Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 420 (2007) 329–338
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the Lyapunov and Stein equations ☆

Fernando C. Silva [a], Rita Simões [b,*]

[a] Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016 Lisboa, Portugal
[b] Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal

Received 30 September 2005; accepted 11 July 2006
Available online 13 October 2006
Submitted by G. de Oliveira

Abstract

Let $L \in \mathbb{C}^{n\times n}$ and let $H, K \in \mathbb{C}^{n\times n}$ be Hermitian matrices. The general inertia theorem gives a complete set of relations between the similarity class of L and the congruence class of H, when the Lyapunov equation $LH + HL^* = K$ is satisfied and $K > 0$.

In this paper, we give some relations between the similarity class of L and the congruence class of K, when the Lyapunov equation is satisfied and $H > 0$.

We also consider the corresponding problem with the Stein equation.

AMS classification: 93D05

Keywords: Lyapunov equation; Stein equation; Inertia of matrices

1. Introduction

Let $L \in \mathbb{C}^{n\times n}$ and let $H, K \in \mathbb{C}^{n\times n}$ be Hermitian matrices.

The *inertia* of $L \in \mathbb{F}^{n\times n}$ is the triple $\mathrm{In}(L) = (\pi(L), \nu(L), \delta(L))$, where $\pi(L)$, $\nu(L)$ and $\delta(L)$ denote, respectively, the number of eigenvalues of L with real positive part, with real negative part and with real part equal to zero. We shall say that $H, H' \in \mathbb{C}^{n\times n}$ are congruent if there exists a

☆ Research done within the activities of the *Centro de Estruturas Lineares e Combinatórias* and supported by *Fundação para a Ciência e a Tecnologia* (FCT). The work of the second author was also supported by FCT grant SFRH/BD/11133/2002.
* Corresponding author. Tel.: +351 234370359; fax: +351 234382014.
E-mail addresses: fcsilva@fc.ul.pt (F.C. Silva), rsimoes@mat.ua.pt (R. Simões).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.07.025

nonsingular matrix $S \in \mathbb{C}^{n\times n}$ such that $H' = SHS^*$. It is well-known that two Hermitian matrices are congruent if and only if they have the same inertia.

The article [3] gives a complete set of relations between the inertias of L, H and K when the Lyapunov equation

$$LH + HL^* = K \tag{1}$$

holds. For every nonsingular matrix $S \in \mathbb{C}^{n\times n}$, (1) is equivalent to

$$(SLS^{-1})(SHS^*) + (SHS^*)(SLS^{-1})^* = SKS^*. \tag{2}$$

From this simple remark, it follows that the main inertia theorem [6,7] gives a complete set of relations between the similarity class of L and the congruence class of H when (1) holds and K is positive definite. More precisely, there exists $L' \in \mathbb{C}^{n\times n}$ similar to L and there exists a Hermitian matrix $H' \in \mathbb{C}^{n\times n}$ congruent to H such that $L'H' + H'L'^*$ is positive definite if and only if $\delta(L) = 0$ and $\mathrm{In}(L) = \mathrm{In}(H)$. These results suggest the following problem.

Problem 1. Find a complete set of relations between the similarity class of L and the congruence classes of H and K when (1) holds.

A complete answer to Problem 1 seems to be very hard. Three partial answers can be found in [4, Theorems 1–3]. In all these theorems, $\min\{\pi(K), \nu(K)\} > 0$, $\delta(H) = 0$, $\delta(L) = n$ and L is nonderogatory of a special type. Another closely related result is [2, Corollary III] that can be found in this paper as Corollary 8.

In this paper, we prove some relations between the similarity class of L and the congruence class of K, when (1) holds and $H > 0$. Corresponding results with the Stein equation can be obtained using a Cayley transform.

2. On the Lyapunov equation

Let $L \in \mathbb{C}^{n\times n}$. Let $i(L)$ be the number of nonconstant invariant polynomials of L. Let $i_+(L)$ (respectively, $i_-(L)$, $i_0(L)$) be the number of invariant polynomials of L with at least one root with positive (respectively, negative, zero) real part. Recall that, if λ is an eigenvalue of L, then the geometric multiplicity of λ is the number of invariant polynomials of L with λ as a root. Let $i_0^2(L)$ be the number of invariant polynomials of L with at least one root with zero real part and multiplicity $\geqslant 2$.

Theorem 2. *Let $L \in \mathbb{C}^{n\times n}$, let $K \in \mathbb{C}^{n\times n}$ be Hermitian and let $H \in \mathbb{C}^{n\times n}$ be positive definite. If $LH + HL^* = K$, then*

$$\pi(K) \geqslant \max\{i_+(L), i_0^2(L)\}, \tag{3}$$

$$\nu(K) \geqslant \max\{i_-(L), i_0^2(L)\}, \tag{4}$$

$$\delta(K) \geqslant 2i_0(L) - n, \tag{5}$$

$$\pi(K) + \delta(K) \geqslant i_0(L), \tag{6}$$

$$\nu(K) + \delta(K) \geqslant i_0(L), \tag{7}$$

and the following special case does not hold:

$$(\mathbf{S}) \quad \delta(L) > \delta(K), \quad i_0^2(L) = 0 \quad \text{and} \quad \min\{\pi(K), \nu(K)\} = 0.$$

The next theorem can be viewed as giving a solution to Problem 1 in the following two cases: (i) $H > 0$ and L is nonderogatory; (ii) $H > 0$ and $K \geqslant 0$. The proofs will be given later.

Theorem 3. *Let $L \in \mathbb{C}^{n\times n}$ and let $K \in \mathbb{C}^{n\times n}$ be Hermitian. Suppose that either L is nonderogatory or K is positive semidefinite. Suppose that the special case* (**S**) *is not satisfied.*

Then there exists a positive definite matrix $H \in \mathbb{C}^{n\times n}$ such that

$$\operatorname{In}(LH + HL^*) = \operatorname{In}(K)$$

if and only if (3)–(7) *are satisfied.*

Remark 4. Theorem 3 is not always true, when L is derogatory and K is not positive semidefinite, as the following example shows:

Suppose that $L = \lambda_1 I_p \oplus \lambda_2 I_q$, with $\lambda_1 \neq \lambda_2$ and $\Re(\lambda_1) = \Re(\lambda_2) = 0$. Suppose that there exists a positive definite matrix

$$H = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{1,2}^* & H_{2,2} \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad \text{where } H_{1,1} \in \mathbb{C}^{p\times p} \tag{8}$$

such that $\operatorname{In}(LH + HL^*) = \operatorname{In}(K)$. Then

$$LH + HL^* = \begin{bmatrix} 0 & (\lambda_1 - \lambda_2)H_{1,2} \\ (\lambda_2 - \lambda_1)H_{1,2}^* & 0 \end{bmatrix}.$$

It follows from [1], that $\pi(K) = \nu(K)$.

Now let $p = 3$, $q = 2$ and let $K \in \mathbb{C}^{5\times 5}$ be a Hermitian matrix such that $\operatorname{In}(K) = (2, 1, 2)$. Then (**S**) is not satisfied and (3)–(7) hold. Therefore Theorem 3 is not true in this case.

As (1) and (2) are equivalent, it follows that, in order to prove Theorems 2 and 3, L can be replaced by any similar matrix. Recall that L is similar to

$$C(f_1) \oplus \cdots \oplus C(f_r), \tag{9}$$

where $f_1|\cdots|f_r$ are the nonconstant invariant polynomials of L and $C(f_i)$ is the companion matrix of f_i, $i \in \{1, \dots, r\}$. Also recall that L is nonderogatory if and only if $i(L) = r = 1$.

Proof of Theorem 2. Suppose that $LH + HL^* = K$.

Proof of $\pi(K) \geqslant i_+(L)$. Let $p = i_+(L)$. Then the Jordan canonical form of L is permutation similar to a matrix of the form

$$L' = \begin{bmatrix} \lambda I_p & 0 \\ * & * \end{bmatrix}, \tag{10}$$

where $\Re(\lambda) > 0$. Without loss of generality, $L = L'$. Partition H as in (8). Then $2\Re(\lambda)H_{1,1}$ is positive definite and is a principal submatrix of K. According to the interlacing inequalities for the eigenvalues, $\pi(K) \geqslant p = i_+(L)$.

Proof of $\pi(K) \geqslant i_0^2(L)$. Let $p = i_0^2(L)$. Then the Jordan canonical form of L is permutation similar to a matrix of the form

$$L' = \begin{bmatrix} \lambda I_p & 0 & 0 \\ I_p & \lambda I_p & 0 \\ * & * & * \end{bmatrix},$$

where $\Re(\lambda) = 0$. Without loss of generality, $L = L'$. Partition H as follows:

$$H = \begin{bmatrix} H_{1,1} & H_{1,2} & H_{1,3} \\ H_{1,2}^* & H_{2,2} & H_{2,3} \\ H_{1,3}^* & H_{2,3}^* & H_{3,3} \end{bmatrix}, \quad \text{where } H_{1,1}, H_{2,2} \in \mathbb{C}^{p\times p}.$$

Then

$$\begin{bmatrix} 0 & H_{1,1} \\ H_{1,1} & H_{1,2} + H_{1,2}^* \end{bmatrix}$$

is a principal submatrix of K and has inertia $(p, p, 0)$. According to the interlacing inequalities for the eigenvalues, $\pi(K) \geqslant p = i_0^2(L)$.

Proof of $\delta(K) \geqslant 2i_0(L) - n$ *and* $\pi(K) + \delta(K) \geqslant i_0(L)$. Let $p = i_0(L)$. Then the Jordan canonical form of L is permutation similar to a matrix of the form (10), where $\Re(\lambda) = 0$. Without loss of generality, $L = L'$. Partition H as in (8). Then 0_p is a principal submatrix of K. It follows that $\operatorname{rank} K \leqslant 2(n - p)$ and $\delta(K) = n - \operatorname{rank} K \geqslant 2p - n = 2i_0(L) - n$. On the other hand, according to the interlacing inequalities for the eigenvalues, $\pi(K) + \delta(K) \geqslant p = i_0(L)$.

Proof that (**S**) *is not satisfied*. By induction on n. If $n = 1$, $\operatorname{In}(L) = \operatorname{In}(LH + HL^*) = \operatorname{In}(K)$ and the result is trivial. Suppose that $n \geqslant 2$. In order to get a contradiction, suppose that (**S**) is satisfied. As $\delta(L) > 0$ and $i_0^2(L) = 0$, L is similar to a matrix of the form $[\lambda] \oplus L_0$, where $\Re(\lambda) = 0$. Without loss of generality, suppose that $L = [\lambda] \oplus L_0$. Suppose that

$$H = \begin{bmatrix} h & g \\ g^* & H_0 \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad \text{where } H_0 \in \mathbb{C}^{(n-1)\times(n-1)}.$$

Then

$$K = LH + HL^* = \begin{bmatrix} 0 & g(\lambda I_{n-1} + L_0^*) \\ (-\lambda I_{n-1} + L_0)g^* & L_0 H_0 + H_0 L_0^* \end{bmatrix}.$$

If $g(\lambda I_{n-1} + L_0^*) \neq 0$, then K contains a 2×2 principal submatrix M with a principal entry equal to zero and its nonprincipal entries different from zero. Then $\operatorname{In}(M) = (1, 1, 0)$. From the interlacing inequalities for the eigenvalues of Hermitian matrices, $\pi(K) \geqslant \pi(M) = 1$ and $\nu(K) \geqslant \nu(M) = 1$, what contradicts (**S**).

Suppose that $g(\lambda I_{n-1} + L_0^*) = 0$. Let $K_0 = L_0 H_0 + H_0 L_0^*$. Note that $i_0^2(L_0) = 0$ and

$$\delta(L_0) = \delta(L) - 1 > \delta(K) - 1 = \delta(K_0).$$

According to the induction assumption, $\min\{\pi(K_0), \nu(K_0)\} > 0$. As $\pi(K) = \pi(K_0)$ and $\nu(K) = \nu(K_0)$, we have again a contradiction. □

Lemma 5. *Let* $\lambda_1, \ldots, \lambda_n$ *be elements of* $\mathbb{C}$ *ordered so that, if* $\lambda_i = \lambda_j$, *for some* $i < j$, *then* $\lambda_i = \lambda_k$, *for every* $k \in \{i, \ldots, j\}$. *Let*

$$T = [t_{i,j}] = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{11}$$

be an upper triangular matrix such that $t_{i,i+1} \neq 0$, *for every* $i \in \{1, \ldots, n-1\}$. *Then* T *is non-derogatory*.

Proof. The number of nonconstant invariant polynomials of T is equal to

$$n - R_{\mathbb{C}}(T), \quad \text{where } R_{\mathbb{C}}(T) = \min_{\lambda\in\mathbb{C}} \operatorname{rank}(\lambda I_n - T)$$

(cf. [5]). Bearing in mind the form of T, $R_{\mathbb{C}}(T) = n - 1$. Therefore $i(T) = 1$, that is, T is nonderogatory. $\square$

Lemma 6. *Let $a, b \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Let $K \in \mathbb{C}^{3\times 3}$ be a Hermitian matrix with* $\mathrm{In}(K) \geqslant (1, 1, 0)$. *Then for every $z \in \mathbb{C} \setminus \{0\}$, there exists $y \in \mathbb{C}$ such that the matrix*

$$T = \begin{bmatrix} ia & z & y \\ 0 & ib & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

satisfies $\mathrm{In}(T + T^*) = \mathrm{In}(K)$.

Proof. Let

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z^{-1} & -\bar{y}\bar{z}^{-1} & 1 \end{bmatrix}.$$

Then $T + T^*$ is congruent to

$$S(T + T^*)S^* = \begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix} \oplus \begin{bmatrix} 2\Re(\lambda) - 2\Re(yz^{-1}) \end{bmatrix}.$$

Clearly y can be chosen so that $\mathrm{In}(T + T^*) = \mathrm{In}(K)$. $\square$

Lemma 7. *Let $L \in \mathbb{C}^{n\times n}$ be nonderogatory with $\nu(L) = 0$. Let $K \in \mathbb{C}^{n\times n}$ be Hermitian. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L ordered so that, if $\lambda_i = \lambda_j$, for some $i < j$, then $\lambda_i = \lambda_k$, for every $k \in \{i, \ldots, j\}$, and, if $\Re(\lambda_i) = 0$, for some i, then $\Re(\lambda_k) = 0$, for every $k \in \{1, \ldots, i\}$. Suppose that the following special case is not satisfied*:

($\mathbf{S}'$) $n \geqslant 2$, $\delta(L) > 0$ *and* $\min\{\pi(K), \nu(K)\} = 0$.

If (3)–(7) *are satisfied, then L is similar to a matrix of the form* (11) *such that $t_{i,i+1} \neq 0$, for every $i \in \{1, \ldots, n-1\}$, and* $\mathrm{In}(T + T^*) = \mathrm{In}(K)$.

Proof. By induction on n. When $n = 1$, (3)–(7) imply that $\mathrm{In}(K) = \mathrm{In}(L) = \mathrm{In}(L + L^*)$.

Suppose that $n = 2$. For every $t \in \mathbb{C} \setminus \{0\}$, L is similar to

$$T = \begin{bmatrix} \lambda_1 & t \\ 0 & \lambda_2 \end{bmatrix}.$$

If $\Re(\lambda_1) = 0$, then, for every $t \in \mathbb{C} \setminus \{0\}$, $\mathrm{In}(T + T^*) = (1, 1, 0) = \mathrm{In}(K)$. Now suppose that $\Re(\lambda_1) > 0$. Then $\Re(\lambda_2) > 0$, (3) implies that $\pi(K) \geqslant 1$ and

$$\mathrm{In}(T + T^*) = \begin{cases} (2, 0, 0), & \text{when } 0 < |t| < 2\sqrt{\Re(\lambda_1)\Re(\lambda_2)}, \\ (1, 1, 0), & \text{when } |t| > 2\sqrt{\Re(\lambda_1)\Re(\lambda_2)}, \\ (1, 0, 1), & \text{when } |t| = 2\sqrt{\Re(\lambda_1)\Re(\lambda_2)}. \end{cases}$$

Suppose that $n \geqslant 3$. Let $L_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ be a nonderogatory matrix with eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. If $n = 3$ and $\mathrm{In}(K) = (1, 1, 1)$, let $K_0 = \mathrm{diag}(1, -1)$; otherwise, let $K_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ be a Hermitian matrix such that

$$\min\{\pi(K), 1\} \leqslant \pi(K_0) \leqslant \pi(K),$$
$$\min\{\nu(K), 1\} \leqslant \nu(K_0) \leqslant \nu(K),$$
$$\min\{\delta(K), 1\} \leqslant \delta(K_0) \leqslant \delta(K).$$

According to the induction assumption, L_0 is similar to a matrix of the form

$$T_0 = [t_{i,j}] = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} * & t \\ 0 & \lambda_{n-1} \end{bmatrix} \in \mathbb{C}^{(n-1)\times(n-1)}, \tag{12}$$

where $t \in \mathbb{C}^{(n-2)\times 1}$, such that $t_{i,i+1} \neq 0$, for every $i \in \{1, \ldots, n-2\}$, and $\mathrm{In}(T_0 + T_0^*) = \mathrm{In}(K_0)$.

Case 1. Suppose that $\Re(\lambda_{n-1}) > 0$. Let

$$X_0 = \begin{bmatrix} I_{n-2} & -(2\Re(\lambda_{n-1}))^{-1}t \\ 0 & 1 \end{bmatrix} \in \mathbb{C}^{(n-1)\times(n-1)}.$$

Then $T_0 + T_0^*$ is congruent to

$$X_0(T_0 + T_0^*)X_0^* = S \oplus [2\Re(\lambda_{n-1})] \in \mathbb{C}^{(n-1)\times(n-1)}$$

for some $S \in \mathbb{C}^{(n-2)\times(n-2)}$. Then $\mathrm{In}(S) = \mathrm{In}(K_0) - \mathrm{In}[\lambda_{n-1}] = \mathrm{In}(K_0) - (1, 0, 0)$. According to the induction assumption, there exists $\nu \in \mathbb{C} \setminus \{0\}$ such that

$$R = \begin{bmatrix} \lambda_{n-1} & \nu \\ 0 & \lambda_n \end{bmatrix}$$

satisfies $\mathrm{In}(R + R^*) = \mathrm{In}(K) - \mathrm{In}(S) \geqslant \mathrm{In}(K_0) - \mathrm{In}(S) = (1, 0, 0)$. Let

$$T = \left[\begin{array}{c|c} T_0 & \begin{matrix} (2\Re(\lambda_{n-1}))^{-1}\nu t \\ \nu \end{matrix} \\ \hline 0 & \lambda_n \end{array}\right].$$

According to Lemma 5, T is nonderogatory. As T and L have the same eigenvalues, they are similar. Let $X = X_0 \oplus [1]$. Then $T + T^*$ is congruent to

$$X(T + T^*)X^* = S \oplus (R + R^*),$$

what shows that $\mathrm{In}(T + T^*) = \mathrm{In}(K)$.

Case 2. Suppose that $\Re(\lambda_{n-1}) = 0$. Note that all the principal entries of $T_0 + T_0^*$ are equal to 0. Partition T_0 as follows:

$$T_0 = \begin{bmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{bmatrix},$$

where $T_{2,2} \in \mathbb{C}^{2\times 2}$. Then $\mathrm{In}(T_{2,2} + T_{2,2}^*) = (1, 1, 0)$. When $n = 3$, we have $T_0 = T_{2,2}$ and the following argument should be adapted accordingly. Let

$$X_0 = \begin{bmatrix} I_{n-3} & -T_{1,2}(T_{2,2} + T_{2,2}^*)^{-1} \\ 0 & I_2 \end{bmatrix} \in \mathbb{C}^{(n-1)\times(n-1)}.$$

Then $T_0 + T_0^*$ is congruent to

$$X_0(T_0 + T_0^*)X_0^* = S \oplus (T_{2,2} + T_{2,2}^*) \in \mathbb{C}^{(n-1)\times(n-1)}$$

for some $S \in \mathbb{C}^{(n-3)\times(n-3)}$. Then $\mathrm{In}(S) = \mathrm{In}(K_0) - \mathrm{In}(T_{2,2} + T_{2,2}^*)$. According to Lemma 6, there exists $y \in \mathbb{C}$ such that

$$R = \left[\begin{array}{c|c} T_{2,2} & \begin{matrix} y \\ 1 \end{matrix} \\ \hline 0 & \lambda_n \end{array}\right]$$

satisfies $\mathrm{In}(R + R^*) = \mathrm{In}(K) - \mathrm{In}(S) \geqslant \mathrm{In}(K_0) - \mathrm{In}(S) = \mathrm{In}(T_{2,2} + T_{2,2}^*) = (1, 1, 0)$. Let

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} & T_{1,2}(T_{2,2} + T_{2,2}^*)^{-1}M \\ 0 & T_{2,2} & M \\ 0 & 0 & \lambda_n \end{bmatrix}, \quad \text{where } M = \begin{bmatrix} y \\ 1 \end{bmatrix}.$$

According to Lemma 5, T is nonderogatory. As T and L have the same eigenvalues, they are similar. Let $X = X_0 \oplus [1]$. Then $T + T^*$ is congruent to

$$X(T + T^*)X^* = S \oplus (R + R^*),$$

which shows that $\mathrm{In}(T + T^*) = \mathrm{In}(K)$. □

Proof of Theorem 3. Suppose that either L is nonderogatory or K is positive semidefinite. Suppose that (**S**) is not satisfied. Bearing in mind Theorem 2, it remains to prove that, if (3)–(7) are satisfied, then there exists a positive definite matrix $H \in \mathbb{C}^{n\times n}$ such that $\mathrm{In}(LH + HL^*) = \mathrm{In}(K)$. This proof is by induction on n. If L is scalar, then (3)–(7) imply that $\mathrm{In}(K) = \mathrm{In}(L) = \mathrm{In}(L + L^*)$. Suppose that L is nonscalar.

Case 1. Suppose that $L \in \mathbb{C}^{n\times n}$ is nonderogatory.

Suppose that $\nu(L) = 0$. If (**S**′) is not satisfied, then, according to Lemma 7, L is similar to a matrix $T = X^{-1}LX$, where $X \in \mathbb{C}^{n\times n}$ is nonsingular, such that $\mathrm{In}(T + T^*) = \mathrm{In}(K)$; then $\mathrm{In}(LH + HL^*) = \mathrm{In}(K)$, with $H = XX^*$.

Now suppose that (**S**′) is satisfied. Then (**S**′) and (3)–(7) imply that $i_0^2(L) = 0$, $i_0(L) = 1$ and $\delta(K) > 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L. Without loss of generality, suppose that $\Re(\lambda_1) = 0$. As (**S**) is not satisfied, $\delta(K) \geqslant \delta(L)$.

Suppose that $n = 2$. Suppose that $\Re(\lambda_2) = 0$. As $i_0^2(L) = 0$, $\lambda_1 \neq \lambda_2$ and L is similar to $\mathrm{diag}(\lambda_1, \lambda_2)$. Without loss of generality, $L = \mathrm{diag}(\lambda_1, \lambda_2)$. Then $\mathrm{In}(L + L^*) = (0, 0, 2) = \mathrm{In}(K)$. Suppose that $\Re(\lambda_2) \neq 0$. As $\nu(L) = 0$, $\Re(\lambda_2) > 0$. Without loss of generality, $L = \mathrm{diag}(\lambda_1, \lambda_2)$. It is easy to see that $\mathrm{In}(L + L^*) = (1, 0, 1) = \mathrm{In}(K)$.

Suppose that $n \geqslant 3$. As $\Re(\lambda_1) = 0$ and $i_0^2(L) = 0$, L is similar to $[\lambda_1] \oplus L_0$, for some $L_0 \in \mathbb{C}^{(n-1)\times(n-1)}$. Without loss of generality, $L = [\lambda_1] \oplus L_0$. Let $K_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ be a Hermitian matrix such that $\mathrm{In}(K_0) = \mathrm{In}(K) - (0, 0, 1)$. According to the induction assumption, there exists a positive definite matrix $H_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ such that $\mathrm{In}(L_0H_0 + H_0L_0^*) = \mathrm{In}(K_0)$. Let $H = [1] \oplus H_0$. Then $\mathrm{In}(LH + HL^*) = \mathrm{In}(K)$.

The proof has been completed when $\nu(L) = 0$. The case $\pi(L) = 0$ is analogous.

Now suppose that $\nu(L) > 0$ and $\pi(L) > 0$. Then $\nu(K) > 0$ and $\pi(K) > 0$ and L is similar to a matrix $L_+ \oplus L_-$, where $L_+ \in \mathbb{C}^{n_+\times n_+}$ is nonderogatory, $\nu(L_+) = 0$, $L_- \in \mathbb{C}^{n_-\times n_-}$ is nonderogatory, $\pi(L_-) = \delta(L_-) = 0$. Let

$$\pi_+ = \max\{1, \min\{\pi(K), n_+\} - 1\},$$

$$\nu_+ = \min\{\nu(K), n_+ - \pi_+\},$$

$$\delta_+ = n_+ - \pi_+ - \nu_+,$$

$$\nu_- = \min\{\nu(K), n_-\},$$

$$\pi_- = \min\{\pi(K), n_- - \nu_-\},$$

$$\delta_- = n_- - \pi_- - \nu_-.$$

It is not hard to see that the numbers $\pi_+, \nu_+, \delta_+, \pi_-, \nu_-, \delta_-$ are nonnegative and the following inequalities are satisfied:

$$\pi_+ \geqslant 1, \tag{13}$$

$$\nu_+ \geqslant 1, \quad \text{unless} \quad n_+ = 1, \tag{14}$$

$$\nu_- \geqslant 1, \tag{15}$$

$$\max\{\pi_+, \pi_-\} \leqslant \pi(K) \leqslant \min\{n_+ + \pi_-, n_- + \pi_+\}, \tag{16}$$

$$\max\{\nu_+, \nu_-\} \leqslant \nu(K) \leqslant \min\{n_+ + \nu_-, n_- + \nu_+\}, \tag{17}$$

$$\pi(K) - \nu(K) \leqslant \pi_+ + \pi_-, \tag{18}$$

$$\nu(K) - \pi(K) \leqslant \nu_+ + \nu_-. \tag{19}$$

According to the induction assumption, there exist positive definite matrices $H_+ \in \mathbb{C}^{n_+ \times n_+}$ and $H_- \in \mathbb{C}^{n_- \times n_-}$ such that $\mathrm{In}(L_+H_+ + H_+L_+^*) = (\pi_+, \nu_+, \delta_+)$ and $\mathrm{In}(L_-H_- + H_-L_-^*) = (\pi_-, \nu_-, \delta_-)$. According to [1], there exists $X \in \mathbb{C}^{n_+ \times n_-}$ such that

$$\begin{bmatrix} L_+H_+ + H_+L_+^* & X \\ X^* & L_-H_- + H_-L_-^* \end{bmatrix} \tag{20}$$

has the same inertia as K. As L_+ and L_- do not have common eigenvalues, L is similar to

$$\begin{bmatrix} L_+ & XH_-^{-1} \\ 0 & L_- \end{bmatrix}.$$

Without loss of generality, suppose that L has this form. Let $H = H_+ \oplus H_-$. Then $LH + HL^*$ has the form (20).

Case 2. Suppose that L is derogatory and $K \geqslant 0$. Without loss of generality, suppose that L has the form (9). Suppose that $L = L_0 \oplus L_r$, where $L_0 = C(f_1) \oplus \cdots \oplus C(f_{r-1}) \in \mathbb{C}^{n_0 \times n_0}$, $L_r = C(f_r) \in \mathbb{C}^{n_r \times n_r}$. Let

$$\pi_0 = \max\{i_+(L_0), \delta(L_r) + \pi(K) - n_r\},$$
$$\delta_0 = n_0 - \pi_0,$$
$$\pi_r = \pi(K) - \pi_0,$$
$$\delta_r = n_r - \pi_r.$$

As $K \geqslant 0$, (4) implies that $i_-(L) = i_0^2(L) = 0$. As (**S**) is not satisfied, $\delta(L) \leqslant \delta(K)$. It is not hard to prove that

$$\pi_0 \geqslant i_+(L_0),$$
$$\delta_0 \geqslant \delta(L_0) \geqslant i_0(L_0) \geqslant 2i_0(L_0) - n_0,$$
$$\pi_r \geqslant i_+(L_r),$$
$$\delta_r \geqslant \delta(L_r) \geqslant i_0(L_r) \geqslant 2i_0(L_r) - n_r.$$

According to the induction assumption, there exist positive definite matrices $H_0 \in \mathbb{C}^{n_0 \times n_0}$ and $H_r \in \mathbb{C}^{n_r \times n_r}$ such that $\mathrm{In}(L_0H_0 + H_0L_0^*) = (\pi_0, 0, \delta_0)$ and $\mathrm{In}(L_rH_r + H_rL_r^*) = (\pi_r, 0, \delta_r)$. Let $H = H_0 \oplus H_r$. Then $\mathrm{In}(LH + HL^*) = \mathrm{In}(K)$. $\square$

Corollary 8 [2, Corollary III]. *There exists a positive definite matrix $H \in \mathbb{C}^{n\times n}$ such that $LH + HL^* \geqslant 0$ if and only if L is positive semistable and L does not have elementary divisors with multiple imaginary roots.*

Proof. Suppose that there exists a positive definite matrix $H \in \mathbb{C}^{n\times n}$ such that $K = LH + HL^*$ is positive semidefinite. According to Theorem 2, $\max\{i_-(L), i_0^2(L)\} = 0$. That is, L is positive semistable and L does not have elementary divisors with multiple imaginary roots.

Conversely, suppose that L is positive semistable and L does not have elementary divisors with multiple imaginary roots. Let $K \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with $\mathrm{In}(K) = (i_+(L), 0, n - i_+(L))$. It is easy to see that (3)–(7) are satisfied and $\delta(L) \leqslant \delta(K)$. According to Theorem 3, there exists a positive definite matrix $H \in \mathbb{C}^{n\times n}$ such that $\mathrm{In}(LH + HL^*) = \mathrm{In}(K)$. Therefore $LH + HL^* \geqslant 0$. $\square$

3. On the Stein equation

In order to study the corresponding problems with the Stein equation, we shall use a Cayley transform. Let $A \in \mathbb{C}^{n\times n}$. Choose a complex number θ of modulus 1 such that $\theta I_n + A$ is nonsingular. Let

$$L_\theta(A) = (\theta I_n + A)^{-1}(\theta I_n - A).$$

Let $\mu_1, \ldots, \mu_t$ be the eigenvalues of A, without repetitions. For every $k \in \{1, \ldots, t\}$, let

$$\lambda_k = (\theta + \mu_k)^{-1}(\theta - \mu_k).$$

Let

$$J = \bigoplus_{j=1}^{s} J_{p_j}(\mu_{k_j})$$

be a Jordan canonical form of A, where $J_{p_j}(\mu_{k_j})$ is the Jordan block of size $p_j \times p_j$ with eigenvalue μ_{k_j}, $j \in \{1, \ldots, s\}$. If $X \in \mathbb{C}^{n\times n}$ is a nonsingular matrix such that $X^{-1}AX = J$, then

$$X^{-1}L_\theta(A)X = L_\theta(J) = \bigoplus_{j=1}^{s} L_\theta(J_{p_j}(\mu_{k_j})).$$

For every $j \in \{1, \ldots, s\}$, the characteristic matrix $xI_{p_j} - L_\theta(J_{p_j}(\mu_{k_j}))$ is equivalent to

$$(\theta I_{p_j} + J_{p_j}(\mu_{k_j}))x - \theta I_{p_j} + J_{p_j}(\mu_{k_j}).$$

The last matrix has two $(p_j - 1) \times (p_j - 1)$ submatrices with determinants $(x + 1)^{p_j - 1}$ and $((\theta + \mu_{k_j})x - \theta + \mu_{k_j})^{p_j - 1}$, respectively. These determinants are relatively prime. It follows that $L_\theta(J_{p_j}(\mu_{k_j}))$ has $p_j - 1$ constant invariant polynomials. Moreover, $L_\theta(J_{p_j}(\mu_{k_j}))$ has characteristic polynomial $(x - \lambda_{k_j})^{p_j}$. Then the elementary divisors of $L_\theta(A)$ are

$$(x - \lambda_{k_1})^{p_1}, \ldots, (x - \lambda_{k_s})^{p_s}.$$

The following proposition follows easily.

Proposition 9. *If A has invariant polynomials $\alpha_1 | \cdots | \alpha_n$, where*

$$\alpha_l = (x - \mu_1)^{q_{l,1}} \cdots (x - \mu_t)^{q_{l,t}}, \quad l \in \{1, \ldots, n\},$$

then $L_\theta(A)$ *has invariant polynomials* $\beta_1|\cdots|\beta_n$, *where*

$$\beta_l = (x-\lambda_1)^{q_{l,1}}\cdots(x-\lambda_t)^{q_{l,t}}, \quad l \in \{1,\ldots,n\},$$

The following propositions are not hard to prove.

Proposition 10. *For every* $k \in \{1,\ldots,t\}$, $|\mu_k| < 1$ (*respectively*, $|\mu_k| = 1$, $|\mu_k| > 1$) *if and only if* $\Re(\lambda_k) > 0$ (*respectively*, $\Re(\lambda_k) = 0$, $\Re(\lambda_k) < 0$).

Proposition 11. *For every Hermitian matrix* $H \in \mathbb{C}^{n\times n}$, $H - AHA^*$ *and* $L_\theta(A)H + H(L_\theta(A))^*$ *are congruent.*

Using these propositions, it is easy to obtain results, analogous to Theorems 2, 3 and Corollary 8, about the Stein equation $H - AHA^* = K$. These results are quite obvious and, therefore, we do not write them here.

References

[1] B.E. Cain, E.M. Sá, The inertia of a Hermitian matrix having prescribed complementary principal submatrices, Linear Algebra Appl. 37 (1981) 161–171.
[2] D. Carlson, H. Schneider, Inertia theorems for matrices: the semidefinite case, J. Math. Anal. Appl. 6 (1963) 430–446.
[3] L.M. DeAlba, C.R. Johnson, Possible inertia combinations in the Stein and Lyapunov equations, Linear Algebra Appl. 222 (1995) 227–240.
[4] L.M. DeAlba, Inertia of the Stein transformation with respect to some nonderogatory matrices, Linear Algebra Appl. 241/243 (1996) 191–201.
[5] G.N. Oliveira, E.M. Sá, J.A. Dias da Silva, On the eigenvalues of the matrix $A + XBX^{-1}$, Linear and Multilinear Algebra 5 (1977) 119–128.
[6] A. Ostrowski, H. Schneider, Some theorems on the inertia of general matrices, J. Math. Anal. Appl. 4 (1962) 72–84.
[7] O. Taussky, A generalization of a theorem of Lyapunov, SIAM J. Appl. Math. 9 (1961) 640–643.