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# Extended-Kalman-Filter-like observers for continuous time systems with discrete time measurements

Vincent Andrieu \*

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## Abstract

In this short note is studied the observer introduced in [1] and [5]. The relationship between the Lipschitz constant and the measurement stepsize is exhibited. For second order system, we evaluate the authorized measurement stepsize.

## 1 Introduction

We consider a continuous time system of the form:

$$\dot{x} = f(x, u) , \quad x = (x_1, \dots, x_n) \quad (1)$$

with the state  $x$  is in  $\mathbb{R}^n$  and the input  $u$  is in  $U \subseteq \mathbb{R}$ . The solutions of this system are denoted as<sup>1</sup>  $x(t)$ . The state of this system is accessible via a discrete time measure

$$y_k = Cx(t_k), \quad (2)$$

where  $(t_k)_{k \in \mathbb{N}}$  is a sequence of positive real number defined as:

$$t_{k+1} = t_k + \delta$$

where  $\delta$  is a positive real number.

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<sup>1</sup>Solution should be written  $x(x_0, t, u(\cdot))$  but to simplify the presentation we prefer this notation.

The problem under consideration is an observer problem. How can we estimate the state of the system knowing only the measurements.

This problem has been addressed in [1] (see also [3] and the related filtering problem in [4]).

Following what has been done for continuous time measurement in [2], we consider the case in which the system can be written (possibly after a change of coordinates) in the form:

$$\dot{x} = Ax + \phi(x, u) \quad (3)$$

with  $Ax = (x_2, \dots, x_{n-1}, 0)$ ,

$$y_k = Cx(t_k) = x_1(t_k)$$

and where the function  $\phi : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$  satisfies an upper triangular globally Lipschitz condition, i.e. is such that for  $i$  in  $\{1, \dots, n\}$  we have:

$$|\phi_i(x + e, u) - \phi_i(x, u)| \leq c_L \sum_{j=1}^i |e_j|, \quad \forall (x, e) \in \mathbb{R}^n \times \mathbb{R}^n, \quad u \in U, \quad (4)$$

where  $c_L$  is a positive real number named the Lipschitz constant of the nonlinear system.

Inspired by [1] and [5], an estimate of the state can be given as any piecewise continuous function  $\hat{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  solution of the following continuous-discrete system<sup>2</sup>:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \phi(\hat{x}, u) \\ \dot{S} = -\theta S - A'S - SA \quad , \quad t \in [k\delta, (k+1)\delta) \\ \hat{x}(t_k) = \hat{x}(t_k^-) - \delta S(t_k)^{-1} C' (C\hat{x}(t_k^-) - y_k) \\ S(t_k) = S(t_k^-) + \delta C' C \end{cases} \quad (5)$$

with  $\theta$  a positive real number.

Note that compared to what has been done in [1], the Matrix updated law is a Lyapunov like one instead of a Riccati like one. This one has been used in [5] since it allows to study analytically its limit and to give a better approximation on the bound involved in the high-gain approach. Compare to [5] the only difference is the fact that the gain matrix depends on  $S$  which is time varying.

## 2 Main Theorem

Inspired by the result of [5], the following theorem can be obtain.

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<sup>2</sup> For a time function  $x(\cdot)$  the notation  $x(t_k^-)$  means (when it exists)  $\lim_{t \rightarrow t_k, t < t_k} x(t)$

**Theorem 1** *There exists a positive real number  $\zeta$  (which depends only on the dimension of the system), such that, for all  $\delta < \frac{\zeta}{c_L}$  there exist two positive real numbers  $\theta_1 < \theta_2$  such that for any  $\theta$  in  $[\theta_1, \theta_2]$ , the estimate  $\hat{x}$  converges asymptotically to the state of the system (3).*

**Proof :** Let  $E = \Theta^{-1}(\hat{x} - x)$  where  $\Theta = \text{diag}\{1, \dots, \theta^{n-1}\}$  with  $\theta$  a positive real number larger than 1 to be specified. The scaled error satisfies for  $t$  in  $[k\delta, (k+1)\delta)$ :

$$\dot{E} = \Theta^{-1}A[\hat{x} - x] + \Theta^{-1}[\phi(\hat{x}, u) - \phi(x, u)] ,$$

and since  $\Theta^{-1}A = \theta A\Theta^{-1}$ , it yields for  $t$  in  $[k\delta, (k+1)\delta)$ :

$$\dot{E} = \theta AE + \Delta_\theta \phi(x, E) ,$$

with  $\Delta_\theta \phi(x, E) = \Theta^{-1}[\phi(\hat{x}, u) - \phi(x, u)]$  such that  $|\Delta_\theta \phi_i(x, E)| \leq c_L \sum_{j=1}^i |E_j|$  since  $\theta > 1$ . Moreover, the scaled error satisfies:

$$\begin{aligned} E(t_k) &= \Theta^{-1} \left[ \hat{x}(t_k^-) - \delta S(t_k)^{-1} C' \left( C \hat{x}(t_k^-) - y_k \right) - x(t_k) \right] \\ &= \left( I - \delta \tilde{S}(t_k)^{-1} C' C \right) E(t_k^-) , \end{aligned}$$

where  $\tilde{S}(\cdot) = \Theta S(\cdot) \Theta$ . Note that  $\tilde{S}$  satisfies:

$$\tilde{S}(t_k) = \tilde{S}(t_k^-) + \delta C' C \quad (6)$$

and for  $t$  in  $[t_k, (k+1)\delta)$ :

$$\begin{aligned} \dot{\tilde{S}} &= -\theta \tilde{S} - \Theta A' \Theta^{-1} \tilde{S} - \tilde{S} \Theta^{-1} A \Theta , \\ &= -\theta \tilde{S} - \theta A' \tilde{S} - \theta \tilde{S} A . \end{aligned} \quad (7)$$

Note that the sequence  $(s_k)_{k \in \mathbb{N}}$  such that  $s_k = \theta \tilde{S}(t_k)$  satisfies

$$s_{k+1} = \exp(-\rho) \exp(-A' \rho) s_k \exp(-A \rho) + \rho C' C ,$$

where  $\rho = \theta \delta$ . Which gives,

$$s_k = \exp(-k\rho) \exp(-A' k\rho) s_0 \exp(-A k\rho) + \sum_{\ell=0}^{k-1} \exp(-\ell\rho) \exp(-A' \ell\rho) \rho C' C \exp(-A \ell\rho) ,$$

Following what has been done in [5], the following Lemma can be shown (its proof is given in Section 3).

**Lemma 1** *There exist two continuous function  $\alpha_1$  and  $\alpha_2$  such that for all  $\rho > 0$ , the matrix series converges to a limit denoted  $s_\infty(\rho)$ , formally defined as:*

$$s_\infty = \sum_{\ell=0}^{+\infty} \exp(-\ell\rho) \exp(-A'\ell\rho) \rho C' C \exp(-A\ell\rho) ,$$

such that

$$\alpha_1(\rho)I \leq s_\infty \leq \alpha_2(\rho)I , \quad (8)$$

where  $\alpha_1$  and  $\alpha_2$  are two strictly positive continuous functions.

The proof is divided in three steps :

**Part 1:** With this lemma in hand, we will first show that for all  $\rho > 0$ , there exists  $t_0$  such that for all  $t > t_0$ , the matrix  $\tilde{S}$  satisfies:

$$\gamma_1(\rho)I < \theta\tilde{S}(t) < \gamma_2(\rho)I \quad (9)$$

where  $\gamma_1$  and  $\gamma_2$  are two continuous positive function defined as,

$$\gamma_1(\rho) = \exp(-\rho)\alpha_1(\rho)c_3^{\mathbf{Int}(\rho)} c_1 , \quad \gamma_2(\rho) = \alpha_2(\rho)c_2c_4^{\mathbf{Int}(\rho)} \quad (10)$$

where  $\mathbf{Int}(\cdot)$  denotes the integer part of a positive real number,  $c_1$  and  $c_2$  are two positive real numbers and  $(c_3, c_4)$  are two positive real numbers such that  $c_3 < 1$  and  $c_4 > 1$ .

Indeed, for  $k$  in  $\mathbb{N}$  and  $s$  in  $[0, \delta)$ ,  $\tilde{S}(k+s)$  is solution of (7), hence,

$$\theta\tilde{S}(k+s) = \exp(-\theta s) \exp(-A'\theta s) s_k \exp(-A\theta s) .$$

With Lemma , for  $k$  sufficiently large, the matrix  $s_k$  is definite positive, it yields that for all  $v$  in  $\mathbb{R}^n$ ,

$$v'\theta\tilde{S}(k+s)v = \exp(-\theta s) \left| s_k^{\frac{1}{2}} \exp(-A\theta s)v \right|^2 .$$

Hence, we get for all  $v$  in  $\mathbb{R}^n$ :

$$v'\theta\tilde{S}(k+s)v \geq \exp(-\rho) \left( \lambda_{\min} \left\{ s_k^{\frac{1}{2}} \right\} \right)^2 |\exp(-A\theta s)v|^2$$

Note that

$$\begin{aligned} |\exp(-A\theta s)v|^2 &= |\exp(-A\theta s)v|^2 \\ &= |\exp(-A\mathbf{Int}(\theta s)) \exp(-A[\theta s - \mathbf{Int}(\theta s)])v|^2 \\ &\geq (\lambda_{\min} \{ \exp(-A)' \exp(-A) \})^{\mathbf{Int}(\theta s)} |\exp(-A[\theta s - \mathbf{Int}(\theta s)])v|^2 \\ &\geq (\lambda_{\min} \{ \exp(-A)' \exp(-A) \})^{\mathbf{Int}(\theta s)} c_1 |v|^2 \end{aligned}$$

where  $c_1 = \min_{r \leq 1} \{\lambda_{\min} \{\exp(-Ar)' \exp(-Ar)\}\}$ . Furthermore, note that  $\text{Det} \{\exp(-A)' \exp(-A)\} = 1$ , hence  $c_3 := \lambda_{\min} \{\exp(-A)' \exp(-A)\} < 1$ . Consequently,

$$|\exp(-A\theta s)v|^2 \geq c_3^{\text{Int}(\rho)} c_1$$

And since,  $\left(\lambda_{\min} \left\{s_k^{\frac{1}{2}}\right\}\right)^2 \geq \lambda_{\min} \{s_k\}$ , it yields for all  $k$  sufficiently large and all  $v$  in  $\mathbb{R}^n$

$$v' \theta \tilde{S}(k+s)v \geq \exp(-\rho) \lambda_{\min} \{s_k\} c_3^{\text{Int}(\rho)} c_1 |v|^2$$

In a same way,

$$v' \theta \tilde{S}(k+s)v \leq \left(\lambda_{\max} \left\{s_k^{\frac{1}{2}}\right\}\right)^2 |\exp(-A\theta s)v|^2$$

and,

$$|\exp(-A\theta s)v|^2 \leq (\lambda_{\max} \{\exp(-A)' \exp(-A)\})^{\text{Int}(\theta s)} c_2 |v|^2$$

where  $c_2 = \max_{r \leq 1} \lambda_{\max} \{\exp(-Ar)' \exp(-Ar)\}$ . Hence we get

$$v' \theta \tilde{S}(k+s)v \leq \lambda_{\max} \{s_k\} c_2 (\lambda_{\max} \{\exp(-A)' \exp(-A)\})^{\text{Int}(\theta s)} |v|^2$$

Due to the fact that  $c_4 := \lambda_{\max} \{\exp(-A)' \exp(-A)\} > 1$ , we get

$$v' \theta \tilde{S}(k+s)v \leq \lambda_{\max} \{s_k\} c_2 c_4^{\text{Int}(\rho)} |v|^2$$

which, with Lemma 2 implies that equation (9) holds and finishes the first part of the proof.

**Part 2:** Let now  $V(E) = E' \tilde{S} E$ . We will now show that it is strictly decreasing providing  $\theta$  and  $\rho$  satisfy an inequality constraint depending on  $c_L$  the Lipschitz constant of the nonlinear system. The function  $V$  satisfies along the trajectories of the system for  $t$  in  $[k\delta, (k+1)\delta)$ :

$$\dot{V}(E) = E' \left[ \theta A' \tilde{S}(t) + \theta \tilde{S}(t) A + \dot{\tilde{S}} \right] E + 2E' \tilde{S} \Delta_{\theta} \phi(x, E) \quad (11)$$

$$= -\theta E' \tilde{S}(t) E + 2E' \tilde{S}(t) \Delta_{\theta} \phi(x, E) . \quad (12)$$

With Schwarz inequality, we have for all  $t > 0$

$$E' \tilde{S}(t) \Delta_{\theta} \phi(x, E) \leq |E| |\tilde{S}(t) \Delta_{\theta} \phi(x, E)| \leq \lambda_{\max} \{\tilde{S}(t)\} |E| |\Delta_{\theta} \phi(x, E)| , \quad (13)$$

and we have<sup>3</sup>

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<sup>3</sup>Here we have used the inequality:

$$(a_1 + \dots a_n)^2 \leq n (a_1^2 + \dots + a_n^2)$$

$$\begin{aligned}
|\Delta_\theta \phi(x, E)|^2 &= \sum_{i=1}^n |\Delta_\theta \phi_i(x, E)|^2 \\
&\leq \sum_{i=1}^n c_L^2 \left( \sum_{j=1}^i |E_j| \right)^2 \leq n c_L^2 \left( \sum_{j=1}^n |E_j| \right)^2 \leq n^2 c_L^2 |E|^2
\end{aligned} \tag{14}$$

Consequently, we get:

$$\dot{V}(E) \leq \left[ \lambda_{\max} \{ \tilde{S}(t) \} 2n c_L - \theta \lambda_{\min} \{ \tilde{S}(t) \} \right] |E|^2. \tag{15}$$

Note that, due to equation (9), we get that if we can select  $\theta$  and  $\delta$  such that

$$\theta > \frac{2\gamma_2(\rho) n c_L}{\gamma_1(\rho)} \tag{16}$$

with  $\rho = \delta\theta$  we would get

$$\lambda_{\max} \{ \tilde{S}(t) \} 2n c_L - \theta \lambda_{\min} \{ \tilde{S}(t) \} < 0,$$

for all  $t$  sufficiently large. Hence we get  $\dot{V}(E) < 0$  for all  $t$  in  $[k\delta, (k+1)\delta)$ , with  $k$  sufficiently large. Moreover, we have

$$\begin{aligned}
V(E)(t_k) &= E(t_k^-)' \left( I - \delta \tilde{S}^{-1} C' C \right)' \tilde{S}(t_k) \left( I - \delta \tilde{S}^{-1} C' C \right) E(t_k^-) \\
&= E(t_k^-)' \left[ \tilde{S}(t_k) - 2\delta C' C + \delta^2 C' C \tilde{S}(t_k)^{-1} C' C \right] E(t_k^-) \\
&= E(t_k^-)' \left[ \tilde{S}(t_k^-) - \delta C' C + \delta^2 C' C \left( \tilde{S}(t_k^-) + \delta C' C \right)^{-1} C' C \right] E(t_k^-) \\
&= V(E)(t_k^-) - E(t_k^-)' C' p C E(t_k^-)
\end{aligned}$$

where,

$$p = \delta - \delta^2 C \left( \tilde{S}(t_k^-) + \delta C' C \right)^{-1} C'.$$

Following [4] (see also [1]), note that if we note  $q = \left[ C \tilde{S}(t_k^-)^{-1} C' + \frac{1}{\delta} \right]$ , we have,

$$\begin{aligned}
pq &= \left[ \delta - \delta^2 C \tilde{S}(t_k^-)^{-1} \left( I + \delta C' C \tilde{S}(t_k^-)^{-1} \right)^{-1} C' \right] \left[ C \tilde{S}(t_k^-)^{-1} C' + \frac{1}{\delta} \right] \\
&= 1
\end{aligned}$$

Hence, we get :

$$V(E)(t_k) = V(E)(t_k^-) - E(t_k^-)' C' q^{-1} C E(t_k^-)$$

with  $q > 0$ . Consequently the function  $V$  is decreasing along the trajectories of the system for  $t$  sufficiently large. Its eigenvalues being lower and upper bounded as shown in the first part of the proof, we get convergence of the estimation error to the origin.

**Part 3:** We now show that the constraint we have on  $\theta$  and  $\rho$  can be given in terms of an upper bound on  $\delta$  depending on  $c_L$ .

To obtain estimation, we have to find  $\delta$  and  $\theta$  such that (16) is satisfied and  $\theta > 1$ . Note that (16) is equivalent with :

$$\delta < \frac{\varphi(\rho)}{c_L} ,$$

with  $\varphi(\rho) = \frac{\rho\gamma_1(\rho)}{2n\gamma_2(\rho)}$  where  $\gamma_1$  and  $\gamma_2$  are defined in (10). Hence, we get:

$$\varphi(\rho) = \rho \exp(-\rho) \frac{\alpha_1(\rho)}{2n\alpha_2(\rho)} \frac{c_1}{c_2} \left( \frac{c_3}{c_4} \right)^{\text{Int}(\rho)} \quad (17)$$

Note that since  $\alpha_1(\rho) \leq \alpha_2(\rho)$  and  $c_3 < 1$  and  $c_4 > 1$  we get:

$$\lim_{\rho \rightarrow 0} \varphi(\rho) = 0 , \quad \lim_{\rho \rightarrow +\infty} \varphi(\rho) = 0 .$$

Hence we can define the positive real number  $\kappa$  as :

$$\kappa = \max_{\rho > 0} \{\varphi(\rho)\} ,$$

This function being continuous, we get that for all  $c_L\delta < \kappa$ , there exists  $\tilde{\rho}_1(\delta)$  and  $\tilde{\rho}_2(\delta)$  such that

$$\varphi(\rho) > c_L\delta , \quad \forall \rho \in (\tilde{\rho}_1(\delta), \tilde{\rho}_2(\delta)) ,$$

and,

$$\varphi(\tilde{\rho}_2(\delta)) = \varphi(\tilde{\rho}_1(\delta)) = c_L\delta .$$

The function  $\varphi$  being definite positive, we get that  $\lim_{\delta \rightarrow 0} \tilde{\rho}_2(\delta) = +\infty$ . This implies that there exists  $0 < c_L\zeta \leq \kappa$  such that for all  $0 < \delta < \zeta$  there exist  $\rho_1(\delta)$  and  $\rho_2(\delta)$  such that we have

$$\delta < \rho_1(\delta) < \rho_2(\delta) , \quad \varphi(\rho) > c_L\delta , \quad \forall \rho \in [\rho_1(\delta), \rho_2(\delta)] .$$

Hence, this implies

$$\frac{\varphi(\rho)}{c_L} > \delta , \quad \forall \theta \in [\theta_1, \theta_2] , \quad 1 < \theta_1 < \theta_2 ,$$

where  $\theta_1 = \frac{\rho_1(\delta)}{\delta}$  and  $\theta_2 = \frac{\rho_2(\delta)}{\delta}$  which concludes the proof.

□



### 3 Proof of Lemma 2

We want to show that the sequence

$$s_k = \sum_{\ell=0}^k \exp(-\ell\rho) \exp(-A'\ell\rho) \rho C' C \exp(-A\ell\rho) , \quad k \in \mathbb{N} ,$$

converges to a limit as  $k$  goes to infinity denoted  $s_\infty$  and such that

$$\alpha_1(\rho)I \leq s_\infty \leq \alpha_2(\rho)I . \quad (18)$$

First, note that  $A$  being nilpotent, it yields<sup>4</sup>

$$C \exp(-A\ell\rho) = \sum_{j=0}^{n-1} C(-A\ell\rho)^j = r_{\ell\rho} S_n ,$$

where  $S_n = \text{diag} \left( 1, \dots, \frac{1}{(n-1)!} \right)$  and,  $r_{\ell\rho}$  is a vector defined as :

$$r_\ell = (1, (-\rho\ell), \dots, (-\rho\ell)^{n-1}) .$$

Consequently, we get

$$s_k = \rho S_n M_k(\rho) S_n ,$$

where  $M_k$  is the symmetric positive (at least) semi-definite matrix defined as

$$M_k(\rho) = \left( \sum_{\ell=0}^k \exp(-\ell\rho) r'_{\ell\rho} r_{\ell\rho} \right) .$$

Hence to get the result it is sufficient to work on the sequence  $M_k$  since we have for all  $k$  in  $\mathbb{N}$  where  $M_k$  is definite positive:

$$v' s_k v = \rho \left| M_k^{\frac{1}{2}}(\rho) S_n v \right|^2$$

---

<sup>4</sup>In fact we have

$$C \exp(-A\ell\rho) = r_{\ell\rho} S_n \mathcal{O}$$

where  $\mathcal{O}$  is the observability matrix associated to the couple  $(A, C)$ , i.e.,

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = I .$$

It yields,

$$\rho \left( \lambda_{\min} \{M_k^{\frac{1}{2}}(\rho)\} \lambda_{\min} \{S_n\} \right)^2 I \leq s_k \leq \rho \left( \lambda_{\max} \{M_k^{\frac{1}{2}}(\rho)\} \lambda_{\max} \{S_n\} \right)^2 I$$

or,

$$\rho \lambda_{\min} \{M_k(\rho)\} (\lambda_{\min} \{S_n\})^2 I \leq s_k \leq \rho \lambda_{\max} \{M_k(\rho)\} \lambda_{\max} (\{S_n\})^2 I$$

Consequently

$$\frac{\rho}{(n-1)!} \lambda_{\min} \{M_k(\rho)\} I \leq s_k \leq \rho \lambda_{\max} \{M_k(\rho)\} I$$

So, provided its limit exists, (18) is satisfied with:

$$\alpha_1(\rho) = \frac{\rho}{(n-1)!} \lambda_{\min} \{M_{\infty}(\rho)\} , \quad \alpha_2(\rho) = \rho \lambda_{\max} \{M_{\infty}(\rho)\} \quad (19)$$

where,

$$M_{\infty}(\rho) = \lim_{k \rightarrow +\infty} M_k(\rho)$$

1. We first show that the for all  $k$  sufficiently large, the matrix  $M_k$  is definite positive. First note that:

$$M_k(\rho) \geq \exp(-(n-1)\rho) \left( \sum_{\ell=0}^{n-1} r'_{\ell\rho} r_{\ell\rho} \right) , \quad \forall k \geq n .$$

Note that for all  $v$  in  $\mathbb{R}^n$ ,

$$v' \left( \sum_{\ell=0}^{n-1} r'_{\ell\rho} r_{\ell\rho} \right) v = \sum_{\ell=0}^{n-1} |r_{\ell\rho} v|^2 = |R(\rho) v|^2 \geq \lambda_{\min} \{R(\rho)' R(\rho)\} |v|^2 ,$$

where  $R$  is a Vandermonde matrix<sup>5</sup>  $R = (r_0, \dots, r_{n-1})'$ . Hence, we get

$$M_k(\rho) \geq \lambda_{\min} \{R(\rho)' R(\rho)\} \exp(-(n-1)\rho) I , \quad \forall k \geq n .$$

which shows that  $M_k$  is definite positive for all  $k \geq n$ .

2. We now show that  $M_{\infty}$  exists and is a continuous matrix function of  $\rho$ . Note that

$$(M_k(\rho))_{i,j} = \sum_{\ell=0}^k \exp(-\ell\rho) (-\ell\rho)^{i+j-2} = (-\rho)^{i+j-2} \frac{\partial^{i+j-2} \pi_k}{\partial \rho^{i+j-2}}(\rho) ,$$

---

<sup>5</sup>This implies that it is a full rank matrix.

where  $\pi_k(\phi) = \sum_{\ell=0}^k \exp(-\ell\rho)$ . Note that we have

$$\pi_\infty(\rho) = \lim_{k \rightarrow +\infty} \pi_k(\rho) = \frac{1}{1 - \exp(-\rho)}$$

Consequently,

$$(M_\infty(\rho))_{i,j} = (-\rho)^{i+j-2} \frac{\partial^{i+j-2} \pi_\infty}{\partial \rho^{i+j-2}}(\rho) ,$$

Note that this function is continuous<sup>6</sup> for all  $\rho > 0$ .

## 4 Numerical evaluation of $\zeta$

In this paragraph, we try to give the value of  $\zeta$  in the simple case where the dimension of the system is  $n = 2$ .

**Computation of  $M_\infty$ ,  $\alpha_1$  and  $\alpha_2$ :** We have,

$$M_\infty(\rho) = \begin{pmatrix} \frac{1}{1 - \exp(-\rho)} & \frac{-\rho \exp(-\rho)}{(1 - \exp(-\rho))^2} \\ \frac{-\rho \exp(-\rho)}{(1 - \exp(-\rho))^2} & \frac{\rho^2 [\exp(-\rho) + \exp(-2\rho)]}{(1 - \exp(-\rho))^3} \end{pmatrix}$$

Note that when  $n = 2$ , we have,

$$\alpha_1(\rho) = \lambda_{\min} \{M_\infty(\rho)\} , \quad \alpha_2(\rho) = \lambda_{\max} \{M_\infty(\rho)\}$$

Employing WolframAlpha website, we are able to give explicitly this eigenvalues as a function of  $\rho$ .

**Computation of  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$ :** We have also,

$$\exp(-A'r) \exp(Ar) = \begin{pmatrix} 1 & -r \\ -r & 1 + r^2 \end{pmatrix} ,$$

---

<sup>6</sup>Moreover, it satisfies:

$$\lim_{\rho \rightarrow 0} [\rho (M_\infty(\rho))_{i,j}] = (-1)^{(i+j-2)} (i+j-2)! ,$$

Hence, we have:

$$\lim_{\rho \rightarrow 0} \alpha_1(\rho) > 0$$

consequently,

$$\lambda_{\min} \{ \exp(-A'r) \exp(Ar) \} = \frac{2 + r^2 - r\sqrt{r^2 + 4}}{2} ,$$

and,

$$\lambda_{\max} \{ \exp(-A'r) \exp(Ar) \} = \frac{2 + r^2 + r\sqrt{r^2 + 4}}{2} .$$

Note that  $r \mapsto \lambda_{\min} \{ \exp(-A'r) \exp(Ar) \}$  is a strictly decreasing function on  $\mathbb{R}_+$  and that  $r \mapsto \lambda_{\max} \{ \exp(-A'r) \exp(Ar) \}$  is a strictly increasing function on  $\mathbb{R}_+$ . Consequently, we obtain

$$c_1 = c_2 = \frac{3 - \sqrt{5}}{2} , c_3 = c_4 = \frac{3 + \sqrt{5}}{2}$$

Note that with these data, the function  $\varphi$  defined in (17) can be computed threw matlab. Its maximal value is reached for

$$\kappa = \max_{\rho > 0} \{ \varphi(\rho) \} = 0.0022 , \quad \rho_{opt} = \text{Argmax}_{\rho > 0} \{ \varphi(\rho) \} = 0.5327 > \kappa .$$

Consequently, in this case,  $\zeta = 0.0022$ .

**Better approximation of  $\gamma_1$  and  $\gamma_2$ :** In the particular case of  $n = 2$  a better approximation can be given. Indeed, we have

$$\min_{s < \delta} \{ \lambda_{\min} \{ \exp(-A\theta s)' \exp(-A\theta s) \} \} = \frac{2 + \rho^2 - \sqrt{(2 + \rho^2)^2 - 4}}{2} , \quad \rho = \theta\delta .$$

Hence, we can take:

$$\gamma_1(\rho) = \exp(-\rho) \alpha_1(\rho) \frac{2 + \rho^2 - \sqrt{(2 + \rho^2)^2 - 4}}{2} , \quad \gamma_2(\rho) = \alpha_2(\rho) \frac{2 + \rho^2 + \sqrt{(2 + \rho^2)^2 - 4}}{2} .$$

The function  $\varphi(\rho) = \frac{\rho \gamma_1(\rho)}{2n\gamma_2(\rho)}$  can be computed threw matlab. Its maximal value is reached for

$$\kappa = \max_{\rho > 0} \{ \varphi(\rho) \} = 0.0148 , \quad \rho_{opt} = \text{Argmax}_{\rho > 0} \{ \varphi(\rho) \} = 0.5313 > \kappa .$$

Consequently, in this case,  $\zeta = 0.0148$ . This implies that given the Lipschitz constant of the system  $c_L$ , the observer which maximizes the time between two measures can be tuned with

$$\delta = \frac{0.0148}{c_L} , \quad \theta = 35.9 \, c_L .$$

## 5 Conclusion

In this short note, the continuous discrete observer presented in [5] has been studied. The maximal time between two measurement has been exhibited as a function of the Lipschitz constant. Even for small dimension, the obtained value seems to be small.

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