Systems Theory

Dr. Steve Weller

steven.weller@newcastle.edu.au
School of Electrical Engineering and Computer Science
The University of Newcastle, Callaghan, NSW 2308
Australia

Controllability, stabilizability and reachability

- Important question that lies at the heart of control using state-space models:

"Can we steer the state, via the control input,

to certain locations in the state space?"

- **Controllability**
  - can initial state be driven back to origin?
- **Stabilizability**
  - can *all* states be taken back to origin?
- **Reachability**
  - can a certain state be reached *from* origin?

Controllability

Issue of *controllability* concerns whether a given initial state x_0 can be steered to the origin in finite time using the input $u(t)$

Definition 1:

- A state x_0 is said to be *controllable* if there exists a finite time interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = 0$
- If all states are controllable, then the system is said to be *completely controllable*

Reachability

Converse to controllability is *reachability*:

Definition 2:

- A state $\bar{x} \neq 0$ is said to be *reachable* (from the origin) if, given $x(0) = 0$, there exist a finite time interval $[0, T]$ and an input $\{u(t),\ \ t \in [0, T]\}$ such that $x(T) = \bar{x}$.

- If all states are reachable, the system is said to be *completely reachable*

Controllability and reachability—not *quite* the same

For continuous-time, linear time-invariant systems:

complete controllability $\Leftrightarrow$ complete reachability

Following example illustrates subtle difference in discrete-time…

- ❖ consider the following shift-operator state space model:

$$x[k+1] = 0$$

- system is completely controllable since every state goes to origin in one time-step
- *but* no non-zero state is reachable, so system is *not* completely reachable

Controllability or reachability?

- In view of the subtle distinction between controllability and reachability in discrete-time, we will use the term *controllability* in the sequel to cover the stronger of the two concepts

  $\Rightarrow$ discrete-time proofs for the results are a little easier

- We will thus present results using the following discrete-time model, written in terms of the delta operator:

$$
\begin{aligned}
\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\
y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]
\end{aligned}
$$

A test for controllability

We now present a simple algebraic test for controllability that can easily be applied to a given state-space model

stated for delta model, but holds for shift and continuous-time models too

Theorem 2: Consider the state-space model

$$x[k] = A\; x[k] + B\; u[k]$$
$$y[k] = C\; x[k] + D\; u[k]$$

(i) The set of all controllable states is the range space of the *controllability matrix* $\Gamma_c[\mathbf{A}, \mathbf{B}]$, where

$$\Gamma_c[A, B] = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

(ii) The model is completely controllable if and only if $\Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank

Example: A completely controllable system

Consider a state-space model with

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The controllability matrix is given by

$$\Gamma_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{A}\mathbf{B}] = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$$

- rank $\Gamma_c[\mathbf{A}, \mathbf{B}] = 2$

$\Rightarrow$ the system is completely controllable

❑ ***Observation***: complete controllability of a system is independent of **C** and **D**

Example: A non-completely controllable system

For

$$A = \begin{matrix} 1 & 1 \\ 2 & 0 \end{matrix}; \quad B = \begin{matrix} 1 \\ 1 \end{matrix}$$

the controllability matrix is given by:

$$_c[A;B] = [B;AB] = \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}$$

- rank Γ_c[**A**, **B**] = 1 since row 2 = -row1

⇒ Γ_c[**A**, **B**] is *not* full row rank

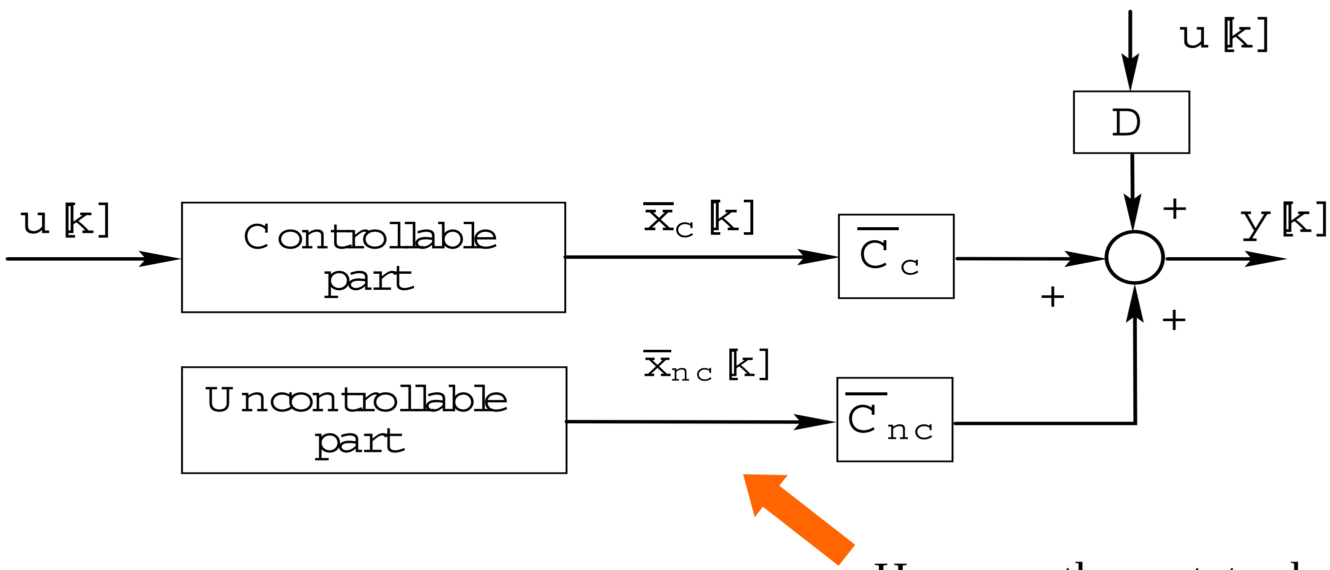
⇒ system is *not* completely controllable

Controllability—a word of caution

- Controllability is a black and white issue: a model either is completely controllable or it's not
- Knowing that a system is controllable (or not) is a valuable piece of information, but…
- knowing that a system is controllable really tells us nothing about the *degree* of controllability, i.e., about the difficulty that might be involved in achieving a certain objective
    - for example: how much energy is required to drive system state to origin?
    - this issue lies at the heart of the fundamental design trade-offs in control

Controllable–uncontrollable decomposition

If a system is not completely controllable, it can be decomposed into a controllable and a completely uncontrollable subsystem



However these states behave, it's independent of input u[k]

Partitioning the state-space model

Key to the controllable–uncontrollable decomposition is the transformation of **A**, **B**, and **C** into suitably partitioned form:

$$\begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{nc} \end{bmatrix} \begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} + \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} \overline{C}_c & \overline{C}_{nc} \end{bmatrix} \begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} + D u[k]$$

Controllable decomposition

Lemma 1: Consider a system having rank$\{\Gamma_c[\mathbf{A}, \mathbf{B}]\} = k < n$. Then there exists a similarity transformation T such that $\bar{x} = \mathbf{T}^{-1}x$,

$$\bar{A} = T^{-1}AT; \qquad \bar{B} = T^{-1}B$$

and $\bar{\mathbf{A}}, \bar{\mathbf{B}}$ have the form

$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{nc} \end{bmatrix}; \qquad \bar{B} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

where $\bar{\mathbf{A}}_c$ has dimension k and $(\bar{\mathbf{A}}_c, \bar{\mathbf{B}}_c)$ is completely controllable.

Details of actually computing matrix T not considered here

Controllable subspace

Output has a component $\mathbf{C}_{nc}\,\bar{x}_{nc}[k]$ that does not depend on the manipulated input $u[k]$, so…

$\Rightarrow$ caution must be exercised when controlling a system which is not completely controllable

- same holds when *model* used for control design is not completely controllable

Definition 3: The *controllable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in $\bar{x}_c$

stability of controllable subspace $\Leftrightarrow$ stability of all eigenvalues of $\bar{\boldsymbol{A}}_c$

Uncontrollable models in control design

▪ Uncontrollable models are often a very convenient way of describing *disturbances* when modeling for control design

Example: constant disturbance can be modeled by the following state-space model:

$$\underline{x}_d = 0$$

- uncontrollable *and* non-stabilizable

$\Rightarrow$ very common to employ uncontrollable models in control-system design

Stabilizability

Definition 4: The *uncontrollable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in $\bar{\boldsymbol{x}}_{nc}$

stability of uncontrollable subspace $\Leftrightarrow$ stability of all eigenvalues of $\bar{\boldsymbol{A}}_{nc}$

A state-space model is said to be *stabilizable* if its uncontrollable subspace is stable.

In other words: system is stabilizable only if those states that cannot be controlled decay to origin "by themselves"

Canonical forms

If a system is completely controllable, there exist similarity transformations that convert it into special “standard forms”, or *canonical forms*:

❑ *controllability canonical form*

❑ *controller canonical form*

▪ These canonical forms present **A** and **B** matrices in highly structured ways

☹ Physical interpretation of states is lost

☺ Can be written directly from knowledge of system poles

Controllability canonical form

Lemma 2: Consider a completely controllable state-space model for a single-input, single-output (SISO) system. Then there exists a similarity transformation that converts the state-space model into the following *controllability canonical form*:

$$A' = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix} \qquad B' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\lambda^n + \alpha_{n-1}\lambda_{n-1} + \ldots + \alpha_1\lambda + \alpha_0 = \det(\lambda \mathbf{I} - \mathbf{A})$ is the characteristic polynomial of **A**.

Controller canonical form

Lemma 3: Consider a completely controllable state-space model for a SISO system. Then there exists a similarity transformation that converts the state-space model into the following *controller canonical form*:

$$A' = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\lambda^n + \alpha_{n-1}\lambda_{n-1} + \ldots + \alpha_1\lambda + \alpha_0 = \det(\lambda\mathbf{I} - \mathbf{A})$ is the characteristic polynomial of $\mathbf{A}$.

Observability, detectability and reconstructibility

$$x[k] = A\ x[k] + B\ u[k]$$

$$y[k] = C\ x[k] + D\ u[k]$$

❑ **Observability**

- What do observations of output tell us about *initial* state of system?

❑ **Detectability**

- Do observations of output tell us everything “important” about the internal state of system? (Yes, if non-observable states decay to origin)

❑ **Reconstructibility**

- Can we establish *current* state of system from past output response?
- Same as observability for continuous-time systems, but subtly different for discrete-time systems

Observability

Observability is concerned with what can be said about the initial state when given measurements of the plant output

Definition 5:

- A state $x_0 \neq 0$ is said to be *unobservable* if, given $x(0) = x_0$, and $u[k] = 0$ for $k \geq 0$, then $y[k] = 0$ for $k \geq 0$
    - state is doing something "interesting", (or at least is non-zero!), yet output is zero
- The system is said to be *completely observable* if there exists no non-zero initial state that it is unobservable

Reconstructability

Reconstructability is concerned with what can be said about $x(T)$, on the basis of the past values of the output, i.e., $y[k]$ for $0 \leq k \leq T$

For *continuous-time* LTI systems:

complete reconstructability $\Leftrightarrow$ complete observability

Observability vs. reconstructibility

Consider discrete-time system:

$$x[k+1] = 0 \qquad\qquad x[0] = x_o$$

$$y[k] = 0$$

❖ we know for certain that $x[T] = 0$ for all $T \geq 1 \Rightarrow$ system is reconstructable

❖ but $y[k] = 0 \ \forall k$, irrespective of the value of $x_0 \Rightarrow$ completely unobservable

In view of the subtle difference between observability and reconstructability, we will use the term observability in the sequel to cover the stronger of the two concepts

A test for observability

A test for observability of a system is established in the following theorem.

Theorem 3: Consider the state model

holds for discrete (shift) and continuous-time models too

$$x[k] = A\, x[k] + B\, u[k]$$

$$y[k] = C\, x[k] + D\, u[k]$$

(*i*) The set of all unobservable states is equal to the null space of the *observability matrix* Γ_0[**A**, **C**], where

$$\Gamma_o[A, C] \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

(*ii*) The system is completely observable if and only if Γ_0[**A**, **C**], has full column rank *n*

Example: A completely observable system

Consider the following state space model:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Then

$$\Gamma_o[A; C] = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 2$, and the system is completely observable.

Example: A non-completely observable system

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Here

$$\Gamma_o[A; C] = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 1 < 2$, and the system is *not* completely observable.

The controllable—observable duality

It's no coincidence that:

complete controllability $\Leftrightarrow$ $\Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank

complete observability $\Leftrightarrow$ $\Gamma_o[\mathbf{A}, \mathbf{C}]$ has full column rank

as the following Theorem shows:

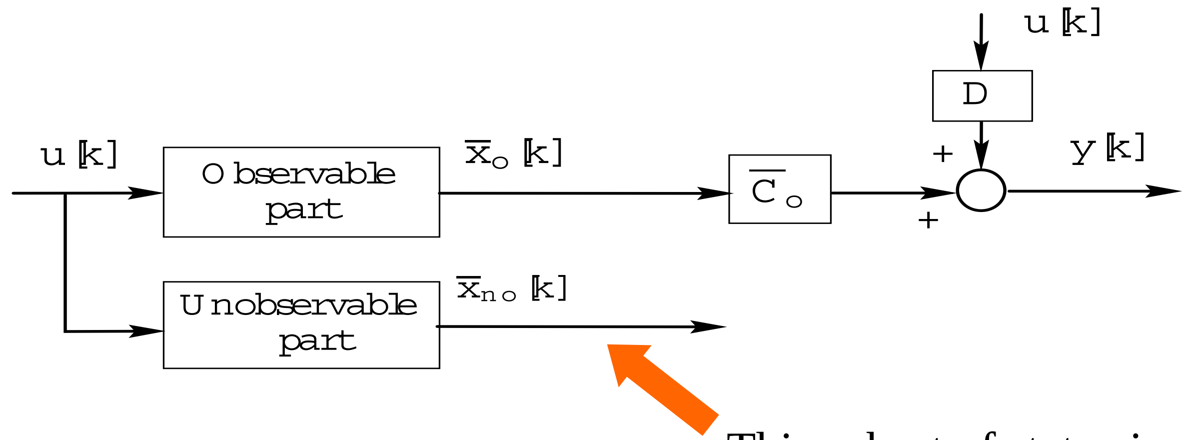
Theorem 4 Consider a state-space model described by the ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$). Then

($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$) completely controllable $\Leftrightarrow$ ($\mathbf{A}^T$, $\mathbf{C}^T$, $\mathbf{B}^T$, $\mathbf{D}^T$) completely observable

the so-called *dual system*

Observable–unobservable decomposition

If a system is not completely observable, it can be decomposed into an observable and a completely unobservable subsystem



This subset of states *in no way* influences output

Partitioning of A, B, and C

Decomposition of state-space into observable and non-observable parts relies on suitable partitioning of (similarity transformed) state-space matrices:

$$\begin{bmatrix} \overline{x}_{o}[k] \\ \overline{x}_{no}[k] \end{bmatrix} = \begin{bmatrix} \overline{A}_{o} & 0 \\ \overline{A}_{21} & \overline{A}_{n0} \end{bmatrix} \begin{bmatrix} \overline{x}_{o}[k] \\ \overline{x}_{no}[k] \end{bmatrix} + \begin{bmatrix} \overline{B}_{o} \\ \overline{B}_{no} \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} \overline{C}_{o} & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_{o}[k] \\ \overline{x}_{no}[k] \end{bmatrix} + D\,u[k]$$

Observable decomposition

Lemma 4: If rank$\{\Gamma_o[\mathbf{A}, \mathbf{C}]\} = k < n$, there exists a similarity transformation T such that with $\bar{x} = \mathbf{T}^{-1}x$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\bar{\mathbf{C}} = \mathbf{C}\mathbf{T}$, then $\bar{\mathbf{C}}$ and $\bar{\mathbf{A}}$ take the form

$$\bar{A} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{no} \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}$$

where $\bar{\mathbf{A}}_0$ has dimension k and the pair $(\bar{\mathbf{C}}_o, \bar{\mathbf{A}}_o)$ is completely observable

Observable subspace

Definition 6: *The observable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in $\overline{X}_0$

stability of controllable subspace $\Leftrightarrow$ stability of all eigenvalues of $\overline{\mathbf{A}}_0$

Detectability

Definition 7: The *unobservable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in $\bar{x}_{n0}$.

$$\begin{array}{ccc} \text{stability of} & & \text{stability of all} \\ \text{unobservable subspace} & \Leftrightarrow & \text{eigenvalues of } \bar{\mathbf{A}}_{n0} \end{array}$$

A state-space model is said to be *detectable* if its unobservable subspace is stable.

In other words: system is observable only if those states that cannot be observed decay to origin "by themselves"

❖ while non-stabilizable models are frequently used to model disturbances in control-system design, this is *not* true for non-detectable models.

Canonical decomposition

- Dual to controller and controllability canonical forms are *observer* and *observability canonical forms*
  - Precise forms aren't important here
- Can also combine controllable and observable decompositions into a *canonical decomposition* with subsystems which are:
  - Controllable and observable $(\boldsymbol{A_{co}}, \boldsymbol{B}_1, \boldsymbol{C}_1)$
  - Controllable, not observable
  - Observable, not controllable
  - Not observable, not controllable

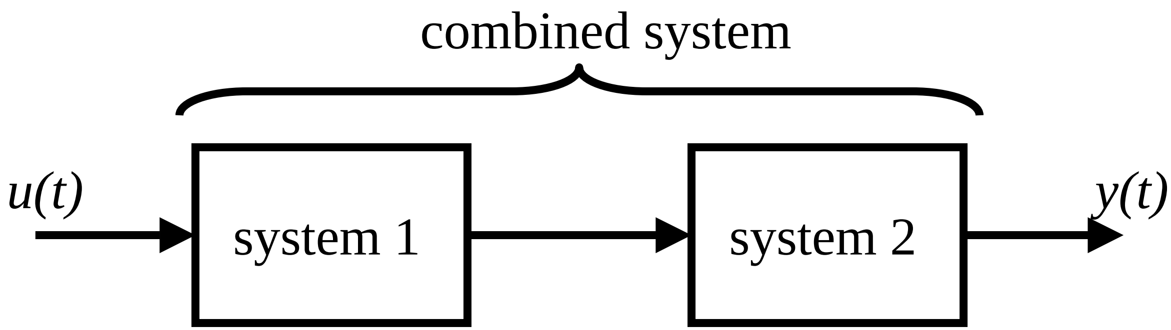
Only controllable *and* observable parts of system appear in transfer function!

$$\boldsymbol{H}(\boldsymbol{s}) = \boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A})^{-1}\boldsymbol{B} = \boldsymbol{C}_1(\boldsymbol{sI} - \boldsymbol{A_{co}})^{-1}\boldsymbol{B}_1$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_{co}} & 0 & \boldsymbol{A}_{13} & 0 \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \boldsymbol{A}_{24} \\ 0 & 0 & \boldsymbol{A}_{33} & 0 \\ 0 & 0 & 0 & \boldsymbol{A}_{44} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & 0 & \boldsymbol{C}_2 & 0 \end{bmatrix}$$

Pole-zero cancellations

Systems which are either non-completely controllable and/or non-completely observable are associated with transfer functions having *pole-zero cancellations*



• zero at α
• pole at β

• zero at β
• pole at α

Then combined system:

❖ has a pole at β that is not observable from *y(t)*

❖ has a zero at α that is not controllable from *u(t)*

The big picture

- *Controllability*: can we use input to steer system state to origin in finite time?
  - *Stabilizability*: not controllable, but uncontrollable states well behaved
- *Observability*: can we infer system state from measurements of output?
  - *Detectability*: not observable, but unobservable states decay to origin
- Algebraic tests for controllability and observability
- Non-observable and/or non-controllable systems have transfer functions with pole-zero cancellations

Drinking from a firehose…