Observability, Eigenvalues, and Kalman Filtering

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In higher order Kalman filtering applications the analyst often has very little insight into the nature of the observability of the system. For example, there are situations where the filter may be estimating certain linear combinations of state variables quite well, but this is not apparent from a glance at the error covariance matrix. It is shown here that the eigenvalues and eigenvectors of the error covariance matrix, when properly normalized, can provide useful information about the observability of the system.

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INTRODUCTION

In the two decades since R.E. Kalman [1] introduced the idea of least square recursive filtering, there have been fantastic advances in computer technology. This has made it feasible to implement considerably more complicated filters than one would have dreamed possible 20 years ago. For example, in the global positioning system (GPS) ground control segment, the state vector being estimated will contain over 200 elements when the system becomes fully operational [2].

In complex estimation applications, it becomes most difficult for the analyst to maintain good physical insight into the problem. The difficulty with most computer analyses of higher order systems is that the analyst is provided with an utter deluge of information without, at the same time, being provided with means of sorting the relatively important information from that which is not. The discussion that follows is intended to help the analyst with this sorting process and to help provide additional insight into complicated estimation problems.

It is certainly apparent to those familiar with estimation and control theory that there is a close connection between observability and estimation. Observability in a deterministic sense simply means that an observation of the output over the time span $(0,t)$ provides sufficient information to determine the initial state of the system [3]. This, along with knowledge of the system driving function, uniquely specifies the state at any other time within the interval. However, there are problems in applying the observability concept to complicated situations. The difficulties are essentially twofold: (1) The test for observability is not easy to apply in higher order systems. In a fixed-parameter system it requires determination of the rank of a large rectangular matrix; in a time-variable system it suffices to say that the usual test is even more difficult to apply [3]. (2) Even when one can apply the test of observability successfully, it only provides a yes–no type answer. It tells one nothing about the *degree* of observability. One might think that these difficulties are solved by doing off-line Kalman filter analysis, and, in a sense, this is true. However, as mentioned before, the problem here is that the computer tells the analyst more than he wants to know. In a few minutes a computer can output more numbers than one can intelligently digest in months of study! Allowing for symmetry, there are $n(n+1)/2$ terms to examine in the error covariance matrix. If $n=200$, the analyst must at least glance at about 20 000 numbers with each step of the recursive process—a most formidable task! If the analyst shortens his sights and looks at just the terms along the major diagonal, he runs the risk of overlooking some cross-correlations that might have significance.

The remainder of this paper shows that the eigenvalues and eigenvectors of the error covariance matrix give useful information about system observability. In particular, with appropriate normalization, they provide insight into the observability of linear combinations of states that might otherwise be overlooked with a casual examination of the major-diagonal terms of the P matrix. The beauty of the eigenvalue approach is that one only has to look at n rather than $n(n+1)/2$ items to gain this insight into the degree of observability of the system.

SIGNIFICANCE OF THE EIGENVALUES AND EIGENVECTORS OF THE ERROR COVARIANCE MATRIX

Kalman filtering [1,4,5] is the basis for what will be termed here the *stochastic approach* to determine the degree of observability in a control system [6]. This approach is developed as follows. Let $\tilde{x}$ be the vector estimation error defined as

$$\tilde{x} = \hat{x} - x$$

where x is the system state vector, and $\hat{x}$ is its optimal estimate. The scalar components of x, $\hat{x}$, and $\tilde{x}$ will be denoted with subscripts. Next, consider a linear combination of $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ defined by

$$\begin{aligned} w &= v_1\tilde{x}_1 + v_2\tilde{x}_2 + \cdots + v_n\tilde{x}_n \\ &= v^{\mathrm{T}}\tilde{x}. \end{aligned} \tag{1}$$

Since $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ are random variables, so is w. Therefore, w has an associated variance given by

$$\begin{aligned} \sigma_w^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j p_{ij} \\ &= v^{\mathrm{T}}Pv \end{aligned} \tag{2}$$

where the p_{ij} are the elements of the error covariance matrix P. Now, our objective will be to determine the particular linear combination in (1) which results in a relatively large σ_w^2 in (2). Since a large σ_w^2 would indicate poor observability, this term should be maximized subject to the constraint

$$v^{\mathrm{T}}v = 1. \tag{3}$$

This is a maxima–minima problem suitable for the Lagrangian multiplier method. The Lagrangian multiplier formulation requires that

$$(\partial / \partial v)\,[\sigma_w^2 - \lambda(v^{\mathrm{T}}v - 1)] = 0 \tag{4}$$

where λ is the Lagrangian multiplier (a scalar). Using (2) this can be rewritten as

$$(\partial / \partial v)\,[v^{\mathrm{T}}Pv - \lambda(v^{\mathrm{T}}v - 1)] = 0. \tag{5}$$

Carrying out the indicated differentiation results in

$$(P - \lambda I)v = 0. \tag{6}$$

This system of equations has a nonzero solution if and only if

$$|P - \lambda I| = 0. \tag{7}$$

Therefore, the solutions to (6) are the eigenvectors of P. Now, multiplying both sides of (6) by the vector v^{T} yields

$$v^{\mathrm{T}}(P - \lambda I)v = v^{\mathrm{T}}Pv - v^{\mathrm{T}}\lambda v = 0. \tag{8}$$

Since $v^{\mathrm{T}}Pv = \sigma_w{}^2$ and $v^{\mathrm{T}}v = 1$, equation (8) becomes

$$\sigma_w^2 = \lambda. \tag{9}$$

Therefore, the largest eigenvalue of P is the variance of the state or linear combination of states that is poorly observable relative to the direction of poor observability. Similarly, the state or linear combination of states that is most observable is indicated by the smallest eigenvalue.

The preceding development suggests that one might be able to gain some insight into the degree of observability of a system by examining the eigenvalues and associated eigenvectors of the P matrix. In particular, these will indicate those linear combinations that are highly observable, as evidenced by small associated eigenvalues. These linear combinations might not be obvious from just a casual look at the major diagonal terms of the P matrix.

NORMALIZATION

If the eigenvalues of P are calculated directly, with no normalization, the resulting range of values can be almost anything. Not only is there no set bound for the eigenvalues, but dimensional homogeneity may not exist among various linear combinations of states in the system. This can cause confusion when making comparisons among the eigenvalues. Thus, some sort of normalization is in order.

The following normalization scheme serves two purposes. First it sets a bound on the eigenvalues, and second, it forces the transformed estimation error vector to be dimensionless. It is assumed here that $P^-(0)$ is diagonal and positive definite. (Superscript minus denotes "a priori" and superscript plus denotes "a posteriori.") The first step in the normalization is to modify P^+ by $P^-(0)$ according to a congruent transformation as per (10):

$$P'^+(k) = (\sqrt{P^-(0)})^{-1}\,P^+(k)\,(\sqrt{P^-(0)})^{-1} =$$

$$\begin{bmatrix} \dfrac{p_{11}}{p_{11}(0)} & \dfrac{p_{12}}{\sqrt{p_{11}(0)p_{22}(0)}} & \cdots & \dfrac{p_{1n}}{\sqrt{p_{11}(0)p_{nn}(0)}} \\ \dfrac{p_{21}}{\sqrt{p_{22}(0)p_{11}(0)}} & \dfrac{p_{22}}{p_{22}(0)} & \cdots & \dfrac{p_{2n}}{\sqrt{p_{22}(0)p_{nn}(0)}} \\ \vdots & & \ddots & \vdots \\ \dfrac{p_{n1}}{\sqrt{p_{nn}(0)p_{11}(0)}} & \dfrac{p_{n2}}{\sqrt{p_{nn}(0)p_{22}(0)}} & \cdots & \dfrac{p_{nn}}{p_{nn}(0)} \end{bmatrix}. \tag{10}$$

In (10) the elements of $P^+(k)$ are denoted as $p_{11}, p_{12}, \ldots$, and the matrix $(\sqrt{P^-(0)})^{-1}$ is a diagonal matrix whose elements are $1/\sqrt{P_{11}(0)}$, $1/\sqrt{P_{22}(0)}$, etc. Under the congruent transformation the eigenvalues are not preserved but their positive nature is. The rank is also preserved; therefore, if $P^+(k)$ is positive definite, so is

$P'^{+}(k)$. The symmetry of $P^{+}(k)$ is also preserved. Note that the matrix defined by (10) has dimensionless elements. To complete the normalization, a bound will be imposed on the eigenvalues of $P'^{+}(k)$. It can be shown that the sum of the eigenvalues of a matrix is equal to its trace [7]. Therefore, if the matrix in (10) is multiplied by $n/\mathrm{Tr}(P'^{+}(k))$, the eigenvalues will be bound between zero and n (the order of the system). Let the elements of $P'^{+}(k)$ be denoted as $p'_{11}, p'_{12}, \ldots$, and let the final normalized matrix be shown as in (11):

$$P^{N+}(k) = \frac{n}{\mathrm{Tr}(P'^{+}(k))}\begin{bmatrix} p'_{11} & p'_{12} & \cdots & p'_{1n} \\ p'_{21} & p'_{22} & \cdots & p'_{2n} \\ \vdots & & \ddots & \vdots \\ p'_{n1} & p'_{21} & \cdots & p'_{nn} \end{bmatrix}. \tag{11}$$

This is one of several methods that can be utilized to normalize the error covariance matrix. This particular method was chosen since the values in the congruent transformation are the system's initial values. Therefore, all resultant eigenvalues and eigenvectors of the normalized matrix are calculated relative to the initial conditions of the system. Another method to normalize the error covariance matrix can be accomplished by transforming the matrix to yield the correlation matrix (this matrix can be generated by replacing the $P^{-}(0)$ matrix in (10) by $P^{+}(k)$. In multivariate statistics the preceding eigenanalysis lies in the realm of principal component analysis [8].

An example will illustrate how the normalized matrix given by (11) can be used to gain some insight into the relative observability of a system.

EXAMPLE

The example to be considered here is an undriven six-state system representing the error propagation in a damped, slow-moving inertial navigation system. The error-propagation equations are given by Pitman [9] and the appropriate Kalman filter model was developed by Bona and Smay [10]. The state and measurement models for this system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & \Omega_z & 0 & 1 & 0 & 0 \\ -\Omega_z & 0 & \Omega_x & 0 & 1 & 0 \\ 0 & -\Omega_x & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}. \tag{13}$$

The parameters Ω_x and Ω_z in (12) have physical significance as the north and vertical components of the Earth rate. For a latitude of 45 degrees, Ω_x and $\Omega_z \approx 7.292 \times 10^{-5}/\sqrt{2}$ rad/s.

The system state variables have the following physical significance:

- x_1 inertial system's east–west position error (in angular measure)
- x_2 inertial system's north–south position error (in angular measure)
- x_3 linear combination of platform azimuth and east–west position errors (in angular measure)
- x_4 gyro drift rate about the north axis
- x_5 gyro drift rate about the west axis
- x_6 gyro drift rate about the vertical axis.

The measurement relationship (13) indicates that a noninertial position measurement and a platform azimuth measurement (perhaps from a celestial source) are available for updating the inertial system.

When the rank of the observability matrix is tested, it indicates that the system is observable, i.e., its rank is 6 (see Chen [3]). Thus one would expect that a discrete Kalman filter would be able to continually improve on its initial estimates as more and more measurement data become available. To verify this the initial error covariance matrix was set at

$$P^{-}(0) = \begin{bmatrix} 2.283 \times 10^{-7}\ \mathrm{rad}^2 & & & & & \cdots \ 0 \\ & 2.283 \times 10^{-7}\ \mathrm{rad}^2 & & & & \cdot \\ \cdot & & 2.283 \times 10^{-7}\ \mathrm{rad}^2 & & & \cdot \\ \cdot & & & 2.350 \times 10^{-15}\ (\mathrm{rad/s})^2 & & \cdot \\ \cdot & & & & 2.350 \times 10^{-15}\ (\mathrm{rad/s})^2 & \\ 0 \cdots & & & & & 2.350 \times 10^{-15}\ (\mathrm{rad/s})^2 \end{bmatrix} \tag{14}$$

and the measurement error covariance matrix was chosen to be

$$R = \begin{bmatrix} 2.283 \times 10^{-9}\ \mathrm{rad}^2 & & \cdots \ 0 \\ \cdot & 2.283 \times 10^{-9}\ \mathrm{rad}^2 & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & & 2.350 \times 10^{-9}\ \mathrm{rad}^2 \end{bmatrix}. \tag{15}$$

In more familiar units, the initial state-estimate uncertainties correspond to 10 000 ft rms for x_1 and x_2, about

1.6 minutes of arc rms for x_3, and 0.01 deg/h rms for x_4, x_5, and x_6. The uncorrelated position-measurement errors were assumed to be about 1000 ft rms, and the independent azimuth measurement was assumed to have an associated error of about 10 seconds of arc rms.

The natural period of oscillation for this system is large (24 h), so the step size for the discrete Kalman filter was chosen to be 1 h. The Kalman filter was cycled through a fairly large number of recursive steps, and the error covariance behaved much as expected. For example, after the 120th step the estimation error given by the terms along the major diagonal of the P matrix were

$$\begin{aligned}
\sigma_{11}^2 &= 5.703 \times 10^{-11} \\
\sigma_{22}^2 &= 3.808 \times 10^{-11} \\
\sigma_{33}^2 &= 5.759 \times 10^{-11} \\
\sigma_{44}^2 &= 0.5121 \times 10^{-19} \\
\sigma_{55}^2 &= 1.027 \times 10^{-19} \\
\sigma_{66}^2 &= 0.5121 \times 10^{-19}.
\end{aligned} \tag{22}$$

These are all considerably less than the initial values, and the first three and latter three are bunched around the same values just as one would expect in this measurement situation. Thus a casual look at the P matrix does not reveal anything startling.

Next, the eigenvalues and eigenvectors for the normalized P matrix resulting after 120 steps were computed using the EISPACK subroutine package [11]. The resulting six eigenvalues and the eigenvector corresponding to the smallest eigenvalue are given in Table I.

TABLE I
Eigenvalues and Eigenvector for Example

Eigenvalues	Eigenvector
0.001060 0.1502 0.1510 1.515 1.529 2.654	$\nu^{(1)} = \begin{bmatrix} -0.2429 \times 10^{-1} \\ -0.3182 \times 10^{-4} \\ -0.2430 \times 10^{-1} \\ 0.7067 \\ -0.6962 \times 10^{-3} \\ 0.7067 \end{bmatrix}$

Recall from preceding comments that a small eigenvalue indicates a high degree of observability, and the corresponding eigenvector gives the "direction" of this high degree of observability. Note in this example that one eigenvalue is conspicuously smaller than the others. This suggests that there is something special about the observability of this system that is not apparent from a casual glance at the P matrix. Proceeding further, we see that the eigenvector associated with the smallest eigenvalue is directed halfway between the x_4 and x_6 axes. This then further suggests that a linear combination of x_4 and x_6 (the north and vertical gyro drift rates) is being observed especially well relative to estimates of the three drift rates when viewed in the x,y,z reference frame. This conclusion was certainly not obvious at the outset, nor was it apparent from a casual look at the terms along the major diagonal of the P matrix. But it is, in fact, true in this example that there is something special about the linear combination $x_4 + x_6$!

We will not go through all the details here, but it can be shown (see Pitman [9]) that special decoupling arises when the inertial error equations are rewritten in a "peq" coordinate frame (p is polar, e is equatorial, and q is mutually orthogonal to p and e). This coordinate transformation was performed for this example, and the Kalman filter was rerun in the new coordinate frame. After 120 steps the normalized eigenvalues were the same as before. This was as expected because the coordinate transformation was orthogonal. However, the diagonal terms of the P matrix were quite different. The latter three, which correspond to the peq gyro drift rates, are of special interest and they worked out to be

$$\begin{aligned}
\sigma_{44}^2 &= 1.241 \times 10^{-21} \\
\sigma_{55}^2 &= 1.027 \times 10^{-19} \\
\sigma_{66}^2 &= 1.012 \times 10^{-19}.
\end{aligned} \tag{23}$$

Note that the P matrix in the transformed system confirms our previous conclusions. The polar component (which is a linear combination of x_4 and x_6 in the original coordinate frame) of the vector gyro drift is, in fact, being observed and estimated much better than the other two components.

CONCLUSIONS

It has been shown that the eigenvalues and eigenvectors of the error covariance matrix can be used to provide additional insight into the degree of observability of a system. In order to provide this insight, it is convenient to normalize the P matrix such that the sum of the eigenvalues is equal to the order of the system. There will, of course, be cases where the eigenvalue distribution is uninteresting and nothing startling is revealed by the distribution. If this is the case, so be it; some computational effort was wasted. On the other hand, wide dispersion of the eigenvalues can indicate exceptionally good (or poor) observability of certain linear combinations of state variables, and this can provide useful information to the analyst, especially in complicated systems where physical insight is difficult. Efficient subroutines for finding eigenvalues and eigenvectors are commonplace now, so this extra insight into system observability is available with only minimal effort. This being the case, one might just as well take advantage of the extra information.

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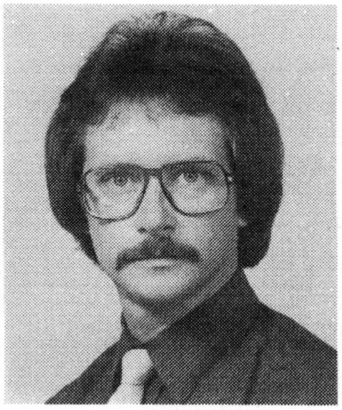

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