Developed Observers for Time-Varying Linear Systems

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Outline

- **The Full-State and Reduced-Order Observer for Time-Varying Systems by Using the Application of Matrix Generalized Inverses.**
- **Finding the Optimal Initial Condition for Discrete Time-Varying Systems by Using the Min Max Optimization.**
- **The Combination of Two kinds of Observers**

The mathematical Background

- For every matrix A, there exists one or more matrices X satisfying $XAX = A$.
- In this case the matrix X is called

$$\{1\}\text{-inverse of } A$$

- Consider the linear system

$$\dot{x} = Ax + Bu$$

which can be solved for u if and only if

$$\left(I - B^{\{1\}}B\right)\left(\dot{x} - Ax\right) = 0$$

Luenberger Observer

- Consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t)$$

- The basic of Luenberger observer can be determined by

$$\dot{z}(t) = F(t)z(t) + G(t)y(t) + T(t)Bu(t)$$

The Full State Observer Using the Application of Matrix Generalized Inverses

- Consider the linear system (all variable are the function of time):

$$\dot{x} = Ax + Bu \qquad (1)$$

and Luenberger observer:

$$\dot{z} = Fz + Gy + TBu \qquad (2)$$

- Premultiplication (1) by T yields

$$T\dot{x} = TAx + TBu \qquad (3)$$

- Now, equation (2) can obviously be rewritten as

$$\dot{z} - \dot{T}x = Fz + Gy + TBu - \dot{T}x \qquad (4)$$

- Then, subtracting (3) from (4), and substituting $\boldsymbol{y = Cx}$ yields as

$$\frac{d}{dt}(z - Tx) = Fz - \left(\dot{T} + TA - GC\right)x \qquad (5)$$

The Full State Observer Using the Application of Matrix Generalized Inverses

$$\frac{d}{dt}(z - Tx) = Fz - \left(\dot{T} + TA - GC\right)x \qquad (5)$$

- Hence, if the matrices F, G, and T are chosen so that

$$\dot{T} + TA - GC = FT \qquad (6)$$

- Then equation (5) becomes

$$\frac{d}{dt}(z - Tx) = F(z - Tx) \qquad (7)$$

- Therefore, $z \rightarrow Tx$ as $t \rightarrow \infty$.
- The estimate of the actual state can be determined from

$$\hat{x}(t) = T^{-1}z(t) \qquad (8)$$

The Full State Observer Using the Application of Matrix Generalized Inverses

- Therefore the key of this observer is in equation (6). Recall equation (6):

$$\dot{T} + TA - GC = FT \qquad (6)$$

- Need to choose the matrices F, G, and T . Rewrite equation (6) as

$$\dot{T} + TA - FT = GC \qquad (9)$$

- The Lovass-Nugy's design procedure is the following step:

1) Choose a constant matrix F having some desirable eigenvalues. For example, if all eigenvalues are real, choose F to be upper triangular with unspecified elements above the main diagonal.
2) Let T be an n x n an unspecified matrix whose elements are initially required only to be differentiable functions of t.

The Full State Observer Using the Application of Matrix Generalized Inverses

3) Assume that G will be computed from A, F, and T by solving equation (9) for G. The theory of matrix generalized inverses yields that equation (9) can be solved for G if and only if

$$\left(\dot{T} + TA - FT\right)\left(I - C^{\{1\}}C\right) = 0$$

4) Use the still unspecified elements of F and T to satisfy the condition

$$\det(T) \neq 0 \quad \text{for all } t$$

5) Thus G can be computed as follows

$$G = \left(\dot{T} + TA - FT\right)C^{\{1\}}$$

6) Finally, construct the final form of the observer as

$$\dot{z} = Fz + Gy + TBu$$

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- For the basic construction of a reduced-order observer,

  if *y(t)* is of dimension *m* , then an observer of order *n - m* is constructed with state *z(t)* that, approximates *Tx(t)* for some *m x n* matrix *T*.

- Then an estimate of *x(t)* can be determined through

$$\hat{x}(t) = \begin{bmatrix} T \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix}$$

  which the indicated partitioned matrix is invertible. Thus the *T* associated with the observer must have *n - m* rows that are linearly independent of the rows of *C*.

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- Again, consider the linear system (all are the function of time):

$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx \qquad (2)$$

- It is more convenient to partition the state vector as

$$x = \begin{bmatrix} y \\ w \end{bmatrix}$$

and then rewrite the system in the form

$$\dot{y} = A_{11}y + A_{12}w + B_1u \qquad (3)$$

$$\dot{w} = A_{21}y + A_{22}w + B_2u \qquad (4)$$

- The idea of the construction is: the vector y is available for measurement, and if we differentiate it, so is dy/dt. Since $u(t)$ is also measurable, equation (3) provides the measurement $A_{12}w$ for the system equation (4) which has state vector w and input $A_{21}y + B_2u$.

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- Rewrite equation (3) as

$$A_{12}w = \dot{y} - A_{11}y - B_1u = \overline{y} \qquad (5)$$

- Therefore, we can apply the generalized inverse method on a system (4)

$$\dot{w} = A_{21}\overline{y} + A_{22}w + B_2u \qquad (6)$$

with the given observation

$$\overline{y} = Cw = A_{12}w = \dot{y} - A_{11}y - B_1u$$

- Premultiplication of the equation of the system (5) by T yields

$$T\dot{w} = TA_{21}\overline{y} + TA_{22}w + TB_2u \qquad (7)$$

- Now, the observer equation can obviously be rewritten as

$$\dot{z} - \dot{T}w = Fz + G\overline{y} + TB_2u - \dot{T}w \qquad (8)$$

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- Then, subtracting (7) from (8), and substituting yields as

$$\dot{z} - \dot{T}w - T\dot{w} = Fz + GCw - \dot{T}w - TA_{21}Cw - TA_{22}w \tag{9}$$

$$\frac{d}{dt}(z - Tw) = Fz - \left(\dot{T} + TA_{22} + TA_{21}C - GC\right)w \tag{10}$$

- Hence, if the matrices F, G, and T are chosen so that

$$\dot{T} + TA_{22} + TA_{21}C - GC = FT \tag{11}$$

- Then equation (10) becomes

$$\frac{d}{dt}(z - Tw) = F(z - Tw)$$

- Rewrite equation (11) in the form

$$\dot{T} + TA_{22} + TA_{21}C - FT = GC \tag{12}$$

- Then we can find F, G and T by the Lovass-Nugy's procedure.

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- Therefore, we can construct the final form of the reduced-order observer as

$$\dot{z} = Fz + G(\dot{y} - A_{11}y - B_1u) + TB_2u \tag{13}$$

where

$$y = \overline{y} = \dot{y} - A_{11}y - B_1u$$

- Now the problem is the differentiation of y. It can be solve by modifying the block diagram in the next slide

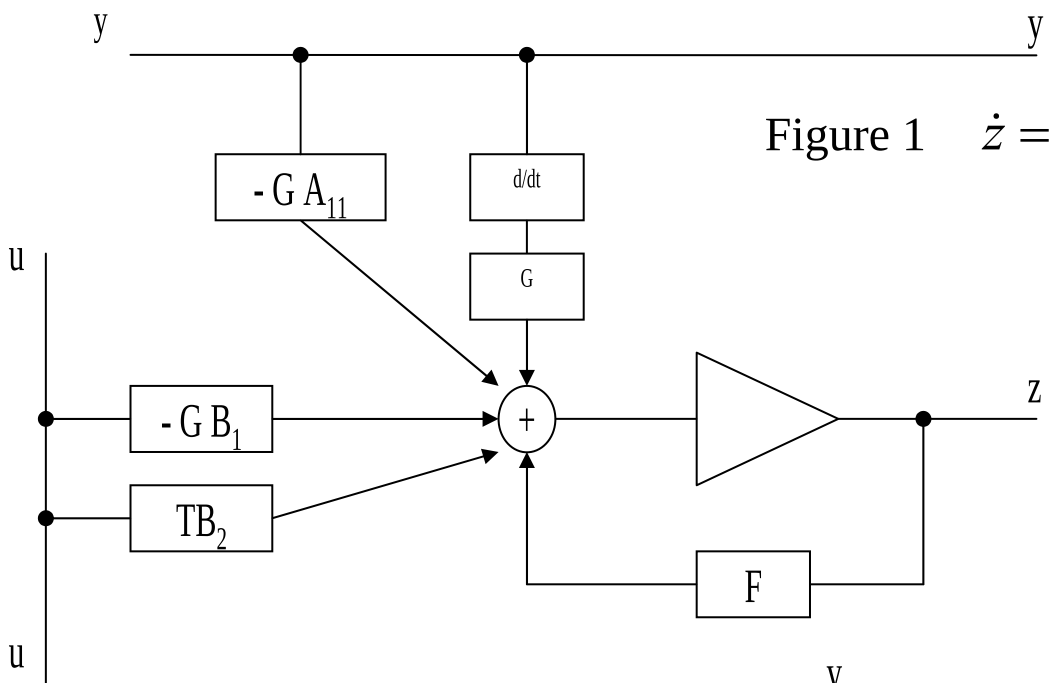


Figure 1 $\dot{z} = Fz + G(\dot{y} - A_{11}y - B_1u) + TB_2u$

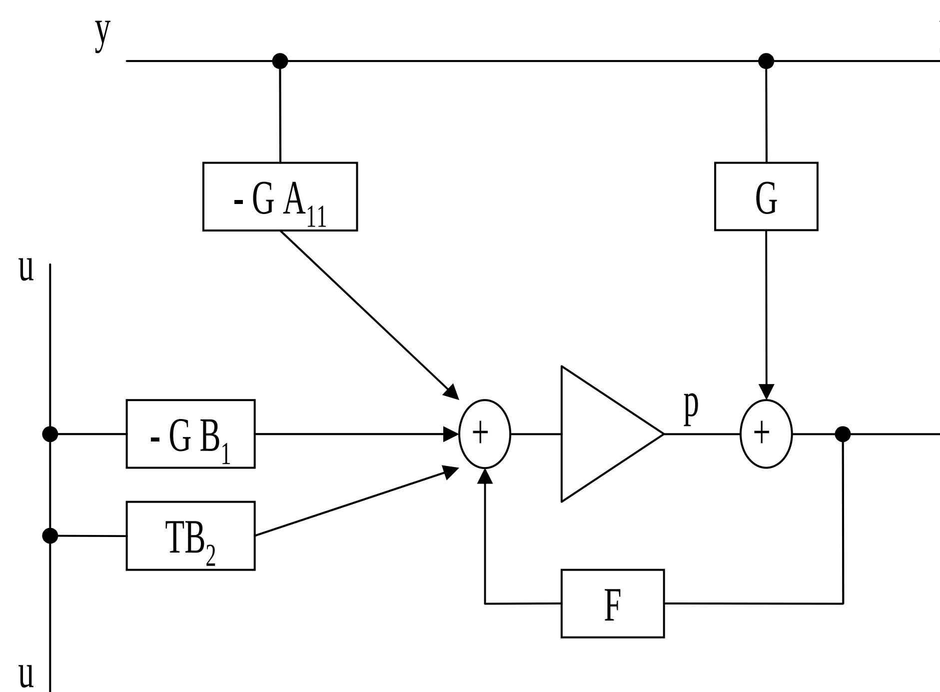


Figure 2

The Reduced-Order Observer Using the Application of Matrix Generalized Inverses

- The required differentiation of y can be avoided by modifying the block diagram of Figure 1 to that of Figure 2, which is equivalent at the point z. This yields the desired final form of the observer, which can be written

$$\dot{p}(t) = F(p(t) + Gy(t)) - GA_{11}y(t) - GB_1u(t) + TB_2u(t)$$
$$z(t) = p + Gy(t)$$

with $\hat{w}(t) = T^{-1}z(t).$

Example

- Consider the forth-order system. Let

$$A(t)=\begin{bmatrix}-10 & -10\alpha & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & -1 & -5 & 0\\ 0 & 0 & -4/\beta & 0\end{bmatrix},\qquad B(t)=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

where

$$\alpha=-0.4\exp(\sin(0.3t)-0.5)$$

$$\beta=0.4\cos(0.3t)-0.5$$

<u>Solution</u>

- Choose $F(t)=\begin{bmatrix}-5 & 0\\ 0 & -5\end{bmatrix}$ and $T(t)=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$

where T_{ij} is some unspecified function of t which is assumed to differentiable for all t.

- Then $C=A_{12}=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$, therefore the chosen $C^{\{1\}}$ is $\begin{bmatrix}0 & 0\\ 0.5 & 0.5\end{bmatrix}$.

Example (continue)

- The matrix G also can be found by

$$G = (\dot{T} + TA_{22} + TA_{21}C - FT)C^{\{1\}}$$

- In the simulation, the initial states are

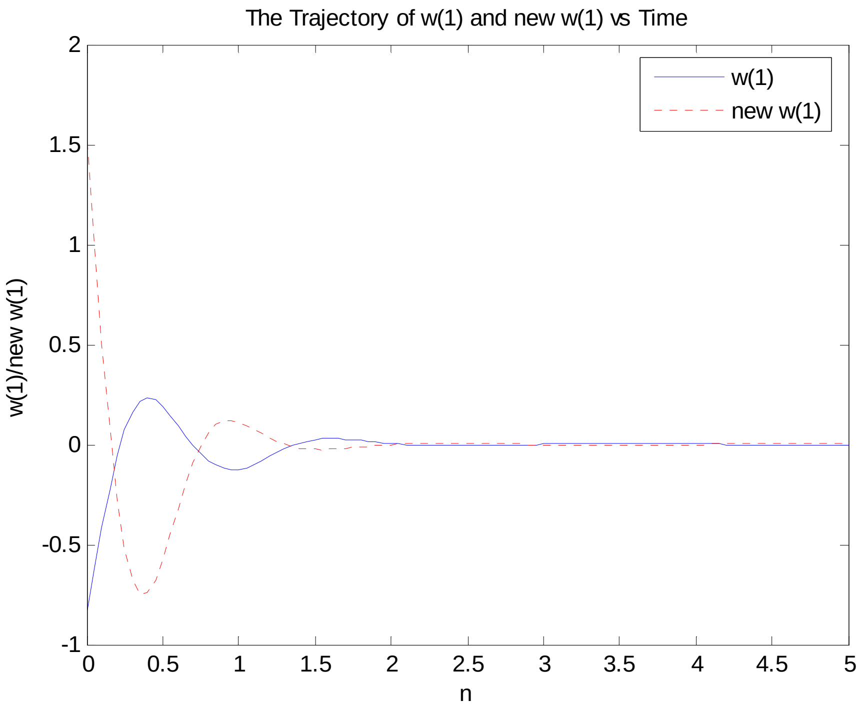
$$y_0 = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$$

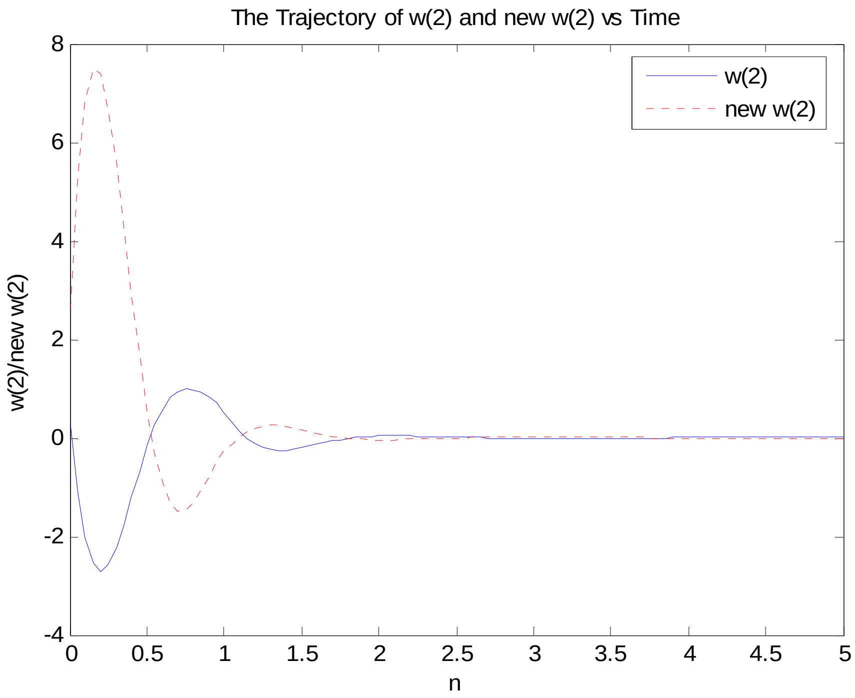
$$w_0 = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$$

$$z_0 = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$$

- The simulation is run for 5 unit time and convergence is observed after 2.5 unit time.

Example (continue)





Finding the Optimal Initial Condition for Time-Varying Systems by Using the Min Max Optimization

- Consider the linear system (discrete time case)

$$x(k+1) = A(k)x(k) + B(k)u(k): \quad x(k_0) = x_{k0}$$
$$y(k) = C(k)x(k)$$

- The problem is how to find $x(k_0)$, the initial condition, from the past data. Then $x(k)$, $k \geq k_0$ can easily be found from the initial condition.
- The basic cost is

$$\sum_{k=k_0}^{k_0+N} \left\| y(k) - C\overline{\Phi}(k, k_0, U, x_0) \right\|_\infty \qquad (1)$$

where $y(k)$ is the measured response of the system.

Finding the Optimal Initial Condition for Time-Varying Systems by Using the Min Max Optimization

- Equation (1) can be written as

$$\left\| \begin{matrix} y(k_0) - C[\Phi(k_0, k_0)x_0 + 0] \\ y(k_0+1) - C[\Phi(k_0+1, k_0)x_0 + \sum_{j=k_0}^{k_0} \Phi(k_0+1, j+1)Bu(j)] \\ \vdots \\ y(k_0+N) - C[\Phi(k_0+N, k_0)x_0 + \sum_{j=k_0}^{k_0+N-1} \Phi(k_0+N, j+1)Bu(j)] \end{matrix} \right\|_\infty$$

$$= \left\| \begin{matrix} y(k_0) - 0 - C\Phi(k_0, k_0)x_0 \\ y(k_0+1) - C\sum_{j=k_0}^{k_0} \Phi(k_0+1, j+1)Bu(j) - C\Phi(k_0+1, k_0)x_0 \\ \vdots \\ y(k_0+N) - C\sum_{j=k_0}^{k_0+N-1} \Phi(k_0+N, j+1)Bu(j) - C\Phi(k_0+N, k_0)x_0 \end{matrix} \right\|_\infty = \left\| \overline{y}(k) - \psi(k)\, x_0 \right\|_\infty$$

where

$$\overline{y}(k) = y(k_0+N) - C\sum_{j=k_0}^{k_0+N-1} \Phi(k_0+N, j+1)Bu(j) \text{ and } \psi(k) = C\Phi(k_0+N, k_0)\,; \quad N = 0,1,2,\ldots$$

Finding the Optimal Initial Condition for Time-Varying Systems by Using the Min Max Optimization

- First, we start with the solution of maximization.
- The simplest useful case is to take amount of errors in the measurement y, therefore the problem can be changed to

$$\max_{\|\delta y(k)\|_\infty \le \varepsilon} \left\| \bar{y}(k) + \delta y(k) - \psi(k)\, x_0 \right\|_\infty \qquad (3)$$

- This is a Linear Program which can be seen as follows

$$\max_{\|\delta y(k)\|_\infty \le \varepsilon} \left\| \bar{y}(k) + \delta y(k) - \psi(k)\, x_0 \right\|_\infty$$

$$= \max_{\substack{|\delta y(k)_i| \le \varepsilon \\ i=1,2,\ldots}} \max_i \left| \bar{y}(k)_i + \delta y(k)_i - (\psi(k)\, x_0)_i \right|$$

$$= \max_i \max_{\substack{|\delta y(k)_i| \le \varepsilon \\ i=1,2,\ldots}} \left| \bar{y}(k)_i - (\psi(k)\, x_0)_i + \delta y(k)_i \right|$$

$$= \max_i \max \left\{ \left| \bar{y}(k)_i - (\psi(k)\, x_0)_i + \varepsilon \right|, \left| \bar{y}(k)_i - (\psi(k)\, x_0)_i - \varepsilon \right| \right\} \qquad (4)$$

Finding the Optimal Initial Condition for Time-Varying Systems by Using the Min Max Optimization

- Now, the solution of minimization will be described next.
- It follows from equation (4) that the minimization problem to be solved is

$$\min_{x_0} \max_{i} \max\left\{\left|\bar{y}(k)_i - (\psi(k)x_0)_i + \varepsilon\right|, \left|\bar{y}(k)_i - (\psi(k)x_0)_i - \varepsilon\right|\right\} \tag{5}$$

- Equation (5) can be written as

$$\begin{aligned}
&\min_{x_0} \theta \\
&\textit{subject to} \\
&-\theta \le \bar{y}(k)_i - (\psi(k)x_0)_i + \varepsilon \le \theta \\
&-\theta \le \bar{y}(k)_i - (\psi(k)x_0)_i - \varepsilon \le \theta, \quad \forall i
\end{aligned}$$

which this optimization problem is now easy to be solved.

The Useful Of Min Max Optimization

- Now I would like to show that the solving of the min max problem is helpful.

<u>*Algorithm*</u>

1) Determine $\min_{x_0} \left\|\overline{y}(k) - \psi(k)x_0\right\|_\infty$ and the minimizing $x_0, \tilde{x}_0$.
2) Determine $\min_{x_0} \max_{\left\|\delta y(k)\right\|_\infty \le \varepsilon} \left\|\overline{y}(k) + \delta y(k) - \psi(k)x_0\right\|_\infty = \hat{\theta}$ and the minimizing $x_0, \hat{x}_0$.
3) Now add to each the entries of *y(k)*, i.e. choosing

$$\delta y(k) = \begin{bmatrix} \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

- If the computation of $\left\|\overline{y}(k) + \delta\hat{y}(k) - \psi\,\hat{x}_0\right\|_\infty \le \left\|\overline{y}(k) + \delta y(k) - \psi\,\tilde{x}_0\right\|_\infty$, then

$$\left\|\overline{y}(k) + \delta y(k) - \psi(k)\,\tilde{x}_0\right\|_\infty - \hat{\theta} \ge 0.$$

- This would prove that the optimal from the min max problem is a better when disturbances are presented.

Example

- Consider the forth-order system. Let

$$A(k)=\frac{1}{20}\begin{bmatrix}10 & 10\alpha & 1 & 0\\ -2 & 0 & -1 & -1\\ 0 & -1 & -5 & 0\\ -1 & 2 & 4/\beta & -5\end{bmatrix}, B(k)=\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \text{ and } C=\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

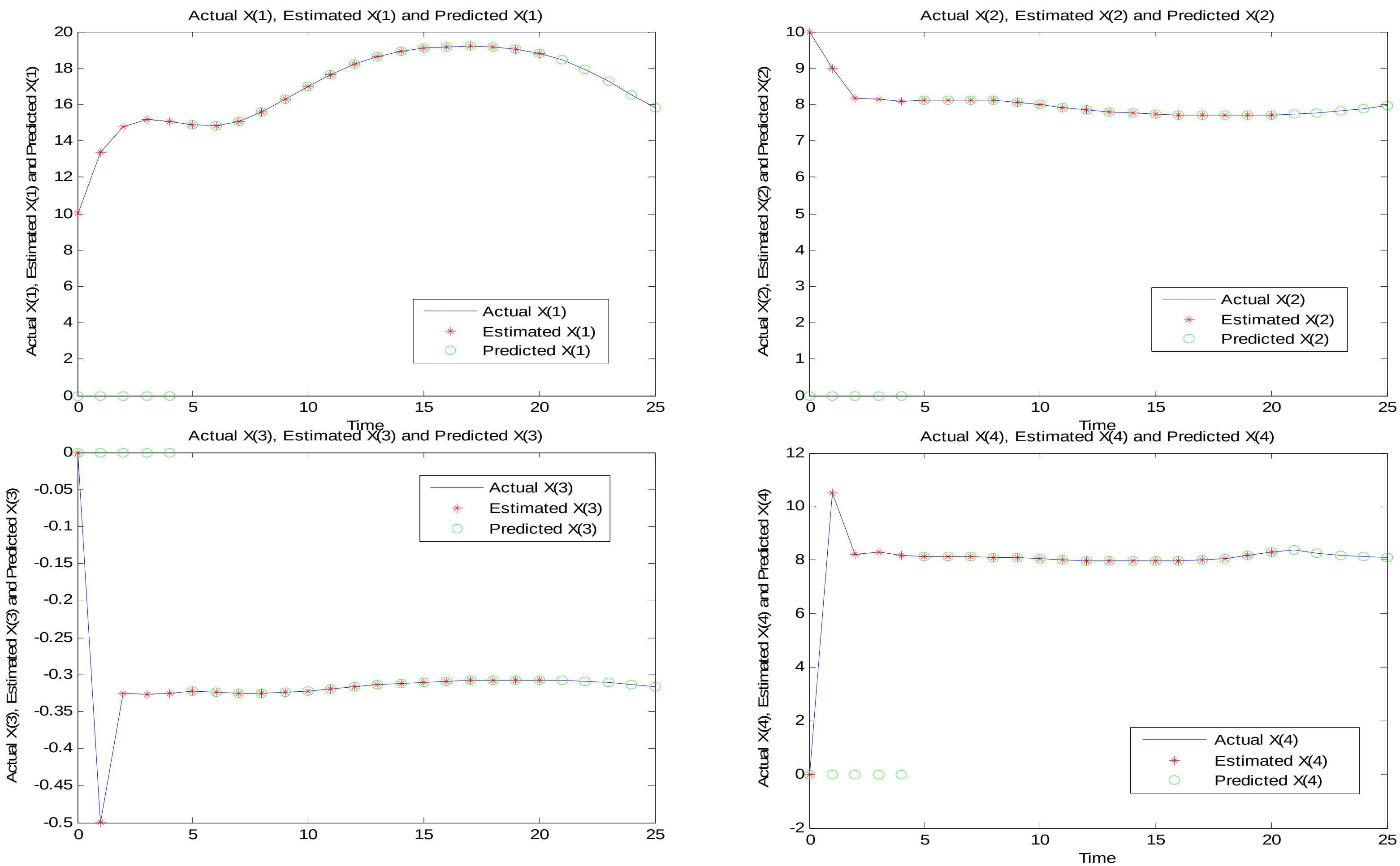
where

$$\alpha=-0.4\exp(\sin(0.3k)-0.5) \text{ and } \beta=0.4\cos(0.3k)-0.5$$

- In the simulation, we choose the initial states are

$$x_0=\begin{bmatrix}10 & 10 & 0 & 0\end{bmatrix}^T$$

Example (continue)



Example (continue)

Figure 3

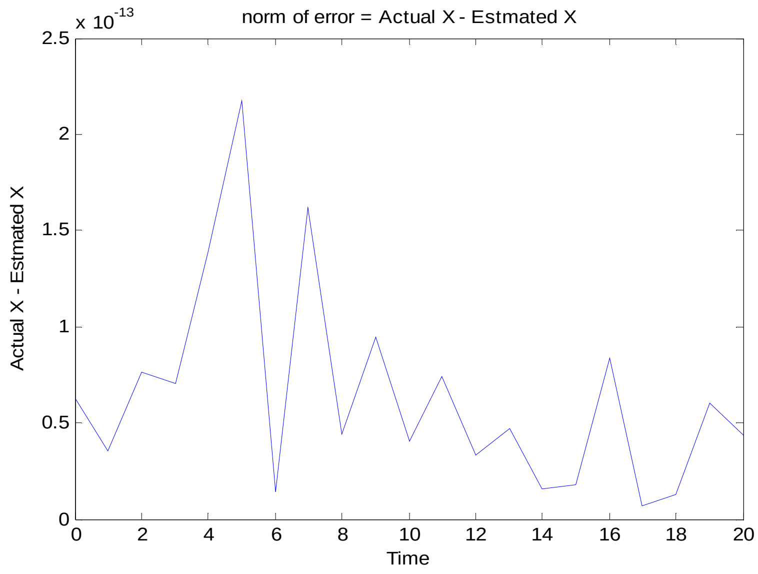
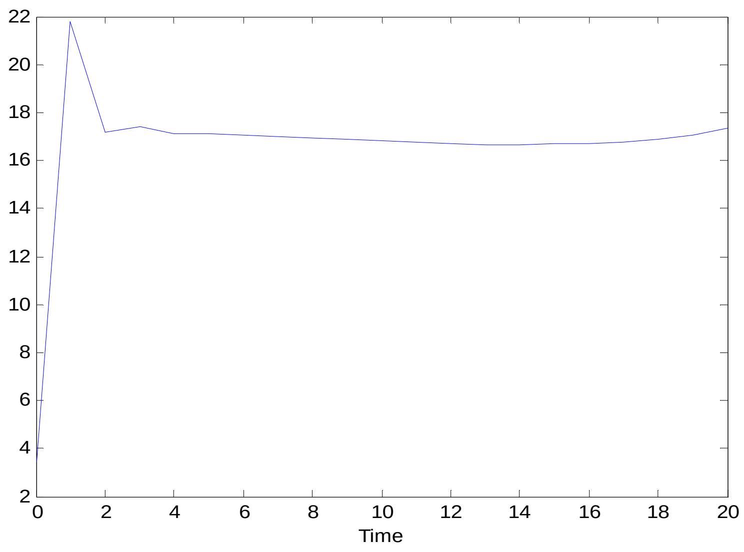


Figure 4



• From the figure 3, it shows the norm error between the actual x and the estimated x which is very small.

• From the figure 4, it shows that

$$\left\| \bar{y}(k) + \delta \hat{y}(k) - \psi(k)\,\tilde{x}_0 \right\|_\infty - \hat{\theta} \geq 0$$

•Therefore this will show that solving the min max problem is helpful and work well.

The Combination of Two kinds of Observers

- Now we have two types of observers as follows

1. Observer that finds by optimization but, even with a min-max formulation does not cope very well with modelling errors.
2. Usual Luenberger- type observers which are fairly robust to errors.

The Combination of Two kinds of Observers

- Then we can combine min-max observer and Luenberger which use the advantage of each observer.
- The observers can be combined in many possible cases which should give us the best of both kinds of observers.
- One of possible cases is Luenberger Observer with the optimal initial $x0$.

Algorithm:

1) Suppose at time t and data is given which can be used for the period $[t - T, t]$
2) Find an approximation of $x(t – T)$ using the min-max observer.
3) Feeding min-max observer into a simulation of Luenberger observer with the optimal $x(t – T)$ found by step 2 which is an initial condition of Luenberger observer.
4) Then simulate it up to time t with that estimate initial condition in order to find $x(t)$.

Example

- Consider the time varying forth-order discrete time system. Let

$$A(k) = \frac{1}{20}\begin{bmatrix} -10 & 10\alpha & 0 & -1 \\ -1 & -10 & -5 & 5 \\ 0 & -1 & -5 & 0 \\ -1 & 0 & -4/\beta & 0 \end{bmatrix}, B(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\alpha = -0.4\exp(\sin(0.3k) - 0.5) \text{ and } \beta = 0.4\cos(0.3k) - 0.5.$$

Example (continue)

- The plant system is $A(k) + P(k)$ where

$$P(k) = 0.02\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

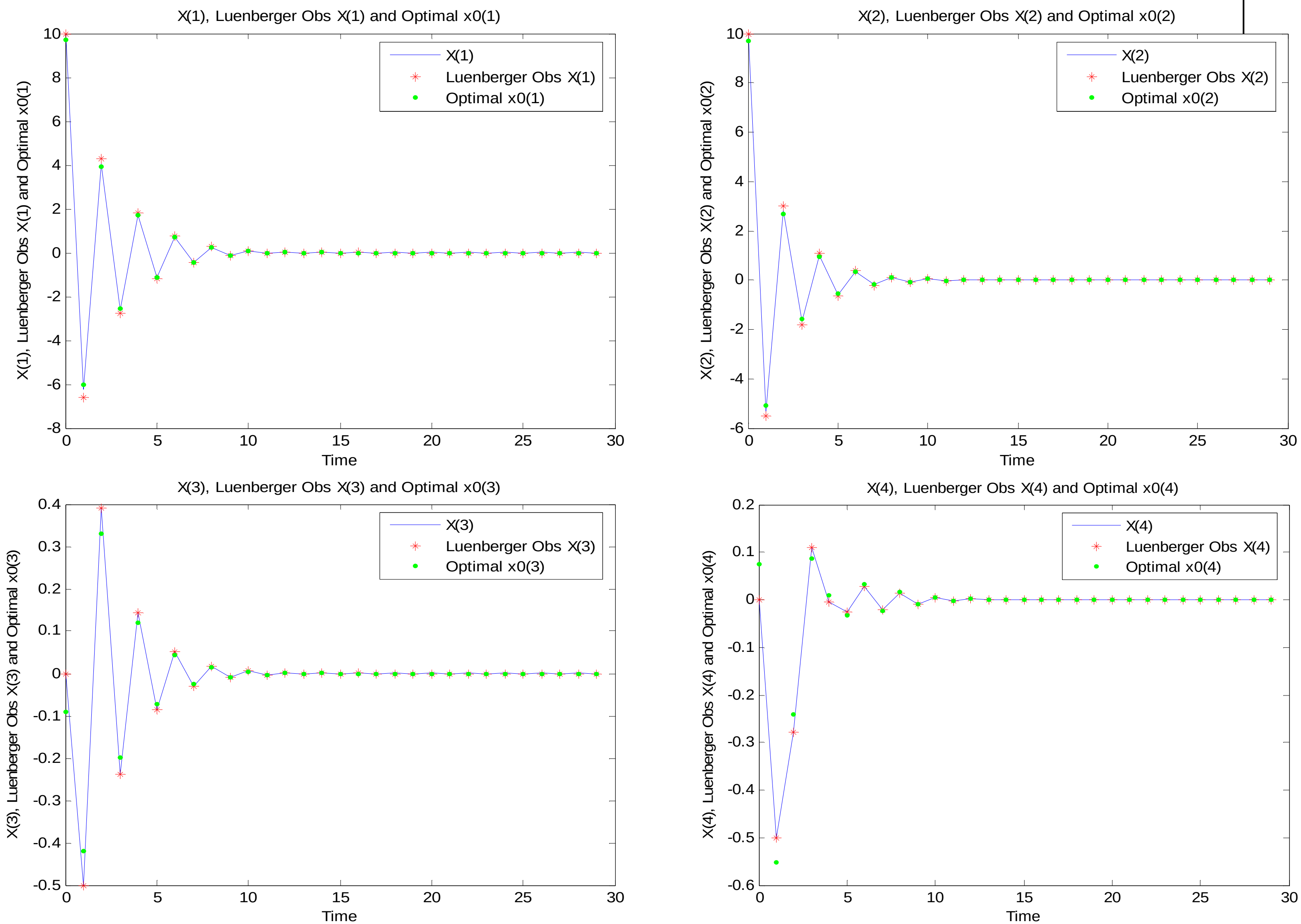
- Matrix F of Luenberger observer is chosen as

$$F = 0.01\begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -40 \end{bmatrix}$$

- In the simulation, we choose the initial states are

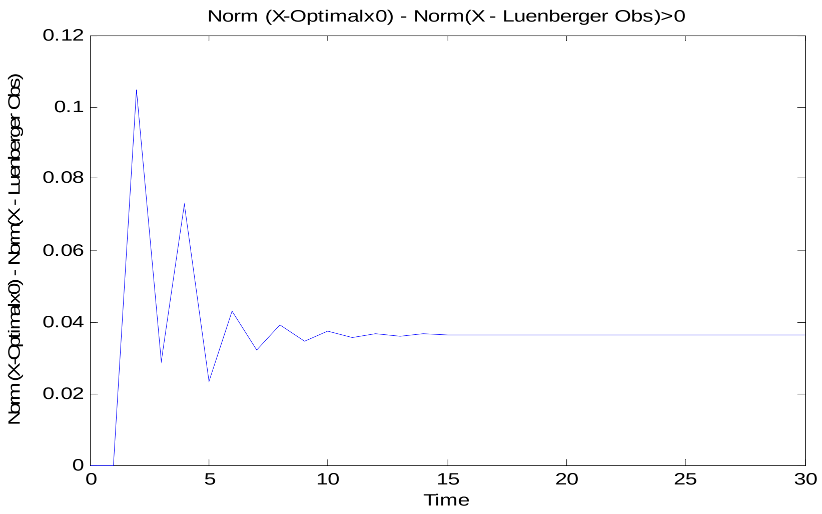
$$x_0 = \begin{bmatrix} 10 & 10 & 0 & 0 \end{bmatrix}^T$$

Example (continue)



Example (continue)

Figure 5



• Moreover, we can check it from the figure 5 which is shown the difference between the norm of (x_0 - x_0 from min max) and the norm of (x_0 - x_0 from Luenberger observer with min max).

• If Luenberger observer with min max can converse faster than the min man observer, the norm of (x_0 - x_0 from Luenberger observer with min max)) should be smaller than the norm of (x_0 - x_0 from min max).

• From the figure 5 shows that the difference is positive that means the Luenberger observer with the optimal initial condition can converse to the actual system faster than the min max observer.

Conclusion

- We can see that the full-state and reduced-order observer for time-varying systems can be designed by the application of matrix generalized inverses.
- Min Max of optimization can find the optimal initial condition for the time-varying system.
- The combination of two kinds of observer can give us the best of both kinds of observers

THANK YOU