Stability, Pole Placement, Observers and Stabilization

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DISC Course Mathematical Models of Systems

Outline

Stability of autonomous Systems

Autonomous systems

Let $P(\xi) \in \mathbb{R}^{q\times q}[\xi]$, with $\det P(\xi) \neq 0$, i.e., $\det P(\xi)$ is not the zero polynomial.
Consider the system of differential equations $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$.
This represents the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with
$\mathfrak{B} = \{w : \mathbb{R} \to \mathbb{R}^q \mid w \text{ satisfies } P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \text{ weakly}\}$.
Since $\det P(\xi) \neq 0$, the resulting system is autonomous. Hence $\mathfrak{B}$ is finite-dimensional and each weak solution of is a strong one.

Stability of autonomous systems

All solutions

All solutions are infinitely differentiable. In fact, $w \in \mathfrak{B}$ if and only if $w = w_1 + w_2 + \cdots + w_N$, with w_ks associated with one of the distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $\det P(\xi)$. This w_k is given by

$$w_k(t) = \left(\sum_{\ell=0}^{n_k - 1} B_{k\ell} t^{\ell} \right) e^{\lambda_k t},$$

where n_k is the multiplicity of the root λ_k of $\det P(\xi)$, $B_{k\ell}$s are suitable constant complex vectors.

Stability of autnomous systems

Stability definitions

The linear dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ is said to be

1. stable if all elements of its behavior $\mathfrak{B}$ are bounded on the half-line $[0, \infty)$, i.e, if $(w \in \mathfrak{B}) \Rightarrow$ (there exists $M \in \mathbb{R}$ such that $\|w(t)\| \leq M$ for $t \geq 0$). Of course, this bound M depends on the particular solution $w \in \mathfrak{B}$,
2. unstable if it is not stable,
3. asymptotically stable if all elements of $\mathfrak{B}$ approach zero for $t \to \infty$ (i.e, if $(w \in \mathfrak{B}) \Rightarrow (w(t) \to 0$ as $t \to \infty$)).

Important note: If Σ is stable or asymptotically stable then it must be autonomous. Hence without loss of generality $\mathfrak{B} = \ker\ P(\frac{\mathrm{d}}{\mathrm{d}t})$ with $P(\xi)$ nonsingular.

Stability of autonomous systems

Definition of simple and semisimple roots

Let $P(\xi) \in \mathbb{R}^{q \times q}[\xi]$ be nonsingular. Then

1. The roots of $P(\xi)$ are defined to be those of the scalar polynomial $\det P(\xi)$. Hence $\lambda \in \mathbb{C}$ is a root of $P(\xi)$ if and only if $\mathrm{rank}\ P(\lambda) < q$,
2. The root λ is called simple if it is a root of $\det P(\xi)$ of multiplicity one,
3. semisimple if the rank deficiency of $P(\lambda)$ equals the multiplicity of λ as a root of $P(\xi)$ (equivalently, if the dimension of $\ker\ P(\lambda)$ is equal to the multiplicity of λ as a root of $\det P(\xi)$).

Clearly, for $q = 1$ roots are semisimple if and only if they are simple, but for $q > 1$ the situation is more complicated.

Stability of autonomous systems

Example

$\lambda = 0$ is a root of multiplicity 2 for both the polynomial matrices

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \text{ and } \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}. \tag{1}$$

This root is semisimple in the first case, but not in the second.

Stability of autonomous systems

Theorem

Let $P \in \mathbb{R}^{q\times q}[\xi]$ be nonsingular. The system represented by $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is:

1. asymptotically stable if and only if all the roots of $\det P(\xi)$ have negative real part;
2. stable if and only if for each $\lambda \in \mathbb{C}$ that is a root of $P(\xi)$, either (i) $\mathrm{Re}\,\lambda < 0$, or (ii) $\mathrm{Re}\,\lambda = 0$ and λ is a semisimple root of $P(\xi)$.
3. unstable if $P(\xi)$ has a root with positive real part and/or a nonsemisimple root with zero real part.

Stability of autonomous systems

Examples

1. Scalar first-order system $aw + \frac{d}{dt}w = 0$. Associated polynomial $P(\xi) = a + \xi$. Root $-a$. Hence this system is asymptotically stable if $a > 0$, stable if $a = 0$, and unstable if $a < 0$. Note: behavior $\mathfrak{B} = \{Ae^{-at} \mid A \in \mathbb{R}\}$.
2. Scalar second-order system $aw + \frac{d^2}{dt^2}w = 0$. Associated polynomial $P(\xi) = a + \xi^2$. Roots $\lambda_{1,2} = \pm\sqrt{-a}$ for $a < 0$, $\lambda_{1,2} = \pm i\sqrt{a}$ for $a > 0$, and $\lambda = 0$ is a double, not semisimple root when $a = 0$. Thus we have ($a < 0 \Rightarrow$ instability), ($a > 0 \Rightarrow$ stability), and ($a = 0 \Rightarrow$ instability). Indeed:
$\mathfrak{B} = \{Ae^{\sqrt{-a}t} + Be^{-\sqrt{-a}t} \mid A, B \in \mathbb{R}\}$, if $a < 0$,
$\mathfrak{B} = \{A\cos\sqrt{a}t + B\sin\sqrt{a}tA \mid A, B \in \mathbb{R}\}$, if $a > 0$,
$\mathfrak{B} = \{A + Bt \mid A, B \in \mathbb{R}\}$, if $a = 0$.

Stability of autonomous systems

Special case: stability of state equations

Autonomous state system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$, with $A \in \mathbb{R}^{n \times n}$.
Polynomial matrix $P(\xi) = I\xi - A$. Roots: the eigenvalues of A.
An eigenvalue λ of A is called semisimple if λ is a semisimple root of $\det(I\xi - A)$, i.e:

$$\text{the multiplicity of } \lambda \text{ is equal to } \dim \ker(\lambda I - A).$$

Note: $\dim \ker(\lambda I - A)$ is always equal to the number of independent eigenvectors associated with the eigenvalue λ.

Stability of autonomous systems

Corollary

The system defined by $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$ is:

1. asymptotically stable if and only if the eigenvalues of A have negative real part,
2. stable if and only if for each $\lambda \in \mathbb{C}$ that is an eigenvalue of A, either (i) $\mathrm{Re}\lambda < 0$, or (ii) $\mathrm{Re}\ \lambda = 0$ and λ is a semisimple eigenvalue of A,
3. unstable if and only if A has either an eigenvalue with positive real part or a nonsemisimple one with zero real part.

The pole placement problem

Consider the linear time-invariant dynamical system in state form described by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu,$$

with $x : \mathbb{R} \to \mathbb{R}^n$ and $u : \mathbb{R} \to \mathbb{R}^m$ the state and the input trajectory, and with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

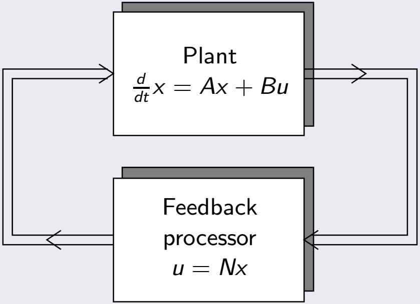
Allow linear state feedback controllers of the form $u = Nx$, with $N \in \mathbb{R}^{m\times n}$ called the feedback gain matrix.

Closed loop equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = (A + BN)x.$$

This represents an autonomous dynamical system.

Stability of Autonomous Systems



The pole placement problem

We call the eigenvalues of $A + BN$ the closed loop poles.
Question: what closed loop pole locations are achievable by choosing the feedback gain matrix N?
Closed loop characteristic polynomial:

$$\chi_{A+BN}(\xi) := \det(\xi I - (A + BN))$$

Pole placement problem

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be given. What is the set of polynomials $\chi_{A+BN}(\xi)$ obtainable by choosing the matrix $N \in \mathbb{R}^{m \times n}$?

The pole placement problem

The pole placement theorem (W.M. Wonham, 1969)

Consider the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$. For any real monic polynomial $r(\xi)$ of degree n there exists $N \in \mathbb{R}^{m \times n}$ such that $\chi_{A+BN} = r(\xi)$ if and only if the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ is controllable.

Monic polynomial of degree n: $p(\xi) = \xi^n + p_{n-1}\xi^{n-1} + \ldots + p_1\xi + p_0$.

W.M. Wonham, "On pole assignment in multi-input controllable linear systems", *IEEE Transactions on Automatic Control*, 1967.

Proof of the pole placement theorem

Proof of the pole placement theorem

System similarity

$\Sigma_{n,m}$:= the family of all systems $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ with n state and m input variables.

We say: $(A, B) \in \Sigma_{n,m}$.

Let $(A_1, B_1), (A_2, B_2) \in \Sigma_{n,m}$. We call (A_1, B_1) and (A_2, B_2) similar if there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that $A_2 = SA_1S^{-1}$ and $B_2 = SB_1$.

If in the state space of $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ we change the coordinate basis by defining $z(t) = Sx(t)$, then the dynamics of z are governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}z = SAS^{-1}z + SBu.$$

Hence similarity corresponds to changing the basis in the state space.

Proof of the pole placement theorem

Lemma

Assume $(A_1, B_1), (A_2, B_2) \in \Sigma_{n,m}$ are similar. Let $r(\xi) \in \mathbb{R}[\xi]$ be a monic polynomial. Then there exists a matrix $N_1 \in \mathbb{R}^{m \times n}$ such that $\chi_{A_1+B_1N_1}(\xi) = r(\xi)$ if and only if there exists a matrix $N_2 \in \mathbb{R}^{m \times n}$ such that $\chi_{A_2+B_2N_2}(\xi) = r(\xi)$.

Proof

Let $A_2 = SA_1S^{-1}$ and $B_2 = SB_1$. Assume $\chi_{A_1+B_1N_1}(\xi) = r(\xi)$. Define $N_2 = N_1S^{-1}$. Then $A_2 + B_2N_2 = S(A_1 + B_1N_1)S^{-1}$, so $A_1 + B_1N_1$ and $A_2 + B_2N_2$ are similar, and therefore $\chi_{A_2+B_2N_2}(\xi) = r(\xi)$ as well. Also the converse.

Proof of the pole placement theorem

Lemma

The system $(A, B) \in \Sigma_{n,m}$ is similar to a system $(A', B') \in \Sigma_{n,m}$ with A', B' of the form

$$A' = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_1 \\ 0 \end{pmatrix}$$

and with (A'_{11}, B'_1) controllable.

Proof of the pole placement theorem

Proof

Two cases:

1. if (A, B) is controllable then $A'_{11} = A' = A$ and $B'_1 = B' = B$, and A'_{12}, A'_{22} and the 0-matrices are 'void'.
2. If (A, B) is not controllable, then $\mathcal{R} := \mathrm{im}(B \; AB \; A^2B \ldots A^{n-1}B) \neq \mathbb{R}^n$, so is a proper subspace of $\mathbb{R}^n$. Choose a basis of $\mathbb{R}^n$ adapted to $\mathcal{R}$. The matrices with respect to this new basis are of the form (A', B') because $A\mathcal{R} \subset \mathcal{R}$ and $\mathrm{im}\ B \subset \mathcal{R}$.

Proof of the pole placement theorem

Pole placement $\Rightarrow$ Controllability

Assume (A, B) is not controllable. Then it is similar to (A', B') of the form

$$A' = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_1 \\ 0 \end{pmatrix}$$

with n_1, the dimension of A'_{11}, less than n. Now apply $N' = (N'_1 \; N'_2)$, with $N'_1 \in \mathbb{R}^{m \times n_1}$ and $N'_2 \in \mathbb{R}^{m \times (n-n_1)}$, on this system. Then

$$A' + B'N' = \begin{pmatrix} A'_{11} + B'_1 N'_1 & A'_{12} + B'_1 N'_2 \\ 0 & A'_{22} \end{pmatrix}.$$

Characteristic polynomial is given by $\chi_{A'_{11}+B'_1N'_1}(\xi)\chi_{A'_{22}}(\xi)$. Therefore, the characteristic polynomial $\chi_{A'+B'N'}(\xi)$, and hence $\chi_{A+BN}(\xi)$, has, regardless of what N is chosen, $\chi_{A'_{22}}(\xi)$ as a factor.

Proof of the pole placement theorem

Theorem

Let $r(\xi)$ be a monic polynomial of degree n. Assume (A, B) controllable, and $m = 1$. Let $F \in \mathbb{R}^{1\times n}$ be the solution of the system of linear equations

$$F(B\ AB \cdots A^{n-2}B\ A^{n-1}B) = (0\ 0 \cdots 0\ 1).$$

Then $N = -F\, r(A)$ yields $\chi_{A+BN}(\xi) = r(\xi)$. Here $r(A) := r_0 I + r_1 A + \cdots + r_{n-1}A^{n-1} + A^n$.

Note: this yields a proof of the implication Controllability $\Rightarrow$ Pole placement for the special case $m = 1$!

Proof of the pole placement theorem

Notation

$\Sigma_{n,m}^{\text{cont}} \subset \Sigma_{n,m}$ are all controllable systems with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$.

Heymann's lemma (1968)

Let $(A, B) \in \Sigma_{n,m}^{\text{cont}}$. Assume $K \in \mathbb{R}^{m \times 1}$ such that $BK \neq 0$.
Then there exists a $N' \in \mathbb{R}^{m \times n}$ such that $(A + BN', BK) \in \Sigma_{1,n}^{\text{cont}}$.

M. Heymann, "Comments to: Pole assignment in multi-input controllable linear systems", *IEEE Transactions on Automatic Control*, 1968.

Proof of the pole placement theorem

Proof of the pole placement theorem

Controllability $\Rightarrow$ Pole placement, $m > 1$: coup de grâce

Choose $K \in \mathbb{R}^{m\times 1}$ such that $BK \neq 0$, by controllability, $B \neq 0$, hence such a K exists.
Choose $N' \in \mathbb{R}^{m\times 1}$ such that $(A + BN', BK)$ controllable.
We are now back in the case $m = 1$.
Choose $N'' \in \mathbb{R}^{1\times n}$ such that $A + BN' + BKN''$ has the desired characteristic polynomial $r(\xi)$.
Finally, take $N = N' + KN''$.
Then $A + BN = A + BN' + BKN''$ so $\chi_{A+BN}(\xi) = r(\xi)$.

The pole placement problem

Algorithm

Data: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, with (A, B) controllable; $r(\xi) \in \mathbb{R}[\xi]$ with $r(\xi)$ monic and of degree n.

Required: $N \in \mathbb{R}^{m\times n}$ such that $\chi_{A+BN}(\xi) = r(\xi)$.

Algorithm:

1. Find $K \in \mathbb{R}^{m\times 1}$ and $N' \in \mathbb{R}^{m\times n}$ such that $(A + BN', BK)$ is controllable.
2. Put $A' = A + BN'$, $B' = BK$, and compute F from
$$F[B', A'B', \ldots, (A')^{n-1}B'] = [0\ 0 \ldots\ 0\ 1].$$
3. Compute $N'' = -F\ r(A')$.
4. Compute $N = N' + KN''$.

Result: N is the desired feedback matrix.

Stabilization

So: if (A, B) is controllable then all polynomials can be "placed".
Question: what polynomials can be placed if (A, B) is not controllable?.
Recall $(A, B) \in \Sigma_{n,m}$ is similar to

$$A' = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_1 \\ 0 \end{pmatrix}.$$

The matrix A'_{22} characterizes the noncontrollable behavior of the system. Its characteristic polynomial $\chi_{A_{22}}(\xi)$ is called the uncontrollable polynomial of the system. Its roots are called the uncontrollable poles ("modes").

Stabilization

Theorem

Consider the system (A, B). Let $\chi_{\mathrm{u}}(\xi)$ be its uncontrollable polynomial. There exists a feedback matrix $N \in \mathbb{R}^{n \times m}$ such that

$$\chi_{A+BN(\xi)} = r(\xi)$$

if and only if $r(\xi)$ is a real monic polynomial of degree n that has $\chi_{\mathrm{u}}(\xi)$ as a factor.

Stabilization

Proof

Take (A', B') similar to (A, B). Partition N' conformably as $N' = (N'_1 \;\; N'_2)$. Then

$$A' + B'N' = \begin{pmatrix} A'_{11} + B'_1 N'_1 & A'_{12} + B'_1 N'_2 \\ 0 & A'_{22} \end{pmatrix}.$$

Hence $\chi_{A'+B'N'}(\xi) = \chi_{A'_{11}+B'_1 N'_1}(\xi)\chi_{A_{22}}(\xi) = \chi_{A'_{11}+B'_1 N'_1}(\xi)\chi_{\mathrm{u}}(\xi)$. Let $r(\xi)$ have $\chi_{\mathrm{u}}(\xi)$ as a factor. Then $r(\xi) = r_1(\xi)\chi_{\mathrm{u}}(\xi)$ for some monic $r_1(\xi)$. Since (A'_{11}, B'_1) is controllable, $\chi_{A'_{11}+B'_1 N'_1}(\xi)$ can be made equal to $r_1(\xi)$. Then with $N' = (N'_1 \;\; 0)$ we have $\chi_{A'+B'N'}(\xi) = r_1(\xi)\chi_{\mathrm{u}}(\xi) = r(\xi)$. Also the converse.

Stabilization

Consider the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ with the control law $u = Nx$. The closed loop system $\frac{\mathrm{d}}{\mathrm{d}t}x = (A + BN)x$ is asymptotically stable if and only if $A + BN$ is Hurwitz.

Question: when does there exist $N \in \mathbb{R}^{m\times n}$ such that $A + BN$ is Hurwitz?

Corollary

There exists N such $A + BN$ is Hurwitz if and only if the uncontrollable polynomial $\chi_{\mathrm{u}}(\xi)$ of (A, B) is Hurwitz.

Stabilization

Definition

We call the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$, or, equivalently, the pair (A, B), stabilizable if its uncontrollable polynomial $\chi_{\mathrm{u}}(\xi)$ is Hurwitz.

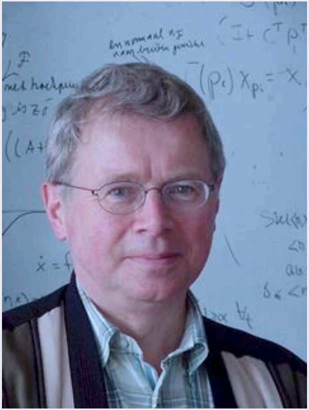
Theorem (Hautus test)

The system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ is stabilizable if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda I - A & B \end{pmatrix} = n$$

for all $\lambda \in \mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$.

Stabilization



State observers

Consider the following plant

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu \ , \ y = Cx$$

x is the state, u the input, and y the output.
System parameters: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$.
Denote the class of such systems by $\Sigma_{n,m,p}$.
We also write $(A, B, C) \in \Sigma_{n,m,p}$.
The external (manifest) signals u and y are measured.
Aim: deduce the internal (latent) signal x from these measurements.

State observers

D. Luenberger, 1963

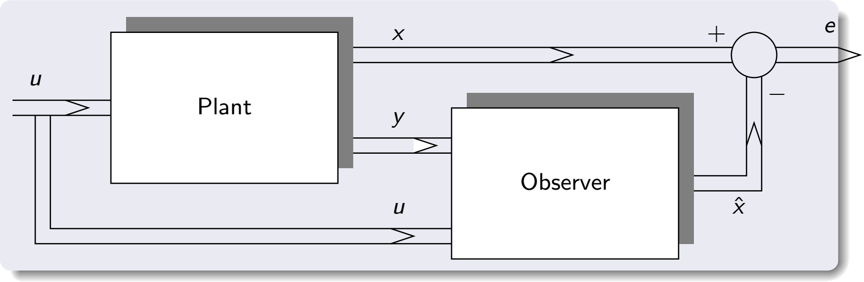
State observers

An algorithm that estimates x from u and y is called a state observer. Denote the estimate of x by $\hat{x}$, and define the estimation error as

$$e := x - \hat{x}.$$

A state observer is a dynamical system with u and y as input, $\hat{x}$ as output, and that makes $e = x - \hat{x}$ small in some sense.
Here we focus on the asymptotic behavior of $e(t)$ for $t \to \infty$.

State observers



State observers

Proposed state observer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = (A - LC)\hat{x} + Bu + Ly,$$

Combining this equation with

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu \ , \ y = Cx$$

yields the following equation for $e = x - \hat{x}$:

$$\frac{d}{dt}e = (A - LC)e.$$

$\frac{\mathrm{d}}{\mathrm{d}t}e = \frac{\mathrm{d}}{\mathrm{d}t}x - \frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = (Ax + Bu) - ((A - LC)\hat{x} + Bu + Ly) = (Ax+Bu)-((A-LC)\hat{x}+Bu+LCx) = (A-LC)(x-\hat{x}) = (A-LC)e.$

State observers

We want that $e(t) \to 0$ as $t \to \infty$ for all $e(0) = x(0) - \hat{x}(0)$, i.e., we want $A - LC$ to be Hurwitz.

Often we need a certain rate of convergence for $e(t)$. This leads to the following question:

What eigenvalues ("observer poles") can we achieve for $A - LC$ by choosing the observer gain matrix L?

In linear algebra terms:

Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$ be given. What is the set of polynomials $\chi_{A-LC}(\xi)$ obtainable by choosing the matrix $L \in \mathbb{R}^{n \times p}$?

State observers

Theorem

Consider the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$, $y = Cx$. There exists for every real monic polynomial $r(\xi)$ of degree n a matrix L such that $\chi_{A-LC}(\xi)$ equals $r(\xi)$ if and only if the system is observable.

Proof

(A, C) observable pair if and only if (A^T, C^T) controllable pair.
Note: for any real square matrix M, $\chi_M(\xi) = \chi_{M^T}(\xi)$. Assume (A, C) observable. By the pole placement theorem, there exists for all $r(\xi)$ a matrix $N \in \mathbb{R}^{p\times n}$ such that $\chi_{A^T+C^TN}(\xi) = r(\xi)$. Thus $\chi_{A-LC}(\xi) = r(\xi)$ with $L = -N^T$.
Conversely: Assume there exists for all $r(\xi)$ a matrix $L \in \mathbb{R}^{n\times p}$ such that $\chi_{A-LC}(\xi) = r(\xi)$. Then $\chi_{A^T+C^T(-L)^T}(\xi) = r(\xi)$ so (A^T, C^T) controllable, whence (A, C) observable.

State observers

What if (A, C) is not observable?

The dynamical systems $(A_1, B_1, C_1) \in \Sigma_{n,m,p}$ and $(A_2, B_2, C_2) \in \Sigma_{n,m,p}$ are called similar if there exist a nonsingular S such that $A_1 = SA_2S^{-1},\ B_1 = SB_2,\ C_1 = C_2S^{-1}$.

Theorem

The system $(A, B, C) \in \Sigma_{n,m,p}$ is similar to a system of the form (A', B', C') in which A' and C' have the following structure:

$$A' = \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix}, \quad C' = (C_1 \ \ 0),$$

with (A_{11}, C_1) observable.

All such decompositions lead to matrices A_{22} that have the same characteristic polynomial.

State observers

The polynomial $\chi_{A_{22}}(\xi)$ is called the unobservable polynomial of (A, C). Notation: $\chi_0(\xi)$
Its roots are called the unobservable eigenvalues (modes).

Theorem

Consider the system $(A, B, C) \in \Sigma_{n,m,p}$. Let $\chi_0(\xi)$ be its unobservable polynomial. There exists $L \in \mathbb{R}^{n\times p}$ such that $\chi_{A-LC}(\xi) = r(\xi)$ if and only if $r(\xi)$ is a real monic polynomial of degree n that has $\chi_0(\xi)$ as a factor.

Proof

$$\begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix} + \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} + L_1 C_1 & 0 \\ A_{12} + L_2 C_1 & A_{22} \end{pmatrix}$$

State observers

Corollary

There exists an observer $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = (A - LC)\hat{x} + Bu + Ly$ such that for all initial states $x(0)$ and $\hat{x}(0)$

$$\lim_{t \to \infty} \hat{x}(t) - x(t) = 0,$$

i.e. such that $A - LC$ is Hurwitz, if and only if the unobservable polynomial $\chi_0(\xi)$ of (A, C) is Hurwitz.

Definition

The system $(A, B, C) \in \Sigma_{n,m,p}$ is called detectable if the the unobservable polynomial $\chi_0(\xi)$ of (A, C) is Hurwitz.

State observers

Theorem (Hautus test)

The system $(A, B, C) \in \Sigma_{n,m,p}$ is detectable if and only if

$$\operatorname{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$$

for all $\lambda \in \mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \geq 0\}$.

Dynamic output feedback

Plant:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu \ , \ y = Cx.$$

Linear time-invariant feedback controller:

$$\frac{d}{dt}z = Kz + Ly \ , \ u = Mz + Ny,$$

with $z : \mathbb{R} \to \mathbb{R}^d$ the state of the controller, $K \in \mathbb{R}^{d\times d}$, $L \in \mathbb{R}^{d\times p}$, $M \in \mathbb{R}^{m\times d}$, and $N \in \mathbb{R}^{m\times p}$ the parameter matrices specifying the controller.

State dimension $d \in \mathbb{N}$ is called the order of the controller. Is a design parameter: typically, we want d to be small.

Dynamic output feedback

By combining the equations of the plant and controller we obtain the closed loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A+BNC & BM \\ LC & K \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}, y = Cx, \; u = Mz + Ny$$

Compact form: with $x_\mathrm{e} := \mathrm{col}(x, z)$ (the extended state), A_e, C_e, and H_e defined in the obvious way, this yields the closed loop system equations

$$\frac{d}{dt}x_\mathrm{e} = A_\mathrm{e}x_\mathrm{e} \; , \; y = C_\mathrm{e}x_\mathrm{e} \; , \; u = H_\mathrm{e}x_\mathrm{e}.$$

This is an autonomous dynamical system. We call the eigenvalues of A_e the closed loop poles and $\chi_{A_\mathrm{e}}(\xi)$ the closed loop characteristic polynomial.

Dynamic output feedback

Question

What closed loop pole locations are achievable by choosing (K, L, M, N)?

More precisely:

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ be given. Determine the set of polynomials $\chi_{A_e}(\xi)$ obtainable by choosing $d \in \mathbb{N}$ and $K \in \mathbb{R}^{d \times d}$, $L \in \mathbb{R}^{d \times p}$, $M \in \mathbb{R}^{m \times d}$, $N \in \mathbb{R}^{m \times p}$, and where A_e is given by

$$A_e = \begin{pmatrix} A + BNC & BM \\ LC & K \end{pmatrix}.$$

Full solution to this problem is unknown. We will describe a very useful partial result.

Dynamic output feedback

We have already seen how to proceed if $C = I$, i.e., if the full state vector is measured. Let

$$u = N'x$$

be a memoryless state feedback control law obtained this way.
We have also seen how we can estimate the state x of from (u, y).
Let

$$\frac{d}{dt}\hat{x} = (A - L'C)\hat{x} + Bu + L'y$$

be a suitable observer.
Separation principle: combine an observer with a state controller and use the same controller gains as in the case in which the state is measured.

Dynamic output feedback

This yields the feedback controller:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = (A - L'C)\hat{x} + BN'\hat{x} + L'y, \qquad u = N'\hat{x}.$$

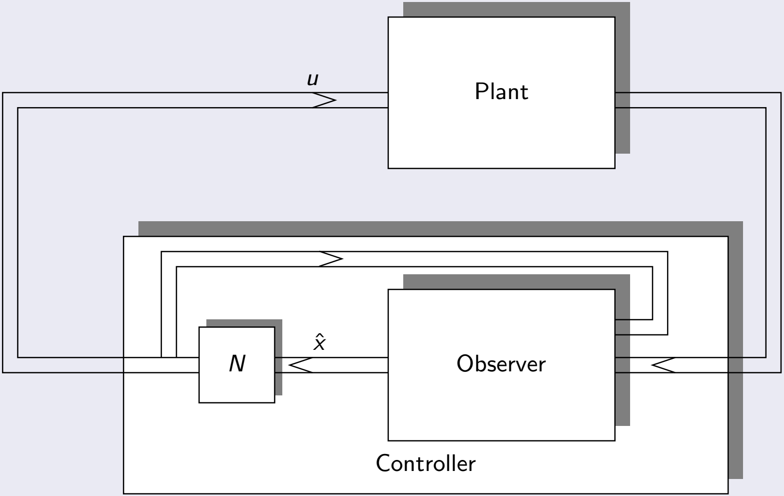
This is, of course, a feedback processor with order $d = n$, $K = A - L'C + BN'$, $L = L'$, $M = N'$, and $N = 0$.

Closed loop system obtained by using this feedback controller:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} A & BN' \\ L'C & A - L'C + BN' \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix},$$

$$u = N'\hat{x}, \qquad y = Cx.$$

Dynamic output feedback



Dynamic output feedback

We are interested in the characteristic polynomial of A_e.
Define a similarity transformation $S \in \mathbb{R}^{2n \times 2n}$ by

$$S := \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}$$

Note: $S^{-1} = S$.
Also

$$\begin{pmatrix} A + BN' & -BN' \\ 0 & A - L'C \end{pmatrix} = S \begin{pmatrix} A & BN' \\ L'C & A - L'C + BN' \end{pmatrix} S^{-1}$$

Hence the closed loop characteristic polynomial equals the product of $\chi_{A+BN'}(\xi)$ and $\chi_{A-L'C}(\xi)$.

Dynamic output feedback

Theorem (Pole placement by dynamic output feedback)

Consider the system (A, B, C) and assume that (A, B) is controllable and that (A, C) is observable.

Then for every real monic polynomial $r(\xi)$ of degree $2n$, factorizable into two real polynomials of degree n, there exists a feedback controller (K, L, M, N) of order n such that the closed loop system matrix A_e has characteristic polynomial $r(\xi)$.

Proof

Take $d = n$, $K = A - L'C + BN'$, $L = L'$, $M = N'$, and $N = 0$. Choose N' such that $\chi_{A+BN'}(\xi) = r_1(\xi)$ and L' such that $\chi_{A-L'C}(\xi) = r_2(\xi)$, where $r_1(\xi)$ and $r_2(\xi)$ are real factors of $r(\xi)$ such that $r(\xi) = r_1(\xi)r_2(\xi)$.

Dynamic output feedback

Theorem (Stabilization by dynamic output feedback)

Consider the system (A, B, C) and let $\chi_u(\xi)$ be its uncontrollable polynomial, $\chi_0(\xi)$ its unobservable polynomial. Then

1. For any real monic polynomials $r_1(\xi)$ and $r_2(\xi)$ of degree n such that $r_1(\xi)$ has $\chi_u(\xi)$ as a factor and $r_2(\xi)$ has $\chi_0(\xi)$ as a factor, there exists a feedback controller (K, L, M, N) of order n such that the closed loop system matrix A_e has characteristic polynomial $r(\xi) = r_1(\xi)r_2(\xi)$.
2. There exists a feedback controller (K, L, M, N) as such that the closed loop system is asymptotically stable if and only if both $\chi_u(\xi)$ and $\chi_0(\xi)$ are Hurwitz, i.e., if and only if (A, B) is stabilizable and (A, C) is detectable.