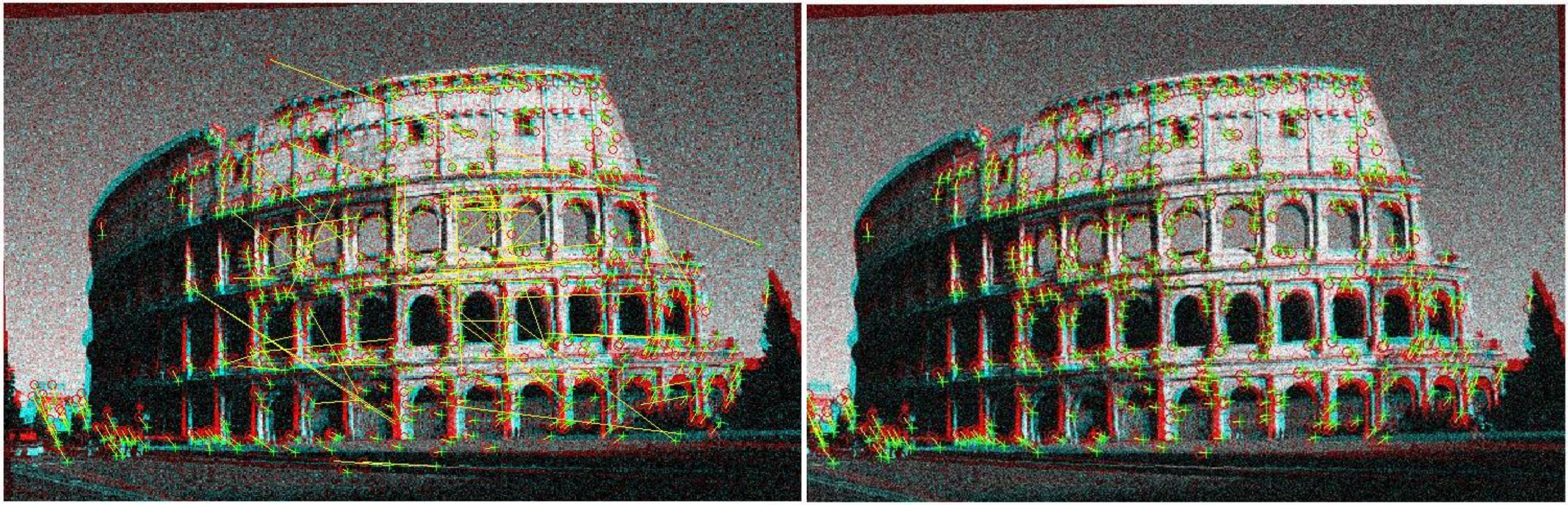
VISUAL NAVIGATION

Estimation



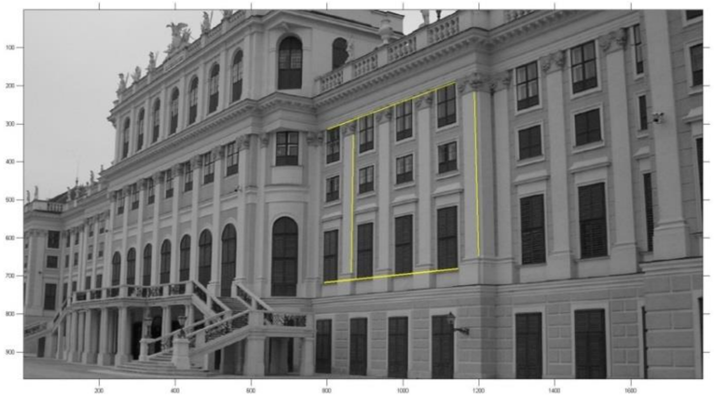
Elements of 3D projective geometry

Lecture outline:

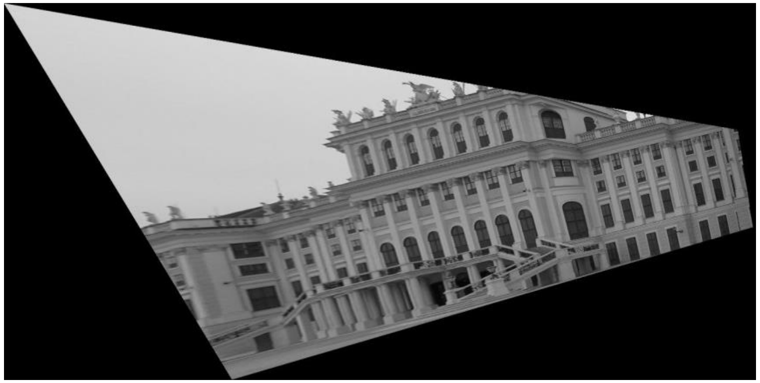
- Measurements and models
- Linear systems
- Linear systems with nonlinear constraints
- Nonlinear systems
- Examples
- Robust estimation

Intro

- How to estimate the homography between two images?



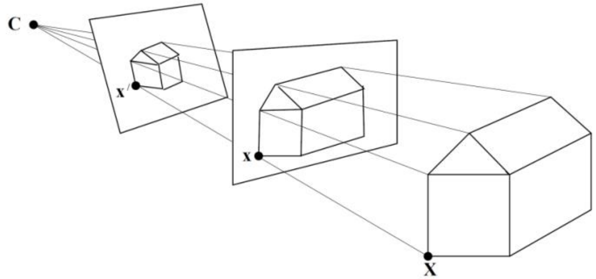
$\mathbf{x}_i$



$\mathbf{x}'_i$

$$\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$$

- How to estimate the 3D reconstructed points from their projections on the camera image plane?



Intro

- Nomenclature

$\mathbf{u}$	-	*measurement*
$\bar{\mathbf{u}}$	-	*true value*
$\hat{\mathbf{u}}$	-	*estimated values*
$\mathcal{E}(\mathbf{y})$	-	*expectation operator (mean)*
$\mathcal{D}(\mathbf{y})$	-	*dispersion operator (variance)*
$P(\mathbf{u})$	-	*probability distribution function*
$\mathbf{u} \sim \mathcal{N}(\bar{\mathbf{u}}, \mathbf{\Sigma})$	-	*(Multivariate) Gaussian pdf*

$$\mathcal{N}(\bar{\mathbf{u}}, \mathbf{\Sigma}) = \frac{1}{\det(2\pi\mathbf{\Sigma})} e^{-\frac{1}{2}\|\mathbf{u}-\bar{\mathbf{u}}\|^2_{\mathbf{\Sigma}}}$$

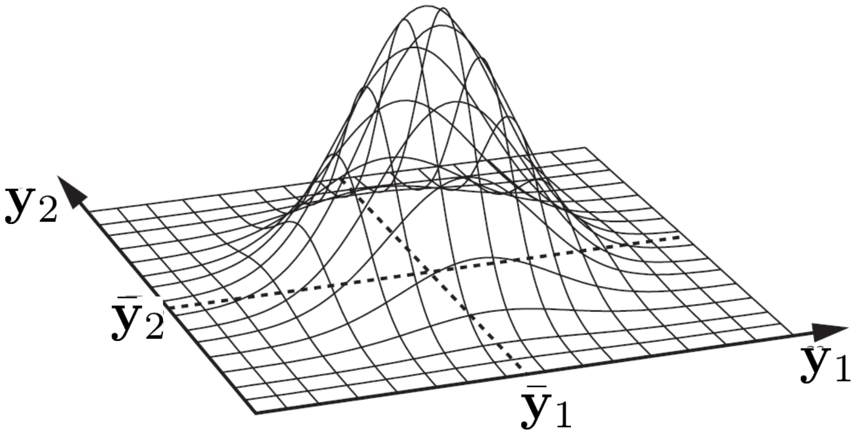
$\|\mathbf{x}-\mathbf{y}\|^2_{\mathbf{\Sigma}}$	-	*Mahalanobis distance*

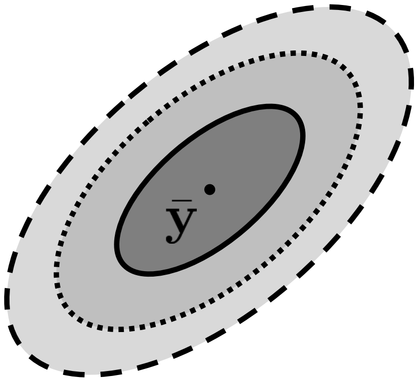
$$\|\mathbf{x}-\mathbf{y}\|^2_{\mathbf{\Sigma}} = (\mathbf{x}-\mathbf{y})^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\mathbf{y})$$

Measurements and models

- Measurement vector $\mathbf{y} \in \mathbb{R}^n$
- Unknown variables $\mathbf{p} \in \mathbb{R}^m$
- Description of measurement error $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \sim \mathcal{N}(\bar{\mathbf{y}}, \mathbf{\Sigma})$

Gaussian distributions widely used because easily manipulated





Measurements and models

- Measurements are usually affected by errors:

$$\mathcal{E}(\mathbf{y}) = \mathbf{f}(\mathbf{p}) \quad , \quad \mathcal{D}(\mathbf{y}) = \mathbf{\Sigma}$$

- *Linear model:* $$\mathcal{E}(\mathbf{y}) = \mathbf{A}_{n\times m}\mathbf{p}$$

- *Linearized model:* $$\mathcal{E}(\mathbf{y}) \approx \mathbf{f}(\tilde{\mathbf{p}}) + \mathbf{J}_{\mathbf{f}|\tilde{\mathbf{p}}}\left(\mathbf{p} - \tilde{\mathbf{p}}\right)$$

with $$\mathbf{J}_{\mathbf{f}|\tilde{\mathbf{p}}} = \begin{bmatrix} \frac{\partial \mathbf{f}_1(\tilde{\mathbf{p}})}{\partial p_1} & \cdots & \frac{\partial \mathbf{f}_1(\tilde{\mathbf{p}})}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \mathbf{f}_n(\tilde{\mathbf{p}})}{\partial p_1} & \cdots & \frac{\partial \mathbf{f}_n(\tilde{\mathbf{p}})}{\partial p_m} \end{bmatrix}$$

Linear systems

- Measurement vector $\mathbf{y} \in \mathbb{R}^n$
- Parameter space (unknowns) $\mathbf{p} \in \mathbb{R}^m$
- Linear model $\mathcal{E}(\mathbf{y}) = \mathbf{A}_{n \times m}\mathbf{p}$

- *If* $n > m$ *: over-determined system*
- *Weighted Least-squares (WLS) solution: minimization of the squared weighted norm of the estimation residuals* $\epsilon = \mathbf{y} - \mathbf{A}\mathbf{p}$

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^m} \left(\|\mathbf{y} - \mathbf{A}\mathbf{p}\|^2_{\mathbf{\Sigma}} \right)$$

Linear systems

- Parameter estimation in linear model:

$$\hat{\mathbf{p}} = (\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{y}$$

- Linear error propagation rule:

$$\mathcal{D}(\hat{\mathbf{p}}) = \boldsymbol{\Sigma}_{\hat{\mathbf{p}}} = (\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}$$

- *Weighted Least-squares (WLS) solution: minimization of the squared weighted norm of the estimation residuals* $\epsilon = \mathbf{y} - \mathbf{A}\mathbf{p}$

$$\hat{\mathbf{p}} = \arg\min_{\mathbf{p}\in\mathbb{R}^m} (\|\mathbf{y} - \mathbf{A}\mathbf{p}\|^2_{\boldsymbol{\Sigma}})$$

Linear systems with nonlinear constraints

- Nonlinear constraints (example):

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p}\in\mathbb{R}^m \;,\; \|\mathbf{p}\|=l} \|\mathbf{y} - \mathbf{A}\mathbf{p}\|_{\mathbf{\Sigma}}^2$$

- Different approaches available, such as variables reparameterization or Lagrangian multiplier method.

- Example of reparameterization: when m=3, we can use spherical coordinates

$$\mathbf{p} = \begin{pmatrix} l \sin\theta \cos\phi \\ l \sin\theta \sin\phi \\ l \cos\theta \end{pmatrix} \quad \Rightarrow \quad \hat{\mathbf{p}}(\hat{\theta}, \hat{\phi}) = \arg \min_{\theta\in[-\pi/2,\pi/2] \;,\; \phi\in[0,2\pi]} \|\mathbf{y} - \mathbf{A}\mathbf{p}(\theta,\phi)\|_{\mathbf{\Sigma}}^2$$

- *System becomes unconstrained, nonlinear*

Linear systems with nonlinear constraints

- Nonlinear constraints (example):

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^m \,,\; \|\mathbf{p}\| = l} \|\mathbf{y} - \mathbf{A}\mathbf{p}\|^2_{\mathbf{\Sigma}}$$

- Lagrangian parameters method: minimization of the Lagrangian function

$$\mathcal{L}(\mathbf{p}, \lambda) = (\mathbf{y} - \mathbf{A}\mathbf{p})^{\mathrm{T}} \mathbf{\Sigma}^{-1} (\mathbf{y} - \mathbf{A}\mathbf{p}) + \lambda(\mathbf{p}^{\mathrm{T}}\mathbf{p} - l^2)$$

- *Solution given by the stationary points of the Lagrangian function*

$$\nabla_{\mathbf{p},\lambda} \mathcal{L}(\mathbf{p}, \lambda) = 0$$

$$\begin{cases} (\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{A} + \lambda\mathbf{I})\mathbf{p} - \mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{y} = \mathbf{0} \\ \mathbf{p}^{\mathrm{T}}\mathbf{p} - l^2 = 0 \end{cases} \quad \Rightarrow \quad \mathbf{p} = (\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{y}$$

$$\Downarrow$$

Secular equation (how to compute λ *):* $\mathbf{p}(\lambda)^{\mathrm{T}}\mathbf{p}(\lambda) - l^2 = 0$

Linear systems with nonlinear constraints

- Nonlinear constraints (example):

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p}\in\mathbb{R}^m \;,\; \|\mathbf{p}\|=l} \|\mathbf{y} - \mathbf{A}\mathbf{p}\|^2_{\boldsymbol{\Sigma}}$$

- *Compute SVD of* $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$
- *Using* $\mathbf{y}' = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{y}$ *the solution* $\mathbf{p}(\lambda)$ *reads* $\mathbf{p}(\lambda) = \mathbf{V}(\mathbf{S}^{\mathrm{T}}\mathbf{S} + \lambda\mathbf{I})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{y}'$
- *Secular equation (from which* $\hat{\lambda}$ *is computed) then is*

$$\mathbf{p}(\lambda)^{\mathrm{T}}\mathbf{p}(\lambda) = (\mathbf{y}')^{\mathrm{T}}\mathbf{U}\mathbf{S}(\mathbf{S}^{\mathrm{T}}\mathbf{S} + \lambda I)^{-2}\mathbf{S}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{y}' = \sum_{i=1}^{m}\left(\frac{\sigma_i(\mathbf{U}^{\mathrm{T}}\mathbf{y}')_i}{\sigma_i^2 + \lambda}\right)^2 = l^2$$

- *And finally:* $\hat{\mathbf{p}}(\hat{\lambda}) = \sum_{i=1}^{m}\frac{\sigma_i(\mathbf{U}^{\mathrm{T}}\mathbf{y})_i}{\sigma_i^2 + \hat{\lambda}}\mathbf{v}_i$

Linear systems with nonlinear constraints

- Homogeneous minimization problem

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^m \;,\; \|\mathbf{p}\| = l} \|\mathbf{A}\mathbf{p}\|^2_{\boldsymbol{\Sigma}}$$

- *Compute SVD of* $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^m \;,\; \|\mathbf{p}\| = l} \|\mathbf{S}\mathbf{V}^{\mathrm{T}}\mathbf{p}\|^2_{I}$$

- *Substituting* $\mathbf{p}' = \mathbf{V}^{\mathrm{T}}\mathbf{p} \quad \Rightarrow \quad \|\mathbf{p}\| = \|\mathbf{p}'\| = l$

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p}' \in \mathbb{R}^m \;,\; \|\mathbf{p}'\| = l} \|\mathbf{S}\mathbf{p}'\|^2_{I} = \mathbf{v}_{\min}$$

- *Minimizer given by the vector* $\mathbf{v}_{\min}$ *associated to the minimum singular value*

Nonlinear systems

- Nonlinear minimization problem

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^m} \|\mathbf{y} - \mathbf{f}(\mathbf{p})\|^2_{\mathbf{\Sigma}}$$

- Solution based on three steps: initialization, linearization and estimation

- ***Linearization*** *of cost function at step i:*

$$\mathcal{E}(\mathbf{y}) \approx \mathbf{f}(\mathbf{p}_i) + \mathbf{J}_{\mathbf{f}|\mathbf{p}_i}(\mathbf{p} - \tilde{\mathbf{p}}) \quad \textit{with} \quad \mathbf{J}_{\mathbf{f}|\mathbf{p}_i} = \begin{bmatrix} \frac{\partial \mathbf{f}_1(\mathbf{p}_i)}{\partial p_1} & \cdots & \frac{\partial \mathbf{f}_1(\mathbf{p}_i)}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \mathbf{f}_n(\mathbf{p}_i)}{\partial p_1} & \cdots & \frac{\partial \mathbf{f}_n(\mathbf{p}_i)}{\partial p_m} \end{bmatrix}$$

$$\hat{\mathbf{p}} \approx \arg \min_{\mathbf{p} \in \mathbb{R}^m} \|\Delta \mathbf{y}_i - \mathbf{J}_{\mathbf{f}|\mathbf{p}_i} \Delta \mathbf{p}_i\|^2_{\mathbf{\Sigma}}$$

- *Now solvable with known linear estimation methods*

Nonlinear systems

- Nonlinear minimization problem

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p}\in\mathbb{R}^m} \|\mathbf{y} - \mathbf{f}(\mathbf{p})\|^2_{\boldsymbol{\Sigma}}$$

- Solution based on three steps: initialization, linearization and **estimation**

$$\Delta\hat{\mathbf{p}}_i = (\mathbf{J}^{\mathrm{T}}_{\mathbf{f}|\mathbf{p}_i}\boldsymbol{\Sigma}^{-1}\mathbf{J}_{\mathbf{f}|\mathbf{p}_i})^{-1}\mathbf{J}^{\mathrm{T}}_{\mathbf{f}|\mathbf{p}_i}\boldsymbol{\Sigma}^{-1}\Delta\mathbf{y}_i$$

$$\boldsymbol{\Sigma}_{\mathbf{p}_{i+1}} = (\mathbf{J}^{\mathrm{T}}_{\mathbf{f}|\mathbf{p}_i}\boldsymbol{\Sigma}^{-1}\mathbf{J}_{\mathbf{f}|\mathbf{p}_i})^{-1}$$

- ***Initialization***: *choice of* $\mathbf{p}_0$

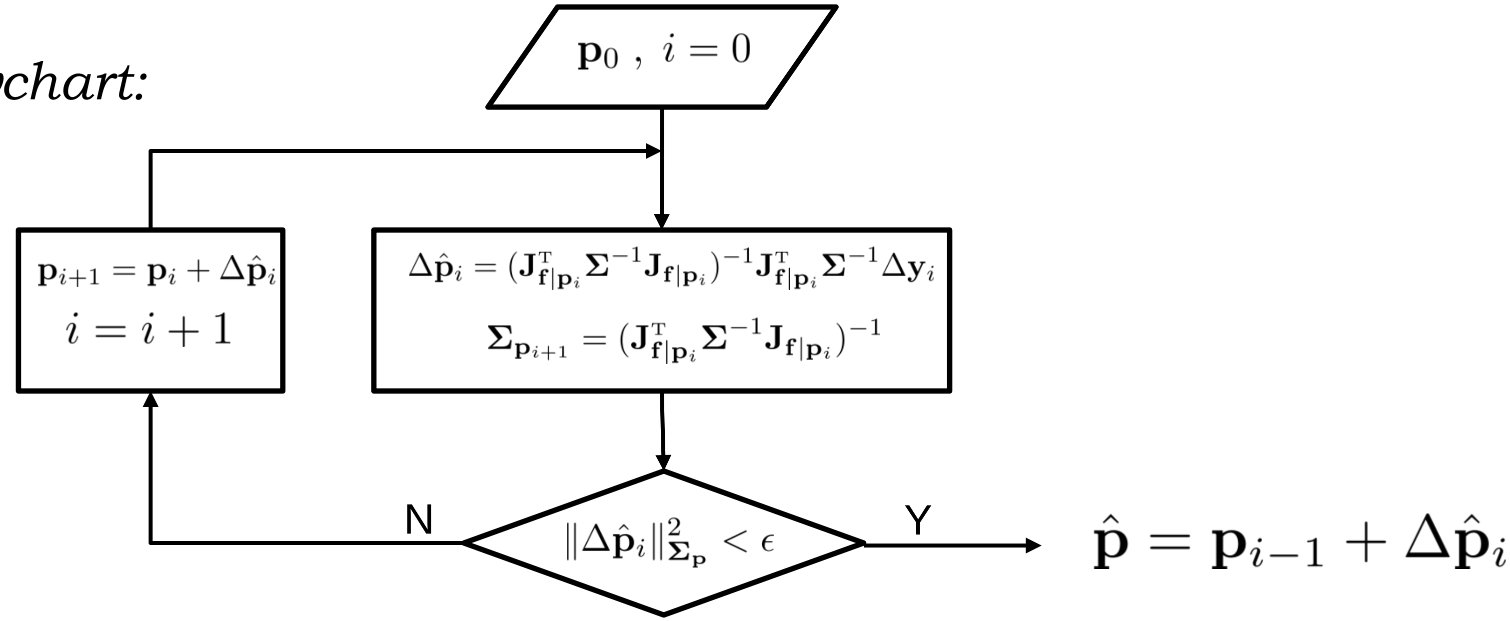
- *Halt criterion:* $\Delta\hat{\mathbf{p}}_i < \epsilon$

Nonlinear systems

- Nonlinear minimization problem

$$\hat{\mathbf{p}} = \arg\min_{\mathbf{p}\in\mathbb{R}^m} \|\mathbf{y} - \mathbf{f}(\mathbf{p})\|^2_{\boldsymbol{\Sigma}}$$

- *Flowchart:*



Examples of estimation problems in CV

- Example: find $\mathbf{H}$ such that $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$ between four corresponding matches in two 2D images.

$$\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i \quad \Leftrightarrow \quad \mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \mathbf{\Omega}_{\mathbf{x}'_i}\mathbf{H}\mathbf{x}_i = \mathbf{0}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}} \\ \mathbf{h}_2^{\mathrm{T}} \\ \mathbf{h}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x'_{i,2}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_3 - x'_{i,3}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_2 \\ x'_{i,3}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_1 - x'_{i,1}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_3 \\ x'_{i,1}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_2 - x'_{i,2}\mathbf{x}_i^{\mathrm{T}}\mathbf{h}_1 \end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^{\mathrm{T}} & -x'_{i,3}\mathbf{x}_i^{\mathrm{T}} & x'_{i,2}\mathbf{x}_i^{\mathrm{T}} \\ x'_{i,3}\mathbf{x}_i^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & -x'_{i,1}\mathbf{x}_i^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{A}_i\mathbf{h} = \mathbf{0}$$

Examples of estimation problems in CV

- With four non-collinear points we find the solution as the null space of

$$\mathbf{A}\mathbf{h} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \\ \mathbf{A}_4 \end{bmatrix}_{8\times 9} \mathbf{h} = \mathbf{0}$$

- *Compute the SVD* $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$

- *The null vector is given by the last column vector of* $\mathbf{V}$

$$\begin{bmatrix} \mathbf{0}^{\mathrm{T}} & -x'_{i,3}\mathbf{x}_i^{\mathrm{T}} & x'_{i,2}\mathbf{x}_i^{\mathrm{T}} \\ x'_{i,3}\mathbf{x}_i^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & -x'_{i,1}\mathbf{x}_i^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{A}_i\mathbf{h} = \mathbf{0}$$

Examples of estimation problems in CV

- With $n > 4$ (non-collinear) points we need to solve for an over-determined system

$$\mathbf{A}_{2n\times 9}\mathbf{h} = \mathbf{0}$$

- If points are known exactly, we can solve for $\mathbf{h}$ up to a scale factor

- Solution: smallest eigenvector of $\mathbf{A}$ (compute SVD and extract unit singular vector corresponding to the smallest singular value – as seen before)

- Usually the points are not know exactly, therefore the need to minimize a cost function

Examples of estimation problems in CV

- With $n > 4$ (non-collinear) points we need to solve for an over-determined system

$$\mathcal{E}(\mathbf{A}_{2n\times 9}\mathbf{h}) = \mathbf{0}$$

- If measurement errors are present: minimization of a cost function

- Solution: smallest eigenvector of $\mathbf{A}$ (compute SVD and extract unit singular vector corresponding to the smallest singular value – as seen before)

$$\hat{\mathbf{h}} = \arg \min_{\mathbf{h}\in\mathbb{R}^m,\ ||\mathbf{h}||=1} ||\mathbf{A}\mathbf{h}||^2 = \mathbf{v}_{\min}$$

- This is known as Direct Linear Transformation (DLT) algorithm

Examples of estimation problems in CV

- The DLT minimizes the squared algebraic distance

$$d^2_{\text{alg}}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) = \|\mathbf{x}'_i - \mathbf{H}\mathbf{x}_i\|^2 = \|\boldsymbol{\epsilon}_i\|^2$$

- The associated minimization problem is

$$\mathbf{h} = \arg \min_{\mathbf{h}\ ,\ \|\mathbf{h}\|=1} \sum_{i=1}^{n} d^2_{\text{alg}}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) = \arg \min_{\mathbf{h}\ ,\ \|\mathbf{h}\|=1} \|\mathbf{A}\mathbf{h}\|^2$$

Examples of estimation problems in CV

- The DLT minimizes the squared algebraic distance
- Warning: to avoid (potentially) large numerical errors, <u>always normalize corresponding images before applying DLT</u>
- *Normalization of each image:*

  - *Translate so centroid of given points is the origin*

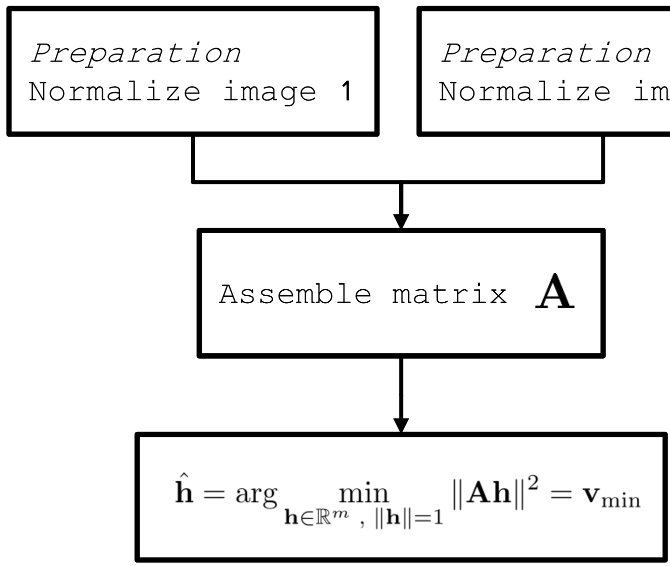
$$\sum_{i=1}^{n} \mathbf{x}_i = \mathbf{0}$$

  - *Scale so average distance to origin is* $\sqrt{2}$

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{d_{\text{alg}}^2(\mathbf{x}_i, \mathbf{0})} = \sqrt{2}$$

Examples of estimation problems in CV

- DLT flowchart



Examples of estimation problems in CV

- Minimization of symmetric transfer error or reprojection error

- *With measurements errors in both images, we can either minimize the error following forward and backward projections:*

$$\min \left(d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) + d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i) \right)$$

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} \left(d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) + d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i) \right)$$

- *OR we can minimize the reprojection error*

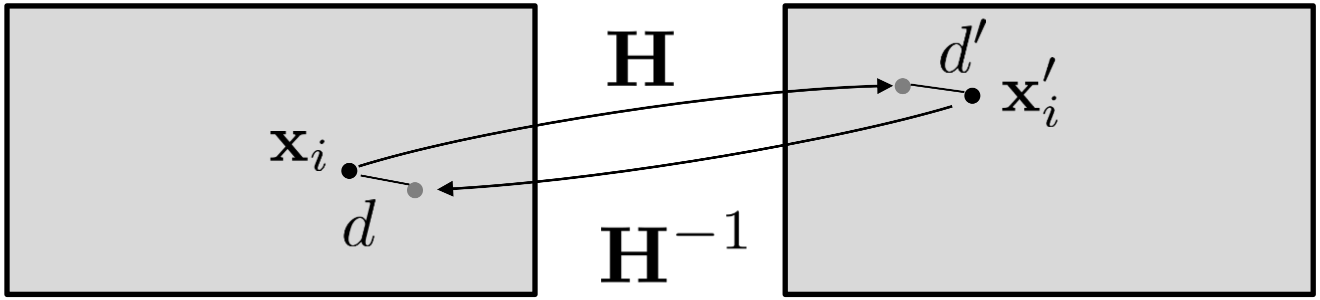
$$\min \left(d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}_i, \hat{\mathbf{x}}_i) + d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}'_i, \hat{\mathbf{x}}'_i) \qquad \text{s.t.} \quad \hat{\mathbf{x}}'_i = \mathbf{H}\hat{\mathbf{x}}_i \right)$$

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}_i, \hat{\mathbf{x}}_i) + d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)$$

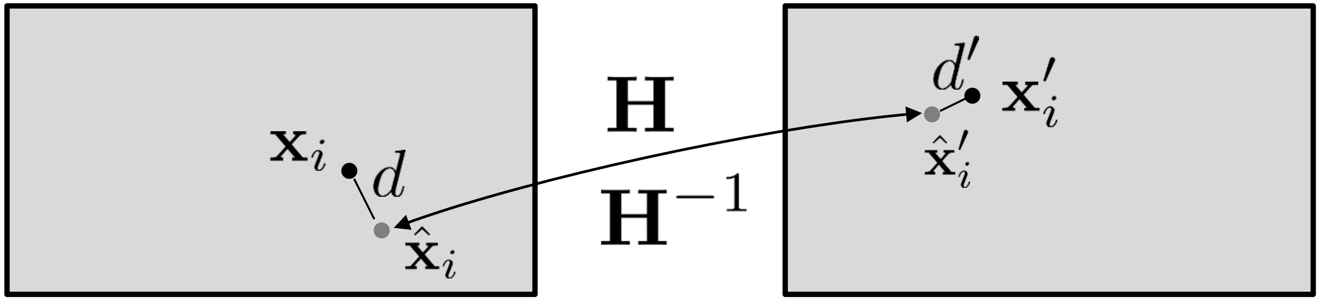
Examples of estimation problems in CV

- Symmetric transfer error VS reprojection error

- *STR:*



- *Reprojection:*



Examples of estimation problems in CV

- Minimization of symmetric transfer error

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} \left(d^2_{\boldsymbol{\Sigma}'_i, \boldsymbol{\Sigma}_i}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) + d^2_{\boldsymbol{\Sigma}'_i, \boldsymbol{\Sigma}_i}(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i) \right)$$

- To be estimated: $\hat{\mathbf{H}}$

- Re-write problem as

$$\|\mathbf{x}'_i - \mathbf{H}\mathbf{x}_i\|^2_{\Sigma'_i} + \|\mathbf{x}_i - \mathbf{H}^{-1}\mathbf{x}'_i\|^2_{\Sigma_i}$$

$$= \Bigg\| \underbrace{\begin{pmatrix} \mathbf{x}'_i \\ \mathbf{x}_i \end{pmatrix}}_{\mathbf{y}_i} - \underbrace{\begin{pmatrix} (\mathbf{x}_i^{\mathrm{T}} \otimes \mathbf{I}_3)\,\mathrm{vec}(\mathbf{H}) \\ (\mathbf{x}_i^{\mathrm{T}} \otimes \mathbf{I}_3)\,\mathrm{vec}(\mathbf{H}^{-1}) \end{pmatrix}}_{\mathbf{f}_i(\mathbf{H})} \Bigg\|^2_{\underbrace{\mathrm{blkdg}(\Sigma'_i, \Sigma_i)}_{\Sigma_{i,ov}}}$$

Examples of estimation problems in CV

- Minimization of symmetric transfer error

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} \left(d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) + d^2_{\mathbf{\Sigma}'_i, \mathbf{\Sigma}_i}(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i) \right)$$

- Associated nonlinear problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{H}} \|\mathbf{y} - \mathbf{f}(\mathbf{H})\|^2_{\mathbf{\Sigma}}$$

with $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)^{\mathrm{T}}$

$$\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)^{\mathrm{T}}$$

$$\mathbf{\Sigma} = \mathrm{blkdg}(\mathbf{\Sigma}_{1,ov}, \ldots, \mathbf{\Sigma}_{n,ov})$$

Examples of estimation problems in CV

- Minimization of reprojection error

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} \left(d^2_{\boldsymbol{\Sigma}_i', \boldsymbol{\Sigma}_i}(\mathbf{x}_i, \hat{\mathbf{x}}_i) + d^2_{\boldsymbol{\Sigma}_i', \boldsymbol{\Sigma}_i}(\mathbf{x}_i', \hat{\mathbf{x}}_i') \right) \qquad \text{s.t.} \quad \hat{\mathbf{x}}_i' = \mathbf{H}\hat{\mathbf{x}}_i$$

- To be estimated: $\hat{\mathbf{H}}$ and $\hat{\mathbf{x}}_i$, $\hat{\mathbf{x}}_i' = \mathbf{H}\hat{\mathbf{x}}_i$

- Re-write problem as $\|\mathbf{x}_i' - \hat{\mathbf{x}}_i'\|^2_{\Sigma_i'} + \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2_{\Sigma_i}$

$$= \|\mathbf{x}_i' - \mathbf{H}\hat{\mathbf{x}}_i\|^2_{\Sigma_i'} + \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2_{\Sigma_i}$$

$$= \left\| \begin{pmatrix} \mathbf{x}_i' \\ \mathbf{x}_i \end{pmatrix} - \begin{pmatrix} (\hat{\mathbf{x}}_i^{\mathrm{T}} \otimes \mathbf{I}_3)\,\mathrm{vec}(\mathbf{H}) \\ \hat{\mathbf{x}}_i \end{pmatrix} \right\|^2_{\mathrm{blkdg}(\Sigma_i', \Sigma_i)}$$

$$\mathbf{y}_i \qquad \mathbf{f}_i(\mathbf{x}_i, \mathbf{H}) \qquad \Sigma_{i,ov}$$

Examples of estimation problems in CV

- Minimization of reprojection error

$$\hat{\mathbf{H}} = \arg\min_{\mathbf{H}} \sum_{i=1}^{n} \left(d^2_{\boldsymbol{\Sigma}'_i, \boldsymbol{\Sigma}_i}(\mathbf{x}_i, \hat{\mathbf{x}}_i) + d^2_{\boldsymbol{\Sigma}'_i, \boldsymbol{\Sigma}_i}(\mathbf{x}'_i, \hat{\mathbf{x}}'_i) \right) \qquad \text{s.t.} \quad \hat{\mathbf{x}}'_i = \mathbf{H}\hat{\mathbf{x}}_i$$

- Associated nonlinear problem:

$$\left\{ \hat{\mathbf{x}}, \hat{\mathbf{H}} \right\} = \arg\min_{\hat{\mathbf{x}}, \mathbf{H}} \|\mathbf{y} - \mathbf{f}(\mathbf{x}, \mathbf{H})\|^2_{\boldsymbol{\Sigma}}$$

with

$$\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)^{\mathrm{T}}$$

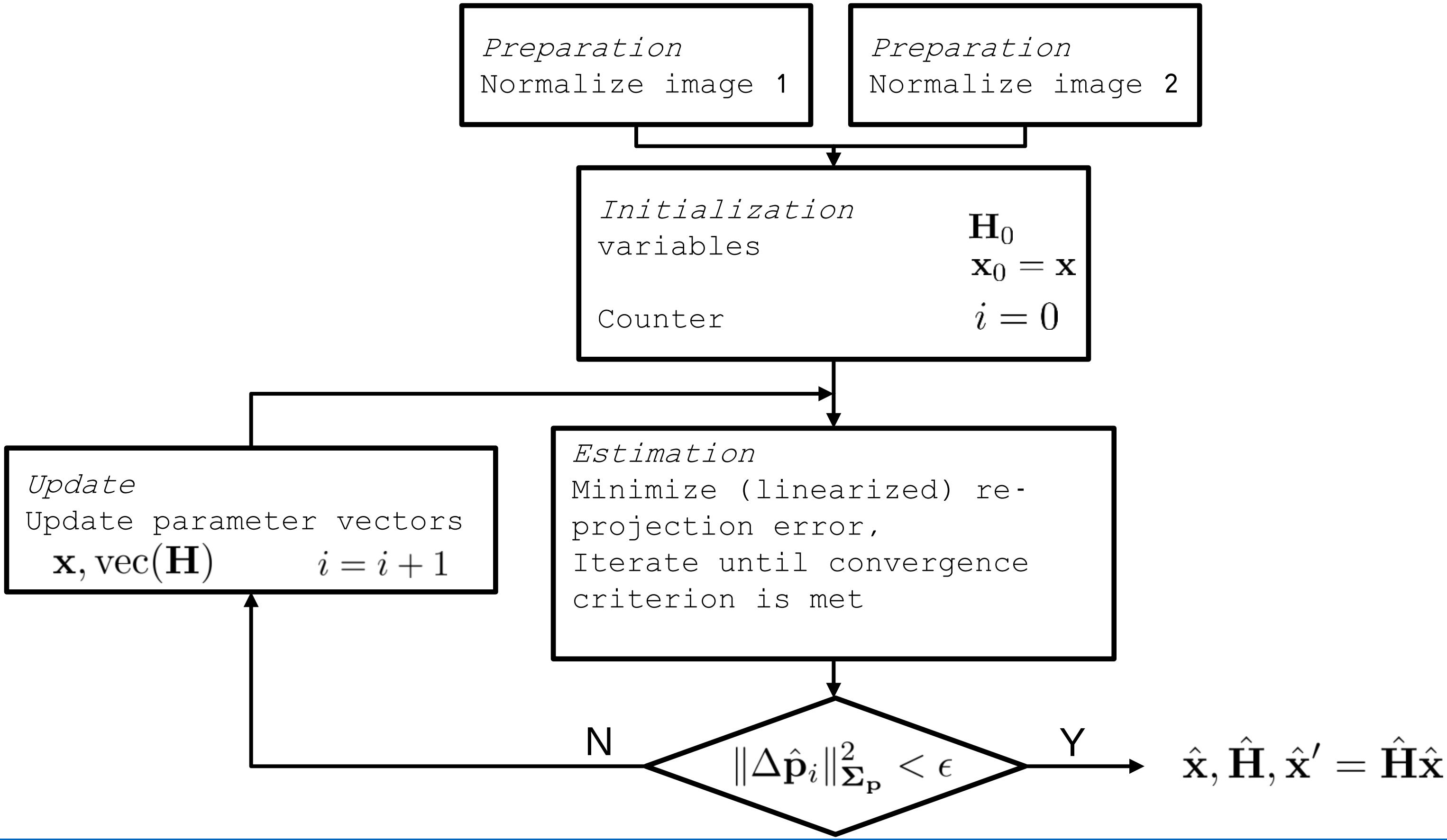
$$\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^{\mathrm{T}}$$

$$\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)^{\mathrm{T}}$$

$$\boldsymbol{\Sigma} = \mathrm{blkdg}(\boldsymbol{\Sigma}_{1,ov}, \ldots, \boldsymbol{\Sigma}_{n,ov})$$

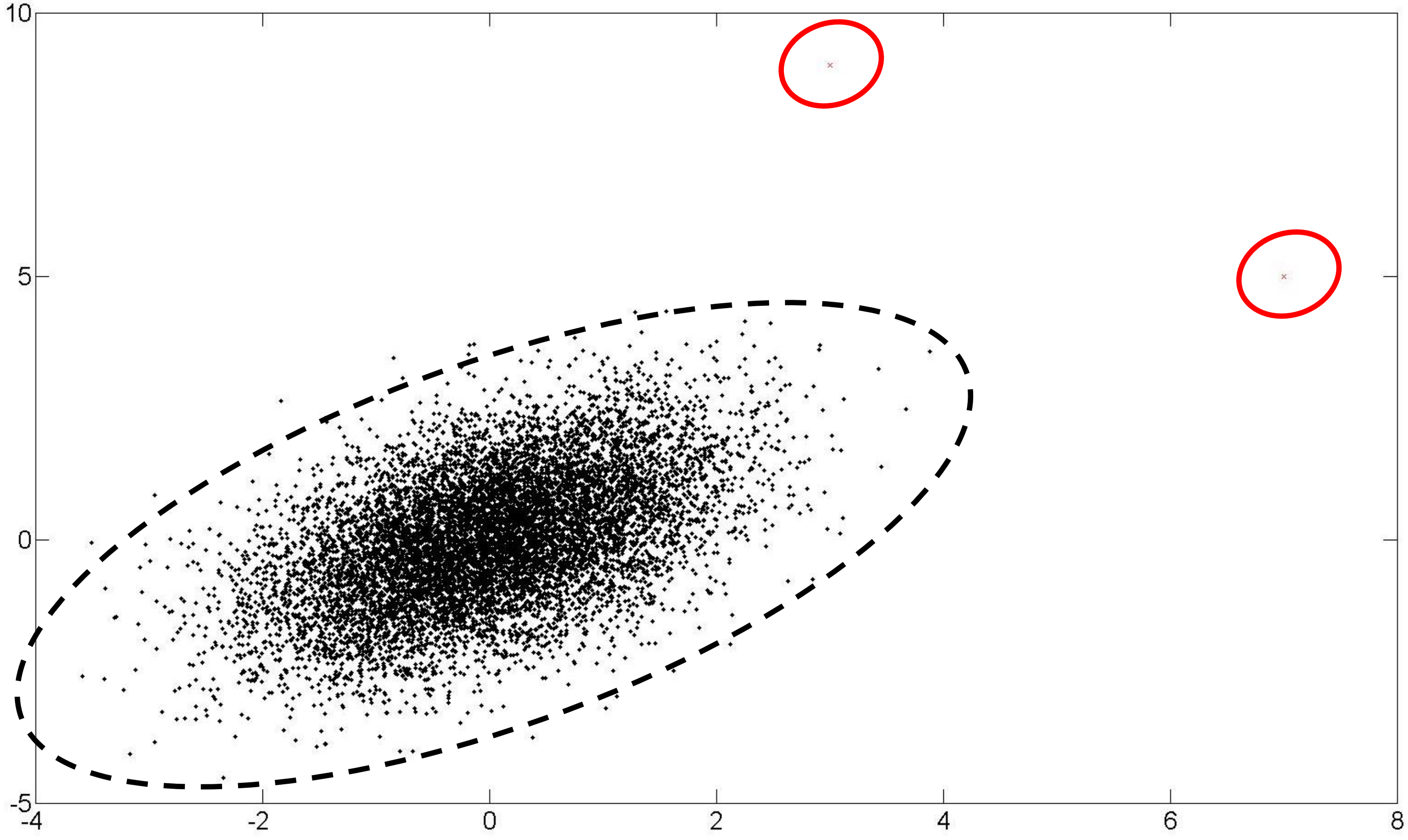
Examples of estimation problems in CV

- Gold-standard algorithm flowchart



Robust estimation

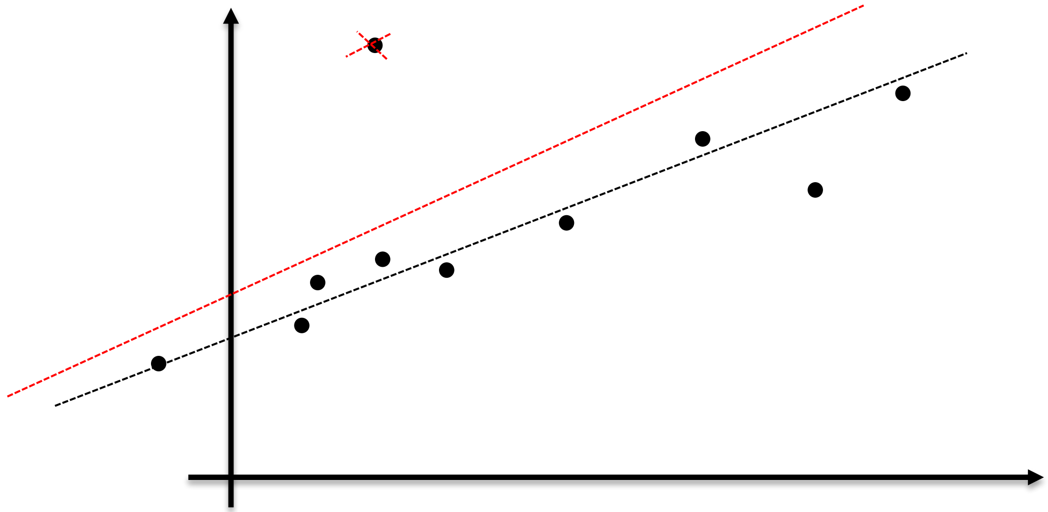
- Measurements outliers



Robust estimation

- Measurements outliers

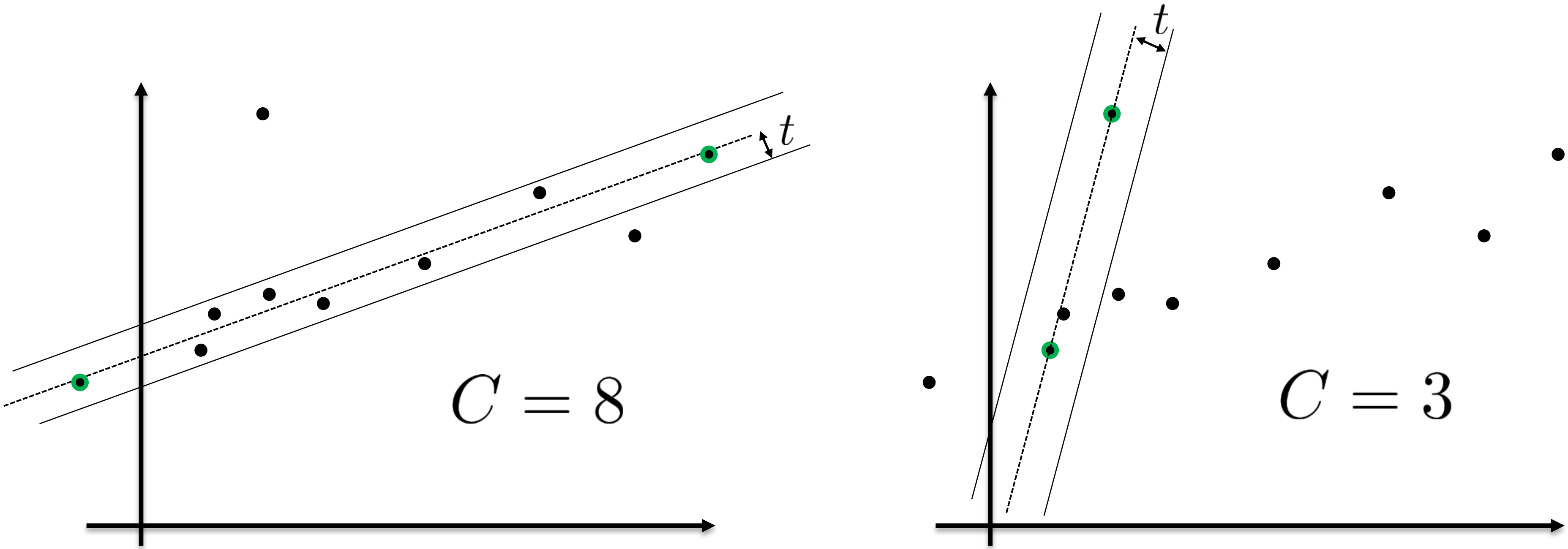
- Robust estimation: identification of outliers in the measurement set through estimation over multiple subsets of the measurement vector
  Example: line parameters estimation



Robust estimation

- Measurements outliers

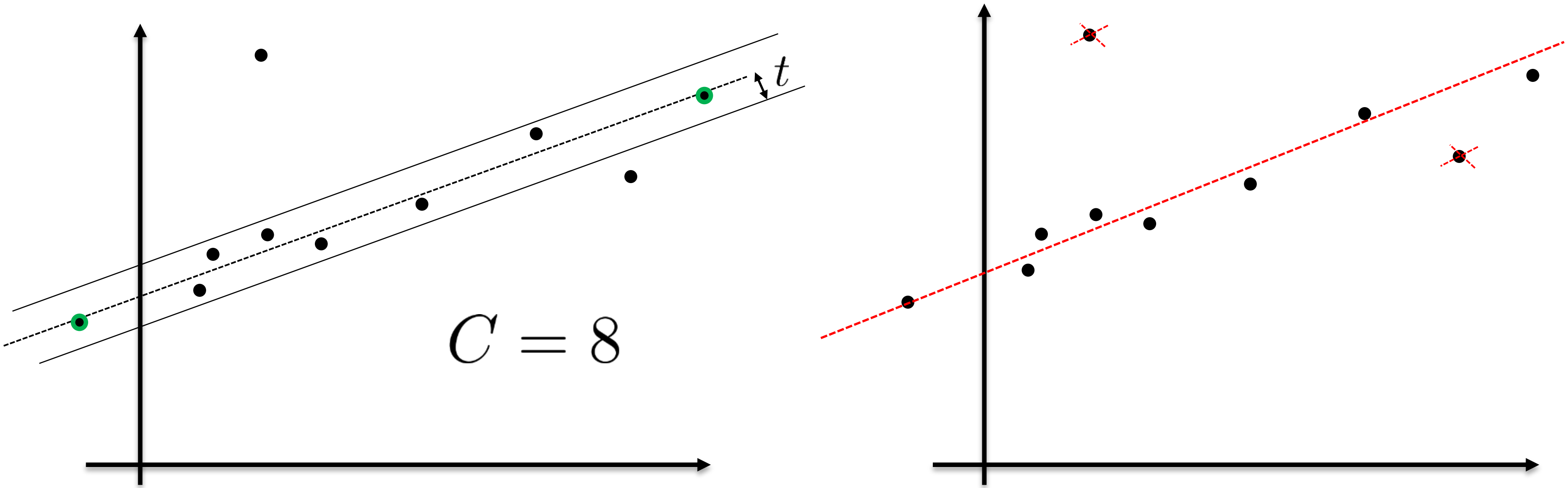
- Correct measurements (inliers) identified based on their "consensus" to the model used to fit the data



Robust estimation

- Measurements outliers

- We try different subsets, each with a small (minimum!) number of points, and eliminate those data points that are outside the "acceptance region" of the subgroup with largest consensus



Robust estimation

- How to choose the threshold?

- Empirically
- Probabilistic evaluation
  Example with line parameter estimation:
  Distance between line from subset s $\mathbf{l}_s = (a, b, c)^{\mathrm{T}}$ *and point* $\mathbf{x}_i = (x_1, x_2, x_3)^{\mathrm{T}}$ *:*

$$d_i^2 = (\mathbf{l}_s^{\mathrm{T}} \mathbf{x}_i)^2$$

- *If points are normally distributed, the squared distance is chi-distributed*

$$d_i^2 \sim \chi_m^2 = \chi_1^2$$

- *Probability that a random* χ_m^2 *-distributed variable does not exceed the value* k^2 *:*

$$P(d_i^2 \leq k^2) = \int_0^{k^2} \chi_1^2(\xi) d\xi$$

Robust estimation

- How to choose the threshold?

$$\mathbf{x}_i \sim \mathcal{N}(\bar{\mathbf{x}}, \sigma^2 \mathbf{I})$$

Codimension m	Model	t^2
1	line, fundamental matrix	$3.84\ \sigma^2$
2	homography, camera matrix	$5.99\ \sigma^2$
3	trifocal tensor	$7.81\ \sigma^2$

- *If points are* <u>*normally distributed*</u>*, the squared distance is chi-distributed*

$$P(d_i^2 \leq t^2) = \alpha = 0.95 \quad \Rightarrow t_\alpha$$

- *Probability that a random* χ_m^2 *-distributed variable does not exceed the value* k^2 *:*

$$P(d_i^2 \leq k^2) = \int_0^{k^2} \chi_1^2(\xi) d\xi$$

Robust estimation

- How many subset samples should one try?
- *Choose number of samples* N *so that at least one sample is free of outliers with probability* p
- *Probability that a point is an inlier:* w *, thus the probability that a point is an outlier is* $\epsilon = 1 - w$
- *Probability that taking* N *different samples of* s *points (assumed independent) returns an outlier-free sample with probability* p *:*

$$\underbrace{(1 - w^s)^N}_{\text{Probability of having an outlier in a single sample of s point}} = \underbrace{1 - p}_{\text{Probability of having all samples affected by outliers}} \Rightarrow \qquad N = \frac{\log(1-p)}{\log(1-w^s)}$$

Robust estimation

- How many subset samples should one try?

- *Number N of samples required to ensure, with a probability p = 0.99, that at least one sample has no outliers for a given size of sample, s, and proportion of outliers,* $\epsilon = 1 - w$

Sample size	Proportion of outliers ϵ						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

$$N = \frac{\log(1-p)}{\log(1-w^s)}$$

Robust estimation

- When to terminate?

- *A rule of thumb is to terminate when finding a sample with a consensus equal to the expected number of outliers.*
  Example. Expected ratio of inliers in the data set of n *points is reached:* $T = wn$
- *Tricky: we often do not know the probability of inliers*

- *Alternative approach: adaptively adjust the expected probability of inliers by starting with 0.5 and adjusting during search*

- This method is named <u>RANdom SAmple Consensus (RANSAC)</u> algorithm. *Alternative robust estimation approach: Least Median of Squares (LMS) , with model scored by taking the median of squared distances (rather than the number of inliers).*

Robust estimation

RANSAC WORKFLOW

→ Hypothesize:

i) *Set threshold according to dimensionality of problem and used distance metric*

ii) *Set required probability* p *of having outlier-free samples*

→ Iterate on random samples:

i) *Randomly select a minimum sample of s data points and estimate model*

ii) *Determine the set of inliers (distance smaller than the threshold set): store sample consensus*

iii) *If current consensus is larger than previous, store new model and inliers, and update ratio of outliers* ϵ

iv) *Repeat for minimum N trials to guarantee that best sample consensus returns likely correct solution*

v) *Re-estimate model with all the inliers*

Robust estimation

- Example: using the RANSAC to estimate a planar homography

- *Minimum number of sample points: s=4*

- *Distance: DLT (easiest) or symmetric transfer error*

$$d^2_{\mathrm{STE}} = d^2_{\mathbf{\Sigma}'_i,\mathbf{\Sigma}_i}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i) + d^2_{\mathbf{\Sigma}'_i,\mathbf{\Sigma}_i}(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i)$$

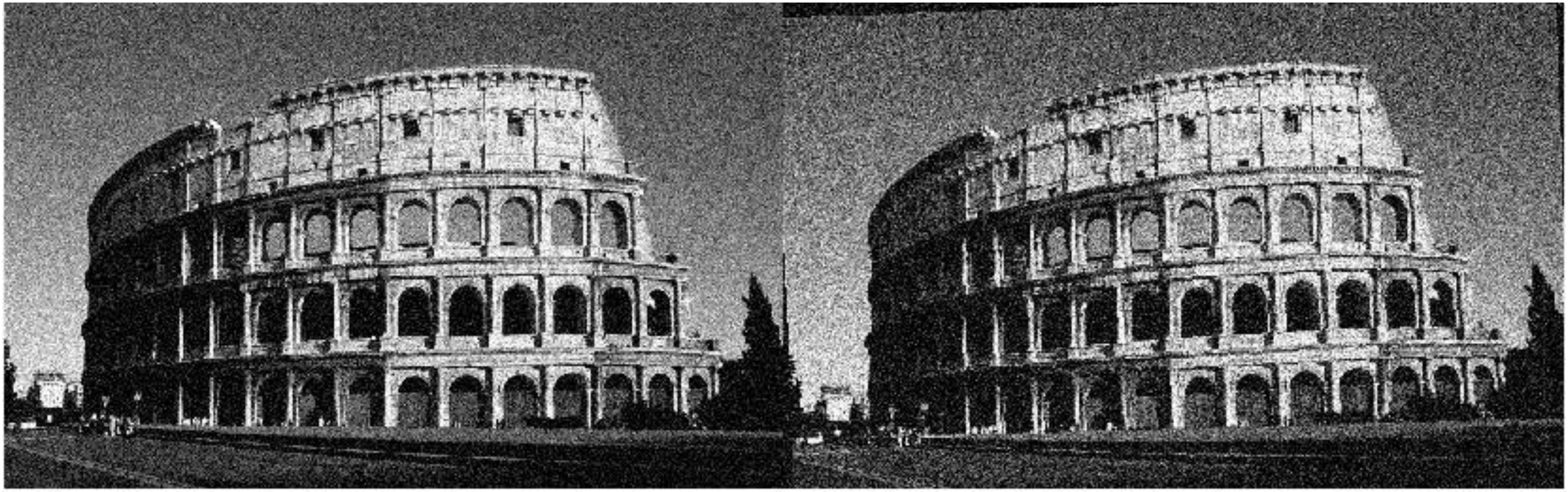
- *Disregard samples with 3 or four collinear points. Choose good spatial distribution (but depends on feature detection and matching results)*

- *Apply RANSAC*

Robust estimation

- Example: using the RANSAC to estimate a homography

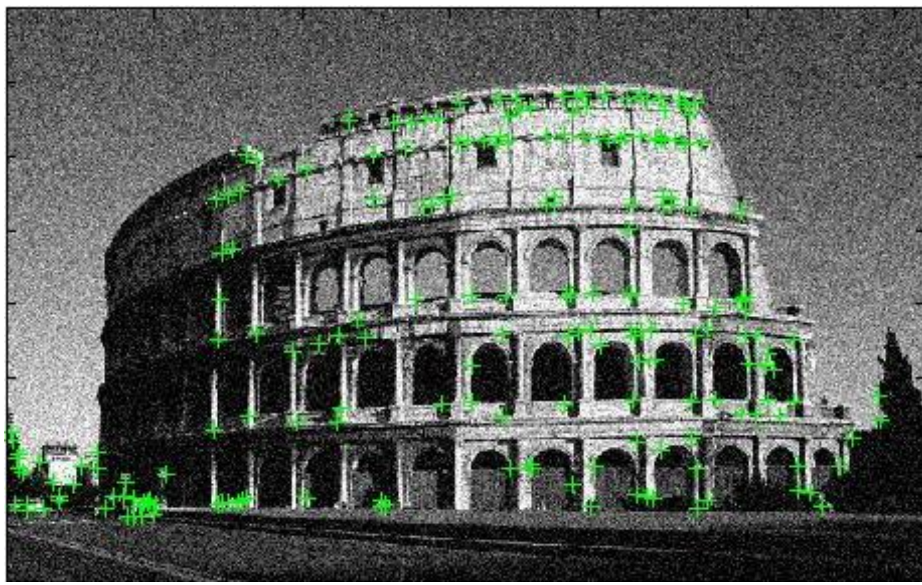
- *Original images*

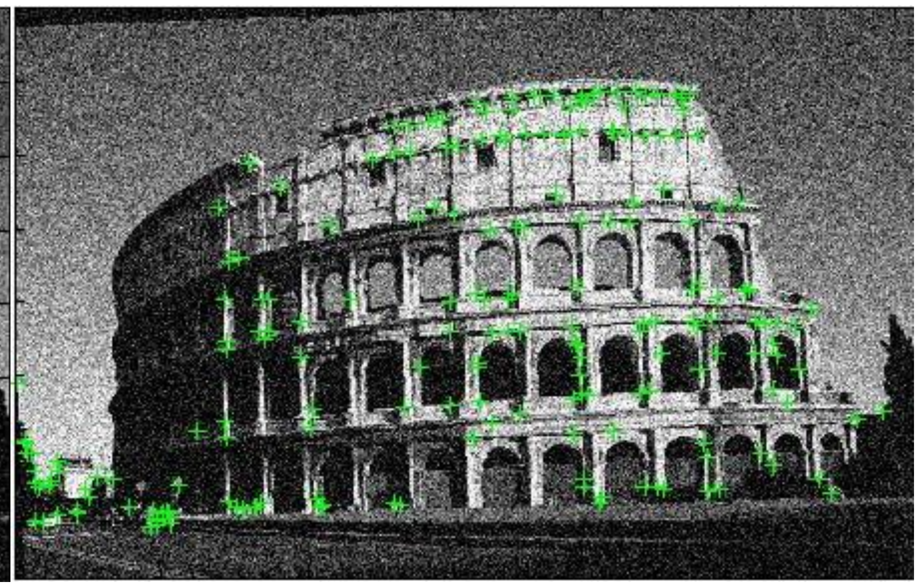


Robust estimation

- Example: using the RANSAC to estimate a homography

- *Feature extraction (corners)*

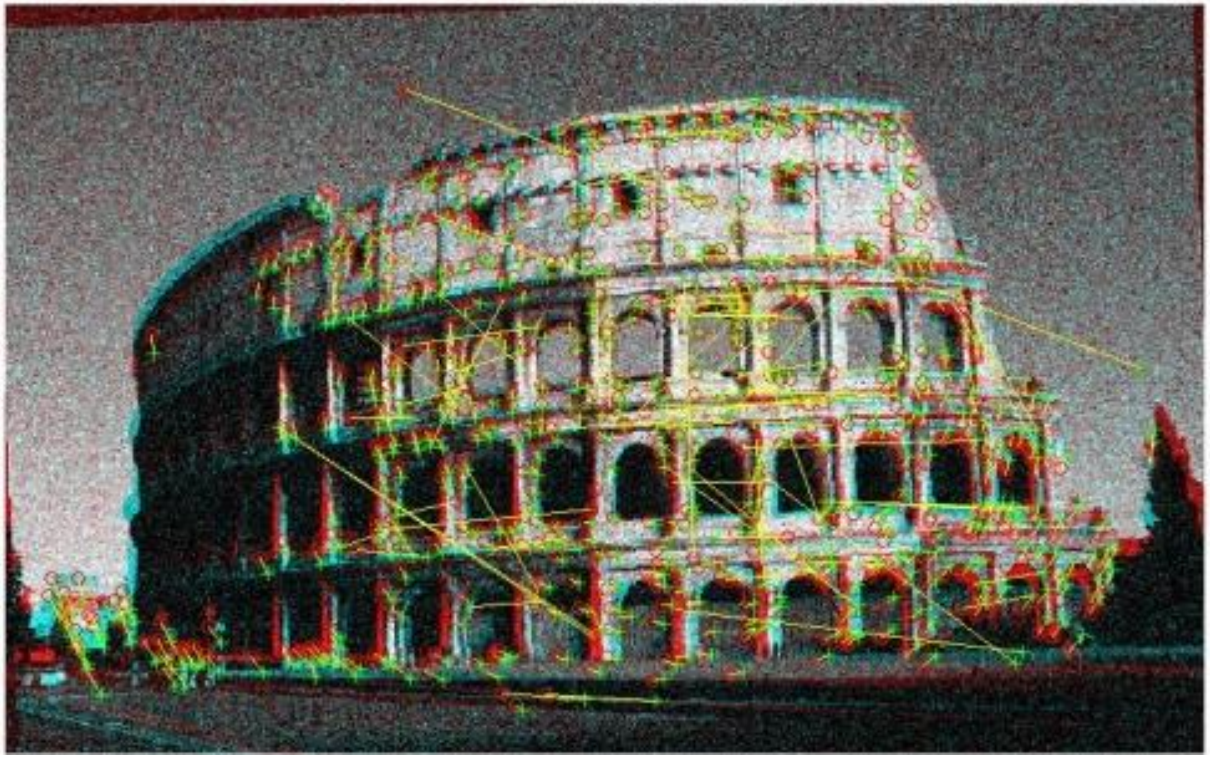




Robust estimation

- Example: using the RANSAC to estimate a homography

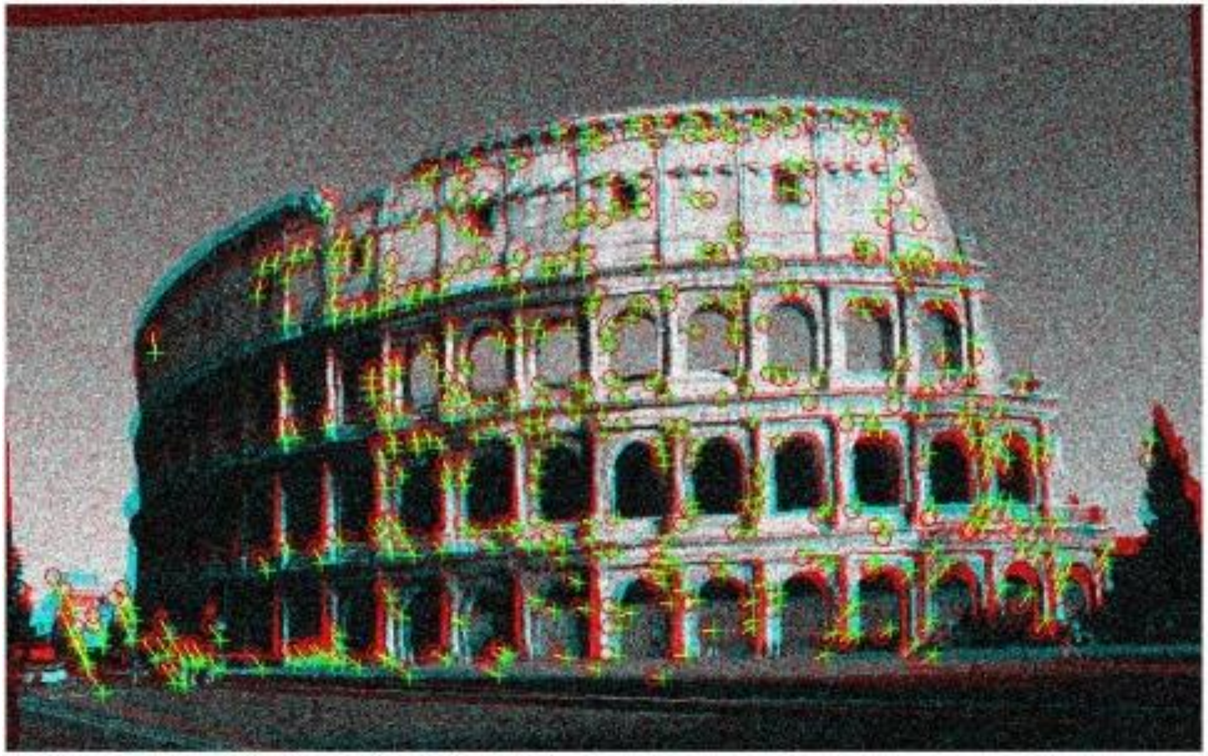
- *Matching (with outliers)*



Robust estimation

- Example: using the RANSAC to estimate a homography

- *Outlier removal with RANSAC*



Appendix: vec operator and Kronecker product

- Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

- vec operator

$$\mathrm{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_{:,1} \\ \vdots \\ \mathbf{a}_{:,n} \end{bmatrix}$$

- Properties

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}} &= \mathbf{A}^{\mathrm{T}} \otimes \mathbf{B}^{\mathrm{T}} \\ (\mathbf{A} \otimes \mathbf{B})^{-1} &= \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \\ (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= \mathbf{AC} \otimes \mathbf{BD} \quad \text{(If conforming matrices)} \\ \mathrm{vec}(\mathbf{ABC}) &= (\mathbf{C}^{\mathrm{T}} \otimes \mathbf{A})\mathrm{vec}\mathbf{B} \end{aligned}$$