EE290T (3D Image Processing and Computer Vision)

Lecture6 (10/06/2009)

Outline:

- 1) Circular points and their duals
- 2) Angle on the projective plane
- 3) Recovery of metric properties from images
- 4) Properties of conics
- 5) Fixed points and lines

1) Circular Points

There is exactly two points on l_∞ which are fixed under any similarity transformation; these are circular points I, J with canonical coordinates $\mathrm{I} = (1,\mathrm{i},0)^{\mathrm{T}}$ and $\mathrm{J} = (1,\text{-i},0)^{\mathrm{T}}$. To verify this observe that (with similar argument for J):

$$I' = H_s I = \begin{bmatrix} s\cos(\theta) & -s\sin(\theta) & t_x \\ s\sin(\theta) & s\cos(\theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = I$$

The name "circular points" arises because every circle intersects l_∞ at the circular points. To see this note that a conic is a circle if a=c and b=0 (with a=1); then

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

This conic intersects l_∞ in the points for which $x_3 = 0$ (line at infinity), namely

$$x_1^2 + x_2^2 = 0$$

with solution $\mathrm{I\,I} = (1,\mathrm{i},0)^{\mathrm{T}}$ and $\mathrm{J} = (1,\text{-i},0)^{\mathrm{T}}$, i.e. any circle intersects l_∞ in the circular points.

The conic dual to the circular points: The conic

$$C_\infty^* = \mathbf{I}\mathbf{J}^{\mathsf{T}} + \mathbf{J}\mathbf{I}^{\mathsf{T}}$$

is dual to the circular points which is a degenerate (rank 2) line conic. In Euclidian coordinate system it is given by:

$$C_\infty^* = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, under the point transformation $x' = H_s x$ we have $C_\infty^{*'} = H_s C_\infty^* H_s^T = C_\infty^*$ (*: adjoint); hence:

- The dual conic C_∞^* is fixed under the projective transformation H iff H is a similarity.

Also, l_∞ is the null vector of C_∞^*.

2) Angles on the projective plane

The Euclidian angle between two lines l and m is computed by

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

or equivalently by equation below (Equ.1):

$$\cos\theta = \frac{\mathbf{l}^\mathsf{T} \mathrm{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T} \mathrm{C}_\infty^* \mathbf{l})(\mathbf{m}^\mathsf{T} \mathrm{C}_\infty^* \mathbf{m})}}$$

Therefore, once you know C_∞^*, use the equation above in order to measure Euclidean angles. We can also measure length rations $\frac{d(b,c)}{d(a,c)} = \frac{\sin(\alpha)}{\sin(\beta)}$ with the following procedure:

Assuming, C_∞^* is known, using Equ.1 both $\cos(\alpha)$ and $\cos(\beta)$ are computed from the lines $l' = a' \times b', m' = c' \times a', n' = b' \times c'$ for any projective frame in which C_∞^* is specified. Consequently, the ratio is computed by $\frac{d(b,c)}{d(a,c)} = \frac{\sin(\alpha)}{\sin(\beta)}$ (Fig.1).

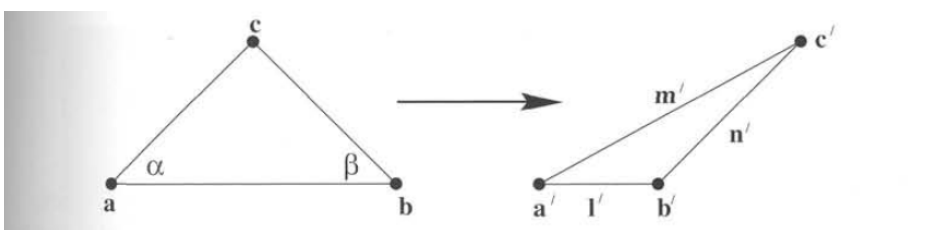

Figure 1: Fig. 2.16. **Length ratios.** *Once C_∞^* is identified the Euclidean length ratio $d(\mathbf{b},\mathbf{c}) : d(\mathbf{a},\mathbf{c})$ may be measured from the projectively distorted figure. See text for details.*

3) Recovery of metric properties from images

We can also recover metric properties from a image of a plane by transforming the circular points to their canonical position. The procedure is as following:

- Identify circular points of an image
- Find transformation H that maps the imaged circular points to their canonical positions $(1, \pm i, 0)^T$
- Rectify the rest of the image with H.

Note that, the dual conic C_∞^* packages all the information required for a metric rectification. It enables both projective and affine components of a projective transformation to be determined up to a similarity transformation.

Now the question is if I have a point transformation $x' = Hx = (H_P H_A H_S)x$,how does C_∞^* gets transformed under this transformation. The answer is:

$$\begin{aligned}
C_\infty^{*}{}' &= (H_P\, H_A\, H_S)\, C_\infty^* \,(H_P\, H_A\, H_S)^T = (H_P\, H_A) \left(H_S\, C_\infty^* H_S^T\right) \left(H_A^T\, H_P^T\right) \\
&= (H_P\, H_A)\, C_\infty^* \left(H_A^T\, H_P^T\right) \\
&= \begin{bmatrix} KK^T & KK^T\mathbf{v} \\ \mathbf{v}^T KK^T & \mathbf{v}^T KK^T \mathbf{v} \end{bmatrix}.
\end{aligned}$$

where **V** is projective component and **K** is affine component. Therefore:

- Once the conic C_∞^* is identified on the projective plane then projective distortion may be rectified up to a similarity (i.e. C_∞^* enables us to determine projective and affine components of our transformation, but not the similarity).

Now suppose that we know the image of C_∞^* in our image plane; how we can find rectifying transformation to recover up to a similarity. Using SVD we can write C_∞^* as

$$C_\infty^{*}{}' = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

then by inspection the rectifying projectivity is H=U up to a similarity.

Example 1: Assume an image has been rectified (up to a similarity). Also, suppose the lines l' and m' in the affinely rectified image correspond to an orthogonal line pair l and m on the world plane (Fig.2); then using the facts $l'^T C_\infty^{*}{}' m' = 0$ and $C_\infty^* = \begin{bmatrix} KK^T = S & 0 \\ 0 & 0 \end{bmatrix}$ (**V**=0) we have:

$$\begin{pmatrix} l_1' & l_2' & l_3' \end{pmatrix} \begin{bmatrix} KK^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = 0$$

which is a linear constraint on S (a symmetric matrix with 3 independent elements and 2 DOF). Using two pairs of orthogonal lines, two linear constraints on S can be determined; consequently, S and K (by Cholesky decomposition) is determined up to a scale and C_∞^* is determined.

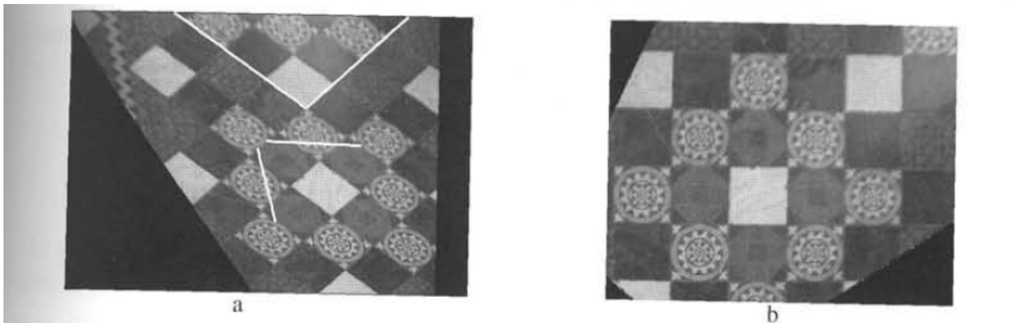

Fig. 2.17. **Metric rectification via orthogonal lines I.** *The affine transformation required to metrically rectify an affine image may be computed from imaged orthogonal lines. (a) Two (non-parallel) line pairs identified on the affinely rectified image (figure 2.13) correspond to orthogonal lines on the world plane. (b) The metrically rectified image. Note that in the metrically rectified image all lines orthogonal in the world are orthogonal, world squares have unit aspect ratio, and world circles are circular.*

Figure 2:

In this example first the projective and subsequently the affine distortions were removed. This two-step approach is termed ***Stratified***.

Example 2: Suppose lines l and m are images of orthogonal lines on the world plane, then $l^T C_\infty^* m = 0$. This provides a linear constraint on the elements of C_∞^*, as

$$(l_1 m_1, (l_1 m_2 + l_2 m_1)/2, l_2 m_2, (l_1 m_3 + l_3 m_1)/2, (l_2 m_3 + l_3 m_2)/2, l_3 m_3)\,\mathbf{c} = 0$$

where $C = (a,b,c,d,e,f)^T$ is the conic matrix of C_∞^* written as 6-vector (with 5 DOF). Having five such constraints using five pair of orthogonal lines (Fig.3) we can determine C_∞^*.

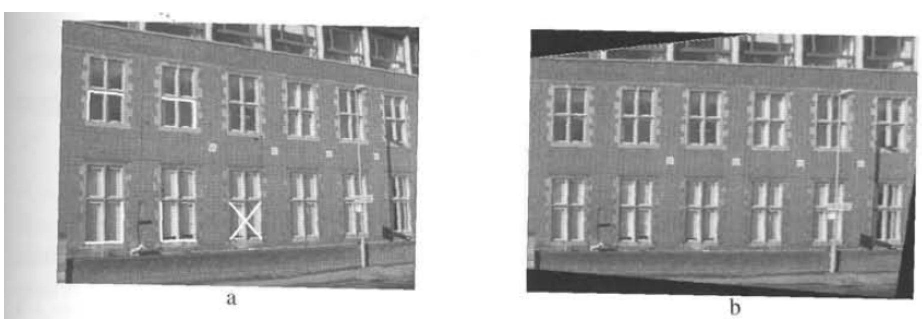

Fig. 2.18. **Metric rectification via orthogonal lines II.** *(a) The conic C_∞^* is determined on the perspectively imaged plane (the front wall of the building) using the five orthogonal line pairs shown. The conic C_∞^* determines the circular points, and equivalently the projective transformation necessary to metrically rectify the image (b). The image shown in (a) is the same perspective image as that of figure 2.4(p35), where the perspective distortion was removed by specifying the world position of four image points.*

Figure 3:

4) Properties of Conics

A point x and conic C define a line $l = Cx$. The line l is called the ***polar*** of x w.r.t. C, and the point x is the pole of l w.r.t. to C. The polar line l=Cx of the point x w.r.t. a conic C intersects the conic in two points. The two lines tangent to C at these points intersects at x; also, if the point x is on C then the polar is the tangent line to the conic at x (Fig.4).

Conic C induces a map between points and lines of P^2; this map between points and lines is a projective transformation called ***correlation*** which is an invertible mapping from points of P^2 to lines of P^2 and it is represented by a 3-by-3 non-singular matrix A as l= Ax (in fact, it provides systematic way to "dualize" relationship between points and lines). If the point y is on the line $l = Cx$ then $y^T l = y^T Cx = 0$. Any two points x, y satisfying $y^T Cx = 0$ are conjugate w.r.t. the conic C. If x is on polar of y then y is on the polar of x.

Since C is a symmetric matrix it has real eigenvalues, and may be decomposed as a product $C = U^T DU$. Suppose we apply the projective transformation U to conic C; then the new conic is $D = U^T CU^{-T}$. Therefore, any conic is equivalent under projective transformation to a conic with diagonal elements. Now apply $\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$ to D to get another conic $\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$ where $\varepsilon_i = \pm 1, 0$. Consequently, the various types of conics can be summarized in table 1.

Diagonal	Equation	Conic type
$(1,1,1)$	$x^2 + y^2 + w^2 = 0$	Improper conic – no real points.
$(1,1,-1)$	$x^2 + y^2 - w^2 = 0$	Circle
$(1,1,0)$	$x^2 + y^2 = 0$	Single real point $(0,0,1)^\mathsf{T}$
$(1,-1,0)$	$x^2 - y^2 = 0$	Two lines $x = \pm y$
$(1,0,0)$	$x^2 = 0$	Single line $x = 0$ counted twice.

Table 1: Table 2.2. **Projective classification of point conics.** *Any plane conic is projectively equivalent to one of the types shown in this table. Those conics for which $\epsilon_i = 0$ for some i are known as degenerate conics, and are represented by a matrix of rank less than 3. The conic type column only describes the real points of the conics – for example as a complex conic $x^2 + y^2 = 0$ consists of the line pair $x = \pm iy$.*

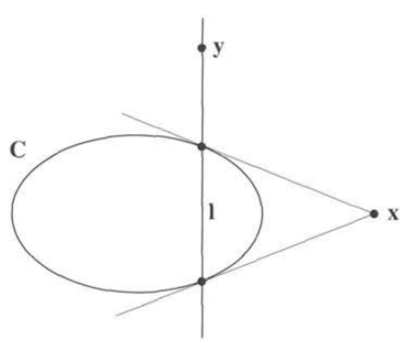

Figure 4: Fig. 2.19. **The pole–polar relationship.** *The line* $\mathrm{l} = \mathrm{Cx}$ *is the* polar *of the point* x *with respect to the conic* C, *and the point* $\mathrm{x} = \mathrm{C}^{-1}\mathrm{l}$ *is the* pole *of* l *with respect to* C. *The polar of* x *intersects the conic at the points of tangency of lines from* x. *If* y *is on* l *then* $\mathrm{y}^\mathsf{T}\mathrm{l} = \mathrm{y}^\mathsf{T}\mathrm{Cx} = 0$. *Points* x *and* y *which satisfy* $\mathrm{y}^\mathsf{T}\mathrm{Cx} = 0$ *are* conjugate.

5) Fixed points and lines

l_∞ and the circular points may be fixed under projective transformation. Lets assume we have projective transformation H (3-by3 matrix). Then, eigenvalues of H are its fixed *points* because $He = \lambda e$ where e and λe are the same point. A similar development can be given for *fixed lines* which corresponds to eigenvectors of H^T. Also, H has up to 3 fixed points if eigenvalues of H are distinct; and if $\lambda_2 = \lambda_3$, but e_2 and e_3 are distinct then the line containing the eigenvectors e_2 and e_3 will be fixed pointwise.