Chapter 6 Controllability and Obervability

- Controllability:

  whether or not the state-space equation can be controlled from input.

- Observability:

  whether or not the initial state can be observed from output.

6.2 Controllability

- Consider the n-dimensional p-input equation

  $$\dot{x} = Ax + Bu$$

- Definition 6.2 The pair (A, B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input that transfer x_0 to x_1 in a finite time.
- Example 6.1: Figure 6.2 (a) and (b) is not controllable.

- Theorem 6.1 The following statement are equivalent.

1. The n-dimensional pair (A, B) is controllable
2. The n×n matrix

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

is nonsingular for any $t > 0$.

3. The n×np controllability matrix

$$C = [B \; AB \; A^2B \; \dots \; A^{n-1}B]$$

has rank n (full row rank).

4. The $n \times (n+p)$ matrix $[A-\lambda I \;\; B]$ has full row rank at every eigenvalue, λ, of A'.

5. If all eigenvalues of A have negative real parts, then the unique solution of

$$AW_c + W_cA^T = -BB^T$$

is positive definite. The solution is called the controllability Gramian and can be expressed as

$$W_c = \int_0^\infty e^{A\tau} BB' e^{A'\tau} d\tau$$

6.2.1 Controllability indices

- Let A and B be $n \times n$ and $n \times p$ constant matrix. If (A, B) is controllable, its controllability matrix

$$C = [B\ AB\ A^2B\ \ldots\ A^{n-1}B]$$

$$= [b_1 \ldots b_p\ Ab_1 \ldots Ab_p\ A^{n-1}b_1 \ldots A^{n-1}b_p]$$

has rank n and consequently, n linearly independent column.

- If C has rank n, then

  $\mu_1 + \mu_2 + \ldots + \mu_p = n.$

  The set $\{\mu_1, \mu_2, \ldots, \mu_p\}$ is called the controllability indices and

  $\mu = \max(\mu_1, \mu_2, \ldots, \mu_p)$

  is called the controllability index of (A, B).
- Consequently, if (A, B) is controllable, the controllability index μ is the least integer such that

  $\rho(C_\mu) = \rho([B \; AB \; A^2B \; \ldots \; A^{\mu-1}B]) = n.$

- The controllability index satifies

$$n/p \le \mu \le \min(\bar{n}, n-p+1)$$

where r(B) = p, and

$$A^{\bar{n}}B \Leftarrow \{B, AB, \ldots, A^{\bar{n}-1}B\} \text{(Linear combination)}$$

- Corollary 6.1 The dimensional pair (A, B) is controllable if and only if the matrix

$$C_{n-p+1} := [B\ AB\ \ldots\ A^{n-p}B]$$

where $\rho(B) = p$, has rank n or the $n \times n$ matrix $C_{n-p+1}C^T_{n-p+1}$ is nonsingular.

- Theorem 6.2 The controllability property is invariant under equivalence transformation.
- Theorem 6.3 The set of controllability indices (A, B) is invariant under any equivalence transformation and any reordering of the columns of B.

6.3 Observability

- The concept of observability is dual to that of controllability.
- Controllability studies the possibility of steering the state from input.
- Observability studies the possibility of estimating the state from output.

- Consider the n-dimensional p-input q-output state equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Definition 6.O1 The above state equation is said to be onservable if for any unknow initial state x(0), there exists a finite $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices t_0 determine uniquely the initial state x(0).

- Theorem 6.4 The state equation is observable if and only if the n×n matrix

$$W_0 = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

- Theorem 6.5 (Theorem of duality) The pair (A, B) is controllable if and only if the pair (A^T, B^T) is observable.

- Theorem 6.O1 The following statement are equivalent.

1. The n-dimensional pair (A, C) is observable.
2. The n×n matrix $W_0 = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau$ is nonsingular for any t > 0.
3. The nq×n observability $O = \begin{bmatrix} C \\ CA \\ \cdot \\ CA^{n-1} \end{bmatrix}$ has rank n (full column rank).
4. The (n+q)×n matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue.

5. If all eigenvalues of A have negative real parts, then the unique solution of

$$A^T W_0 + W_0 A^T = -C^T C$$

is positive definite.

- If (A, C) is observable, its observability matrix O has rank n.
- Let v_m be the number of the linearly independent rows associated with c_m. If O has rank n, then

$$v_1 + v_2 + \ldots + v_q = n$$

- The set $\{v_1, v_2, \ldots, v_q\}$ is called observablity indices and $v = \max(v_1, v_2, \ldots, v_q)$ is called the observablility index of (A, C).
- If (A, C) is observable, it is the least integer such that

$$\rho(O_v) := \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{v-1} \end{bmatrix} = n$$

- Dual to the controllability part,

$$n / q \leq v \leq \min(\bar{n}, n - q + 1)$$

where r(C)=q and $\bar{n}$ is the degree of the minimal polynomial of A.

- Corollary 6.O1 The dimensional pair (A, C) is observable if and only if the matrix

$$O_{n-q+1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{n-q} \end{bmatrix}$$

has full rank or the n×n matrix $O^T_{n-q+1} O_{n-q+1}$ is nonsingular.

- Theorem 6.O2 The observability property is invariant under any equivalence transformation.
- Theorem 6.O3 The set of observability indices of (A, C) is invariant under any equivalence transformation and any reordering of the rows of C.

6.4 Canonical decomposition

- Theorem 6.6 Consider the n-dimensional state equation with

  $\rho(C) = \rho([B\ AB\ \ldots\ A^{n-1}B]) = n_1 < n.$

  We form the n×n matrix

  $P^{-1} := [q_1\ \ldots\ q_{n1}\ \ldots\ q_n]$

  where the first n_1 columns are any linearly independent columns of C, and the remaining columns can arbitrarily be chosen as long as P is nonsingular.

- Then the equivalent transformation will obtain

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

where $\bar{A}_c$ is $n_1 \times n_1$ and $\bar{A}_{\bar{c}}$ is $(n-n_1) \times (n-n_1)$, and the n_1-dimensional subequation is controllable.

- Theorem 6.O6 Consider the n-dimensional state equation with

$$\rho(O) = \rho\begin{bmatrix} C \\ CA \\ \cdot \\ CA^{n-1} \end{bmatrix} = n_2 < n$$

We form the nn matrix $P = \begin{bmatrix} p_1 \\ \cdot \\ p_{n_2} \\ \cdot \\ p_n \end{bmatrix}$ where n_2 rows are any n_2 linearly independent rows of O.

- Then the equivalence transformation will obtain

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{12} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Du$$

where $\bar{A}_o$ is $n_2 \times n_2$ and $\bar{A}_{\bar{o}}$ is $(n-n_2) \times (n-n_2)$, and the n_2-dimensional subequation is observable.

- Theorem 6.7 Every state-space equation can be transformed into the canonical form

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

6.5 Conditions in Jordan-form equations

- If a state equation is transformed into Jordan form, then the controllability and observability conditions become very simple.
- Consider the state equation

$$\dot{x} = Jx + Bu$$

$$y = Cx$$

- Assume that J has only two distinct eigenvalues and can be written as $J = \mathrm{diag}(J_1, J_2)$

- Again assume that J_1 has three Jordan blocks and J_2 has two Jordan blocks or

  $J_1 = \mathrm{diag}(J_{11}, J_{12}, J_{13}) \quad J_2 = \mathrm{diag}(J_{21}, J_{22})$
- The row of B corresponding to the last row of J_{ij} is denoted by b_{lij}.
- The column of C corresponding to the first column of J_{ij} is denoted by c_{fij}.

- Theorem 6.8

1. The state equation in Jordan form is controllable if and only if $\{b_{l11}, b_{l12}, b_{l13}\}$ are linearly independent and $\{b_{l21}, b_{l22}\}$ are linearly independent.
2. That is observable if and only if $\{c_{f11}, c_{f12}, c_{f13}\}$ are linearly independent and $\{c_{f21}, c_{f22}\}$ are linearly independent.

6.6 Discrete-time state equations

- Consider the n-dimensional p-input q-output state equation

  $x[k+1] = Ax[k] + Bu[k]$

  $y[k] = Cx[k]$

- Definition 6.D1 The discrete-time state equation or the pair (A, B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input sequence of finite length that transfers x_0 to x_1.

- Theorem 6.D1 The following statements are equivalent:

1. The n-dimensional pair (A, B) is controllable.
2. The n×n matrix

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} (A)^m BB'(A')^m$$

is nonsingular.

3. n×np controllability matrix

$$C_d = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

has rank n (full row rank).

4. The $n \times (n+p)$ matrix $[A-\lambda I \;\; B]$ has full row rank at every eigenvalue, λ, of A.

5. If alleigenvalues of A has magnitudes less than 1, then the unique solution of

$$W_{dc} - AW_{dc}A^T = BB^T$$

(Controllability Gramian) is positive definite.

- Definition 6.D2 The discrete-time state equation or the pair (A, C) is said to be observable if for any unknown initial state x[0], there exists a finite integer $k_1 > 0$, such that the knowledge of the input sequence u[k] and the output sequence y[k] from k = 0 to k_1 suffices to determine uniquely the initial state x[0].

- Theorem 6.DO1 The following statements are equivalent:

1. The n-dimensional pair (A, C) is observable.
2. The n╳n matrix

$$W_{do}[n-1] = \sum_{m=0}^{n-1} (A')^m C'CA^m$$

   is nonsingular (positive definite).
3. The nq×n observability matrix

$$O_d = \begin{bmatrix} C \\ CA \\ . \\ CA^{n-1} \end{bmatrix}$$

   has rank n (full column rank).

4. The $(n+q)\times n$ matrix

$$\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$$

has full column rank at every eigenvalue, λ, of A.

5. If all eigenvalues of A have magnitude less than 1, then the unique solution of

$$W_{do} - A^T W_{do} A = C^T C$$

(observability Gramian) is positive definite.

- There are three different controllability definitions:

1. Definition 6.D1.
2. Controllability to the origin: transfer any state to the zero state.
3. Reachability (controllability from the origin): Transfer the zero state to any state.

- In continuous-time case, because e^{At} is nonsingular, the three definitions are equivalent.

- In discrete-time case, if A is nonsingular, the three definitions are again equivalent.
- But if A is singular, then 1 and 3 are equivalent, but not 2 and 3.

6.7 Controllability after sampling

- Consider a continuous-time state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- If the input is piecewise constant or

  $u[k] := u[kT] = u(t)$ for $kT \le t < (k+1)T$

  then the equation can be described by

$$\bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}u[k]$$

  with

$$\bar{A} = e^{AT} \qquad \bar{B} = \left(\int_0^T a^{At} dt\right)B =: MB$$

- Theorem 6.9 suppose (6.65) is controllable. A sufficient condition for its discretized equation in (6.66), with sampling period T, to be controllable is that $|\mathrm{Im}[\lambda_i-\lambda_j]| \neq 2\pi m/T$ for m=1, 2, …, whenever $\mathrm{Re}[\lambda_i-\lambda_j] = 0$.
- Theorem 6.10 If a continuous-time linear time-invariant state equation is not controllable, then its discretized state equation, with any sampling period, is not controllable.

6.8 LTV state equations

- Consider the n-dimensional p-input q-output state equation

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x$$

- In the time-varying case, the specification of t_0 and t_1 is crucial.

- Theorem 6.11 The n-dimensional pair (A(t), B(t)) is controllable at time t_0 if and only if there exists a finite $t_1 > t_0$ such that n×n matrix

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B'(\tau) \Phi'(t_1, \tau) d\tau$$

where $\Phi(t, \tau)$: the state transition matrix is nonsingular.

- The controllability condition without involving $\Phi(t, \tau)$:

  Define $M_0(t) = B(t)$, and then define recursively a sequence of n×p matrices $M_m(t)$ as

$$M_{m+1}(t) := -A(t)M_m(t) + \frac{d}{dt}M_m(t)$$

- we have

$$\Phi(t_2, t)B(t) = \Phi(t_2, t)M_0(t)$$

- Using $\frac{\partial}{\partial t}\Phi(t_2, t) = -\Phi(t_2, t)A(t)$, we compute

$$\frac{\partial}{\partial t}[\Phi(t_2, t)B(t)] = \frac{\partial}{\partial t}[\Phi(t_2, t)]B(t) + \Phi(t_2, t)\frac{d}{dt}B(t)$$

$$= \Phi(t_2, t)[-A(t)M_0(t) + \frac{d}{dt}M_0(t)] = \Phi(t_2, t)M_1(t)$$

- Thus, we have

$$\frac{\partial^m}{\partial t^m}\Phi(t_2,t)B(t)=\Phi(t_2,t)M_m(t)$$

- Theorem 6.12 Let A(t) and B(t) be n-1 times continuously differentiable. Then the n-dimensional pair (A(t), B(t)) is controllable at t_0 if there exists a finite $t_1>t_0$ such that

$$\operatorname{rank}\begin{bmatrix}M_0(t_1) & M_1(t_1) & \ldots & M_{n-1}(t_1)\end{bmatrix}=n$$

- Theorem 6.O11 The pair(A(t), C(t)) is observable at time t_0 if there exists a finite $t_1>t_0$ such that the n×n matrix

$$W_o(t_0,t_1)=\int_{t_0}^{t_1}\Phi'(\tau,\tau_0)C'(\tau)C(\tau)\Phi(\tau,t_0)d\tau$$

- Theorem 6.O12 Let A(t) and C(t) be n-1 times continuously differentiable. Then the n-dimensional pair (A(t), C(t)) is observable at t_0 if there exists a finite $t_1 > t_0$ such that

$$\operatorname{rank}\begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \cdot \\ N_{n-1}(t_1) \end{bmatrix} = n$$

where

$$N_{m+1}(t) = N_m(t)A(t) + \frac{d}{dt}N_m(t)$$

with $N_0 = C(t)$