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Controllability in Nonlinear Systems*

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In this paper we obtain an explicit expression for the reachable set for a class of nonlinear systems. This class is described by a chain condition on the Lie algebra of vector fields associated with each nonlinear system. These ideas are used to obtain a generalization of a controllability result for linear systems in the case where multiplicative controls are present.

1. INTRODUCTION

The purpose of this paper is to study the controllability of nonlinear systems of the form $dx/dt = f(x, \mathbf{u})$ where the state space is a differentiable manifold M. Our goal is to identify a large class of nonlinear systems for which there is an explicit expression for the reachable set from any initial state x_0 at time t. Recent work by Sussman and Jurdjevic [2] has established some of the basic properties of the reachable set in the case where M is a real analytic manifold but, except for two special classes of systems (Brockett's Theorem 7 of [1] and symmetric systems, i.e., $f(\cdot, \mathbf{u}) = -f(\cdot, -\mathbf{u})$) the problem of obtaining an explicit expression for the reachable set from x at time t remains open.

We shall assume, as in [2], that the vector field on $M, f(\cdot, \mathbf{u})$, corresponding to a constant control $\mathbf{u}$, is complete. One nice consequence of this assumption is the existence of a 1-parameter group $X_t^{\mathbf{u}}$ for each vector field $f(\cdot, \mathbf{u})$ where $\mathbf{u}$ is a constant control. The collection of all such 1-parameter groups and their products (under composition) is a group G of diffeomorphisms of M. For the systems which we shall be considering the reachable set from a state x at time t can be expressed as the orbit of the point x under a certain subset of the group G. Thus one can study the properties of the reachable set from x

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by examining the structure of the orbits of x under G and certain subsets of G (see, for example, Lobry [3]). The orbit of x under G and the Lie algebra $\mathscr{L}$ of vector fields generated by the vector fields $f(\cdot, \mathbf{u})$ corresponding to constant controls are closely related. In the case where M and $f(\cdot, \mathbf{u})$ are real analytic, the orbit of x under G is the integral manifold of $\mathscr{L}$ through x (cf. [3]). Thus the structure of the reachable set from x can be studied by examining the structure of the Lie algebra of vector fields $\mathscr{L}$ associated with a system.

Our approach will be to study the structure of the group of diffeomorphisms G directly. In general, this global approach is ineffective because of the lack of structure in G, but in the case where $\mathscr{L}$ is finite dimensional G can be given the structure of a Lie group with Lie algebra isomorphic to $\mathscr{L}$. This result, which is due to Palais [7], enables us to reformulate the original control system as a nonlinear system on the Lie group G for which the vector fields corresponding to constant controls are right-invariant. For these reasons we will restrict our attention to nonlinear systems for which the associated Lie algebra of vector fields $\mathscr{L}$ is finite dimensional. For a subclass of these systems the associated Lie algebra $\mathscr{L}$ of vector fields satisfies a certain chain condition. For these systems we are able to obtain an explicit description of the reachable set at time t from any initial state x, subject to one additional constraint on the associated transformation group G. For systems of the form

$$\frac{dx}{dt} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

on M, where A, $B_1, \ldots, B_m$ are smooth vector fields on M, this additional constraint on G can be dropped.

This paper is organized as follows: In Section 2 we introduct notations and present some of the known controllability results. In Section 3 we describe some of Palais' results on Lie transformation groups, prove our main result, and present an example. In Section 4 these results are applied to obtain a generalization of Brockett's Theorem 7 of [1] and a generalization of the exact time controllability result for linear systems where multiplication controls are present.

2. Preliminaries

We shall assume that the reader is familiar with the basic notions of differential geometry and Lie theory (cf. [4, 11]). The following notation will be used:

M a Hausdorff differentiable manifold,

$V(M)$ the set of smooth vector fields on M,

diff(M)	the group of diffeomorphisms of M,
M_x	the tangent space to M at $x \in M$,
TM	the tangent bundle of M,
$\mathbf{G}$	a Lie group,
$\mathscr{L}(\mathbf{G})$	the Lie algebra of right-invariant vector fields on $\mathbf{G}$,
$\mathbf{R}$	the real numbers,
$\mathbf{R}^m$	m-dimensional Euclidean space,
$\mathbf{R}^+$	the open interval $(0, \infty)$, a semigroup under addition.

exp: $\mathscr{L}(\mathbf{G}) \to \mathbf{G}$ is the standard exp map (cf. [4]).
Suppose $H \subset \mathbf{G}$ and $h \subset \mathscr{L}(\mathbf{G})$.

$\{H\}_G$	the group generated by H in $\mathbf{G}$,
$\{h\}_{LA}$	the Lie subalgebra of $\mathscr{L}(\mathbf{G})$ generated by h,
$\{h\}_{LS}$	the linear span of h in the vector space $\mathscr{L}(\mathbf{G})$,
cl H	the closure of H in $\mathbf{G}$,
int H	the interior of H in $\mathbf{G}$.

We regard $V(M)$ as a Lie algebra over $\mathbf{R}$. For all $X, Y \in V(M)$, we define the Lie bracket of X and Y by

$$[X, Y](m) = X(m)Y - Y(m)X \qquad \text{(cf. [4])},$$

and denote the linear map: $Y \to [X, Y]$ of $V(M)$ into $V(M)$ by ad_X. Then $ad_X{}^2 = ad_X \cdot ad_X$ and $ad_X{}^0 =$ identity map. If $M = \mathbf{R}^n$ we identify M_x with $\mathbf{R}^n$ for all $x \in \mathbf{R}^n$ and then

$$[X, Y](m) = (dY)_m X(m) - (dX)_m Y(m),$$

where X and Y are treated as maps from $\mathbf{R}^n$ into $\mathbf{R}^n$ and $(dX)_m$ and $(dY)_m$ are the Jacobians of the maps X and Y, respectively.

We shall consider nonlinear systems of the form

$$\frac{dx}{dt}(t) = f(x(t), \mathbf{u}(t)), \tag{†}$$

where $x \in M$ and

(i) The functions $\mathbf{u}(\cdot)$ are contained in $\mathscr{P}^{(m)}$, the class of piecewise constant functions from $[0, \infty)$ into $\mathbf{R}^m$.

(ii) For each $\mathbf{u} \in R^m$, $f(\cdot, \mathbf{u})$ is a complete vector field, i.e., the integral curve for $f(\cdot, \mathbf{u})$ is defined for all $t \in \mathbf{R}$. Thus for all $\mathbf{u}(\cdot) \in \mathscr{P}^{(m)}$ the corresponding solution to (†) with $x(0) = x$, $\pi(x, \mathbf{u}(t), t)$, exists for all $t \in \mathbf{R}^+$.

(iii) The Lie algebra generated by the collection of vector fields $D = \{f(\cdot, \mathbf{u}): \mathbf{u} \in \mathbf{R}^m\} \subset V(M)$ is finite dimensional.

Remark. Property (ii) implies that for each vector field $f(\cdot, \mathbf{u}) \in D$ there is a 1-parameter group of $f(\cdot, \mathbf{u}), X_t^{\mathbf{u}}$. Thus for all $t_1, t_2 \in \mathbf{R}, X_{t_1}^{\mathbf{u}}, X_{t_2}^{\mathbf{u}} \in \operatorname{diff}(M)$, $X_{t_1}^{\mathbf{u}} \circ X_{t_2}^{\mathbf{u}} = X_{t_1+t_2}^{\mathbf{u}}$, and for all $p \in M$, $(d/dt)\, X_t^{\mathbf{u}}(p) = f(X_t^{\mathbf{u}}(p), \mathbf{u})$ and $X_0^{\mathbf{u}}(p) = p$.

DEFINITION. A point $y \in M$ is *reachable from x M at time t* iff there exists $\mathbf{u}(\cdot) \in \mathscr{P}^{(m)}$ such that $y = \pi(x, \mathbf{u}(t), t)$. We define the *reachable set from x at time t, $\mathscr{R}_t(x)$*, as the set of points in M reachable from x at time t. We define the *reachable set from x, $\mathscr{R}(x)$*, to be $\bigcup_{t>0} \mathscr{R}_t(x)$.

We are interested in obtaining global results and will consider the problem of determining the following subsets of $\operatorname{diff}(M)$:

$$G = \{X_{t_1}^{\mathbf{u}_1} \circ X_{t_2}^{\mathbf{u}_2} \circ \cdots \circ X_{t_n}^{\mathbf{u}_n} : \mathbf{u}_i \in \mathbf{R}^m,\ t_i \in \mathbf{R},\ n = 1, 2,...\}$$

$$G^+ = \{X_{t_1}^{\mathbf{u}_1} \circ \cdots \circ X_{t_n}^{\mathbf{u}_n} : \mathbf{u}_i \in \mathbf{R}^m,\ t_i \in \mathbf{R}^+,\ n = 1, 2,...\}$$

$$G_t = \left\{X_{t_1}^{\mathbf{u}_1} \circ \cdots \circ X_{t_n}^{\mathbf{u}_n} : \mathbf{u}_i \in \mathbf{R}^m,\ t_i \in \mathbf{R}^+,\ \sum_{i=1}^{n} t_i = t,\ n = 1, 2,...\right\}.$$

For all $x \in M$, $G \cdot x$, $G^+ \cdot x$, $G_t \cdot x$ will denote the orbits through x of G, G^+, G_t, respectively, i.e., $G_t \cdot x = \{g(x): g \in G_t\}$. We will be interested in obtaining an explicit expression for G_t. This will result in an explicit expression for $\mathscr{R}_t(x)$, since for all $x \in M$, $t \in \mathbf{R}^+$, $\mathscr{R}_t(x) = G_t \cdot x$. The orbits of G_t and the structure of the finite dimensional Lie algebra $\mathscr{L}$ generated by D in $V(M)$ are closely related. In particular the following decomposition of $\mathscr{L}$ reveals some of the basic properties of the orbits of G_t in the case where M is a real analytic manifold and the vector fields $f(\cdot, \mathbf{u})$ in D are real analytic:

$$\mathscr{L} = \{X: X \in D\}_{LA}$$

$$\mathscr{B} = \{r_1 X_1 + r_2 X_2 + r_n X_n: r_i \in \mathbf{R}, r_1 + \cdots + r_n = 0, X_i \in D, n = 1, 2,...\}_{LA}$$

$$\mathscr{L}_0 = \text{the ideal generated by } \mathscr{B} \text{ in } \mathscr{L}.$$

For all $x \in M$, $I(\mathscr{L}, x)$, $I(\mathscr{L}_0, x)$, and $I(\mathscr{B}, x)$ will denote the maximal integral manifolds through x of $\mathscr{L}$, $\mathscr{L}_0$, and $\mathscr{B}$, respectively, i.e., $I(\mathscr{L}, x)$ is the largest connected submanifold N of M such that $N_y = \{X(y): X \in \mathscr{L}\}$ for all $y \in N$. Its existence, in the analytic case, is a consequence of the global version of Frobenius' theorem [5].

DEFINITION. For all $x \in M$, $t \in \mathbf{R}^+$, $I^t(\mathscr{L}_0, x) \triangleq X_t^{\mathbf{u}}(I(\mathscr{L}_0, x))$, where $X_t^{\mathbf{u}}$ is the 1-parameter group associated with $f(\cdot, \mathbf{u}) \in D$. Lemma 3.6 of [2] states that $I^t(\mathscr{L}_0, x)$ is well defined, i.e., independent of the choice of $\mathbf{u} \in \mathbf{R}^m$.

THEOREM 2.1 (Sussman and Jurdjevic [2]). *Consider the nonlinear system* (†) *with the associated triple of Lie algebras* $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$. *Then for all* $t \in \mathbf{R}^+$, $x \in M$, $G_t \cdot x \subset I^t(\mathscr{L}_0, x)$, *and the interior of* $G_t \cdot x$ *is dense in* $G_t \cdot x$ *with respect to the topology of* $I^t(\mathscr{L}_0, x)$.

Remark. This result holds even when $\mathscr{L}$ is infinite dimensional, but Theorem 2.1 requires that M be real analytic and that D consists of real analytic vector fields.

In the case where D is *symmetric*, i.e., $-X \in D$ whenever $X \in D$, and everything is real analytic, Chow's theorem results in an explicit expression for $\mathscr{R}(x)$.

THEOREM 2.2 (see Lemma 2.5 of [2]). *Consider the nonlinear system* (†) *where* D *is a symmetric collection of vector fields and* $\mathscr{L} = \{D\}_{LA}$. *Then*

$$\mathscr{R}(x) = I(\mathscr{L}, x) \qquad \text{for all} \quad x \in M.$$

In the case where M is a Lie group $\mathbf{G}$ and D is a collection of right-invariant vector fields on $\mathbf{G}$ (i.e., $D \subset \mathscr{L}(\mathbf{G})$) the analyticity assumptions are automatically satisfied (cf. [11, 2]) and $\mathscr{L} = \{D\}_{LA}$ will be finite dimensional. For a subclass of these systems there is an *explicit* expression for G_t if the Lie algebra $\mathscr{L}$ satisfies a certain chain condition (cf. [6, 10]). We will now describe this algebraic condition where M is the Lie group $\mathbf{G}$.

Suppose $\mathscr{Q}$ is a Lie subalgebra of $\mathscr{L}$ and $A \in D$ is a right-invariant vector field on $\mathbf{G}$. Consider the chain of Lie subalgebras of $\mathscr{L}$,

$$\mathscr{Q} \subset \tilde{\mathscr{Q}} \subset \{\mathscr{Q}, A\}_{LA}, \tag{*}$$

where $\tilde{\mathscr{Q}}$ is the ideal generated by $\mathscr{Q}$ in $\{\mathscr{Q}, A\}_{LA}$.

DEFINITION. The chain (*) is called an *A-chain* if $\mathscr{Q}$ is an ideal in $\tilde{\mathscr{Q}}$. If $\mathscr{Q}$ is contained in $\mathscr{h}$, a Lie subalgebra of $\mathscr{L}$, and (*) is an A-chain, we will call (*) an *A-chain from* $\mathscr{h}$.

DEFINITION. Suppose that $\mathscr{h}$ is a Lie subalgebra of $\mathscr{L}$ and (*) is an A-chain from $\mathscr{h}$. Then the Lie algebra $\{\mathscr{h}, \tilde{\mathscr{Q}}\}_{LA}$ is said to be *A-generated from* $\mathscr{h}$.

DEFINITION. Suppose $\mathscr{B}_0$, $\mathscr{B}_n$ are Lie subalgebras of $\mathscr{L}_0$ and $\mathscr{B}_0 \subset \mathscr{B}_n$. A chain of Lie subalgebras

$$\mathscr{B}_0 \subset \mathscr{B}_1 \subset \cdots \subset \mathscr{B}_{n-1} \subset \mathscr{B}_n$$

is called an *A-series for* $\mathscr{B}_0$ *terminating at* $\mathscr{B}_n$ if $\mathscr{B}_{i+1}$ is A-generated from $\mathscr{B}_i$ for $i = 0, 1,\ldots, n-1$. The A-radical for $\mathscr{B}_0$, $\mathscr{R}(A; \mathscr{B}_0)$, is the largest Lie subalgebra $\mathscr{h}$ of $\mathscr{L}_0$ with the property that there exists an A-series for $\mathscr{B}_0$ terminating at $\mathscr{h}$. $\mathscr{R}(A; \mathscr{B}_0)$ is well defined and unique as a consequence of the definitions and the finite dimensionality of $\mathscr{L}$ (see [6]).

THEOREM 2.3 (Hirschorn [6, 10]). *Consider the system on the Lie group* $\mathbf{H}$,

$$\frac{dx}{dt} = A(x) + \sum_{i=1}^{m} u_i B_i(x); \qquad x(0) = x_0 \in \mathbf{H},$$

where $u_i \in \mathscr{P}^{(1)}$ *and* $\{A, B_1,\ldots, B_m\} \subset \mathscr{L}(\mathbf{H})$. *Associated with this system is the triple of Lie algebras* $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$. *Suppose* $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$. *Then for all* $t \in \mathbf{R}^+$, $\mathscr{R}_t(x_0) = G_t \cdot x_0 = I^t(L_0, x_0) = \{\exp \mathscr{L}_0\}_G \cdot (\exp tA) \cdot x_0$.

Remark. For the class of systems described in the above theorem a straightforward computation shows that

$$\mathscr{L} = \{A, B_1,\ldots, B_m\}_{LA}$$

$$\mathscr{B} = \{B_1, B_2,\ldots, B_m\}_{LA}$$

and

$$\mathscr{L}_0 = \{ad_A{}^k B_i : i = 1, 2,\ldots, m \text{ and } k = 0, 1,\ldots\}_{LA}.$$

Here $\mathscr{L}$, $\mathscr{L}_0$ and $\mathscr{B}$ are Lie subalgebras of $\mathscr{L}(\mathbf{H})$ and we can associate with these systems the triple of groups $(\mathbf{G}, \mathbf{G}_0, \mathbf{B})$ where $\mathbf{G} = \{\exp \mathscr{L}\}_G$, $\mathbf{G}_0 = \{\exp \mathscr{L}_0\}_G$ and $\mathbf{B} = \{\exp \mathscr{B}\}_G$. A standard result in Lie theory states that $\mathbf{G}$, $\mathbf{G}_0$, and $\mathbf{B}$ are connected Lie subgroups of $\mathbf{H}$ with Lie algebras $\mathscr{L}$, $\mathscr{L}_0$ and $\mathscr{B}$, respectively [4]. Thus the above theorem states that if $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$ then $\mathscr{R}_t(e) = \mathbf{G}_0 \cdot \exp tA$. Note that if $\mathscr{B}$ is an ideal in $\mathscr{L}_0$, then $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$. This follows directly from the definitions.

In the next section we show that a result analogous to that of Theorem 2.3 holds for systems of the form

$$\frac{dx}{dt} = A(x) + \sum_{i=1}^{m} u_i B_i(x),$$

where $x \in M$, a Hausdorff differentiable manifold, $A, B_1,\ldots, B_m$ are complete vector fields, and $\{A, B_1,\ldots, B_m\}_{LA}$ is finite dimensional.

3. Dynamical Systems With Lie Transformation Groups

For the nonlinear system (†) on M the assumption that $\mathscr{L} = \{D\}_{LA}$ is finite dimensional results in many simplifications. In particular, the subgroup $G \subset \operatorname{diff}(M)$ which contains G_t and G^+ can be given the structure of a Lie group with Lie algebra isomorphic to $\mathscr{L}$. This result, due to Palais, enables us to reformulate (†) as a nonlinear system on the Lie group $\mathbf{G}$ and determine G_t explicitly for a large class of nonlinear systems.

Suppose X, $Y \in V(M)$ are complete vector fields. If M is not compact then neither $[X, Y]$ nor $X + Y$ need be complete (see Palais [7]). We have the following result:

Theorem 3.1 (Palais [7]). *Let $\mathscr{L}$ be a finite dimensional Lie algebra of vector fields on the Hausdorff differentiable manifold M. Then the following conditions are equivalent*:

(A) *Every $L \in \mathscr{L}$ is complete.*

(B) *The set of $L \in \mathscr{L}$ which are complete generate the Lie algebra $\mathscr{L}$.*

Definition (Palais). A finite dimensional Lie algebra of vector fields on a Hausdorff differentiable manifold M will be called an *infinitesimal group* of M if it satisfies conditions (A) or (B) of Theorem 3.1.

Thus if D is the set of complete vector fields associated with the nonlinear system (†) and $\mathscr{L} = \{D\}_{LA}$, then $\mathscr{L}$ is an infinitesimal group of M as a consequence of property (iii) of (†).

Definition (Palais). Let $\mathbf{H}$ be a connected Lie group whose underlying group is a subgroup of $\operatorname{diff}(M)$. We shall call $\mathbf{H}$ a *connected Lie transformation group of M* if the mapping $\phi: (h, p) \to h(p)$ of $\mathbf{H} \times M \to M$ is smooth. We call ϕ the *natural global $\mathbf{H}$-transformation group*. For all $p \in M$ we define the mapping $\phi_p: \mathbf{H} \to M$ by setting $\phi_p(h) = h(p)$ and define the mapping $\phi^+: \mathscr{L}(H) \to V(M)$ by setting $\phi^+(L)(p) = (d\phi_p)_e(L(e))$ for all $L \in \mathscr{L}(\mathbf{H})$ $p \in M$. The range of ϕ^+ is called the *infinitesimal group of* $\mathbf{H}$.

Remark. Palais shows that ϕ^+ is an isomorphism from $\mathscr{L}(H)$ onto $\mathscr{L}$, the infinitesimal group of $\mathbf{H}$. Thus for all $L \in \mathscr{L} = \phi^+(\mathscr{L}(\mathbf{H}))$ there exists a *unique* vector field $\tilde{L} \in \mathscr{L}(\mathbf{H})$ with the property that $\phi^+(\tilde{L}) = L$ and $\exp t\tilde{L}$ is the 1-parameter group of L.

The following interesting result is due to Palais [7]:

Theorem 3.2. *Every infinitesimal group of M is the infinitesimal group of a* unique *connected Lie transformation group of M.*

If D is the set of complete vector fields associated with a nonlinear system then $\mathscr{L} = \{D\}_{LA}$ is an infinitesimal group of M. Theorem 3.2 implies the existence of a unique connected Lie transformation group $\mathbf{G}$ of M with infinitesimal group $\mathscr{L}$. As a consequence of the above remark, $\mathscr{L}$ is isomorphic to $\mathscr{L}(\mathbf{G})$, i.e., $\phi^+: \mathscr{L}(\mathbf{G}) \to \mathscr{L}$ is an isomorphism. For all $L \in \mathscr{L}$ we set $\tilde{L} = (\phi^+)^{-1}(L)$. Then the group of transformations G, defined in Section 2, can be expressed as

$$G = \{\exp t_1 L_1 \exp t_2 L_2 \cdots \exp t_n L_n : t_i \in \mathbf{R}, L_i \in \mathscr{L}(\mathbf{G}), \phi^+(L_i) \in D \subset \mathscr{L}, n = 1, 2, \ldots\}.$$

Since $\exp t_i L_i \in \mathbf{G}$ for $t_i \in \mathbf{R}$, $L_i \in \mathscr{L}(\mathbf{G})$, we see that $G \subset \mathbf{G}$.

Claim. $G = \mathbf{G}$.

Proof. Let $\tilde{D} = (\phi^+)^{-1}(D) \subset \mathscr{L}(\mathbf{G})$. Then $D_s = \{\pm X : X \in \tilde{D}\}$ is a symmetric collection of right-invariant vector fields on $\mathbf{G}$. If we consider D_s to be the collection of complete vector fields arising from a nonlinear system (†) on $\mathbf{G}$, then $G = \mathscr{R}(e)$, the reachable set from the identity element in $\mathbf{G}$. Now $\{D_s\}_{LA} = \mathscr{L}(\mathbf{G})$ because $\{D\}_{LA} = \mathscr{L}$ and $I(\mathscr{L}(\mathbf{G}), e) = \mathbf{G}$ because $\mathbf{G}$ is a connected Lie group with Lie algebra $\mathscr{L}(\mathbf{G})$, thus Theorem 2.2 implies that $\mathscr{R}(x) = G = I(\mathscr{L}(\mathbf{G}), e) = \mathbf{G}$. This proves the assertion.

Thus G is the connected Lie group $\mathbf{G}$, and G^+ and G_t are subsets of $\mathbf{G}$. Associated with each nonlinear system (†) is a triple of Lie algebras $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$ and a triple of connected Lie transformation groups $(\mathbf{G}, \mathbf{G}_0, \mathbf{B})$ where $\mathscr{L}$, $\mathscr{L}_0$, and $\mathscr{B}$ are the infinitesimal groups of $\mathbf{G}$, $\mathbf{G}_0$, and $\mathbf{B}$, respectively. Here $\mathbf{G} \supset \mathbf{G}_0 \supset \mathbf{B}$ and $\mathbf{G}_0$ is a normal subgroup of $\mathbf{G}$ since $\mathscr{L}_0$ is an ideal in $\mathscr{L}$ (cf. [4]). We now use these ideas to obtain an explicit expression for G_t and $\mathscr{R}_t(x)$ for a class of nonlinear systems.

THEOREM 3.3. *Consider the nonlinear system* (†) *on M with the associated triple of Lie algebras $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$ and connected Lie transformation groups $(\mathbf{G}, \mathbf{G}_0, \mathbf{B})$. Let $A \in \mathscr{L}$ be the vector field $A(\cdot) = f(\cdot, \mathbf{0})$ and let A^0 be the isomorphic image of A in $\mathscr{L}(\mathbf{G})$. Suppose that the A-radial for $\mathscr{B}$ is $\mathscr{L}_0$ and that for all $t \in \mathbf{R}^+$, $\mathbf{B} \cdot \exp tA^0 \subset \operatorname{cl} G_t$. Then for all $t \in \mathbf{R}^+$, $x \in M$, $G_t = \mathbf{G}_0 \cdot \exp tA^0$ and*

$$\mathscr{R}_t(x) = \mathbf{G}_0 \cdot \exp tA^0 \cdot x = I^t(\mathscr{L}_0, x).$$

Before proving this result we prove the following Lemma:

LEMMA 3.1. *Consider the system on the Lie group* $\mathbf{G}$,

$$\frac{dx}{dt} = A(x) + u_1 B_1(x) + \cdots + u_m B_m(x); \qquad x(0) = e \in \mathbf{G},$$

where $u_i \in \mathscr{P}^{(1)}$ *and* $A, B_1, B_2, \ldots, B_m \in \mathscr{L}(\mathbf{G})$. *Associated with this system is the triple of Lie algebras* $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$ *and Lie groups* $(\mathbf{G}, \mathbf{G}_0, \mathbf{B})$ *where* $\mathbf{G} = \{\exp \mathscr{L}\}_G$, $\mathbf{G}_0 = \{\exp \mathscr{L}_0\}_G$ *and* $\mathbf{B} = \{\exp \mathscr{B}\}_G$. *For all* $t \in \mathbf{R}^+$ *let*

$$Q_t = \{(b_1 \exp t_1 A)(b_2 \exp t_2 A) \cdots (b_n \exp t_n A): \\ b_i \in \mathbf{B},\ t_i \in \mathbf{R}^+, \text{ and } t_1 + \cdots + t_n = t\}.$$

Then for all $t \in \mathbf{R}^+$, $\operatorname{cl} Q_t = \operatorname{cl} G_t = \operatorname{cl} \mathscr{R}_t(e)$.

Proof. First we show that for all $t \in \mathbf{R}^+$, $Q_t \subset \operatorname{cl} G_t$, which implies that $\operatorname{cl} Q_t \subset \operatorname{cl} G_t$. Suppose that n, l are positive integers; $k_1, \ldots, k_l \in \{1, 2, \ldots, m\}$; $r_1, \ldots, r_l \in \mathbf{R}$ and $t \in \mathbf{R}^+$. We let

$$p(n) = \left(\exp \frac{t}{n}\left(A + \frac{nr_1}{t} B_{k_1}\right)\right) \cdots \left(\exp \frac{t}{n}\left(A + \frac{nr_l}{t} B_{k_l}\right)\right) \\ \times \left(\exp\left(t - \frac{lt}{n}\right) A\right).$$

Then for $n > l$, $p(n) \in G_t$ and

$$\lim_{n\to\infty} p(n) = (\exp r_1 B_{k_1}) \cdots (\exp r_l B_{k_l})(\exp tA) \in \operatorname{cl} G_t,$$

hence

$$\{\exp r_i B_i : r_i \in \mathbf{R},\ 1 \leqslant i \leqslant m\}_G \cdot (\exp tA) \subset \operatorname{cl} G_t.$$

Theorem 1 of [1] implies that

$$\{\exp r_i B_i : r_i \in \mathbf{R},\ i = 1, 2, \ldots, m\}_G = \{\exp\{B_1, \ldots, B_m\}_{LA}\}_G = \{\exp \mathscr{B}\}_G = \mathbf{B},$$

and thus $\mathbf{B} \cdot \exp tA \subset \operatorname{cl} G_t$ and so $Q_t \subset \operatorname{cl} G_t$.

We now show that $G_t \subset \operatorname{cl} Q_t$ which implies that $\operatorname{cl} G_t \subset \operatorname{cl} Q_t$ and so completes the proof. Since every element of G_{t_1} is the product of elements of the form $p = \exp t(A + r_1 B_1 + \cdots + r_m B_m)$ where $t \in \mathbf{R}^+$, $r_i \in \mathbf{R}$, it suffices to show that $p \in \operatorname{cl} Q_t$. Let m be a positive integer and set

$$p(n) = \left(\exp \frac{t}{n}(r_1 B_1 + \cdots + r_m B_m) \exp \frac{t}{n} A\right)^n \in Q_t.$$

Since $\lim_{n\to\infty} p(n) = p$, an easy consequence of the Campbell–Baker–Hausdorff (cf. [11]), $p \in \operatorname{cl} Q_t$. Since t is an arbitrary positive real number this implies that $G_t \subset \operatorname{cl} Q_t$, which completes the proof. Q.E.D.

Proof. Consider the system defined on the Lie group $\mathbf{G}$,

$$\frac{dx}{dt} = f^0(x, \mathbf{u}), \qquad x(0) = e, \tag{††}$$

where e is the identity in $\mathbf{G}$, $\mathbf{u} \in \mathscr{P}^{(m)}$ and for all $\mathbf{u} \in \mathbf{R}^m$, $f^0(\cdot, \mathbf{u})$ is the isomorphic image of $f(\cdot, \mathbf{u})$ in $\mathscr{L}(\mathbf{G})$. This new system is related to the original system as follows: if $x^0(e, \mathbf{u}, t)$ is a trajectory for (††) and $x(x_1, \mathbf{u}, t)$ is the trajectory for (†) for the same control $\mathbf{u}$, then $x(x_1, \mathbf{u}, t) = x^0(e, \mathbf{u}, t)(x_1)$—note that for each $t > 0$, $x^0(e, \mathbf{u}, t) \in \mathbf{G} \subset \operatorname{diff}(M)$. It follows directly from the definitions that $(\mathscr{L}(\mathbf{G}), \mathscr{L}(\mathbf{G}_0), \mathscr{L}(\mathbf{B}))$ is the triple of Lie algebras associated with (††),

$$\mathscr{R}_t(e) = G_t \subset \mathbf{G}_0 \cdot \exp tA^0,$$

and

$$I^t(\mathscr{L}(\mathbf{G}_0), e) = \mathbf{G}_0 \cdot \exp tA^0 \qquad \text{(cf. [2, 3, 6])}.$$

Theorem 2.1 states that int G_t is dense in G_t with respect to the topology of $\mathbf{G}_0 \cdot \exp tA^0$. Thus we can set $G_t = P_t \exp tA^0$ where int P_t is dense in P_t with respect to the topology of $\mathbf{G}_0$, and $P_{t_2} \supset P_{t_1}$ for $t_2 > t_1 > 0$, i.e., if $p \in P_{t_1}$ then $p = \pi(e, \mathbf{u}, t_1) \cdot \exp - t_1 A^0$ and if we set $\mathbf{v}(t) = 0$ for $0 \leqslant t \leqslant t_2 - t_1$ then $\pi(e, \mathbf{v}, t_2) = p \cdot \exp t_2 A^0$. Similarly int $P_{t_2} \supset$ int P_{t_1} for $t_2 > t_1 > 0$.

Claim. cl P_t = cl $\mathbf{G}_0$, i.e., int P_t is dense in $\mathbf{G}_0$: First note that for all $t \in \mathbf{R}^+$, $\mathbf{B} \in$ cl P_t since $\mathbf{B} \cdot \exp tA^0 \subset$ cl G_t by assumption. Consider the following system on $\mathbf{G}$,

$$\frac{dx}{dt} = A^0(x) + u_1 B_1{}^0(x) + \cdots + u_m B_m{}^0(x); \qquad x(0) = e, \tag{*}$$

where $u_i \in \mathscr{P}^{(1)}$; $A^0, B_1{}^0, \ldots, B_m{}^0 \in \mathscr{L}(\mathbf{G})$ and $\{B_1{}^0, \ldots, B_m{}^0\}$ is a basis for $\mathscr{L}(\mathbf{B})$. Let $(\mathscr{L}^0, \mathscr{L}_0{}^0, \mathscr{B}^0)$ be the triple of Lie algebras associated with this system. Then $\mathscr{L}^0 = \mathscr{L}(\mathbf{G})$, $\mathscr{L}_0{}^0 = \mathscr{L}(\mathbf{G}_0)$ and $\mathscr{B}^0 = \mathscr{L}(\mathbf{B})$ as a direct consequence of the definitions, and $\mathscr{R}(A^0, \mathscr{B}^0) = \mathscr{L}_0{}^0$ since $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$ and $\mathscr{L}$ is isomorphic to $\mathscr{L}(\mathbf{G}) = \mathscr{L}^0$. Let $\mathscr{R}^0(e)$ denote the reachable set from e for the system (*). Theorem 2.3 implies that $\mathscr{R}^0(e) = \mathbf{G}_0 \exp tA^0$. Letting

$$Q_t = \{(b_1 \exp t_1 A) \cdots (b_n \exp t_n A) \colon b_i \in \mathbf{B}, t_i \in \mathbf{R}^+, t_1 + \cdots + t_n = t\},$$

it follows that $Q_t \subset$ cl G_t. Lemma 3.1 implies that cl Q_t = cl $\mathscr{R}^0(e)$ = cl $\mathbf{G}_0 \cdot \exp tA^0 \subset$ cl $G_t \subset$ cl $\mathbf{G}_0 \cdot \exp tA^0$. Thus cl G_t = (cl P_t) $\exp tA^0$ = (cl $\mathbf{G}_0$) $\exp tA^0$ and cl P_t = cl $\mathbf{G}_0$. Thus cl(int P_t) = cl $\mathbf{G}_0$.

We complete the proof by showing that cl P_t = cl $\mathbf{G}_0$ implies that $P_t = \mathbf{G}_0$.

We begin by forming $\mathbf{H} = \mathbf{G}_0 \times_\eta \mathbf{R}$, the semidirect product of $\mathbf{G}_0$ and $\mathbf{R}$, where $(g_0, r_0) \cdot (g_1, r_1) = (g_0(\exp r_0 A^0) g_1(\exp -r_0 A^0), r_0 + r_1)$. With this multiplication $\mathbf{H}$ is a Lie group (cf. [11]) and $S_0 = \{(p, t): p \in \operatorname{int} P_t, t \in \mathbf{R}^+\} \subset \mathbf{H}$ is a subsemigroup of $\mathbf{H}$. If we give S_0 the relative topology from $\mathbf{H}$ then S_0 becomes a topological semigroup, a semigroup with a continuous multiplication. If we let $S = \{(p, t): p \in \mathbf{G}_0, t \in \mathbf{R}^+\}$, then $S_0 \subset S \subset \mathbf{H}$, S_0 and S are open subsemigroups in $\mathbf{H}$, $\operatorname{cl} S_0 = \operatorname{cl} S$ as $\operatorname{cl}(\operatorname{int} P_t) = \operatorname{cl} \mathbf{G}_0$, and $(e, 0)$, the identity in $\mathbf{H}$, is contained in $\operatorname{cl} S_0$. There is a result for topological semigroups which states that if P is an open subsemigroup in a topological group J and the identity of J is contained in $\operatorname{cl} P$, then $\operatorname{int}(\operatorname{cl} P) = P$ (cf. [6, 10]). Thus $\operatorname{int}(\operatorname{cl} S_0) = S_0 = \operatorname{int}(\operatorname{cl} S) = S$ which implies that $P_t = \mathbf{G}_0$ for all $t \in R^+$. Thus for all $t \in \mathbf{R}^+$, $G_t = P_t \exp tA^0 = \mathbf{G}_0 \cdot \exp tA^0$. Q.E.D.

COROLLARY. *Consider the nonlinear system on* M

$$\frac{dx}{dt} = A(x(t)) + u_1(t)\, B_1(x(t), + \cdots + u_m(t)\, B_m(x(t)),$$

where $u_i(\cdot) \in \mathscr{P}^{(1)}$, $A, B_1, \ldots, B_m$ *are complete vector fields in* $V(M)$, *and* $\{A, B_1, \ldots, B_m\}_{LA}$ *is finite dimensional. Associated with this system is the triple of finite dimensional Lie algebras* $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$, *connected Lie transformation groups* $(\mathbf{G}, \mathbf{G}_0, \mathbf{B})$, *and the Lie subalgebra* $\mathscr{R}(A; \mathscr{B})$ *of* $\mathscr{L}_0$. *Then* $\mathscr{B} = \{B_1, \ldots, B_m\}_{LA}$, $\mathscr{L} = \{A, B_1, \ldots, B_m\}_{LA}$, *and if* $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$, *then for all* $x \in M$, $t \in \mathbf{R}^+$,

$$G_t \cdot x = \mathscr{R}_t(x) = \mathbf{G}_0 \exp tA^0 \cdot x,$$

where A^0 *is the isomorphic image of* A *in* $\mathscr{L}(\mathbf{G})$.

Proof. If $\{A, B_1, \ldots, B_m\}_{LA}$ is finite dimensional, then for all $u_1, u_2, \ldots, u_m \in \mathbf{R}$ $A + u_1 B_1 + \cdots + u_m B_m$ is a complete vector field as a consequence of Theorem 3.1. Thus

$$\mathscr{L} = \{A + u_1 B_1 + \cdots + u_m B_m : u_i \in \mathbf{R}\}_{LA} = \{A, B_1, \ldots, B_m\}_{LA}$$

is finite dimensional. That $\mathscr{B} = \{B_1, \ldots, B_m\}_{LA}$ follows directly from the definitions. The second assertion follows from the observation that if $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$, then this system satisfies the conditions of Theorem 3.3. Q.E.D.

Remark. Theorem 3.3 and its Corollary remain valid in the case where $u \in U$, where $\mathscr{U}$ is any class of controls which includes all of the piecewise constant controls, i.e., $\mathscr{P}^{(m)} \subset \mathscr{U}$.

EXAMPLE. Consider the nonlinear system on $\mathbf{R}^4$:

$$\frac{dx}{dt}(t) = A(x(t)) + u_1(t)\, B_1(x(t)) + u_2 B_2(x(t)),$$

where for all

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbf{R}^4, \qquad A(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ 1 \\ x_1 \\ x_2 \end{pmatrix}, \qquad B_1(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$B_2(\mathbf{x}) = \begin{pmatrix} 4 \\ 0 \\ -2x_2 \\ 0 \end{pmatrix}.$$

It is straigthforward to verify that A, B_1 and B_2 are complete vector fields and that

$$ad_A B_1(\mathbf{x}) = \begin{pmatrix} 4x_2 \\ 0 \\ -x_2^2 \\ 1 \end{pmatrix}, \qquad ad_A^2 B_1(\mathbf{x}) = \begin{pmatrix} 4 \\ 0 \\ -6x_2 \\ 0 \end{pmatrix},$$

$$B_3(\mathbf{x}) = [B_1, B_2](\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix},$$

$$ad_A B_3 = [ad_A B_1, B_3] = [ad_A B_1, B_2] = [B_1, B_3] = [B_2, B_3] = 0,$$
$$ad_A B_2 = [ad_A^2 B_1, B_1] = -3B_3, \qquad [ad_A B_1, B_1] = B_2,$$

and

$$ad_A^3 B_1 = -5B_3.$$

Thus if $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$ is the triple of Lie algebras associated with this system then $\mathscr{L}$ has a basis $\{A, B_1, B_2, B_3, ad_A B_1, ad_A^2 B_1\}$, $\mathscr{B}$ has a basis $\{B_1, B_2, B_3\}$, and $\mathscr{L}_0$ has a basis $\{B_1, B_2, B_3, ad_A B_1, ad_A^2 B_1\}$. The bracket structure of $\mathscr{L}$, which is displayed above, implies that $\mathscr{B}$ is an ideal in $\mathscr{L}_0$. Thus $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$ and since $\mathscr{L}$ is finite dimensional this system satisfies the conditions of the Corollary to Theorem 3.3 and $G_t \cdot x_0 = \mathscr{R}_t(\mathbf{x}_0) = \mathbf{G}_0 \exp \cdot tA^0\mathbf{x}_0 = \mathbf{G}_0 \cdot \mathbf{z}_0$ for $\mathbf{z}_0 \in \mathbf{R}^4$. It is easy to see that $\mathbf{G}_0 \cdot \mathbf{z}_0 = \mathbf{R}^4$; for all $\mathbf{z}_0 = (z_1, z_2, z_3, z_4)$

and $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4$ the integral curve through $\mathbf{z}_0$ for $ad_A B_1$ at time $(y_4 - z_4)$ is $\mathbf{p} = (p_1, p_2, p_3, y_4) \in \mathbf{R}^4$. The integral curve through $\mathbf{p}$ for B_1 at time $(p_2 - y_2)$ is $\mathbf{q} = (q_1, y_2, q_3, y_4)$; the integral curve through q for B_2 at time $(y_1 - q_1)/4$ is $\mathbf{r} = (y_1, y_2, r_3, y_4)$; and the integral curve through $\mathbf{r}$ for B_3 at time $(y_3 - r_3)/2$ is $(y_1, y_2, y_3, y_4) = \mathbf{y}$. Thus for all $\mathbf{x}_0 \in \mathbf{R}^4$, $t \in \mathbf{R}^+$, $\mathscr{R}_t(\mathbf{x}_0) = G_t \cdot \mathbf{x}_0 = \mathbf{G}_0 \cdot \exp tA^0 \cdot \mathbf{x}_0 = \mathbf{R}^4$.

4. Linear Systems With Multiplicative Controls

In this section we will consider an application of Theorem 3.3 to the controllability problem for a class of linear systems with multiplicative controls. The following result is due to Brockett [9]. Consider the system on $\mathbf{R}^n$:

$$\frac{dx}{dt}(t) = (A + u_1(t) B_1 + \cdots + u_m(t) B_m)\, x(t) + F\mathbf{v}(t); \qquad x(0) = \mathbf{0},$$

where $A, B_1, ..., B_m$ are n by n matrices over $\mathbf{R}$, F is an n by l matrix over $\mathbf{R}$ and $u_i(\cdot) \in \mathscr{P}^{(1)}$, $\mathbf{v}(\cdot) = (v_1(\cdot), ..., v_l(\cdot)) \in \mathscr{P}^{(l)}$. Let $\mathbf{f}_i \in \mathbf{R}^n$ be the ith column of F and set $\mathscr{L} = \{A, B_1, ..., B_m\}_{LA}$ and

$$\mathscr{L}^i = \{L_{k_1} L_{k_2} \cdots L_{k_i} : L_{k_j} \in \mathscr{L}, \text{ for } j = 1, ..., l\}_{LS}.$$

Then

$$\mathscr{R}_t(\mathbf{0}) = \{\mathscr{L}^i f_j : i = 0, 1, ... \text{ and } j = 1, ..., l\}_{LS}.$$

For the case where $B_1 = \cdots = B_m = 0$ $\mathscr{R}_t(\mathbf{0}) = \operatorname{range}(F, AF, ..., A^{n-1}F)$ which is the usual result for linear system (cf. [8]). This result, which can be proved using Theorem 3.3, depends on the fact that $\mathscr{R}_t(\mathbf{0})$ is a vector space when $x(0) = \mathbf{0}$ (cf. [6]). The case where $x(0) \neq \mathbf{0}$ is treated by the corollary to the following theorem:

Theorem 4.1. *Consider the nonlinear system on M*

$$\frac{dx}{dt} = A(x) + \sum_{i=1}^{m} u_i(t)\, B_i(x) + \sum_{i=1}^{l} v_i(t)\, F_i(x),$$

where $A, B_1, ..., B_m, F_1, ..., F_l$ are complete vector fields on M and $u_i(\cdot)$, $v_i(\cdot) \in \mathscr{P}^{(1)}$. Associated with this system is the triple of Lie algebras $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$, where $\mathscr{B} = \{B_1, ..., B_m, F_1, ..., F_l\}_{LA}$, $\mathscr{L} = (A, \mathscr{B})_{LA}$, and $\mathscr{L}_0$ is the ideal generated by $\mathscr{B}$ in $\mathscr{L}$. Consider the Lie algebras $(\tilde{\mathscr{L}}, \tilde{\mathscr{L}}_0, \tilde{\mathscr{B}})$, where $\tilde{\mathscr{B}} =$

$\{B_1, \ldots, B_m\}_{LA}$, $\tilde{\mathscr{L}} = \{A, \mathscr{B}\}_{LA}$, *and* $\mathscr{L}_0$ *is the ideal generated by* $\tilde{\mathscr{B}}$ *in* $\tilde{\mathscr{L}}$. *Suppose* $\mathscr{L}$ *is finite dimensional,* $\mathscr{R}(A; \tilde{\mathscr{B}}) = \tilde{\mathscr{L}}_0$, *and*

$$[ad_{X_1} ad_{X_2} \cdots ad_{X_n} F_i, F_j] = 0, \qquad \text{for} \quad X_i \in \{A, B_1, \ldots, B_m\},$$
$$i, j = 1, 2, \ldots, l \text{ and } n = 0, 1, \ldots .$$

Then for all $x \in M$, $t \in \mathbf{R}^+$,

$$\mathscr{R}_t(x) = G_t \cdot x = I^t(\mathscr{L}_0, x).$$

Proof. We will show that $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$. This theorem then follows directly from Theorem 3.3. Set

$$\mathscr{Q}_0 = \{ad_{X_1} ad_{X_2} \cdots ad_{X_n} F_m : n = 0, 1, \ldots; m = 1, \ldots, l; X_i \in \tilde{\mathscr{B}}\}_{LS}.$$

Then $\mathscr{Q}_0 \subset \mathscr{B}$ and if $\tilde{\mathscr{Q}}_0$ is the ideal generated by $\mathscr{Q}_0$ in $\{\mathscr{Q}_0, A\}_{LA}$, $[\mathscr{Q}_0, \tilde{\mathscr{Q}}_0] = 0$. Thus $\mathscr{Q}_0 \subset \tilde{\mathscr{Q}}_0 \subset \{\mathscr{Q}_0, A\}_{LA}$ is an A-chain from $\mathscr{B}$. Set $\mathscr{B}_1 = \{\mathscr{B}, \tilde{\mathscr{Q}}_0\}_{LA}$ and

$$\mathscr{Q}_1 = \{ad_{X_1} \cdots ad_{X_n} ad_A{}^k ad_{Y_1} \cdots ad_{Y_m} F_j : n, m, k = 0, 1, \ldots; X_i, Y_i \in \tilde{\mathscr{B}},$$
$$\text{and } j = 1, 2, \ldots, l\}_{LS}.$$

If $\tilde{\mathscr{Q}}_1$ is the ideal generated by $\mathscr{Q}_1$ in $\{\mathscr{Q}_1, A\}_{LA}$, then $\tilde{\mathscr{Q}}_1$ is an Abelian Lie algebra as a consequence of the hypothesis on the bracket structure of $\mathscr{L}$, and $\mathscr{Q}_1 \subset \tilde{\mathscr{Q}}_1 \subset \{\mathscr{Q}_1, A\}_{LA}$ is an A-chain from $\mathscr{B}_1$. We set $\mathscr{B}_2 = \{\mathscr{B}_1, \tilde{\mathscr{Q}}_1\}_{LA}$ and continue this process until $\mathscr{B}_n = \mathscr{B}_{n+1}$ for some positive integer n, whose existence is guaranteed by the finite dimensionality of $\mathscr{L}$. There exists an A-series from $\tilde{\mathscr{B}}$ terminating at $\tilde{\mathscr{L}}_0$, $\tilde{\mathscr{B}} \subset \tilde{\mathscr{B}}_1 \subset \tilde{\mathscr{B}}_2 \subset \cdots \subset \tilde{\mathscr{B}}_m = \tilde{\mathscr{L}}_0$, as $\mathscr{R}(A; \tilde{\mathscr{B}}) = \tilde{\mathscr{L}}_0$ by assumption. Setting $\mathscr{B}_{n+i} = \{\tilde{\mathscr{B}}_i, \mathscr{B}_n\}_{LA}$ it follows that $\mathscr{B} \subset \mathscr{B}_1 \subset \cdots \subset \mathscr{B}_n \subset \mathscr{B}_{n+1} \subset \cdots \subset \mathscr{B}_{n+m} = \mathscr{L}_0$ is an A-series from $\mathscr{B}$ terminating at $\mathscr{L}_0$; hence $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$ and the proof is complete. Q.E.D.

COROLLARY. *Consider the system on* $\mathbf{R}^n$

$$\frac{dx}{dt}(t) = (A + u_1(t) B_1 + \cdots + u_m(t) B_m) x(t) + F\mathbf{v}(t),$$

where $A, B_1, \ldots, B_m$ *are* n *by* n *matrices over* $\mathbf{R}$, $F = (\mathbf{f}_1, \ldots, \mathbf{f}_l)$ *where* $\mathbf{f}_i \in \mathbf{R}^n$ *is the* ith *column of* F, *and* $u_i(\cdot) \in \mathscr{P}^{(1)}$, $\mathbf{v}(\cdot) = (v_1(\cdot), \ldots, v_l(\cdot)) \in \mathscr{P}^{(l)}$. *Associated with this system is the triple of Lie algebras* $(\mathscr{L}, \mathscr{L}_0, \mathscr{B})$ *where* $\mathscr{B} = \{B_1, \ldots, B_m\}_{LA}$, $\mathscr{L} = \{\mathscr{B}, A\}_{LA}$ *and* $\mathscr{L}_0$ *is the ideal generated by* $\mathscr{B}$ *in* $\mathscr{L}$. *Suppose* $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$. *Then for all* $x \in \mathbf{R}^n$, $t \in \mathbf{R}^+$,

$$\mathscr{R}_t(x) = e^{tA}\{e^{\mathscr{L}_0}\}_G \mathbf{x} + \{\mathscr{L}^i \mathbf{f}_j : i = 0, 1, \ldots \text{ and } j = 1, \ldots, l\}_{LS}.$$

Proof. Let

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{F}_i = \begin{pmatrix} 0 & \mathbf{f}_i \\ 0 & 0 \end{pmatrix},$$

be a collection of $n + 1$ by $n + 1$ matrices over $\mathbf{R}$. It is easily verified by direct computation that the system

$$\frac{dX}{dt}(t) = (\tilde{A} + u_1(t)\tilde{B}_1 + \cdots + u_m(t)\tilde{B}_m + v_1(t)\tilde{F}_1 + \cdots + v_l(t)\tilde{F}_l)X(t),$$

satisfies the conditions of Theorem 4.1. Thus for this bilinear system the reachable set from the identity matrix I_{n+1} at time t is

$$\tilde{\mathscr{R}}_t(I_{n+1}) = e^{t\tilde{A}}\{e^{\tilde{\mathscr{L}}_0}\}_G = I^t(\tilde{\mathscr{L}}_0, I_{n+1}).$$

By direct computation

$$\tilde{\mathscr{L}}_0 = \left\{\begin{pmatrix} \mathscr{L}_0 & \mathscr{L}^i\mathbf{f}_j \\ 0 & 0 \end{pmatrix} : i = 0, 1,\dots \text{ and } j = 1, 2,\dots, l\right\}_{LS}.$$

Thus if $\pi_n: (x_1, \dots, x_{n+1}) \to (x_1, \dots, x_n)$ is the projection from $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$,

$$\begin{aligned} \mathscr{R}_t(\mathbf{x}_0) = \pi_n\left(\tilde{\mathscr{R}}_t(I_{n+1})\begin{pmatrix}\mathbf{x}_0 \\ 1\end{pmatrix}\right) &= e^{tA}\{e^{\mathscr{L}_0}\}_G\,\mathbf{x}_0 \\ &+ \{\mathscr{L}^i\mathbf{f}_j : i = 0, 1,\dots \text{ and } j = 1,\dots, l\}_{LS}. \end{aligned}$$ Q.E.D.

Theorem 4.1 is a generalization of Brockett's Theorem 7 of [1]. For the case where $B_1 = \cdots = B_m = 0$ we have $\mathscr{L}_0 = \mathscr{B} = \{0\}$, $\mathscr{R}(A; \mathscr{B}) = \mathscr{L}_0$, and the corollary states that

$$\begin{aligned} \mathscr{R}_t(x) &= e^{tA}x + \{A^i\mathbf{f}_j : i = 0, 1,\dots \text{ and } j = 1,\dots, l\}_{LS} \\ &= e^{tA}x + \operatorname{range}(F, AF,\dots, A^{n-1}F), \end{aligned}$$

which is the usual result for linear systems (cf. [8]).

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References

1. R. W. Brockett, System theory on group manifolds and coset spaces, *SIAM J. Contr.* **10** (May 1972).
2. H. J. Sussman and V. Jurdjevic, Controllability of nonlinear systems, *J. Diff. Eq.* **12** (July 1972).
3. C. Lobry, Geometrical structure of orbits of dynamical polysystems, Control Theory Center, Univ. of Warwick, R 19, July 1972.
4. R. W. Warner, "Foundations of Differentiable Manifolds and Lie Groups," Scott, Foresman, Glenview, Illinois, 1971.
5. C. Lobry, Contrôlabilité des systèmes non linéaires, *SIAM J. Contr.* **8**, 573–605 (1970).
6. R. M. Hirschorn, Topological semigroups and controllability in bilinear systems, Ph.D. Thesis, Div. of Eng. and Appl. Phys., Harvard Univ., Sept. 1973.
7. R. S. Palais, A global formulation of the Lie theory of transformation groups, Mem. of the Amer. Math. Soc. **22** (1957).
8. R. W. Brockett, "Finite Dimensional Linear Systems," Wiley, New York, 1970.
9. R. W. Brockett, Control theory on Lie groups, Proc. NATO Advanced Study Institute, Imperial College, London, 1973.
10. R. M. Hirschorn, Topological semigroups, sets of generators, and controllability, *Duke Math. J.* (Dec. 1973).
11. G. Hochschild, "The Structure of Lie Groups," Holden–Day, San Francisco, 1965.
12. H. J. Sussman and V. Jurdjevic, Control systems on Lie groups, *J. Diff. Eq.* (Sept. 1972).