Time-Varying Systems and Computations
Lecture 5

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Realization Theory for Finite Matrices (cont'd)

Factorization of the Hankel operator

We recall the Hankel operator $\mathcal{H}_k$, which we can factor in the product of Observability and Controllability as

$$\mathcal{H}_k = \begin{bmatrix} C_k B_{k-1} & C_k A_{k-1} B_{k-2} & C_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+1} A_k B_{k-1} & C_{k+1} A_k A_{k-1} B_{k-2} & C_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+2} A_{k+1} A_k B_{k-1} & C_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+2} A_{k+1} A_k A_{k-1} A_{k-2} B_{k-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{O}_k \cdot \mathcal{C}_k$$

where

$$\mathcal{O}_k = \begin{bmatrix} C_k \\ C_{k+1} A_k \\ C_{k+2} A_{k+1} A_k \\ \vdots \end{bmatrix}, \ \mathcal{C}_k = \begin{bmatrix} B_{k-1} & A_{k-1} B_{k-2} & A_{k-1} A_{k-2} B_{k-3} & \cdots \end{bmatrix}.$$

We can easily read off the matrices B_{k-1} and C_k from any minimal factorization of the Hankel matrix $\mathcal{H}_k$, either as the first column of the Controllability matrix or as the first row of the Observability matrix, respectively. However, it takes extra effort to extract the matrix A_k or A_{k-1}. To this end, we exploit a special property of the Hankel matrices, which we denote as *shift invariance.*

Shift-Invariance

For extracting the matrix A_k from the observabilty matrix $\mathcal{O}_k$ or for extracting the matrix A_{k-1} from the controllability matrix $\mathcal{C}_k$ we consider shifted versions of the Hankel matrix $\mathcal{H}_k$.

Up-Shifted Version

We take an up-shifted version of the Hankel operator, i.e. we shift all rows of $\mathcal{H}_k$ up by one notch and dropping the first row. Hence the up-shifted version of the Hankel operator reads as

$$\mathcal{H}_k\!\uparrow = \begin{bmatrix} C_{k+1} A_k B_{k-1} & C_{k+1} A_k A_{k-1} B_{k-2} & C_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+2} A_{k+1} A_k B_{k-1} & C_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+2} A_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+3} A_{k+2} A_{k+1} A_k B_{k-1} & C_{k+3} A_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+3} A_{k+2} A_{k+1} A_k A_{k-1} A_{k-2} B_{k-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$=$

$$= \begin{bmatrix} C_{k+1} \\ C_{k+2}A_{k+1} \\ C_{k+3}A_{k+2}A_{k+1} \\ \vdots \end{bmatrix} \cdot A_k \cdot \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \cdots \end{bmatrix}$$

$$= \mathcal{O}_{k+1} \cdot A_k \cdot \mathcal{C}_k.$$

Obviously, shifting the Hankel matrix up and dropping the first row results in a matrix $\mathcal{H}_k\uparrow$ with a row space that lies within the row space of the original matrix $\mathcal{H}_k$, that is, we have the situation

$$\text{row}(\mathcal{H}_k\uparrow) \subset \text{row}(\mathcal{H}_k),$$

which denotes that the row space of $\mathcal{H}_k$ is a shift-invariant subspace.

We can observe the relation

$$\begin{array}{rlcl} & \mathcal{O}_k\uparrow & = & \mathcal{O}_{k+1}A_k \\ \Rightarrow & \mathcal{O}_{k+1}^T\mathcal{O}_k\uparrow & = & \mathcal{O}_{k+1}^T\mathcal{O}_{k+1}A_k \\ \Rightarrow & (\mathcal{O}_{k+1}^T\mathcal{O}_{k+1})^{-1}\mathcal{O}_{k+1}^T\mathcal{O}_k\uparrow & = & A_k \\ \Rightarrow & \mathcal{O}_{k+1}^\dagger\mathcal{O}_k\uparrow & = & A_k \end{array}, \tag{1}$$

where $(\mathcal{O}_{k+1}^T\mathcal{O}_{k+1})^{-1}$ must exist, indicating that the factorization is minimal, or, in other words, that the Observability matrix has full column rank. Using the factored form of the up-shifted Hankel-operator provides us with a method to determine the elements of the state-space realization for index k by identifying

$$\begin{array}{rcl} A_k & = & \mathcal{O}_{k+1}^\dagger\mathcal{O}_k\uparrow \\ B_k & = & \text{first column of } \mathcal{C}_{k+1} \\ C_k & = & \text{first row of } \mathcal{O}_k. \end{array} \tag{2}$$

Left-shifted Version

The left-shifted version of Hankel operator is given as

$$\overleftarrow{\mathcal{H}_k} = \begin{bmatrix} C_kA_{k-1}B_{k-2} & C_kA_{k-1}A_{k-2}B_{k-3} & C_kA_{k-1}A_{k-2}A_{k-3}B_{k-4} & \cdots \\ C_{k+1}A_kA_{k-1}B_{k-2} & C_{k+1}A_kA_{k-1}A_{k-2}B_{k-3} & C_{k+1}A_kA_{k-1}A_{k-2}A_{k-3}B_{k-4} & \cdots \\ C_{k+2}A_{k+1}A_kA_{k-1}B_{k-2} & C_{k+2}A_{k+1}A_kA_{k-1}A_{k-2}B_{k-3} & C_{k+2}A_{k+1}A_kA_{k-1}A_{k-2}A_{k-3}B_{k-4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix} \cdot A_{k-1} \cdot \begin{bmatrix} B_{k-2} & A_{k-2}B_{k-3} & A_{k-2}A_{k-3}B_{k-4} & \cdots \end{bmatrix}$$

$$= \mathcal{O}_k \cdot A_{k-1} \cdot \mathcal{C}_{k-1}.$$

Obviously, shifting the Hankel matrix to the left and dropping the first column results in a matrix $\overleftarrow{\mathcal{H}_k}$ with a column space that lies with the column space of the original matrix $\mathcal{H}_k$, that is we have the situation that

$$\text{col}(\overleftarrow{\mathcal{H}_k}) \subset \text{col}(\mathcal{H}_k),$$

which denotes that the column space of $\mathcal{H}_k$ is a shift-invariant subspace. We can observe the relation

$$\begin{aligned} \overleftarrow{\mathcal{C}}_k &= A_{k-1} \cdot \mathcal{C}_{k-1} && (3) \\ \Rightarrow \quad \overleftarrow{\mathcal{C}}_k \cdot \mathcal{C}_{k-1}^T &= A_{k-1} \cdot \mathcal{C}_{k-1} \cdot \mathcal{C}_{k-1}^T \\ \Rightarrow \quad \overleftarrow{\mathcal{C}}_k \mathcal{C}_{k-1}^T (\mathcal{C}_{k-1}\mathcal{C}_{k-1}^T)^{-1} &= A_{k-1} \\ \Rightarrow \quad \overleftarrow{\mathcal{C}}_k \mathcal{C}_{k-1}^\dagger &= A_{k-1}, && (4) \end{aligned}$$

where $(\mathcal{C}_{k-1}\mathcal{C}_{k-1}^T)^{-1}$ must exist, indicating that the factorization is minimal, or, in other words, that the controllability matrix has full row rank. Using the factored form of the up-shifted Hankel-operator provides us with a method to determine the elements of the state-space realization for index $k-1$ by identifying

$$\begin{aligned} A_{k-1} &= \overleftarrow{\mathcal{C}}_k \mathcal{C}_{k-1}^\dagger && (5) \\ B_{k-1} &= \text{first column of } \mathcal{C}_k \\ C_{k-1} &= \text{first column of } \mathcal{O}_{k-1} \end{aligned}$$

Matrix Factorizations

There exists an infinite number of ways that we can factor a given matrix into the product of two matrices. Some of the more well-known factorizations are the LU-factorization, the Cholesky factorization (for symmetric positive definite matrices), the QR factorization, the polar decomposition, to name just a few. We can add the eigenvalue decomposition and the singular value decomposition, even though there are three matrices involved in the decomposition. Each factorization of the Hankel operator corresponds to one state-space realization for the given matrix. The set of all realizations $\mathbf{\Sigma}$ for a transfer operator T is parameterized by the set of all admissible (non-singular) state-transformations $\boldsymbol{R}$.

Singular Value Decomposition

We use the Singular Value Decomposition (SVD) as a tool to compute a factorization of the Hankel operator. However, the SVD creates a factorization into 3 factors

$$\mathcal{H}_k = \mathcal{O}_k\mathcal{C}_k = USV^T, \quad U^TU = 1, \quad VV^T = 1, \quad S = diag\,\{\sigma_i\}, i = 1, 2, \ldots, n.$$

We now have three choices to recombine two matrices in the SVD to assign them to Observability and Controllability matrices.

1. We talk about an *Input Normal Realization* if the Controllability matrix is orthogonal, i.e. if we have

   $$\mathcal{C}_k\mathcal{C}_k^T = 1$$

   which we achieve by identifying the Controllability matrix and the Observability as

   $$\mathcal{O}_k = US, \quad \mathcal{C}_k = V^T$$

2. Combining the elements of the SVD to form the Observability and Controllability matrices as

   $$\mathcal{O}_k = U, \quad \mathcal{C}_k = SV^T,$$

   produces an *Output Normal Realization*, which satisfies $\mathcal{O}_k^T\mathcal{O}_k = 1$.

3. Finally, assigning the square roots of the singular values to both matrices according to

$$\mathcal{O}_k = U\sqrt{S}, \quad \mathcal{C}_k = \sqrt{S}V^T,$$

creates a *Balanced Realization*, which is characterized by the relation

$$\mathcal{O}_k^T \mathcal{O}_k = \mathcal{C}_k \mathcal{C}_k^T = S$$

QR Decomposition

Using the QR decomposition instead of the SVD for factoring the Hankel operator

$$\mathcal{H}_k = QR, \quad Q^T Q = 1, \quad R \text{ upper triangular}$$

provides us with an Output Normal Realization.

To create an Input Normal Realization using the QR approach we change the standard method to produce

$$\mathcal{H}_k = LQ, \quad Q^T Q = 1, \quad L \text{ lower triangular.}$$

Realization of a lower triangular matrix

As an example for the realization algorithm for causal time-varying systems we consider the Toeplitz-operator given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .800 & .900 & 0 & 0 & 0 & 0 \\ .200 & .600 & .800 & 0 & 0 & 0 \\ .050 & .240 & .500 & .700 & 0 & 0 \\ .013 & .096 & .250 & .400 & .600 & 0 \\ .003 & .038 & .125 & .240 & .300 & .500 \end{bmatrix}.$$

We can determine the Hankel operators and their factorization into observability and controllability by using the singular value decomposition as

$$\mathcal{H}_i = \mathcal{O}_i \cdot \mathcal{C}_i = (U_i \Sigma_i) \cdot (V_i^T), \quad i = 0, 1, 2, \ldots 6$$

This way we get the following Hankel operators and their corresponding factorizations

$$\mathcal{H}_0 = [.]$$

$$\mathcal{H}_1 = \begin{bmatrix} .800 \\ .200 \\ .050 \\ .013 \\ .003 \end{bmatrix} = \begin{bmatrix} 0.968 \\ 0.242 \\ 0.060 \\ 0.016 \\ 0.004 \end{bmatrix} \cdot 0.8262 \cdot 1$$

$$\mathcal{H}_2 = \begin{bmatrix} .600 & .200 \\ .240 & .050 \\ .096 & .013 \\ .038 & .003 \end{bmatrix} = \begin{bmatrix} 0.922 & 0.375 \\ 0.356 & -0.735 \\ 0.139 & -0.501 \\ 0.054 & -0.261 \end{bmatrix} \cdot \begin{bmatrix} 0.685 & 0 \\ 0 & 0.032 \end{bmatrix} \cdot \begin{bmatrix} 0.955 & 0.298 \\ -0.298 & 0.955 \end{bmatrix}$$

$$\mathcal{H}_3 = \begin{bmatrix} .500 & .240 & .050 \\ .250 & .096 & .013 \\ .125 & .038 & .003 \end{bmatrix} =$$

$$= \begin{bmatrix} -0.882 & 0.448 & 0.144 \\ -0.424 & -0.622 & -0.658 \\ -0.205 & -0.642 & 0.739 \end{bmatrix} \cdot \begin{bmatrix} 0.631 & 0 & 0 \\ 0 & 0.029 & 0 \\ 0 & 0 & 0.001 \end{bmatrix} \cdot \begin{bmatrix} -0.907 & -0.412 & -0.080 \\ -0.405 & 0.808 & 0.428 \\ 0.112 & -0.420 & 0.900 \end{bmatrix}$$

$$\mathcal{H}_4 = \begin{bmatrix} .400 & .250 & .096 & .013 \\ .240 & .125 & .038 & .003 \end{bmatrix}$$

$$= \begin{bmatrix} -0.870 & -0.493 \\ -0.493 & 0.870 \end{bmatrix} \cdot \begin{bmatrix} 0.553 & 0 \\ 0 & 0.024 \end{bmatrix} \cdot \begin{bmatrix} -0.843 & -0.505 & -0.185 & -0.023 \\ 0.498 & -0.606 & -0.599 & -0.160 \end{bmatrix}$$

$$\mathcal{H}_5 = \begin{bmatrix} .300 & .240 & .125 & .038 & .003 \end{bmatrix} = 1 \cdot 0.406 \cdot \begin{bmatrix} 0.739 & 0.591 & 0.308 & 0.094 & 0.007 \end{bmatrix}$$

$$\mathcal{H}_6 = [.]$$

The non-zero singular values of the Hankel operators of T can be summarized by

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_6$
σ_1	0.826	0.685	0.631	0.553	0.406	
σ_2		0.032	0.029	0.024		
σ_3			0.001			

.

The dimension of the state-space is increasing from zero to three and the shrinking back to zero again. This way we can identify the sequence of state-space realizations, which we will denote as

$$\mathbf{\Sigma}_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i = 0, 1, \ldots 6.$$

The individual state-space realization matrices are then given by

$$\mathbf{\Sigma}_0 = \left[\begin{array}{c|c} \cdot & 1.000 \\ \hline \cdot & 1.000 \end{array}\right]$$

$$\mathbf{\Sigma}_1 = \left[\begin{array}{r|r} .298 & .955 \\ -.955 & .298 \\ \hline .800 & .900 \end{array}\right]$$

$$\mathbf{\Sigma}_2 = \left[\begin{array}{rr|r} .417 & 0.47 & .908 \\ -.899 & .167 & .405 \\ -.133 & -.985 & .112 \\ \hline .632 & -.012 & .800 \end{array}\right]$$

$$\mathbf{\Sigma}_3 = \left[\begin{array}{rrr|r} .536 & .045 & -.000 & .843 \\ -.810 & .308 & .040 & .498 \\ \hline .557 & -.013 & .000 & .700 \end{array}\right]$$

$$\mathbf{\Sigma}_4 = \left[\begin{array}{rr|r} .671 & .051 & .739 \\ \hline .481 & -.012 & .600 \end{array}\right]$$

$$\mathbf{\Sigma}_5 = \left[\begin{array}{r|r} \cdot & \cdot \\ \hline .406 & .500 \end{array}\right]$$

Looking at the third singular value of $\mathcal{H}_3$ reveals that this matrix is almost singular. This opens the opportunity to approximate the original system with a system of lower degree (= lower complexity) if we take a rank 2 approximation $\hat{\mathcal{H}}_3$ by omitting the third singular value and its associated colums/rows in the SVD.

Semi-Separable Matrix Structure

General Toeplitz Operator

State equations for causal and anti-causal system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k & x'_k &= A'_k x'_{k+1} + B'_k u_k \\ y_k^{(1)} &= C_k x_k + D_k u_k & y_k^{(2)} &= C'_k x'_{k+1} \\ y_k &= y_k^{(1)} + y_k^{(2)} \end{aligned} \tag{6}$$

We consider now a more general matrix T, which is no longer constraint to be lower triangular. However, we still want to consider such a matrix to consist of a collection of time-varying impulse responses, which may also be anti-causal. That is, we write for a Toeplitz-Operator as

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & D_1 & C'_1B'_2 & C'_1A'_2B'_3 & C'_1A'_2A'_3B'_4 & C'_1A'_2A'_3A'_4B'_5 & \dots \\ \dots & C_2B_1 & D_2 & C'_2B'_3 & C'_2A'_3B'_4 & C'_2A'_3A'_4B'_5 & \dots \\ \dots & C_3A_2B_1 & C_3B_2 & D_3 & C'_3B'_4 & C'_3A'_4B'_5 & \dots \\ \dots & C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & D_4 & C'_4B'_5 & \dots \\ \dots & C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & D_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

The individual matrix entries $T_{i,j}$ are given as

$$T_{ij} = \begin{cases} D_i & \text{for} \quad i = j, \\ C_iA_{i-1}\dots A_{j+1}B_j & \text{for} \quad i < j, \\ C'_iA'_{i+1}\dots A'_{j-1}B'_j & \text{for} \quad i > j, \end{cases}$$

The lower triangular part of T, including the main diagonal blocks D_{ii} correspond to a causal linear time-varying system. We can interpret the strictly upper triangular part of T as containing the impulse responses of a strictly anti-causal linear time-varying system. If a matrix T can be represented in this particular way we call it a *semi-separable* matrix. The corresponding state-space realization structure is depicted in Figure 1.

Hankel Matrices

We take the example of a finite 5×5 matrix

$$T = \begin{bmatrix} D_1 & C'_1B'_2 & C'_1A'_2B'_3 & C'_1A'_2A'_3B'_4 & C'_1A'_2A'_3A'_4B'_5 \\ C_2B_1 & D_2 & C'_2B'_3 & C'_2A'_3B'_4 & C'_2A'_3A'_4B'_5 \\ C_3A_2B_1 & C_3B_2 & D_3 & C'_3B'_4 & C'_3A'_4B'_5 \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & D_4 & C'_4B'_5 \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & D_5 \end{bmatrix}.$$

Lower/Causal Part

For this matrix we can identify the Hankel matrices of the lower triangular (= causal) part as

$$\mathcal{H}_0 = [\cdot]$$

$$\mathcal{H}_1 = \begin{bmatrix} C_2B_1 \\ C_3A_2B_1 \\ C_4A_3A_2B_1 \\ C_5A_4A_3A_2B_1 \end{bmatrix},$$

$$\mathcal{H}_2 = \begin{bmatrix} C_3B_2 & C_3A_2B_1 \\ C_4A_3B_2 & C_4A_3A_2B_1 \\ C_5A_4A_3B_2 & C_5A_4A_3A_2B_1 \end{bmatrix}$$

$$\mathcal{H}_3 = \begin{bmatrix} C_4B_3 & C_4A_3B_2 & C_4A_3A_2B_1 \\ C_5A_4B_3 & C_5A_4A_3B_2 & C_5A_4A_3A_2B_1 \end{bmatrix},$$

$$\mathcal{H}_4 = \begin{bmatrix} C_5B_4 & C_5A_4B_3 & C_5A_4A_3B_2 & C_5A_4A_3A_2B_1 \end{bmatrix}$$

$$\mathcal{H}_5 = [\cdot]$$

Upper/Anti-Causal Part

For the given 5×5 the Hankel matrices corresponding to the anti-causal (strict upper triangular) part look like

$$\mathcal{H}'_0 = [\cdot]$$

$$\mathcal{H}'_1 = \begin{bmatrix} C'_1B'_2 & C'_1A'_2B'_3 & C'_1A'_2A'_3B'_4 & C'_1A'_2A'_3A'_4B'_5 \end{bmatrix}$$

$$\mathcal{H}'_2 = \begin{bmatrix} C'_2B'_3 & C'_2A'_3B'_4 & C'_2A'_3A'_4B'_5 \\ C'_1A'_2B'_3 & C'_1A'_2A'_3B'_4 & C'_1A'_2A'_3A'_4B'_5 \end{bmatrix}$$

$$\mathcal{H}'_3 = \begin{bmatrix} C'_3B'_4 & C'_3A'_4B'_5 \\ C'_2A'_3B'_4 & C'_2A'_3A'_4B'_5 \\ C'_1A'_2A'_3B'_4 & C'_1A'_2A'_3A'_4B'_5 \end{bmatrix}$$

$$\mathcal{H}'_4 = \begin{bmatrix} C'_4B'_5 \\ C'_3A'_4B'_5 \\ C'_2A'_3A'_4B'_5 \\ C'_1A'_2A'_3A'_4B'_5 \end{bmatrix}$$

$$\mathcal{H}'_5 = [\cdot]$$

Special Case: Linear Time Invariant Systems

Factorization of the Hankel matrix

In case of a linear time-invariant system we have the Hankel matrix

$$\mathcal{H} = \begin{bmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & CA^3B & \cdots \\ CA^2B & CA^3B & CA^4B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{O} \cdot \mathcal{C}, \tag{8}$$

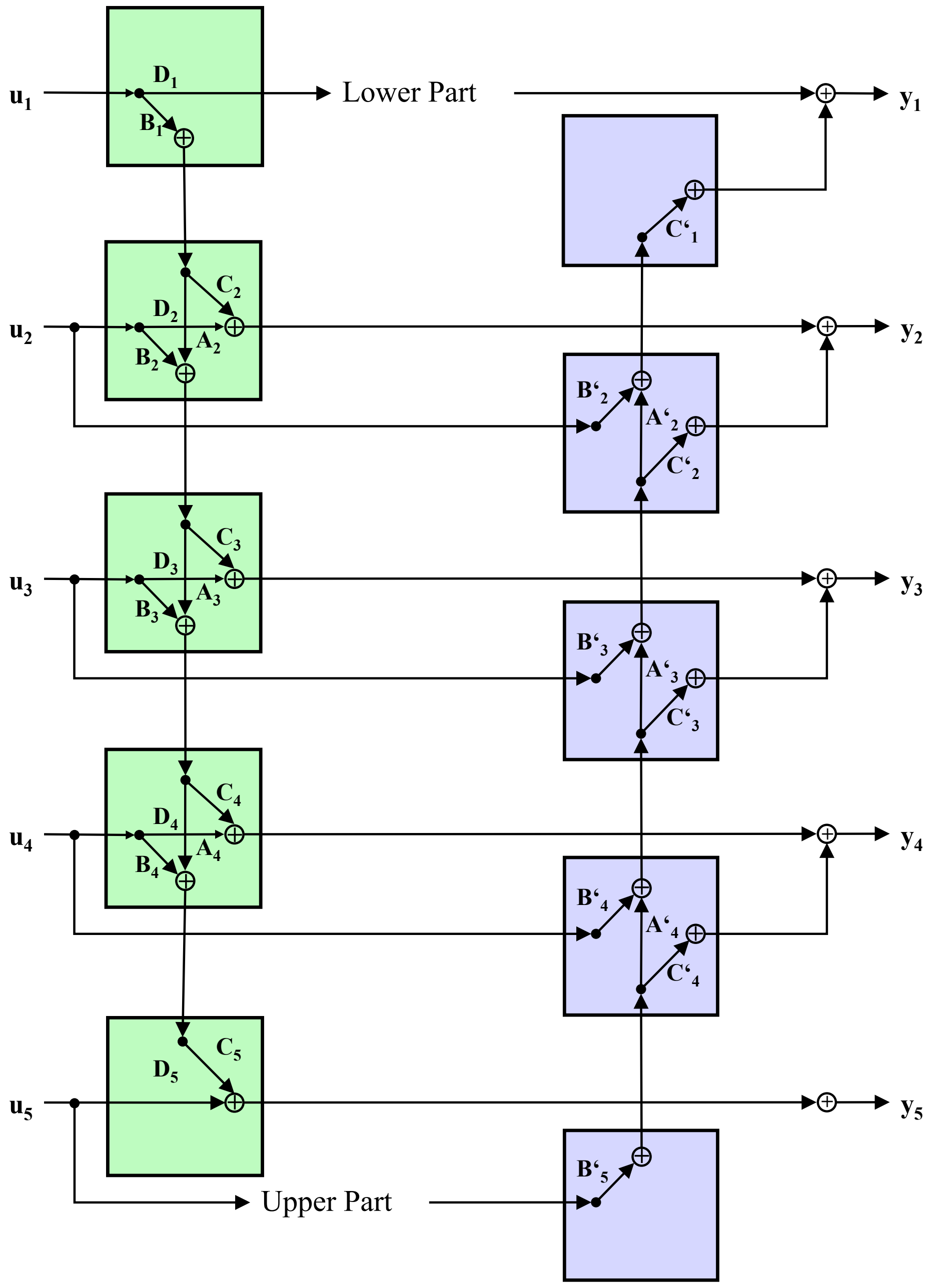

Figure 1: State-Space Realization Structurev for mixed causal/anti-causal system corresponding to a full 5×5 matrix T

where we have the Observability and the Controllability matrices as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}. \tag{9}$$

In case of LTI systems the shift-invariance property of the Hankel matrix is slightly more obvious, i.e. we have

$$\mathcal{H}\!\uparrow = \overleftarrow{\mathcal{H}} = \begin{bmatrix} CAB & CA^2B & CA^3B & \cdots \\ CA^2B & CA^3B & CA^4B & \cdots \\ CA^3B & CA^4B & CA^5B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{O}A\mathcal{C} = \mathcal{O}\!\uparrow \mathcal{C} = A\overleftarrow{\mathcal{C}}.$$

We can exploit the shift-invariance of the Hankel matrix $\mathcal{H}$ to gain three equations to identify the matrix A directly as

$$A = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\mathcal{H}\!\uparrow \mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1} = \mathcal{O}^{\dagger}\mathcal{H}\!\uparrow \mathcal{C}\dagger$$

or via the Observabillity matrix

$$\mathcal{O}\!\uparrow = \mathcal{O}A \quad \Rightarrow \quad A = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\mathcal{O}\!\uparrow = \mathcal{O}^{\dagger}\mathcal{O}\!\uparrow$$

or via the Controllability matrix

$$\overleftarrow{\mathcal{C}} = A\mathcal{C} \quad \Rightarrow \quad A = \overleftarrow{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1} = \overleftarrow{\mathcal{C}}\mathcal{C}^{\dagger}.$$

For this approach to work we need the Observability and the Controllability matrices to have full column rank and full row rank, respectively. This requirement is identical to a system that is fully observable and fully controllable.

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