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Elements of linear algebra

Elements of linear algebra

Lecture outline

- Vector space, matrices and linear equations
- Euclidean spaces, orthogonality and projectors
- Eigenvalues, Eigenvectors and matrix decompositions
- Quadratic forms

Vector space, matrices and linear equations

- Vector space $\mathcal{V}$: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$
  - $\mathbf{u} + \mathbf{v} \in \mathcal{V} \quad ; \quad c\mathbf{u} \in \mathcal{V} \quad , \quad \forall c \in \mathbb{R}$
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad ; \quad (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $\exists \mathbf{0} \in \mathcal{V} : \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - $\exists (-\mathbf{u}) \in \mathcal{V} : (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
  - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \quad ; \quad (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \quad , \quad \forall a, b \in \mathbb{R}$
  - $(ab)\mathbf{u} = a(b\mathbf{u}) \quad , \quad \forall a, b \in \mathbb{R}$
- *Example of vector space:* $\mathbb{R}^n$

Vector space, matrices and linear equations

- Subspace $\mathcal{W} \subset \mathcal{V}$ iff $\mathbf{u}, \mathbf{v} \in \mathcal{W} \Rightarrow a\mathbf{u} + b\mathbf{v} \in \mathcal{W}$
- Linear combination $\mathbf{u}_i \in \mathcal{V}$, $a_i \in \mathbb{R}$ $\Rightarrow$ $\sum_{i=1}^{n} a_i \mathbf{u}_i$
- The set of all linear combinations, denoted as $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, defines a subspace of $\mathcal{V}$
- If every vector in $\mathcal{V}$ can be obtained as a linear combination of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ the latter *spans* vector space $\mathcal{V}$: $\mathcal{V} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$
- Linear independence: vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ are independent iff

$$\sum_{i=1}^{n} a_i \mathbf{u}_i = \mathbf{0} \Rightarrow (a_1, \ldots, a_n)^{\mathrm{T}} = \mathbf{0}$$

Vector space, matrices and linear equations

- A set of linear independent vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ that span $\mathcal{V}$ is a **base** of $\mathcal{V}$

Vector space, matrices and linear equations

- Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

- *Properties*

  - $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \qquad c\mathbf{A} = [c \cdot a_{ij}] \quad , \quad \forall c \in \mathbb{R}$
  - $\mathbf{C}_{(m \times o)} = \mathbf{A}_{(m \times n)} \mathbf{B}_{(n \times o)} = \left[c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \right]$

Vector space, matrices and linear equations

➢ *Properties (continues)*:

- $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \quad , \quad \mathbf{A}\mathbf{B}\neq\mathbf{B}\mathbf{A}$
- $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B})+\mathbf{C}$
- $a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B} \quad , \quad \forall a\in\mathbb{R}$
- $(a+b)\mathbf{A}=a\mathbf{A}+b\mathbf{A} \quad , \quad \forall a,b\in\mathbb{R}$
- $\mathbf{A}-\mathbf{A}=\mathbf{A}+(-\mathbf{A})=\mathbf{0}$
- $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{A}\mathbf{C}+\mathbf{B}\mathbf{C} \qquad \mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}$
- $(\mathbf{A}\mathbf{B})\mathbf{C}=\mathbf{A}(\mathbf{B}\mathbf{C})$

Vector space, matrices and linear equations

- Transpose $\mathbf{A} = [a_{ij}]$, $\mathbf{A}^\mathrm{T} = [a_{ji}]$

  Properties

  - $(a\mathbf{A})^\mathrm{T} = a\mathbf{A}^\mathrm{T}$ $(a\mathbf{A} + b\mathbf{B})^\mathrm{T} = a\mathbf{A}^\mathrm{T} + b\mathbf{B}^\mathrm{T}$
  - $(\mathbf{AB})^\mathrm{T} = \mathbf{B}^\mathrm{T}\mathbf{A}^\mathrm{T}$

- Trace $\mathrm{tr}(\mathbf{A}_{m\times m}) = \sum_{i=1}^{m} a_{ii}$

  Properties

  - $\mathrm{tr}(\mathbf{A}^\mathrm{T}) = \mathrm{tr}(\mathbf{A})$
  - $\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$
  - $\mathrm{tr}(c\mathbf{A}) = c\,\mathrm{tr}(\mathbf{A})$
  - $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$

Vector space, matrices and linear equations

- Symmetric matrix (square) $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$
- Skew-symmetric matrix (square) $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$
- Skew decomposition:

$$\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}})}_{\textit{Skew}} + \underbrace{\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}})}_{\textit{Symmetric}}$$

- Matrix inverse: $\mathbf{A}_{m\times m}^{-1}\mathbf{A}_{m\times m} = \mathbf{I}_m$

Property $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Vector space, matrices and linear equations

- Inverse of partitioned matrices:

$$\left(\begin{bmatrix}\mathbf{A} & \mathbf{B}) \\ \mathbf{C} & \mathbf{D})\end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

Corollaries:

$$(\mathbf{M}+\mathbf{N})^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1}(\mathbf{M}^{-1}+\mathbf{N}^{-1})^{-1}\mathbf{M}^{-1}$$

$$(\mathbf{M}+\mathbf{b}\mathbf{c}^{\mathrm{T}})^{-1} = \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{b}\mathbf{c}^{\mathrm{T}}\mathbf{M}^{-1}}{1+\mathbf{c}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{b}}$$

with $\mathbf{b}, \mathbf{c}$ *vectors*

Vector space, matrices and linear equations

- Determinant (Laplace formula)

$$\det(\mathbf{A}_{m\times m}) = \sum_{j=1}^{m} a_{ij}(-1)^{i+j} m_{ij} = \sum_{i=1}^{m} a_{ij}(-1)^{i+j} m_{ij}$$

- *Having chosen a column* j *or a row* i *:*

a_{ij} : *ij component of* $\mathbf{A}$

m_{ij} : *determinant of ij minor of* $\mathbf{A}$

- Cofactor of $\mathbf{A}$: $(-1)^{i+j} m_{ij}$
- Adjugate (or adjoint) of $\mathbf{A}$: $\mathrm{adj}(\mathbf{A}) = \left[(-1)^{i+j} m_{ij}\right]^{\mathrm{T}}$

Vector space, matrices and linear equations

- Determinant (Laplace formula)

$$\det(\mathbf{A}_{m\times m}) = \sum_{j=1}^{m} a_{ij}(-1)^{i+j} m_{ij} = \sum_{i=1}^{m} a_{ij}(-1)^{i+j} m_{ij}$$

Properties

- $\det(c\mathbf{A}_{m\times m}) = c^m \det(\mathbf{A}_{m\times m})$
- $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$ $\quad\quad$ $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
- $\mathbf{A}$ is invertible iff $\det(\mathbf{A}) \neq 0$
- $\mathbf{A}^{-1} = \dfrac{\mathrm{adj}(\mathbf{A})}{\det(\mathbf{A})}$

Vector space, matrices and linear equations

- Determinant (Laplace formula)

$$\det(\mathbf{A}_{m\times m}) = \sum_{j=1}^{m} a_{ij}(-1)^{i+j} m_{ij} = \sum_{i=1}^{m} a_{ij}(-1)^{i+j} m_{ij}$$

Properties (cont.)

- $$\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{B}) \\ \mathbf{C} & \mathbf{D}) \end{bmatrix}\right) = \det(\mathbf{A})\det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})$$
  $$= \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})$$

- $$\det(\mathbf{A} + \mathbf{B}\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})$$

- $$\det(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathrm{T}}) = \det(\mathbf{A})\det(1 + \mathbf{c}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{b})$$

Vector space, matrices and linear equations

- System of linear equations:

$$\underbrace{\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}}$$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Vector space, matrices and linear equations

- Range space

$$\mathcal{R}(\mathbf{A}_{(m\times n)}) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

- Null space

$$\mathcal{N}(\mathbf{A}_{(m\times n)}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subset \mathbb{R}^n$$

- Rank: maximum number of independent column vectors of $\mathbf{A}$

Regular or nonsingular matrix $\mathrm{rk}(\mathbf{A}_{m\times m}) = m$

Rank-deficiency $\mathrm{rk}(\mathbf{A}_{m\times n}) < \min(m, n)$

Vector space, matrices and linear equations

- Range space

$$\mathcal{R}(\mathbf{A}_{(m\times n)}) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

- Null space

$$\mathcal{N}(\mathbf{A}_{(m\times n)}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subset \mathbb{R}^n$$

- Rank: maximum number of independent column vectors of $\mathbf{A}$

Properties

- $\mathrm{rk}(\mathbf{A}\mathbf{B}) \leq \min(\mathrm{rk}(\mathbf{A}), \mathrm{rk}(\mathbf{B}))$
- $\mathrm{rk}(\mathbf{A}_{(m\times n)}\mathbf{B}_{(n\times p)}) \geq \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) - n$

Vector space, matrices and linear equations

- Solvability of linear systems of equations
- Homogeneous system: $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{0}$

  Trivial solution $\mathbf{x} = \mathbf{0}$

  Nontrivial solution iff $\mathrm{rk}(\mathbf{A}) < n$ *or* $\mathcal{N}(\mathbf{A}) \neq \{\mathbf{0}\}$
- Inhomogeneous system: $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{y}$

  If a solution exists, the system is <u>*consistent*</u>

  Consistency is guaranteed when $\mathrm{rk}(\mathbf{A}) = m$ *(full row rank)*

  Consistency is not guaranteed when $\mathrm{rk}(\mathbf{A}) < m$

Vector space, matrices and linear equations

- Solvability of linear systems of equations

$$\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{y}$$

- *System overdetermined if* $\mathrm{rk}(\mathbf{A}) < m$
- *System underdetermined if* $\mathrm{rk}(\mathbf{A}) < n$
- *Redundancy:* $\rho = m - r$
- *Parameter deficiency:* $n - r$

Vector space, matrices and linear equations

- Definite and indefinite matrices

- *Positive definite* $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$
- *Positive semidefinite* $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geq 0 \quad \forall \mathbf{x} \neq \mathbf{0}$
- *Negative defnite* $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} < 0 \quad \forall \mathbf{x} \neq \mathbf{0}$
- *Negative semidefinite* $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \leq 0 \quad \forall \mathbf{x} \neq \mathbf{0}$
- *Indefinite otherwise*
- *Tests:* $\mathbf{A}$ *is positive definite iff* $\forall \det(\mathbf{A}_i) > 0 \quad , \quad i = 1, \ldots, m$

Euclidean spaces, orthogonality and projectors

- Euclidean space

An Euclidean space is a vector space with inner product $(\mathbf{u},\mathbf{v})$

- $(\mathbf{u},\mathbf{v}) = (\mathbf{v},\mathbf{u}) \qquad \forall \mathbf{u},\mathbf{v} \in \mathcal{V}$
- $(c\mathbf{u},\mathbf{v}) = c(\mathbf{v},\mathbf{u}) \qquad \forall \mathbf{u},\mathbf{v} \in \mathcal{V}\ ,\ \forall c \in \mathbb{R}$
- $(\mathbf{u},\mathbf{u}) > 0 \quad \forall \mathbf{u} \neq \mathbf{0} \quad , \qquad (\mathbf{u},\mathbf{u}) = 0 \Rightarrow \mathbf{u} = \mathbf{0}$
- $(\mathbf{u}+\mathbf{v},\mathbf{w}) = (\mathbf{u},\mathbf{w}) + (\mathbf{v},\mathbf{w}) \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{V}$

Euclidean spaces, orthogonality and projectors

- Euclidean space
- *We define* $(\mathbf{u},\mathbf{v})_{\mathbf{W}} = \mathbf{u}^{\mathrm{T}}\mathbf{W}\mathbf{v}$ *with* $\mathbf{W}$ *<u>symmetric</u>*
- *Length:* $\|\mathbf{u}\|_{\mathbf{W}} = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{W}\mathbf{u}}$
- *Angle between two vectors:* $\cos(\theta_{\mathbf{W}}) = \dfrac{\mathbf{u}^{\mathrm{T}}\mathbf{W}\mathbf{v}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{W}\mathbf{u}}\sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{W}\mathbf{v}}}$

Euclidean spaces, orthogonality and projectors

- Euclidean space

  - *Orthogonality:* $\mathbf{u}^{\mathrm{T}}\mathbf{W}\mathbf{v} = 0$

  - *Orthogonal set* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *if* $\mathbf{u}_i^{\mathrm{T}}\mathbf{W}\mathbf{u}_j = 0 \quad , \quad i \neq j$

  - *Orthonormal set* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *if* $\mathbf{u}_i^{\mathrm{T}}\mathbf{W}\mathbf{u}_j = \delta_{ij}$

- Orthonormal matrices: column vectors are orthonormal vectors

  Properties:

  - $\det(\mathbf{U}) = \pm 1$ $\qquad$ $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$

Euclidean spaces, orthogonality and projectors

- Euclidean space

  - *Cross product in* $\mathbb{R}^3$ *:*

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

  - *Alternative notation:*

$$\mathbf{u} \times \mathbf{v} = \mathbf{\Omega}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \mathbf{v}$$

Euclidean spaces, orthogonality and projectors

- Projectors (or idempotent matrices) $\mathbf{P}\mathbf{P} = \mathbf{P}$

  Properties

  - $\mathbf{I} - \mathbf{P}$ *is a projector*
  - $\mathcal{R}(\mathbf{P}) = \mathcal{N}(\mathbf{I} - \mathbf{P})$
  - $\mathcal{N}(\mathbf{P}) = \mathcal{R}(\mathbf{I} - \mathbf{P})$

- Orthogonal projector (relative to the range space of $\mathbf{A}$):

$$\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}$$

$$\mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I}$$

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}$$

Euclidean spaces, orthogonality and projectors

- Pythagorean theorem

$$\|\mathbf{y}\|_{\mathbf{W}}^2 = \|\mathbf{P}_{\mathbf{A}}\mathbf{y}\|_{\mathbf{W}}^2 + \|\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{y}\|_{\mathbf{W}}^2$$

- Orthogonal projector (relative to the range space of $\mathbf{A}$):

$$\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}$$

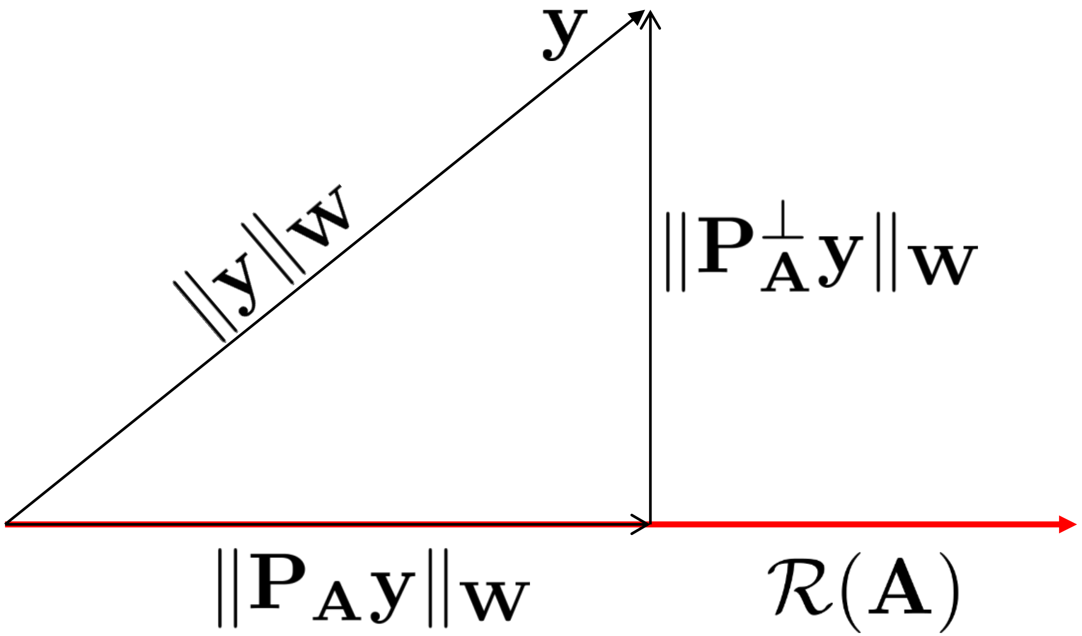
$$\mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I}$$

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}$$

Euclidean spaces, orthogonality and projectors

- Pythagorean theorem

$$\|\mathbf{y}\|_{\mathbf{W}}^2 = \|\mathbf{P_A}\mathbf{y}\|_{\mathbf{W}}^2 + \|\mathbf{P_A^\perp}\mathbf{y}\|_{\mathbf{W}}^2$$

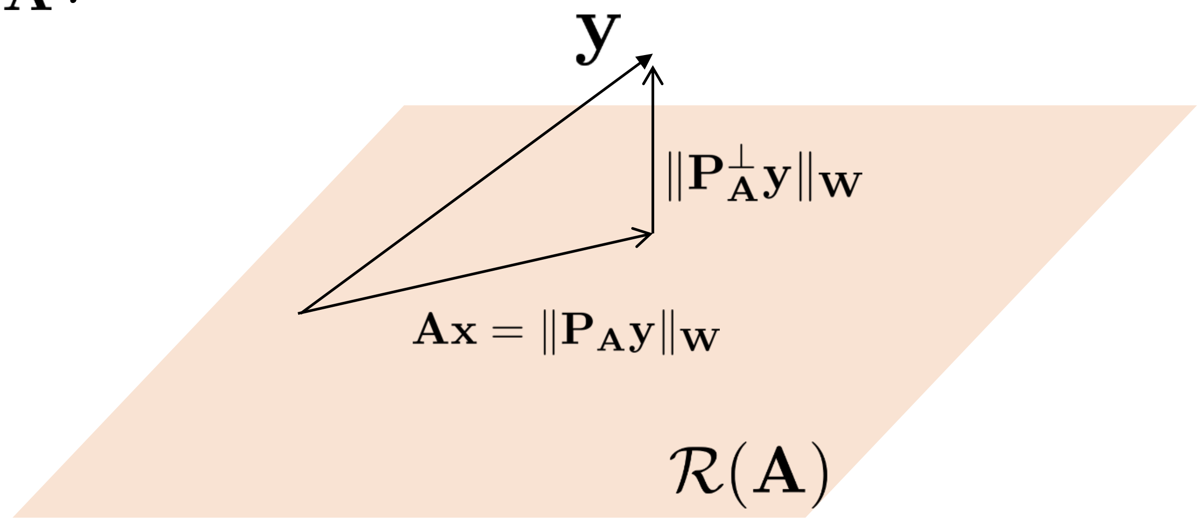


Euclidean spaces, orthogonality and projectors

- Pythagorean theorem

$$\|\mathbf{y}\|^2_{\mathbf{W}} = \|\mathbf{P}_{\mathbf{A}}\mathbf{y}\|^2_{\mathbf{W}} + \|\mathbf{P}^{\perp}_{\mathbf{A}}\mathbf{y}\|^2_{\mathbf{W}}$$

- *How do we find the nearest vector to* $\mathbf{y}$ *belonging to the range space of* $\mathbf{A}$ *?*



Euclidean spaces, orthogonality and projectors

- Pythagorean theorem

$$\|\mathbf{y}\|_{\mathbf{W}}^2 = \|\mathbf{P}_{\mathbf{A}}\mathbf{y}\|_{\mathbf{W}}^2 + \|\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{y}\|_{\mathbf{W}}^2$$

- *Application: solution of a weighted least-squares problem:*

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 &= \min_{\mathbf{x}\in\mathbb{R}^n} \left[\|\mathbf{P}_{\mathbf{A}}\mathbf{y}-\mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \|\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{y}\|_{\mathbf{W}}^2\right] \\ &= \min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{P}_{\mathbf{A}}\mathbf{y}-\mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 = 0\end{aligned}$$

$$\Updownarrow$$

$$\hat{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{y}$$

Eigenvalues, Eigenvectors and matrix decompositions

- Eigenvalues and eigenvectors

Consider solving the linear system of equations

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

Unknowns: $\mathbf{x} \in \mathbb{R}^n \quad , \quad \lambda \in \mathbb{R}$

Nontrivial solutions iff $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ *(characteristic equation)*

- *Eigenvalues:* λ_i
- *Eigenvectors:* $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = 0$

Eigenvalues, Eigenvectors and matrix decompositions

- Eigenvalues and eigenvectors

Consider solving the linear system of equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Unknowns: $\mathbf{x} \in \mathbb{R}^n \quad , \quad \lambda \in \mathbb{R}$

Properties

\- *There exist* n *eigenvalues (not necessarily distinct)*
\- *Eigenvalues of diagonal and triangular matrices are the diagonal entries*
\- *Eigenvalues of symmetric matrix are real*

Eigenvalues, Eigenvectors and matrix decompositions

- Eigenvalues and eigenvectors

Consider solving the linear system of equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Unknowns: $\mathbf{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

Properties (continue)

- *Eigenvalues of positive definite matrix are positive*

- *Eigenvalues of orthonormal matrix are* $= \pm 1$

- *Eigenvalues of a projector:* $\{0, 1\}$

Eigenvalues, Eigenvectors and matrix decompositions

- Eigenvalues and eigenvectors

Consider solving the linear system of equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

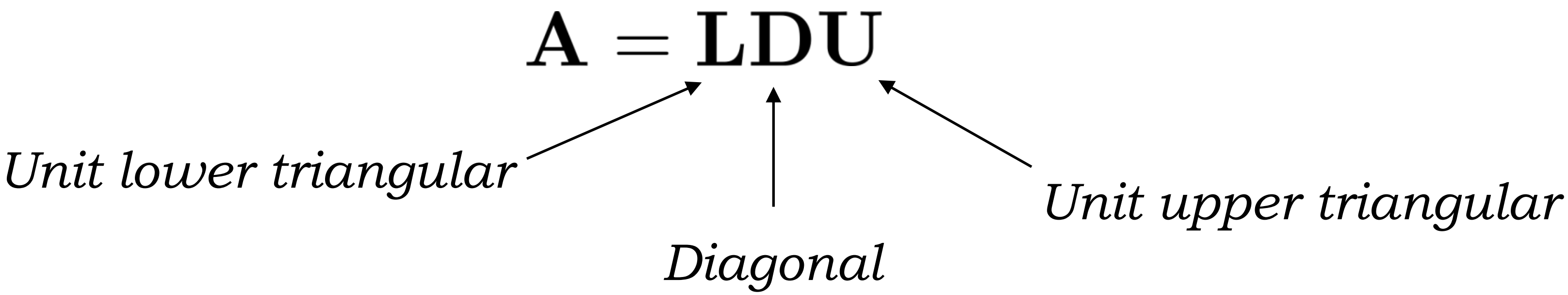
Unknowns: $\mathbf{x} \in \mathbb{R}^n \quad , \quad \lambda \in \mathbb{R}$

Properties (continue)

- $\det(\mathbf{A}_{n\times n}) = \lambda_1\lambda_2\ldots\lambda_n$
- $\mathrm{tr}(\mathbf{A}_{n\times n}) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

Eigenvalues, Eigenvectors and matrix decompositions

- Matrix decompositions: LDU / LDL decompositions



$$\mathbf{A} = \mathbf{LDU}$$

Unit lower triangular

Diagonal

Unit upper triangular

- *Exists if* $\det(\mathbf{A}_i) \neq 0 \quad , \quad i = 1, \ldots, (n-1)$
- *If* $\mathbf{A}$ *is symmetric:* $\mathbf{A} = \mathbf{LDL}^{\mathrm{T}}$
- *If* $\mathbf{A}$ *is positive definite, the decomposition* $\mathbf{A} = \mathbf{LDL}^{\mathrm{T}}$ *is unique, and the entries of* $\mathbf{D}$ *are all positive*

Eigenvalues, Eigenvectors and matrix decompositions

- Matrix decompositions: Cholesky decomposition

  If $\mathbf{A}$ *is positive definite,*

$$\mathbf{A} = \mathbf{G}^{\mathrm{T}}\mathbf{G}$$

  with $\mathbf{G}$ *a unique upper triangular matrix.*

- *Derived from the LDL decomposition as*

$$\mathbf{G} = \mathbf{D}^{\frac{1}{2}}\mathbf{L}$$

Eigenvalues, Eigenvectors and matrix decompositions

- Matrix decompositions: Singular Value Decomposition (SVD)

$$\mathbf{A}_{m\times n} = \mathbf{U}_m \mathbf{S}_{m\times n} \mathbf{V}_n^{\mathrm{T}}$$

with $\mathbf{U}$, $\mathbf{V}$ orthonormal and $\mathbf{S}$ “diagonal” (entries: singular values).

- Main application: solution of homogeneous system $\mathbf{A}\mathbf{x} = 0$

- Null vector is the vector column of $\mathbf{V}^{\mathrm{T}}$ corresponding to the singular value equal to zero

Eigenvalues, Eigenvectors and matrix decompositions

- Matrix decompositions: Eigenvalue Decomposition (SVD)

$$\mathbf{A}_{m\times m} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}}$$

with $\mathbf{U}$ orthonormal and $\mathbf{\Lambda}$ diagonal.

- $\mathbf{\Lambda}$ contains the eigenvalues
- $\mathbf{U}$ contains the corresponding eigenvectors

Eigenvalues, Eigenvectors and matrix decompositions

- Matrix decompositions: Orthogonal decomposition (QR factorization)

  Any matrix $\mathbf{A}_{m\times n}$ can be factorized as (for $m \geq n$)

  $$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

  with $\mathbf{Q}_{m\times m}$ orthonormal w.r.t. $\mathbf{W}$: $\mathbf{Q}^{\mathrm{T}}\mathbf{W}\mathbf{Q}$

  and $\mathbf{R}_{m\times n}$ triangular

- Cholesky decomposition of $\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A}$

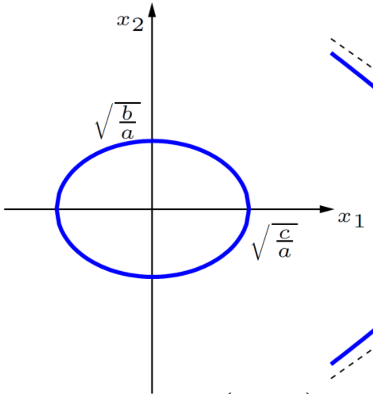
$$\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A} = \mathbf{R}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{W}\mathbf{Q}\mathbf{R} = \mathbf{R}^{\mathrm{T}}\mathbf{R}$$

Quadratic forms

- Quadratic forms $$p(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

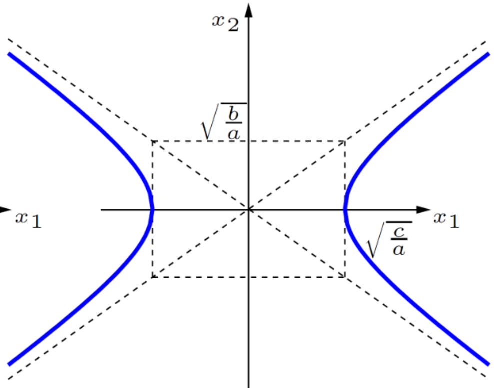
  with $\mathbf{A}$ *symmetric.*

- *Points for which* $p(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = c$ *describe different geometrical figures. Example in 2D:*



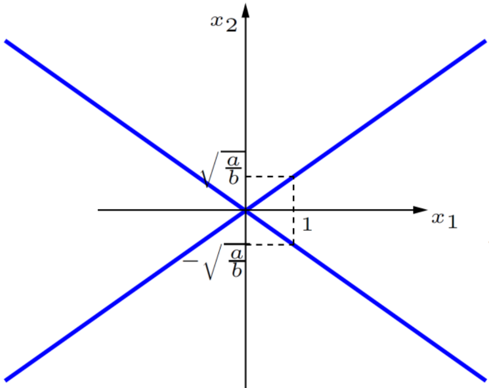
$\mathbf{A} = \mathrm{diag}(a, b)$

$a, b, c > 0$



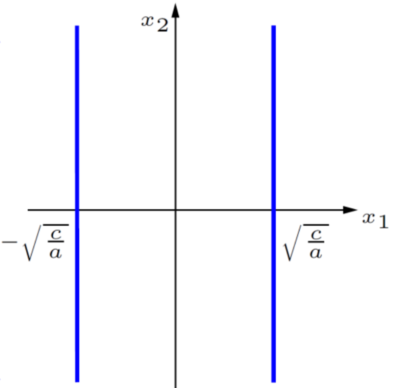
$\mathbf{A} = \mathrm{diag}(a, -b)$

$a, b, c > 0$



$\mathbf{A} = \mathrm{diag}(a, -b)$

$a, b > 0 \quad c = 0$



$\mathbf{A} = \mathrm{diag}(a, b)$

$a, c > 0 \quad b = 0$

Quadratic forms

- (Hyper)ellipsoid and principal axes

  If $\mathbf{A}$ is positive definite $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = c$ is the equation of a (hyper)ellipsoid

- SVD decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$ gives principal axes :

  length of semiaxis: $\sqrt{\frac{c}{\lambda_i}}$

  direction of semiaxis: $\mathbf{u}_i$