Time-Varying Systems and Computations
Lecture 2

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1 Linear Time-Invariant Systems

State-Space Model for Linear Systems

A linear input-output system with input signal u_k, impulse response t_k and the corresponding y_k is typically represented by the schematic drawing in Figure 1, where T denotes the Toeplitz matrix of

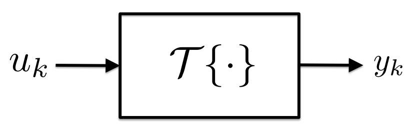

Figure 1: A linear input-output system.

the linear convolution operator function for the impulse response t_k. In a next step the drawing of the system is slightly redrawn to produce the representation in Figure 2 The internals of the systems

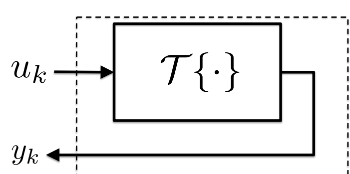

Figure 2: A slightly redrawn version of the linear system.

inside the dashed box will be further structured while the input-output relation $\boldsymbol{T}$ stays unchanged. The process of structuring we are about to embark has been called *Reactance extraction* by Dante Youla [4]. Reactance extraction means that the internals of the block are separated into two *cascaded* sub-blocks as shown in Figure 3. The first block on left, labeled Σ contains non-dynamic components only, i.e. in an analog electronics world this amounts to resistors, ideal transformers and gyrators. In the realm of digital technology, the block Σ contains only arithmetic operators (multiplication, division, addition, subtraction, square-roots, etc.). Hence the function of the block Σ can be mathematically described by constant matrices over the real or complex field. The second block, labeled Z contains all dynamic

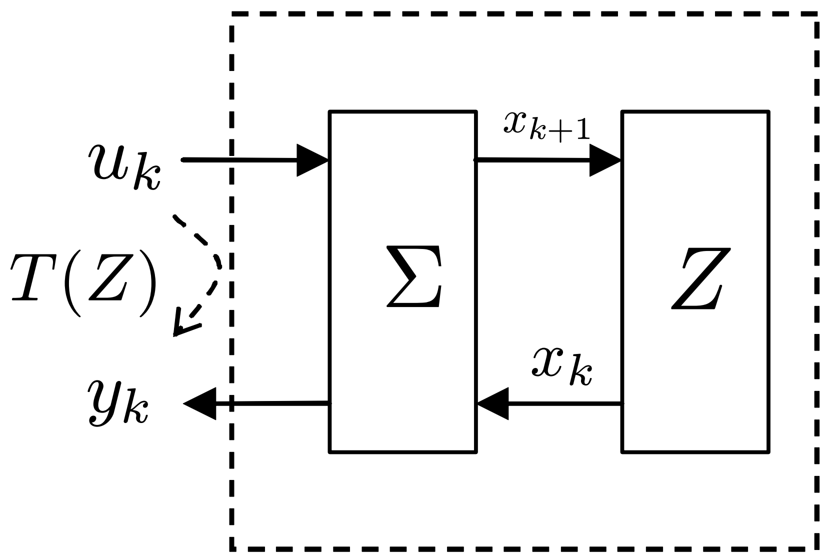

Figure 3: Reactance Extraction: Structuring internals of Linear Systems.

components of the linear system. Again, in an analog electronics world this amounts to inductances and capacitances, which are commonly referred to as 'reactances' in classical network theory, hence the name '*reactance extraction*'. Using digital technology the dynamic components of the systems are delays, storages, registers, latches etc. Such a cascaded structuring of the system introduces new variables at the interconnections between Σ and Z, which we denote by $\boldsymbol{x}_{k+1}$ and $\boldsymbol{x}_k$. This definition of variables also expresses the function of the Z-block, which is to delay signals by one time-unit. If we were to deal with continuous time linear systems, these variables would be replaced by $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}$, respectively, and the function of the Z-block being an integration. The description for our linear system which results from this approach is commonly referred to as the *Kalman state-space description.*

For the sake of completeness we mention that there exists an alternative decomposition for the system $\boldsymbol{T}$. This alternative decomposition is called a *Darlington model*, where the Z-block now comprises all non-dynamic elements of the system, which consume or produce energy, i.e. (positive and negative) resistors in an analog electronics world or simple real numbers in a digital domain. In correspondence to the previous paragraph we can name this approach a '*resistance extraction*'. The matrices, which are used to describe such a Z-block are constant matrices. If all elements, which consume or produce energy are comprised in the Z-block then the block Σ is left as a *lossless* transformation, which also contains all dynamic, or frequency-dependent parts. This can be translated into the Σ-block consisting of ideal transformers, gyrators, inductances and capacitances for an analog electronics box. The corresponding mathematical description is based on matrices over the field of rational functions, where the notion of losslessness is expressed by the property that the corresponding matrices are para-unitary. We won't go any further in this direction here and leave it as such.

Both possible decompositions of the system amount to very specific representations or parameterizations for rational functions. In the following we will make use of the state-space description according to a reactance extraction approach.

State-Space Equations

The Σ-block in Figure 3, which contains the non-dynamic components is further elaborated such that we arrive at a internal description as given in Figure 4. From the signal flow diagram in Figure 4 we can read off a set of equations, which describe the internal workings of the system. The equations are

$$\begin{aligned} x_{k+1} &= A \cdot x_k + B \cdot u_k \\ y_k &= C \cdot x_k + D \cdot u_k, \end{aligned} \tag{1}$$

using the matrices $A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}$. Here, at a time instant k, the system is assumed to have m inputs ($\boldsymbol{u}_k \in \mathbb{R}^m$) and p outputs ($\boldsymbol{y}_k \in \mathbb{R}^p$), while n is the dimension of the state-vector ($x_k, x_{k+1} \in \mathbb{R}^n$), which again corresponds with the dynamic degree for this system (determined by the dynamic sub-block). We conveniently combine the state-equations (1) into a more compact matrix notation

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2}$$

which expresses the signals coming out of the Σ-block in terms of signals going in the block. We can associate the interpretation of the matrix Σ given in Equation 2 as being the scattering matrix of the multi-port Σ.

Transfer Function

The transfer function describes the mapping of the input vector u onto the output vector y (complete sequences). In this section we derive a representation of the transfer function T, which is parametrized in terms of the quantities of the state-space model, i.e. in terms of the matrices A, B, C, D.

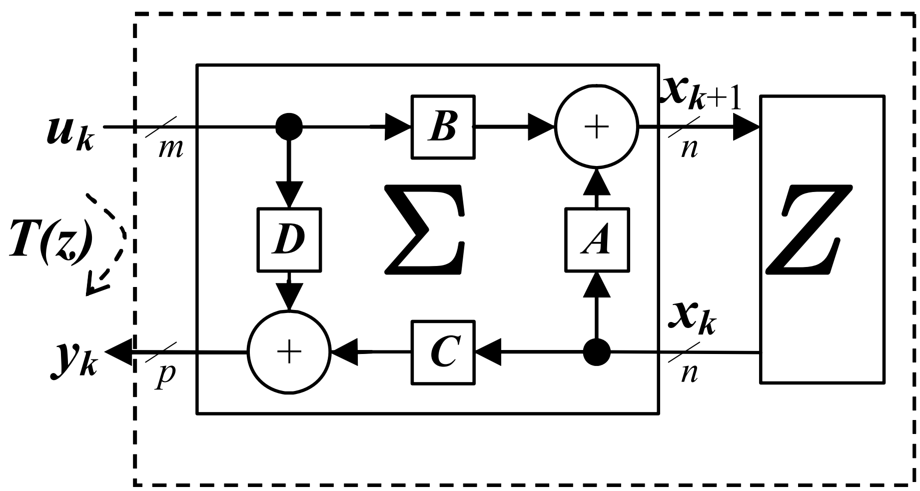

Figure 4: State-Space Model based on Reactance Extraction.

We want to use only purely purely algebraic concepts to arrive at a compact representation of the Toeplitz operator T, such that we can replace the complex-valued transfer function $T(z)$ (z-transformation), which is restricted to the time-invariant case. We first acknowledge that we need a block diagonal expansion of the state-space equations all diagonal blocks are identical, i.e. we have

$$\boldsymbol{A} = \begin{bmatrix} \ddots & & & & \\ & A & & & \\ & & \boxed{A} & & \\ & & & A & \\ & & & & \ddots \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \ddots & & & & \\ & B & & & \\ & & \boxed{B} & & \\ & & & B & \\ & & & & \ddots \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \ddots & & & & \\ & C & & & \\ & & \boxed{C} & & \\ & & & C & \\ & & & & \ddots \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} \ddots & & & & \\ & D & & & \\ & & \boxed{D} & & \\ & & & D & \\ & & & & \ddots \end{bmatrix},$$

and that the shift operator basically stays the same (except that all identity matrices have identical dimension)

$$\boldsymbol{Z} = \begin{bmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & \mathbf{1} & \boxed{0} & & \\ & & \mathbf{1} & 0 & \\ & & & \mathbf{1} & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Using these conventions we build an infinite-dimensional vector $\boldsymbol{x}$ which contains a sequence of state-vectors, i.e.

$$\boldsymbol{x} = \begin{bmatrix} \vdots \\ x_{k-2} \\ x_{k-1} \\ \boxed{x_k} \\ x_{k+1} \\ \vdots \end{bmatrix},$$

where the square around the vector x_k indicates the position of the state-vector pertaining to the current time index. We employ the shift operator Z, which denotes the pushing down of the elements in a vector by one notch, i.e. we use it to push the vector $\boldsymbol{x}$ up such as

$$\begin{bmatrix} \vdots \\ x_{k-1} \\ x_k \\ \boxed{x_{k+1}} \\ x_{k+2} \\ \vdots \end{bmatrix} = \boldsymbol{Z}^{-1} \cdot \begin{bmatrix} \vdots \\ x_{k-2} \\ x_{k-1} \\ \boxed{x_k} \\ x_{k+1} \\ \vdots \end{bmatrix} = \boldsymbol{x} \uparrow .$$

Reviewing Equations (1) leads us now to the formulation

$$\begin{aligned} \boldsymbol{Z}^{-1}\boldsymbol{x} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \end{aligned} \tag{3}$$

where the vectors $\boldsymbol{y}$ and $\boldsymbol{u}$ denote infinite dimensionally extended versions of the output vector an the input vector, respectively. We need to eliminate the state-variable $\boldsymbol{x}$ from Equation 3 to arrive at a formula for $\boldsymbol{T}\boldsymbol{u} = \boldsymbol{y}$. From the first equation we can determine

$$(\boldsymbol{I}_n - \boldsymbol{Z}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{Z}\boldsymbol{B}\boldsymbol{u} \quad \rightarrow \quad \boldsymbol{x} = (\boldsymbol{I}_n - \boldsymbol{Z}\boldsymbol{A})^{-1}\boldsymbol{Z}\boldsymbol{B}\boldsymbol{u}$$

Inserting this equation into the second part of Equation (3) we arrive at

$$\boldsymbol{y} = \left[\boldsymbol{D} + \boldsymbol{C}(\boldsymbol{I}_n - \boldsymbol{Z}\boldsymbol{A})^{-1}\boldsymbol{Z}\boldsymbol{B}\right] \cdot \boldsymbol{u},$$

from which we can directly read off the transfer function $\boldsymbol{T}$ which is a compact representation of the Toeplitz operator T in terms of the linear fractional map

$$\boldsymbol{T} = \boldsymbol{D} + \boldsymbol{C}(\boldsymbol{I}_n - \boldsymbol{Z}\boldsymbol{A})^{-1}\boldsymbol{Z}\boldsymbol{B}. \tag{4}$$

In order to verify the validity of the representation in Equation (4), we first want to deliberate on the resolvent

$$\begin{aligned} (\boldsymbol{1} - \boldsymbol{Z}\boldsymbol{A})^{-1} &= \boldsymbol{1} + \boldsymbol{Z}\boldsymbol{A} + (\boldsymbol{Z}\boldsymbol{A})^2 + (\boldsymbol{Z}\boldsymbol{A})^2 + \dots && (5)\\ &= \boldsymbol{1} + \boldsymbol{Z}\boldsymbol{A} + \boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}\boldsymbol{A} + \boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}\boldsymbol{A} + \dots && (6)\\ &= \boldsymbol{1} + \boldsymbol{Z}\boldsymbol{A} + \boldsymbol{Z}^2\boldsymbol{A}^2 + \boldsymbol{Z}^3\boldsymbol{A}^3 + \dots, && (7) \end{aligned}$$

where we use the Neumann expansion $(1 - X)^{-1} = 1 + X + X^2 + X^3 + \dots, \|X\| < 1$, which structurally resembles the geometric series. The series in Equation 7 converges if the matrix $\boldsymbol{A}$ has all its eigenvalues inside the unit disk. Now, in the case of time-invariant systems with constant block-diagonals the shift-operator $\boldsymbol{Z}$ commutes with the block-diagonal matrix $\boldsymbol{A}$, i.e.

$$\boldsymbol{Z}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{Z},$$

because in this case, shifting down of the block-diagonal matrix (pre-multiplication with $\boldsymbol{Z}$) has the same effect then shifting left (post-multiplication with $\boldsymbol{Z}$), i.e.

$$\boldsymbol{Z}\boldsymbol{A} = \begin{bmatrix} \ddots & & & & \\ \ddots & \downarrow & & & \\ & A & \downarrow & & \\ & & A & \downarrow & \\ & & & A & \ddots \\ & & & & \ddots \end{bmatrix}, \quad \boldsymbol{A}\boldsymbol{Z} = \begin{bmatrix} \ddots & & & & \\ \ddots & \leftarrow & & & \\ & A & \leftarrow & & \\ & & A & \leftarrow & \\ & & & A & \ddots \\ & & & & \ddots \end{bmatrix}$$

Putting together the series expansion in Equation (7) we generate a matrix representation

$$(\boldsymbol{1} - \boldsymbol{Z}\boldsymbol{A})^{-1} = \begin{bmatrix} \ddots & & & & & \\ \ddots & \boldsymbol{1} & & & & \\ \ddots & A & \boxed{\boldsymbol{1}} & & & \\ \ddots & A^2 & A & \boldsymbol{1} & & \\ \ddots & A^3 & A^2 & A & \boldsymbol{1} & \\ & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

With this intermediate result we can evaluate the explicit formula

$$\boldsymbol{C}\,(\mathbf{1}-\boldsymbol{ZA})^{-1}\,\boldsymbol{ZB} =$$

$$= \begin{bmatrix} \ddots & & & & \\ & C & & & \\ & & \boxed{C} & & \\ & & & C & \\ & & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & & & \\ \ddots & \mathbf{1} & & & & \\ \ddots & A & \boxed{\mathbf{1}} & & & \\ \ddots & A^2 & A & \mathbf{1} & & \\ \ddots & A^3 & A^2 & A & \mathbf{1} & \\ & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & B & \boxed{0} & & \\ & & B & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ \ddots & CB & \boxed{0} & & & \\ \ddots & CAB & CB & 0 & & \\ \ddots & CA^2B & CAB & CB & 0 & \\ & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Adding to his intermediate result the diagonal block matrix containing the Da completes the derivation, i.e. we have

$$\begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ \ddots & CB & \boxed{0} & & & \\ \ddots & CAB & CB & 0 & & \\ \ddots & CA^2B & CAB & CB & 0 & \\ & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & & & \\ & D & & & \\ & & \boxed{D} & & \\ & & & D & \\ & & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & \\ \ddots & D & & & & \\ \ddots & CB & \boxed{D} & & & \\ \ddots & CAB & CB & D & & \\ \ddots & CA^2B & CAB & CB & D & \\ & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We have seen that the purely algebraic representation of the Toeplitz operator in terms of the linear fractional map works well for time-invariant systems and also generalizes well to the time-varying case, if we take into account the slightly more complicated notation in the context of the modified Neumann expansion.

State-Space Equivalence

Transformation

It is permissible to transform the state-space with a non-singular transformation R,

$$x'_{k+1} = R \cdot x_{k+1}, \quad \text{or} \quad R^{-1} \cdot x'_k = x_k.$$

Using

$$\boldsymbol{x}' = \boldsymbol{R} \cdot \boldsymbol{x} \quad \Rightarrow \quad \begin{bmatrix} \vdots \\ x'_{k-1} \\ \boxed{x'_k} \\ x'_{k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & R & & \\ & & \boxed{R} & \\ & & & R & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{k-1} \\ \boxed{x_k} \\ x_{k+1} \\ \vdots \end{bmatrix}.$$

The corresponding blocks are inserted in the system model and shown in Figure 5. Substituting the

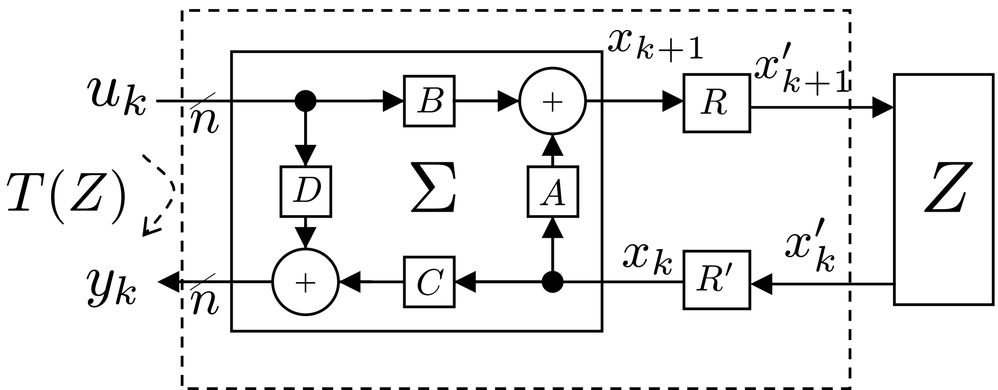

Figure 5: Transformation of State-Space

transformed state-variables in the state-equations results in

$$\begin{aligned} R^{-1} \cdot x'_{k+1} &= A \cdot R^{-1} x'_k + Bu_k \\ y_k &= C \cdot R^{-1} x'_k + Du_k, \end{aligned}$$

which will then directly lead to the transformed state-space representation

$$\begin{bmatrix} x'_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} R & \\ & \mathbf{1} \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} R^{-1} & \\ & \mathbf{1} \end{bmatrix} \cdot \begin{bmatrix} x'_k \\ u_k \end{bmatrix}, \tag{8}$$

with the transformed realization matrix

$$\Sigma' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} RAR^{-1} & RB \\ CR^{-1} & D \end{bmatrix}.$$

We can see that the state-space transformation induces a similarity transformation RAR^{-1} on the matrix A. Similarity transformations leave the eigenvalues of a matrix invariant, i.e. the matrices A and RAR^{-1} have the same eigenvalues.

Invariance of transfer function and impulse response

The transfer function T is invariant under non-singular transformations of the state-space with R, i.e. the matrices Σ and Σ' are realizations for the same transfer function T. This can be shown by the following computation

$$\begin{aligned} T' &= D' + C'(\mathbf{1} - ZA')^{-1}ZB' \\ &= D + CR^{-1}(\mathbf{1} - ZRAR^{-1})^{-1}ZRB \\ &= D + C(\mathbf{1} - ZA)^{-1}ZB \\ &= T. \end{aligned}$$

The invariance of the transfer function can also be shown by showing the invariance of the corresponding impulse response. The impulse response t'_k can be read off from the block diagram in Figure (5), which includes the state transformation R. The impulse response t'_k is compared to the impulse response t_k identified by inspection of the system shown in Figure (4)

k	0	1	2	3	4	...
t'_k	D	$CR^{-1}RB$	$CR^{-1}RAR^{-1}RB$	$CR^{-1}(RAR^{-1})^2RB$	$CR^{-1}(RAR^{-1})^3RB$	... .
t_k	D	CB	CAB	CA^2B	CA^3B	...

It is clearly visible, that the effect of the transform R cancels out leaving the impulse response and hence the transfer function invariant. The effect of the transformation R is similar to an *all-pass filter*, which can not be identified by looking at the input-output map. The transfer function of the system stays identical, yet the realization can change. Hence, there is no unique realization for a given transfer function. The set of all possible realizations for a given transfer function is parameterized in terms of the non-singular matrix R. Since there exist infinitely many non-singular matrices R there also exist infinitely many realizations for a particular transfer function. This has a very important consequence - starting out with any realization Σ for a given T the transformation R can be employed in an optimization scheme to find an alternative realization Σ', which minimizes e.g. the arithmetic cost, or which minimizes round-off noise, or coefficient sensitivity or any other conceivable cost function.

State-Space Model to Toeplitz-Matrix

The notion of a Toeplitz matrix, which describes the external system behavior, i.e. the input-output map of a linear time-invariant system has been established earlier. In the following we will revisit this topic.

We can measure the impulse response of a causal LTI system and enter the measurements in a lower triangular Toeplitz matrix. If we employ a system model according to Figure (4) or equivalently, according to Equation (1), we can write down the impulse response t_k expressed in terms of the state-space parameter A, B, C, D

$$t_k = \begin{cases} 0 & \text{for} \quad k < 0 \\ D & \text{for} \quad k = 0 \\ CA^{k-1}B & \text{for} \quad k > 0 \end{cases} \tag{9}$$

and enter them into a matrix to arrive at the Toeplitz matrix

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \ddots & 0 & \vdots & \vdots & \\ \ddots & \boxed{t_0} & 0 & \vdots & \\ \ddots & t_1 & \boxed{t_0} & 0 & \\ \ddots & t_2 & t_1 & \boxed{t_0} & \ddots \\ \ddots & t_3 & t_2 & t_1 & \ddots \\ & t_4 & t_3 & t_2 & \ddots \\ & \vdots & t_4 & t_3 & \ddots \\ & \vdots & \vdots & t_4 & \ddots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \ddots & 0 & \vdots & \vdots & \\ \ddots & \boxed{\mathrm{D}} & 0 & \vdots & \\ \ddots & CB & \boxed{\mathrm{D}} & 0 & \\ \ddots & CAB & CB & \boxed{\mathrm{D}} & \ddots \\ \ddots & CA^2B & CAB & CB & \ddots \\ & CA^3B & CA^2B & CAB & \ddots \\ & \vdots & CA^3B & CA^2B & \ddots \\ & \vdots & \vdots & CA^3B & \ddots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{10}$$

where the boxed matrix elements identify the main diagonal. That means that if we have a state-space realization for a linear time-invariant system given in terms of a realization matrix Σ we can immediately write down the impulse response t_k of the system and hence the Toeplitz matrix T.

Literatur

[1] G. Strang. *Computational Science and Engineering*. Wellesley-Cambridge Press, 2007.

[2] P.Regalia. *Adaptive IIR Filtering in Signal Processing and Control*. CRC Press, 1995.

[3] T.Kailath. *Linear Systems*. Prentice Hall, 1980.

[4] D.Youla, P.Tissi. *n-Post Synthesis via Reactance Extraction - Part I*. IEEE International Convention Record, 1966, p.183-208.