1(x_1) = \hat{x}_2 + x_1^2$$

$$\dot{x}_1 = -x_1 + x_1^2(\hat{x}_2 + x_1^2 + \tilde{x}_2)$$

$$\dot{x}_1 = -x_1 + x_1^2(z + \tilde{x}_2)$$

$$\dot{z} = -cz - x_1^3(1 + 2\tilde{x}_2)$$

$$\tilde{x}_2(t) = e^{-kt}\tilde{x}_2(0)$$

Can show that, even with $z = 0$, the first state variable diverges: $\dot{x}_1 = -x_1 + x_1^2 e^{-kt}\tilde{x}_2(0)$

Divergence

Consider $\dot{x} = -x + x^2 a e^{-kt}$

Change of variable $w = 1/x$,

$$\dot{w} = -\frac{1}{x^2}\dot{x} = w - ae^{-kt}$$

$$x = \frac{(1+k)x(0)}{[1 + k - x(0)a]e^t + x(0)ae^{-kt}}$$

$$x(t) \to \infty \; as \; t \to t_f$$

$$t_f = \frac{1}{1+k}\ln\left\{\frac{x(0)a}{x(0)a - (1+k)}\right\}$$