Chapter 2
Lie Derivatives

In this chapter several additional useful concepts are introduced, which will be extensively employed in the second half of this book. It is shown that there is a one-to-one relation between vector fields on a manifold and families of transformations of the manifold onto itself. This relation is essential in the study of various symmetries, as shown in Chaps. 4, 6, and 8, and in the relationship of a Lie group with its Lie algebra, treated in Chap. 7.

2.1 One-Parameter Groups of Transformations and Flows

Definition 2.1 Let M be a differentiable manifold. A *one-parameter group of transformations*, φ, on M, is a differentiable map from $M \times \mathbb{R}$ onto M such that $\varphi(x, 0) = x$ and $\varphi(\varphi(x, t), s) = \varphi(x, t + s)$ for all $x \in M$, $t, s \in \mathbb{R}$.

If we define $\varphi_t(x) \equiv \varphi(x, t)$, then, for each $t \in \mathbb{R}$, φ_t is a differentiable map from M onto M and $\varphi_{t+s}(x) = \varphi(x, t + s) = \varphi(\varphi(x, t), s) = \varphi(\varphi_t(x), s) = \varphi_s(\varphi_t(x)) = (\varphi_s \circ \varphi_t)(x)$, that is,

$$\varphi_{t+s} = \varphi_s \circ \varphi_t = \varphi_t \circ \varphi_s$$

(since $t + s = s + t$). φ_0 is the identity map of M since $\varphi_0(x) = \varphi(x, 0) = x$ for all $x \in M$. We have then $\varphi_t \circ \varphi_{-t} = \varphi_{-t} \circ \varphi_t = \varphi_0$, which means that each map φ_t has an inverse, φ_{-t}, which is also differentiable. Therefore, each φ_t is a diffeomorphism of M onto itself. Thus, the set of transformations $\{\varphi_t \mid t \in \mathbb{R}\}$ is an Abelian group of diffeomorphisms of M onto M, and the map $t \mapsto \varphi_t$ is a homomorphism from the additive group of the real numbers into the group of diffeomorphisms of M.

Each one-parameter group of transformations φ on M determines a family of curves in M (the *orbits* of the group). The map $\varphi_x : \mathbb{R} \to M$ given by $\varphi_x(t) = \varphi(x, t)$ is a differentiable curve in M for each $x \in M$. Since $\varphi_x(0) = \varphi(x, 0) = x$, the tangent vector to the curve φ_x at $t = 0$ belongs to $T_x M$. The *infinitesimal generator* of φ is the vector field $\mathbf{X}$ such that $\mathbf{X}_x = (\varphi_x)'_0$. In other words, the infinitesimal

G.F. Torres del Castillo, *Differentiable Manifolds*,
DOI 10.1007/978-0-8176-8271-2_2,

generator of φ is a vector field tangent to the curves generated by the one-parameter group of transformations.

Example 2.2 Let $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ and let $\varphi : M \times \mathbb{R} \to M$ be given by

$$\varphi\big((x_0, y_0), t\big) = \frac{(2x_0, 2y_0 \cos t + (1 - {x_0}^2 - {y_0}^2) \sin t)}{1 + {x_0}^2 + {y_0}^2 + (1 - {x_0}^2 - {y_0}^2) \cos t - 2y_0 \sin t}. \tag{2.1}$$

The map (2.1) is differentiable because it is the composition of differentiable functions and the denominator does not vanish for $x_0 \neq 0$ (it can be verified that the denominator in (2.1) is equal to $2[(x_0 \sin(t/2))^2 + (y_0 \sin(t/2) - \cos(t/2))^2]$). Furthermore, $\varphi((x_0, y_0), t) \in M$ for any $(x_0, y_0) \in M$, $t \in \mathbb{R}$, and $\varphi((x_0, y_0), 0) = (x_0, y_0)$. Finally, a direct but lengthy computation shows that (2.1) satisfies the relation $\varphi(\varphi((x_0, y_0), t), s) = \varphi((x_0, y_0), t + s)$, and therefore we have a one-parameter group of transformations on M.

For $(x_0, y_0) \in M$ fixed, $\varphi_{(x_0,y_0)}(t) \equiv \varphi((x_0, y_0), t)$ is a differentiable curve in M whose tangent vector at $t = 0$ can be obtained using (1.20), that is,

$$(\varphi_{(x_0,y_0)})'_0 = \frac{\mathrm{d}}{\mathrm{d}t}(x \circ \varphi_{(x_0,y_0)})\bigg|_{t=0} \left(\frac{\partial}{\partial x}\right)_{(x_0,y_0)} + \frac{\mathrm{d}}{\mathrm{d}t}(y \circ \varphi_{(x_0,y_0)})\bigg|_{t=0} \left(\frac{\partial}{\partial y}\right)_{(x_0,y_0)}$$

with

$$\begin{aligned}
(x \circ \varphi_{(x_0,y_0)})(t) &= \frac{2x_0}{1 + {x_0}^2 + {y_0}^2 + (1 - {x_0}^2 - {y_0}^2) \cos t - 2y_0 \sin t}, \\
(y \circ \varphi_{(x_0,y_0)})(t) &= \frac{2y_0 \cos t + (1 - {x_0}^2 - {y_0}^2) \sin t}{1 + {x_0}^2 + {y_0}^2 + (1 - {x_0}^2 - {y_0}^2) \cos t - 2y_0 \sin t}
\end{aligned} \tag{2.2}$$

[see (2.1)]. Calculating the derivatives of the expressions (2.2) with respect to t at $t = 0$, one finds that the infinitesimal generator of the one-parameter group (2.1), $\mathbf{X}$, is given by

$$\begin{aligned}
\mathbf{X}_{(x_0,y_0)} \equiv (\varphi_{(x_0,y_0)})'_0 &= x_0 y_0 \left(\frac{\partial}{\partial x}\right)_{(x_0,y_0)} + \frac{1 - {x_0}^2 + {y_0}^2}{2} \left(\frac{\partial}{\partial y}\right)_{(x_0,y_0)} \\
&= \left(xy \frac{\partial}{\partial x} + \frac{1 - x^2 + y^2}{2} \frac{\partial}{\partial y}\right)_{(x_0,y_0)}
\end{aligned}$$

[see (1.32)]; thus,

$$\mathbf{X} = xy \frac{\partial}{\partial x} + \frac{1 - x^2 + y^2}{2} \frac{\partial}{\partial y}. \tag{2.3}$$

The (images of the) curves defined by the one-parameter group (2.1), to which $\mathbf{X}$ is tangent, are circle arcs. In order to simplify the notation, we shall write x and y in place of $x \circ \varphi_{(x_0,y_0)}$ and $y \circ \varphi_{(x_0,y_0)}$, respectively; then, from (2.2), eliminating the parameter t, we see that

$$\left(x - \frac{1 + {x_0}^2 + {y_0}^2}{2x_0}\right)^2 + y^2 = \left(\frac{1 + {x_0}^2 + {y_0}^2}{2x_0}\right)^2 - 1,$$

which is the equation of a circle centered at a point of the x axis.

Exercise 2.3 Show that the following families of maps $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ form one-parameter groups of transformations and find their infinitesimal generators:

(a) $\varphi_t(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t)$.
(b) $\varphi_t(x, y) = (x + at, y + bt)$, with $a, b \in \mathbb{R}$.
(c) $\varphi_t(x, y) = (\mathrm{e}^{at} x, \mathrm{e}^{bt} y)$, with $a, b \in \mathbb{R}$.

Exercise 2.4 Let φ be a one-parameter group of transformations on M and let $\mathbf{X}$ be its infinitesimal generator. Show that if $y = \varphi_x(t_0)$, for some $t_0 \in \mathbb{R}$, then $(\varphi_x)'_{t_0} = (\varphi_y)'_0$ and, therefore, $(\varphi_x)'_{t_0} = \mathbf{X}_{\varphi_x(t_0)}$.

Given a differentiable vector field, $\mathbf{X}$, on M, there does not always exist a one-parameter group of transformations whose infinitesimal generator is $\mathbf{X}$; it is said that $\mathbf{X}$ is *complete* if such a one-parameter group of transformations exists.

Integral Curves of a Vector Field

Definition 2.5 Let $\mathbf{X}$ be a vector field on M. A curve $C : I \to M$ is an *integral curve* of $\mathbf{X}$ if $C'_t = \mathbf{X}_{C(t)}$, for $t \in I$. If $C(0) = x$ we say that C starts at x. (According to Exercise 2.4, if φ is a one-parameter group of transformations and $\mathbf{X}$ is its infinitesimal generator, then the curve φ_x is an integral curve of $\mathbf{X}$ that starts at x.)

If $(x^1, x^2, \ldots, x^n)$ is a local coordinate system on M and $\mathbf{X}$ is expressed in the form $\mathbf{X} = X^i (\partial/\partial x^i)$, the condition that C be an integral curve of $\mathbf{X}$ amounts to the system of ordinary differential equations (ODEs) [see (1.20)]

$$\frac{\mathrm{d}(x^i \circ C)}{\mathrm{d}t} = X^i \circ C. \tag{2.4}$$

More explicitly, writing the right-hand side of the previous equation in the form

$$\begin{aligned}
(X^i \circ C)(t) &= (X^i \circ \phi^{-1})\big(\phi(C(t))\big) \\
&= (X^i \circ \phi^{-1})\big(x^1(C(t)), x^2(C(t)), \ldots, x^n(C(t))\big) \\
&= (X^i \circ \phi^{-1})\big((x^1 \circ C)(t), (x^2 \circ C)(t), \ldots, (x^n \circ C)(t)\big),
\end{aligned}$$

one finds that equations (2.4) correspond to the (autonomous) system of equations

$$\frac{\mathrm{d}(x^i \circ C)}{\mathrm{d}t} = (X^i \circ \phi^{-1})(x^1 \circ C, x^2 \circ C, \ldots, x^n \circ C) \tag{2.5}$$

for the n functions $x^i \circ C$ of $\mathbb{R}$ to $\mathbb{R}$. (Note that each composition $X^i \circ \phi^{-1}$ is a real-valued function defined in some subset of $\mathbb{R}^n$.) According to the fundamental theorem for systems of ODEs, given $x \in M$, there exists a unique integral curve of $\mathbf{X}$, C, starting at x. (That is, if D is another integral curve of $\mathbf{X}$ starting at x, then $D = C$ in the intersection of their domains.)

Let C be an integral curve of $\mathbf{X}$ starting at x, and let $\varphi(x,t) \equiv C(t)$. The curve D defined by $D(t) \equiv C(t+s)$, with s fixed, is an integral curve of $\mathbf{X}$, since for an arbitrary function $f \in C^\infty(M)$

$$\begin{aligned} D'_t[f] &= \lim_{h\to 0} \frac{f(D(t+h)) - f(D(t))}{h} \\ &= \lim_{h\to 0} \frac{f(C(t+h+s)) - f(C(t+s))}{h} \\ &= C'_{t+s}[f] = \mathbf{X}_{C(t+s)}[f] = \mathbf{X}_{D(t)}[f]. \end{aligned}$$

The curve D starts at $D(0) = C(s)$ and by virtue of the uniqueness of the integral curves, we have

$$D(t) = \varphi\big(C(s), t\big) = \varphi\big(\varphi(x,s), t\big).$$

On the other hand, from the definition of D,

$$D(t) = C(t+s) = \varphi(x, t+s);$$

therefore,

$$\varphi\big(\varphi(x,s), t\big) = \varphi(x, t+s) \tag{2.6}$$

(cf. Definition 2.1).

In some cases φ is not defined for all $t \in \mathbb{R}$, and for that reason it is not a one-parameter group of transformations. However, for each $x \in M$ there exist a neighborhood, U of x and an $\varepsilon > 0$ such that φ is defined on $U \times (-\varepsilon, \varepsilon)$ and is differentiable. The map φ is called a *flow* or *local one-parameter group of transformations* and $\mathbf{X}$ is its infinitesimal generator.

If $\mathbf{X}$ is the infinitesimal generator of a one-parameter group of transformations or a flow, the transformations φ_t are also denoted by $\exp t\mathbf{X}$. Then, the relation (2.6) is expressed as $\exp t\mathbf{X} \circ \exp s\mathbf{X} = \exp(t+s)\mathbf{X}$.

Example 2.6 Let $M = \mathbb{R}$ with the usual coordinate system, $x = \mathrm{id}$. The integral curves of the vector field $\mathbf{X} = x^2\, \partial/\partial x$ are determined by the single differential equation [see (2.4)]

$$\frac{\mathrm{d}(x \circ C)}{\mathrm{d}t} = x^2 \circ C = (x \circ C)^2 \tag{2.7}$$

[the previous equality follows from (1.7), which gives $x^2(p) = (x(p))^2$; hence, $(x^2 \circ C)(t) = x^2(C(t)) = [x(C(t))]^2 = ((x \circ C)(t))^2 = (x \circ C)^2(t)$]. The solution of (2.7) is $(x \circ C)(t) = -1/(t+a)$, where a is a constant or, simply, since

$x = \mathrm{id}$, $C(t) = -1/(t+a)$. If the integral curve of $\mathbf{X}$ starts at x_0, then $C(0) = -1/a = x_0$, i.e., $a = -1/x_0$. Since φ_{x_0} is the integral curve of $\mathbf{X}$ starting at x_0 (see Definition 2.5), we have

$$\varphi_{x_0}(t) = -\frac{1}{t - 1/x_0} = \frac{x_0}{1 - x_0 t},$$

and therefore

$$\varphi(x_0, t) = \frac{x_0}{1 - x_0 t} \tag{2.8}$$

is the local one-parameter group generated by $x^2\,\partial/\partial x$.

The expression (2.8) is not defined for $t = 1/x_0$, and therefore we are not dealing with a one-parameter group of transformations, despite the fact that $\mathbf{X}$ is differentiable. However, the flow (2.8) satisfies the relation (2.6), since, according to (2.8),

$$\begin{aligned}\varphi\big(\varphi(x_0, s), t\big) &= \frac{\varphi(x_0, s)}{1 - \varphi(x_0, s)t} = \frac{x_0/(1 - x_0 s)}{1 - t x_0/(1 - x_0 s)} = \frac{x_0}{1 - x_0(t + s)} \\ &= \varphi(x_0, t + s),\end{aligned}$$

whenever all the expressions involved are defined.

Example 2.7 Let $M = \mathbb{R}^2$ and let $\mathbf{X} = y\,\partial/\partial x + x\,\partial/\partial y$, where (x, y) are the usual coordinates of $\mathbb{R}^2$. Equations (2.4) are in this case

$$\frac{\mathrm{d}(x \circ C)}{\mathrm{d}t} = y \circ C, \qquad \frac{\mathrm{d}(y \circ C)}{\mathrm{d}t} = x \circ C.$$

By adding and subtracting these equations we obtain

$$\frac{\mathrm{d}(x \circ C + y \circ C)}{\mathrm{d}t} = x \circ C + y \circ C, \qquad \frac{\mathrm{d}(x \circ C - y \circ C)}{\mathrm{d}t} = -(x \circ C - y \circ C),$$

whose solutions are $(x \circ C + y \circ C)(t) = (x_0 + y_0)\,\mathrm{e}^t$ and $(x \circ C - y \circ C)(t) = (x_0 - y_0)\,\mathrm{e}^{-t}$, where x_0 and y_0 are the initial values of $x \circ C$ and $y \circ C$, respectively. Hence, $(x \circ C)(t) = x_0 \cosh t + y_0 \sinh t$, $(y \circ C)(t) = x_0 \sinh t + y_0 \cosh t$, and

$$\varphi_{(x_0, y_0)}(t) = (x_0 \cosh t + y_0 \sinh t, x_0 \sinh t + y_0 \cosh t). \tag{2.9}$$

Since $(x \circ C)^2 - (y \circ C)^2 = {x_0}^2 - {y_0}^2$, the (images of the) integral curves of $\mathbf{X}$ are hyperbolas or straight lines. The expression (2.9) is defined for all $t \in \mathbb{R}$, and therefore it corresponds to a one-parameter group of transformations. Substituting (2.9) into (2.6) one finds the well-known addition formulas

$$\begin{aligned}\cosh(t + s) &= \cosh t \cosh s + \sinh t \sinh s, \\ \sinh(t + s) &= \sinh t \cosh s + \cosh t \sinh s.\end{aligned}$$

Exercise 2.8 Let $\psi : M_1 \to M_2$ be a differentiable map and let φ_1 and φ_2 be one-parameter groups of transformations or flows on M_1 and M_2, respectively. Show that if $\varphi_{2t} \circ \psi = \psi \circ \varphi_{1t}$, then the infinitesimal generators of φ_1 and φ_2 are ψ-related, i.e., show that $\psi_{*x}\mathbf{X}_x = \mathbf{Y}_{\psi(x)}$, where $\mathbf{X}$ and $\mathbf{Y}$ are the infinitesimal generators of φ_1 and φ_2, respectively.

Example 2.9 An integration procedure distinct from that employed in the preceding examples is illustrated by considering the vector field $\mathbf{X} = \frac{1}{2}(x^2 - y^2)\,\partial/\partial x + xy\,\partial/\partial y$ on $M \equiv \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. (The one-parameter group generated by this vector field is also found, by another method, in Example 6.12.) The system of equations (2.4) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\left(x^2 - y^2\right), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = xy, \tag{2.10}$$

where, in order to simplify the notation, we have written x and y in place of $x \circ C$ and $y \circ C$, respectively. Eliminating the variable t from these equations (with the aid of the chain rule) we obtain the ODE $\mathrm{d}y/\mathrm{d}x = 2xy/(x^2 - y^2)$. Noting that the right-hand side of the last equation is the quotient of two homogeneous functions of the same degree, it is convenient to introduce $u \equiv y/x$, so that $\mathrm{d}u/\mathrm{d}x = u(1 + u^2)/[x(1 - u^2)]$, which by the standard procedures leads to

$$\frac{\mathrm{d}x}{x} = \frac{(1 - u^2)\,\mathrm{d}u}{u(1 + u^2)} = \left(\frac{1}{u} - \frac{2u}{1 + u^2}\right)\mathrm{d}u,$$

whose solution is given by $x = cu/(1 + u^2) = cy/[x(1 + y^2/x^2)]$, where c is some constant. Hence $x^2 + y^2 = cy$, which corresponds to the circle centered at $(0, c/2)$ and radius $c/2$.

In order to obtain the parametrization of these curves, one can substitute $x = \pm\sqrt{cy - y^2}$ into the second of equations (2.10), which yields $\mathrm{d}y/\mathrm{d}t = \pm y\sqrt{cy - y^2}$, or, putting $v = 1/y$, $\mathrm{d}v/\mathrm{d}t = \mp\sqrt{cv - 1}$; hence $2\sqrt{cv - 1} = \mp c(t - t_0)$, where t_0 is a constant. Thus, from the foregoing relations we find that

$$y = \frac{4c}{4 + c^2(t - t_0)^2}, \qquad x = -\frac{2c^2(t - t_0)}{4 + c^2(t - t_0)^2}. \tag{2.11}$$

For the integral curve of $\mathbf{X}$ starting at (x_0, y_0), from (2.11) we have $y_0 = 4c/(4 + c^2{t_0}^2)$ and $x_0 = 2c^2 t_0/(4 + c^2{t_0}^2)$, which imply that

$$c = \frac{{x_0}^2 + {y_0}^2}{y_0}, \qquad t_0 = \frac{2x_0}{{x_0}^2 + {y_0}^2}$$

and, substituting these expressions into (2.11), we obtain

$$\varphi\big((x_0, y_0), t\big) = \frac{2({x_0}^2 + {y_0}^2)(2x_0 - ({x_0}^2 + {y_0}^2)t, 2y_0)}{[({x_0}^2 + {y_0}^2)t - 2x_0]^2 + 4{y_0}^2}$$

$$= \frac{(x_0 - ({x_0}^2 + {y_0}^2)t/2, y_0)}{(1 - x_0 t/2)^2 + {y_0}^2(t/2)^2}. \tag{2.12}$$

From (2.12) we see that the integral curves of $\mathbf{X}$ are defined for all $t \in \mathbb{R}$, and therefore $\mathbf{X}$ is complete and (2.12) corresponds to a one-parameter group of transformations. Further examples are given in Examples 4.1, 6.11, 6.12, 6.20, 7.40, and 7.41.

Exercise 2.10 Find the integral curves of the vector field $\mathbf{X} = \frac{1}{x^2+y^2}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ and the one-parameter group of diffeomorphisms generated by $\mathbf{X}$.

From equations (2.10) one notices that if one looks for the integral curves of $f\mathbf{X}$, where f is some real-valued differentiable function, on eliminating the variable t the function f disappears and one obtains the same equation for $\mathrm{d}y/\mathrm{d}x$ as obtained in the preceding example. Therefore, the same circles are obtained. For any vector field $\mathbf{X}$, the integral curves of $\mathbf{X}$ and $f\mathbf{X}$, with $f \in C^\infty(M)$, only differ in the parametrization. If φ_t denotes the flow or one-parameter group generated by $\mathbf{X}$ and σ is a function of some open subset of $\mathbb{R}$ in the domain of the curve φ_x, then the tangent vector to the curve $\psi_x \equiv \varphi_x \circ \sigma$ satisfies, for $g \in C^\infty(M)$,

$$\begin{aligned}
(\psi_x)'_{t_0}[g] &= \frac{\mathrm{d}}{\mathrm{d}t} g\bigl(\psi_x(t)\bigr)\Big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}\bigl((g \circ \varphi_x) \circ \sigma\bigr)(t)\Big|_{t=t_0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(g \circ \varphi_x)\Big|_{\sigma(t_0)} \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Big|_{t_0} = (\varphi_x)'_{\sigma(t_0)}[g] \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Big|_{t_0} \\
&= \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Big|_{t_0} \mathbf{X}_{\varphi_x(\sigma(t_0))}[g],
\end{aligned} \tag{2.13}$$

where we have made use of the chain rule for functions from $\mathbb{R}$ into $\mathbb{R}$ and of the result of Exercise 2.4. The expression (2.13) coincides with $(f\mathbf{X})_{\varphi(\sigma(t_0))}[g]$ if we choose σ in such a way that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = f\bigl(\varphi_x\bigl(\sigma(t)\bigr)\bigr). \tag{2.14}$$

Hence, if additionally we impose the condition $\sigma(0) = 0$, the curve $\psi_x = \varphi_x \circ \sigma$ is an integral curve of $f\mathbf{X}$ starting at x.

Example 2.11 The integral curves of $f\mathbf{X}$, where $\mathbf{X}$ is the vector field considered in Example 2.9 and f is any function belonging to $C^\infty(M)$, can be obtained by solving equation (2.14) with φ_x given by (2.12), i.e.,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}t} = f\biggl(&\frac{({x_0}^2 + {y_0}^2)[4x_0 - 2({x_0}^2 + {y_0}^2)\sigma(t)]}{[({x_0}^2 + {y_0}^2)\sigma(t) - 2x_0]^2 + 4{y_0}^2}, \\
&\frac{4y_0({x_0}^2 + {y_0}^2)}{[({x_0}^2 + {y_0}^2)\sigma(t) - 2x_0]^2 + 4{y_0}^2}\biggr).
\end{aligned} \tag{2.15}$$

If we take, for example, $f(x, y) = y^{-1}$, equation (2.15) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{[({x_0}^2 + {y_0}^2)\sigma(t) - 2x_0]^2 + 4{y_0}^2}{4y_0({x_0}^2 + {y_0}^2)}$$

and with the change of variable $({x_0}^2 + {y_0}^2)\sigma(t) - 2x_0 = 2y_0 \tan u$ we have $\mathrm{d}u/\mathrm{d}t = 1/2$. Hence $u = (t - t_0)/2$, where t_0 is some constant and

$$\sigma(t) = \frac{2x_0 + 2y_0 \tan \frac{1}{2}(t - t_0)}{{x_0}^2 + {y_0}^2}. \tag{2.16}$$

The condition $\sigma(0) = 0$ amounts to $0 = x_0 - y_0 \tan \frac{1}{2} t_0$, which, substituted into (2.16), yields

$$\sigma(t) = \frac{2 \tan \frac{1}{2} t}{y_0 + x_0 \tan \frac{1}{2} t} = \frac{2 \sin \frac{1}{2} t}{x_0 \sin \frac{1}{2} t + y_0 \cos \frac{1}{2} t}. \tag{2.17}$$

Thus, the flow generated by $f\mathbf{X} = y^{-1}[\frac{1}{2}(x^2 - y^2)\, \partial/\partial x + xy\, \partial/\partial y]$ is given by $\psi((x_0, y_0), t) = \varphi((x_0, y_0), \sigma(t))$, where φ is the one-parameter group generated by $\mathbf{X}$, given by (2.12), and σ is the function (2.17), i.e.,

$$\begin{aligned} &\psi\big((x_0, y_0), t\big) \\ &\quad = \frac{1}{2y_0}\big(0, {x_0}^2 + {y_0}^2\big) \\ &\qquad + \frac{1}{2y_0}\big(2x_0 y_0 \cos t - \big({y_0}^2 - {x_0}^2\big) \sin t, \big({y_0}^2 - {x_0}^2\big) \cos t + 2x_0 y_0 \sin t\big) \end{aligned} \tag{2.18}$$

[cf. (2.12)]. Even though the expression (2.18) is defined for all $t \in \mathbb{R}$, the variable t has to be restricted to some open interval of length 2π where $\psi((x_0, y_0), t) \neq (0, 0)$, taking into account that the manifold being considered is $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. It may be noticed that $f\mathbf{X}$ is differentiable on M because y does not vanish there. Whereas $\mathbf{X}$ is complete, $f\mathbf{X}$ is not. The expression (2.18) shows that the images of the integral curves of $f\mathbf{X}$ (and of $\mathbf{X}$) are arcs of circles.

Second-Order ODEs A vector field $\mathbf{X}$ on the tangent bundle TM such that, for $v \in TM$,

$$\pi_{*v}\mathbf{X}_v = v, \tag{2.19}$$

where π is the canonical projection of TM on M, corresponds to a system of second-order ODEs. (Equation (2.19) makes sense because v is a tangent vector to M at $\pi(v)$, that is, $v \in T_{\pi(v)}M$, and π_{*v} applies $T_v(TM)$ into $T_{\pi(v)}M$.) In effect, using the local expression $\mathbf{X} = A^i\, \partial/\partial q^i + B^i\, \partial/\partial \dot{q}^i$ as well as (1.29) and (1.27), the relation

(2.19) amounts to

$$A^i(v)\left(\frac{\partial}{\partial x^i}\right)_{\pi(v)} = \dot{q}^i(v)\left(\frac{\partial}{\partial x^i}\right)_{\pi(v)},$$

that is, $A^i = \dot{q}^i$. Hence, any vector field on TM satisfying (2.19) locally is of the form

$$\mathbf{X} = \dot{q}^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial \dot{q}^i}$$

in a coordinate system induced by a coordinate system on M (see Sect. 1.2), where the B^i are n arbitrary real-valued functions defined on TM. The integral curves of $\mathbf{X}$ are determined by the equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \dot{q}^i, \qquad \frac{\mathrm{d}\dot{q}^i}{\mathrm{d}t} = B^i,$$

which are equivalent to the system of n second-order ODEs

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = \left(B^i \circ \overline{\phi}^{-1}\right)\left(q^1, \ldots, q^n, \frac{\mathrm{d}q^1}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}q^n}{\mathrm{d}t}\right).$$

Exercise 2.12 Let $\varphi(x, y, t) = (F_1(x, y, t), F_2(x, y, t))$ be a one-parameter group of transformations on $\mathbb{R}^2$ [which, among other things, implies that F_1 and F_2 are differentiable functions from $\mathbb{R}^3$ into $\mathbb{R}$ such that $F_1(x, y, 0) = x$ and $F_2(x, y, 0) = y$], and let

$$F_3(x, y, z, t) \equiv \frac{D_1 F_2 + z D_2 F_2}{D_1 F_1 + z D_2 F_1}, \tag{2.20}$$

where D_i represents partial differentiation with respect to the ith argument. Show that $\varphi^{(1)}(x, y, z, t) \equiv (F_1(x, y, t), F_2(x, y, t), F_3(x, y, z, t))$ is a (possibly local) one-parameter group of transformations on $\mathbb{R}^3$ (known as the extension or *first prolongation* of φ). Show that if $\xi\,(\partial/\partial x) + \eta\,(\partial/\partial y)$ is the infinitesimal generator of φ, then the infinitesimal generator of $\varphi^{(1)}$ is

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \left[\eta_x + z(\eta_y - \xi_x) - z^2 \xi_y\right] \frac{\partial}{\partial z}, \tag{2.21}$$

where the subscripts denote partial differentiation (e.g., $\eta_x \equiv \partial\eta/\partial x$). [Strictly speaking, in (2.21), in place of x, y, ξ, η, their pullbacks under the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ should appear.] The prolongation of a one-parameter group of diffeomorphisms is employed in the study of the symmetries of an ODE; see, e.g., Hydon (2000).

Canonical Lift of a Vector Field A differentiable mapping $\psi : M_1 \to M_2$ gives rise to a differentiable mapping $\overline{\psi} : TM_1 \to TM_2$, defined by

$$\overline{\psi}(v_p) \equiv \psi_{*p}(v_p), \quad \text{for } v_p \in T_p(M_1).$$

Since $\psi_{*p}(v_p) \in T_{\psi(p)}M_2$, we see that $\pi_2 \circ \overline{\psi} = \psi \circ \pi_1$, where π_1 is the canonical projection of TM_1 on M_1 and, similarly, π_2 is the canonical projection of TM_2 on M_2. Making use of the chain rule (1.25), one can readily verify that if $\psi_1 : M_1 \to M_2$ and $\psi_2 : M_2 \to M_3$ are two differentiable mappings, then $\overline{(\psi_2 \circ \psi_1)} = \overline{\psi}_2 \circ \overline{\psi}_1$. Hence, if $\{\varphi_t\}$ is a one-parameter group of diffeomorphisms on a manifold M, the mappings $\overline{\varphi_t}$ form a one-parameter group of diffeomorphisms on TM.

The local expression of the transformations φ_t is given by the functions $\varphi_t{}^*x^i$, where the x^i form some coordinate system on M. Then, in terms of the coordinates q^i, $\dot{q}^i$ induced on TM by the x^i, the transformations $\overline{\varphi_t}$ are locally given by the functions $\overline{\varphi_t}{}^*q^i$ and $\overline{\varphi_t}{}^*\dot{q}^i$. Since $\pi \circ \overline{\varphi_t} = \varphi_t \circ \pi$ and, by definition, $q^i = \pi^*x^i$, we obtain

$$\overline{\varphi_t}{}^*q^i = (\overline{\varphi_t}{}^* \circ \pi^*)x^i = (\pi \circ \overline{\varphi_t})^*x^i = (\varphi_t \circ \pi)^*x^i = \pi^*\big(\varphi_t{}^*x^i\big)$$

and, making use of the definitions of $\overline{\varphi_t}$ and of the coordinates $\dot{q}^i$ [see (1.27)], we find that

$$\begin{aligned}\big(\overline{\varphi_t}{}^*\dot{q}^i\big)(v_p) &= \dot{q}^i\big(\overline{\varphi_t}(v_p)\big) = \dot{q}^i\big(\varphi_{t*p}(v_p)\big) = \big(\varphi_{t*p}(v_p)\big)\big[x^i\big] = v_p\big[\varphi_t{}^*x^i\big] \\ &= \dot{q}^j(v_p)\left(\frac{\partial}{\partial x^j}\right)_p\big[\varphi_t{}^*x^i\big] = \left[\dot{q}^j\,\pi^*\left(\frac{\partial(\varphi_t{}^*x^i)}{\partial x^j}\right)\right](v_p),\end{aligned}$$

i.e.,

$$\overline{\varphi_t}{}^*\dot{q}^i = \dot{q}^j\,\pi^*\left(\frac{\partial(\varphi_t{}^*x^i)}{\partial x^j}\right).$$

Recalling that the infinitesimal generator, $\mathbf{X}$, of φ_t, is given by $\mathbf{X} = X^i\,\partial/\partial x^i$ with $X^i = (\mathrm{d}/\mathrm{d}t)(\varphi_t{}^*x^i)|_{t=0}$, from the expressions obtained above we find that the infinitesimal generator, $\overline{\mathbf{X}}$, of $\overline{\varphi_t}$ is locally given by

$$\overline{\mathbf{X}} = \big(\pi^*X^i\big)\frac{\partial}{\partial q^i} + \dot{q}^j\,\pi^*\left(\frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial \dot{q}^i}. \qquad (2.22)$$

The vector field $\overline{\mathbf{X}}$ is called the *canonical lift* of $\mathbf{X}$ to TM.

Exercise 2.13 Find the one-parameter group of diffeomorphisms on the tangent bundle $T\mathbb{R}^2$ induced by the one-parameter group of diffeomorphisms on $\mathbb{R}^2$ defined by $\varphi_t(x, y) = (\mathrm{e}^{at}x, \mathrm{e}^{bt}y)$, with $a, b \in \mathbb{R}$. Show that its infinitesimal generator is

$$aq^1\frac{\partial}{\partial q^1} + bq^2\frac{\partial}{\partial q^2} + a\frac{\partial}{\partial \dot{q}^1} + b\frac{\partial}{\partial \dot{q}^2},$$

where the q^i and $\dot{q}^i$ are the coordinates on $T\mathbb{R}^2$ induced by the Cartesian coordinates x, y.

Exercise 2.14 Show that $\overline{[\mathbf{X}, \mathbf{Y}]} = [\overline{\mathbf{X}}, \overline{\mathbf{Y}}]$, for $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$.

Exercise 2.15 A (time-independent) Lagrangian is a real-valued function defined in TM. A differentiable curve C in M is a solution of the Euler–Lagrange equations corresponding to the Lagrangian L if, locally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}^i}\big(\overline{C}(t)\big)\right] - \frac{\partial L}{\partial q^i}\big(\overline{C}(t)\big) = 0, \quad i = 1, 2, \ldots, n,$$

where $\overline{C}$ is the curve in TM defined by $\overline{C}(t) = C'_t$. The vector field $\mathbf{X}$ on M represents a *symmetry* of the Lagrangian L if $\overline{\mathbf{X}}L = 0$. Show that if $\mathbf{X}$ represents a symmetry of L, then

$$X^i\big(C(t)\big)\frac{\partial L}{\partial \dot{q}^i}\big(\overline{C}(t)\big)$$

is a *constant of motion*, i.e., it does not depend on t. (Note that $\pi(\overline{C}(t)) = C(t)$, hence $q^i(\overline{C}(t)) = x^i(C(t))$, and that, according to (1.28) and (1.20), $\dot{q}^i(\overline{C}(t)) = C'_t[x^i] = \mathrm{d}(x^i \circ C)/\mathrm{d}t = \mathrm{d}(q^i(\overline{C}(t)))/\mathrm{d}t$.)

2.2 Lie Derivative of Functions and Vector Fields

Let φ be a one-parameter group of transformations or a flow on M. As pointed out above, the map $\varphi_t : M \to M$, defined by $\varphi_t(x) = \varphi(x, t)$, is a differentiable mapping. For $f \in C^\infty(M)$, $\varphi_t^* f = f \circ \varphi_t$ also belongs to $C^\infty(M)$; the limit $\lim_{t\to 0} \frac{\varphi_t^* f - f}{t}$ represents the rate of change of the function f under the family of transformations φ_t.

If $\mathbf{X}$ is the infinitesimal generator of φ, the curve φ_x given by $\varphi_x(t) = \varphi(x, t)$ is the integral curve of $\mathbf{X}$ that starts at x; therefore

$$\begin{aligned}
\left(\lim_{t\to 0}\frac{\varphi_t^* f - f}{t}\right)(x) &= \lim_{t\to 0}\frac{f(\varphi_t(x)) - f(x)}{t} \\
&= \lim_{t\to 0}\frac{f(\varphi(x,t)) - f(x)}{t} \\
&= \lim_{t\to 0}\frac{f(\varphi_x(t)) - f(\varphi_x(0))}{t} \\
&= (\varphi_x)'_0[f] = \mathbf{X}_x[f] \\
&= (\mathbf{X}f)(x),
\end{aligned}$$

which shows that, for any differentiable function, the limit $\lim_{t\to 0} \frac{\varphi_t^* f - f}{t}$ exists and depends on φ only through its infinitesimal generator. This limit is called the *Lie derivative* of f with respect to $\mathbf{X}$ and is denoted by $\pounds_{\mathbf{X}} f$. From the expression

$$\pounds_{\mathbf{X}} f = \mathbf{X} f \tag{2.23}$$

one can derive the properties of the Lie derivative of functions.

Exercise 2.16 Show that if $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, then $\pounds_{\mathbf{X}}(\pounds_{\mathbf{Y}} f) - \pounds_{\mathbf{Y}}(\pounds_{\mathbf{X}} f) = \pounds_{[\mathbf{X},\mathbf{Y}]} f$.

Let M and N be differentiable manifolds and let $\psi : M \to N$ be a diffeomorphism. If $\mathbf{X}$ is a vector field on N, then there exists a unique vector field $\mathbf{Y}$ on M such that $\mathbf{Y}$ and $\mathbf{X}$ are ψ-related. Indeed, since $\psi^{-1} \circ \psi$ is the identity map of M, using the chain rule (1.25) we find that $(\psi^{-1})_{*\psi(x)}$ is the inverse of ψ_{*x} and, therefore, the condition that $\mathbf{Y}$ and $\mathbf{X}$ be ψ-related (i.e., $\psi_{*x}\mathbf{Y}_x = \mathbf{X}_{\psi(x)}$) has a unique solution, given by

$$\mathbf{Y}_x = \left(\psi^{-1}\right)_{*\psi(x)} \mathbf{X}_{\psi(x)}.$$

The vector field $\mathbf{Y}$ is, by definition, the *pullback* of $\mathbf{X}$ under ψ and will be denoted by $\psi^*\mathbf{X}$, that is,

$$(\psi^*\mathbf{X})_x \equiv \left(\psi^{-1}\right)_{*\psi(x)} \mathbf{X}_{\psi(x)}. \tag{2.24}$$

Note that since $\psi^*\mathbf{X}$ and $\mathbf{X}$ are ψ-related,

$$(\psi^*\mathbf{X})(\psi^* f) = \psi^*(\mathbf{X} f), \tag{2.25}$$

for $f \in C^\infty(N)$ [see (1.40)].

Exercise 2.17 Show that $\psi^*(f\mathbf{X}) = (\psi^* f)(\psi^*\mathbf{X})$ and that $\psi^*(a\mathbf{X} + b\mathbf{Y}) = a\psi^*\mathbf{X} + b\psi^*\mathbf{Y}$ for $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(N)$, $f \in C^\infty(N)$, and $a, b \in \mathbb{R}$.

Exercise 2.18 Show that if $\psi : M \to N$ is a diffeomorphism and φ is a one-parameter group of transformations on N whose infinitesimal generator is $\mathbf{X}$, then $\chi_t \equiv \psi^{-1} \circ \varphi_t \circ \psi$ is a one-parameter group of transformations on M whose infinitesimal generator is $\psi^*\mathbf{X}$ (cf. Exercise 2.8).

Exercise 2.19 Show that if $\psi_1 : M_1 \to M_2$ and $\psi_2 : M_2 \to M_3$ are diffeomorphisms, then $(\psi_2 \circ \psi_1)^*\mathbf{X} = (\psi_1{}^* \circ \psi_2{}^*)\mathbf{X}$, for $\mathbf{X} \in \mathfrak{X}(M_3)$.

Let φ be a one-parameter group of transformations or a flow on M and let $\mathbf{X}$ be its infinitesimal generator. For any vector field $\mathbf{Y}$ on M, the limit $\lim_{t\to 0} \frac{\varphi_t^*\mathbf{Y} - \mathbf{Y}}{t}$, if it exists, is called the *Lie derivative* of $\mathbf{Y}$ with respect to $\mathbf{X}$ and is denoted by $\pounds_{\mathbf{X}}\mathbf{Y}$.

Proposition 2.20 *Let* $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$; *then the Lie derivative of* $\mathbf{Y}$ *with respect to* $\mathbf{X}$ *exists and is equal to the Lie bracket of* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof Let f be an arbitrary differentiable function, then, using (2.25),

$$\begin{aligned}\pounds_{\mathbf{X}}(\mathbf{Y} f) &= \lim_{t\to 0} \frac{\varphi_t^*(\mathbf{Y} f) - \mathbf{Y} f}{t} \\ &= \lim_{t\to 0} \frac{(\varphi_t^*\mathbf{Y})(\varphi_t^* f) - \mathbf{Y} f}{t}\end{aligned}$$

$$= \lim_{t\to 0}\left[(\varphi_t^*\mathbf{Y})\frac{\varphi_t^* f - f}{t} + \frac{\varphi_t^*\mathbf{Y} - \mathbf{Y}}{t} f \right]$$
$$= \mathbf{Y}(\pounds_{\mathbf{X}} f) + (\pounds_{\mathbf{X}}\mathbf{Y}) f, \tag{2.26}$$

but $\pounds_{\mathbf{X}} f = \mathbf{X} f$; therefore

$$\mathbf{X}(\mathbf{Y}f) = \pounds_{\mathbf{X}}(\mathbf{Y}f) = \mathbf{Y}(\mathbf{X}f) + (\pounds_{\mathbf{X}}\mathbf{Y}) f,$$

hence

$$(\pounds_{\mathbf{X}}\mathbf{Y}) f = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f) = [\mathbf{X}, \mathbf{Y}] f,$$

which means that

$$\pounds_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]. \tag{2.27}$$

□

As in the case of the relation (2.23), the formula (2.27) allows us to readily obtain the properties of the Lie derivative of vector fields. Furthermore, the relation (2.27) allows us to give a geometrical meaning to the Lie bracket.

Exercise 2.21 Show that if $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, then $\pounds_{\mathbf{X}}(f\mathbf{Y}) = f\,\pounds_{\mathbf{X}}\mathbf{Y} + (\pounds_{\mathbf{X}} f)\mathbf{Y}$ [cf. (2.26)]. Also show that $\pounds_{\mathbf{X}}(\mathbf{Y} + \mathbf{Z}) = \pounds_{\mathbf{X}}\mathbf{Y} + \pounds_{\mathbf{X}}\mathbf{Z}$. (*Hint*: use (2.23), (2.27), and Exercise 1.22.)

Exercise 2.22 Show that if $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$, then $\pounds_{\mathbf{X}}(\pounds_{\mathbf{Y}}\mathbf{Z}) - \pounds_{\mathbf{Y}}(\pounds_{\mathbf{X}}\mathbf{Z}) = \pounds_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$ (cf. Exercise 2.16).

Example 2.23 The Lie derivative frequently appears in connection with symmetries. The vector field $\mathbf{Y} \in \mathfrak{X}(M)$ is invariant under the one-parameter group of diffeomorphisms φ_t if $\pounds_{\mathbf{X}}\mathbf{Y} = 0$, where $\mathbf{X}$ is the infinitesimal generator of φ_t. For instance, in order to find all the vector fields on $\mathbb{R}^2$ invariant under rotations about the origin, it is convenient to employ polar coordinates (r, θ), so that, locally, $\mathbf{X} = \partial/\partial\theta$. The condition $\pounds_{\mathbf{X}}\mathbf{Y} = 0$ amounts to

$$0 = \left[(\partial/\partial\theta), Y^1(\partial/\partial r) + Y^2(\partial/\partial\theta)\right] = \frac{\partial Y^1}{\partial\theta}\frac{\partial}{\partial r} + \frac{\partial Y^2}{\partial\theta}\frac{\partial}{\partial\theta},$$

where Y^1, Y^2 are the components of $\mathbf{Y}$ with respect to the natural basis induced by the coordinates (r, θ). Hence, $\mathbf{Y}$ is invariant under rotations about the origin if and only if Y^1, Y^2 are functions of r only.

Exercise 2.24 Show that if φ_t and ψ_t are two one-parameter groups of diffeomorphisms on M that commute with each other, i.e., $\varphi_t\psi_s = \psi_s\varphi_t$ for all $t, s \in \mathbb{R}$, then the Lie bracket of their infinitesimal generators is equal to zero (cf. Exercise 2.18). (The converse is also true: two vector fields $\mathbf{X}, \mathbf{Y}$ on M such that $[\mathbf{X}, \mathbf{Y}] = 0$ generate (local) one-parameter groups that commute.)

2.3 Lie Derivative of 1-Forms and Tensor Fields

Let $\psi : M \to N$ be a differentiable map. If t is a tensor field of type $\binom{0}{k}$ on N, the pullback of t under ψ, $\psi^* t$, is the tensor field on M such that

$$(\psi^* t)_p(u_p, \ldots, w_p) \equiv t_{\psi(p)}(\psi_{*p} u_p, \ldots, \psi_{*p} w_p), \tag{2.28}$$

for $u_p, \ldots, w_p \in T_p M$, $p \in M$. Given that ψ_{*p} is a linear transformation, it can readily be verified that effectively $\psi^* t$ is a tensor field of type $\binom{0}{k}$ on M.

Exercise 2.25 Let $\psi : M \to N$ be a differentiable map and let α be a linear differential form on N. Show that

$$\int_C \psi^* \alpha = \int_{\psi \circ C} \alpha,$$

for any differentiable curve C in M (see Example 1.28).

If $f \in C^\infty(N)$, the differential of f, $\mathrm{d}f$, is a tensor field of type $\binom{0}{1}$. Therefore, from (2.28)

$$(\psi^* \mathrm{d}f)_p(v_p) = \mathrm{d}f_{\psi(p)}(\psi_{*p} v_p),$$

for $v_p \in T_p M$. But from the definitions of $\mathrm{d}f$ and of the Jacobian [see (1.41) and (1.23)], we have

$$\mathrm{d}f_{\psi(p)}(\psi_{*p} v_p) = \psi_{*p} v_p[f] = v_p[\psi^* f] = \mathrm{d}(\psi^* f)_p(v_p).$$

Thus

$$\psi^* \mathrm{d}f = \mathrm{d}(\psi^* f). \tag{2.29}$$

If t and s are tensor fields of type $\binom{0}{k}$ on N and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&\big(\psi^*(at + bs)\big)_p(u_p, \ldots, w_p) \\
&\quad = (at + bs)_{\psi(p)}(\psi_{*p} u_p, \ldots, \psi_{*p} w_p) \\
&\quad = (at_{\psi(p)} + bs_{\psi(p)})(\psi_{*p} u_p, \ldots, \psi_{*p} w_p) \\
&\quad = at_{\psi(p)}(\psi_{*p} u_p, \ldots, \psi_{*p} w_p) + bs_{\psi(p)}(\psi_{*p} u_p, \ldots, \psi_{*p} w_p) \\
&\quad = a(\psi^* t)_p(u_p, \ldots, w_p) + b(\psi^* s)_p(u_p, \ldots, w_p) \\
&\quad = (a\psi^* t + b\psi^* s)_p(u_p, \ldots, w_p),
\end{aligned}$$

for $u_p, \ldots, w_p \in T_p M$, that is,

$$\psi^*(at + bs) = a\psi^* t + b\psi^* s. \tag{2.30}$$

Similarly, if $f : N \to \mathbb{R}$

$$
\begin{aligned}
\big(\psi^*(ft)\big)_p(u_p, \ldots, w_p) &= (ft)_{\psi(p)}(\psi_{*p}u_p, \ldots, \psi_{*p}w_p) \\
&= f\big(\psi(p)\big)t_{\psi(p)}(\psi_{*p}u_p, \ldots, \psi_{*p}w_p) \\
&= (\psi^* f)(p)(\psi^* t)_p(u_p, \ldots, w_p) \\
&= \big((\psi^* f)(\psi^* t)\big)_p(u_p, \ldots, w_p);
\end{aligned}
$$

hence

$$\psi^*(ft) = (\psi^* f)(\psi^* t). \tag{2.31}$$

Finally, if t and s are tensor fields of type $\binom{0}{k}$ and $\binom{0}{l}$ on N, respectively, we have

$$
\begin{aligned}
\big(\psi^*(t \otimes s)\big)_p &= (t \otimes s)_{\psi(p)}(\psi_{*p}u_p, \ldots, \psi_{*p}w_p) \\
&= t_{\psi(p)}(\psi_{*p}u_p, \ldots)\, s_{\psi(p)}(\ldots, \psi_{*p}w_p) \\
&= (\psi^* t)_p(u_p, \ldots)(\psi^* s)_p(\ldots, w_p) \\
&= \big((\psi^* t) \otimes (\psi^* s)\big)_p(u_p, \ldots, w_p),
\end{aligned}
$$

for $u_p, \ldots, w_p \in T_pM$, and therefore

$$\psi^*(t \otimes s) = (\psi^* t) \otimes (\psi^* s). \tag{2.32}$$

Exercise 2.26 Let $\psi_1 : M_1 \to M_2$ and $\psi_2 : M_2 \to M_3$ be differentiable maps. Show that $(\psi_2 \circ \psi_1)^* t = ({\psi_1}^* \circ {\psi_2}^*)t$, for $t \in T_k^0(M_3)$.

Thus, if t is a tensor field of type $\binom{0}{k}$ on N, given locally by $t = t_{i\ldots j}\, \mathrm{d}y^i \otimes \cdots \otimes \mathrm{d}y^j$, the pullback of t under ψ is given by

$$
\begin{aligned}
\psi^* t &= \psi^*\big(t_{i\ldots j}\, \mathrm{d}y^i \otimes \cdots \otimes \mathrm{d}y^j\big) \\
&= (\psi^* t_{i\ldots j})\big(\psi^*\, \mathrm{d}y^i\big) \otimes \cdots \otimes \big(\psi^* \mathrm{d}y^j\big) \\
&= (\psi^* t_{i\ldots j})\, \mathrm{d}\big(\psi^* y^i\big) \otimes \cdots \otimes \mathrm{d}\big(\psi^* y^j\big).
\end{aligned}
$$

But $\mathrm{d}(\psi^* y^i) = (\partial(\psi^* y^i)/\partial x^l)\, \mathrm{d}x^l$, where $(x^1, \ldots, x^n)$ is a coordinate system on M; hence

$$\psi^* t = (\psi^* t_{i\ldots j}) \frac{\partial(\psi^* y^i)}{\partial x^l} \cdots \frac{\partial(\psi^* y^j)}{\partial x^m}\, \mathrm{d}x^l \otimes \cdots \otimes \mathrm{d}x^m. \tag{2.33}$$

This expression shows that $\psi^* t$ is differentiable if t is.

Example 2.27 In the standard treatment of ODEs one encounters expressions of the form $P\, \mathrm{d}x + Q\, \mathrm{d}y = 0$. The left-hand side of this equation can be regarded as a

1-form on some manifold, M, with local coordinates (x, y) (assuming that the functions P and Q are differentiable) and the equality to zero is to be understood considering curves, $C : I \to M$, such that $C^*(P\,\mathrm{d}x + Q\,\mathrm{d}y) = 0$. (That is, $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is not equal to zero as a covector field on M; it is only its pullback under C that vanishes.) Then, for one of these curves, using the properties (2.29), (2.30), and (2.31), we have

$$(P \circ C)\,\mathrm{d}(x \circ C) + (Q \circ C)\,\mathrm{d}(y \circ C) = 0. \tag{2.34}$$

Since $x \circ C$ and $y \circ C$ (as well as $P \circ C$ and $Q \circ C$) are functions from I to $\mathbb{R}$, we can write [see (1.52)]

$$\mathrm{d}(x \circ C) = \frac{\mathrm{d}(x \circ C)}{\mathrm{d}t}\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}(y \circ C) = \frac{\mathrm{d}(y \circ C)}{\mathrm{d}t}\,\mathrm{d}t,$$

where t is the usual coordinate of $\mathbb{R}$. Hence, from (2.34), we get the equivalent expression

$$(P \circ C)\frac{\mathrm{d}(x \circ C)}{\mathrm{d}t} + (Q \circ C)\frac{\mathrm{d}(y \circ C)}{\mathrm{d}t} = 0.$$

This equation alone does not determine the two functions $x \circ C$ and $y \circ C$. If, for instance, $\mathrm{d}(x \circ C)/\mathrm{d}t \neq 0$ in I (which holds if Q does not vanish), using the chain rule (regarding $x \circ C$ as the independent variable instead of t), one finds that

$$\frac{\mathrm{d}(y \circ C)}{\mathrm{d}(x \circ C)} = -\frac{P \circ C}{Q \circ C}.$$

In this manner, writing x in place of $x \circ C$ and similarly for the other functions, one obtains the first-order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{P(x, y)}{Q(x, y)}, \tag{2.35}$$

where it is assumed that y is a function of x. According to the existence and uniqueness theorem for the solutions of the differential equations, through each point of M there passes one of these curves. In this way, equation (2.35) corresponds to the expression $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$.

Now we want to find one-parameter groups of diffeomorphisms, φ_s, on M such that, when applied to a solution curve of the differential equation expressed in the usual form, $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$, they yield another solution curve. More precisely, this corresponds to finding the one-parameter groups of diffeomorphisms such that if $C^*\alpha = 0$, where $\alpha \equiv P\,\mathrm{d}x + Q\,\mathrm{d}y$, then $(\varphi_s \circ C)^*\alpha = 0$, for all $s \in \mathbb{R}$. The previous equality amounts to $C^*({\varphi_s}^*\alpha) = 0$ (see Exercise 2.26), which is equivalent to the existence of a function $\chi_s \in C^\infty(M)$ (which may depend on s) such that ${\varphi_s}^*\alpha = \chi_s\alpha$. A one-parameter group of diffeomorphisms, φ_s, such that ${\varphi_s}^*\alpha = \chi_s\alpha$ is a *symmetry of the equation* $\alpha = 0$. (As shown in Sect. 4.3, knowing a symmetry of the equation $\alpha = 0$, or its infinitesimal generator, allows us to find the solution of the differential equation.)

Let φ be a one-parameter group of transformations or a flow on M with infinitesimal generator $\mathbf{X}$, and let t be a tensor field of type $\binom{0}{k}$ on M. If the limit $\lim_{h\to 0}\frac{\varphi_h^* t - t}{h}$ exists, it is called the Lie derivative of t with respect to $\mathbf{X}$ and is denoted by $£_{\mathbf{X}} t$. The properties of the Lie derivative of tensor fields of type $\binom{0}{k}$ follow from the properties of the pullback of tensor fields. That is, given two tensor fields of type $\binom{0}{k}$ on M, s, and t, it follows from (2.32) that

$$\begin{aligned}
£_{\mathbf{X}}(t\otimes s) &= \lim_{h\to 0}\frac{\varphi_h^*(t\otimes s) - t\otimes s}{h}\\
&= \lim_{h\to 0}\frac{(\varphi_h^* t)\otimes(\varphi_h^* s) - t\otimes s}{h}\\
&= \lim_{h\to 0}\left[(\varphi_h^* t)\otimes\frac{\varphi_h^* s - s}{h} + \frac{\varphi_h^* t - t}{h}\otimes s\right]\\
&= t\otimes(£_{\mathbf{X}} s) + (£_{\mathbf{X}} t)\otimes s.
\end{aligned}\tag{2.36}$$

If t and s are of type $\binom{0}{k}$ and $a, b\in\mathbb{R}$, by (2.30) we have

$$\begin{aligned}
£_{\mathbf{X}}(at+bs) &= \lim_{h\to 0}\frac{\varphi_h^*(at+bs) - (at+bs)}{h}\\
&= \lim_{h\to 0}\frac{a\varphi_h^* t + b\varphi_h^* s - at - bs}{h}\\
&= a£_{\mathbf{X}} t + b£_{\mathbf{X}} s.
\end{aligned}\tag{2.37}$$

For $f\in C^\infty(M)$, using (2.31) we have

$$\begin{aligned}
£_{\mathbf{X}}(ft) &= \lim_{h\to 0}\frac{\varphi_h^*(ft) - ft}{h}\\
&= \lim_{h\to 0}\frac{(\varphi_h^* f)(\varphi_h^* t) - ft}{h}\\
&= \lim_{h\to 0}\left[\varphi_h^* f\,\frac{\varphi_h^* t - t}{h} + \frac{\varphi_h^* f - f}{h}t\right]\\
&= f(£_{\mathbf{X}} t) + (£_{\mathbf{X}} f)t.
\end{aligned}\tag{2.38}$$

Furthermore, by (2.29), the Lie derivative of $\mathrm{d}f$ with respect to $\mathbf{X}$ is

$$\begin{aligned}
£_{\mathbf{X}}\,\mathrm{d}f &= \lim_{h\to 0}\frac{\varphi_h^*\,\mathrm{d}f - \mathrm{d}f}{h}\\
&= \lim_{h\to 0}\frac{\mathrm{d}(\varphi_h^* f) - \mathrm{d}f}{h}\\
&= \mathrm{d}(£_{\mathbf{X}} f).
\end{aligned}\tag{2.39}$$

Using these properties of the Lie derivative we can find the components of the Lie derivative of any tensor field of type $\binom{0}{k}$. If t is given locally by $t = t_{i\dots j}\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j$, we have

$$\begin{aligned}
\pounds_{\mathbf{X}} t &= \pounds_{\mathbf{X}}\big(t_{i\dots j}\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j\big) \\
&= (\pounds_{\mathbf{X}} t_{i\dots j})\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j \\
&\quad + t_{i\dots j}\big(\pounds_{\mathbf{X}}\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j + \cdots + \mathrm{d}x^i \otimes \cdots \otimes \pounds_{\mathbf{X}}\,\mathrm{d}x^j\big) \\
&= (\mathbf{X} t_{i\dots j})\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j \\
&\quad + t_{i\dots j}\big[\mathrm{d}\big(\pounds_{\mathbf{X}} x^i\big) \otimes \cdots \otimes \mathrm{d}x^j + \cdots + \mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}\big(\pounds_{\mathbf{X}} x^j\big)\big].
\end{aligned}$$

Expressing $\mathbf{X}$ in the form $\mathbf{X} = X^l(\partial/\partial x^l)$ and using (2.23) we find that

$$\pounds_{\mathbf{X}} x^i = \mathbf{X} x^i = X^l\left(\frac{\partial}{\partial x^l}\right) x^i = X^i;$$

hence, $\mathrm{d}(\pounds_{\mathbf{X}} x^i) = \mathrm{d}X^i = (\partial X^i/\partial x^l)\,\mathrm{d}x^l$, and

$$\begin{aligned}
\pounds_{\mathbf{X}} t &= (\mathbf{X} t_{i\dots j})\,\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j \\
&\quad + t_{i\dots j}\left(\frac{\partial X^i}{\partial x^l}\,\mathrm{d}x^l \otimes \cdots \otimes \mathrm{d}x^j + \cdots + \mathrm{d}x^i \otimes \cdots \otimes \frac{\partial X^j}{\partial x^l}\,\mathrm{d}x^l\right) \\
&= \left(X^l \frac{\partial t_{i\dots j}}{\partial x^l} + t_{l\dots j}\frac{\partial X^l}{\partial x^i} + \cdots + t_{i\dots l}\frac{\partial X^l}{\partial x^j}\right)\mathrm{d}x^i \otimes \cdots \otimes \mathrm{d}x^j.
\end{aligned} \tag{2.40}$$

Example 2.28 According to the results of Example 2.27, if $\mathbf{X}$ is the infinitesimal generator of a one-parameter group of diffeomorphisms that maps solutions of the differential equation $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$ into solutions of the same equation, then $\pounds_{\mathbf{X}}(P\,\mathrm{d}x + Q\,\mathrm{d}y) = \nu(P\,\mathrm{d}x + Q\,\mathrm{d}y)$, where ν is some real-valued function. Writing

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y},$$

by means of the relation (2.40) we find that

$$\begin{aligned}
\xi\frac{\partial P}{\partial x} + \eta\frac{\partial P}{\partial y} + P\frac{\partial \xi}{\partial x} + Q\frac{\partial \eta}{\partial x} &= \nu P, \\
\xi\frac{\partial Q}{\partial x} + \eta\frac{\partial Q}{\partial y} + Q\frac{\partial \eta}{\partial y} + P\frac{\partial \xi}{\partial y} &= \nu Q,
\end{aligned}$$

which can be conveniently expressed in the form (eliminating the unknown function ν)

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) f - \frac{\partial \xi}{\partial y} f^2, \tag{2.41}$$

where $f \equiv -P/Q$ [cf. (2.35)]. This equation, for the two functions ξ and η, has infinitely many solutions and turns out to be more convenient for finding the symmetries of the differential equation $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$ than the condition ${\varphi_s}^*\alpha = \chi_s\alpha$. This is so since, whereas φ must satisfy the conditions defining a one-parameter group of diffeomorphisms, the functions ξ and η only have to be differentiable. A practical way of finding some solution of (2.41) consists in proposing expressions for ξ and η containing some constants to be determined (see, e.g., Hydon 2000).

If t is a tensor field of type $\binom{0}{k}$ on M and $\mathbf{X}$ is a vector field on M, the *contraction* of t with $\mathbf{X}$, denoted by $\mathbf{X} \lrcorner t$, is the tensor field of type $\binom{0}{k-1}$ on M given by

$$(\mathbf{X} \lrcorner t)_p(v_p, \dots, w_p) \equiv k\, t_p(\mathbf{X}_p, v_p, \dots, w_p), \tag{2.42}$$

for $v_p, \dots, w_p \in T_pM$ (the constant factor k appearing on the right-hand side is introduced for later convenience). If t is a tensor field of type $\binom{0}{0}$ on M, that is, t is a function from M into $\mathbb{R}$, we define $\mathbf{X} \lrcorner t \equiv 0$. Note that if α is a 1-form on M, $\mathbf{X} \lrcorner \alpha$ is the function $\alpha(\mathbf{X})$ [see (1.43)].

The contraction commutes with the pullback under diffeomorphisms; for if $\psi : M \to N$ is a diffeomorphism, t a tensor field of type $\binom{0}{k}$ on N, and $\mathbf{X}$ a vector field on N, then, since $\psi^*\mathbf{X}$ and $\mathbf{X}$ are ψ-related, we have

$$\begin{aligned}
\big[(\psi^*\mathbf{X}) \lrcorner (\psi^* t)\big]_p(v_p, \dots, w_p) &= k(\psi^* t)_p\big((\psi^*\mathbf{X})_p, v_p, \dots, w_p\big) \\
&= k\, t_{\psi(p)}\big(\psi_{*p}(\psi^*\mathbf{X})_p, \psi_{*p}v_p, \dots, \psi_{*p}w_p\big) \\
&= k\, t_{\psi(p)}(\mathbf{X}_{\psi(p)}, \psi_{*p}v_p, \dots, \psi_{*p}w_p) \\
&= (\mathbf{X} \lrcorner t)_{\psi(p)}(\psi_{*p}v_p, \dots, \psi_{*p}w_p) \\
&= \big[\psi^*(\mathbf{X} \lrcorner t)\big]_p(v_p, \dots, w_p),
\end{aligned}$$

for $v_p, \dots, w_p \in T_pM$, that is,

$$\psi^*(\mathbf{X} \lrcorner t) = (\psi^*\mathbf{X}) \lrcorner (\psi^* t). \tag{2.43}$$

Hence, for $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ and $t \in T^0_k(M)$, we have

$$\pounds_{\mathbf{X}}(\mathbf{Y} \lrcorner t) = (\pounds_{\mathbf{X}}\mathbf{Y}) \lrcorner t + \mathbf{Y} \lrcorner (\pounds_{\mathbf{X}} t). \tag{2.44}$$

Thus, if $t \in T^0_k(M)$ and $\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_k \in \mathfrak{X}(M)$, repeatedly applying this relation, we obtain

$$\begin{aligned}
&\mathbf{X}\big(t(\mathbf{Y}_1, \dots, \mathbf{Y}_k)\big) \\
&\quad = \pounds_{\mathbf{X}}\big(t(\mathbf{Y}_1, \dots, \mathbf{Y}_k)\big) \\
&\quad = \frac{1}{k!}\pounds_{\mathbf{X}}(\mathbf{Y}_k \lrcorner \mathbf{Y}_{k-1} \lrcorner \cdots \lrcorner \mathbf{Y}_1 \lrcorner t)
\end{aligned}$$

$$= \frac{1}{k!}\big[(\pounds_{\mathbf{X}}\mathbf{Y}_k)\lrcorner\mathbf{Y}_{k-1}\lrcorner\cdots\lrcorner\mathbf{Y}_1\lrcorner t + \mathbf{Y}_k\lrcorner(\pounds_{\mathbf{X}}\mathbf{Y}_{k-1})\lrcorner\mathbf{Y}_{k-2}\lrcorner\cdots\lrcorner\mathbf{Y}_1\lrcorner t$$
$$+ \cdots + \mathbf{Y}_k\lrcorner\mathbf{Y}_{k-1}\lrcorner\cdots\lrcorner(\pounds_{\mathbf{X}}\mathbf{Y}_1)\lrcorner t + \mathbf{Y}_k\lrcorner\mathbf{Y}_{k-1}\lrcorner\cdots\lrcorner\mathbf{Y}_1\lrcorner(\pounds_{\mathbf{X}}t)\big]$$
$$= t(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \pounds_{\mathbf{X}}\mathbf{Y}_k) + t(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \pounds_{\mathbf{X}}\mathbf{Y}_{k-1}, \mathbf{Y}_k) + \cdots$$
$$+ t(\pounds_{\mathbf{X}}\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_k) + (\pounds_{\mathbf{X}}t)(\mathbf{Y}_1, \ldots, \mathbf{Y}_k)$$
$$= (\pounds_{\mathbf{X}}t)(\mathbf{Y}_1, \ldots, \mathbf{Y}_k) + \sum_{i=1}^{k} t(\mathbf{Y}_1, \ldots, \pounds_{\mathbf{X}}\mathbf{Y}_i, \ldots, \mathbf{Y}_k),$$

that is,

$$(\pounds_{\mathbf{X}}t)(\mathbf{Y}_1, \ldots, \mathbf{Y}_k) = \mathbf{X}\big(t(\mathbf{Y}_1, \ldots, \mathbf{Y}_k)\big) - \sum_{i=1}^{k} t\big(\mathbf{Y}_1, \ldots, [\mathbf{X}, \mathbf{Y}_i], \ldots, \mathbf{Y}_k\big). \quad (2.45)$$

Exercise 2.29 Show that all the properties of the Lie derivative of tensor fields of type $\binom{0}{k}$ follow from (2.45).

Exercise 2.30 Show that if $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ and $t \in T_k^0(M)$, then $\pounds_{\mathbf{X}}(\pounds_{\mathbf{Y}}t) - \pounds_{\mathbf{Y}}(\pounds_{\mathbf{X}}t) = \pounds_{[\mathbf{X},\mathbf{Y}]}t$.

Exercise 2.31 Show that if $\mathbf{X} \in \mathfrak{X}(M)$ and $t \in T_k^0(M)$, then $\pounds_{\mathbf{X}}(\mathbf{X}\lrcorner t) = \mathbf{X}\lrcorner(\pounds_{\mathbf{X}}t)$.

Exercise 2.32 Let t be a differentiable tensor field of type $\binom{k}{l}$ on M. Assuming that the first k arguments of t are covectors and defining $\pounds_{\mathbf{X}}t$ by

$$\begin{aligned}
&(\pounds_{\mathbf{X}}t)(\alpha_1, \ldots, \alpha_k, \mathbf{Y}_1, \ldots, \mathbf{Y}_l)\\
&\quad \equiv \mathbf{X}\big(t(\alpha_1, \ldots, \alpha_k, \mathbf{Y}_1, \ldots, \mathbf{Y}_l)\big)\\
&\qquad - \sum_{i=1}^{k} t(\alpha_1, \ldots, \pounds_{\mathbf{X}}\alpha_i, \ldots, \alpha_k, \mathbf{Y}_1, \ldots, \mathbf{Y}_l)\\
&\qquad - \sum_{i=1}^{l} t(\alpha_1, \ldots, \alpha_k, \mathbf{Y}_1, \ldots, \pounds_{\mathbf{X}}\mathbf{Y}_i, \ldots, \mathbf{Y}_l),
\end{aligned}$$

for $\alpha_1, \ldots, \alpha_k \in \Lambda^1(M)$, $\mathbf{Y}_1, \ldots, \mathbf{Y}_l \in \mathfrak{X}(M)$, show that $\pounds_{\mathbf{X}}t$ is a differentiable tensor field of type $\binom{k}{l}$ and that $\pounds_{\mathbf{X}}(t \otimes s) = (\pounds_{\mathbf{X}}t) \otimes s + t \otimes (\pounds_{\mathbf{X}}s)$ for any pair of mixed tensor fields.

http://www.springer.com/978-0-8176-8270-5

Differentiable Manifolds
A Theoretical Physics Approach
Torres del Castillo, G.F.
2012, VIII, 275 p. 20 illus., Hardcover
ISBN: 978-0-8176-8270-5
A product of Birkhäuser Basel