Robust Extended Kalman Filtering

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Abstract— **Linearization errors inherent in the specification of an extended Kalman filter (EKF) can severely degrade its performance. This correspondence presents a new approach to the robust design of a discrete-time EKF by application of the robust linear design methods based on the H_∞ norm minimization criterion. The results of simulations are presented to demonstrate an advantage for signal demodulation and nonlinear equalization applications.**

I. INTRODUCTION

The use of the extended Kalman filter (EKF) is heavily entrenched in nonlinear signal processing applications. The EKF is derived via a linearization procedure, which is presented in [1]. Briefly, the nonlinear model describing the measurements is successively linearized about each current state estimate, and the (linear) Kalman filter is then applied to produce the next state estimate. Suboptimal behavior can arise when there are large deviations of the estimated state trajectory from the nominal trajectory. In such cases, the nonlinear signal model is less accurately approximated by the Taylor series expansion about the conditional mean, and the higher order terms become more significant. However, in the EKF, the higher order terms are ignored. The approach taken in this correspondence is not to neglect the higher order terms of the Taylor series expansions but rather assume them to be functions of the state estimation error and the exogenous inputs which have bounded H_∞ norm. This approach naturally leads to a minimax estimation problem that can be treated using standard H_∞ methods [2]–[11]. The chosen norm bounds cannot, in general, be precomputed; therefore, they are regarded as "tuning" parameters for the resulting filter. The choice of zero as the norm bound results in the standard EKF.

The original contribution of this work is the application of linear H_∞ techniques to arrive at an "extended H_∞ filter" and the description of examples that demonstrate some advantages compared with the EKF. An extended H_∞ filter has also been described elsewhere [10]. Our approach differs from [10] in the respect that here, it is recognized that linearization can give rise to modeling errors. This correspondence describes some extensions to [11] and is organized as follows. Some linear, discrete-time H_∞ results are described in Section II. In Section III, we state a nonlinear filtering problem and a conventional EKF solution. In Section IV, we motivate the linearization errors as a model uncertainty problem and set out an extended H_∞ filter. Some extensions of the EKF and extended H_∞ filter are detailed in Section V. The results of some simulation studies are presented in Section VI.

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II. PRELIMINARIES

In this section, we state a game-theoretic H_∞ solution for a linear filtering problem. The H_∞ filter has been variously reported in, e.g., [2], [4]–[6], [8]–[10]. We begin by considering the linear discrete-time signal model $x_{t+1} = A_t x_t + B_t w_t, y_t = C_t x_t + v_t$, where $x_t \in \boldsymbol{R}^N, w_t \in \boldsymbol{R}^M, v_t, y_t \in \boldsymbol{R}^P, A_t, B_t$, and C_t are matrices of appropriate dimension; w_t and v_t are uncorrelated zero mean white noise processes having known covariances Q_t and R_t, respectively. We seek to find a linear filter that estimates the quantity $z_t = H_t x_t \in R^Q$ from the observed data y_t such that the error $\tilde{z}_{t/t} = H_t x_t - \hat{z}_{t/t}$ satisfies a worst-case performance criterion.

It is desired to minimize the maximum overall input noises $(w_t$ and $v_t)$, the fraction $\|\tilde{z}_{t/t}\|_2^2 / \|w_t\|_2^2 + \|v_t\|_2^2$, which is the energy of the estimation error normalized by the energy of the input noises. The so-called minimax problem is to bound the above fraction by some $\gamma^2, \gamma \in \boldsymbol{R}$ for the worst-case scenario, i.e., the least favorable noises w_t and v_t. That is, we seek a filter that achieves the performance $\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2(\|w_t\|_2^2 + \|v_t\|_2^2)$ for some minimum γ^2. The H_∞ solution arises as a saddle-point strategy in game theory (see [2]–[10]) and follows from the solution of a Riccati equation. In the stationary case, an interpretation of the H_∞ criterion is that the maximum magnitude of the error power spectrum density is bounded above by γ^2.

The filtering problem may be written in a generalized regulator framework [3], [8] as

$$
\begin{aligned}
x_{t+1} &= A_t x_t + B_t w_t \\
\tilde{z}_{t/t} &= H_t x_t - \hat{y}_{t/t} \\
y_t &= C_t x_t + v_t.
\end{aligned} \tag{1}
$$

For ease of comparison with Kalman filters, we introduce scaling of the input noise sequences. We restrict our attention to problems where the input noises are uncorrelated and denote $Q_t = \operatorname{diag}\{(\sigma_{w_t}^{(1)})^2, \cdots, (\sigma_{w_t}^{(M)})^2\}$ and $R_t = \operatorname{diag}\{(\sigma_{v_t}^{(1)})^2, \cdots, (\sigma_{v_t}^{(P)})^2\}$ as the covariances of w_t and v_t, respectively. These scalings may be interpreted as simple weights because H_∞ filtering does not rely on the availability of statistical information on w_t and v_t [8]. The corresponding performance is

$$
\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2 \left(\sum_{i=1}^{M} (\sigma_{w_t}^{(i)})^{-2} \|w_t\|_2^2 + \sum_{i=1}^{P} (\sigma_{v_t}^{(i)})^{-2} \|v_t\|_2^2 \right).
$$

A solution to the H_∞ filtering problem (1) can be found via a simplification of the control results in [3]. It is given by

$$
\begin{aligned}
\hat{x}_{t+1/t} &= A_t \hat{x}_{t/t-1} + A_t L_t \tilde{z}_{t/t-1} \\
\hat{z}_{t/t} &= H_t \hat{x}_{t/t-1} + H_t L_t \tilde{z}_{t/t-1}
\end{aligned} \tag{2}
$$

where $L_t = \Sigma_{t/t-1} C_t^T (C_t \Sigma_{t/t-1} C_t^T + R_t)^{-1}$ is the predictor gain, $\tilde{z}_{t/t-1} = z_t - H_t \hat{x}_{t/t-1}$ is the prediction error, and $\Sigma_{t/t-1}$ arises from the Riccati difference equation (RDE)

$$
\begin{aligned}
\Sigma_{t/t} = \Sigma_{t/t-1} - \Sigma_{t/t-1} &[-H_t^T \ C_t^T] \\
&\cdot \begin{bmatrix} H_t \Sigma_{t/t-1} H_t^T - \gamma^2 I & -H_t \Sigma_{t/t-1} C_t^T \\ -C_t \Sigma_{t/t-1} H_t^T & C_t \Sigma_{t/t-1} C_t^T + R_t \end{bmatrix}^{-1} \\
&\cdot \begin{bmatrix} -H_t \\ C_t \end{bmatrix} \Sigma_{t/t-1}
\end{aligned} \tag{3}
$$

$$
\Sigma_{t+1/t} = A_t \Sigma_{t/t} A_t^t + B_t Q_t B_t^T. \tag{4}
$$

A minimum value for γ^2 needs to be found by searching over $\gamma \ne 0$ such that $\Sigma_{t+1/t} > 0 \ \forall \{w_t\} \in l_2, \ \{v_t\} \in l_2$. It can be easily shown that with $\gamma = \infty$, the RDE (3) reverts to the Kalman RDE, which is given by (11). Thus, the γ may be thought of as a tuning parameter to control the trade off between H_∞ performance and minimum variance performance.

III. Nonlinear Problem Statement

We consider the nonlinear, discrete-time signal model

$$\begin{aligned} x_{t+1} &= a_t(x_t) + b_t(x_t)w_t \\ y_t &= c_t(x_t) + v_t \end{aligned} \tag{5}$$

where $w_t \in \boldsymbol{R}^M, v_t, y_t \in \boldsymbol{R}^P$, $a, b,$ and c are sufficiently smooth functions of appropriate dimension, and w_t and v_t are uncorrelated zero mean white noise processes having known covariances Q_t and R_t, respectively. The objective is to arrive at a causal filter that produces approximate conditional mean estimates of x_t, given the measurements of y_t. The nonlinear functions $a_t, b_t,$ and c_t can be expanded in a Taylor series about the filtered and predicted estimates $\hat{x}_{t/t}$ and $\hat{x}_{t/t-1}$ as

$$\begin{aligned} a_t(x_t) &= a_t(\hat{x}_{t/t}) + \boldsymbol{D}a_t(\hat{x}_{t/t})(x_t - \hat{x}_{t/t}) \\ &\quad + \Delta_1(x_t - \hat{x}_{t/t}) \\ b_t(x_t) &= b_t(\hat{x}_{t/t}) + \Delta_2(x_t - \hat{x}_{t/t}) \\ c_t(x_t) &= c_t(\hat{x}_{t/t-1}) + \boldsymbol{D}c_t(\hat{x}_{t/t-1})(x_t - \hat{x}_{t/t-1}) \\ &\quad + \Delta_3(x_t - \hat{x}_{t/t-1}) \end{aligned} \tag{6}$$

where $\boldsymbol{D}$ denotes the gradient. Here, $\Delta_i, i = 1, 2, 3$ are assumed to be continuous operators from $l_2 \to l_2$ with induced norms bounded by δ_i, respectively. The Δ_i represent the higher order terms of the Taylor series expansions (which are functions of $x_{t/t}$ and $x_t - \hat{x}_{t/t}$) and have not been explicitly included in the formulation of the EKF. Denoting the filter state error $x_t - \hat{x}_{t/t}$ by $\tilde{x}_{t/t}$ and the predictor state error $x_t - \hat{x}_{t/t-1}$ by $\tilde{x}_{t/t-1}$ and substituting (6) into (5) gives

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t + p_t + \Delta_1(\tilde{x}_{t/t}) + \Delta_2(\tilde{x}_{t/t})w_t \\ y_t &= C_t x_t + v_t + q_t + \Delta_3(\tilde{x}_{t/t-1}) \end{aligned} \tag{7}$$

where $A_t = \boldsymbol{D}a_t(\hat{x}_{t/t}), B_t = b_t(\hat{x}_{t/t}), C_t = \boldsymbol{D}c_t(\hat{x}_{t/t-1})$, and the known inputs are $p_t = a_t(\hat{x}_{t/t}) - A_t\hat{x}_{t/t}$ and $q_t = c_t(\hat{x}_{t/t-1}) - C_t\hat{x}_{t/t-1}$. The EKF arises by determining the Kalman filter for the linearized system (7) with the higher order terms set to zero as

$$\hat{x}_{t/t} = \hat{x}_{t/t-1} + L_t(y_t - c_t(\hat{x}_{t/t-1})) \tag{8}$$

$$\hat{x}_{t+1/t} = a_t(\hat{x}_{t/t}) \tag{9}$$

$$L_t = \Sigma_{t/t-1}\, C_t^T (C_t \Sigma_{t/t-1} C_t^T + R_t)^{-1} \tag{10}$$

$$\begin{aligned} \Sigma_{t/t} &= \Sigma_{t/t-1} - \Sigma_{t/t-1} C_t^T (C_t \Sigma_{t/t-1} C_t^T + R_t)^{-1} \\ &\quad \cdot C_t \Sigma_{t/t-1} \end{aligned} \tag{11}$$

$$\Sigma_{t+1/t} = A_t \Sigma_{t/t} A_t^T + B_t Q_t B_t^T. \tag{12}$$

These iterations need to be suitably initialized at $t = 0$.

IV. Extended H_∞ Filter

We now describe the main result of this correspondence, namely, a H_∞ filter for the nonlinear signal model (5). The objective is to solve a H_∞ optimization problem, namely, we desire to estimate the states in (5) and satisfy a H_∞ performance criterion for all uncertainties $\Delta_1, \Delta_2,$ and Δ_3 satisfying their respective norm bounds. In lieu of the problem (7) possessing uncertainties, we pose an auxiliary problem defined as

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t + p_t + s_t \\ \tilde{x}_{t/t} &= x_t - \hat{x}_{t/t} \\ y_t &= C_t x_t + v_t + q_t + r_t \end{aligned} \tag{13}$$

where $s_t = \Delta_1(\tilde{x}_{t/t}) + \Delta_2(\tilde{x}_{t/t})w_t$ and $r_t = \Delta_3(\tilde{x}_{t/t}) \approx \Delta_3(\tilde{x}_{t/t-1})$ are additional exogenous inputs satisfying $\|s_t\|_2^2 \le \delta_1^2\|\tilde{x}_{t/t}\|_2^2 + \delta_2^2\|w_t\|_2^2$ and $\|r_t\|_2^2 \le \delta_3^2\|\tilde{x}_{t/t}\|_2^2$. A solution to (13) can be obtained by solving another problem in which w_t and v_t are scaled in lieu of the extra inputs s_t and r_t. Consider the scaled H_∞ problem

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t c_w w_t + p_t \\ \tilde{x}_{t/t} &= x_t - \hat{x}_{t/t} \\ y_t &= C_t x_t + c_v v_t + q_t \end{aligned} \tag{14}$$

where $c_v^2 = 1 - \gamma^2\delta_1^2 - \gamma^2\delta_3^2$, and $c_w^2 = c_v^2(1 + \delta_2^2)^{-1}$.

Lemma 1: Suppose for a given $\gamma \ne 0$ that the scaled H_∞ optimization problem (14) is solvable, i.e., $\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2(c_w^{-2}\|w_t\|_2^2 + c_v^{-2}\|v_t\|_2^2)$. Then, this guarantees the performance $\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2(\|w_t\|_2^2 + \|s_t\|_2^2 + \|v_t\|_2^2 + \|r_t\|_2^2)$ for the solution of the auxiliary H_∞ optimization problem (13).

Proof: From the assumption that problem (14) is solvable, it follows that $\|\tilde{x}_{t/t}\|_2^2 \le \gamma^2(c_w^{-2}\|w_t\|_2^2 + c_v^{-2}\|v_t\|_2^2)$. By substituting for c_w^2 and c_v^2, we obtain $(1 - \gamma^2\delta_1^2 - \gamma^2\delta_3^2)\|\tilde{x}_{t/t}\|_2^2 \le \gamma^2((1 + \delta_2^2)\|w_t\|_2^2 + \|v_t\|_2^2)$. Rearranging and using the assumed bounds for $\|s_t\|_2^2$ and $\|r_t\|_2^2$ yields $\|\tilde{x}_{t/t}\|_2^2 \le \gamma^2(\|w_t\|_2^2 + \|s_t\|_2^2 + \|r_t\|_2^2 + \|v_t\|_2^2)$.

The extended H_∞ filter is the standard H_∞ filter for the problem given by (14), which, by virtue of the lemma, implies a filter for (13). The filter has the structure of the EKF (8)–(12), except that the approximate error covariance correction (11) is instead given by (3), in which $H_t = I$, the w_t, and v_t are scaled by c_w and c_v, respectively. It follows that when the state error and the process noise power are negligibly small, then at $\gamma = \infty$, the extended H_∞ filter reverts to the EKF. The approximations in the EKF are reasonable when the state error is small (so that $\Delta_1(\tilde{x}_{t/t}) \approx 0$ and $\Delta_3(\tilde{x}_{t/t-1}) \approx 0$), and the process noise power is small (so that $\Delta_2(\tilde{x}_{t/t})w_t \approx 0$). Thus, for the linearized signal model, with $\delta_1 = \delta_2 = \delta_3 = 0$, we simply have the application of a H_∞ filter, which reverts to the Kalman filter at $\gamma = \infty$.

The above scalings only guarantee that a H_∞ performance criterion is met. The design may well be too conservative, and it is prudent to explore the merits of using values for $\delta_1, \delta_2,$ and δ_3 less than the norm bounds on the respective model uncertainties. The use of the minimum possible γ may also be too conservative—too much emphasis may be placed in accommodating the worst case input conditions at the cost of mean square error (MSE). It may be advantageous to choose a γ greater than the minimum possible value and realize a compromise between MSE and H_∞ error performance.

A conventional approach that attempts to mitigate against linearization errors is the *second-order EKF* [12]. This filter arises by retaining second-order terms of the Taylor series expansions of the plant parameters about the conditional mean. Another approach is the *iterated filter* [12], [13], where the EKF recursions are iterated at each observation until it is deemed that better estimates have been obtained. Both of these methods increase the computational overhead, whereas the procedure outlined here does not.

V. Extensions

In the sequel, we consider the signal model

$$\begin{aligned} x_{t+1} &= a_t(x_t) + b_t(x_t)w_t \\ y_t &= c_t(x_t) + d_t(x_t)w_t + v_t \end{aligned} \tag{15}$$

where the direct feedthrough function $d_t(x_t)$ is assumed to be sufficiently smooth so that it may be expanded in a Taylor series about the predicted estimate as

$$d_t(x_t) = d_t(\hat{x}_{t/t-1}) + \Delta_4(x_t - \hat{x}_{t/t-1}). \tag{16}$$

Substituting (6) and (16) into (15) yields the linearized system

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t + p_t + \Delta_1(\tilde{x}_{t/t}) + \Delta_2(\tilde{x}_{t/t})w_t \\ y_t &= C_t x_t + D_t w_k + v_t + q_t + \Delta_3(\tilde{x}_{t/t-1}) \\ &\quad + \Delta_4(\tilde{x}_{t/t-1})w_t \end{aligned} \tag{17}$$

where $D_t = d_t(\hat{x}_{t/t-1})$. The EKF corresponding to (15) arises by determining the Kalman filter for (17) with the Δ_i set to zero. A Kalman filter that accommodates the direct feedthrough term D_t can be deduced via either a correlated noise argument (see [1], [13]) or the solution of the general regulator problem [3]. The predictor step then follows by making use of p_t and q_t, i.e.,

$$\hat{x}_{t+1/t} = a(\hat{x}_{t/t}) + B_t Q_t D_t^T \overline{R}_t^{-1}(y_t - c_t(\hat{x}_{t/t})) \tag{18}$$

where $\overline{R}_t = R_t + D_t Q_t D_t^T$. It is easily shown (see [1, p. 117]) that the approximate error covariance prediction is given by

$$\begin{aligned} \Sigma_{t+1/t} &= (A_t - B_t Q_t D_t^T \overline{R}_t^{-1} C_t)\Sigma_{t/t}(A_t - B_t Q_t d_t^T \overline{R}_t^{-1} C_t)^T \\ &\quad + B_t(Q_t - Q_t D_t^T \overline{R}_t^{-1} D Q_t)B_t^T. \end{aligned} \tag{19}$$

The corrector step is unchanged. The filter output is given by

$$\hat{y}_{t/t} = c(\hat{x}_{t/t}) + D_t Q_t D_t^T \overline{R}_t^{-1}(y_t - c_t(\hat{x}_{t/t})). \tag{20}$$

An extended H_∞ filter for the model (15) may be constructed by approximating (17) by an auxiliary problem defined by

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t + p_t + s_t \\ \tilde{x}_{t/t} &= x_t - \hat{x}_{t/t} \\ y_t &= C_t x_t + D_t w_t + v_t + q_t + r_t \end{aligned} \tag{21}$$

where r_t is an additional input satisfying $\|r_t\|_2^2 \le \delta_3^2\|\tilde{x}_{t/t}\|_2^2 + \delta_4^2\|w_t\|_2^2$. Rather than address the auxiliary problem (21), which has additional exogenous inputs, we consider instead the scaled H_∞ problem

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t c_w w_t + p_t \\ \tilde{x}_{t/t} &= x_t - \hat{x}_{t/t} \\ y_t &= C_t x_t + D_t c_w w_t + c_v v_t + q_t \end{aligned} \tag{22}$$

where $c_w^2 = c_v^2(1+\delta_2^2+\delta_4^2)^{-1}$ in which $c_v^2 = 1 - \gamma^2\delta_1^2 - \gamma^2\delta_3^2$.

Lemma 2: Suppose that for a given $\gamma \neq 0$, the scaled H_∞ optimization problem (22) is solvable, and $\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2(c_w^{-2}\|w_t\|_2^2 + |c_v^{-2}\|v_t\|_2^2)$. Then, this guarantees the performance $\|\tilde{z}_{t/t}\|_2^2 \le \gamma^2(\|w_t\|_2^2 + \|s_t\|_2^2 + \|v_t\|_2^2 + \|r_t\|_2^2)$ for the solution of the auxiliary H_∞ optimization problem (21).

Proof: The result follows by making use of $c_w^2 = c_v^2(1+\delta_2^2+\delta_4^2)^{-1}$ in the proof of Lemma 1.

VI. Applications

Example 1: The EKF is commonly applied to demodulate frequency modulated (FM) signals [1]. For this application, the signal is usually modeled such that the signal nonlinearity is present only in the output mapping. Here, we replace the pure integrator in the models of [1] with a saturating nonlinearity in an endeavor to better approximate some practical systems. The FM model is given by

$$\begin{aligned} \omega_{t+1} &= \mu\omega_t + w_t \\ \varphi_{t+1} &= \arctan(\lambda\varphi_t + \omega_t) \\ y_t^{(1)} &= \cos\varphi_t + v_t^{(1)} \\ y_t^{(2)} &= \sin\varphi_t + v_t^{(2)}. \end{aligned} \tag{23}$$

Let σ_w^2 and σ_v^2 denote the variances of w_t and v_t, respectively. When σ_w^2 is small, the model output is an FM signal. Alternatively, when σ_w^2 is sufficiently large, the model output resembles a phase shift keyed signal. The objective is to estimate the frequency message ω_t from the noisy in-phase and quadrature observations $y_t^{(1)}$, $y_t^{(2)}$. Choosing

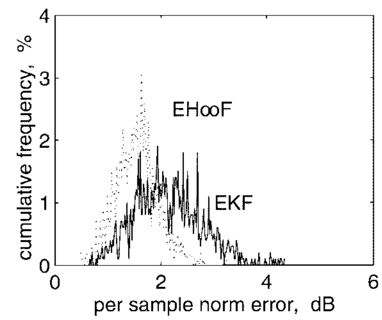

Fig. 1. Error histogram for Example 1.

$$x_t = \begin{bmatrix}\varphi_t \\ \omega_t\end{bmatrix}$$

yields

$$A_t = \begin{bmatrix} \dfrac{\lambda}{(\lambda\varphi_r+\omega_t)^2+1} & \dfrac{1}{(\lambda\varphi_t+\omega_t)^2+1} \\ 0 & \mu \end{bmatrix}$$

for the state linearization, and

$$C_t(\hat{x}_t) = \begin{bmatrix} -\sin(\hat{\varphi}_t) & 0 \\ \cos(\hat{\varphi}_t) & 0 \end{bmatrix}$$

for the output mapping. Simulations were conducted with $\mu = 0.9, \lambda = 0.99$, and $\sigma_v^2 = 0.001$. For $\sigma_w < 0.1$, where the state behavior is almost linear, the extended H_∞ filter was not found to produce a performance benefit compared with the EKF. However, when $\sigma_w^2 = 1$, the state behavior is substantially nonlinear, and then, the extended H_∞ filter may offer some robustness to linearization errors, which can plague the EKF. A robust design was implemented using $\delta_1 = 0.1, \delta_2 = 4.5$, and $\delta_3 = 0.001$. It was found that $\gamma = 1.38$ is sufficient for Σ_t of (3) to always be positive definite. Although both filters were observed to yield frequency outliers, those of the EKF, while infrequent, tend to be more severe. A histogram of the frequency estimation errors is shown in Fig. 1. For 1000 realizations of 1000 data points each, the EKF is observed to produce outliers that are up to 1.1 dB worse than the robust version. It turns out here that the extended H_∞ filter also exhibits 0.5 dB less MSE than the EKF.

The solution set out in Section V pertains to output estimation, where the objective is to estimate the output of the plant (14). The task of estimating the input to the plant (14), which is known as equalization, can be solved as follows. Since the measurements y_t are unchanged, the state observer is the same; therefore, (18) and (19) remain applicable. The only difference arises in the output mapping; instead of requiring an estimate of $C_t x_t + D_t w_t$, we seek an estimate of w_t. The required simplication of (20) is

$$\hat{y}_{t/t} = Q_t D_t^T \overline{R}_t^{-1}(y_t - c_t(\hat{x}_{t/t})). \tag{24}$$

Example 2: We consider a nonlinear channel having the state space realization

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 1.2 & -.36 \\ 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_t \\ y_t &= [12.1\ -22.1]x_t + 14\ \arctan([0\ 1])x_t)w_t. \end{aligned} \tag{25}$$

The task of equalizing (25) is a difficult problem because the nonlinear direct feedthrough function $d_t(\hat{x}_t) = \arctan([0\ \ 1]x_t)$ depends on the state estimate that can be error prone since the

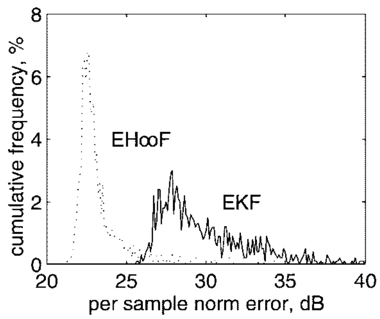

Fig. 2. Error histogram for Example 2.

linearized channel model tends to be nonminimum phase. The H_∞ equalizer follows from a straightforward application of (3), (8), (10), (18), (19), and (24). The results of simulations with $\gamma = 10.1, \delta_3 = 0,$ and $\delta_4 = .008$ are shown in Fig. 2. It turns out that for 1000 realizations of 1000 data points each, the MSE exhibited by the H_∞ equalizer is about 6 dB less than that of the EKF. This example demonstates that the extended H_∞ filter can be advantageous because it possesses parameters (γ, δ_3, and δ_4), which may be tuned to achieve a trade off between MSE and the H_∞ performance criterion.

VII. Conclusions

The pursuit of a robust EKF has been motivated as a model uncertainty problem. An auxiliary H_∞ filtering problem has been posed, wherein additional inputs are introduced to account for the higher order terms that are neglected in the formulation of the EKF. There are many possible ways in which an auxiliary problem can be constructed, and an H_∞ solution can be found. The procedure outlined here guarantees that the auxiliary H_∞ problem is solvable and the resulting design, albeit conservative, has been demonstrated to be advantageous in demodulation and equalization applications.

When the state dynamics are linear and the EKF works well, then the conservativeness of the H_∞ approach is likely to yield degraded performance. Conversely, when the problem is sufficiently nonlinear, it has been demonstrated that the so-called extended H_∞ filter can offer a performance benefit. There are additional tuning parameters that need to be determined by trial and error. These parameters may be used to control the tradeoff between the two performance criteria and the scaling of the inputs in order to accommodate linearization errors. Since the H_∞ filter has the same structure as the Kalman filter, it is not surprising that the extended H_∞ filter arrived at here has precisely the structure of the EKF. The computational overheads are not significantly affected since the order of the problem remains unchanged.

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Reply to "Comments on 'Stability and Absence of Overflow Oscillations for 2-D Discrete-Time Systems'"

Chengshan Xiao and David J. Hill

***Index Terms*—Overflow oscillations, stability, two-dimensional systems.**

We wish to clarify that our main result in [1, Th. 2], which was incorrectly commented in [2] as "unappreciated" as the erroneous result in [3, Th. 3], remains valid. However, the first two sentences in the Conclusion of [1], which were based on the incorrect result of [3, Th. 3], are inaccurate.

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