AUTOMATIC CONTROL AND SYSTEM THEORY

CONTROLLABILITY & OBSERVABILITY

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Controllability and Reachability

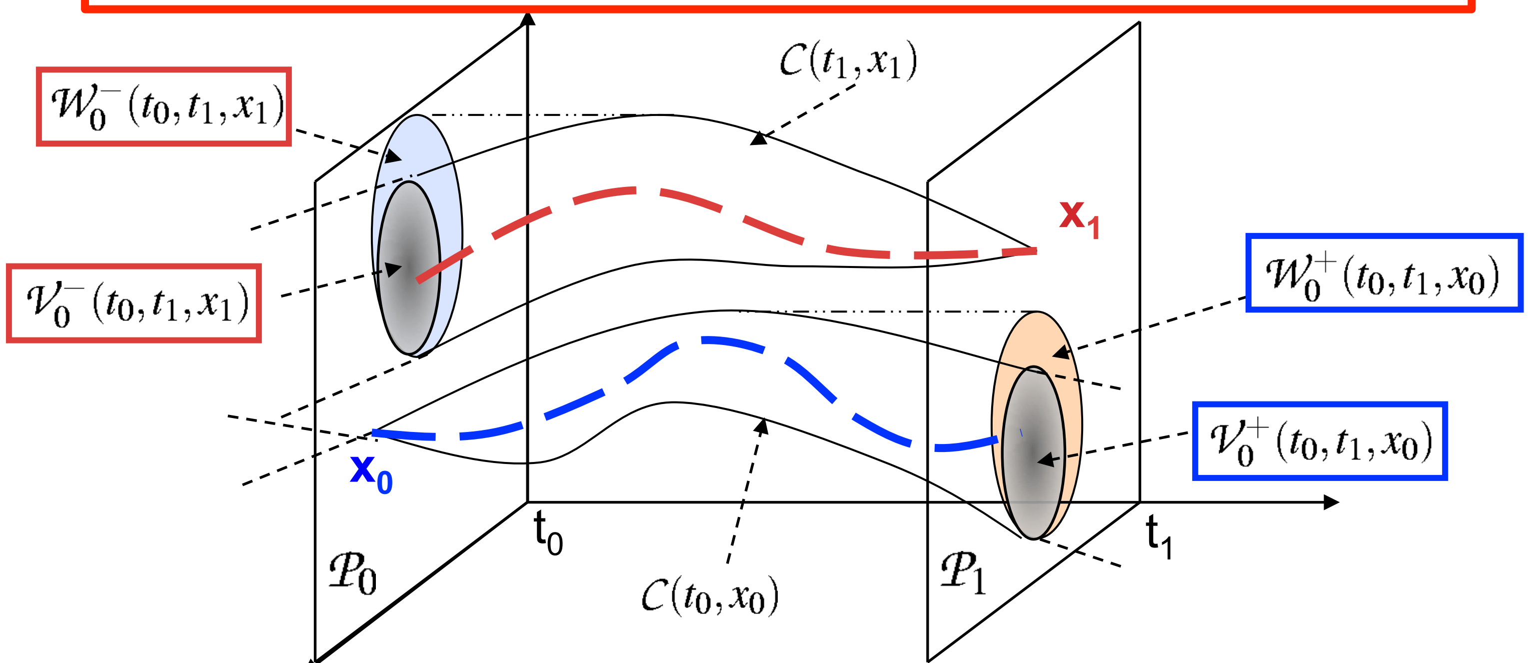
Controllability and Reachability

Set of *states reachable* *at time t_1 starting from state x_0 at time instant t_0*

$$\boxed{\mathcal{V}_0^+(t_0,t_1,x_0) \;=\; \{x_1 : x_1 = \varphi(t_1,t_0,x_0,u(\cdot)),\; u(\cdot)\in\mathcal{U}_f\}}$$

Set of *states controllable* *to state x_1 at time t_1 from time instant t_0*

$$\boxed{\mathcal{V}_0^-(t_0,t_1,x_1) \;=\; \{x_0 : x_1 = \varphi(t_1,t_0,x_0,u(\cdot)),\; u(\cdot)\in\mathcal{U}_f\}}$$



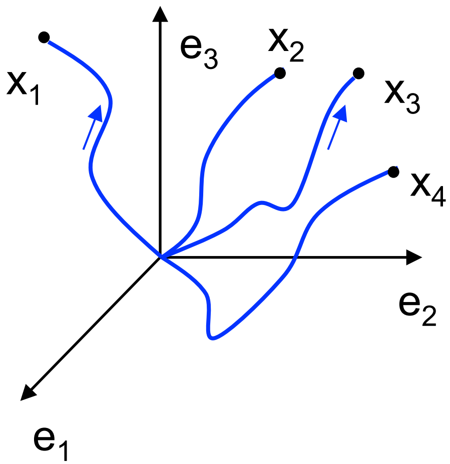
Controllability and Reachability

The linear, time invariant MIMO system (continuous- or discrete-time):

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad \left(\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned} \right)$$

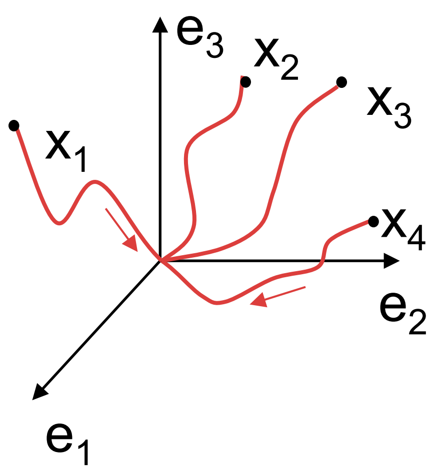
is completely reachable if, starting from the origin, every state can be reached in a finite amount of time with a proper input control action

$$\mathcal{V}_0^+(t_0, t_1, x_0) = \mathbb{R}^n$$



is completely controllable if, starting from any state, it is possible to reach the origin in a finite amount of time by applying a proper input control action

$$\mathcal{V}_0^-(t_0, t_1, x_0) = \mathbb{R}^n$$



Controllability and Reachability

Consider a discrete-time system

$$x(k+1) \;=\; A\,x(k)+B\,u(k), \qquad x(0)=0$$

then:

$$\begin{aligned}
x(1) &= B\,u(0) \\
x(2) &= Ax(1)+Bu(1) = ABu(0)+Bu(1) \\
x(3) &= Ax(2)+Bu(2) = A^2Bu(0)+ABu(1)+Bu(2) \\
&\dots \\
x(k) &= \sum_{i=0}^{k-1} A^{k-i-1}Bu(i) = [B\;\; AB\;\; A^2B\;\; \dots\;\; A^{k-1}B]\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}
\end{aligned}$$

State reachable in k steps from "0"

- By changing the sequence u(0), u(1), … u(k-1) we obtain the set of states reachable in k steps

$$\mathcal{V}_k^+(0) = im\{P_1(k)\}$$

$$P_1(k) = [B\;\; AB\;\; A^2B\;\; \dots\;\; A^{k-1}B]$$

Reachable states:
a subspace of R^n

Controllability and Reachability

- Example:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ a & a_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$im\{P_1(k)\} = im\{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & a_2 & \dots & a_2^{k-1} \end{bmatrix}\} = span\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

- If

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \mathcal{V}_2^+\{0\} = im\{\begin{bmatrix} 1 & a_1 \\ 0 & a \end{bmatrix}\} = \mathbb{R}^2$$

- Note that in this case rank{V_2^+} = 2

Controllability and Reachability

Note that the subspaces $V^+_k\{0\}$ satisfy the sequence

$$\mathcal{V}_1^+ \subseteq \mathcal{V}_2^+ \subseteq \ldots \subseteq \mathcal{V}_k^+$$

This sequence, because of the Cayley-Hamilton Theorem, is stationary for k >= n steps, where n is the dimension of the state space.

$$\mathcal{V}_1^+\{0\} \subseteq \mathcal{V}_2^+\{0\} \subseteq \ldots \mathcal{V}_n^+\{0\} = \mathcal{V}_{n+1}^+\{0\}$$

Then:

→ *If a state is reachable, it can be reached in n steps at the most.*

Note that the inclusion sequence for $V^+_k\{0\}$ can stop for k < n.
We define as *the reachability index* the smallest integer r such that

rank $\{[B, \; A B, A^2 B, \ldots A^{r-1} B]\} = n$

We define matrix P_1

$$P_1 = [B, \; A B, A^2 B, \ldots A^{n-1} B]$$

as the *reachability matrix.*

Controllability and Reachability

Notice that the reachability index is less than *n* (i.e. *r* < *n*) if the system has more than an input, and at least two inputs are independent, i.e. that matrix B has at least two independent columns. In other words, if rank{B} = m, then r <= n-m+1.

Moreover:

1. V^+ is the smallest A-invariant subspace containing im{B}

   $A\ P_1 \subseteq V^+ \qquad \mathrm{im}\{B\} \subseteq V^+$

2. The subspace V^+ is invariant with respect to changes of basis in the state space:

$$x = T\ \bar{x} \quad \longrightarrow \quad \mathcal{V}^+ = T\ \bar{\mathcal{V}}^+$$

$$A' = T^{-1}AT$$

$$B' = T^{-1}B$$

$$\begin{aligned}
P_1 &= [B,\ AB,\ A^2B, \dots A^{n-1}B] \\
P_1' &= [B', A'B', A'^2B', \dots A'^{n-1}B'] \\
&= [T^{-1}B,\ \ T^{-1}AT\,T^{-1}B,\ \ T^{-1}A^2T\,T^{-1}B, \dots T^{-1}A^{n-1}T\,T^{-1}B] \\
&= [T^{-1}B,\ \ T^{-1}AB,\ \ T^{-1}A^2B, \dots T^{-1}A^{n-1}B] \\
&= T^{-1}[B,\ \ AB,\ \ A^2B, \dots A^{n-1}B] \\
&= T^{-1}P_1
\end{aligned}$$

$$\longrightarrow \quad P_1 = TP_1'$$

Controllability and Reachability

Let us consider the discrete-time system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

A state x(0) is controllable to "0" in k steps if:

$$x(k) = 0 \;=\; A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i) \quad \rightarrow \quad -A^k x(0) = \sum_{i=0}^{k-1} A^{k-i-1} B u(i)$$

The state x(0) is controllable in k steps if the state $-A^k$ x(0) is reachable in k steps from the state 0

$$A^k x \subseteq im\{[B \;\; AB \;\;\; A^2 B \;\;\; \dots \; A^{k-1} B]\} \;=\; \mathcal{V}_k^+(0)$$

equivalent to: $$im\{A^k\} \subseteq im\{[B \;\; AB \;\;\; A^2 B \;\;\; \dots \; A^{k-1} B]\} \;=\; \mathcal{V}_k^+(0)$$

Set of states controllable in k steps:

$$\mathcal{V}_k^- = \{x \,:\, A^k x \in \mathcal{V}_k^+\} \quad \rightarrow$$

Controllable states:
a subspace of R^n

Controllability and Reachability

Let us consider the discrete-time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

Theorem: If $P_1 = [B,\ A\,B,\ A^2\,B,\ \ldots\ A^{n-1}\,B]$, the set of reachable states V^+ in any finite interval of time is

$$\mathcal{V}^+ = \mathcal{R} = im\{P_1\}$$

The set of controllable states V^- in any finite time interval is

$$\mathcal{V}^- = A^{-n} im\{P_1\}$$

$$\longrightarrow \quad im\{A^n\} \subseteq im\{P_1\}$$

The system is completely reachable iff $im\{P_1\} = R^n$, that is if P_1 is full rank.

- Reachability implies controllability (if $im\{P_1\} = R^n$, then it contains $im\{A^n\}$).
- Controllability does not imply reachability (if $dim[im\{A^n\}] < n$).
- These conditions are equivalent if and only of A is full rank.

Controllability and Reachability

Given the continuous-time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

Theorem: If $P_1 = [B,\ A B,\ A^2 B,\ \dots\ A^{n-1} B]$, the set of the reachable states V^+ in any finite interval of time is

$$\mathcal{V}^+ = \mathcal{R} = im\{P_1\}$$

The set of controllable states V^- in any finite time interval is

$$\mathcal{V}^- = \mathcal{R} = im\{P_1\}$$

Therefore, the system is completely controllable and reachable if and only if

$$im\{P_1\} = \mathbb{R}^n$$

i.e. if P_1 is full rank.

- $\mathcal{R}$ is the minimum A-invariant subspace containing im{B}
- Matrix P_1 is the so-called reachability matrix

Controllability and Reachability

It is possible to prove that V^- , the subspace of the states controllable to the origin in any limited interval of time:

- for continuous-time systems coincides with $\mathcal{R}$ (controllability = reachability)
- for discrete-time systems in general contains $\mathcal{R}_d$ and coincides with $\mathcal{R}_d$ if A_d is non-singular.

Controllability and Reachability

For discrete-time systems, the reachability and controllability properties are not equivalent.

Reachability implies controllability: in fact

$$Im\{[B, AB, A^2B, \cdots, A^{n-1}B]\} = \mathcal{V}_n^+ = \mathbb{R}^n$$

From which

$$Im\{A^n\} \subseteq Im\{[B, AB, A^2B, \cdots, A^{n-1}B]\} = \mathcal{V}^+ = \mathbb{R}^n$$

Controllability does not imply reachability: for example assume that A = 0 and that rank{B} < n, then

$$Im\{A^n\} \subseteq Im\{P_1\}$$

$$rank\{[B, AB, A^2B, \cdots, A^{n-1}B]\} = rank\{[B, 0, 0, \cdots, 0]\} < n$$

If *rank{A}* = *n*, then the two properties are equivalent.

Controllability and Reachability

The main difference between reachability and controllability for discrete-time systems is that a state x(0) can be controlled to zero also without applying any input. This is due to the fact that if A is not full rank, it has at least a null eigenvalues, and therefore the corresponding eigenspace tends "naturally" to zero in a finite number of steps.

Example:

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) \qquad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; x(1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; x(2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; x(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Controllability and Reachability

Example: Let consider a diagonalizable system with a single input.

$$x(k+1) = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ & & \vdots & \\ & & & \lambda_n \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(k)$$

In this case:

$$P_1 = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \dots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \dots & \lambda_2^{n-1} b_2 \\ \vdots & & & & \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \dots & \lambda_n^{n-1} b_n \end{bmatrix} = \begin{bmatrix} b_1 & 0 & & \\ 0 & b_2 & & \\ & & \vdots & \\ & & & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & & & & \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}$$

Therefore the system is completely reachable iff:

1) All the elements b_i are not null
2) All eigenvalues are not null

Controllability and Reachability

Example: Let consider the system:

$$x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$$

In this case:

$$\mathcal{V}_1^+ = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{V}_2^+ = \mathcal{V}_3^+ = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The system is controllable in two steps, being

$$Im\{A^2\} = Im\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

However, the system is not controllable in one step, since:

$$Im\{A\} = Im\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \not\subseteq span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = \mathcal{V}_1^+$$

Controllability and Reachability

Example: Let consider the system:

$$x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$$

The subspaces V^-_1 and V^-_2 are:

$$\mathcal{V}_1^- = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{V}_2^- = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note that V^+_1 and V^-_1 do not coincide!

Controllability and Reachability

Example

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [B,\ AB,\ A^2B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

System non fully controllable; The controllable subspace is defined by $x = [1, 0, 0]^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [B,\ AB,\ A^2B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

Fully controllable system

Controllability for discrete-time systems

- **Problem:** Given a discrete-time system, compute the input sequence *u(0), u(1), …, u(k-1)* in order to reach the state *x(k)* from *x(0).*

- Necessary and sufficient condition for the problem to have a solution is that

$$x(k) - A^k x(0) \in im\{[B \;\; AB \;\; \dots \;\; A^{k-1}B]\} = \mathcal{V}_k^+$$

that is that the state *x(k)- A^k x (0)* is reachable from zero in *k* steps. If this is the case, then the solution is given by the n equations:

$$x(k) - A^k x(0) = [B \;\; AB \;\; \dots \;\; A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

in the (k x m) unknowns:

$$\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix} = \begin{bmatrix} u_1(k-1) \\ \dots \\ u_m(k-1) \\ \dots \\ u_1(0) \\ \dots \\ u_m(0) \end{bmatrix}$$

In general, the solution is not unique (n equations in k x m unknowns):

$$u = \bar{u} + v, \qquad v : P_1 v = 0$$

$$\bar{u} = P_1^+ (x(k) - A^k x(0))$$

Minimum norm solution

Controllability for discrete-time systems

- Since there are more solutions, it is possible to choose the "best one" according to some criteria.

For example, the minimum norm solution minimizes the control energy necessary to achieve the desires state transition

$$\bar{u} = P_1^+ (x(k) - A^k x(0))$$

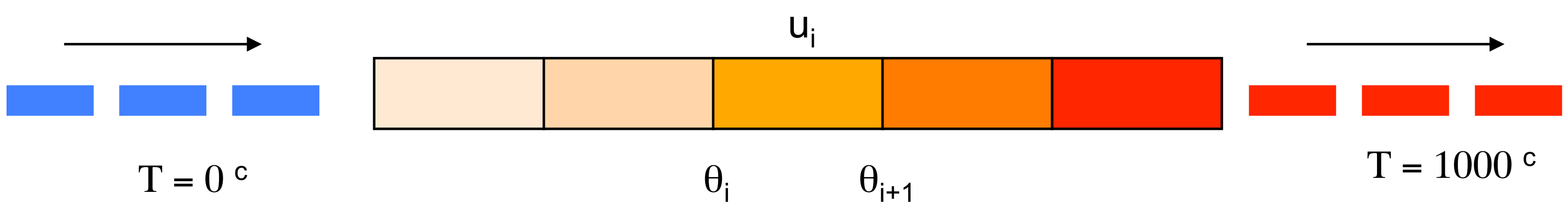
$$u = \bar{u} + v, \qquad v : P_1 v = 0$$

$$\xrightarrow{\text{minimization of}} \quad \|u\| = \sqrt{\sum_{i=0}^{k-1} u^T(i) u(i)}$$

This solution, based on the (pseudo)inverse of matrix P_1, is not used frequently in practice since it requires a *perfect knowledge* of the model parameters (matrices A and B) and *no disturbances* applied to the system.

Controllability for discrete-time systems - Example

- *Example:* It is desired to heat up to 1000^c some steel bars through an oven with five stages and increasing temperature. The cost for heating each stage is proportional to the square of its temperature. Compute the five stage temperatures in order to minimize the overall cost $C = (\Sigma_\iota\, u_i^2)^{1/2}$.



- Let θ_I and θ_{i+1} i = 1, …, 5 be the input/output temperatures of the bars in each stage, and let u_i be the temperature of the i-th stage.

- The heating dynamics in each stage can be described by:

$$\theta_{i+1} = \theta_i + \frac{u_i - \theta_i}{2} \quad \Longleftrightarrow \quad \theta_{i+1} = \frac{1}{2}\theta_i + \frac{1}{2}u_i, \qquad A = \frac{1}{2}, \quad B = \frac{1}{2}$$

- The 5-step reachability matrix is:

$$P_1(5) = R_5 = \left[\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}\right]$$

Controllability for discrete-time systems - Example

- Assuming that the initial temperature of the steel bars is 0ᶜ , then:

$$1000 = R_5\, u = \left[\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}\right] \begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

- In this case, matrix R_5 is not invertible and it is necessary to use the pseudo-inverse (right pseudo-inverse, since R_5 is "lower rectangular")

$$u = R_5^{+} 1000 = R_5^T (R_5 R_5^T)^{-1} 1000 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \frac{1}{32} \end{bmatrix} \cdot 3.0029 \cdot 1000 = \begin{bmatrix} 1501.5 \\ 750.7 \\ 375.4 \\ 187.7 \\ 93.8 \end{bmatrix}$$

$$\longrightarrow \quad \theta = [0,\ 46.9,\ 117.3,\ 246,3,\ 498.5,\ 1000]$$

Controllability for discrete-time systems - Example

- Since R_5 is a 1 x 5 lower rectangular matrix, a 4-dimensional null space exists defined by:

$$N = \begin{bmatrix} -0.4332 & -0.2166 & -0.1083 & -0.0542 \\ 0.8994 & -0.0503 & -0.0251 & -0.0126 \\ -0.0503 & 0.9749 & -0.0126 & -0.0063 \\ -0.0251 & -0.0126 & 0.9937 & -0.0031 \\ -0.0126 & -0.0063 & -0.0031 & 0.9984 \end{bmatrix}$$

$$0 = R_5\, v = R_5\, (N\, y), \quad \forall y$$

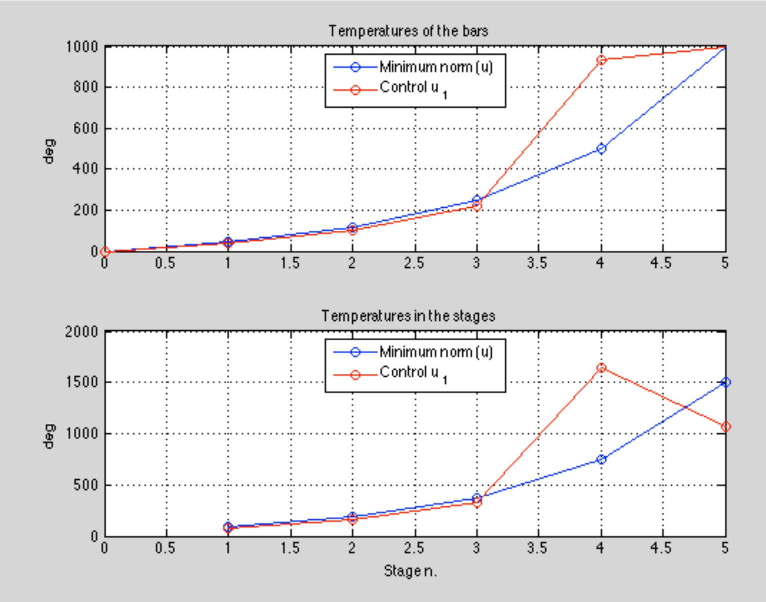
Therefore, if for example $y = [1000, 10, 1, 1]^T$, we obtain

$$u_1 = u + v = \begin{bmatrix} 1065.9 \\ 1649.6 \\ 334.8 \\ 163.4 \\ 82.2 \end{bmatrix}$$

Same final value

$$\theta_1 = [0, \ 41.1, \ 102.3, \ 218.5, \ 934.1, \ 1000]$$

and the cost increases from $C = 1732.9$ to $C_1 = 2000.8$



Reachability for continuous-time systems

- Consider the continuous-time system described by;

$$\dot{x}(t) = A\ x(t) + B\ u(t) \qquad\qquad x(0) = 0$$

- The state x(t) is reachable at time instant t if an input function u(·) exists such that the forced motion results

$$x(t) = \int_0^t e^{A(t-\tau)} B\ u(\tau) d\tau$$

- Consider the linear operator

$$R_t\ :\ u(\cdot) \rightarrow x(t) = \int_0^t e^{A(t-\tau)} B\ u(\tau) d\tau$$

- Then x(t) is reachable at time t iff x(t) belongs to the range space of R_t, that represents the reachability subspace $V^+(t)$ at time t

Reachability for continuous-time systems

- By defining the reachability matrix $P_1 = [B,\ A B, A^2 B, \dots A^{n-1} B]$, where n is the dimension of the state space (order of matrix A), we have the following:

- **Property:** The subspace reachable at time instant t > 0, $V^+(t)$, is the image of the reachability matrix P_1

- Note: for continuous-time systems, the reachability subspace does not depend on the length of the time interval [0, t] in which the input function u(·) is active (for discrete-time systems, V^+_k depends on the time interval).

- Note: This is true if the amplitude of the input signal may have arbitrarily large values. In practice, this cannot be true, and therefore the set of reachable states depend on the time interval and the input amplitude. The “theoretical” result is not affected, since in this case the input set is not a vector space and therefore one of the main (theoretical) assumptions is not verified.

Controllability for continuous-time systems

- Consider the continuous-time system described by;

$$\dot{x}(t) = A\, x(t) + B\, u(t) \qquad\qquad x(0) = 0$$

- **Property:** The controllabilty subspace $V^-(t)$, does not depend on the time interval [0, t] in which the control input is active. Moreover, $V^-(t) = V^+(t)$.

- **Problem:** Given a continuous-time system, compute the input function *u(•)* in order to reach the state *x(t)* from *x(0).*
- The problem has solution if

$$x(t) - e^{At}x(0) \in \mathcal{V}^+$$

- Then, the states that can be reached at time t from x(0) belong to the set

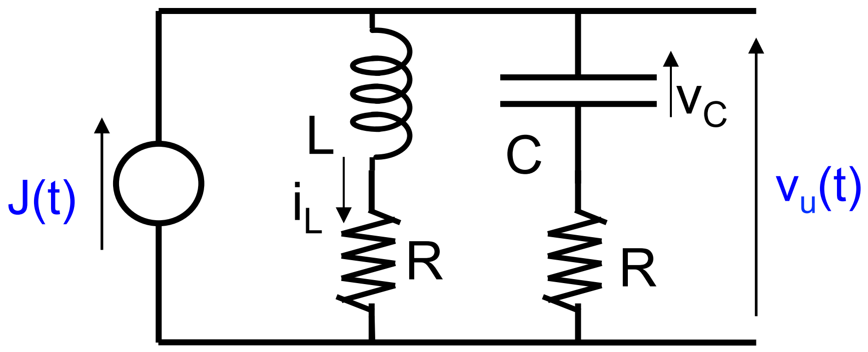
$$e^{At}x(0) + \mathcal{V}^+$$

- The input function can be computed by solving the equation

$$x(t) - e^{At}x(0) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = R_t\; u(\cdot)$$

Controllability and Reachability - Example

- Consider the electric circuit:



$$
\begin{aligned}
L\frac{di_L}{dt} &= v_C + R(J - i_L) - Ri_L \\
C\frac{dv_C}{dt} &= J - i_L \\
v_u &= v_C + R(J - i_L)
\end{aligned}
$$

- In matrix form:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -\frac{2R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} J \\
v_u &= \begin{bmatrix} -R & 1 \end{bmatrix} x + [R] J
\end{aligned}
$$

$$
P_1 = \begin{bmatrix} \frac{R}{L} & \frac{1}{LC} - \frac{2R^2}{L^2} \\ \frac{1}{C} & -\frac{R}{LC} \end{bmatrix}
$$

$$
det(P_1) = \frac{1}{LC}\left[\frac{R^2}{L} - \frac{1}{C}\right]
$$

The system is not completely controllable if det(P_1) = 0, that is if R C = L/R (the same time constant)

Observability

- A system is said *completely observable* if, knowing the input and output functions $u|_{[0,\, t_1]}$, $y|_{[0,\, t_1]}$ in any finite interval of time it is possible to determine the state x(0)

- **Theorem:** By defining

$$P_2 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

the set of *unobservable states* in any finite interval of time is

$$Q = ker\{P_2\}$$

The system is completely observable iff $Ker\{P_2\} = 0$ that is if P_2 has full rank. Q is a subspace of $\mathbf{R}^n$

Matrix P_2 *is the observability matrix.*

Observability

- *Example:*

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = [0, 1, 0] \qquad \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

System non fully observable: states defined by $Ker\{P_2\} = [-1, 0, 1]^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = [0, 1, 1] \qquad \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix},$$

Fully observable system: ($Ker\{P_2\} = \{0\}$)

Observability

- Define $$P_2^T = [C^T \;\; A^T C^T \;\; \dots \;\; (A^T)^{n-1} C^T]$$

  $Q = ker\{P_2\}$ is the maximum A-invariant subspace contained in $ker\{C\}$: for every initial state in Q, the related free motion does not affect the output y.

- For any matrix P:

$$ker\, P = (im\, P^T)^{\perp}, \qquad im\, P = (ker\, P^T)^{\perp}$$

  That is:
  - the null space of any matrix P is equal to the orthogonal complement of the image of P^T
  - the image of any matrix P is equal to the orthogonal complement of the null space of P^T

Equivalent systems

- Two systems Σ *(A, B, C, D)* and Σ' *(A', B', C', D')* are *equivalent* if a non singular matrix T exists such that *x = T x'* and

$$A' = T^{-1}AT \qquad B' = T^{-1}B$$

$$C' = CT \qquad D' = D$$

- For the reachability property, we have that:

$$P_1' = T^{-1}P_1 \qquad (P_1 = TP_1')$$

- If the two systems are reachable, then

$$T = P_1(P_1')^{+}$$

and, in case of a single input

$$T = P_1(P_1')^{-1}$$

Equivalent systems

- For the reachability property, we have that:

$$P_1' = T^{-1} P_1 \qquad\qquad (P_1 = T P_1')$$

- As a matter of fact, the reachable subspaces in k steps for the two systems are

$$\begin{aligned} V_k'^{+} &= Im\{[B', \; A'B', \; A'^2B', \cdots, \; A'^{k-1}B']\} \\ &= Im\{[T^{-1}B, \; T^{-1}ATT^{-1}B, \; \cdots, \; T^{-1}A^{k-1}TT^{-1}B]\} \\ &= Im\{T^{-1}[B, \; AB, \; \cdots, \; A^{k-1}B]\} \\ &= T^{-1}V_k^{+} \end{aligned}$$

- From this equation it follows that $P'_1 = T^{-1} P_1$

Equivalent systems

- *Example:*

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix},$$

$$P_1' = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$T = P_1 (P_1')^{-1} = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 5/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Check:

$$T^{-1}AT = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

Dual systems

- Consider a linear time-invariant system Σ

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad (1)$$

The *dual system* Σ_D is defined as

$$\begin{aligned} \dot{z}(t) &= A^T z(t) + C^T v(t) \\ \rho(t) &= B^T z(t) + D^T v(t) \end{aligned} \qquad (2)$$

Number of input of Σ = Number of output of Σ_D

Number of output of Σ = Number of input of Σ_D

Dual systems

Properties of dual systems:

Σ stable	$\Leftrightarrow$	Σ_D stable
Σ completely controllable	$\Leftrightarrow$	Σ_D completely reconstructable
Σ completely reconstructable	$\Leftrightarrow$	Σ_D completely controllable
Σ completely observable	$\Leftrightarrow$	Σ_D completely reachable
Σ completely reachable	$\Leftrightarrow$	Σ_D completely observable

The proof of these properties can be easily obtained from the following equations:

$$\lambda_i\{A\} = \lambda_i\{A^T\}, \qquad i = 1, \ldots, n$$

$$\begin{aligned} P_{1,D}^T &= P_2 \\ P_{2,D}^T &= P_1 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} P_1^T &= P_{2,D} \\ P_2^T &= P_{1,D} \end{aligned}$$

Standard reachability form

Consider the MIMO linear time-invariant system (continuous- or discrete-time):

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad \left(\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned} \right)$$

with $rank(P_1) = \rho < n$ ($dim\ \mathcal{R} = \rho < n$): the system is NOT completely reachable.

Consider the transformation matrix

$$T = [T_1 \quad T_2]$$

where the $n \times \rho$ matrix T_1, is a basis matrix of V^+ and the $n \times (n-\rho)$ matrix T_2 makes T non singular. The following equivalent (and simpler) form is obtained for the system:

$$A' = T^{-1}AT = \left[\begin{array}{cc} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{array} \right] \qquad B' = T^{-1}B = \left[\begin{array}{c} B'_1 \\ 0 \end{array} \right]$$

$$C' = CT = [C'_1, C'_2] \qquad D' = D$$

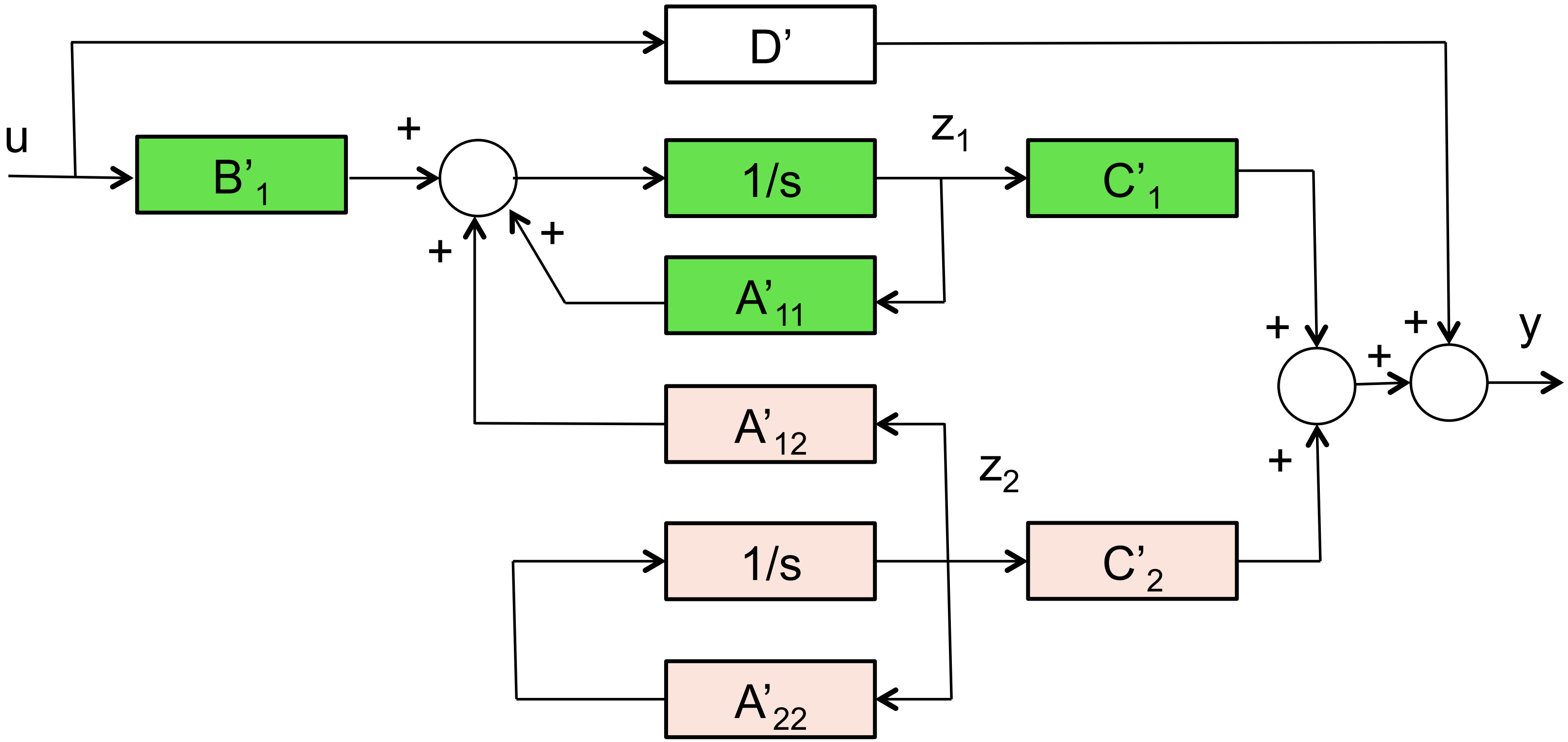
$$dim\{A'_{11}\} = \rho \times \rho, \qquad dim\{B'_1\} = \rho \times m$$

This is the standard reachability form, since the reachable part of the system is clearly identified by matrices *(A'_{11}, B'_1, C'_1).*

Standard reachability form

- **Note:** By acting through the input, it is possible to modify only the first ρ components of the new state $z = T\,x$. It is not possible to apply any control action on the remaining n - ρ components: it they are not stable, an unstable behavior will always take place.

$$
\begin{aligned}
\dot{z}_1(t) &= A'_{11} z_1(t) + A'_{12} z_2(t) + B'_1 u(t) \\
\dot{z}_2(t) &= A'_{22} z_2(t) \\
y(t) &= C'_1 z_1(t) + C'_2 z_2(t) + D' u(t)
\end{aligned}
$$



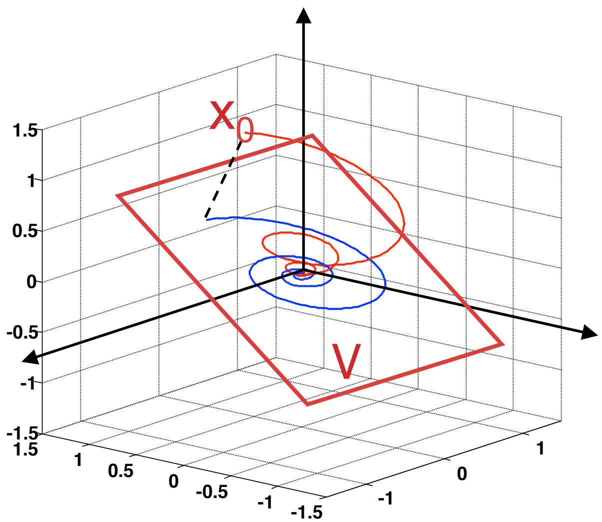
Standard reachability form

Consider the homogeneous system

$$\dot{x}(t) = A\, x(t), \qquad x(0) = x_0, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

A trajectory originated on an invariant will remain on it. Therefore, since the reachability subspace is A-invariant, if matrix A_{11} is stable, a trajectory originated in this subspace will tend to the origin. Moreover, a trajectory originated out of this invariant tends to it if matrix A_{22} is stable.

- If A_{11} is stable, the invariant is called *internally stable*
- If A_{22} is stable, the invariant is called *externally stable*



If A_{22} is stable:
- x_2 tends to 0
- x tends to V

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + \cancel{A_{12}x_2} + B_{11}u \\ \dot{x}_2 &= \cancel{A_{22}x_2} \end{aligned}$$

Standard observability form

Consider the MIMO linear time-invariant system (continuous- or discrete-time):

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad \left(\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}\right)$$

with *rank*(P_2) = $\rho < n$ (dim $\mathcal{Q}$ = $n - \rho > 0$. *The system is NOT completely observable.*
A transformation matrix *M* exists such that

$$A' = M^{-1}AM = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \qquad B' = M^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix}$$

$$C' = CM = [C'_1, 0] \qquad D' = D$$

$$dim\{A'_{11}\} = \rho \times \rho, \qquad dim\{C'_1\} = p \times \rho$$

This is the standard observability form, since the observable part of the system is clearly identified by matrices *(A'_{11}, B'_1, C'_1).*

From the properties of dual systems we have $\boxed{M = (T^{-1})^T = (T^T)^{-1}}$

where T is the matrix that transforms the system in the standard reachability form.

Standard observability form

The following relationship holds for the transformation matrices :

$$\boldsymbol{M} = (\boldsymbol{T}^{-1})^T = (\boldsymbol{T}^T)^{-1}$$

As a matter of fact, if for one of the systems we have

$$\Sigma = (A,\ B,\ C,\ D) \xrightarrow{\text{T}} \Sigma' = (A',\ B',\ C',\ D') = (T^{-1}AT,\ T^{-1}B,\ CT,\ D)$$

Dual

Dual

Then, for its dual system:

$$\Sigma_D = (A^T,\ C^T,\ B^T,\ D^T) \xrightarrow{\text{T}^{-T}} \Sigma'_D = (T^T A^T T^{-T},\ T^T C^T,\ T^{-T} B^T,\ D^T)$$

Kalman decomposition

- Consider the system ∑ = *(A, B, C, D)*. Let *R* be the reachable subspace and *Q* the unobservable subspace.

  Consider the state transformation defined by the matrix

$$T = [T_1 \;\; T_2 \;\; T_3 \;\; T_4]$$

- Where:

$$\begin{array}{rcl} im\{T_1\} & = & \mathcal{R} \cap \mathcal{Q} \\ im\{[T_1,\; T_2]\} & = & \mathcal{R} \\ im\{[T_1,\; T_3]\} & = & \mathcal{Q} \\ T_4 & : & T \;\; \text{non singular} \end{array}$$

Basis of the reachable and non observable subspace

Basis of the reachable subspace

Basis of the non observable subspace

- The system ∑’ = *(A’, B’, C’, D’)* is obtained, where

$$A' = T^{-1}AT \qquad B' = T^{-1}B$$

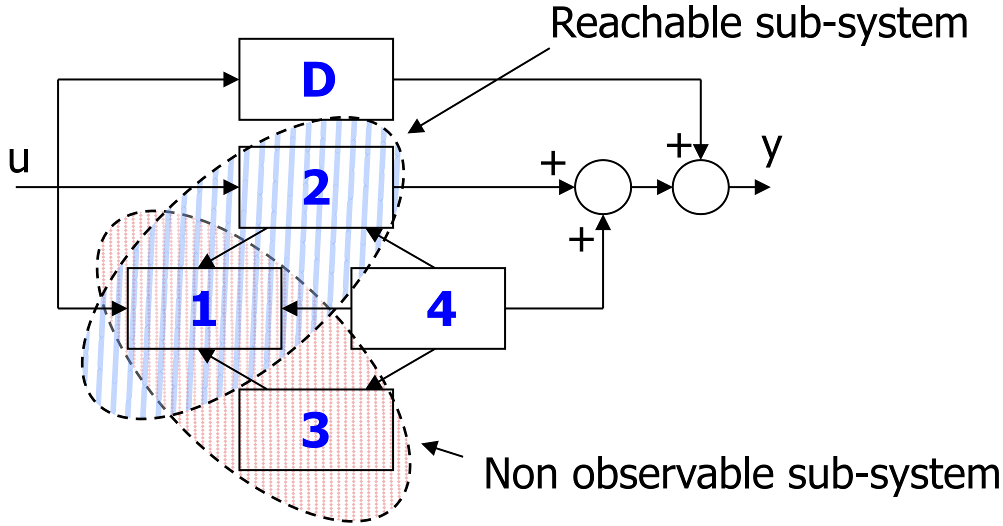
$$C' = CT \qquad D' = D$$

Kalman decomposition

$$A' = T^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14} \\ 0 & A'_{22} & 0 & A'_{24} \\ 0 & 0 & A'_{33} & A'_{34} \\ 0 & 0 & 0 & A'_{44} \end{bmatrix} \qquad B' = T^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C' = CT = [0 \;\; C'_2 \;\; 0 \;\; C'_4] \qquad D' = D$$

- $\Sigma_2 = (A'_{22}, B'_2, C'_2)$ reachable and observable
- $\Sigma_1 = (A'_{11}, B'_1, 0)$ reachable and non observable
- $\Sigma_4 = (A'_{44}, 0, C'_4)$ non reachable and observable
- $\Sigma_3 = (A'_{33}, 0, 0)$ non reachable and non observable



From state space to transfer functions

- Consider the SISO linear time-invariant system

$$
\begin{aligned}
\dot{x}(t) &= A\,x(t) + b\,u(t), \qquad x(0) = x_0 \\
y(t) &= c\,x(t) + d\,u(t)
\end{aligned}
$$

$$
\rightarrow \quad
\begin{aligned}
s\,x(s) &= A\,x(s) + b\,u(s) + x_0 \\
y(s) &= c\,x(s) + d\,u(s)
\end{aligned}
$$

$$
\rightarrow \quad
\begin{aligned}
x(s) &= (s\,I - A)^{-1}(b\,u(s) + x_0) \\
y(s) &= c\,(s\,I - A)^{-1} b\,u(s) + c\,(s\,I - A)^{-1} x_0 + d\,u(s)
\end{aligned}
$$

- From the latter equations, we can identify:

  - the input-output transfer function $\quad G(s) = c\,(s\,I - A)^{-1} b$

  - the free response term $\quad c\,(s\,I - A)^{-1} x_0$

From state space to transfer functions

- Similar results hold for MIMO systems

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0 \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

In fact, one obtains

$$\begin{aligned} x(s) &= (s\,I - A)^{-1}(B\,u(s) + x_0) \\ y(s) &= C\,(s\,I - A)^{-1}B\,u(s) + C\,(s\,I - A)^{-1}x_0 + D\,u(s) \end{aligned}$$

where matrix

$$G(s) = C\,(s\,I - A)^{-1}B + D$$

is *the system transfer matrix.*

It is simple to verify that the system transfer matrix depends only on the reachable and observable part of the system.

$$G(s) = C\,(s\,I - A)^{-1}B + D = C_2\,(s\,I - A_{22})^{-1}B_2 + D$$