Linear Control Systems
Lecture # 13
Eigenvalue-Eigenvector Placement

Characterization of Closed-loop Eigenvectors

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

with the state feedback control

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{v}$$

The closed-loop system is

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v}$$

Let $\boldsymbol{\lambda}_i$ be an eigenvalue of $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})$ and $\boldsymbol{v}_i$ be the corresponding eigenvector

$$(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{v}_i = \boldsymbol{\lambda}_i \boldsymbol{v}_i$$

$$(\boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v}_i - \boldsymbol{B}\boldsymbol{F}\boldsymbol{v}_i = \mathbf{0}$$

Let $\boldsymbol{q}_i = \boldsymbol{F}\boldsymbol{v}_i$

$$(\boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v}_i - \boldsymbol{B}\boldsymbol{q}_i = \mathbf{0}$$

$$\begin{bmatrix} \boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A}, & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_i \\ -\boldsymbol{q}_i \end{bmatrix} = \mathbf{0}$$

These equations hold for $\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{n}$. Hence

$$\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_n \end{bmatrix} = \boldsymbol{F} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots \boldsymbol{v}_n \end{bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{F}\boldsymbol{V}$$

Can we reverse these expressions to assign eigenvectors?

Assume $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Then

$$\textbf{rank} \left[\begin{array}{cc} \boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A}, & \boldsymbol{B} \end{array} \right] = \boldsymbol{n}, \;\; \forall\, \boldsymbol{\lambda}_i$$

Let the $(\boldsymbol{n} + \boldsymbol{m}) \times \boldsymbol{m}$ matrix $\boldsymbol{Y}_i$ span the null space of $\left[\begin{array}{cc} \boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A}, & \boldsymbol{B} \end{array} \right]$. Partition $\boldsymbol{Y}_i$ as $\boldsymbol{Y}_i = \left[\begin{array}{c} \boldsymbol{M}_i \\ -\boldsymbol{D}_i \end{array} \right]$, where $\boldsymbol{M}_i$ is $\boldsymbol{n} \times \boldsymbol{m}$ and $\boldsymbol{D}_i$ is $\boldsymbol{m} \times \boldsymbol{m}$

$$\left[\begin{array}{c} \boldsymbol{v}_i \\ -\boldsymbol{q}_i \end{array} \right] \in \mathcal{N}\left(\left[\begin{array}{cc} \boldsymbol{\lambda}_i \boldsymbol{I} - \boldsymbol{A}, & \boldsymbol{B} \end{array} \right] \right) \Rightarrow \textbf{There is } \boldsymbol{\xi}_i \textbf{ such that}$$

$$\left[\begin{array}{c} \boldsymbol{v}_i \\ -\boldsymbol{q}_i \end{array} \right] = \boldsymbol{Y}_i \boldsymbol{\xi}_i = \left[\begin{array}{c} \boldsymbol{M}_i \\ -\boldsymbol{D}_i \end{array} \right] \boldsymbol{\xi}_i$$

$$\Rightarrow \quad v_i = M_i \xi_i, \quad q_i = D_i \xi_i$$

We must choose $v_i \in \mathcal{R}(M_i)$. This choice determines ξ_i. Then we calculate q_i from $q_i = D_i \xi_i$, construct the matrices

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}, \; V = \begin{bmatrix} v_1 & v_2 & \cdots v_n \end{bmatrix}$$

and solve the equation

$$Q = FV$$

This equation has a unique solution if and only if V is nonsingular, which is the case when the eigenvectors are linearly independent

Algorithm for Eigenvalue-Eigenvector Placement

Let (A, B) be controllable, where A is $n \times n$ and B is $n \times m$

1. Choose a self conjugate set of desired (distinct) closed-loop eigenvalues $\lambda_1, \ldots, \lambda_n$
2. For every i, form the matrix $\left[\begin{array}{cc} \lambda_i I - A, & B \end{array}\right]$
3. Find an $(n+m) \times m$ matrix Y_i whose columns form a basis of the null space of $\left[\begin{array}{cc} \lambda_i I - A, & B \end{array}\right]$
4. Partition Y_i as $Y_i = \left[\begin{array}{c} M_i \\ -D_i \end{array}\right]$, where M_i is $n \times m$ and D_i is $m \times m$

5. Choose a set of desired closed-loop eigenvectors $v_1, \ldots, v_n$ such that
   - v_i belongs to the range space of M_i
   - $v_i = \bar{v}_j$ whenever $\lambda_i = \bar{\lambda}_j$
   - $v_1, \ldots, v_n$ are linearly independent
6. For every i, find a vector ξ_i such that $v_i = M_i \xi_i$
7. Determine $q_i = D_i \xi_i$
8. Form the matrices Q and V as

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & v_2 & \cdots v_n \end{bmatrix}$$

9. Determine F from $F = QV^{-1}$

Example:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 2 & -2 \\ -1 & 0 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$$

Desired eigenvalues: $\lambda_1 = -1,\ \lambda_{2,3} = -1 \pm j$

$$\begin{bmatrix} \lambda_1 I - A, & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 2 & -3 & 2 & 0 & 2 \\ 1 & 0 & -4 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_2 I - A, & B \end{bmatrix} = \begin{bmatrix} j & 0 & -1 & 1 & 0 \\ 2 & -3+j & 2 & 0 & 2 \\ 1 & 0 & -4+j & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{Y_1} = \begin{bmatrix} \mathbf{-0.8364} & \mathbf{-0.0735} \\ \mathbf{-0.4424} & \mathbf{0.5734} \\ \mathbf{-0.1106} & \mathbf{0.1434} \\ \mathbf{-0.1106} & \mathbf{0.1434} \\ \mathbf{0.2833} & \mathbf{0.7903} \end{bmatrix}$$

$$\boldsymbol{M_1} = \begin{bmatrix} \mathbf{-0.8364} & \mathbf{-0.0735} \\ \mathbf{-0.4424} & \mathbf{0.5734} \\ \mathbf{-0.1106} & \mathbf{0.1434} \end{bmatrix}$$

$$\boldsymbol{D_1} = \begin{bmatrix} \mathbf{0.1106} & \mathbf{-0.1434} \\ \mathbf{-0.2833} & \mathbf{-0.7903} \end{bmatrix}$$

$$
\boldsymbol{Y_2} = \begin{bmatrix} -0.1947 + 0.6655j & -0.0645 - 0.0835j \\ -0.2368 + 0.2864j & 0.5539 + 0.1218j \\ -0.1519 + 0.0336j & 0.1624 + 0.0006j \\ 0.5137 + 0.2283j & 0.0789 + 0.0651j \\ 0.1345 - 0.1511j & 0.7939 - 0.0113j \end{bmatrix}
$$

$$
\boldsymbol{M_2} = \begin{bmatrix} -0.1947 + 0.6655j & -0.0645 - 0.0835j \\ -0.2368 + 0.2864j & 0.5539 + 0.1218j \\ -0.1519 + 0.0336j & 0.1624 + 0.0006j \end{bmatrix}
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$$

$$\boldsymbol{\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow v_1 = M_1\xi_1 = \begin{bmatrix} -0.8364 \\ -0.4424 \\ -0.1106 \end{bmatrix}}$$

$$\boldsymbol{\xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow v_2 = M_2\xi_2 = \begin{bmatrix} -0.2592 + 0.5820j \\ 0.3171 + 0.4082j \\ 0.0106 + 0.0342j \end{bmatrix}}$$

$$\boldsymbol{v_3 = \bar{v}_2 = \begin{bmatrix} -0.2592 - 0.5820j \\ 0.3171 - 0.4082j \\ 0.0106 - 0.0342j \end{bmatrix}}$$

$$V = \begin{bmatrix} -0.8364 & -0.2592 + 0.5820j & -0.2592 - 0.5820j \\ -0.4424 & 0.3171 + 0.4082j & 0.3171 - 0.4082j \\ -0.1106 & 0.0106 + 0.0342j & 0.0106 - 0.0342j \end{bmatrix}$$

$$\text{rank}\, V = 3$$

$$q_1 = D_1 \xi_1 = \begin{bmatrix} 0.1106 \\ -0.2833 \end{bmatrix}$$

$$q_2 = D_2 \xi_2 = \begin{bmatrix} -0.5926 - 0.2934j \\ -0.9284 + 0.1624j \end{bmatrix}$$

$$q_3 = \bar{q}_2 = \begin{bmatrix} -0.5926 + 0.2934j \\ -0.9284 - 0.1624j \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.1106 & -0.5926 - 0.2934j & -0.5926 + 0.2934j \\ -0.2833 & -0.9284 + 0.1624j & -0.9284 - 0.1624j \end{bmatrix}$$

$$F = QV^{-1} = \begin{bmatrix} 0.4795 & -1.5263 & 1.4791 \\ 1.5469 & -1.5676 & -2.8653 \end{bmatrix}$$

Calculation of the Null space

Example: Find the null space of

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{2} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{0} & \mathbf{-4} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{2} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{0} & \mathbf{-4} & \mathbf{-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \\ \boldsymbol{x_5} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\begin{aligned} -x_3 + x_4 &= 0 \\ 2x_1 - 3x_2 + 2x_3 + 2x_5 &= 0 \\ x_1 - 4x_3 - x_4 + x_5 &= 0 \end{aligned}$$

$$x_3 = x_4, \quad x_1 = 4x_3 + x_4 - x_5 = 5x_4 - x_5$$

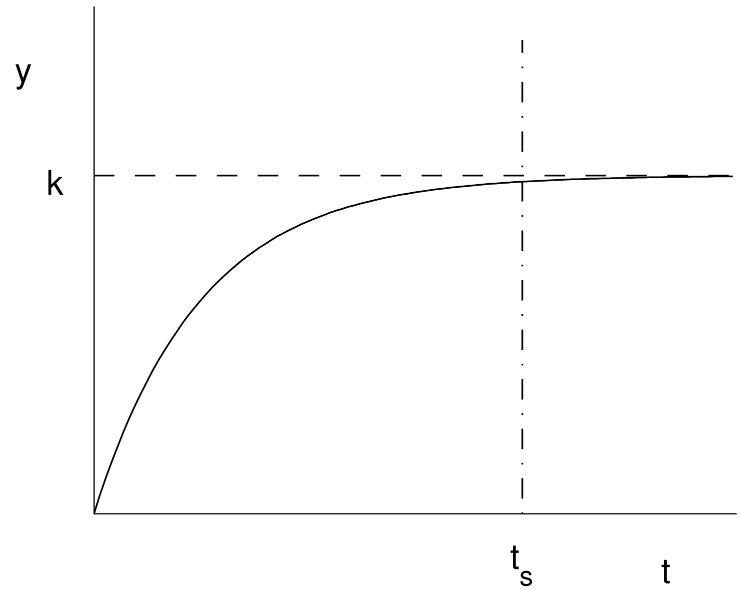
$$x_2 = \tfrac{2}{3}(x_1 + x_3 + x_5) = \tfrac{2}{3}(5x_4 - x_5 + x_4 + x_5) = 4x_4$$

$$x = \begin{bmatrix} 5x_4 - x_5 \\ 4x_4 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

Selection of the Eigenvalues

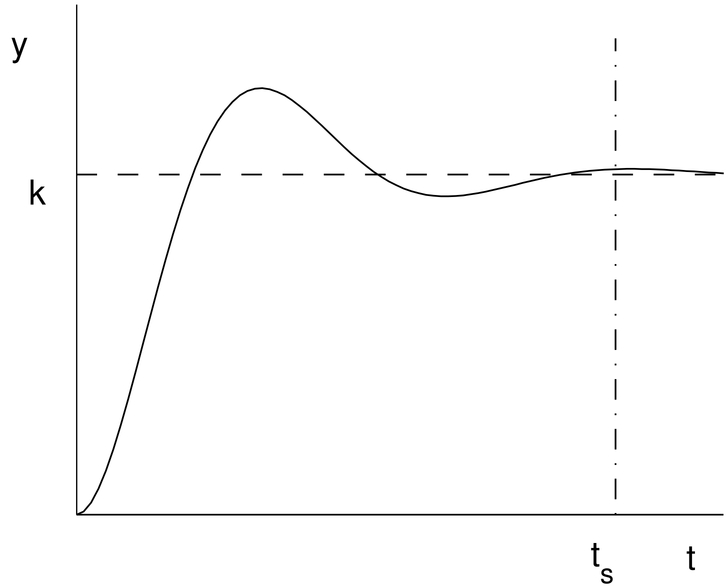
Review of the step response of SISO transfer functions

$$H(s) = \frac{Ka}{s+a}$$



$$2\% \text{ Settling time } t_s = \frac{4}{a}$$

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



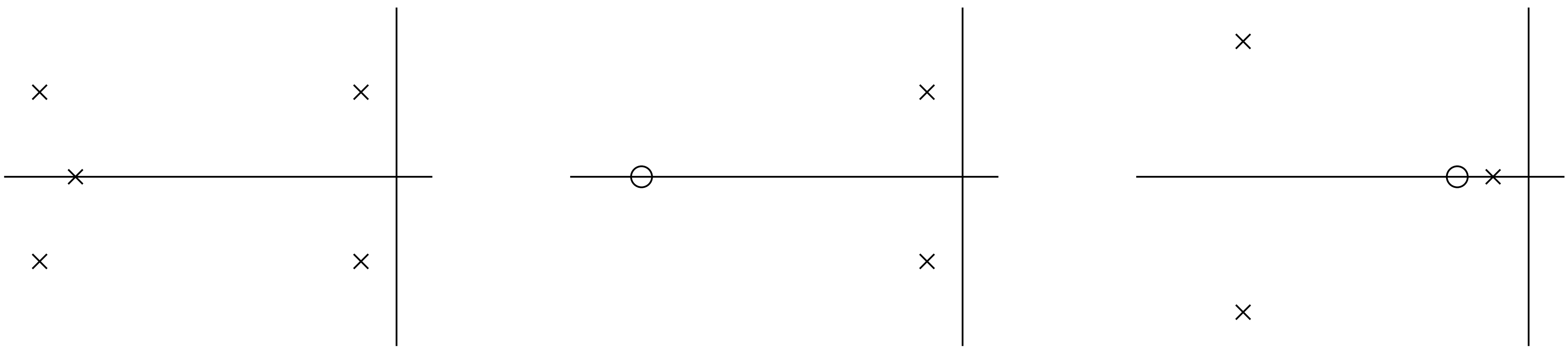
$$\text{Percent Overshoot } = 100 \times e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\zeta = 0.5 \;\rightarrow\; P.O. = 16\%, \quad \zeta = 0.7 \;\rightarrow\; P.O. = 5\%$$

$$2\% \text{ Settling time } t_s = \frac{4}{\zeta\omega_n}$$

The step response relations for the transfer function $k\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ will be good approximations for other transfer functions that have a pair of dominant complex poles, such as

- Other poles are far to the left (by a factor of 10 or higher)
- a zero is far to the left (by a factor of 10 or higher)
- Almost pole-zero cancellation



The transient response of the closed-loop system

$$\dot{x} = (A + BF)x + Bv, \quad y = (C + DF)x + Dv$$

is studied by simulating the step response or the zero input response

General Guidelines:

- The settling time of each mode is $t_s = 4/|R_e[\lambda_i]|$. To achieve faster response, move eigenvalues to the left
- For complex eigenvalues, the response is more oscillatory for smaller ζ
- If the eigenvalues are clustered into slow and fast ones, the slow eigenvalues are dominant

Can we make the response arbitrarily fast by moving the eigenvalues far enough to the left?

In theory, yes, but this will require large feedback gains and hence large control effort since $\boldsymbol{u}(\boldsymbol{t}) = \boldsymbol{F}\boldsymbol{x}(\boldsymbol{t})$

The choice of eigenvalues is a tradeoff between the transient response and the control effort

Examine the open-loop eigenvalues carefully and do not make unnecessary changes in their locations

Example:

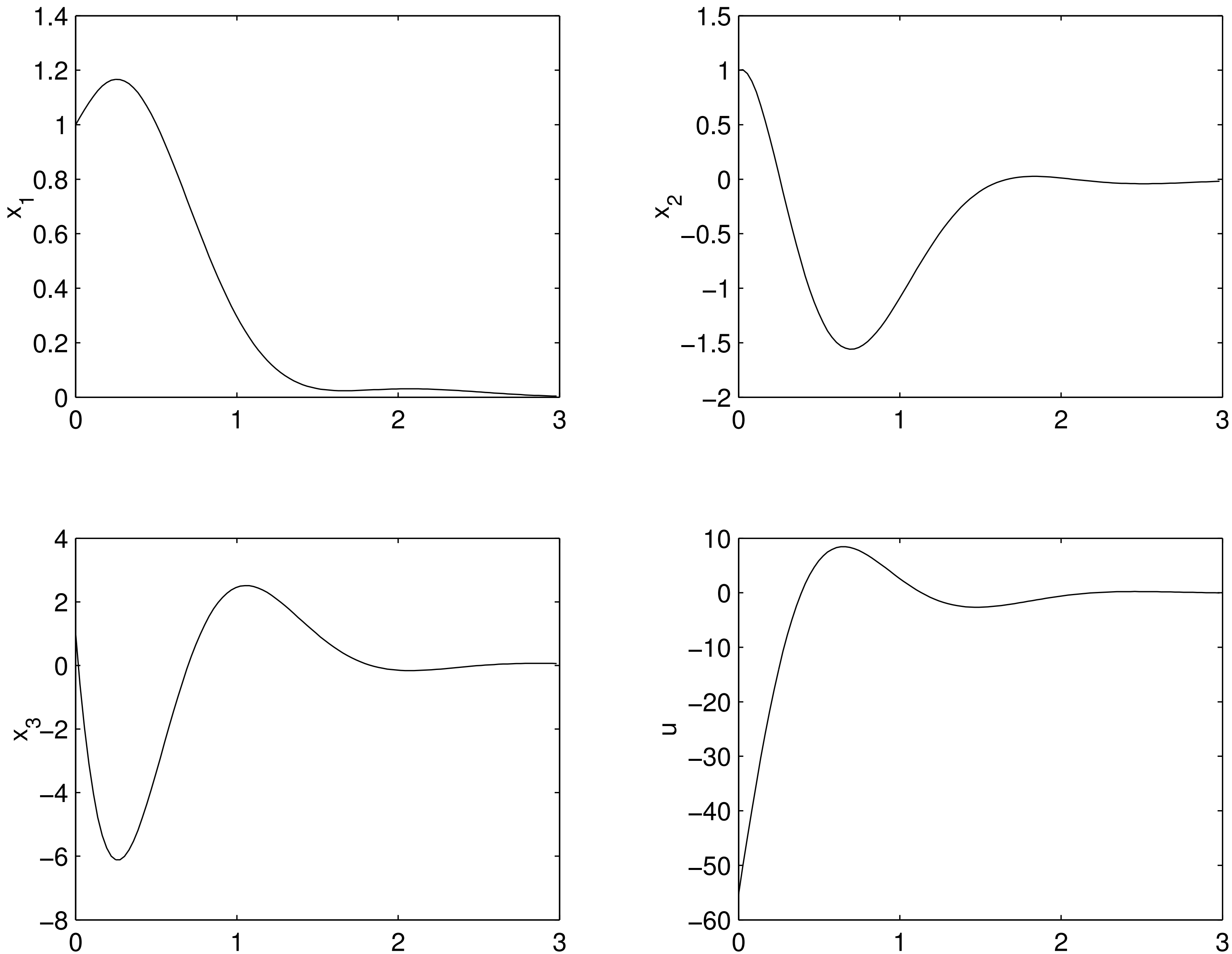
$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{-0.4} & \mathbf{-4.2} & \mathbf{-2.1} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Open-loop eigenvalues are: $\mathbf{-0.1,\ -1 \pm 1.7321}\boldsymbol{j}$
Desired settling time is $\mathbf{4}$ sec.

Choose the closed-loop eigenvalues as $\mathbf{-2,\ -2 \pm 2\sqrt{3}}\boldsymbol{j}$ to achieve a settling time of about $\mathbf{2}$ sec.

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{-31.6} & \mathbf{-19.8} & \mathbf{-3.9} \end{bmatrix}$$

Calculate the zero-input response to $\boldsymbol{x}(\mathbf{0}) = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}^{\boldsymbol{T}}$

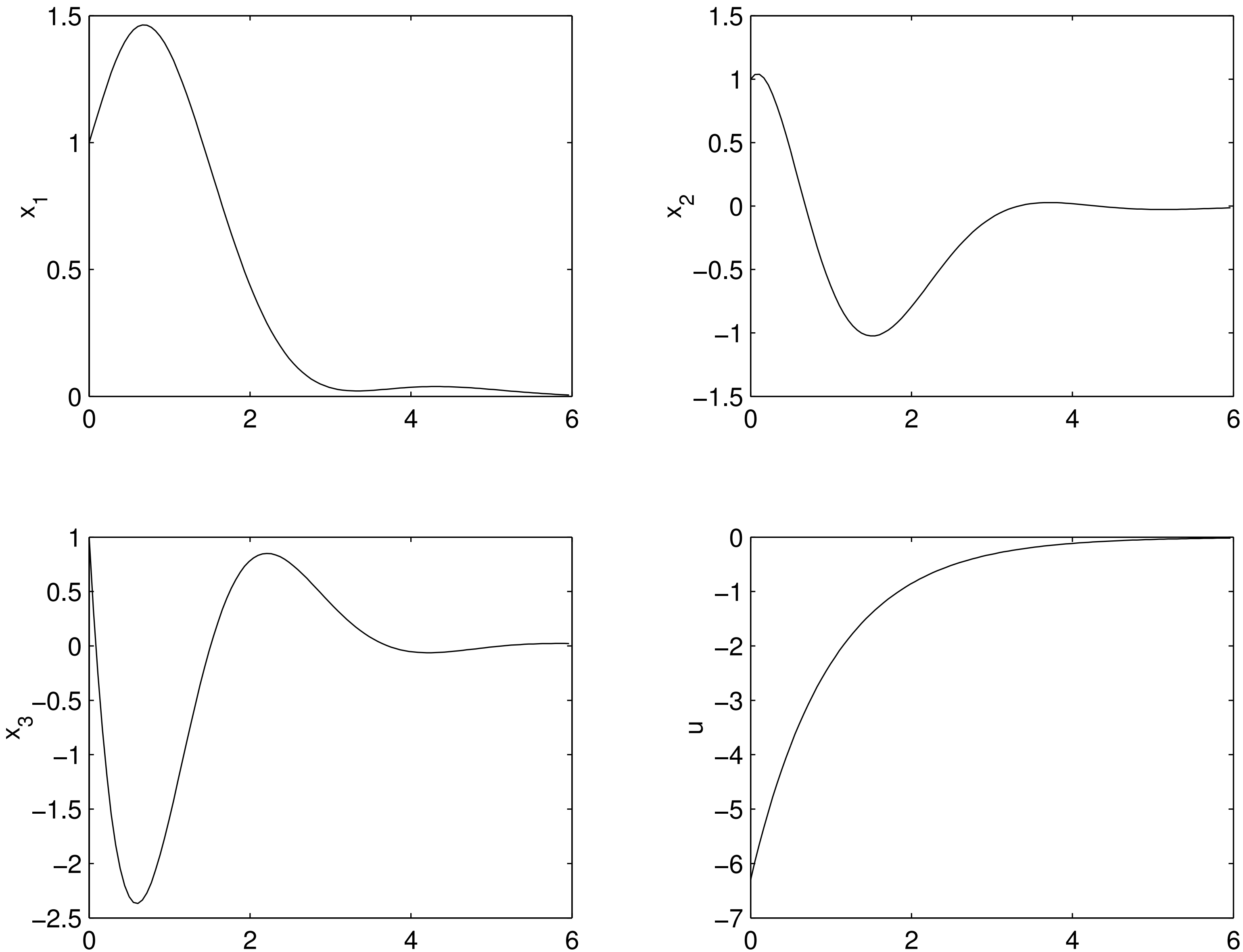

x_1
x_2
x_3
u

Move the real parts of the eigenvalues to -1 to achieve a settling time of about 4 sec. Keep the damping ratio of the complex eigenvalues the same. The new choice is
$\boldsymbol{-1,\ -1 \pm \sqrt{3}j}$

$$\boldsymbol{F = \left[\begin{array}{ccc} -3.6 & -1.8 & -0.9 \end{array}\right]}$$

Compare with the previous

$$\boldsymbol{F = \left[\begin{array}{ccc} -31.6 & -19.8 & -3.9 \end{array}\right]}$$



x1
x2
x3
u
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−4
−5
−6
−7
2
4
6

Matlab Commands:

```
» A = [0 1 0;0 0 1;-0.4 -4.2 -2.1]; B = [0;0;1];
» eig(A)
» p = [-2;-2+2*sqrt(3)*i;-2-2*sqrt(3)*i];
» K = place(A,B,p); F = - K;
» sys = ss(A+B*F,B,F,0); [U,T,X] = initial(sys,[1;1;1]);
» subplot(2,2,1), plot(T,X(:,1)), ylabel('x1')
» subplot(2,2,2), plot(T,X(:,2)), ylabel('x2')
» subplot(2,2,3), plot(T,X(:,3)), ylabel('x3')
» subplot(2,2,4), plot(T,U), ylabel('u')
» p = [-1;-1+sqrt(3)*i;-1-sqrt(3)*i];
» K = place(A,B,p); F = - K;
```