Survey of Nonlinear Observer Design Techniques

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Abstract

This paper presents three "nonlinear" approaches to the problem of nonlinear observer design. Beginning with a brief introduction to the state estimation (observer) problem, we present approaches based on Lyapunov methods, the method of extended linearization, and a Lie-algebraic approach. The main issues in each technique are briefly explained and explored, as well as the applicability of each method.

1 Introduction

The problem of estimating the state of a dynamical system from outputs and inputs, (commonly known as "observing the state", hence the name "observer") is an important problem in the theory of systems. For linear systems, it has been extensively studied, and has proven extremely useful, especially for control applications such as observer-based-control design. For nonlinear systems, the theory of observers is not nearly as complete nor successful as it is for linear systems. Applying linear observer theory to nonlinear problems has had success as exemplified by the extended Kalman filter, but has by no means closed the book on nonlinear observer design. Instead, attempts continue to be made to construct nonlinear observers using tools from nonlinear systems theory. This paper presents a brief introduction to some of these nonlinear approaches to the problem of observer design.

Section 2and 3 provide a brief introduction to linear observer theory and the problems encountered when it is applied to nonlinear systems. Section 4 introduces some of the first attempts to apply nonlinear techniques, specifically Lyapunov stability theory, to the observer problem. Two early results in this development are presented, attributable to Kou (1975) [9], and Thau (1973) [14]. Section 5 presents an approach to designing observers for use in observer based controllers for nonlinear systems due to Baumann and Rugh (1986) [4]. Finally, the last sections introduce a Lie-algebraic approach to nonlinear observer design, which was developed by primarily by Krener [13], Zeitz [6], and others. It relies on linear observer theory, but nonlinear state transformations to convert nonlinear systems to systems where linear theory is applicable. Finally, we conclude by discussing the advantages and shortcomings of all three approaches.

2 Linear Observer Design

Consider the following uncontrolled linear system:

$$\begin{aligned} \dot{x} &= Ax, \quad && x \in \mathbf{R}^n \\ y &= Cx, \quad && y \in \mathbf{R}^p \end{aligned} \tag{1}$$

The problem of generating an estimate of the state, x, in (1) is usually referred to as the problem of designing an *observer* for (1).

In the early 1960's, Luenberger gave the following result for the construction of an observer for (1):

Let $\hat{x}$ be our estimate of the true state, x, and assume $\hat{x}$ obeys the following dynamics.

$$\begin{aligned} \hat{x} &= A\hat{x} + L(\hat{y} - y), \quad && \hat{x} \in \mathbf{R}^n \\ \hat{y} &= C\hat{x}, && \hat{y} \in \mathbf{R}^p \end{aligned} \tag{2}$$

where $L \in \mathbf{R}^{n \times p}$. Then if (C, A) is an observable pair, L may be chosen such that $\hat{x}$ will converge to x arbitrarily exponentially fast.

To see that this is the case, we consider the dynamics of the error between the state estimate $\hat{x}$ and the true state x. Denoting this error by $e = \hat{x} - x$, we calculate the dynamics of e by subtracting (1) from (2) to obtain:

$$\dot{e} = (A + LC)e \tag{3}$$

These error dynamics are linear. Because of this, it is clear that if (C, A) is an observable pair, then the eigenvalues of $A + LC$ can be arbitrarily assigned, and hence placed as far into the left half plane as desired, causing the error to decay to zero at any desired exponential rate.

At this point it is important to note the form of our observer in (2). Careful inspection of (2) will reveal that out observer dynamics are exactly those of the true system (1) except with a linear function, L, of the difference between the estimated and true output, $\hat{y} - y$, injected into the dynamics. This is a standard trick used throughout observer design, and is commonly referred to as *output injection.*

3 Nonlinear Systems: A Linear Approach

Consider the nonlinear system,

$$\begin{aligned} \dot{x} &= f(x) \quad && x \in \mathbf{R}^n \\ y &= h(x) \quad && y \in \mathbf{R}^p \end{aligned} \tag{4}$$

As a first attempt at an observer design, we might consider copying our technique for linear systems in (2) and design an observer with linear output injection of the form:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + L(\hat{y} - y) \\ \hat{y} &= h(\hat{x}) \end{aligned} \tag{5}$$

where once again $L \in \mathbf{R}^{n \times p}$ is a matrix. Again we consider the error between our estimated state and the true state, $e = \hat{x} - x$, and wish to determine if it can be made to decay to zero,

$$\dot{e} = f(\hat{x}) - f(x) + L(h(\hat{x}) - h(x)) \tag{6}$$

This time, our error dynamics are nonlinear. The stability of the error dynamics are now unclear.

Motivated by the fact that stability of a linearized system about a fixed point, implies local stability of the corresponding nonlinear system about that fixed point, we might attempt to linearize the error dynamics about its fixed point, $e = 0$,

$$\begin{aligned} \dot{e} &= f(\hat{x}) - f(x) + L(h(\hat{x}) - h(x)) \\ &= f(x+e) - f(x) + L(h(x+e) - h(x)) \\ &= (\tfrac{\partial f}{\partial x}(x) + L\tfrac{\partial h}{\partial x}(x))e + O(e^2) \end{aligned} \tag{7}$$

So the linearized system is,

$$\dot{e} = (\frac{\partial f}{\partial x}(x) + L\frac{\partial h}{\partial x}(x))e \tag{8}$$

Unfortunately, the above linearization is a function of the true state x which, first of all is not a fixed quantity, and secondly, is unknown to us.

It is now clear that techniques beyond the realm of linear theory are needed to analyze the error dynamics in (6). This leads to our next section.

4 Lyapunov-based Methods

In an attempt to analyze and determine conditions under which the nonlinear error dynamics in (6) have a stable fixed point at the origin ($e = 0$), Lyapunov stability theory was employed. This approach leads to results giving sufficient conditions for existence of observers for classes of nonlinear systems, but unfortunately rarely gives insight into methods for constructing such an observer.

As a glimpse into the kinds of results available, we will present two early results in this development, the first of which is due to Kou et al. (1975) [9].

Theorem 1 *Consider the nonlinear system (4), the nonlinear observer (5) and the corresponding error dynamics (6). If there exists a constant $n \times m$ matrix L and a positive definite, symmetric $n \times n$ matrix P such that*

$$P(\frac{\partial f}{\partial x}(x) - L\frac{\partial h}{\partial x}(x)) < 0 \quad \forall x \tag{9}$$

(i.e. it is uniformly negative-definite for all x), then the dynamical system (5), with the matrix L satisfying (9), and with any initial estimate, $\hat{x}(t_0)$, is an exponential observer for the system (4) and

$$\|\hat{x}(t) - x(t)\| \leq \alpha_1 \|\hat{x}(t_0) - x(t_0)\| \exp[-\alpha_2(t - t_0)] \tag{10}$$

for all $t \geq t_0$ and some positive numbers α_1 and α_2

Proof: We will only prove the first part of the theorem, namely that (5) is a global asymptotic observer.

Consider the following positive-definite Lyapunov function for (6)

$$V(e) = e^T P e \tag{11}$$

then

$$\begin{aligned} \dot{V}(e) &= 2e^T P \dot{e} \\ &= 2e^T P(f(\hat{x}) - f(x) + L(h(\hat{x}) - h(x))) \end{aligned} \tag{12}$$

Now consider the curve $c(t) = t\hat{x} + (1 - t)x \quad t \in [0, 1]$. This curve traces the straight line from x to $\hat{x}$ as t goes from 0 to 1. Then we have the following,

$$\begin{aligned} f(\hat{x}) - f(x) &= \int_0^1 \frac{\partial f}{\partial x}(c(t))c'(t)dt \\ &= \int_0^1 (\frac{\partial f}{\partial x}(t\hat{x} + (1 - t)x))(\hat{x} - x)dt \end{aligned} \tag{13}$$

Similarly we have,

$$h(\hat{x}) - h(x) = \int_0^1 (\frac{\partial h}{\partial x}(t\hat{x} + (1 - t)x))(\hat{x} - x)dt \tag{14}$$

Using this, we can continue with our above calculation of $\dot{V}(e)$,

$$\begin{aligned} \dot{V}(e) &= 2e^T P \dot{e} \\ &= 2e^T P(f(\hat{x}) - f(x) + L(h(\hat{x}) - h(x))) \\ &= 2e^T P \int_0^1 (\frac{\partial f}{\partial x}(t\hat{x} + (1 - t)x) + L\frac{\partial h}{\partial x}(t\hat{x} + (1 - t)x))(\hat{x} - x)dt \\ &= 2 \int_0^1 e^T (P(\frac{\partial f}{\partial x} + L\frac{\partial h}{\partial x}))e dt \\ &< 0. \qquad\qquad (since \;\; P(\frac{\partial f}{\partial x} + L\frac{\partial h}{\partial x}) < 0) \end{aligned} \tag{15}$$

Therefore $V(e)$ is a Lyapunov function for (6) and $e = 0$ is a globally stable fixed point. $\diamond$

As a side note of interest, we present the following corollary to the above result

Theorem 2 *Consider the nonlinear system*

$$\dot{x} = f(x), \quad f(0) = 0 \tag{16}$$

If

$$\frac{\partial f}{\partial x}(x) < 0 \quad \forall x \tag{17}$$

ie. the jacobian matrix of f is negative-definite for all x, then $x = 0$ is a global, asymptotically stable fixed point of (16).

The preceding results are very strong indeed. They apply to general nonlinear systems and ensure global observability in the first case, and stability in the later. It is only natural that we would find it very difficult to find systems that satisfy the conditions of the above theorems, which is indeed the case.

As an example of a slightly less ambitious result, we present the following observer construction due to Thau (1973) [14].

Consider a nonlinear system of the form

$$\begin{array}{ll} \dot{x} = Ax + f(x) & x \in \mathbf{R}^n \\ y = Cx & y \in \mathbf{R}^p \end{array} \tag{18}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, $A \in \mathbf{R}^{n\times n}$, and $C \in \mathbf{R}^{p\times n}$. The nonlinear function f *may also contain linear terms in x.* We also assume that the pair (C, A) is observable. This allows us to find an $L \in \mathbf{R}^{nxp}$ such that the eigenvalues of $A + LC$ are in the open left half plane.

Our observer is built as

$$\begin{array}{l} \dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(\hat{y} - y) \\ \hat{y} = C\hat{x} \end{array} \tag{19}$$

Once again let e denote that error between the true state and our estimated state, $e = \hat{x} - x$ so e satisfies

$$\dot{e} = (A + LC)e + f(\hat{x}) - f(x) = (A + LC)e + f(x + e) - f(x) \tag{20}$$

As stated previously, it is a well known result that linear stability implies nonlinear stability in a neighborhood of the fixed point of a nonlinear system. In this case, the fixed point of (20) is $e = 0$. Hence if $f(x)$ contained no linear terms then by choosing L so that $A + LC$ was stable, we could guarantee that our error would be stable (ie. go to zero) in a neighborhood of the origin. But in Thau's formulation, he allowed for linear terms appearing $f(x)$ as well. Hence local stability is not guaranteed by the stability of $A + LC$.

In this case, to find conditions under which our observer error dynamics will be stable, we must once again turn to Lyapunov theory.

Since $A + LC$ is stable, then for any positive definite $Q \in \mathbf{R}^{n\times n}$ there exists a unique positive definite $P \in \mathbf{R}^{n\times n}$ such that

$$(A + LC)^T P + P(A + LC) = -2Q \tag{21}$$

Consider the following positive definite Lyapunov function candidate

$$V(e) = e^T P e \tag{22}$$

The derivative of V(e) evaluated along the solution of the error differential equation (20) is given by

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e} = -2e^T Q e + 2e^T P[f(x+e) - f(x)] \tag{23}$$

To ensure stability, we would like the above derivative of V(e) to be negative. Toward this goal, we impose the additional constraint that the function f is *locally* Lipschitz about the origin; ie. there exists a positive constant M such that

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\| \tag{24}$$

for all x_1, x_2 in some neighborhood W of the origin. Therefore if e is contained in W then the following inequalities are valid:

$$\dot{V}(e) \leq -2e^T Q e + 2M\|Pe\|\|e\| \leq (-2\sigma_{min}(Q) + 2M\sigma_{max}(P))\|e\|^2 \tag{25}$$

where $\sigma_{min}(Q)$ is the minimum singular value of Q and $\sigma_{max}(P)$ is the maximum singular value of P.

Hence if

$$\frac{\sigma_{min}(Q)}{\sigma_{max}(P)} > M \tag{26}$$

then $e = 0$ is an asymptotically stable equilibrium point of (20) and we have shown the our observer converges to the true state locally.

Two points associated with the application of the so called direct stability method of Lyapunov are worth noting. First, these theorems provide sufficient conditions for an asymptotic observer and these conditions tend to be conservative. Secondly, satisfaction of the theorems does not of itself constitute a constructive procedure for determining a stabilizing gain matrix L. The choice of L in order to satisfy the theorems is a trial and error process that may practically be impossible for systems of high order. These limitations are part of the difficulty of analyzing general non-linear systems.

The results presented in this section by no means cover all the results using Lyapunov methods in observer design. They are, however, representative of the types of results available in the area, and point out their limited usefulness.

In the next section we present, in some sense, a counterpart to the method just explored.

5 The Method of Extended Linearization

In the previous sections, we used linear output injection (L) and nonlinear techniques to determine the stability of the error dynamics. In this section, we present the method of extended linearization due to Baumann and Rugh (1986) [4] which uses nonlinear output injection in order to make the linearized error dynamics have locally constant eigenvalues.

At this point it is important to point out that the observer presented in this section is appropriate in the context of designing an observer to be used in an observer based controller. While in the previous sections, we considered uncontrolled dynamics, (although we could have added an input with no added complications), when considering the method of extended linearization, it is *only* sensible in the presence of controlled dynamics as will be seen presently.

Consider the single input system described by

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{27}$$

where f and h are analytic and

$$f(\cdot, \cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n, \quad \text{with} \quad f(0, 0) = 0$$

$$h(\cdot) : \mathbf{R}^n \to \mathbf{R}^p, \quad \text{with} \quad h(0) = 0$$

We shall assume that (27) has a family of fixed points denoted by $u = \epsilon$ and $x = x_\epsilon = x(\epsilon)$ where x_ϵ is an analytic function of ϵ in a neighborhood of $\epsilon = 0$. That is,

$$f(x_\epsilon, \epsilon) = 0 \tag{28}$$

Let the observer estimate $\hat{x}$ obey

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) + g(y) - g(\hat{y}) \\ \hat{y} &= h(\hat{x}) \end{aligned} \tag{29}$$

and $g(\cdot) : \mathbf{R}^p \to \mathbf{R}^n$ is analytic with $g(0) = 0$. If we denote $e = x - \hat{x}$ then

$$\dot{e} = f(x, u) - f(x - e, u) - g(y) + g(\hat{y}) \tag{30}$$

Note that if $u = \epsilon$ then $x = x_\epsilon$ and $\hat{y}_\epsilon = y_\epsilon = h(x_\epsilon) = h(x(\epsilon))$ is an equilibrium point of (27) and (30). At this point, we also call to the readers attention that we use the subscript ϵ to denote when the subscripted quantity is being considered as a function of ϵ (ex. $x_\epsilon = x(\epsilon)$). This notation will be used throughout this section.

Expanding (30) about this equilibrium point and neglecting higher-order terms yields

$$\dot{e} = [\frac{\partial f(x_\epsilon, \epsilon)}{\partial x} - \frac{\partial g(y_\epsilon)}{\partial y} \frac{\partial h(x_\epsilon)}{\partial x}] e \tag{31}$$

Let us impose the following constraints,

(i) $(\frac{\partial f(0,0)}{\partial x})^{-1}$ exists
(ii) $(\frac{\partial f(0,0)}{\partial x}, \frac{\partial h(0)}{\partial x})$ is observable
(iii) $\frac{\partial y_\epsilon}{\partial \epsilon}|_{\epsilon=0} = \frac{\partial h(0)}{\partial x}\frac{\partial x_\epsilon(0)}{\partial \epsilon} = -\frac{\partial h(0)}{\partial x}(\frac{\partial f(0,0)}{\partial x})^{-1}\frac{\partial f(0,0)}{\partial u} \neq 0$

The aim of this observer design is to find an analytic g such that the eigenvalues of (31) are locally invariant with respect to ϵ.

Note that the eigenvalues of (31) are the same as the eigenvalues of

$$[\frac{\partial f(x_\epsilon,\epsilon)}{\partial x}^T - \frac{\partial h(x_\epsilon)}{\partial x}^T \frac{\partial g(y_\epsilon)}{\partial y}^T] \tag{32}$$

Recall from constraint (ii) that the pair $(\frac{\partial f(0,0)}{\partial x}, \frac{\partial h(0)}{\partial x})$ is observable. Therefore the pair $(\frac{\partial f(0,0)}{\partial x}^T, \frac{\partial h(0)}{\partial x}^T)$ is controllable. This implies that we may use Ackermann's (1977) synthesis formula to find $C(\epsilon)$, where $C(\cdot) : \mathbf{R} \to \mathbf{R}^{p\times n}$ such that

$$\frac{\partial f(x_\epsilon,\epsilon)}{\partial x}^T - \frac{\partial h(x_\epsilon)}{\partial x}^T C(\epsilon) \tag{33}$$

has prespecified eigenvalues that are independent of ϵ, for ϵ sufficiently small.

Thus, if we can find an analytic g such that the following partial differential equation is satisfied,

$$\frac{\partial g(y_\epsilon))}{\partial y}^T = C(\epsilon) \tag{34}$$

then the linearized observer error equation (31) will have eigenvalues that are locally independent of ϵ.

Let us define

$$C^T(\epsilon) = [c_1(\epsilon), ..., c_p(\epsilon)] \tag{35}$$

where $c_i(\epsilon)$ are $(n \times 1)$ column vectors and

$$y = [y_1, ..., y_p] = [h_1(x), ..., h_p(x)] \tag{36}$$

and when we are considering y as a function of ϵ, we denote it as

$$y_\epsilon = [y_{1\epsilon}, ..., y_{p\epsilon}] \tag{37}$$

Without loss of generality, we may assume that the first element of the p-vector $\frac{\partial y_\epsilon}{\partial \epsilon}$ in constraint (iii) is non-zero. (i.e. $\frac{\partial y_{1\epsilon}}{\partial \epsilon} \neq 0$). Thus by the inverse function theorem, we can find $y_{1\epsilon}^{-1}$ such that

$$y_{1\epsilon}^{-1}(y_{1\epsilon}) = \epsilon \tag{38}$$

Let $g_\epsilon = g(h(x_\epsilon))$, which implies

$$\frac{\partial g_\epsilon}{\partial \epsilon} = \frac{\partial g(y_\epsilon)}{\partial y}\frac{\partial h(x_\epsilon)}{\partial \epsilon} = C^T(\epsilon)\frac{\partial h(x_\epsilon)}{\partial \epsilon} \tag{39}$$

Since we know the right side of (39) (we can find $\frac{\partial h(x_\epsilon)}{\partial \epsilon}$ in a manner similar to (iii)), then by integrating (39), we can find g_ϵ. Finally, we claim that

$$g(y) = g_\epsilon(y_{1\epsilon}^{-1}(y_1)) + \sum_{j=2}^{p} c_j(y_{1\epsilon}^{-1}(y_1))[y_j - y_{j\epsilon}(y_{1\epsilon}^{-1}(y_1))] \tag{40}$$

To substantiate this claim, it can be shown by direct differentiation that

$$\frac{\partial g}{\partial y}|_{y=y_\epsilon} = C(\epsilon)^T \tag{41}$$

When $g(y)$ cannot be evaluated in closed form, Baumann and Rugh offer the

following approach. The coefficients in the power series expansion for g about $y = 0$ can be computed as follows. From (40) the expansion has the form

$$g(y) = \sum_{i=1}^{p} \sum_{j=0}^{\infty} g_{ij} y_i y_1^j \tag{42}$$

Writing (34) in series form,

$$\frac{\partial g}{\partial y_i}(y_\epsilon) = \sum_{j=0}^{\infty} c_{ij} \epsilon^j, \quad i = 1, ..., p \tag{43}$$

and using (42) gives the requirement

$$\sum_{j=0}^{\infty} (j+1) g_{1j} y_1^j + \sum_{i=2}^{p} \sum_{j=1}^{\infty} j g_{ij} y_i y_1^{j-1} = \sum_{j=0}^{\infty} c_{1j} \epsilon^j \tag{44}$$

$$\sum_{j=0}^{\infty} g_{ij} y_1^j = \sum_{j=0}^{\infty} c_{ij} \epsilon^j, \quad i = 2, ..., n. \tag{45}$$

The above series, (44) and (45) allow for the successive, unique determination of the coefficients g_{ij} (see [4] for details). Hence we may determine $g(y)$ up to any order, say m, and define the polynomial $\tilde{g}(y)$ such that $\frac{\partial \tilde{g}}{\partial y}(y_\epsilon)$ agrees with $\frac{\partial g}{\partial y}(y_\epsilon)$ up to order $m-1$ in ϵ.

The advantage of using this construction of observer as given by Baumann and Rugh is that a similar construction can be used in designing a controller that has locally invariant eigenvalues about a fixed point (see [4] for details). When combining the above observer with the similarly designed controller to stabilize a system about its fixed point, locally the linearized closed loop system will exhibit the well known eigenvalue separation property. (i.e. the eigenvalues of the closed loop system are equal to those of the observer plus those of the controller). Baumann and Rugh demonstrate this design technique on the inverted pendulum and obtain results superior to those obtained by linearizing the dynamics and designing an observer based controller using linear theory. (see [4])

Once again, it is important to point out the limitations of the above observer construction. This observer is designed to operate only locally about a fixed point of the original system (27). Secondly, we have given the result for a single input system. The generalization to the multi-input case is not completely straightforward, and the construction of the observer is much more difficult. (see [5]) Those familiar with the theory of pseudolinearization may recognize the similarities between extended linearization and pseudolinearization. For details on this, once again we refer the reader to works by Baumann and Rugh.

In the following sections, we present the third method for nonlinear observer design based on Lie-algebraic methods. Before embarking, we first need to develop some notation.

6 Notation

Let h be a real valued function, and f and g be vector fields, all of which are defined on a subset U of $\mathbf{R}^n$. The following definitions and notation are standard.

The differential or gradient of a real valued function h is defined as

$$dh = \left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n} \right) \tag{46}$$

This is sometimes denoted

$$dh = \frac{\partial h}{\partial x} \tag{47}$$

The *Lie derivative* of h with respect to f is defined as

$$\mathcal{L}_f h = \langle dh, f \rangle = \frac{\partial h}{\partial x} f = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i \tag{48}$$

We denote repeated applications of the Lie derivative by

$$\mathcal{L}_f^i h = \mathcal{L}_f(\mathcal{L}_f^{i-1} h) \tag{49}$$

The *Lie bracket* between two vector fields f and g is defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{50}$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the Jacobian matrices of the mappings g and f respectively.

When repeated bracketing of a vector field g with the same vector field f is needed, we use the following notation

$$
\begin{aligned}
&ad_f^k g = [f, ad_f^{k-1} g] \qquad for\ k \geq 1 \\
&ad_f^0 g = g
\end{aligned}
\tag{51}
$$

The Lie derivative of dh with respect to f is defined as

$$\mathcal{L}_f dh = (\frac{\partial dh^T}{\partial x} f)^T + dh \frac{\partial f}{\partial x} \tag{52}$$

with this definition, the following can be verified

$$d\mathcal{L}_f h = \mathcal{L}_f dh \tag{53}$$

Finally, we note that Lie derivatives obey the following

$$\mathcal{L}_{[f,g]} h = \mathcal{L}_g \mathcal{L}_f - \mathcal{L}_f \mathcal{L}_g h \tag{54}$$

7 The Lie-Algebraic Approach

The Lie-algebraic approach is based upon the following idea.

Consider the *single output* system as given below,

$$
\begin{aligned}
\dot{z} &= Az + \phi(y), \quad & z \in \mathbf{R}^n \\
y &= Cz, & y \in \mathbf{R}
\end{aligned}
\tag{55}
$$

Where ϕ is an arbitrary function of y, then using linear output injection to design an observer of the form

$$
\begin{aligned}
\dot{\hat{z}} &= A\hat{z} + \phi(y) + L(\hat{y} - y), \quad & \hat{z} \in \mathbf{R}^n \\
\hat{y} &= C\hat{z}, & y \in \mathbf{R}
\end{aligned}
\tag{56}
$$

will produce linear error dynamics ($e = \hat{z} - z$) of the form

$$\dot{e} = (A + LC)e \tag{57}$$

If (C, A) is an observable pair, then we can freely assign the eigenvalues of $A + LC$, and cause the error to go to zero exponentially fast.

The Lie-algebraic approach is based on the idea the given an arbitrary non-linear system

$$
\begin{aligned}
\dot{x} &= f(x) \quad & x \in \mathbf{R}^n \\
y &= h(x) & y \in \mathbf{R}
\end{aligned}
\tag{58}
$$

if there exists a state transformation $x = T(z)$ that takes (58) to (55), then we can transform (58) into (55), design an observer for (55) as in (56) by using linear observer results, and then transform back to obtain an observer for the nonlinear system (58).

From this point on, we will, without loss of generality, assume that (55) is in what will be referred to as *generalized observer canonical form* **(gocf)** which is shown below

$$
\dot{z} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} z + \begin{pmatrix} a_1(y) \\ \vdots \\ \vdots \\ a_n(y) \end{pmatrix} \tag{59}
$$

$$
y = (0, \dots, 0, 1)\, z
$$

In linear systems theory, it is known that any observable system can be turned into *observer canonical form* through a change of state. Similarly, any observable system of the form (55) can be converted to (gocf)(59) by a linear state transformation.

Now it is clear that the class of nonlinear systems for which the above outlined observer construction will work is exactly the class of nonlinear systems that can be transformed into (gocf) (59).

The following well known result characterizes this class of systems. [1] [2].

Theorem 3 *The nonlinear system (58) is locally (about x^0) equivalent to a linear system in (gocf) (59) under a change of state coordinates $x = T(z)$ iff the following two conditions hold in a neighborhood V of x^0,*

(a) $dim[span\{dh(x), \dots, d\mathcal{L}_f^{n-1}h(x)\}] = n, \qquad \forall x \in V$

(b) The vector field g defined on V via.

$$
\mathcal{L}_g \mathcal{L}_f^j h(x) = \begin{cases} 0, & j = 0, \dots, n-2 \\ 1, & j = n-1 \end{cases}
$$

satisfies

$$
[g, ad_f^k g](x) = 0 \quad k = 1, 3, 5, \dots, 2n-3 \qquad \forall x \in V \tag{\dagger}
$$

Remark: Using the Jacobi identity for the Lie bracket of vector fields, it can be shown that (†) implies $[ad_f^k, ad_f^l] = 0, \qquad 0 \le k + l \le 2n - 2$.

We shall not prove this theorem, for a proof see [1], [2] or [3]. Instead lets try to make some sense of the seemingly mysterious conditions (a) and (b).

Lets consider the case in which (58) is just a linear system,

$$
\begin{aligned} \dot{x} &= Ax \quad (= f(x)) \\ y &= Cx \quad (= h(x)) \end{aligned} \tag{60}
$$

and ask when we can transform it to (gocf) (which is just observer canonical form for a linear system).

$$\begin{aligned} \dot{z} &= \begin{pmatrix} 0 & \cdots & 0 & a_1 \\ 1 & & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_n \end{pmatrix} z \\ y &= (0, \ldots, 0, 1)\, z \end{aligned} \tag{61}$$

Condition (a) of the theorem tells us that

$$\begin{pmatrix} dh \\ \vdots \\ d\mathcal{L}_f^{n-1} h \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{62}$$

must have dimension n (or full rank) which just means that (60) is observable (which, of course, we already knew). Hence we may interpret

$$\begin{pmatrix} dh \\ \vdots \\ d\mathcal{L}_f^{n-1} h \end{pmatrix} \tag{63}$$

as a generalized observability matrix, and condition (a) as an observability condition.

To understand condition (b), lets once again go back to linear systems.

In linear theory, if we would like to determine the transformation from (60) to (61) we consider the observability matrices of both systems and determine that if $x = Tz$ $(= T(z))$ transforms (60) to (61) then their observability matrices must satisfy the following easily derived relation

$$\mathcal{O}_{60} T = \mathcal{O}_{61} \tag{64}$$

from this it is clear to see that since (60) is observable (and hence $\mathcal{O}_{60}$ is nonsingular), then T is given by $T = \mathcal{O}_{60}^{-1} \mathcal{O}_{61}$.

Lets see what happens if we try the same trick for our nonlinear system. The generalized observability matrix for a system in (gocf) is of the form

$$\begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ \cdot & & & \cdot & \\ \cdot & & \cdot & & * \\ \cdot & \cdot & & & \\ 1 & & * & & * \end{pmatrix} \tag{65}$$

Now, if $x = T(z)$ takes (58) to (gocf) then we must have, by comparing generalized observability matrices, that

$$\begin{pmatrix} dh \\ \vdots \\ d\mathcal{L}_f^{n-1}h \end{pmatrix} \frac{\partial T}{\partial z} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ \cdot & & & \cdot & \\ \cdot & & \cdot & & * \\ \cdot & \cdot & & & \\ 1 & & * & & * \end{pmatrix} \tag{66}$$

Comparing this equation, (66), to what we would have in the linear case (64) we note two new complications. First off, in the linear case, the observability matrix $\mathcal{O}_{61}$ for the system in observer canonical form (61) is known, since $a_1, ..., a_n$ are the coefficients of the characteristic equation for A in (60) which can easily be calculated. In the nonlinear case we cannot apriori calculate the generalized observability matrix corresponding to (gocf) (59) since $a_1(\cdot), ..., a_n(\cdot)$ cannot be easily determined apriori (although we will see that they do correspond to a generalized characteristic equation whose solution is a major topic of research). Hence without knowledge of the $a(\cdot)'s$, the $*'s$ in (66) are true unknowns.

Secondly, in the linear case, we knew that the state transformation was linear, $x = Tz$ which meant that $\frac{\partial T}{\partial z} = T$. Hence from (64) we could solve for the desired transformation trivially. But in the nonlinear case, we do not apriori, know the form of the state transformation $x = T(z)$. Provided we can find a $\frac{\partial T}{\partial z}$ that solves (66), we are still left with the onerous task of integrating $\frac{\partial T}{\partial z}$ to find the state transformation $T(z)$.

With the preceding comments in mind, lets return to our task of interpreting condition (b) of the theorem. If we stare at (66) long enough, we might recognize that it is very similar in form to a very important equation that arises in the theory of feedback linearization (see [2, 3]), namely that if there exists a $g(x) \in \mathbf{R}^p$ such that $\mathcal{L}_g h = \ldots = \mathcal{L}_g \mathcal{L}_f^{n-2} h = 0, \quad \mathcal{L}_g \mathcal{L}_f^{n-1} = 1$ then the following equation holds true,

$$\begin{pmatrix} dh \\ \vdots \\ \mathcal{L}_f^{n-1}h \end{pmatrix} (g \quad -ad_f g \quad \ldots \quad \pm\, ad_f^{n-1} g) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ \cdot & & & \cdot & \\ \cdot & & \cdot & & * \\ \cdot & \cdot & & & \\ 1 & & * & & * \end{pmatrix} \tag{67}$$

((67) is easily derived once it is established that $\mathcal{L}_g h = \ldots = \mathcal{L}_g \mathcal{L}_f^{n-2} h = 0, \quad \mathcal{L}_g \mathcal{L}_f^{n-1} = 1 \quad \Leftrightarrow \quad \mathcal{L}_g h = \mathcal{L}_{ad_f} g = \ldots = \mathcal{L}_{ad_f^{n-2}} g = 0, \quad \mathcal{L}_{ad_f^{n-1}} g = 1$ which follows without too much trouble from (54))

Comparing (67) to (66) we see that it is of the proper form if

$$\frac{\partial T}{\partial z} = (g \quad -ad_f g \quad \ldots \quad \pm\, ad_f^{n-1} g) \tag{68}$$

So the above is an excellent candidate solution. A necessary and sufficient condition for the solution of a partial differential equation of the above form (68) (see [2]) is that

$$[ad_f^i g, ad_f^j g] = 0, \quad i, j \leq n-1 \tag{69}$$

We now recognize this as the second half of condition (b) in the theorem (i.e. (†)). (*It is important to realize that this is a very restrictive condition in the sense that it is non-generic. Hence the nonlinear systems that are amenable to the Lie-algebraic method of observer design represent a rather small class nonlinear systems.)*

The first part of condition (b) comes by considering the first column of (67).

$$\begin{pmatrix} dh \\ \vdots \\ \mathcal{L}_f^{n-1} h \end{pmatrix} g = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{70}$$

Hopefully, now the theorem seems a little less cryptic.

If we attempt to use the results of the theorem to find the state transformation T, then we are faced with the daunting task of integrating the n coupled partial differential equations in (68). Hence, an alternate approach may be pursued. In order to get a handle on T, it would be useful to have some knowledge of the functions $a_1(\cdot), ..., a_n(\cdot)$.

To do this we begin by differentiating the output y in (58) and in (gocf) (59),

$$\begin{aligned} & y = h(x) = z_n \\ & \dot{y} = \mathcal{L}_f h = \dot{z}_n = z_{n-1} - a_n(z_n) \\ & \ddot{y} = \mathcal{L}_f^2 h = z_{n-2} - a_{n-1}(z_n) - \mathcal{L}_f a_n(z_n) \\ & \vdots \\ & y^n = \mathcal{L}_f^n h = -\mathcal{L}_f^0 a_1(z_n) - \mathcal{L}_f^1 a_2(z_n) - \cdots - \mathcal{L}_f^{n-1} a_n \end{aligned} \tag{71}$$

Rewriting the last equation gives

$$\mathcal{L}_f^n h + \mathcal{L}_f^{n-1} a_n + \cdots + \mathcal{L}_f^1 a_2 + \mathcal{L}_f^0 a_1 = 0 \tag{72}$$

The above equation (72) is known as the *generalized characteristic equation.*

To give an indication of why the above can be considered as a generalized characteristic equation, consider once again the linear system

$$\begin{aligned} \dot{z} &= Az \\ y &= Cz \end{aligned} \tag{73}$$

and assume the it is in observer canonical form as in (61). Calculating the generalized characteristic equation for this system gives,

$$C^T[A^n + a_n A^{n-1} + \cdots + a_2 A + a_1 I]z = 0 \tag{74}$$

Looking just at the part of (74) in $[\cdots]$, we see from the Cayley-Hamilton theorem that $a_1, \ldots, a_n$ are the coefficients of the characteristic equation for A. Due to these considerations, it is clear the generalized characteristic equation is indeed a generalization of the standard characteristic equation $det(sI - A) = 0$.

If it is possible to solve the generalized characteristic equation for $a_1, \ldots, a_n$ then we can determine the necessary state transformation to bring the system to (gocf) by using (71)

$$\begin{aligned} & z_n = h(x) \ (= T_n^{-1}(x)) \\ & z_{n-1} = \mathcal{L}_f(h(x)) + a_n(h(x)) \ (= T_{n-1}^{-1}(x)) \\ & \vdots \end{aligned} \tag{75}$$

So another characterization of when a system can be transformed to (gocf) is if and only if the generalized characteristic equation (72) has a solution.

The question remains as to how to solve the generalized characteristic equation. The fact is that this is a very difficult problem. Keller [13] (considering systems with inputs) derived necessary and sufficient conditions under which a solution to the generalized characteristic equation is possible. He converts the problem of solving the generalized characteristic equation into the problem of solving a number of partial differential equations. For a second order system, his conditions give 3 pde's, for a third order system, we are left to solve 6 pde's, and so on. With increasing order, the number of pde's rises sharply, leaving this approach tractable only for systems of low order.

A more practical approach has been proposed by Bortoff and Lynch [12]. They use spline functions to solve the generalized characteristic equation approximately. An advantage to this approach is that is may also be used as an attempt to solve the problem in which condition (b) (specifically (†)) of theorem 3 is not satisfied, which is the generic case, by also solving the generalized characteristic equation approximately (in this case, there is no exact solution). The weakness in this approach is that by only using an approximate solution to the generalized characteristic equation, it can no longer be guaranteed that the observer state will converge to the true state.

Before we conclude this section, a couple of notes are in order. Once again, the Lie-algebraic approach to observer design, produces an observer only locally. Specifically where the state transformation $x = T(z)$ is valid. In contrast to the observer designed by the method of extended linearization, which had to be designed about a fixed point of the system, an observer designed using Lie-algebraic methods is valid in any region where a state transformation to (gocf) can be found. Yet this transformation may only exist locally around our area of interest. In this sense, the design methods presented in this section produce only local results.

Finally, it is important to state that the results in this section can be extended to systems with multiple inputs and outputs (see [11, 13, 6, 7]).

8 Conclusions

In this paper, we presented three methods for designing observers for nonlinear systems. The first method, based on Lyapunov stability results, is able to produce strong sufficient results concerning the existence of observers, and even when they can be guaranteed global. Yet, as is the complication with all results relying on Lyapunov theory, they provide no constructive procedure for determining Lyapunov functions nor for designing an observer. For low order systems it may be possible to use trial an error to design an observer that meets the conditions of the theorems, but for high order system, this task may be virtually impossible.

Next, we presented observer design by the method of extended linearization. This observer works well for its intended use, which is that of an observer to be used in an observer based controller. But its applications do not extend much further since it is designed to operate only locally about a fixed point. Furthermore, even when used as intended, it is not trivial to extend to multi-input system.

Finally, we explored Lie-algebraic techniques for nonlinear observer design. The advantage of these techniques, were that they attempt to exploit our knowledge of linear observer design by reducing a nonlinear observer problem to one that can be handled by linear techniques. The drawbacks are that for this technique to be applicable, the nonlinear system must satisfy a non-generic condition ((b) of the theorem), and even when this is so, finding the necessary state transformation will most likely be a difficult problem in itself. Approximate techniques may be used, but then guarantees of an asymptotic observer may be lost.

For a comparison of these three techniques, see the paper by Walcott, Corless, and Zak [10]. In their study a very simple nonlinear system was used in order that all three techniques could simultaneously be applied. In reality, each technique is suited only for problems with special structure, and comparisons are difficult to make. What is clear though, is that nonlinear observer design is still an open area for research, especially in attempting to broaden and adapt the above techniques so that they may apply to larger classes of nonlinear systems.

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