Local Observability of Nonlinear Differential-Algebraic Equations (DAEs) From the Linearization Along a Trajectory

William J. Terrell

***Abstract*—The main results of this note establish sufficient rank conditions for local observability of nonlinear differential-algebraic equations (DAEs) systems near a known trajectory associated with a given control. Verification of the observability rank condition is addressed, and the relationship with previous work on observability of DAEs is indicated.**

***Index Terms*—Differential-algebraic equations, linearization, observability.**

I. Introduction

The local observability concept we consider in this note deals with the ability to "instantaneously" distinguish states from their neighbors, throughout a neighborhood of a given solution curve of a nonlinear differential-algebraic equation (DAE) system. Thus, the intuitive idea is similar to the intuition behind the notion of *local weak observability* in the fundamental work [8] on systems of nonlinear ordinary differential equations. Ideas related to those in [8] are developed in [7] using methods of differential algebra. A somewhat different notion of local observability (again for explicit ordinary differential systems), is the L*-observability* concept in [12] that is more closely related to output stabilization problems, and intuitively says that nearby states are distinguishable without using controls that produce "large excursions." Using the notion of fiber bundle, reference [15] gives a very general definition of a nonlinear system having local representation as a system of explicit ordinary differential equations, and discusses a natural observability concept which, in many cases, implies the notion of local weak observability (see [8]) for these systems.

In the case of implicit (DAE) models, observability conditions based on system Jacobians are important because they can provide useful system-theoretic information especially in situations where various possible models of a system are being examined.

Systems of DAEs appear frequently as models for dynamical systems in many application areas [1], [9]–[11]. There has been much work on the solutions and control-theoretic properties of DAE systems which are linear and time-invariant (for example, see [6]). By contrast, relatively few results are available on control and systems theory for nonlinear DAEs.

A verifiable condition for observability of linear time-varying DAE systems was established in [5]. This condition generalized classical matrix rank criteria for observability of nonsingular systems. The condition was extended in [13] to give an observability test for systems described by a general class of nonlinear DAEs.

It is known that time-invariant linearizations (for example, the frozen system at an initial time t_0), are often not appropriate models for nonlinear DAE system behavior [2]. On the other hand, the linearization of a DAE along a given trajectory is generally an implicit, time-varying linear system, and we will show that observability of this time-varying linearization does imply local observability of the nonlinear system near the trajectory. Thus, our results overcome the problem of the inadequacy of time-invariant linearization for the determination of observability. Our sufficient conditions for local observability may be verified using symbolic software and numerical linear algebra. Thus, these results can be the basis for algorithms for determining observability of nonlinear DAEs. These new results also draw a nice connection between the results of [5] and [13]. Specifically, we consider the following question: If the observability Jacobian of the linearization along a trajectory satisfies the observability condition of [5], is the nonlinear system locally observable in a neighborhood of the trajectory; in particular, does the Jacobian of the nonlinear system satisfy the generalized observability condition of [13]? We give a positive answer to this question in Theorem 2. This note provides a more detailed discussion of this question than that given in [14].

Section II gives the preliminaries needed to place this note in the context of previous work. Section III discusses linearization, and Section IV presents conditions on a linearization that are sufficient for local observability of a nonlinear DAE system in a neighborhood of a given trajectory. Section IV also provides an example in which observability of the nonlinear system is not correctly determined by the usual time-invariant linearization, but the theory of this note applies.

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The author is with the Department of Mathematical Sciences, Virginia Commonwealth University, Richmond, VA 23284-2014 USA (e-mail: wterrell@atlas.vcu.edu).

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II. Preliminaries

Nonlinear DAEs: Consider a nonlinear DAE system of the form

$$F(x',\, x,\, t) = D(t,\, u) \tag{1}$$

$$y = H(x,\, t) \tag{2}$$

where (2) describes the output of the system, with $x \in R^n$, $F(\cdot,\, \cdot,\, \cdot) \in R^n$, $y \in R^p$, $u \in R^m$, and $\partial F/\partial x'$ is singular. The DAE (1) is defined for $t \in [a,\, b] \equiv \mathcal{I}$.

If (1) is differentiated j times with respect to t we get the $(j+1)n$ equations

$$\tilde{F}_j(x,\, x',\, w,\, t) = \begin{bmatrix} F(x',\, x,\, t) \\ F_t + F_x x' + F_{x'} x'' \\ \vdots \\ \dfrac{d^j}{dt^j}\,[F(x',\, x,\, t)] \end{bmatrix} = \mathbf{u} \tag{3}$$

where $w = [x^{(2)},\, \ldots,\, x^{(k+1)}]$ with $x^{(i)} = d^i x/dt^i$, and $\mathbf{u}$ is a vector of time derivatives of the right side of (1).

If the output (2) is differentiated k times with respect to t we get the $(k+1)p$ equations

$$\mathcal{H} \equiv \tilde{H}_j(x,\, x',\, \overline{w},\, t) \equiv \begin{bmatrix} H(x,\, t) \\ H_t(x,\, t) + H_x(x,\, t)x' \\ \vdots \\ \dfrac{d^k}{dt^k}\,[H(x,\, t)] \end{bmatrix} \equiv \mathbf{y} \tag{4}$$

where $\overline{w} = [x^{(2)},\, \ldots,\, x^{(j)}]$; since we may have $j \neq k+1$, we will simply write $w = [x^{(2)},\, \ldots,\, x^{(\sigma)}]$ where $\sigma = \max\{k,\, j+1\}$. Then we write the combination of equation (3) with equation (4) as

$$\mathcal{O}(x,\, x',\, w,\, t) = \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix} \tag{5}$$

where $\mathbf{y} = [y,\, y',\, y'',\, \ldots,\, y^{(k)}]$.

Definition 1: If a matrix A is column-partitioned as $A = [A_1 \;\; A_2]$, with A_1 the first n columns, we say that A is 1-full (with respect to the first n columns) if

$$rank\, A = rank[A_1 \;\; A_2] = n + rank\, A_2. \tag{6}$$

Thus, the first n columns are linearly independent, and linearly independent of the remaining columns. Equivalently, any vector in the null space of A has the first n components equal to zero.

Our assumptions concerning equation (1) are as follows:

A1) F is sufficiently differentiable in its arguments.

A2) $\mathcal{G} \equiv \tilde{F}_j = 0$ is consistent as an algebraic equation, that is, solutions exist.

A3) $[\mathcal{G}_{x'} \;\; \mathcal{G}_w]$ is 1-full with respect to x' and has constant rank independent of (x, x', w, t).

A4) $[\mathcal{G}_x \;\; \mathcal{G}_{x'} \;\; \mathcal{G}_w]$ has full row rank independent of (x, x', w, t).

These conditions guarantee that solutions exist for all sufficiently differentiable u, and solutions are uniquely determined by consistent initial conditions (x_0, t_0) [4]. The *uniform differentiation index* ν is defined as the least value of j for which A2)–A4) hold [3].

Smooth Observability: We assume that H is sufficiently smooth so that all of the derivatives which appear below are continuous.

Definition 2: System (1), (2) is *smoothly observable on* $[a, b]$ if there exist nonnegative integers j, k and a differentiable function $R(t, \cdot, \ldots, \cdot)\colon [a, b] \times R^{(k+1)p} \times R^{(j+1)m} \to R^n$ such that

$$x(t) = R\left(t, y(t), \ldots, y^{(k)}(t), u(t), \ldots, u^{(j)}(t)\right)$$

for every smooth solution $x(t)$ of (1).

Definition 2 may be modified to give a definition of *smooth observability in a neighborhood* $\mathcal{W}$ of a given trajectory $\overline{x}(t)$, for example by restricting the variables $(x, x', \ldots, x^{(j+1)}) \in R^{(j+2)n}$ to an open neighborhood of $(\overline{x}(t), \overline{x}'(t), \ldots, \overline{x}^{(j+1)}(t))$ for each t.

The Jacobian matrix of $\mathcal{O}(x, x', w, t)$ [see (5)] with respect to (x, x', w) is, with reference to our previous notation in A2)–A4)

$$J_{\mathcal{O}} \equiv \begin{bmatrix} \mathcal{G}_x & \mathcal{G}_{x'} & \mathcal{G}_w \\ \mathcal{H}_x & \mathcal{H}_{x'} & \mathcal{H}_w \end{bmatrix}. \tag{7}$$

Theorem 1 [13]: Given equations (5), suppose A1)–A4) hold in a set $\mathcal{W}$ (which is open in $R^{(j+2)n+1}$ and) containing a given extended trajectory $p(t) \triangleq (\overline{x}(t), \overline{x}'(t), \overline{w}(t))$, for $t \in \mathcal{I}$. Suppose the Jacobian $J_{\mathcal{O}}$ in (7) of system (1), (2) satisfies

A5) $rank\; J_{\mathcal{O}} = n + rank \begin{bmatrix} \mathcal{G}_{x'} & \mathcal{G}_w \\ \mathcal{H}_{x'} & \mathcal{H}_w \end{bmatrix}$, for $(x, x', w, t) \in U$.

A6) $J_{\mathcal{O}}$ has constant rank on U.

Then system (1), (2) is smoothly observable in a neighborhood about $\overline{x}(t)$.

Section IV discusses the verification of A5) and A6).

III. Linearization Along a Trajectory

A development of linearizations of nonlinear DAEs appears in [2]. Suppose $\overline{x}$ is a solution of (1) corresponding to an input $\overline{u}(t)$. The time-varying linearization of system (1), (2) along the solution $\overline{x}$ near the control $\overline{u}(t)$ is defined for $t \in \mathcal{I}$ by [2]

$$A(t)\hat{x}' + B(t)\hat{x} + \tilde{D}(t)\hat{u} = 0 \tag{8}$$

$$y = C(t)\hat{x} \tag{9}$$

where $\hat{x} = x - \overline{x}$, $\hat{u} = u - \overline{u}$, and

$$\begin{aligned} A(t) &= F_{x'}(\overline{x}'(t), \overline{x}(t), t) \\ B(t) &= F_x(\overline{x}'(t), \overline{x}(t), t) \\ \tilde{D}(t) &= -D_u(t, \overline{u}(t)) \\ C(t) &= H_x(\overline{x}(t), t). \end{aligned}$$

By setting $\hat{x} = x - \overline{x}$, $\hat{x}^{(i)} = x^{(i)} - \overline{x}^{(i)}$ in the left side of (5), the Taylor expansion argument in [2] implies that the Jacobian $J_{\mathcal{O}}$ in (7) [taken now with respect to $(\hat{x}, \hat{x}', \ldots, \hat{x}^{(j+1)})$], when evaluated at $(\overline{x}(t), \overline{x}'(t), \overline{w}(t), t)$, has the form

$$\begin{bmatrix} F_x & F_{x'} & 0 & 0 & \cdot \\ F_{x,t} & F_x + F_{x',t} & F_{x'} & 0 & \cdot \\ F_{x,t^2} & 2F_{x,t} + F_{x',t^2} & F_x + 2F_{x',t} & F_{x'} & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_x & 0 & 0 & 0 & \cdot \\ H_{x,t} & H_x & 0 & 0 & \cdot \\ H_{x,t^2} & 2H_{x,t} & H_x & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{10}$$

where the notation R_{t^i} denotes the result of taking the ith derivative with respect to t of an expression $R(t, x, x', \ldots, x^{(\sigma)})$ thinking of t and the $x^{(k)}$ as independent variables. When thus evaluated, this last Jacobian is exactly the observability array from [5] for the time-varying linearization (8), (9) along $\overline{x}$

$$\mathcal{O}_{j,k} = \left[\begin{array}{c|c} \mathcal{B}_j & \mathcal{A}_j \\ \tilde{\mathcal{C}}_k & [\,\hat{\mathcal{C}}_k \quad 0_{(k+1)p\times(j+1-k)n}\,] \end{array}\right] \tag{11}$$

where

$$\mathcal{B}_j = \begin{bmatrix} B \\ B' \\ B'' \\ \vdots \\ B^{(j)} \end{bmatrix}$$

$$\mathcal{A}_j = \begin{bmatrix} A & 0 & \cdot & 0 \\ A' + B & A & \cdot & \cdot \\ A'' + 2B' & 2A' + B & \cdot & \cdot \\ \vdots & & & 0 \\ A^{(j)} + jB^{(j-1)} & & \cdot & A \end{bmatrix}$$

and

$$\mathcal{C}_k = \left[\begin{array}{c|ccccc} C & 0 & \cdot & \cdot & 0 \\ C' & C & 0 & \cdot & \cdot \\ C'' & 2C' & C & \ddots & \cdot \\ \vdots & \vdots & * & \ddots & 0 \\ C^{(k)} & kC^{(k-1)} & * & * & C \end{array}\right] = \left[\tilde{\mathcal{C}}_k \;\mid\; \hat{\mathcal{C}}_k\right].$$

From [5], the time-varying linearization (8), (9) is smoothly observable if $\mathcal{O}_{j,k}$ is 1-full with respect to $\hat{x}$ and constant rank on $\mathcal{I}$. Also from [5], the matrix $\mathcal{A}_j$ is 1-full with respect to $(\hat{x}', \hat{x}'', \ldots, \hat{x}^{(k)})$ when $j \geq \nu + k - 1$, because the uniform differentiation index of (8) is less than or equal to ν [2].

IV. Main Results

We consider here whether the full Jacobian of the nonlinear DAE system is 1-full with respect to x if the Jacobian (11) of the linearization has that property. By considering simple matrix examples, one can see that in general, the 1-full condition (6) is not preserved by small changes in a matrix. Our job here is to show that the 1-full condition is indeed preserved by the structure of a DAE array, under very general conditions on the DAE system.

We present two lemmas before stating the main results.

Lemma 1: Let $J(t, \epsilon)$ be a matrix function of $(t, \epsilon) \in \mathcal{I} \times R^{\sigma}$, and write

$$J(t, \epsilon) = [J_1(t, \epsilon) \quad J_2(t, \epsilon)]$$

where $J_1(t, \epsilon)$ has n columns. Suppose $rank\, J(t_0, \epsilon_0) = n + rank\, J_2(t_0, \epsilon_0)$ for some (t_0, ϵ_0). Then

i) $rank\, J(t, \epsilon)$ is constant in some open neighborhood of (t_0, ϵ_0) if and only if $rank\, J_2(t, \epsilon)$ is constant in some open neighborhood of (t_0, ϵ_0).

ii) If $rank\, J(t, \epsilon)$ is constant in an open neighborhood of (t_0, ϵ_0), then $rank\, J(t, \epsilon) = n + rank\, J_2(t, \epsilon)$ for all (t, ϵ) in an open neighborhood of (t_0, ϵ_0).

iii) If $\overline{\epsilon}(t)$ is a smooth curve for which $rank\, J(t, \overline{\epsilon}(t)) = n + rank\, J_2(t, \overline{\epsilon}(t))$ for all $t \in \mathcal{I}$, then there is an open set $\mathcal{W}$ containing the curve $(t, \overline{\epsilon}(t))$ for $t \in \mathcal{I}$, that is topologically equivalent to a cylinder $\mathcal{I} \times U \subset R \times R^{\sigma}$ with U open in R^{σ}, on which we have $rank\, J(t, \epsilon) = n + rank\, J_2(t, \epsilon)$ for all $(t, \epsilon) \in \mathcal{W}$.

Proof: Consider statement i) first. We must have $rank\, J_1(t, \epsilon) = n$ for all (t, ϵ) in some open set W containing (t_0, ϵ_0). We also have $rank\, J_2(t, \epsilon) \geq rank\, J_2(t_0, \epsilon_0)$ for all $(t, \epsilon) \in \tilde{W} \subset W$. Thus, statement i) follows by considering open neighborhoods $\mathcal{W}$ of (t_0, ϵ_0) with $\mathcal{W} \subset \tilde{W}$.

Statement ii) follows immediately, since we may take the constant rank assumption to hold in some open neighborhood $\mathcal{W}$ of (t_0, ϵ_0) as just described.

Statement iii) follows from ii) and the assumption that the interval $\mathcal{I}$ is compact. □

Lemma 2: For any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|x(t_0) - \overline{x}(t_0)\| < \delta$ then solutions of (1) satisfy $\|x^{(i)}(t) - \overline{x}^{(i)}(t)\| < \epsilon$ for all $t \in \mathcal{I}$, and for $0 \leq i \leq j+1$.

Proof: This follows from our smoothness assumptions, the continuous dependence of solutions of (1) on initial conditions, and the fact that the time interval $\mathcal{I}$ is compact. □

Theorem 2: Assume A6) holds. Assume also that the linearization (8), (9) of system (1), (2) satisfies the condition that the matrix $\mathcal{O}_{j,\,k}$ in (11) is 1-full with respect to x on $\mathcal{I}$. Then the full Jacobian (7) of the nonlinear system remains 1-full with respect to x in a neighborhood $\mathcal{W}$ of the extended trajectory, and system (1), (2) is smoothly observable in $\mathcal{W}$. In fact, the linearized output function along the trajectory is sufficient for observing states within $\mathcal{W}$.

Proof: Lemma 1 applies to the Jacobian $J_{\mathcal{O}}$; a desired neighborhood $\mathcal{W}$ consisting of trajectories exists by Lemma 1 iii) with the help of Lemma 2. Thus, only the statement concerning observability with a linear output remains to be proved. By hypothesis, the linearization satisfies the observability rank condition, and thus the perturbation to the Jacobian for the full nonlinear system supplied with that linear output, must maintain the condition of being 1-full with respect to x (and constant rank) in such a neighborhood $\mathcal{W}$. □

The assumptions of Theorem 2 (and Theorem 3 below) may be verified using symbolic software and numerical linear algebra; for additional details, see [4]. Thus, these results can be the basis for algorithms for determining observability of nonlinear DAEs.

The practicality of Theorem 2 is supported by Theorem 3, which says that sufficient differentiation, combined with the smooth observability condition on the linearization, guarantees constant rank for the observability Jacobian of a solvable nonlinear DAE.

Theorem 3: Suppose k differentiations of the output and $j \geq \nu$ differentiations of the DAE produce a Jacobian for the linearization along $\overline{x}$ that is 1-full with respect to x on $\mathcal{I}$. Then system (1), (2) is smoothly observable in an open set $\mathcal{W}$ about $\overline{x}$ by means of a linear output function.

Proof: There are two cases: a) $j \geq \nu+k-1$, and b) $j < \nu+k-1$. In case a), all derivatives of x that appear in the differentiated output equations are uniquely determined in terms of consistent (x, t) by (3); thus, the constant rank statement follows from the constant rank of the DAE Jacobian [assumption A3)]. In case b), simply differentiate the DAE an additional $(\nu + k - 1) - j$ times to achieve case a). Thus, the observability Jacobian $J_{\mathcal{O}}$ can be made constant rank in an open set $\mathcal{W}$ containing $p(t) = (\overline{x}(t), \overline{x}'(t), \overline{w}(t)), t \in \mathcal{I}$, with no more than $(\nu+k-1)-j$ additional differentiations of the DAE. The conclusion follows. □

As promised, we give here an example which shows that the time-invariant linearization (frozen at time t_0) is inadequate for determining observability, but our new theory applies and decides the issue.

Example: Consider the nonlinear DAE system with output

$$\begin{aligned}
x_1' &= x_1 - tx_3 + f_1(x, t) + u_1(t)\\
x_2' &= x_1 - x_2 + f_2(x, t) + u_2(t)\\
0 &= x_2 + x_3 + f_3(x, t) + u_3(t)\\
y &= x_1
\end{aligned}$$

where the nonlinear functions f_1, f_2 and f_3 are smooth and of at least second order in the components of x. Thus, $x = 0$ is an equilibrium solution of the unforced system. We verify below that this system is smoothly observable on $[0, T]$ in a neighborhood of the equilibrium solution $\overline{x} = 0$ (we drop the barred notation for the remainder of the example). Rather than displaying the full Jacobian array in this verification, we make the example more intuitive by showing that differentiation of the output, combined with substitutions from the differential equations, allows us to recover the state x from a knowledge of the input and output.

Notice first that the time-invariant linearization at $t_0 = 0$ does not reveal the observability. The time-invariant linearization at $t_0 = 0$ for the equilibrium solution $x = 0$ is

$$\begin{aligned}
x_1' &= x_1 + u_1(t)\\
x_2' &= x_1 - x_2 + u_2(t)\\
0 &= x_2 + x_3 + u_3(t)\\
y &= x_1.
\end{aligned}$$

This time-invariant linearized system is not observable, because neither x_2 nor x_3 can be observed. However, the time-varying linearization given by

$$\begin{aligned}
x_1' &= x_1 - tx_3 + u_1(t)\\
x_2' &= x_1 - x_2 + u_2(t)\\
0 &= x_2 + x_3 + u_3(t)\\
y &= x_1
\end{aligned}$$

is smoothly observable. We verify this by noticing that three differentiations of the output and substitutions from the DAE equations produces

$$\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -t \\ (1+t) & -t & -(1+t) \\ (3+t) & -2 & -(t^2+t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \hat{U}(t, u, u', u'')$$

where $\hat{U}(t, u, u', u'')$ indicates terms dependent only on the input $u = [u_1, u_2, u_3]^T$ and its' derivatives up to second order in t. Notice that the time-varying linearization allows x_2 into the equation for

y'' (except at $t = 0$). One additional differentiation to get y''' guarantees full column rank equal to three for the coefficient matrix of vector x. For the example, this rank condition is equivalent to the 1-full condition on the Jacobian $\mathcal{O}_{j,k}$ of (11) for the time-varying linearization. (We may take $j = 1$ and $k = 3$.) Thus we have smooth observability on the interval $[0, T]$ for the time-varying linearized system. Then, provided A6) holds (perhaps after additional differentiation, see Theorem 3), Theorem 2 applies and guarantees smooth observability of the nonlinear system on $[0, T]$ in a neighborhood of $x = 0$. □

Based on Theorem 2, many special results may now be deduced, involving conditions on a linearization (8), (9), or on submatrices of the full Jacobian (7), that guarantee smooth observability in $\mathcal{W}$ about a trajectory, in particular for semi-explicit systems where (7) simplifies somewhat. For example, the analysis in [13] shows that the hypotheses in Theorem 2 will hold for Hessenberg DAE systems supplied with an effective output function, that is, an output for which $\mathcal{O}_{j,k}$ in (11) is 1-full with respect to x on $\mathcal{I}$.

Finally, the special u-dependence dealt with in (1) is not a real restriction: it is still possible to apply the condition (6) to (7), allowing Theorems 2 and 3 of Section IV to be extended to the case of a DAE with the more general u-dependence, $F(x, x', t, u) = 0$, in place of (1). The linearization is still given by (8), (9), but in this case the Jacobian (7) also depends on the variables $u^{(i)}$ for $0 \leq i \leq j$.

V. Conclusion

We have established some sufficient conditions for local observability of nonlinear DAE systems near a known trajectory. We indicated by an example the importance of these results in overcoming the inadequacy of time-invariant linearizations for determining observability of nonlinear DAEs. Our sufficient conditions for smooth observability are verifiable and are strong enough to guarantee that the full system observability Jacobian satisfies the smooth observability condition in [13]. The results of this note can provide a basis for the future development of algorithms for determining observability of nonlinear DAE systems.

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On the Convergence Rate of Ordinal Comparisons of Random Variables

Michael C. Fu and Xing Jin

***Abstract*—The asymptotic exponential convergence rate of ordinal comparisons follows from well-known results in large deviations theory, where the critical condition is the existence of a finite moment generating function. In this note, we show that this is both a necessary and sufficient condition, and also show how one can recover the exponential convergence rate in cases where the moment generating function is not finite. In particular, by working with appropriately truncated versions of the original random variables, the exponential convergence rate can be recovered.**

***Index Terms*—Large deviations, ordinal optimization, stochastic simulation.**

I. Introduction

Estimation of the mean of a random variable by Monte Carlo simulation has convergence rate $1/\sqrt{n}$, where n is the number of samples taken. On the other hand, often one is not so interested in the actual value of the mean in the absolute sense as in its value relative to other means, e.g., if one is comparing various designs in order to select the best one. Determination of the best is carried out by using some surrogate metric for performance evaluation. In using the sample mean to decide the best design, the probability of correctly selecting the best design often exhibits an asymptotically exponential convergence rate.

Specifically, our problem setting is as follows. Among m designs, we wish to determine the one with minimum mean; doing so is called "correct selection." Without loss of generality, suppose

$$EX_1 < EX_2 < \cdots < EX_m.$$

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The authors are with R. H. Smith School of Business, University of Maryland, College Park, MD 20742-1815 USA (e-mail: mfu@rhsmith.umd.edu; xjin@rhsmith.umd.edu).

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