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Kalman and $\mathcal{H}_\infty$ Optimal Filtering for a Class of Kinematic Systems

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Abstract: This paper presents a set of optimal filtering results for a class of kinematic systems with particular application to the estimation of linear quantities in Integrated Navigation Systems for mobile platforms. At the core of the proposed methodology there is a time varying orthogonal Lyapunov coordinate transformation that renders the overall system dynamics linear time invariant (LTI). The design is based on the Kalman or $\mathcal{H}_\infty$ filtering steady state solutions for an equivalent LTI system and allows for the natural use of frequency weights to explicitly achieve adequate disturbance rejection and attenuation of the noise of the sensors on the state estimates. Afterwards, the resulting solution is converted back to the original coordinate space, yielding a globally stable time varying optimal estimator for the problem at hand. A simple example of practical importance in marine systems is provided that demonstrates the applicability of the proposed design methodologies and simulation results are included to illustrate the filtering achievable performance.

1. INTRODUCTION

This paper presents a set of optimal time varying filtering solutions for a class of kinematic systems with direct application to the estimation of linear quantities in precise Integrated Navigation Systems.

The design of Navigation Systems plays an important role in a great variety of applications. Naturally, many strategies have been suggested in the literature to tackle the problem. In Fossen and Strand (1999) a globally exponentially stable (GES) observer for ships (in two-dimensions) that includes features such as wave filtering and bias estimation is proposed and in H. Nijmeijer and T. I. Fossen (Eds) (1999) an extension to this result with adaptive wave filtering is available. An alternative filter is presented in Pascoal et al. (2000) where the problem of estimating the position and velocity of an autonomous vehicle in three-dimensions was solved resorting to special bilinear time varying complementary filters. Refsnes et al. (2006) presents a pair of coworking GES observers for underwater vehicles that includes the ocean current in the model plant to improve the performance of the observer. A passivity based controller-observer design for robots is proposed in Berghuis and Nijmeijer (1993) and a sliding mode observer for robotic manipulators is reported in C. De Wit and J.-J. Slotine (1991). The development of nonlinear observers for Euler-Lagrange systems has been addressed in Skjetne and Shim (2001) and Ortega et al. (1998). In these approaches robustness to environmental disturbances and/or noise of the sensors is considered but no optimal results are provided.

The filter design techniques presented in this paper are based on previous work by the authors, see Batista et al. (2006), where a globally stable ocean current observer was designed to feed a nonlinear sensor based integrated guidance and control law. This state feedback law was designed to drive an underactuated autonomous underwater vehicle to a neighborhood of a fixed target, in 3D, using the information provided by an Ultra-Short Baseline positioning system. At the core of the observer design methodology there was a time varying orthogonal Lyapunov coordinate transformation that rendered the observer dynamics linear time invariant (LTI), and global exponential stability was attained for the observer error. This transformation is now exploited to derive a set of optimal filtering solutions for a much larger variety of dynamic systems. The proposed methodology is based on the Kalman or $\mathcal{H}_\infty$ filtering steady state solutions for an equivalent LTI system and allows for the natural use of frequency weights to explicitly achieve adequate disturbance rejection and attenuation of the noise of the sensors on the state estimates. Afterwards, the resulting solution is converted back to the original coordinate space, yielding a globally stable time varying optimal estimator for the problem at hand.

The paper is organized as follows. Section 2 introduces the class of dynamic systems and the filtering problem addressed in this work. In Section 3 the proposed design technique is presented and the Kalman filter derived. The

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$\mathcal{H}_\infty$ optimal filtering solution is outlined in Section 4 and Section 5 references some extensions and properties of the proposed solutions. A simple example of application is presented and simulation results are discussed in Section 6 and finally Section 7 summarizes the main contributions of the paper.

2. PROBLEM STATEMENT

Consider the class of dynamic systems

$$\begin{cases}\dot{\boldsymbol{\eta}}_p(t) = \mathbf{A}_p\boldsymbol{\eta}_p(t) - \mathbf{M}_S\left(\boldsymbol{\omega}(t)\right)\boldsymbol{\eta}_p(t) + \mathbf{B}_p(t)\mathbf{u}(t) \\ \boldsymbol{\psi}(t) = \boldsymbol{C}_p\boldsymbol{\eta}_p(t)\end{cases}, \quad (1)$$

where $\boldsymbol{\eta}_p(t) = \left[\boldsymbol{\eta}_1^T(t)\ \dots\ \boldsymbol{\eta}_N^T(t)\right]^T$, with $\boldsymbol{\eta}_i(t) \in X_i \subseteq \mathbb{R}^3, i = 1,\dots,N$, is the system state, $\boldsymbol{\psi}(t) \in \mathbb{R}^3$ is the system output, $\mathbf{u}(t)$ is a deterministic system input, $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is a continuous bounded function of t, $\mathbf{M}_S\left(\boldsymbol{\omega}(t)\right)$ is the block diagonal matrix $\mathbf{M}_S\left(\boldsymbol{\omega}(t)\right) := \operatorname{diag}\left(\mathbf{S}(\boldsymbol{\omega}(t)),\dots,\mathbf{S}(\boldsymbol{\omega}(t))\right)$, where $\mathbf{S}(\boldsymbol{\omega}(t))$ is a skew-symmetric matrix that verifies $\mathbf{S}\left(\mathbf{a}\right)\mathbf{b} = \mathbf{a}\times\mathbf{b}$, with $\times$ denoting the cross product, and that satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}(\boldsymbol{\omega}(t)),$$

where $\mathbf{R}(t) \in \left\{\mathbf{R}\in\mathbb{R}^{3\times3} : \mathbf{R}\mathbf{R}^T = \mathbf{I}_3,\ \det(\mathbf{R}) = 1\right\}$, i.e., $\mathbf{R}(t)$ is a proper rotation matrix,

$$\mathbf{A}_p = \begin{bmatrix} \mathbf{0} & \gamma_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \vdots & & & \ddots & \gamma_{N-1}\mathbf{I} \\ \mathbf{0} & \dots & \dots & \dots & \mathbf{0}\end{bmatrix}, \quad (2)$$

$\gamma_i \in \mathbb{R}$, $\gamma_i \neq 0$, $i = 1,\dots,N-1$, and $\mathbf{C}_p = [\mathbf{I}_3\ \mathbf{0}_{3\times3}\ \dots\ \mathbf{0}_{3\times3}]$. It is assumed that $\mathbf{R}(t)$ and $\boldsymbol{\omega}(t)$ are known over time. Finally, suppose that there exist system disturbances $\boldsymbol{\xi}(t)$ and noise of the sensors $\boldsymbol{\theta}(t)$, as depicted in Fig. 1, where $\mathbf{M}_p$ is assumed to be a full row-rank matrix.

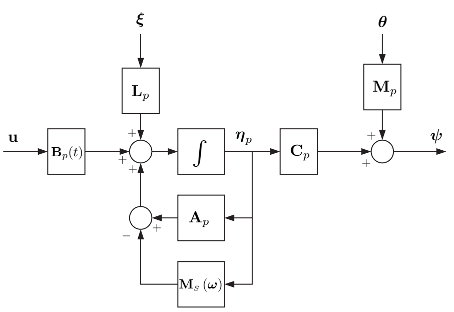


Fig. 1. Generalized system dynamics

The filtering problem considered in the paper can be stated as follows.

Problem statement: Consider the dynamic system (1), as previously described, subject to system disturbances and measurement noise, as depicted in Fig. 1. Design an optimal filter that minimizes the impact of the noise of the sensors and the system disturbances on the state estimates.

Throughout the paper the symbol $\mathbf{0}_{n\times m}$ denotes an $n\times m$ matrix of zeros, $\mathbf{I}_n$ an identity matrix with dimension $n\times n$, and $\operatorname{diag}(\mathbf{A}_1,\dots,\mathbf{A}_n)$ a block diagonal matrix. When the dimensions are omitted the matrices are assumed of appropriate dimensions. The usual Hilbert space of square integrable functions is denoted by $\mathcal{L}_2$.

3. KALMAN OPTIMAL FILTER

This section presents the derivation of the time varying Kalman filter for the dynamic system (1), subject to system disturbances and measurement noise. The main idea, which is also at the core of the $\mathcal{H}_\infty$ filter design, consists of converting the linear time varying (LTV) dynamic system into an equivalent LTI system. A classic Kalman filter is then applied to the resulting LTI system, in its more general setup with frequency weights to model the system disturbances and noise of the sensors. Afterwards, the solution is transformed back to the original coordinate space, yielding the equations and structure of the filter. Finally, a brief interpretation on the proposed solution is given.

Without loss of generality, consider the dynamic system (1) where the deterministic input $\mathbf{u}(t)$ is omitted, and define $\mathbf{x}_p(t) = \left[\mathbf{x}_1(t)^T\ \dots\ \mathbf{x}_N^T(t)\right]^T$ as

$$\mathbf{x}_p(t) := \mathbf{T}(t)\boldsymbol{\eta}_p(t), \quad (3)$$

where $\mathbf{T}(t)$ is the coordinate transformation matrix defined by

$$\mathbf{T}(t) := \operatorname{diag}\left(\mathbf{R}(t),\dots,\mathbf{R}(t)\right).$$

Notice that (3) is a Lyapunov transformation (see Brockett (1970)) as

- $\mathbf{T}(t)$ is continuous differentiable for all t;
- $\mathbf{T}(t)$ and $\dot{\mathbf{T}}(t)$ are bounded for all t, where $\dot{\mathbf{T}}(t) = \mathbf{T}(t)\mathbf{M}_S(\boldsymbol{\omega}(t))$;
- $\det\left[\mathbf{T}(t)\right] = 1$.

Define also a new system output as

$$\mathbf{y}(t) := \mathbf{R}(t)\boldsymbol{\psi}(t)$$

It is straightforward to show that after these coordinate transformations the resulting system dynamics are LTI. Adding system disturbances $\mathbf{d}$ and noise of the sensors $\mathbf{n}$ to this new coordinate space, the resulting dynamics can be written as

$$\begin{cases}\dot{\mathbf{x}}_p(t) = \mathbf{A}_p\mathbf{x}_p(t) + \mathbf{L}_p\mathbf{d}(t) \\ \mathbf{y}(t) := \boldsymbol{C}_p\mathbf{x}_p(t) + \mathbf{M}_p\mathbf{n}(t)\end{cases}. \quad (4)$$

It is now immediate to design a Kalman filter for the LTI system (4). In particular, this design technique allows for the natural use of dynamic systems to model both the system disturbances and the noise of the sensors. To that purpose, consider the block diagram depicted in Fig. 2, where the LTI system dynamics are shown together with LTI weight filters $\mathcal{W}_d$ and $\mathcal{W}_n$. In the figure, $\mathbf{w}(t) = \left[\mathbf{w}_1^T(t)\,\mathbf{w}_2^T(t)\right]^T$ represents the generalized disturbance vector, assumed to be continuous-time zero-mean unit intensity white noise.

Define $\mathbf{x}(t) = \left[\mathbf{x}_p^T(t)\,\mathbf{x}_d^T(t)\,\mathbf{x}_n^T(t)\right]^T$, where $\mathbf{x}_d(t)$ and $\mathbf{x}_n(t)$ denote the internal states of state space realizations $(\mathbf{A}_d,\mathbf{B}_d,\mathbf{C}_d,\mathbf{D}_d)$ and $(\mathbf{A}_n,\mathbf{B}_n,\mathbf{C}_n,\mathbf{D}_n)$ of $\mathcal{W}_d$ and $\mathcal{W}_n$,

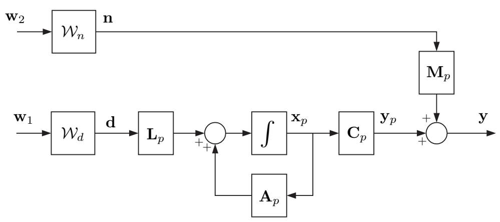


Fig. 2. Generalized LTI Plant

respectively. Then, the augmented plant can be written, in a compact form, as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{w}(t) \end{cases},$$

where

$$\mathbf{A} := \begin{bmatrix} \mathbf{A}_p & \mathbf{L}_p\mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_n \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} \mathbf{L}_p\mathbf{D}_d & 0 \\ \mathbf{B}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_n \end{bmatrix},$$

$\mathbf{C} = [\mathbf{C}_p \, \mathbf{0} \, \mathbf{M}_p\mathbf{C}_n]$, and $\mathbf{D} = [\mathbf{0} \, \mathbf{M}_p\mathbf{D}_n]$. It is assumed that the Kalman filtering problem is well-posed (this only depends, in this case, on the choice of the frequency weights and the matrices $\mathbf{L}_p$ and $\mathbf{M}_p$). Define $\mathbf{V}$ as

$$\mathbf{V} := \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T \, \mathbf{D}^T \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{xx} & \mathbf{V}_{xy} \\ \mathbf{V}_{xy}^T & \mathbf{V}_{yy} \end{bmatrix}.$$

Notice that since $\mathbf{D}$ has full row-rank, $\mathbf{V}_{yy}$ is positive definite, from which follows that it admits inverse. The Kalman filter for this system is given by (A. Gelb (Ed) (1974))

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{K}_2(t)\left[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right]$$

where $\mathbf{K}_2(t)$ is the Kalman gain matrix, given by

$$\mathbf{K}_2(t) = \left[\mathbf{P}_2(t)\mathbf{C}^T + \mathbf{V}_{xy}\right]\mathbf{V}_{yy}^{-1},$$

where $\mathbf{P}_2(t)$ is the solution of the matrix differential Riccati equation

$$\begin{aligned} \dot{\mathbf{P}}_2(t) = {} & \mathbf{A}_e\mathbf{P}_2(t) + \mathbf{P}_2(t)\mathbf{A}_e^T - \mathbf{P}_2(t)\mathbf{C}^T\mathbf{V}_{yy}^{-1}\mathbf{C}\mathbf{P}_2(t) \\ & + \mathbf{V}_{xx} - \mathbf{V}_{xy}\mathbf{V}_{yy}^{-1}\mathbf{V}_{xy}^T, \end{aligned} \tag{5}$$

with $\mathbf{A}_e = \mathbf{A} - \mathbf{V}_{xy}\mathbf{V}_{yy}^{-1}\mathbf{C}$. The initial condition $\mathbf{P}_2(t_0)$ will be given later on.

In order to recover the filter equations in the appropriate coordinate space, consider the coordinate transformation

$$\hat{\boldsymbol{\eta}}(t) = \mathbf{T}_c^T(t)\hat{\mathbf{x}}(t), \tag{6}$$

where the matrix $\mathbf{T}_c$ is given by

$$\mathbf{T}_c(t) := \operatorname{diag}\left(\mathbf{T}(t), \mathbf{I}, \mathbf{I}\right).$$

Computing the time derivative of (6), and after a few straightforward algebraic manipulations, the final filter equations can be written as

$$\dot{\hat{\boldsymbol{\eta}}}(t) = \boldsymbol{\mathcal{A}}(t)\hat{\boldsymbol{\eta}}(t) + \boldsymbol{\mathcal{B}}_p(t)\mathbf{u}(t) + \boldsymbol{\mathcal{K}}_2(t)\left[\boldsymbol{\psi}(t) - \boldsymbol{\mathcal{C}}(t)\hat{\boldsymbol{\eta}}(t)\right], \tag{7}$$

where

$$\boldsymbol{\mathcal{A}}(t) = \begin{bmatrix} \mathbf{A}_p - \mathbf{M}_S(\boldsymbol{\omega}(t)) & \mathbf{T}^T(t)\mathbf{L}_p\mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_n \end{bmatrix}, \quad \boldsymbol{\mathcal{B}}_p(t) = \begin{bmatrix} \mathbf{B}_p(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{\mathcal{C}}(t) = \left[\mathbf{C}_p \,|\, \mathbf{0} \,|\, \mathbf{R}^T(t)\mathbf{M}_p\mathbf{C}_n\right],$$

and

$$\boldsymbol{\mathcal{K}}_2(t) = \mathbf{T}_c^T(t)\mathbf{K}_2(t)\mathbf{R}(t). \tag{8}$$

Notice that the deterministic input term $\boldsymbol{\mathcal{B}}_p(t)\mathbf{u}(t)$ was added to complete the filter dynamics. Fig. 3 presents the corresponding original generalized system. As it can be seen, the description of the system disturbances and the noise of the sensors may not be exact due to the coordinate transformations $\mathbf{T}^T(t)$ on the system disturbances and $\mathbf{R}^T(t)$ on the noise of the sensors. Indeed, while in the original framework presented in Fig. 1 the term $\mathbf{L}_p\boldsymbol{\xi}(t)$ affects the system state and the term $\mathbf{M}_p\boldsymbol{\theta}(t)$ represents the noise of the sensors, now $\mathbf{T}^T(t)\mathbf{L}_p\mathbf{d}(t)$ acts on the system state and $\mathbf{R}^T(t)\mathbf{M}_p\mathbf{n}(t)$ models the noise of the sensors. It so happens in practice that sometimes the available description of the system disturbances or the noise of the sensors is defined on the transformed (LTI) space, for which this structure is appropriate. Nevertheless, when that is not the case, it should be noted that the aforementioned coordinate transformation preserves the norm. Thus, only the directionality is affected over time.

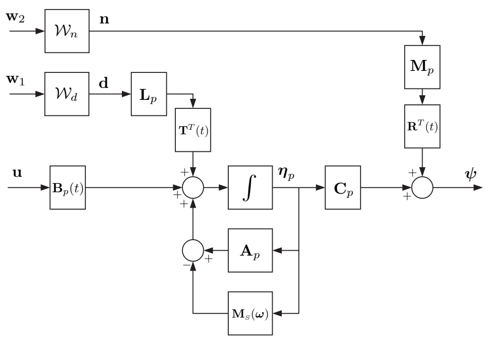


Fig. 3. Kalman filter design setup

The following theorem is the main result of this section.

Theorem 1. Consider the generalized system dynamics as depicted in Fig. 3, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are continuous-time zero-mean unit intensity white noises. Let $\boldsymbol{\mathcal{P}}_0$ be the initial covariance matrix of the system states $\boldsymbol{\eta}$. Then, the optimal Kalman filter is given by (7), where the initial condition for the differential equation (5) is given by

$$\mathbf{P}_2(t_0) := \mathbf{T}_c(t_0)\boldsymbol{\mathcal{P}}_0\mathbf{T}_c^T(t_0).$$

Proof. The augmented system corresponding to the generalized dynamics, as depicted in Fig. 3, can be written as

$$\begin{cases} \dot{\boldsymbol{\eta}}(t) = \boldsymbol{\mathcal{A}}(t)\boldsymbol{\eta}(t) + \boldsymbol{\mathcal{B}}_p(t)\mathbf{u}(t) + \boldsymbol{\mathcal{B}}(t)\mathbf{w}(t) \\ \boldsymbol{\psi}(t) = \boldsymbol{\mathcal{C}}(t)\boldsymbol{\eta}(t) + \boldsymbol{\mathcal{D}}(t)\mathbf{w}(t) \end{cases},$$

where $\boldsymbol{\mathcal{A}}(t)$, $\boldsymbol{\mathcal{B}}_p(t)$, and $\boldsymbol{\mathcal{C}}(t)$ are as previously defined, $\boldsymbol{\mathcal{B}}(t) = \mathbf{T}_c^T(t)\mathbf{B}$, and $\boldsymbol{\mathcal{D}}(t) = \mathbf{R}^T(t)\mathbf{D}$. The optimal Kalman filter has the structure of (7). The corresponding differential Riccati equation is given by

$$\begin{aligned} \dot{\boldsymbol{\mathcal{P}}}_2(t) = {} & \boldsymbol{\mathcal{A}}_e(t)\boldsymbol{\mathcal{P}}_2(t) + \boldsymbol{\mathcal{P}}_2(t)\boldsymbol{\mathcal{A}}_e^T(t) + \boldsymbol{\mathcal{V}}_{xx}(t) \\ & - \boldsymbol{\mathcal{P}}_2(t)\boldsymbol{\mathcal{C}}^T(t)\mathbf{V}_{yy}^{-1}(t)\boldsymbol{\mathcal{C}}(t)\boldsymbol{\mathcal{P}}_2(t) - \boldsymbol{\mathcal{V}}_{xy}(t)\boldsymbol{\mathcal{V}}_{yy}^{-1}(t)\boldsymbol{\mathcal{V}}_{xy}^T(t), \end{aligned} \tag{9}$$

with $\boldsymbol{\mathcal{P}}_2(t_0) = \boldsymbol{\mathcal{P}}_0$, $\boldsymbol{\mathcal{A}}_e(t) = \boldsymbol{\mathcal{A}}(t) - \boldsymbol{\mathcal{V}}_{xy}(t)\boldsymbol{\mathcal{V}}_{yy}^{-1}(t)\boldsymbol{\mathcal{C}}(t)$, and

$$\boldsymbol{\mathcal{V}}(t) := \begin{bmatrix} \boldsymbol{\mathcal{B}}(t) \\ \boldsymbol{\mathcal{D}}(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{B}}^T(t) \, \boldsymbol{\mathcal{D}}^T(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{V}}_{xx}(t) & \boldsymbol{\mathcal{V}}_{xy}(t) \\ \boldsymbol{\mathcal{V}}_{xy}^T(t) & \boldsymbol{\mathcal{V}}_{yy}(t) \end{bmatrix}.$$

It is straightforward to show that

$$\boldsymbol{\mathcal{P}}_2(t) := \mathbf{T}_c^T(t)\mathbf{P}_2(t)\mathbf{T}_c(t)$$

satisfies the differential Riccati equation (9). The Kalman gain follows as

$$\boldsymbol{\mathcal{K}}_2(t) = \left[\boldsymbol{\mathcal{P}}_2(t)\boldsymbol{\mathcal{C}}^T(t) + \boldsymbol{\mathcal{V}}_{xy}\right]\boldsymbol{\mathcal{V}}_{yy}^{-1}(t),$$

which gives (8) after a few algebraic simplifications.

Remark 1. The proposed Kalman filter covariance matrix has a limit solution, although the system at hand is not LTI. Indeed, as t approaches infinity, $\mathbf{P}_2(t)$ converges to the solution $\mathbf{P}_2^\infty$ of the matrix Riccati equation

$$\mathbf{A}_e\mathbf{P}_2^\infty + \mathbf{P}_2^\infty\mathbf{A}_e^T - \mathbf{P}_2^\infty\mathbf{C}^T\mathbf{V}_{yy}^{-1}\mathbf{C}\mathbf{P}_2^\infty + \mathbf{V}_{xx} - \mathbf{V}_{xy}\mathbf{V}_{yy}^{-1}\mathbf{V}_{xy}^T = \mathbf{0}.$$

Thus, as t approaches infinity, the covariance matrix converges to the limit solution

$$\lim_{t\to\infty}\boldsymbol{\mathcal{P}}_2(t) = \mathbf{T}_c^T(t)\mathbf{P}_2^\infty\mathbf{T}_c(t).$$

4. $\mathcal{H}_\infty$ OPTIMAL FILTER

This section introduces the $\mathcal{H}_\infty$ optimal filter for the class of dynamic systems (1). To that purpose, consider Fig. 4, which depicts the general $\mathcal{H}_\infty$ filtering framework for the class of systems addressed in the paper. In the figure $\mathbf{w}(t) = \left[\mathbf{w}_1^T(t)\,\mathbf{w}_2^T(t)\right]^T$ represents the generalized disturbance vector, assumed to be square integrable, i.e., $\mathbf{w} \in \mathcal{L}_2$. The dynamic systems $\mathcal{W}_d$ and $\mathcal{W}_n$ denote, once again, weights that shape both the system disturbances and the noise of the sensors. The matrix $\mathbf{L}_\infty$ weights the states and defines the performance variable

$$\boldsymbol{\zeta}(t) := \mathbf{L}_\infty\mathbf{T}(t)\boldsymbol{\eta}_p = \boldsymbol{\mathcal{L}}(t)\boldsymbol{\eta},$$

with $\boldsymbol{\mathcal{L}}(t) := \mathbf{L}\mathbf{T}_c(t)$, $\mathbf{L} := [\mathbf{L}_\infty\,\mathbf{0}\,\mathbf{0}]$. The goal is to design a filter to obtain an estimate $\hat{\boldsymbol{\zeta}}(t)$ of $\boldsymbol{\zeta}(t)$, using the measurements $\boldsymbol{\psi}(t)$, that minimizes

$$J_\infty := \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_0,\mathbf{w})\in\mathbb{R}^{3N}\times\mathcal{L}_2}\frac{\left\|\boldsymbol{\zeta}-\hat{\boldsymbol{\zeta}}\right\|^2}{\|\mathbf{w}\|^2 + \boldsymbol{\eta}_0^T\mathbf{R}_0\boldsymbol{\eta}_0}$$

with $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$, $\mathbf{R}_0 = \mathbf{R}_0^T \succ \mathbf{0}$. The solution to

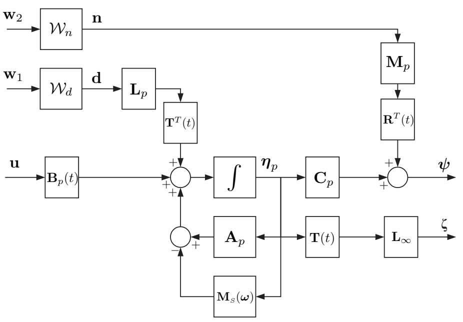


Fig. 4. $\mathcal{H}_\infty$ filter design setup

this problem is well known (see Nagpal and Khargonekar (1991)) and, as the design follows similar steps as for the Kalman filter, previously derived in detail, only the final result is summarized in the next theorem.

Theorem 2. Consider the generalized dynamics presented in Fig. 4, where $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{L}_2$. There exists a filter such that $J_\infty < \gamma^2$ if and only if there exits a symmetric positive definite matrix $\mathbf{P}_\infty(t)$ that satisfies

$$\dot{\mathbf{P}}_\infty(t) = \mathbf{A}_e\mathbf{P}_\infty(t) + \mathbf{P}_\infty(t)\mathbf{A}_e^T - \mathbf{P}_\infty(t)\mathbf{C}^T\mathbf{V}_{yy}^{-1}\mathbf{C}\mathbf{P}_\infty(t) + \frac{1}{\gamma^2}\mathbf{P}_\infty(t)\mathbf{L}^T\mathbf{L}\mathbf{P}_\infty(t) + \mathbf{V}_{xx} - \mathbf{V}_{xy}\mathbf{V}_{yy}^{-1}\mathbf{V}_{xy}^T,$$

with $\mathbf{P}_\infty(t_0) = \mathbf{T}_c(t_0)\mathbf{R}_0^{-1}\mathbf{T}_c^T(t_0)$, and such that the unforced linear time varying system

$$\dot{\mathbf{p}}_\infty(t) = \left[\mathbf{A}_e - \mathbf{P}_\infty(t)\left(\mathbf{C}^T\mathbf{V}_{yy}^{-1}\mathbf{C} - \frac{1}{\gamma^2}\mathbf{L}^T\mathbf{L}\right)\right]\mathbf{p}_\infty(t) \qquad (10)$$

is exponentially stable. Moreover, the filter structure is equal to the structure of the Kalman filter, with gain

$$\boldsymbol{\mathcal{K}}_\infty(t) = \left[\boldsymbol{\mathcal{P}}_\infty(t)\boldsymbol{\mathcal{C}}^T(t) + \boldsymbol{\mathcal{V}}_{xy}(t)\right]\boldsymbol{\mathcal{V}}_{yy}^{-1}(t),$$

where

$$\boldsymbol{\mathcal{P}}_\infty(t) = \mathbf{T}_c^T(t)\mathbf{P}_\infty(t)\mathbf{T}_c(t). \qquad (11)$$

Proof. It is straightforward to show that (11) is the solution of the differential Riccati equation

$$\begin{aligned}\dot{\boldsymbol{\mathcal{P}}}_\infty(t) &= \boldsymbol{\mathcal{A}}_e(t)\boldsymbol{\mathcal{P}}_\infty(t) + \boldsymbol{\mathcal{P}}_\infty(t)\boldsymbol{\mathcal{A}}_e^T(t) + \boldsymbol{\mathcal{V}}_{xx}(t)\\ &- \boldsymbol{\mathcal{P}}_\infty(t)\boldsymbol{\mathcal{C}}^T(t)\mathbf{V}_{yy}^{-1}(t)\boldsymbol{\mathcal{C}}(t)\boldsymbol{\mathcal{P}}_\infty(t) - \boldsymbol{\mathcal{V}}_{xy}(t)\boldsymbol{\mathcal{V}}_{yy}^{-1}(t)\boldsymbol{\mathcal{V}}_{xy}^T(t)\\ &+ \frac{1}{\gamma^2}\boldsymbol{\mathcal{P}}_\infty(t)\boldsymbol{\mathcal{L}}^T(t)\boldsymbol{\mathcal{L}}(t)\boldsymbol{\mathcal{P}}_\infty(t),\end{aligned}$$

with $\boldsymbol{\mathcal{P}}_\infty(t_0) = \mathbf{R}_0^{-1}$. Moreover, if the LTV system (10) is exponentially stable, so is the LTV unforced system

$$\dot{p}_\infty(t) = \left[\boldsymbol{\mathcal{A}}_e(t) - \boldsymbol{\mathcal{P}}_\infty(t)\left(\boldsymbol{\mathcal{C}}^T(t)\boldsymbol{\mathcal{V}}_{yy}^{-1}(t)\boldsymbol{\mathcal{C}}(t) - \frac{1}{\gamma^2}\boldsymbol{\mathcal{L}}^T(t)\boldsymbol{\mathcal{L}}(t)\right)\right]p_\infty(t),$$

as $p_\infty(t) = \mathbf{T}_c^T(t)\mathbf{p}_\infty(t)$, which is a Lyapunov coordinate transformation (see Brockett (1970)). This suffices to complete the proof, see Nagpal and Khargonekar (1991) for details.

Remark 2. As with the Kalman filter, the $\mathcal{H}_\infty$ optimal filter also achieves a limit solution. Indeed, as discussed in Nagpal and Khargonekar (1991), as t approaches infinity, $\mathbf{P}_\infty(t)$ converges to the solution $\mathbf{P}_\infty^\infty$ of the Riccati equation

$$\mathbf{A}_e\mathbf{P}_\infty^\infty + \mathbf{P}_\infty^\infty\mathbf{A}_e^T - \mathbf{P}_\infty^\infty\mathbf{C}^T\mathbf{V}_{yy}^{-1}\mathbf{C}\mathbf{P}_\infty^\infty + \frac{1}{\gamma^2}\mathbf{P}_\infty^\infty\mathbf{L}^T\mathbf{L}\mathbf{P}_\infty^\infty + \mathbf{V}_{xx} - \mathbf{V}_{xy}\mathbf{V}_{yy}^{-1}\mathbf{V}_{xy}^T = \mathbf{0}.$$

Thus,

$$\lim_{t\to\infty}\boldsymbol{\mathcal{P}}_\infty(t) = \mathbf{T}_c^T(t)\mathbf{P}_\infty^\infty\mathbf{T}_c(t)$$

and the filter asymptotically converges to the corresponding linear time varying filter that is obtained from the linear time invariant $\mathcal{H}_\infty$ filtering solution when the initial condition is known.

Remark 3. If $\mathbf{L}$ commutes with $\mathbf{T}(t)$, then the $\mathcal{H}_\infty$ performance variable can be rewritten as $\boldsymbol{\zeta}(t) = \mathbf{L}\boldsymbol{\eta}_p(t)$, which weights directly the system states. This is often the case, e.g., when $\mathbf{L}$ is of the form

$$\mathbf{L} = [l_1\mathbf{I}_3\ \ldots\ l_N\mathbf{I}_3],\ l_i \in \mathbb{R},\ i = 1,\ldots,N,$$

the natural form to equally weight each set of states $\boldsymbol{\eta}_i$.

5. EXTENSIONS AND PROPERTIES

The class of systems proposed in Section 2 has some constraints that can be lessened. The system matrix $\mathbf{A}_p$, previously defined as (2), may be of the more general form

$$\mathbf{A}_p = \begin{bmatrix}\mathbf{0} & \gamma_{12}\mathbf{I} & \ldots & \ldots & \gamma_{1N}\mathbf{I}\\ \gamma_{21}\mathbf{I} & \ddots & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \gamma_{N-1,N}\mathbf{I}\\ \gamma_{N1}\mathbf{I} & \ldots & \ldots & \gamma_{N,N-1}\mathbf{I} & \mathbf{0}\end{bmatrix},$$

as long as the appropriate stabilizability and detectability conditions remain untouched. This broadens the applicability of this work. It is also easy to see that the output matrix $\mathbf{C}_p$ admits more general forms. Finally, the problem could have been defined for state variables $\boldsymbol{\eta}_i(t)$ living in a space of different dimension, e.g., $\boldsymbol{\eta}_i(t) \in X_i \subseteq \mathbb{R}^2, i = 1, \ldots, N$.

The following theorem addresses the robustness of the proposed filters with respect to disturbances in the rotation matrix $\mathbf{R}$ and the vector $\boldsymbol{\omega}$. This is of particular interest as these may be quantities provided by external sensor suites, which is the case of the example provided in the paper.

Theorem 3. Suppose that $\boldsymbol{\omega}$ and $\mathbf{R}$ are only known up to some error, i.e., the filters operate with $\boldsymbol{\omega}_m = \boldsymbol{\omega} + \tilde{\boldsymbol{\omega}}$ and $\mathbf{R}_m = \left[\mathbf{I} + \mathbf{S}(\tilde{\boldsymbol{\lambda}})\right]$, where $\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{\lambda}}$ parameterize the errors. Further assume that $\boldsymbol{\eta}$ remains bounded for all t. Then, the estimation errors of both the Kalman and the $\mathcal{H}_\infty$ filters are locally input-to-state stable from the input $(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}})$.

Proof. Since the structure of both filters is identical, the proof is presented for the Kalman filter. If the filter operates with $\boldsymbol{\omega}_m$ and $\mathbf{R}_m$, the dynamics of the filter error can be written as

$$\dot{\tilde{\boldsymbol{\eta}}} = \mathbf{f}\left(t, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right), \tag{12}$$

with

$$\begin{aligned}
\mathbf{f}\left(t, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right) = \; & \left[\boldsymbol{\mathcal{A}}(t) - \boldsymbol{\mathcal{K}}_2(t)\boldsymbol{\mathcal{C}}(t)\right]\tilde{\boldsymbol{\eta}}(t) \\
& - \tilde{\boldsymbol{\mathcal{A}}}\left(t, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right)\hat{\boldsymbol{\eta}}(t) \\
& + \left[\mathbf{T}_c^T(t) + \tilde{\mathbf{T}}_c^T(t, \tilde{\boldsymbol{\lambda}})\right]\mathbf{K}_2(t)\mathbf{R}(t)\left[\mathbf{I} + \mathbf{S}(\tilde{\boldsymbol{\lambda}})\right]\tilde{\boldsymbol{\mathcal{C}}}(t, \tilde{\boldsymbol{\lambda}})\hat{\boldsymbol{\eta}}(t) \\
& - \tilde{\mathbf{T}}_c^T(t, \tilde{\boldsymbol{\lambda}})\mathbf{K}_2(t)\mathbf{R}(t)\left[\mathbf{I} + \mathbf{S}(\tilde{\boldsymbol{\lambda}})\right]\tilde{\boldsymbol{\mathcal{C}}}(t, \tilde{\boldsymbol{\lambda}})\tilde{\boldsymbol{\eta}}(t) \\
& - \mathbf{T}_c^T(t)\mathbf{K}_2(t)\mathbf{R}(t)\mathbf{S}(\tilde{\boldsymbol{\lambda}})\tilde{\boldsymbol{\mathcal{C}}}(t, \tilde{\boldsymbol{\lambda}})\tilde{\boldsymbol{\eta}}(t),
\end{aligned}$$

where $\tilde{\mathbf{T}}_c(t, \tilde{\boldsymbol{\lambda}}) = \operatorname{diag}\left(\mathbf{T}(\mathrm{t})\mathbf{M}_{\mathrm{S}}(\tilde{\lambda}), \mathbf{0}, \mathbf{0}\right)$,

$$\tilde{\boldsymbol{\mathcal{A}}}\left(t, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right) = \begin{bmatrix} -\mathbf{M}_S(\tilde{\boldsymbol{\omega}}) & \mathbf{T}(t)\mathbf{M}_S(\tilde{\boldsymbol{\lambda}})\mathbf{L}_p\mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $\tilde{\boldsymbol{\mathcal{C}}}(t, \tilde{\boldsymbol{\lambda}}) = \left[\mathbf{0} | \mathbf{0} | - \mathbf{S}(\tilde{\boldsymbol{\lambda}})\mathbf{R}^T(t)\mathbf{M}_p\mathbf{C}_n\right]$. The function $\mathbf{f}\left(t, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right)$ is continuously differentiable and, assuming that $\boldsymbol{\eta}$ remains bounded for all t, and as $\boldsymbol{\omega}(t)$ is also assumed to be a bounded function of t, it follows that the Jacobian matrices $[\partial \mathbf{f} / \partial \tilde{\boldsymbol{\eta}}]$ and $\left[\partial \mathbf{f} / \partial \left(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\lambda}}\right)\right]$ are bounded, uniformly in t. As the system $\dot{\tilde{\boldsymbol{\eta}}} = \mathbf{f}\left(t, \tilde{\boldsymbol{\eta}}, \mathbf{0}, \mathbf{0}\right)$ has a uniformly asymptotically stable equilibrium point at the origin $\tilde{\boldsymbol{\eta}} = \mathbf{0}$, then the system (12) is locally input-to-state stable (Lemma 5.4, Khalil (1996)).

6. EXAMPLE OF APPLICATION

This section presents an example of practical interest in marine applications that demonstrates the applicability of the proposed methodologies.

Consider an autonomous surface craft (ASC) equipped with a GPS (Global Positioning System) that measures the position of the vehicle with respect to an inertial coordinate frame $\{I\}$, an AHRS (Attitude and Heading Reference System), which provides the attitude and the angular velocity of the vehicle, and a Doppler velocity log that gives the velocity of the vehicle relative to the water. Assume that the vehicle is moving in the presence of constant unknown ocean currents. The problem here considered is that of estimate the velocity of the ocean current as well as the filtered vehicle position.

In order to properly cast the estimation problem into the class of systems addressed in the paper, consider a fixed mission reference point placed arbitrarily in the mission scenario, denoted by ${}^I(\mathbf{r})$ in inertial coordinates. Denote also by ${}^I(\mathbf{p})$ the position of the vehicle in inertial coordinates, as provided by the GPS. Then, the position of the mission reference point with respect to the origin of the body-fixed coordinate system $\{B\}$, described in body-fixed coordinates, can be written as $\mathbf{r} = \mathbf{R}^T \left[{}^I(\mathbf{r}) - {}^I(\mathbf{p})\right]$, where $\mathbf{R}$ is the rotation matrix from body-fixed coordinates to inertial coordinates. The time derivative of $\mathbf{r}$ can be written as

$$\dot{\mathbf{r}} = -\mathbf{v}_r - \mathbf{v}_c - \mathbf{S}\left(\boldsymbol{\omega}\right),$$

where $\mathbf{v}_r$ is the velocity of the vehicle relative to the water, $\mathbf{v}_c$ is the velocity of the ocean current, and $\boldsymbol{\omega}$ is the angular velocity of the vehicle, all expressed in body-fixed coordinates. Since the ocean current is assumed to be constant in the inertial frame, the time derivative of $\mathbf{v}_c$ is simply given by $\dot{\mathbf{v}}_c = -\mathbf{S}\left(\boldsymbol{\omega}\right)\mathbf{v}_c$. Clearly, the problem of estimating of $\mathbf{r}$ and $\mathbf{v}_c$ falls into the class of problems addressed in the paper, with $\boldsymbol{\eta}_1 = \mathbf{r}$, $\boldsymbol{\eta}_2 = \mathbf{v}_c$,

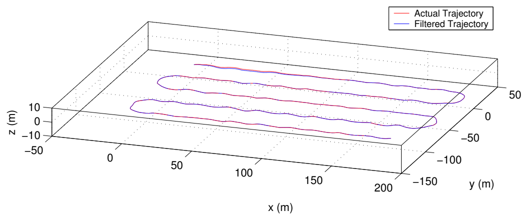
$$\mathbf{A}_p = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_p(t) = \begin{bmatrix} -\mathbf{I}_3 \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{u} = \mathbf{v}_r$. Thus, it is possible to design a position and velocity Navigation filter using the methodology introduced in the previous sections and, once an estimate of $\mathbf{r}$ is obtained, an estimate of the position of the vehicle is given by

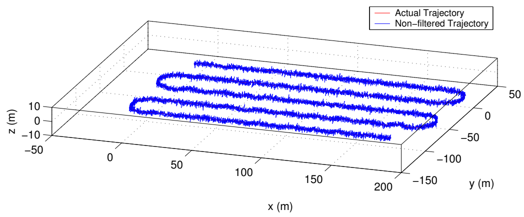
$${}^I(\mathbf{p}) = {}^I(\mathbf{r}) - \mathbf{R}\mathbf{r}.$$

To illustrate the performance of the proposed Navigation system a set of simulations was carried out with a simplified model of the ASC DELFIM, Alves et al. (2006). In the simulations the GPS noise was assumed Gaussian with standard deviation of 1 m. In addition, the measurements of the vehicle velocity relative to the water and the vehicle angular velocity were assumed to be corrupted by Gaussian noise, with standard deviations of 0.01 m/s and 0.02 °/s, respectively. The system disturbances and sensors noise weight matrix transfer functions were chosen as $\mathbf{W}_d(s) = 0.01\mathbf{I}_6$ and $\mathbf{W}_n(s) = \mathbf{I}_3$, respectively, and the filter initial conditions were selected as to reflect the a priori knowledge of the position of the vehicle. Due to the lack of space only the Kalman filter results are presented and discussed.

The vehicle described a typical surveying trajectory, which is shown in Fig. 5, along with the estimated trajectory and the non-filtered trajectory as provided by the GPS. The filter error variables are shown in Fig. 6, where the initial transients arise due to the mismatch of the initial conditions of the filter states and can be seen as a Navigation filter warming up time. Notice that the



(a) Estimated trajectory



(b) Non-filtered trajectory

Fig. 5. Trajectory described by the vehicle

magnitude of these transients can be highly reduced by proper initialization of the filter state variables. The filter

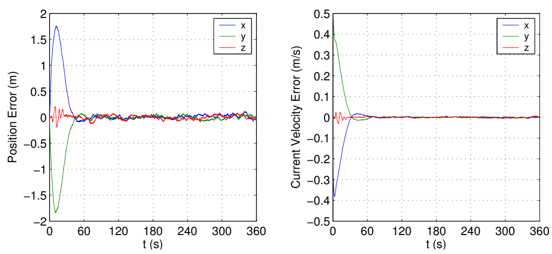


Fig. 6. Evolution of the Kalman filter error variables

error variables are depicted in greater detail in Fig. 7. From the figures it can be concluded that the noise of the sensors is highly attenuated by the proposed Navigation System, producing very accurate estimates of the velocity of the current and the position of the vehicle.

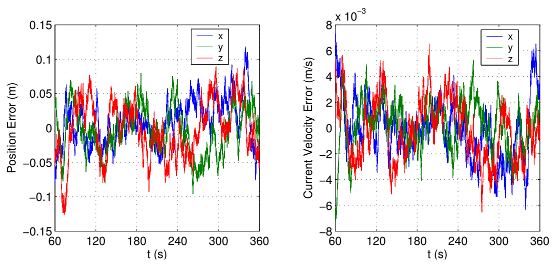


Fig. 7. Detailed evolution of the Kalman filter error variables

7. CONCLUSIONS

This paper presented optimal filtering results for a class of kinematic systems with application to the design of Integrated Navigation Systems. At the core of the proposed design techniques there is a time varying orthogonal Lyapunov coordinate transformation that renders the system dynamics linear time invariant (LTI). Resorting to classic optimal filtering techniques, filters were derived in this new LTI coordinate space. Afterwards, the resulting estimation solutions were converted back to the original coordinate space, yielding interpretations for the different proposed structures. The design includes frequency weights to shape both the system disturbances and the noise of the sensors. A brief case study of practical interest in marine applications was presented along with simulations that clearly illustrate the usefulness and performance of the proposed filtering solutions. Other applications can be foreseen in the design of Navigation Systems for other classes of vehicles, including, e.g., in-door and aerospace vehicles.

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