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High-gain observers in nonlinear feedback control

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SUMMARY

In this document, we present the main ideas and results concerning high-gain observers and some of their applications in control. The introduction gives a brief history of the topic. Then, a motivating second-order example is used to illustrate the key features of high-gain observers and their use in feedback control. This is followed by a general presentation of high-gain-observer theory in a unified framework that accounts for modeling uncertainty, as well as measurement noise. The paper concludes by discussing the use of high-gain observers in the robust control of minimum-phase nonlinear systems. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The use of high-gain observers in feedback control appeared first in the context of linear feedback as a tool for robust observer design. In their celebrated work on loop transfer recovery [1], Doyle and Stein used high-gain observers to recover, with observers, frequency-domain loop properties achieved by state feedback. The investigation of high-gain observers in the context of robust linear control continued in the 1980s, as seen in the work of Petersen and Hollot [2] on H_∞ control. The use of high-gain observers in nonlinear feedback control started to appear in the late 1980s in the works of Saberi [3, 4], Tornambe [5], and Khalil [6]. Two key papers, published in 1992, represent the beginning of two schools of research on high-gain observers. The work by Gauthier, Hammouri, and Othman [7] started a line of work that is exemplified by [8–13]. This line of research covered a wide class of nonlinear systems and obtained global results under global growth conditions. The work by Esfandiari and Khalil [14] brought attention to the peaking phenomenon as an important feature of high-gain observers. Although this phenomenon was observed earlier in the literature [15, 16], the paper [14] showed that the interaction of peaking with nonlinearities could induce finite escape time. In particular, it showed that, in the lack of global growth conditions, high-gain observers could destabilize the closed-loop system as the observer gain is driven sufficiently high. It proposed a seemingly simple solution for the problem. It suggested that the control should be designed as a globally bounded function of the state estimates so that it saturates during the peaking period. Because the observer is much faster than the closed-loop dynamics under state feedback, the peaking period is very short relative to the time scale of the plant variables, which

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remain very close to their initial values. Teel and Praly [17, 18] built on the ideas of Esfandiari and Khalil [14] and earlier work by Tornambe [19] to prove the first nonlinear separation principle and develop a set of tools for semiglobal stabilization of nonlinear systems. Their work drew attention to Esfandiari and Khalil [14], and soon afterwards, many leading nonlinear control researchers started using high-gain observers (cf. [20–43]). These papers have studied a wide range of nonlinear control problems, including stabilization, regulation, tracking, and adaptive control. They also explored the use of time-varying high-gain observers. Khalil and his coworkers continued to investigate high-gain observers in nonlinear feedback control for about 20 years converging a wide range of problems (cf. [44–59]). Atassi and Khalil [60] proved a separation principle that adds a new dimension to the result of Teel and Praly [17]; namely, the combination of fast observer with control saturation enables the output feedback controller to recover the trajectories of the state feedback controller as the observer gain is made sufficiently high.

To illustrate the key properties of high-gain observers, we start with a motivating example in Section 2. This is followed by a more general presentation of the theory in Section 3. The nonlinear separation principle is presented in Section 4. As an example of the use of high-gain observers in nonlinear feedback control, we discuss robust control of minimum-phase systems in Section 5

Warning : In order to keep this presentation not too obscure, we may take some liberties with rigor and precision. We refer the reader to the references for precise correct statements and proofs.

2. MOTIVATING EXAMPLE

Consider the second-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,u,w,d) \\ y &= x_1 \end{aligned} \tag{1}$$

where $x = [x_1, x_2]^T$ is the state vector, u is the control input, y is the measured output, d is a vector of disturbance inputs, and w is a vector of known exogenous signals. The function f is locally Lipschitz in (x,u) and continuous in (d,w). We assume that $d(t)$ and $w(t)$ are bounded measurable functions of time. Suppose the state feedback control $u = \gamma(x,w)$ stabilizes the origin $x = 0$ of the closed-loop system,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,\gamma(x,w),w,d) \end{aligned} \tag{2}$$

uniformly in (w,d), where $\gamma(x,w)$ is locally Lipschitz in x and continuous in w. To implement this feedback control using only measurements of the output y, we use the observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + h_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{f}(\hat{x},u,w) + h_2(y - \hat{x}_1) \end{aligned} \tag{3}$$

where $\hat{f}(x,u,w)$ is a model of $f(x,u,w,d)$, and take

$$u = \gamma(\hat{x},w) \ . \tag{4}$$

If f is a known function of (x,u,w), we can take $\hat{f} = f$. We may also take $\hat{f} = 0$ if no model of f is available. The estimation error

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix}$$

satisfies the equation

$$\begin{aligned} \dot{\tilde{x}}_1 &= -h_1\tilde{x}_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -h_2\tilde{x}_1 + \delta(x,\tilde{x},w,d), \end{aligned} \tag{5}$$

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where

$$\delta(x, \tilde{x}, w, d) = f(x, \gamma(\hat{x}, w), w, d) - \hat{f}(\hat{x}, \gamma(\hat{x}, w), w).$$

In the absence of δ, asymptotic error convergence is achieved when the matrix

$$\begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}$$

is Hurwitz, which is the case for any positive constants h_1 and h_2. In the presence of δ, we design h_1 and h_2 with the additional goal of rejecting the effect of δ on $\tilde{x}$. This is ideally achieved if the transfer function

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2} \begin{bmatrix} 1 \\ s + h_1 \end{bmatrix}$$

from δ to $\tilde{x}$ is identically zero. Although this is not possible, we can try to make $\sup_{\omega \in R} \|G_o(j\omega)\|$ arbitrarily small. Because we can rewrite

$$G_o(s) = \frac{\frac{1}{\sqrt{h_2}}}{\left(\frac{s}{\sqrt{h_2}}\right)^2 + \frac{h_1}{\sqrt{h_2}} \frac{s}{\sqrt{h_2}} + 1} \begin{bmatrix} \frac{1}{\sqrt{h_2}} \\ \frac{s}{\sqrt{h_2}} + \frac{h_1}{\sqrt{h_2}} \end{bmatrix},$$

this objective is met when the ratio $\frac{h_1}{\sqrt{h_2}}$ is chosen as some fixed positive real number, and we let h_2 go to infinity. This motivates us for taking

$$h_1 = \frac{\alpha_1}{\varepsilon}, \quad h_2 = \frac{\alpha_2}{\varepsilon^2} \tag{6}$$

for some positive constants α_1 and α_2, and with ε arbitrarily small. In this way, the observer eigenvalues are assigned at $1/\varepsilon$ times the roots of the polynomial $s^2 + \alpha_1 s + \alpha_2$. Therefore, by choosing ε small, we make the observer dynamics much faster than the dynamics of the closed-loop system under state feedback (2).

The disturbance rejection property of the high-gain observer, and its fast dynamics, can be also seen in the time domain by using the scaled estimation errors

$$\zeta_1 = \frac{\tilde{x}_1}{\varepsilon}, \quad \zeta_2 = \tilde{x}_2, \tag{7}$$

which satisfy the singularly perturbed equation

$$\begin{aligned} \varepsilon\dot{\zeta}_1 &= -\alpha_1\zeta_1 + \zeta_2 \\ \varepsilon\dot{\zeta}_2 &= -\alpha_2\zeta_1 + \varepsilon\delta(x, \tilde{x}, w, d). \end{aligned} \tag{8}$$

This equation shows that reducing ε diminishes the effect of δ and makes the dynamics of ζ much faster than those of x. However, the scaling (7) shows that the transient response of the high-gain observer suffers from a *peaking phenomenon*. The initial condition $\zeta_1(0)$ could be $O(1/\varepsilon)$ when $x_1(0) \neq \hat{x}_1(0)$. Consequently, the transient response of (8) could contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$ for some $a > 0$. Although this exponential mode decays rapidly, it exhibits an impulsive-like behavior where the transient peaks to $O(1/\varepsilon)$ values before it decays rapidly towards zero. In fact, the function $(1/\varepsilon)e^{-at/\varepsilon}$ approaches an impulse function as ε tends to zero. In addition to inducing unacceptable transient response, the peaking phenomenon could destabilize the closed-loop nonlinear system ([61, Section 14.6]). This phenomenon is an artifact of the high-gain observer. This being known, we should disregard the large, unrealistic values of the state estimate. To do so, we can design the control law $\gamma(\hat{x}, w)$ and the function $\hat{f}(\hat{x}, u, w)$ to be globally bounded in $\hat{x}$, that is, bounded for all $\hat{x}$ when w is bounded. This property can be always achieved by saturating u and/or $\hat{x}$ outside compact sets of interest. The global boundedness of γ and $\hat{f}$ in $\hat{x}$ provides a buffer that protects the plant from peaking because during the peaking period, the control $\gamma(\hat{x}, w)$

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saturates. Because the peaking period shrinks to zero as ε tends to zero, for sufficiently small ε, the peaking period is so small that the state of the plant x remains close to its initial value. After the peaking period, the estimation error becomes of the order $O(\varepsilon)$, and the feedback control $\gamma(\hat{x}, w)$ becomes $O(\varepsilon)$ close to $\gamma(x, w)$. Consequently, the trajectories of the closed-loop system under the output feedback controller approach its trajectories under the state feedback controller as ε tends to zero. This leads to recovery of the performance achieved under state feedback.

Let us now analyze the closed loop system we have obtained by designing the output feedback as a state feedback fed with state estimates given by a high-gain observer. We start by observing that this system can be represented in the singularly perturbed form

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x, \gamma(\hat{x}, w), w, d) \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} \varepsilon\dot{\zeta}_1 &= -\alpha_1\zeta_1 + \zeta_2 \\ \varepsilon\dot{\zeta}_2 &= -\alpha_2\zeta_1 + \varepsilon\delta(x, \tilde{x}, w, d) \end{aligned}\right\}, \tag{10}$$

where $\hat{x}_1 = x_1 - \varepsilon\zeta_1$ and $\hat{x}_2 = x_2 - \zeta_2$. The slow equation (9) coincides with the closed-loop system under state feedback (2) when $\zeta = 0$. The homogeneous part of the fast equation (10) is $\varepsilon\dot{\zeta} = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix}\zeta \stackrel{\text{def}}{=} A_0\zeta$. Let $V(x)$ be a Lyapunov function for the slow subsystem, which is guaranteed to exist for any stabilizing state feedback control by the converse Lyapunov theorem [61, Theorem 4.17]. Let $W(\zeta) = \zeta^T P_0 \zeta$ be a Lyapunov function for the fast subsystem, where P_0 is the solution of the Lyapunov equation $P_0 A_0 + A_0^T P_0^T = -I$. Define the sets Ω_c and Σ by $\Omega_c = \{V(x) \leqslant c\}$ and $\Sigma = \{W(\zeta) \leqslant \sigma\varepsilon^2\}$, where for any $c > 0$, Ω_c is in the interior of the region of attraction of (2). The analysis is divided in two steps. In the first step, we show that, for appropriately chosen σ, there is $\varepsilon_1^* > 0$ such that, for each $0 < \varepsilon < \varepsilon_1^*$, the origin of the closed-loop system is asymptotically stable, and the set $\Omega_c \times \Sigma$ is a positively invariant subset of the region of attraction. The proof makes use of the fact that ζ is $O(\varepsilon)$ in $\Omega_c \times \Sigma$. In the second step, we show that for any compact sets $C \subset \mathbb{R}^2$ and $\Omega_b = \{V(x) \leqslant b\}$, with $0 < b < c$, there is $\varepsilon_2^* > 0$ such that, for $0 < \varepsilon < \varepsilon_2^*$, $\hat{x}(0) \in C$ and $x(0) \in \Omega_b$, the trajectory enters the set $\Omega_c \times \Sigma$ in finite time. The proof makes use of the fact that Ω_b is in the interior of Ω_c and $\gamma(\hat{x}, w)$ is globally bounded. There exists $T_1 > 0$, independent of ε, such that any trajectory starting in Ω_b remains in Ω_c for all $t \in [0, T_1]$. Using the fact that ζ decays faster than an exponential mode of the form $(1/\varepsilon)e^{-at/\varepsilon}$, we can show that the trajectory enters the set $\Omega_c \times \Sigma$ within a time interval $[0, T(\varepsilon)]$ where $\lim_{\varepsilon\to 0} T(\varepsilon) = 0$. Thus, by choosing ε small enough, we can ensure that $T(\varepsilon) < T_1$. Figure 1 illustrates the fast convergence of the trajectories to the set $\Omega_c \times \Sigma$. Furthermore, because of the global boundedness of the right-hand side of (9) uniformly in ε, by choosing ε small enough, we can make the difference $|x(T(\varepsilon)) - x(0)|$ arbitrarily small. Using this together with the fact that for $t \geqslant T(\varepsilon)$ ζ is $O(\varepsilon)$, it can be shown that the trajectories of x under state and output feedback can be made arbitrarily close to each other for all $t \geqslant 0$.

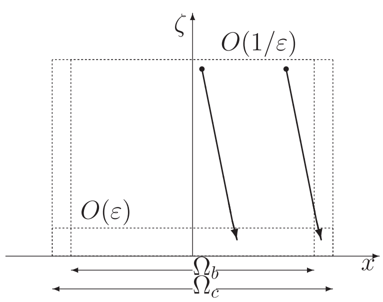

Figure 1. Illustration of fast convergence to the set $\Omega_c \times \Sigma$.

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The foregoing discussion shows that the design of the output feedback controller (4) is based on a separation procedure, whereby the state feedback controller is designed as if the whole state was available for feedback, followed by an observer design that is independent of the state feedback control. By choosing ε small enough, the output feedback controller recovers the stability and performance properties of the state feedback controller. This is the essence of the separation principle that is discussed in Section 3. The separation principle is known in the context of linear systems where the closed-loop eigenvalues under an observer-based controller are the union of the eigenvalues under state feedback and the observer eigenvalues; hence, stabilization under output feedback can be achieved by solving separate eigenvalue placement problems for the state feedback and the observer. Over the last two decades, there have been several results that present forms of the separation principle for classes of nonlinear systems. It is important to emphasize that the separation principle in the case of high-gain observers has a unique feature that does not exist in other separation-principle results, linear systems included, and that is the recovery of state trajectories by making the observer sufficiently fast. This feature has significant practical implications because it allows the designer to design the state feedback controller to meet transient response specification and/or constraints on the state or control variables. Then, by saturating the state estimate $\hat{x}$ and/or the control u outside compact sets of interest to make the functions $\gamma(\hat{x}, w)$ and $\hat{f}(\hat{x}, u, w)$ globally bounded in $\hat{x}$, he/she can proceed to tune the parameter ε by decreasing it monotonically to bring the trajectories under output feedback close enough to the ones under state feedback. This feature is achieved not only by making the observer fast but also by combining this property with the global boundedness of γ and $\hat{f}$ in $\hat{x}$. We illustrate this point by considering the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned} \tag{11}$$

which is a special case of (1) with $f = u$. A linear state feedback that assigns the eigenvalues at $-1 \pm j$ is given by $u = -2x_1 - 2x_2$. The observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + (3/\varepsilon)(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= u + (2/\varepsilon^2)(y - \hat{x}_1) \end{aligned} \tag{12}$$

is a special case of (3) with $\hat{f} = u$. It assigns the observer eigenvalues at $-1/\varepsilon$ and $-2/\varepsilon$. The observer-based controller assigns the closed-loop eigenvalues at $-1 \pm j$, $-1/\varepsilon$ and $-2/\varepsilon$. The closed-loop system under output feedback is asymptotically stable for all $\varepsilon > 0$. As we decrease ε, we make the observer dynamics faster than the closed-loop dynamics under state feedback. Will the trajectories of the system under output feedback approach those under state feedback as ε approaches zero? The answer is shown in Figure 2, where the state x is shown under state feedback and under output feedback for $\varepsilon = 0.1$ and 0.01. The initial conditions of the simulation are $x_1(0) = 1$ and $x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. Contrary to what intuition may suggest, we see that the trajectories under output feedback do not approach the ones under state feedback as ε decreases. In Figure 3, we repeat the same simulation when the control is saturated at ± 4; that is, $u = 4\mathrm{sat}((-2\hat{x}_1 - 2\hat{x}_2)/4)$. The saturation level 4 is chosen such that $4 > \max_\Omega |-2x_1 - 2x_2|$, where $\Omega = \left\{1.25x_1^2 + 0.5x_1x_2 + 0.375x_2^2 \leqslant 1.4\right\}$ is an estimate of the region of attraction under state feedback control that includes the initial state $(1, 0)$ in its interior. This choice of the saturation level saturates the control outside Ω. Figure 3 shows a reversal of the trend we saw in Figure 2. Now the trajectories under output feedback approach those under state feedback as ε decreases. This is a manifestation of the performance recovery property of high-gain observers when equipped with a globally bounded control. Figure 4 shows the control signal u with and without saturation during the peaking period for $\varepsilon = 0.01$. It demonstrates the role of saturation in suppressing the peaking phenomenon.

One of the challenges in the use of high-gain observers is the effect of measurement noise. This stems from the fact that the high-gain observer (3) is an approximate differentiator, which can be easily seen in the special case when $\hat{f} = 0$; for then the observer is linear and the transfer function

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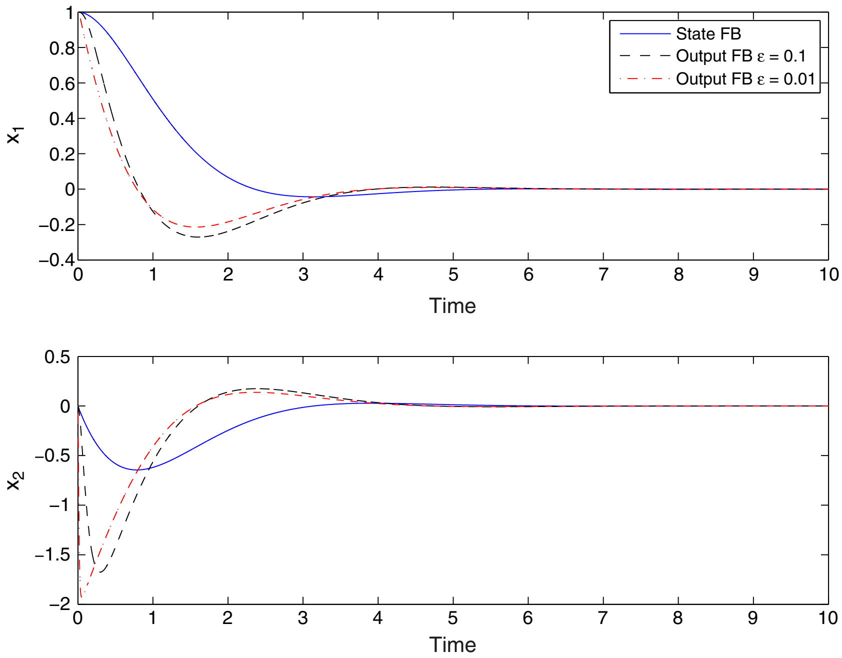

Figure 2. The state trajectories under state and output feedback for linear example without saturated control.

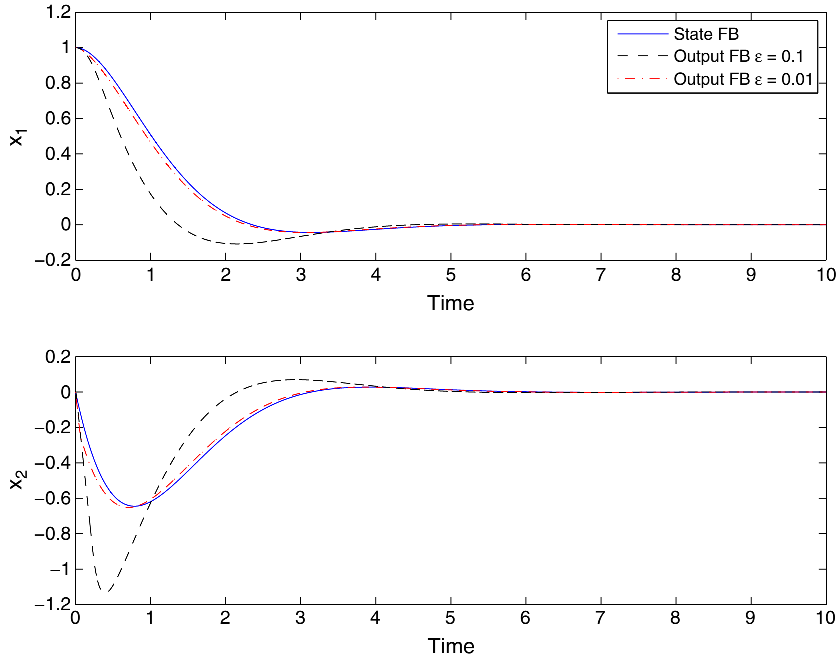

Figure 3. The state trajectories under state and output feedback for linear example with saturated control.

from y to $\hat{x}$ is given by

$$\frac{\alpha_2}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} 1 + (\varepsilon\alpha_1/\alpha_2)s \\ s \end{bmatrix} \xrightarrow{\varepsilon \to 0} \begin{bmatrix} 1 \\ s \end{bmatrix}.$$

In the presence of measurement noise, the output equation $y = x_1$ in (1) changes to $y = x_1 + v$. Before we get to the main issue of concern, let us note that if v is a low-frequency (slow) bounded signal, such as a constant bias in measurements, its effect can be handled by the state feedback

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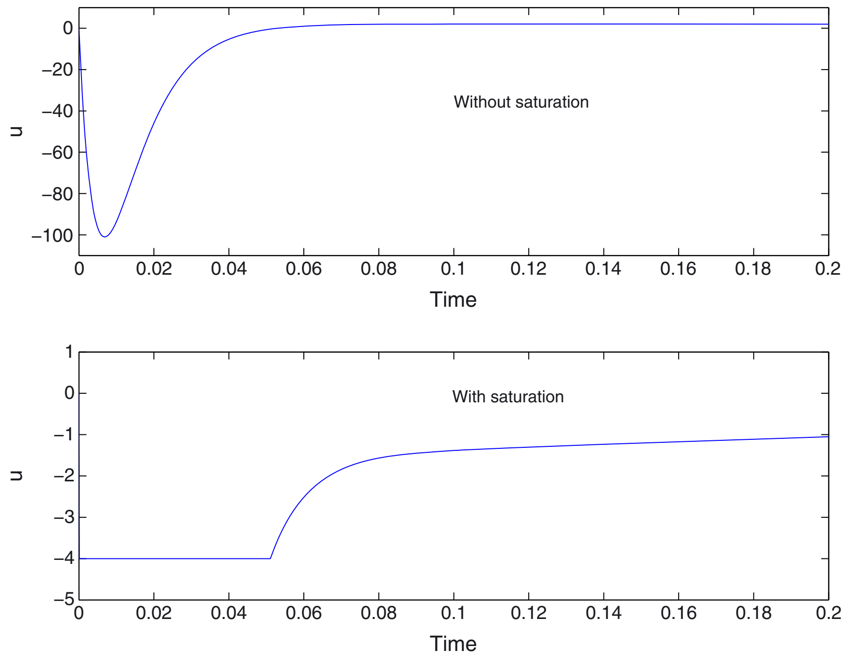

Figure 4. The control signal for the linear example with and without control saturation when $\varepsilon = 0.01$.

design. This can be done by redefining the state variables as the output and its derivative, that is, $z_1 = x_1 + v$ and $z_2 = x_2 + \dot{v}$, resulting in the state model

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= f(z_1 - v, z_2 - \dot{v}, u, w, d) + \ddot{v} \\
y &= z_1.
\end{aligned}$$

Provided the derivatives $\dot{v}$ and $\ddot{v}$ are appropriately bounded, their impact can be handled by the design of the feedback control. The main concern, however, is in the more typical case when measurement noise takes the form of a low-amplitude, high-frequency fluctuating signal. Differentiation of the output in this case leads to a major deterioration in the signal-to-noise ratio. Assuming that v is a bounded measurable signal, the closed-loop equation takes the form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x, \gamma(\hat{x}, w), w, d) \\
\varepsilon\dot{\zeta}_1 &= -\alpha_1\zeta_1 + \zeta_2 - (\alpha_1/\varepsilon)v \\
\varepsilon\dot{\zeta}_2 &= -\alpha_2\zeta_1 + \varepsilon\delta(x, \tilde{x}, w, d) - (\alpha_2/\varepsilon)v.
\end{aligned}$$

In this case, $\|x - \hat{x}\|$ satisfies an inequality of the form

$$\|x(t) - \hat{x}(t)\| \leqslant c_1\varepsilon + c_2\frac{\mu}{\varepsilon}, \qquad \forall\ t \geqslant T \tag{13}$$

for some positive constants c_1, c_2, and T, where $\mu = \sup_{t \geqslant 0} |v(t)|$. This ultimate bound, sketched in Figure 5, shows that the presence of measurement noise puts a lower bound on the choice of ε. For higher values of ε, we can reduce the steady-state error by reducing ε, but ε should not be reduced lower than $c_a\sqrt{\mu}$ because the steady-state error will increase significantly beyond this point. Another trade-off we face in the presence of measurement noise is the one between the steady-state error and the speed of state recovery. For small ε, ζ will be much faster than x. Fast convergence of ζ plays an important role in recovering the performance of the state feedback controller. The presence of measurement noise prevents us from making the observer as fast as we wish. How to let ε vary to improve performance is discussed in Section 3.2.3.

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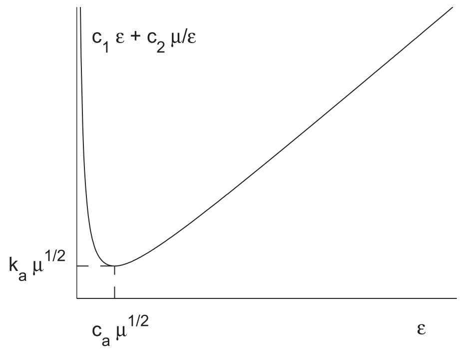

Figure 5. A sketch of $c_1\varepsilon + c_2\mu/\varepsilon$; $c_a = \sqrt{c_2/c_1}$, $k_a = 2\sqrt{c_1 c_2}$.

3. DIFFERENTIAL OBSERVABILITY AND HIGH-GAIN OBSERVERS

3.1. Differential observability of order m

The following paragraph is inspired by [62].

In the previous section, we have seen that high-gain observers provide a very 'natural' solution to the observer problem when the system dynamics is poorly known. Moreover, they can be appropriately combined with state feedback to give output feedback. Let us make clear now when such a solution is possible. For this, we consider the problem of estimating the n-dimensional state x of a dynamical system whose evolution with respect to time t is dictated by the following ordinary differential equation:

$$\dot{x} = f(x, t, u(t)), \tag{14}$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^n$ and $u : \mathbb{R} \to \mathbb{R}^p$ are sufficiently smooth known functions. The information we have for this estimation is the knowledge of the functions f and the value at each time t of $u(t)$ and of

$$y(t) \;=\; h(x, t, u(t)), \tag{15}$$

where $h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^q$ is a sufficiently smooth function.

We denote by $X(x, t, s; u)$ the solution of (14) at time s passing through x at time t and generated using the function u. The estimation problem at time t is, given the a posteriori information on some time window, that is, the function $s \in (t-T, t] \mapsto (u(s), y(s))$, and knowing the function f and h, find x possibly solution of :

$$y(s) \;=\; h(X(x, t, s; u), s, u(s)) \qquad \forall s \in (t-T, t]\,.$$

Assuming there is absolutely no error in the modeling, data acquisition, or whatever, we know that there exists at least one x solution to these equations. It is the one that created y. So the true issue is the uniqueness of this x, or in other words, do we have injectivity of the function

$$\mathcal{H}_t : x \quad \longmapsto \quad (s \in (t-T, t] \mapsto h(X(x, t, s; u), s, u(s)))\ ?$$

To study such a property, it is fruitful to consider the case where the length T of the observation time window is very small. Indeed in this case, we can write a Taylor expansion :

$$h(X(x, t, s; u), s, u(s)) \;=\; \sum_{i=0}^{m-1} h_i(x, t, \bar{u}_i(t))\, \frac{(s-t)^i}{i!} \;+\; o((s-t)^{m-1})$$

where

$$\bar{u}_i(t) \;=\; \left(u(t), \ldots, u^{(i)}(t)\right) \tag{16}$$

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and h_i is a function obtained recursively starting from

$$h_0(x,t,\bar{u}_0) = h(x,t,u)$$

and using

$$\begin{aligned} h_{i+1}(x,t,\bar{u}_{i+1}) &= \dot{\overbrace{h_i(x,t,\bar{u}_i)}} , \\ &= \frac{\partial h_i}{\partial x}(x,t,\bar{u}_i) f(x,t,u) + \frac{\partial h_i}{\partial t}(x,t,\bar{u}_i) + \frac{\partial h_i}{\partial \bar{u}_i}(x,t,\bar{u}_i)\dot{\bar{u}}_i . \end{aligned}$$

It follows that, if there exists an integer m such that, in some uniform way with respect to t, we have that the map

$$x \mapsto H_m(x,t,\bar{u}_{m-1}) = \begin{pmatrix} h_0(x,t,\bar{u}_0) \\ \vdots \\ h_{m-1}(x,t,\bar{u}_{m-1}) \end{pmatrix}$$

is injective, then we do have the injectivity of $\mathcal{H}_t$ for all $t > 0$. We say, in this case, the system is differentially observable of order m. It means that we can reconstruct x from the knowledge of y and u and their $m-1$ first derivatives :

$$x(t) = \Phi\left(t, \bar{y}_{m-1}(t), \bar{u}_{m-1}(t)\right) , \tag{17}$$

with the notation :

$$\bar{y}_{m-1}(t) = \left(y(t), \dots, y^{(m-1)}(t)\right) . \tag{18}$$

We should not be misled by the way (17) is written. The function Φ is not defined for all vectors $\bar{y}_{m-1}$ of dimension $m * q$. It is at most defined on $H_m(\mathbb{R}^n, t, \bar{u}_{m-1})$ only, that is, only when y is given by (15), and $y^{(i)}$ is its ith derivative using (14). This implies in particular that a rigorous, but heavy, writing of (17) is :

$$x(t) = \Phi\left(t, \left(h_0(x,t,\bar{u}_0(t)), \dots, h_{m-1}(x,t,\bar{u}_{m-1}(t))\right), \bar{u}_{m-1}(t)\right) .$$

To go on, we assume that we have chosen an extension[‡] Φ_e of Φ with $\mathbb{R} \times \mathbb{R}^{mq} \times \mathbb{R}^{mp}$ as domain of definition. Our main interest in this function follows from the fact that to each solution of (14) and (15), there corresponds at least one solution of :

$$\begin{gathered} \dot{\eta} = A_m \eta + E_m \varphi_{(m+1)e}(\eta, t, \bar{u}_m(t)) , \quad y(t) = E_1^T \eta(t) , \\ x(t) = \Phi_e\left(t, \eta(t), \bar{u}_{m-1}(t)\right) , \end{gathered} \tag{19}$$

with the notations

$$\varphi_{(m+1)e}(\eta, t, \bar{u}_m) = h_m\left(\Phi_e\left(t, \eta, \bar{u}_{m-1}\right), t, \bar{u}_m\right)$$

and

$$A_m = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & I \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} , \qquad E_i = \left.\begin{pmatrix} 0 \\ \vdots \\ I \\ \vdots \\ 0 \end{pmatrix} \qquad \right\} mq \text{ rows} .$$

[‡]See Tietze extension theorem, Kirszbaum extension theorem, and Whitney extension theorem.

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Actually, we can say more when f is affine in u, h does not depend on u, both do not depend on t, and $m = n$, $q = 1$, that is when we have

$$\dot{x} = a(x) + b(x)\,u \;, \quad y = h(x)\;. \tag{20}$$

Indeed in this case, we can find a function Φ, which does not depend on $(t, \bar{u})$ and $m+1$ functions φ_i satisfying

$$\begin{aligned} \varphi_1(h(x)) &= L_b h(x)\;, \\ \varphi_2(h(x), L_a h(x)) &= L_b L_a h(x)\;, \\ &\vdots \\ \varphi_m(h(x), L_a h(x), \ldots, L_a^{m-1} h(x)) &= L_b L_a^{m-1} h(x)\;, \\ \varphi_{m+1}\left(h(x), L_a h(x), \ldots, L_a^{m-1} h(x)\right) &= L_a^m h(x)\;, \\ x &= \Phi\left(h(x), L_a h(x), \ldots, L_a^{m-1} h(x)\right)\;. \end{aligned}$$

Again these functions φ_i are not defined on $\mathbb{R}, \mathbb{R}^2, \ldots$ but only on $h(\mathbb{R}^n)$, $(h(\mathbb{R}^n) \times L_a h(\mathbb{R}^n)), \ldots$. However, once we have chosen extensions φ_{ie} and Φ_e on $\mathbb{R}^i$ and $\mathbb{R}^n$, we get that to each solution of (20), there corresponds at least one solution of

$$\begin{aligned} \dot{\eta} &= A_m\,\eta + E_m\,\varphi_{(m+1)e}(\eta) + \sum_{i=1}^{m} E_i\,\varphi_{ie}(\eta)\,u\;, \\ y(t) &= E_1^T\,\eta(t)\;, \quad x(t) = \Phi_e\left(\eta(t)\right)\;. \end{aligned} \tag{21}$$

3.2. High-gain observer for the η variables

3.2.1. High-gain observer design The following paragraph is inspired by the many publications dealing with the almost disturbance decoupling problem in state observation. See [63, Theorem 13] or [64, Section 4.4] for instance.

With (19) or (21), we have made an important step towards the design of an observer for x. Indeed these two systems can be seen as linear systems disturbed by $\mathcal{NL}$, which collects all the nonlinearities, that is :

$$\dot{\eta} = A_m\,\eta + \mathcal{NL}(t, \eta, \bar{u}_m(t))\;, \quad y = E_1^T\,\eta\;. \tag{22}$$

See Figure 6. In the following, we restrict ourselves with looking only at the class of observers made of a copy of the dynamics plus a linear correction term, that is to observers of the form

$$\dot{\hat{\eta}} = A_m\,\hat{\eta} + \widehat{\mathcal{NL}}(t, \hat{\eta}, \bar{u}_m(t)) + K\,\left(y - E_1^T\,\hat{\eta}\right)\;, \quad \hat{x}(t) = \hat{\Phi}_e(t, \hat{\eta}(t), \bar{u}_{m-1}(t)) \tag{23}$$

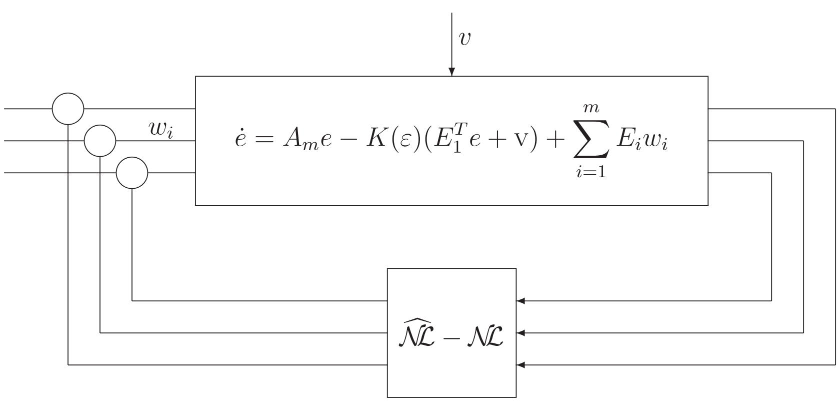

Figure 6. Block representation of the error system generated by (22) and (23).

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where $(\hat{\eta}, \hat{x})$ is an estimate of (η, x), and $\widehat{\mathcal{NL}}$ and $\hat{\Phi}_e$ are approximations of $\mathcal{NL}$ and Φ_e, respectively. Note that, in the case of (19), this requires that, not only u but also its m first-time derivatives are known. Now, we are left with the design of the correction gain K.

To motivate the following, we first concentrate our attention on (19). Let $\tilde{\eta} = \hat{\eta} - \eta$ be the estimation error for η. We have

$$\dot{\tilde{\eta}} = \left(A_m - KE_1^T\right) \tilde{\eta} + E_m \left(\hat{\varphi}_{(m+1)e}(\tilde{\eta} + \eta, t, \bar{u}_m(t)) - \varphi_{(m+1)e}(\eta, t, \bar{u}_m(t))\right) .$$

We see this system as the interconnection of a linear system

$$\dot{\tilde{\eta}} = \left(A_m - KE_1^T\right) \tilde{\eta} + E_m w_m \tag{24}$$

with a static nonlinear one

$$w_m = \hat{\varphi}_{(m+1)e}(\tilde{\eta} + \eta, t, \bar{u}_m(t)) - \varphi_{(m+1)e}(\eta, t, \bar{u}_m(t)) .$$

Assume for the time being that the latter has a linear gain L with an offset $\mathfrak{d}$, that is there exist two real numbers L and $\mathfrak{d}$ such that

$$\left|\hat{\varphi}_{(m+1)e}(\tilde{\eta} + \eta, t, \bar{u}_m) - \varphi_{(m+1)e}(\eta, t, \bar{u}_m)\right| \leqslant \mathfrak{d} + L\,|\tilde{\eta}| \qquad \forall(\tilde{\eta}, \eta, t, \bar{u}_m) . \tag{25}$$

L here plays the role of a Lipschitz constant. Then it follows from the small-gain theorem that, if the H^∞ gain of $\left(sI - \left(A_m - KE_1^T\right)\right)^{-1} E_m$ is strictly smaller than L, then there exists a ball centered at the origin with radius proportional to $\mathfrak{d}$, which is asymptotically stable uniformly in $(t, \eta, \bar{u}_m)$. To design a gain vector K to match this small-gain condition, we may follow the bounded real Lemma, which says that it is sufficient to find a triple (P, K, q) of a non-negative symmetric matrix, a gain vector, and a strictly positive real number satisfying the following matrix inequality

$$P\left(A_m - KE_1^T\right) + \left(A_m - KE_1^T\right)^T P + qI + \frac{1}{q\gamma^2} PE_mE_m^TP \leqslant 0 \tag{26}$$

where γ satisfies

$$\gamma L < 1 . \tag{27}$$

A key remark for getting such a triple (P, K, q) is to observe that we have

$$\varepsilon\, \mathtt{diag}(I, \ldots, \varepsilon^{m-1} I)\, A_m = A_m\, \mathtt{diag}(I, \ldots, \varepsilon^{m-1} I) ,$$
$$\varepsilon^{i-1} E_i = \mathtt{diag}(I, \ldots, \varepsilon^{m-1} I)\, E_i .$$

Because the pair $\left(A_m, E_1^T\right)$ is observable, there exists a pair (P_0, K_0), with P_0 positive definite, satisfying

$$P_0\left(A_m - K_0E_1^T\right) + \left(A_m - K_0E_1^T\right)^T P_0 + I = 0$$

then a triple satisfying (26) and (27) for any $\varepsilon \leqslant \frac{1}{1+\frac{\lambda_{\max}(P_0)^2}{\gamma^2}}$ is

$$\begin{aligned} P(\varepsilon) &= \mathtt{diag}(I, \ldots, \varepsilon^{m-1} I)\, P_0\, \mathtt{diag}(I, \ldots, \varepsilon^{m-1} I) , \\ K(\varepsilon) &= \frac{1}{\varepsilon}\mathtt{diag}(I, \ldots, \varepsilon^{m-1} I)^{-1} K_0 , \\ q(\varepsilon) &= \varepsilon^{2(m-1)} . \end{aligned} \tag{28}$$

Actually, this triple $(P(\varepsilon), K(\varepsilon), and q(\varepsilon))$ previously discussed gives us more. To see this, consider the following system, more general than (24) :

$$\dot{\tilde{\eta}} = A_m \tilde{\eta} - K(\varepsilon)\left(E_1^T \tilde{\eta} + \mathrm{v}(t)\right) + \sum_{i=1}^{m} E_i w_i(t) \tag{29}$$

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where v and w_i are exogenous inputs, the former capturing the effect of a measurement noise, whereas the latter captures the effect of the unmodeled and/or non linear terms on $\dot{\tilde{\eta}}_i$. We get

$$
\begin{aligned}
&\overbrace{\tilde{\eta}^T P(\varepsilon)\,\tilde{\eta}}^{\cdot} \\
&= \frac{2}{\varepsilon}\tilde{\eta}^T \mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)P_0\left(A_m - K_0E_1^T\right)\mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)\tilde{\eta} \\
&\qquad -\frac{2}{\varepsilon}\tilde{\eta}^T \mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)P_0K_0\mathrm{v}(t) \\
&\qquad +2\sum_{i=1}^{m}\varepsilon^{i-1}\tilde{\eta}^T \mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)P_0E_iw_i(t) \\
&\leq -\frac{1}{\varepsilon}\left|\mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)\tilde{\eta}\right|^2 \\
&\qquad +\frac{1}{3\varepsilon}\left|\mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)\tilde{\eta}\right|^2 + \frac{3}{\varepsilon}\left|P_0K_0\mathrm{v}(t)\right|^2 \\
&\qquad +\sum_{i=1}^{m}\left(\frac{1}{3m}\left|\mathtt{diag}(I,\ldots,\varepsilon^{m-1}I)\tilde{\eta}\right|^2 + 3m\varepsilon^{2(i-1)}\left|P_0E_iw_i(t)\right|^2\right) \\
&\leq -\frac{1}{3\varepsilon\lambda_{max}(P_0)}\,\tilde{\eta}^T P(\varepsilon)\,\tilde{\eta} + \frac{3|P_0K_0|^2}{\varepsilon}\,|\mathrm{v}(t)|^2 + 3\varepsilon m\sum_{i=1}^{m}\varepsilon^{2(i-1)}\,|P_0E_i|^2\,|w_i(t)|^2
\end{aligned}
$$

$$
|\tilde{\eta}_j|^2 \leq \frac{1}{\varepsilon^{2(j-1)}}\,\frac{\tilde{\eta}^T P(\varepsilon)\,\tilde{\eta}}{\lambda_{min}(P_0)} .
$$

This establishes that (29) is input-to-state stable (ISS) with linear gain,

$$
\sqrt{\frac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)}}\frac{3|P_0K_0|}{\varepsilon^{j-1}} \text{ from v to } \tilde{\eta}_j
$$

and

$$
\sqrt{m\frac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)}}\frac{3|P_0E_i|}{\varepsilon^{j-i-1}} \text{ from } w_i \text{ to } \tilde{\eta}_j .
$$

Mimicking (25), assume w_i is produced by some nonlinear system with $\tilde{\eta}$ as input such that there exist nonnegative real numbers L_{il} and $\overline{w_i}$, and time functions w_i such that w_i satisfies :

$$
|w_i(t)|^2 \leqslant \mathrm{w}_i(t)^2 + \sum_{l=1}^{i} L_{il}^2\,|\tilde{\eta}_l(t)|^2 , \quad |\mathrm{w}_i(t)| \leqslant \overline{w_i} . \tag{30}
$$

Assume also that v is bounded, that is, there exists a nonnegative real number $\overline{v}$ satisfying

$$
|\mathrm{v}(t)| \leqslant \overline{v} \qquad \forall t .
$$

Then, because this other system does not depend on ε and ε is to be small, the small-gain theorem implies the existence of an asymptotically stable ball centered at the origin. Indeed, for any ν in

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(0, 1), we can find c such that, for all ε sufficiently small to satisfy[§]

$$(1>) \quad \nu \geqslant \max\left\{1\,,\ \frac{9m\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)} \max_{l\in\{1,\dots,m\}} \sum_{i=l}^{m} L_{il}^2 |P_0 E_i|^2\right\} \varepsilon^2\,, \tag{31}$$

we have, for all $t \geqslant s$,

$$\begin{aligned} |\tilde{\eta}_j(t)|^2 \leqslant{}& \frac{1}{\varepsilon^{2(j-1)}} \frac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)} \exp\left(-\frac{c(t-s)}{\varepsilon}\right) |\tilde{\eta}(s)|^2 \\ &+ \frac{1}{\varepsilon^{2(j-1)}} \frac{3|P_0 K_0|^2}{c\lambda_{\min}(P_0)} \sup_{r\in[s,t]} |\mathrm{v}(r)|^2 \\ &+ \varepsilon^2 \sum_{i=1}^{m} \varepsilon^{2(i-j)} \frac{3m|P_0 E_i|^2}{c\lambda_{\min}(P_0)} \sup_{r\in[s,t]} |\mathrm{w}_i(r)|^2\,. \end{aligned} \tag{32}$$

We will come back to this inequality in the succeeding text. For the time being, let us study the implication of the fact that ε should be small enough to satisfy inequality (31). In the case of the model (19), if the extended function $\varphi_{(m+1)e}$ and its model $\widehat{\varphi}_{(m+1)e}$ are close enough to satisfy, with some nonnegative real numbers $L_{(m+1)l}$,

$$\left|\widehat{\varphi}_{(m+1)e}(\tilde{\eta}+\eta, t, \bar{u}_m) - \varphi_{(m+1)e}(\eta, t, \bar{u}_m)\right|^2 \leqslant \bar{w}^2 + \sum_{l=1}^{m} L_{(m+1)l}^2 |\tilde{\eta}_l(t)|^2\,,$$

for all $(\eta, \tilde{\eta}, t, \bar{u}_m)$ in the domain of interest, inequality (31) is satisfied when ε satisfies

$$1 > \nu \geqslant \max\left\{1\,,\ \frac{9m\lambda_{max}(P_0)|P_0 E_m|^2}{\lambda_{min}(P_0)} \max_{l\in\{1,\dots,m\}} L_{(m+1)l}^2\right\} \varepsilon^2\,.$$

In the case of the model (21), if the extended function φ_{ie} and its model $\widehat{\varphi}_{ie}$ are close enough to satisfy, with some nonnegative real numbers L_{il},

$$\begin{aligned} |\widehat{\varphi}_{ie}(\tilde{\eta}+\eta, u) - \varphi_{ie}(\eta)u|^2 &\leqslant \frac{\bar{w}^2}{(m+1)^2} + \sum_{l=1}^{m} L_{il}^2 |\tilde{\eta}_l(t)|^2\,, \\ \left|\widehat{\varphi}_{(m+1)e}(\tilde{\eta}+\eta) - \varphi_{(m+1)e}(\eta)\right|^2 &\leqslant \frac{\bar{w}^2}{(m+1)^2} + \sum_{l=1}^{m} L_{(m+1)l}^2 |\tilde{\eta}_l(t)|^2\,, \end{aligned}$$

for all $(\eta, \tilde{\eta}, u)$ in the domain of interest, then the parameter ε has to satisfy

$$1 > \nu \geqslant \max\left\{1\,, \frac{9m\lambda_{max}(P_0)}{\lambda_{min}(P_0)} \max\left\{\max_{l\in\{1,\dots,m\}} \sum_{i=l}^{m+1} L_{il}^2|P_0 E_i|^2\,,\ \max_{l\in\{1,\dots,m\}} L_{(m+1)l}^2 |P_0 E_m|^2\right\}\right\} \varepsilon^2$$

This says that, when the model functions φ_{ie} are bounded on the domain of interest, their approximating functions $\widehat{\varphi}_{ie}$ can very well be taken as constant. But in this case, the estimation error may not converge because of the term $\bar{w}$. When the model functions are Lipschitz, their approximations can simply be copies. But in this case, the parameter ε has to be small enough with respect to the Lipschitz constant.

[§]
$$\begin{aligned} \sum_{i=1}^{m} \varepsilon^{2(i-1)} |P_0 E_i|^2 \sum_{l=1}^{i} L_{il}^2 |\tilde{\eta}_l(t)|^2 &\leqslant \sum_{l=1}^{m} \left(\varepsilon^{2(l-1)} |\tilde{\eta}_l(t)|^2 \left(\sum_{i=l}^{m} \varepsilon^{2(i-l)} |P_0 E_i|^2 L_{il}^2\right)\right) \\ &\leqslant \max_{l\in\{1,\dots,m\}} \left(\sum_{i=l}^{m} \varepsilon^{2(i-l)} |P_0 E_i|^2 L_{il}^2\right) \frac{\tilde{\eta}^T P(\varepsilon)\tilde{\eta}}{\lambda_{min}(P_0)} \end{aligned}$$

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3.2.2. Some words on performance. Inequality (32) is nothing but the general version of what we saw in (13). Each of the three terms in its right-hand side is meaningful.

1. The last term with ε^2 in factor, that is,

$$\varepsilon^2 \sum_{i=1}^{m} \varepsilon^{2(i-j)} \frac{3m|P_0 E_i|^2}{c\lambda_{\min}(P_0)} \sup_{r\in[s,t]} |\mathrm{w}_i(r)|^2$$

shows that, by picking ε smaller, we can almost decouple each estimation error component $\tilde{\eta}_j$ from each dynamic noise w_i but only for i satisfying the triangular restriction $i \geqslant j$. This establishes that we can reconstruct the vector η in a robust way with respect to dynamic noise as long as the aforementioned triangular restriction is satisfied.

2. There is a price to pay for this robustness. It is shown in the second term, that is

$$\frac{1}{\varepsilon^{2(j-1)}} \frac{3|P_0 K_0|^2}{c\lambda_{\min}(P_0)} \sup_{r\in[s,t]} |\mathrm{v}(r)|^2 .$$

By decreasing ε, we make stronger the effect of the measurement noise on each component of the estimation error, this being even worse as j increases and so, in particular, for higher dimensional systems. With the previous point, this is the trade-off in the choice of ε discussed after (13).

Note also that smaller ε implies higher sampling rates. Indeed as ε is reduced, the observer bandwidth is increased. This requires higher sampling rates in digital implementation, and consequently larger word length [65].

3. Finally, in the case where there is no measurement nor dynamic noise, we get from the first term that, for each t strictly positive

$$\lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon^{2(j-1)}} \frac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)} \exp\left(-\frac{c(t-s)}{\varepsilon}\right) |\tilde{\eta}(s)|^2 \right) = 0 .$$

This shows that the estimation error can be made arbitrarily small after an arbitrarily small amount of time. This property is known as the tunability property. But here again, there is a price to pay for this. As already observed after equation (8), it is the peaking phenomenon. The jth component of the estimation error may have a peak of size $\frac{1}{\varepsilon^{2(j-1)}} \frac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)} |\tilde{\eta}(0)|^2$ during this amount of time. And the faster we want the transient or the smaller we want the estimation error, the bigger this peak may be.

In summary, smaller ε is needed for estimation error boundedness, which is good for reducing dynamic noise effects and increasing convergence speed, but bad as measurement noise effects; peaking may increase, and digital implementation is more demanding.

Peaking can be rounded by using a priori information on the state to be estimated. As we mentioned already, we can disregard the large values of the state estimate because they are unrealistic.

With the advancement of technology, the digital implementation issue will be less of a challenge over time.

Dealing with measurement noise remains the key issue.

3.2.3. Observer parameter adaptation. In many practical problems, the measurement-noise level is much smaller than the initial conditions of the estimation error; hence, the transient response is dominated by the effect of initial conditions. Therefore, a sound strategy to achieve fast convergence while reducing the impact of measurement noise at steady state is to use a smaller ε during the transient time and then increase it at steady state. But, for this, we need to know the upperbound on ε required to guarantee at least estimate boundedness, that is, for (31) to be satisfied.

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Several observer parameter adaptation strategies have been proposed. They differ mainly on a priori information the designer has.

- When no bounds are known a priori on the nonlinearities nor their (local) Lipschitz constants nor on the measurement noise [3, 66–68], the task is difficult, and there is no solution of interest in practice. Indeed in this case, we can only let ε decrease. But then we are facing the same kind of problems as those described in the adaptive control literature for online tuning of control parameters. In this context, it is known that a gain adaptation may lead to serious growth problems when perturbations such as measurement noise are present (see, e.g., [69, Example 4.2], [70, Figure 6.a], and [71]).
- When a bound on the level of measurement noise is available, the aforementioned strategy of decreasing ε can be modified depending on whether or not the estimation error $|y - \hat{\eta}_1|$ is larger or smaller than the bound. This is the so-called dead-zone technique [66, 69, 70]. It can be improved by letting the parameter ε decrease when this estimation error is larger than a threshold, and increase if not [72]. But in this case again, there may be unsatisfactory behavior like the so-called bursting phenomenon, which results from a possible destabilization when ε gets too large.
- When a bound on the nonlinearities or the (local) Lipschitz constants is known as function of the measurement or is observable, a possible strategy consists in letting ε follows this bound, somehow making (31) an equality [20, 36, 68]. But in doing so, only estimate boundedness is considered, nothing is performed directly to improve performance. Performance may still be improved because the strategy aims at keeping ε as large as possible.
- When bounds both on the nonlinearities or the (local) Lipschitz constants and on the measurement-noise level are known, the strategy consists in letting ε evolve between an upper value, given by the former bound and a lower value given by the latter [57, 73, 74].

3.3. Estimation of x

The estimation of η is not our ultimate goal, but we want an estimation of x. For this, as written in (23), we use an approximation $\hat{\Phi}_e$ of the function Φ_e in both (19) and (21). The convergence property of $\hat{\eta}$ to η or the smallness of the corresponding estimation error is transferred into the same property of $\hat{x}$ with respect to x at least when there exist a real number $\overline{x}$ and a class $\mathcal{K}$ function such that we have

$$\left|\hat{\Phi}_e(t, \tilde{\eta} + \eta, \bar{u}_m) - \Phi_e(t, \eta, \bar{u}_m)\right| \leqslant \overline{x} + \rho(|\tilde{\eta}|)$$

for all $(t, \eta, \tilde{\eta}, \bar{u}_m)$ in the domain of interest. It must be noticed that, if $\overline{x}$ is nonzero or ρ is not linearly bounded around 0, that is, if Φ_e is not Lipschitz, the possible exponential convergence of $\hat{\eta}$ to η may not be preserved for $\hat{x}$ going to x.

4. SEMIGLOBAL SEPARATION PRINCIPLE

The following paragraph is inspired by [17, 60]. Many other very significant contributions are available on this topic.

Assume the system,

$$\dot{x} = f(x, u) \ , \quad y = h(x), \tag{33}$$

with state x in $\mathbb{R}^n$, control u in $\mathbb{R}$, and measured output y in $\mathbb{R}$, is such that:

1. We know integers m_y and m_u and a continuous function Φ such that [¶,‖]

$$x = \Phi(\bar{y}_{m_y-1}, \bar{u}_{m_u-1}).$$

¶See the comment following (18).

‖This needs to be satisfied only on $\mathcal{A}$.

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2. We know a state feedback law $\alpha : \mathbb{R}^n \times \mathbb{R}^{m_u} \to \mathbb{R}$ stabilizing a point used as origin of the coordinates for the extended system, controlled by u_{m_u},

$$\dot{x} = f\left(x, E_1^T \bar{u}_{m_u-1}\right) \quad , \quad \dot{\bar{u}}_{m_u-1} = A_{m_u} \bar{u}_{m_u-1} + E_{m_u-1} u_{m_u}$$

with domain of attraction (containing) the open set $\mathcal{A}$.

Let C be an arbitrary compact set contained in $\mathcal{A}$. Let C_0 be a compact neighborhood of the origin, also contained in $\mathcal{A}$. Let C_e be a compact set whose interior contains $C_0 \cup C$. Let Δ be a strictly positive real number such that C_e is forward invariant and C_0 is asymptotically stable with domain of attraction containing C_e for the system

$$\dot{x} = f\left(x, E_1^T \bar{u}_{m_u-1}\right) , \quad \dot{\bar{u}}_{m_u-1} = A_{m_u} \bar{u}_{m_u-1} + E_{m_u-1}\alpha\left((x, \bar{u}_{m_u-1}) + \delta_{u_{m_u}}\right),$$

where $\delta_{u_{m_u}} \in \mathbb{R}^{n+m_u}$ is a disturbance whose norm is smaller than Δ. Such a Δ exists because of the robustness of the asymptotic stability property.

Let B_Δ be the ball in $\mathbb{R}^{n+m_u-1}$, centered at the origin, with radius Δ, and $\overline{\alpha}$ be defined as :

$$\overline{\alpha} = \max_{((x,\bar{u}_{m_u-1}),\delta_{u_{m_u}}) \in C_e \times B_\Delta} \left|\alpha\left((x, \bar{u}_{m_u-1}) + \delta_{u_{m_u}}\right)\right|$$

There exists a real number F, such that, as long as the control u_{m_u} takes its values in $[-\overline{\alpha}, \overline{\alpha}]$, we have :

$$\left|\begin{pmatrix} \dot{x} \\ \dot{\bar{u}}_{m_u-1} \end{pmatrix}\right| \leqslant F \qquad \forall (x, \bar{u}_{m_u-1}) \in C_e$$

So, because C is in the interior of C_e, there exists a time T such that any solution, initialized in C, remains in C_e at least on $[0, T]$ provided its control is in $[-\overline{\alpha}, \overline{\alpha}]$. From this definition of T and the property of Δ, it follows that if, we have an observer able to deliver an estimation of $(x, \bar{u}_{m_u-1})$ with estimation error smaller than Δ after the time T, that is, with the tunability property, we have seen the high-gain observer has, and if we enforce the control u_{m_u} to remain in $[-\overline{\alpha}, \overline{\alpha}]$ then, by using this estimation instead of the true value as argument of α, we are guaranteed that any solution initialized in C will converge to C_0. Hence, we get something like the converging input-bounded state property [75], which, together with the convergence to zero of the estimation error (when there is no modeling error), implies the asymptotic stability.

However, we should not forget that the tunability property of the high-gain observer holds only when the functions are globally Lipschitz on the domain of interest and that we have the peaking phenomenon. The two problems can be rounded in the same way, that is, modify the functions when their values or their arguments do not comply with the data. For instance, we can arbitrarily deform the state function outside the compact set C_e because the solutions will remain in this set (which has to be proved). Similarly, if the estimated state takes values outside C_e, we can modify it. This kind of argument leads, for instance, to the following output feedback :

$$\begin{aligned}
\dot{\bar{u}}_{m-1} &= A_{m_u} \bar{u}_{m-1} + E_{m_u-1} u_{m_u} , \\
\dot{\hat{\eta}} &= A_{m_y} \hat{\eta} + E_{m_y} \varphi_{m_y e}(\hat{\eta}, (\bar{u}_{m-1}, u_{m_u})) + K(\varepsilon)\left(y - E_1^T \hat{\eta}\right) , \\
\hat{x} &= \Phi_e\left(\hat{\eta}, \bar{u}_{m_u-1}\right) , \\
u_{m_u} &= \alpha_e(\hat{x}, \bar{u}_{m-1}) ,
\end{aligned}$$

where $K(\varepsilon)$ is given by (28), the functions $\varphi_{m_y e}$, Φ_e and α_e are modified extensions of the functions φ_{m_y} and Φ, given by the system, and α given by the state feedback. As discussed after (8), a usual modification of α is simply :

$$\alpha_e(x, \bar{u}_{m-1}) = \bar{\alpha}\, \mathtt{sat}\left(\frac{\alpha(x, \bar{u}_{m-1})}{\bar{\alpha}}\right)$$

where `sat` is the standard saturation function.

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The output feedback we have obtained does depend on our choice of C_0, C, and C_e compact subsets of $\mathcal{A}$. Because C and C_e can be as close as we want from $\mathcal{A}$, the asymptotic stabilization is called semiglobal.

Another property of this particular output feedback we have illustrated in Section 2 is that we can recover the performance of the state feedback itself for sufficiently small observer parameter ε, that is as ε approaches zero; the trajectory obtained with the output feedback approaches the one given by the state feedback.

5. ROBUST CONTROL OF MINIMUM-PHASE SYSTEMS

We can go beyond the semiglobal separation principle stated in the previous section when the system has more structure. Under geometric and smoothness conditions, that can be found in Isidori [76] or Marino and Tomei [77], for example, the system (31), with f affine in u, can be transformed into the normal form

$$\begin{aligned}\dot{\xi} &= q(\xi,\eta)\\ \dot{\eta}_i &= \eta_{i+1}, \quad \text{for } 1 \leqslant i \leqslant r-1\\ \dot{\eta}_r &= b(\xi,\eta) + a(\xi,\eta)u\\ y &= \eta_1\end{aligned}$$

where $\eta = col(\eta_1,\ldots,\eta_r)$ and $a(\cdot) \neq 0$ over a domain $\mathcal{D}$. This form is similar to what we have in (19), except for the presence of the extra dynamics $\dot{\xi} = q(\xi,\eta)$ and for having u simply instead of $\bar{u}_{m-1} = \left(u,\ldots,u^{(m-1)}\right)$.

As observed previously, if the goal is to stabilize the system at an equilibrium point, the coordinates can be chosen such that this point is the origin ($\xi = 0, \eta = 0$) for the normal form in open-loop, and it is in the interior of the domain $\mathcal{D}$.

A system that can be written in such a form is said to have relative degree r ($r \leqslant n$), and the equation $\dot{\xi} = q(\xi,0)$ is called the zero dynamics, as it represents the internal dynamics of the system when the output is constrained to zero; equivalently, $\eta(t) \equiv 0$. The system is said to be minimum phase if the zero dynamics have an asymptotically stable equilibrium point at $\xi = 0$. When the functions f and h depend on a vector p of unknown constant parameters in a way that does not change the relative degree of the system and in the presence of matched disturbances/uncertainties, the normal form takes the perturbed form

$$\begin{aligned}\dot{\xi} &= q(\xi,\eta,p)\\ \dot{\eta}_i &= \eta_{i+1}, \quad \text{for } 1 \leqslant i \leqslant r-1\\ \dot{\eta}_r &= b(\xi,\eta,p) + a(\xi,\eta,p)[u+\delta(\cdot)]\\ y &= \eta_1\end{aligned} \tag{34}$$

where $\delta(\cdot) = \delta(t,\xi,\eta,u,p)$.

In Section 4, we were assuming we were given a state feedback for the general system (33). Here instead, we can take advantage of the special structure of the normal form to actually design it. The design of robust state and even output feedback control for the system (34) is well developed in the special case when the system has relative degree one and is minimum phase; that is,

$$\begin{aligned}\dot{\xi} &= q(\xi,y,p)\\ \dot{y} &= b(\xi,y,p) + a(\xi,y,p)[u+\delta(\cdot)]\end{aligned} \tag{35}$$

In this case, a feedback control that depends only on y can be designed to dominate the nonlinear functions b and δ, and bring y arbitrarily close to the origin over a finite time, whereas the minimum-phase property ensures a similar behavior for ξ. Nonlinear control techniques that can achieve this goal using a Lipschitz feedback control law include high-gain feedback, Lyapunov redesign, and continuously implemented sliding mode control [61]. We illustrate the idea by describing the continuously implemented sliding mode control. Suppose the system $\dot{\xi} = q(\xi,y,p)$ is ISS uniformly

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in p; that is, there are class $\mathcal{K}$ function α and class $\mathcal{KL}$ function β, independent of p, such that the solution of $\dot{\xi} = q(\xi, \eta, p)$ satisfies

$$|\xi(t)| \leqslant \beta(|\xi(t_0)|, t - t_0) + \alpha\left(\sup_{\tau \geqslant t_0} |y(\tau)|\right)$$

Suppose further that the functions a, b, and δ satisfy the inequalities

$$a(\cdot) \geqslant c_1, \quad |\delta(\cdot)| \leqslant c_2 + c_3|u|$$

$$\left|\frac{c_2 + |b(\cdot)/a(\cdot)|}{(1 - c_3)}\right| \leqslant K - c_4$$

uniformly in p, for some known positive constants K and c_1 to c_4 with $c_3 < 1$ and $K > c_4$, over a compact subset of $\mathcal{D}$ and for all $t \geqslant 0$. Then the control

$$u = -K \text{ sat}\left(\frac{y}{\mu}\right), \quad \mu > 0 \tag{36}$$

ensures that y is bounded and reaches the set $\{|y| \leqslant \mu\}$ in finite time because when $|y| \geqslant \mu$ the derivative of $\frac{1}{2}y^2$ satisfies

$$y\dot{y} \leqslant -c_1 c_4(1 - c_3)|y|$$

The ISS property of $\dot{\xi} = q(\xi, y, p)$ ensures that ξ also will be bounded and will, in finite time, reach a set of the form $\{|\xi| \leqslant \rho(\mu)\}$ for some class $\mathcal{K}$ function ρ. By choosing μ small enough, the set $\Omega_\mu = \{|\xi| \leqslant \rho(\mu)\} \times \{|y| \leqslant \mu\}$ can be contained inside any given neighborhood of the origin, showing that the control (36) achieves practical stabilization. If the function δ vanishes at the origin, that is, $\delta(t, 0, 0, 0, p) = 0$, some additional assumptions would ensure that the control law stabilizes the origin. The simplest case is when all functions are locally Lipschitz in (ξ, η, u) and the origin of $\dot{\xi} = q(\xi, 0, p)$ is locally exponentially stable uniformly in p. In that case, we can use a Lyapunov function of the form $V(\xi, y) = V_0(\xi) + y^2$, where $V_0(\xi)$ is obtained from the converse Lyapunov theorem [61, Theorem 4.17], to show that, for sufficiently small μ, the origin is exponentially stable and Ω_μ is a subset of its region of attraction.

Turning now to the relative-degree r system (34), we note that if the vector η had been available for feedback, we could have designed a partial state feedback control, dependent only on η, that reproduces the results we have just described for relative-degree-one systems. This is so because if we define

$$s = k_1\eta_1 + k_2\eta_2 + \cdots + k_{r-1}\eta_{r-1} + \eta_r$$

and perform a change of variables to replace η_r by s, the system (34) can be rewritten as

$$\begin{aligned} \dot{z} &= \bar{q}(z, s, p) \\ \dot{s} &= \bar{b}(z, s, p) + \bar{a}(z, s, p)[u + \bar{\delta}(\cdot)] \end{aligned} \tag{37}$$

where $z = col(\xi, \eta_1, \ldots, \eta_{r-1})$,

$$\bar{q}(z, s, p) = \left.\begin{bmatrix} q(\xi, \eta, p) \\ \eta_2 \\ \vdots \\ \eta_{r-1} \\ \eta_r \end{bmatrix}\right|_{\eta_r = s - \sum_{i=1}^{r-1} k_i\eta_i}$$

and $\bar{a}$, $\bar{b}$, and $\bar{\delta}$ are new notations for a, $b + \sum_{i=1}^{r-1} k_i\eta_{i+1}$, and δ, respectively, when η_r is replaced by $s - \sum_{i=1}^{r-1} k_i\eta_i$. The system (37) has relative degree one when s is viewed as the output. By designing k_1 to k_{r-1} such that the polynomial

$$\lambda^{r-1} + k_{r-1}\lambda^{r-2} + \cdots + k_2\lambda + k_1$$

is Hurwitz, it follows from asymptotic stability of the origin of $\dot{\xi} = q(\xi, 0, p)$ and smoothness conditions that the origin of $\dot{z} = \bar{q}(z, 0, p)$ is asymptotically stable. Moreover, if the system $\dot{\xi} = q(\xi, \eta, p)$ is ISS uniformly in p, so is the system $\dot{z} = \bar{q}(z, s, p)$.

In output feedback control, the vector η is not available for feedback; only the output $y = \eta_1$ is measured. Here comes the role of the high-gain observer. The vector η satisfies equation (19) of the previous section with $m = r$. We saw there how to design a high-gain observer to estimate η by $\hat{\eta}$ in such a way that the estimation error $(\eta - \hat{\eta})$ can be made arbitrarily small over an arbitrarily small period. In the current notation, the observer is given by

$$\begin{aligned} \dot{\hat{\eta}}_i &= \hat{\eta}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - \hat{\eta}_1), \quad \text{for } 1 \leqslant i \leqslant r-1 \\ \dot{\hat{\eta}}_r &= \hat{\varphi}(t, \hat{\eta}, u) + \frac{\alpha_r}{\varepsilon^r}(y - \hat{\eta}_1) \end{aligned} \tag{38}$$

where α_1 to α_r are chosen such that the polynomial

$$\lambda^r + \alpha_1 \lambda^{r-1} + \cdots + \alpha_{r-1}\lambda + \alpha_r$$

is Hurwitz, $\hat{\varphi}(t, \eta, u)$ is an approximation $b(\xi, \eta, p) + a(\xi, \eta, p)[u + \delta(\cdot)]$, which is not allowed to depend on ξ, and ε is the observer parameter that is chosen sufficiently small. We saw in the previous section also that if the state feedback control is saturated outside a compact set of interest, to overcome the peaking phenomenon, then the output feedback control recovers the stability properties of the state feedback control when ε is chosen small enough. In particular, if the state feedback control brings the trajectories to a compact set Ω_μ in finite time, so will the output feedback control. If $\delta(\cdot)$ vanishes at the origin and the conditions for exponential stability are satisfied, the output feedback control will ensure that the origin of the closed-loop system is exponentially stable. Finally, by choosing ε small enough, we can ensure that the trajectories of (ξ, η) under output and state feedback control can be made arbitrarily close to each other for all $t \geqslant 0$.

The foregoing discussion lays down a procedure for designing output feedback control for minimum-phase systems of the form (34), which is comprised of the following steps:

1. Design a partial state feedback control that uses only measurements of η to achieve the control objective and meet the design specifications/constraints.
2. Estimate a compact positively invariant set to which the state trajectories will be confined under state feedback and saturate the control u and the function $\hat{\varphi}$ outside this set; control saturation can be achieved by saturating the control signal or saturating the estimates $\hat{\eta}_i$.
3. Implement the high-gain observer, replace η by $\hat{\eta}$ in the control law, and tune the observer parameter ε, by monotonically decreasing it, to recover the performance of the state feedback control.

Steps 2 and 3 are exactly the same as for the general case of Section 4, whereas Step 1 was assumed to be done already.

The foregoing procedure for the design of output feedback control for minimum-phase systems has been applied to a number of control problems beyond stabilization. In the remainder of this section, we briefly describe the tracking and regulation problems. In the tracking problem, the output y is required to asymptotically track a reference signal $\mathcal{R}(t)$, while maintaining boundedness of all state variables. It is assumed that $\mathcal{R}$ and its derivatives up to $\mathcal{R}^{(r)}$ are bounded and $\mathcal{R}^{(r)}$ is piece-wise continuous function of time. In the error coordinates

$$e_i = \eta_i - \mathcal{R}^{(i-1)}, \quad \text{for } 1 \leqslant i \leqslant r$$

equation (34) takes the form

$$\begin{aligned} \dot{\xi} &= q(\xi, \eta, p) \\ \dot{e}_i &= e_{i+1}, \quad \text{for } 1 \leqslant i \leqslant r-1 \\ \dot{e}_r &= b(\xi, \eta, p) + a(\xi, \eta, p)[u + \delta(\cdot)] - \mathcal{R}^{(r)} \\ y_m &= e_1 \end{aligned} \tag{39}$$

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with $e_1 = y - \mathcal{R}$ as the measured signal. From this point on, the design of output feedback control proceeds as in the stabilization case with two main differences. First, it is sufficient to the require the system $\dot{\xi} = q(\xi, \eta, p)$ to be bounded-input–bounded-state stable instead of the stronger ISS condition because η is required only to be bounded, as opposed to requiring its convergence to zero in the stabilization problem. Second, because the function $\delta(\cdot)$ is not likely to vanish at $e = 0$, it is typical that we can only achieve practical tracking where $|e(t)|$ can be made arbitrarily small over a finite time.

In the regulation problem, the output feedback controller is designed to achieve asymptotic tracking of a reference signal $\mathcal{R}$ and/or rejection of disturbance signals, all of which are generated by a known dynamical model, referred to as the exosystem. This of course is to be achieved while maintaining boundedness of all variables in the closed-loop system. The exosystem is a neutrally stable system $\dot{w} = s(w)$ whose initial conditions belong to a compact set.** A neutrally stable system has bounded solutions that are persistent in time; that is, they do not converge to zero as time tends to infinity. A linear system $\dot{w} = Sw$ is neutrally stable if the matrix S has simple eigenvalues on the imaginary axis, which can model constant signals as well as a finite number of sinusoids of known frequencies. We have already seen how we can use a high-gain observer to reduce a relative-degree r problem to a relative-degree-one problem. So for convenience, we will describe the solution of the regulation problem only for relative-degree-one systems of the form (35), which we rewrite as

$$\begin{aligned} \dot{\xi} &= q(\xi, e, w, p) \\ \dot{e} &= b(\xi, e, w, p) + a(\xi, e, w, p)[u + \delta(\xi, e, w, p)] \end{aligned} \tag{40}$$

where $e = y - \mathcal{R}$ is the regulation error and $w(t)$ is generated by the exosystem. It is seen from (40) that at steady state, when $e(t) \equiv 0$, $\xi(t)$ must satisfy the equation $\dot{\xi} = q(\xi, 0, w, p)$, and the steady-state control $u_{ss}(t)$ must satisfy

$$u_{ss} = -\delta(\xi, 0, w, p) - \frac{b(\xi, 0, w, p)}{a(\xi, 0, w, p)}$$

Suppose there is a map $\tau(w, p)$ that satisfies the partial differential equation

$$\frac{\partial \tau}{\partial w} s(w) = q(\tau(w, p), 0, w, p)$$

and set

$$\chi(w, p) = -\delta(\tau(w, p), 0, w, p) - \frac{b(\tau(w, p), 0, w, p)}{a(\tau(w, p), 0, w, p)},$$

then $u_{ss} = \chi(w, p)$. Any controller that solves the regulation problem must contain a dynamical model that generates the steady-state control u_{ss}, a property known as the *internal model principle* [78, 79]. The controller, in fact, has to define an invariant subset of $\{e = 0\}$ where $u = \chi(w, p)$ and has to stabilize this invariant set to ensure that $e(t)$ converges to zero. A special case where the internal model is linear occurs when the exosystem is $\dot{w} = Sw$ and $\chi(w, p)$ is a polynomial in the components of w with coefficients dependent on p. In this case, there is a single-output observable pair (Φ, Γ), independent of p, with Φ having simple eigenvalues on the imaginary axis, and a mapping $\rho(w, p)$ such that $\chi(w, p)$ is generated by the system

$$\dot{\rho} = \Phi\rho, \quad \chi = \Gamma\rho$$

The spectrum of Φ contains the eigenvalues of S. Moreover, as a number of higher order harmonics of the sinusoidal components of w may be generated by the nonlinearities of the system, the spectrum of Φ may contain eigenvalues that are multiples of the eigenvalues of S. Let (F, G) be a single-input controllable pair in which F is Hurwitz and has the same dimensions as Φ. Let Ψ

**See [76] for the definition of neutral stability.

be the unique matrix that assigns the eigenvalues of $(F + G\Psi)$ at the spectrum of Φ. Augment the system (40) with the internal model

$$\dot{\sigma} = (F + G\Psi)\sigma + Ge$$

and define $s = e + \Psi\sigma$ and $\zeta = \xi - \tau(w, p)$. The augmented system is given by

$$\begin{aligned} \dot{\zeta} &= \tilde{q}(\zeta, s - \Psi\sigma, w, p) \\ \dot{\sigma} &= F\sigma + Gs \\ \dot{s} &= b(\cdot) + a(\cdot)[u + \delta(\cdot)] \end{aligned} \tag{41}$$

where $\tilde{q}(\zeta, e, w, p) = q(\zeta + \tau, e, w, p) - q(\tau, e, w, p)$. When s is viewed as its output, the augmented system (41) has relative degree one and its zero dynamics are given by

$$\dot{\zeta} = \tilde{q}(\zeta, -\Psi\sigma, w, p), \quad \dot{\sigma} = F\sigma \tag{42}$$

It can be easily seen that if the origin $\zeta = 0$ is an asymptotically stable equilibrium point of $\dot{\zeta} = \tilde{q}(\zeta, 0, w, p)$ uniformly in (w, p), which is a minimum-phase condition, so is the origin $(\zeta = 0,\ \sigma = 0)$ of (42). On the basis of our earlier discussion of the stabilization of relative-degree-one minimum-phase systems, it is now clear that regulation can be achieved by a high-gain feedback control of the form $u = -Ks$ or a continuously implemented sliding mode control of the form $u = -K\ \mathrm{sat}(y/\mu)$ with positive constants K and μ.

REFERENCES

1. Doyle JC, Stein G. Robustness with observers. *IEEE Transactions on Automatic Control* 1979; **AC-24**(4):607–611.
2. Petersen IR, Holot CV. High-gain observers applied to problems in disturbance attenuation, H-infinity optimization and the stabilization of uncertain linear systems. *Proceedings of American Control Conference*, Atlanta, GA, June 1988; 2490–2496.
3. Khalil HK, Saberi A. Adaptive stabilization of a class of nonlinear systems using high-gain feedback. *IEEE Transactions on Automatic Control* 1987; **AC-32**(11):1031–1035.
4. Saberi A, Sannuti P. Observer design for loop transfer recovery and for uncertain dynamical systems. *IEEE Transactions on Automatic Control* 1990; **35**(8):878–897.
5. Tornambe A. Use of asymptotic observers having high gains in the state and parameter estimation. *Proceedings of IEEE Conference on Decision and Control*, Austin, Texas, 1988; 1791–1794.
6. Esfandiari F, Khalil HK. Observer-based design of uncertain systems: recovering state feedback robustness under matching conditions. *Proceedings of Allerton Conference*, Monticello, IL, September 1987; 97–106.
7. Gauthier JP, Hammouri H, Othman S. A simple observer for nonlinear systems application to bioreactors. *IEEE Transactions on Automatic Control* 1992; **37**(6):875–880.
8. Besançon G. High-gain observation with disturbance attenuation and application to robust fault detection. *Automatica* 2003; **39**:1095–1102.
9. Busawon K, Farza M, Hammouri H. Observer design for a special class of nonlinear systems. *International Journal of Control* 1998; **71**:405–418.
10. Deza F, Busvelle E, Gauthier JP, Rakotopara J. High gain estimation for nonlinear systems. *System & Control Letters* 1992; **18**:295–299.
11. Gauthier JP, Kupka IAK. Observability and observers for nonlinear systems. *SIAM Journal on Control and Optimization* 1994; **32**:975–994.
12. Hammouri H, Targui B, Armanet F. High-gain observer based on a triangular structure. *International Journal of Robust and Nonlinear Control* 2002; **12**:497–518.
13. Viel F, Busvelle E, Gauthier JP. Stability of polymerization reactors using I/O linearization and a high-gain observer. *Automatica* 1995; **31**:971–984.
14. Esfandiari F, Khalil HK. Output feedback stabilization of fully linearizable systems. *International Journal of Control* 1992; **56**:1007–1037.
15. Mita T. On zeros and responses of linear regulators and linear observers. *IEEE Transactions on Automatic Control* 1977; **AC-22**(3):423–428.
16. Polotskii VN. On the maximal errors of an asymptotic state identifier. *Automation and Remote Control* 1979; **11**:1116–1121.
17. Teel A, Praly L. Global stabilizability and observability imply semi-global stabilizability by output feedback. *System & Control Letters* 1994; **22**:313–325.
18. Teel A, Praly L. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization* 1995; **33**:1443–1488.

DOI: 10.1002/rnc

19. Tornambé A. Output feedback stabilization of a class of nonminimum-phase nonlinear systems. *System & Control Letters* 1992; **19**:193–204.
20. Andrieu V, Praly L, Astolfi A. High gain observers with updated gain and homogeneous correction terms. *Automatica* 2009; **45**(2):422–428.
21. Alvarez-Ramirez J, Fermat R. Robust PI stabilization of a class of chemical reactors. *System & Control Letters* 1999; **38**:219–225.
22. Byrnes CI, Celani F, Isidori A. Omega-limit sets of a class of nonlinear systems that are semiglobally practically stabilized. *International Journal of Robust and Nonlinear Control* 2005; **15**:315–333.
23. Gonzalez-Trejo J, Alvarez Ramirez J, Fernandez G. Robust control with uncertainty estimation for feedback linearizable systems: application to control of distillation columns. *Journal of Process Control* 1999; **9**:221–231.
24. Imsland L, Findeisen R, Bullinger E, Allgower F, Foss BA. A note on stability, robustness and performance of output feedback nonlinear model predictive control. *Journal of Process Control* 2003; **13**:633–644.
25. Isidori A. A remark on the problem of semiglobal nonlinear output regulation. *IEEE Transactions on Automatic Control* 1997; **42**(12):1734–1738.
26. Isidori A. A tool for semiglobal stabilization of uncertain nonminimum-phase nonlinear systems via output feedback. *IEEE Transactions on Automatic Control* 2000; **45**:1817–1827.
27. Janković M. Adaptive output feedback control of nonlinear feedback linearizable systems. *International Journal of Adaptive Control and Signal Processing* 1996; **10**:1–18.
28. Jiang Z-P, Hill DJ, Guo Y. Stabilization and tracking via output feedback for the nonlinear benchmark system. *Automatica* 1998; **34**:907–915.
29. Krishnamurthy P, Khorrami F. Dynamic high-gain scaling: state and output feedback with application to systems with ISS appended dynamics driven by all states. *IEEE Transactions on Automatic Control* 2004; **49**:2219–2239.
30. Lei H, Lin W. Universal adaptive control of nonlinear systems with unknown growth rate by output feedback. *Automatica* 2006; **42**:1783–1789.
31. Lin W, Qian C. Semiglobal robust stabilization of nonlinear systems by partial state and output feedback. *Proceedings of IEEE Conference on Decision and Control*, Tampa, FL, December 1998; 3105–3110.
32. Lin Z, Saberi A. Low-and-high gain design technique for linear systems subject to input saturation–a direct method. *International Journal of Robust and Nonlinear Control* 1997; **7**:1071–1101.
33. Maggiore M, Passino K. Output feedback tracking: a separation principle approach. *IEEE Transactions on Automatic Control* 2005; **50**:111–117.
34. Mahmoud MS, Zribi M. Output feedback stabilization of systems with nonlinear actuators. *International Journal of Control* 2000; **73**:187–196.
35. Praly L. Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate. *IEEE Transactions on Automatic Control* 2003; **48**(6):1103–1108.
36. Praly L, Jiang ZP. Linear output feedback with dynamic high gain for nonlinear systems. *System & Control Letters* 2004; **53**:107–116.
37. Qian C. Semi-global stabilization of a class of uncertain nonlinear systems by linear output feedback. *IEEE Transactions on Circuits and Systems II: Express Briefs* 2005; **52**(4):218–222.
38. Saberi A, Lin Z, Teel AR. Control of linear systems with saturating actuators. *IEEE Transactions on Automatic Control* 1996; **41**:368–378.
39. Serrani A, Isidori A, Marconi L. Semiglobal nonlinear output regulation with adaptive internal model. *IEEE Transactions on Automatic Control* 2001; **46**:1178–1194.
40. Shim H, Son YI, Seo JH. Semi-global observer for multi-output nonlinear systems. *System & Control Letters* 2000; **42**:233–244.
41. Shim H, Teel AR. Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback. *Automatica* 2003; **39**:441–454.
42. Singh SN, Zhang R, Chandler P, Banda S. *Decentralized nonlinear robust control of UAVs in close formation*, Vol. 13, 2003.
43. Ye X. Semiglobal output feedback control of uncertain nonlinear systems with unknown high frequency gain sign. *IEEE Transactions on Automatic Control* 2000; **45**:2402–2405.
44. Khalil HK, Esfandiari F. Semiglobal stabilization of a class of nonlinear systems using output feedback. *IEEE Transactions on Automatic Control* 1993; **38**(9):1412–1415.
45. Oh S, Khalil HK. Nonlinear output feedback tracking using high-gain observer and variable structure control. *Automatica* 1997; **33**:1845–1856.
46. Khalil HK. Robust servomechanism output feedback controllers for a class of feedback linearizable systems. *Automatica* 1994; **30**(10):1587–1599.
47. Khalil HK. On the design of robust servomechanisms for minimum-phase nonlinear systems. *International Journal of Robust and Nonlinear Control* 2000; **10**:339–361.
48. Mahmoud NA, Khalil HK. Asymptotic regulation of minimum phase nonlinear systems using output feedback. *IEEE Transactions on Automatic Control* 1996; **41**(10):1402–1412.
49. Seshagiri S, Khalil HK. Robust output feedback regulation of minimum-phase nonlinear systems using conditional integrators. *Automatica* 2005; **41**:43–54.
50. Seshagiri S, Khalil HK. Robust output regulation of minimum phase nonlinear systems using conditional servocompensators. *International Journal of Robust and Nonlinear Control* 2005; **15**:83–102.

DOI: 10.1002/rnc

51. Singh A, Khalil HK. Regulation of nonlinear systems using conditional integrators. *International Journal of Robust and Nonlinear Control* 2005; **15**:339–362.
52. Aloliwi B, Khalil HK. Robust adaptive output feedback control of nonlinear systems without persistence of excitation. *Automatica* 1997; **33**:2025–2032.
53. Khalil HK. Adaptive output feedback control of nonlinear systems represented by input–output models. *IEEE Transactions on Automatic Control* 1996; **41**:177–188.
54. Seshagiri S, Khalil HK. Output feedback control of nonlinear systems using RBF neural networks. *IEEE Transactions on Neural Networks* 2000; **11**:69–79.
55. Freidovich LB, Khalil HK. Lyapunov-based switching control of nonlinear systems using high-gain observers. *Automatica* 2007; **43**:150–157.
56. Dabroom AM, Khalil HK. Output feedback sampled-data control of nonlinear systems using high-gain observers. *IEEE Transactions on Automatic Control* 2001; **46**:1712–1725.
57. Ahrens JH, Khalil HK. High-gain observers in the presence of measurement noise: a switched-gain approach. *Automatica* 2009; **45**:936–943.
58. Vasiljevic LK, Khalil HK. Error bounds in differentiation of noisy signals by high-gain observers. *System & Control Letters* 2008; **57**:856–862.
59. Ahrens AH, Khalil HK. Closed-loop behavior of a class of nonlinear systems under EKF-based control. *IEEE Transactions on Automatic Control* 2007; **52**:536–540.
60. Atassi AN, Khalil HK. A separation principle for the stabilization of a class of nonlinear systems. *IEEE Transactions on Automatic Control* 1999; **44**:1672–1687.
61. Khalil HK. *Nonlinear Systems*, (3rd edn). Prentice Hall: Upper Saddle River, New Jersey, 2002.
62. Gauthier J-P, Kupka I. *Deterministic Observation Theory and Applications*. Cambridge University Press: Cambridge, United Kingdom, 2001.
63. Willems J. Almost invariant subspaces: an approach to high gain feedback design–Part I: almost controlled invariant subspaces. *IEEE Transactions on Automatic Control* 1981; **26**(1):235–252. DOI: 10.1109/TAC.1981.1102551.
64. Saberi A, Chen BM, Sannuti P. *Loop Transfer Recovery: Analysis and Design*, Communications and control engineering series. Springer-Verlag: New York, New York, 1993.
65. Franklin GF, Powell D, Workman ML. *Digital Control of Dynamic Systems*. Addison Wesley: Reading, Massachusetts, 1990.
66. Bullinger E, Allgower F. An adaptive high-gain observer for nonlinear systems. *Proceedings of 36th IEEE Conference on Decision and Control*, San Diego, California, 1997; 4348–4353.
67. Lei H, Wei J, Lin W. A global observer for observable autonomous systems with bounded solution trajectories. *Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05*, Seville, Spain, 2005; 1911–1916.
68. Astolfi A, Praly L. Global complete observability and output-to-state stability imply the existence of a globally convergent observer. *Mathematical Control Signals Systems* 2005; **18**(1):1–34.
69. Egardt B. *Stability of Adaptive Controllers*. Springer Verlag: New York, New York, 1979.
70. Peterson BB, Narendra K. Bounded error adaptive control. *IEEE Transactions on Automatic Control* 1982; **AC-27**(6):1161–1168.
71. Mareels IMY, Gils SV, Polderman JW, Ilchmann A. Asymptotic dynamics in adaptive gain control. *Advances in Control, Highlights of ECC '99*, Budapest, Hungry, 1999; 29–63.
72. Sanfelice RG, Teel AR. On the performance of high-gain observers with sign-indefinite gain adaptation under measurement noise. *Automatica* 2011; **47**(10):2141–2330.
73. Boizot JN, Busvelle E, Gauthier J-P. An adaptive high-gain observer for nonlinear systems. *Automatica* 2010; **46**(9):1483–1488.
74. Ball AA, Khalil HK. A nonlinear high-gain observer for systems with measurement noise. *IEEE Transactions on Automatic Control* 2013; **58**:569–580.
75. Sontag ED. Remarks on stabilization and input-to-state stability. *Proceedings of the 28th IEEE Conference on Decision and Control*, Tampa, Florida, 1989; 1376–1378.
76. Isidori A. *Nonlinear Control Systems*, (3rd edn). Springer-Verlag: New York, 1995.
77. Marino R, Tomei P. *Nonlinear Control Design*. Prentice Hall International: London, 1995.
78. Byrnes CI, Priscoli FD, Isidori A. *Output Regulation of Uncertain Nonlinear Systems*. Birkhauser: Boston, 1997.
79. Huang J. *Nonlinear Output Regulation: Theory and Applications*. SIAM: Philadelphia, 2004.

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