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Stochastic observability, reconstructibility, controllability, and reachability

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Abstract:
This thesis formulates versions of observability, reconstructibility, controllability, and reachability for stochastic linear and nonlinear systems. The concepts of observability and reconstructibility concern whether the measurements of a system suffice to construct a complete characterization of the system behavior while the concepts of controllability and reachability concern whether the actuation of the system suffices to cause the system to behave according to various user specifications. Thus, these concepts are fundamental to the design of control algorithms. In deterministic linear systems, linear algebraic conditions specify whether an unknown state can be exactly reconstructed from the measurements over a finite time interval, and whether there exists an input sequence over a finite time interval which can steer the state to a desired endpoint; thus, the concepts of observability, reconstructibility, controllability, and reachability are straightforwardly defined. The extension to linear stochastic systems is not obvious. While the deterministic matrix conditions have significance in applications such as Kalman filtering and linear-quadratic -optimal control, the presence of noise generally prevents exact reconstruction of the state via the measurements and exact placement of the state via the control inputs. This ambiguity in interpreting the matrix conditions has lead to a multitude of extensions in the literature. Nonlinear behavior introduces further difficulties; even in nonlinear deterministic systems, the generalization of the linear conditions is not immediate; for instance, whereas observability and reconstructibility does not depend on the control input in linear systems, this separation of control and estimation questions need not hold for nonlinear systems. Our purpose is to make precise the stochastic versions of observability, reconstructibility, controllability, and reachability; in the process, we obtain the expected matrix conditions for stochastic linear systems, which arise both in deterministic linear systems analysis and in Kalman filtering theory and linear -quadratic-optimal-control theory. Perhaps unexpectedly, we also obtain an analogous rank condition for the finite-

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Stochastic Observability, Reconstructibility, Controllability, and Reachability

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Andrew R. Liu

Committee in charge:

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Professor Miroslav Krstić
Professor William McEneaney
Professor David Sworder
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2011

The dissertation of Andrew R. Liu is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

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Chapter 2 includes reprints of the following papers:

Andrew R. Liu and Robert R. Bitmead - *Stochastic Observability in Network State Estimation and Control*, Automatica, Vol 47, pp 65-78, 2011;

Andrew R. Liu and Robert R. Bitmead - *Reconstructibility and Estimability*, Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, December, 2009.

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The dissertation author was the primary author in these publications and Professor Bitmead directed and supervised the research.

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ABSTRACT OF THE DISSERTATION

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by

Andrew R. Liu

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

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Professor Robert R. Bitmead, Chair

This thesis formulates versions of observability, reconstructibility, controllability, and reachability for stochastic linear and nonlinear systems. The concepts of observability and reconstructibility concern whether the measurements of a system suffice to construct a complete characterization of the system behavior while the concepts of controllability and reachability concern whether the actuation of the system suffices to cause the system to behave according to various user specifications. Thus, these concepts are fundamental to the design of control algorithms.

In deterministic linear systems, linear algebraic conditions specify whether an unknown state can be exactly reconstructed from the measurements over a finite time interval, and whether there exists an input sequence over a finite time interval

which can steer the state to a desired endpoint; thus, the concepts of observability, reconstructibility, controllability, and reachability are straightforwardly defined. The extension to linear stochastic systems is not obvious. While the deterministic matrix conditions have significance in applications such as Kalman filtering and linear-quadratic-optimal control, the presence of noise generally prevents exact reconstruction of the state via the measurements and exact placement of the state via the control inputs. This ambiguity in interpreting the matrix conditions has lead to a multitude of extensions in the literature.

Nonlinear behavior introduces further difficulties; even in nonlinear deterministic systems, the generalization of the linear conditions is not immediate; for instance, whereas observability and reconstructibility does not depend on the control input in linear systems, this separation of control and estimation questions need not hold for nonlinear systems.

Our purpose is to make precise the stochastic versions of observability, reconstructibility, controllability, and reachability; in the process, we obtain the expected matrix conditions for stochastic linear systems, which arise both in deterministic linear systems analysis and in Kalman filtering theory and linear-quadratic-optimal-control theory. Perhaps unexpectedly, we also obtain an analogous rank condition for the finite-state hidden Markov model. We show important roles of reconstructibility: in linear systems, it corresponds to minimality of the Kalman filter; in nonlinear systems, it is necessary for performance improvement via output feedback over open-loop control. The role of observability in the stability of optimal filters is discussed. Additionally, we demonstrate a connection between stochastic controllability/reachability and Granger causality and its generalizations from the statistics and econometrics literature. The ideas are explored via simulation of a finite-state hidden Markov model for the network congestion control problem.

Chapter 1

Introduction

This thesis extends well-known definitions of observability, reconstructibility, controllability, and reachability to stochastic, nonlinear systems. In deterministic systems, observability and reconstructibility describe whether output measurements of a system suffice to determine exactly the state of the system; typically, this condition is necessary for design of effective output-feedback control algorithms. Similarly, deterministic controllability and reachability describe whether the control inputs to the system suffice to move the state to desired destinations. In control design for stochastic systems, the ability to determine the state and to steer it to desirable points remain primary goals. Complicating the problem is noise, which often prevents, for the observability and reconstructibility questions, the exact recovery of the state from the outputs, and for the controllability and reachability questions, the exact placement of the state at a specified reference point or trajectory. Extending the definitions to stochastic systems chiefly concerns identifying how the approximate satisfaction of these goals is meaningful.

The development focuses on the state-space model, sometimes also known as the (generalized) hidden Markov model. A key concept for models of this class is the state. This concept coincides with that from deterministic systems in that the state "summarizes all the features of the past behavior of the system that, together with future inputs, determines its future state and output behavior" [Cai88]. Questions of whether the behavior of the system can be estimated or controlled thus reduce to questions of whether the state as defined by Caines

[Cai88] can be estimated or controlled.

Information theory is the unifying framework for the formulations. We demonstrate how information-theoretic interpretations reduce to well-known deterministic rank conditions for linear systems under natural conditions on the noise and also show consistency between linear and nonlinear definitions. Additionally, we show connections between our formulation and system properties typically necessary for control design. These include the capacity to improve on an optimal open-loop control sequence using output feedback, the causal relation between the control and the state, and the convergence properties of a class of filters which propagate the conditional distribution such as the Kalman filter, the (finite-state) hidden Markov model filter, and the nonlinear (optimal) filter.

1.1 Motivation

In control design, one expects the controlled input to have influence over the state evolution. Likewise, in the design of output-feedback algorithms, one expects the output to contain relevant information about the state. Conceptually, the questions of controllability/reachability address whether the input fully affects the state and the questions of observability/reconstructibility address whether the output contains sufficient information about the state; thus, in some sense, these properties are self-evidently necessary for control design. The concepts translate readily to mathematical statements in linear deterministic systems:

Definitions ([FH77, Kir70]). *Given $x_k \in \mathbb{R}^n$ with $k = 1, 2, \ldots,$ and*

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{1.1}$$

*a state $\bar{x}$ is said to be **controllable** if for some finite time N and initial state $x_0 = \bar{x}$, there exists an input sequence $\{u_k\}_0^{N-1}$ such that $x_N = 0$; similarly, state $\bar{x}$ is said to be **reachable** if for some finite time N and initial state $x_0 = 0$, there exists an input sequence $\{u_k\}_0^{N-1}$ such that $x_N = \bar{x}$. The system (1.1) is **completely controllable** if every state is controllable and **completely reachable** if every state is reachable.*

*A state $\bar{x}$ is said to be **observable** if for some finite m and any $N > m$, given input sequence $\{u_k\}_0^{N-1}$, the output sequence $\{y_k\}_0^{N-1}$ uniquely determines $x_0 = \bar{x}$, and likewise, a state $\bar{x}$ is said to be **reconstructible** if for any some finite m and any $N > m$, given input sequence $\{u_k\}_0^{N-1}$, the output sequence $\{y_k\}_0^{N}$ uniquely determines $x_N = \bar{x}$. The system (1.1) is **completely observable** if every state is observable and **completely reconstructible** if every state is reconstructible.*

These give rise to the well-known matrix conditions.

Theorem ([AM79])**.** *The system (1.1) is completely controllable if*

$$\textit{range}\ A^n \subset \textit{range}\left[\begin{array}{c} B, AB, \ldots, A^{n-1}B \end{array}\right].$$

It is completely reachable if

$$\textit{rank}\left[\begin{array}{cccc} B & AB & \ldots & A^{n-1}B \end{array}\right] = n.$$

It is completely reconstructible if

$$\textit{range}\ A^{T^n} \subset \textit{range}\left[\begin{array}{cccc} C^T & A^TC^T & \ldots & A^{T^{n-1}}C^T \end{array}\right].$$

It is completely observable if

$$\textit{rank}\left[\begin{array}{cccc} C^T & A^TC^T & \ldots & A^{T^{n-1}}C^T \end{array}\right] = n.$$

The inclusion of noise, and the complexities it entails, is best understood in the context of Kalman filtering and linear-quadratic-optimal-control theory. Here, the matrix conditions continue to correspond to interesting properties and are often used as the definitions. On the other hand, in the presence of noise, one is unable to reconstruct the exact state from the measurements or to maneuver the state to an exact point using the inputs, so the matrix conditions no longer agree with the definitions, and it is questionable whether they meaningfully describe the concepts intended by controllability, observability, reachability, and reconstructibility (CORR).

In linear deterministic systems, the minimality of the system follows from complete observability and reachability. Minimality, the property that the dimension of an initial state is the smallest possible such that, with the inputs, the exact sequence of outputs can be generated, is connected to many topics in realization theory and is arguably more fundamental for state-space systems than CORR [Kai96, For75, SG00]. As with the stochastic extension problem of CORR, the appropriate definition of minimality is unclear in the presence of noise, but due to the close connection between minimality and complete observability/reachability in the deterministic case, one may view the stochastic extensions of complete observability and reachability as a first step to approaching the definition of stochastic minimality.

A further difficulty is due to system nonlinearities. Even in deterministic systems, the adjective "complete" in complete observability/reconstructibility/etc. entails additional technical considerations; for instance, in a version of nonlinear observability due to Hermann and Krener [HK77, NvdS90], the linear rank condition generalizes to a condition over a linear space of functions and repeated Lie derivatives with respect to the state. Additionally, observability and reconstructibility properties of a system may depend on the input sequence to the system. Not surprisingly, these hurdles carry over to the analysis for stochastic systems.

The development presented here is driven by two criteria. First, the formulation should coincide with the linear, deterministic matrix conditions subject to meaningful assumptions on noise signals. Second, the extensions to nonlinear, stochastic systems should admit statistical interpretations under minimal assumptions about the structure of the state distribution; thus, we accommodate nonlinear or non-Gaussian systems where, for instance, the state variance may be a meaningless quantity. The notions of entropy and mutual information in information theory provide the suitable tools to meet these requirements.

The problem of defining stochastic observability is not new. Classical works, such as the textbook by Jazwinski [Jaz98] present (without quantification) the notion that observability is tied to the information conveyed about the state by the output sequence. Chen [Che77, Che79, Che85] defines a version of observability for linear systems in terms of the state-estimator-error covariance; our analysis follows his closely in the linear case.

More recently, Ugrinovskii [Ugr03] considers the observability question for linear stochastic uncertain systems, which extends the observability analysis of linear deterministic uncertain systems due to Petersen [Pet02]. This is a wider problem for linear systems than that which we consider here, since it involves the consideration of whether the bounded set of system disturbance signals admits elements which are capable of causing a loss of observability; this captures the *uncertainty* idea. For the nominal (i.e. not uncertain) linear system, his analysis is comparable to our own and, indeed, his definition of unobservability coincides with a continuous-time variant of that presented here for linear stochastic discrete-time systems; unobservability coincides (after a measure transformation) with the independence of the output $y(t)$ from the initial state $x(0)$ and from the process noise $w(t)$ for t in a finite time interval $t \in [0, T]$. Ugrinovskii's results are not limited to Gaussian linear systems, although absolute continuity of the distributions relative to Gaussian is required, which permits the transformation of the probability measure to the Gaussian case. Interpreting these results in terms of the current work, it is interesting to note the movement from uncorrelatedness to independence with the change from Gaussian systems.

Dragan and Morozan [DM06] and Li et al. [LWZD09] define a version of observability for linear systems with multiplicative noise and Markovian jumping. These are conceptually similar to one another and to our definition, but are difficult to extend to nonlinear systems.

In contrast to research in linear systems, the stochastic observability of nonlinear systems has received little attention. We have already referred to the investigations conducted by Mohler and Hwang. In addition, Xie et al. [XUP08] have defined a version of observability for finite-state hidden Markov models (HMMs). Their definition is a technical condition for the problem of robust estimation of HMMs, which is more akin to the idea of disturbability due to Bryson [Bry78], and which we will not study here.

Where we diverge from the preceding work is in our consideration of the observability of measurable functions of the state, from which we are naturally lead to define a notion of *complete* observability – the circumstance when all measurable functions of the state are observable. The question of observability of functions of the state arises both by analogy to deterministic, nonlinear extensions of observability [HK77, NvdS90], and in our observability investigations of finite-state hidden Markov models. Although the state for such systems is typically scalar, or bijectively mapped to a scalar, the evolution equations for the state and output probabilities,

$$\begin{aligned} \Pi_{k+1} &= A\Pi_k, \quad \text{with state } x_k \sim \Pi_k, \\ \Psi_k &= C\Pi_k, \quad \text{with output } y_k \sim \Psi_k, \end{aligned} \tag{1.2}$$

initialized by $x_0 \sim \Pi_0$, suggest an observability question by their similarity to the linear-system equations.

In studying the finite-state hidden Markov model filter, probability distributions are the quantities of interest. Here, our analysis intersects work by van Handel [vH09a, vH09b], who addresses observability as a property of the probability distributions. This thesis devotes more detail to van Handel's formulation, as his observability coincides with stability of the nonlinear filter; that is, the conditional probability distribution of the state given the output eventually forgets the initial state distribution, accounting for potential application problems where a filter is initialized with a poor choice of initial conditions.

In comparison to the work on stochastic observability, the recent research on stochastic controllability is better established; we review a representative, but small, sample of the literature here. The basic ideas stem from work by Sunahara, et al. [SaA$^+$74] who, given an initial condition, define the capacity to steer the state variance to within an ϵ neighborhood of the origin with a given probability.

Sunahara's work is extended by Mahmudov and his collaborators [Mah01, SKM06] in a series of papers, which include results for certain classes of nonlinear systems as well as systems with time delays. Perhaps because the literature almost entirely focuses on continuous-time systems, Mahmudov does not distinguish between controllability and reachability[1]; in fact, his notion of controllability is more comparable to what we call reachability in this thesis and for which Sunahara's definition is a special case. Being a study on the extensions of the deterministic reachability definitions, Mahmudov's analysis also recovers the better known reachability rank condition. Further extensions to wider classes of nonlinear systems have been approached by Muthukumar and Balasubramaniam [MB09]; here, as in earlier works, there is an emphasis on the idea that reachability describes the ability to steer the state closer to any desired point in the state space.

Ugrinovskii [Ugr05] approaches the controllability question from a different direction. His formulation concerns the robust control of linear uncertain systems and delineates controls which bound the $\mathcal{L}_2$ norm of the state over a class of permissible uncertainties, which are described by their distributions. In this context, he shows necessary and sufficient conditions for complete controllability using game-type Ricatti differential equations and recovers the deterministic results under natural assumptions.

Within the literature that specifically refers to stochastic reachability, we point out work by Bujorianu [Buj09] and by Lygeros and collaborators [BL07, APLS08, SL10]. Bujorianu establishes the primacy of the use of sets and derives results on the probability of the state being in desired sets; this foreshadows our inspection of the reachability of measurable functions, as the set is a building block for measurable functions. Lygeros, et al. apply the concepts of Bujorianu to a number of applications and determine computational methods which supplement Bujorianu's theoretical results.

Because the existing literature on stochastic controllability and reachability is quite established and primarily deals with continuous-time systems – whereas our interests lie with discrete-time systems – rather than engaging in detailed comparison between our formulation and existing definitions, our development proceeds by analogy to the information theoretic formulation for observability.

[1] In continuous-time, time-invariant, deterministic systems, the two are the same.

1.2 Overview

Our analysis focuses on discrete-time systems to avoid obfuscation by the technical difficulties of continuous-time stochastic processes.

1.2.1 Observability and Reconstructibility

We begin with the observability question for linear Gaussian (or Gauss-Markov) systems of the form

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad x_0 \sim N(\bar{x}_0, \Sigma_0) \tag{1.3}$$

$$y_k = Cx_k + v_k, \tag{1.4}$$

where w_k and v_k are zero-mean, normally distributed, white noise sequences which are independent from one another and from the initial state x_0. For (1.3-1.4), the Kalman filter and smoother propagate conditional probability distributions, and the conditional estimator-error covariance provides a measure of the reduction in the uncertainty about the unknown state given the measurement sequence $\{y_k\}_0^N$. By defining stochastic observability as the strict reduction of the error-covariance for the estimator of the initial state x_0, where reduction is possible, we may derive the deterministic observability rank condition.

Some effort is devoted to distinguishing reconstructibility from observability. Whereas observability corresponds to the reduction of uncertainty about the initial state from proceeding measurements, the problem of reconstructibility asks whether the current state uncertainty can be reduced via current and past measurements. Even in the discrete-time, linear deterministic case, an observable system need not be reconstructible. Furthermore, from the perspective of control design, reconstructibility appears to be the more important property. As with the our analysis of observability, we define reconstructibility in terms of reduction of the state estimator-error covariance and recover the deterministic condition under suitable conditions on the noise. Finally, we draw a connection to the concept of estimability defined by Baram and Kailath [BK88], and in the process, a connection is established between stochastic reconstructibility and the irreducibility of the Kalman filter.

In nonlinear, non-Gaussian systems, the variance may not be a meaningful measure of uncertainty. A frequently studied example is the finite-state hidden Markov model, which is governed by (1.2). Because the state and output spaces have finite-cardinality, the indexing for the state and output is not unique, so the variance computation may not have physical significance. A further difficulty of nonlinear systems is that even a scalar random variable may not be observable in a complete sense. For instance, suppose a random variable takes values in $x \in \{-2, -1, 1, 2\}$ while the measurement is $y = |x|$. Then the measurement evidently contains some information about x as it allows exact determination of the magnitude of x; on the other hand, the output may provide no information about the sign of x.

The information entropy and conditional information entropy provide the necessary generalizations of the covariance for nonlinear systems since, since for Gaussian distributions, the entropy is an exponential function of the variance. Following the linear systems analysis, we approach observability and reconstructibility by asking whether the conditional entropy of the state can be reduced given the inputs and outputs compared to the conditional entropy of the state given only the inputs. To negotiate incomplete observability, we generalize observability of the state to observability of scalar measurable functions of the state. After formulating the nonlinear definitions, we demonstrate that the linear definitions can be recovered from their nonlinear versions.

A hidden Markov model for the Internet network congestion problem is introduced to explore observability concepts numerically. In this problem, a source computer sends data through a route in the network which includes a bottleneck router which limits the throughput of the source. Since the source computer has limited observation of the network in typical congestion control problem scenarios, we investigate whether the measurement is informative of the complete state of the bottleneck router or perhaps of only a component of the state. For instance, an output may reduce the uncertainty about the queue length in a bottleneck router while conveying no information about the traffic intensity at that router. By analogy to linear systems, the structure of (1.2) suggests a connection between complete observability of the hidden Markov model and a matrix rank condition on the probability transition matrices (A, C). We additionally establish the suspected rank condition. The network example and the entropy formulation additionally invite us to investigate quantitatively the value of allowing more informative outputs which may be achieved via architectural changes to Internet control protocols.

To validate our observability formulations further, we explore two consequences. First, we show an observability condition for performance improvement over open-loop control via output feedback. The analysis studies the observability of the cost/pay-off function of the state, reinforcing the conceptual basis of our specification of *complete* observability. A second consequence of observability lies at the intersection of our work and van Handel's research. Van Handel defines an observable system as one in which the output probability distribution eventually forgets the initial state distribution. We demonstrate that this definition concurs with our entropy formulation under reasonable conditions; thus, our observability also has implications for stability of the nonlinear filter.

1.2.2 Controllability and Reachability

Our approach to stochastic controllability is by analogy with our development of stochastic observability definitions. As before, we begin with the linear system (1.3). Stochastic controllability and reachability for the linear Gaussian system are defined in terms of appropriate covariances and the corresponding deterministic conditions are recovered.

The extension to nonlinear systems uses the *relative* entropy; unlike observability, the (self-information) entropy is not an adequate generalization of the covariance for analyzing controllability. Roughly speaking, the difference between observability and controllability highlights a difference between estimation and control. Whereas more data is never harmful for estimation (in an information-theoretic sense), more control excitation can deteriorate the system performance depending on nonlinearities and the design objective.

The controllability analysis also provides an opportunity to consider a second difference between control and estimation. Whereas the conditional probability distribution of the state given the outputs is meaningful for estimation, the conditional probability of the state given the inputs may not be so. The reason is that an input may have no causal influence over the state, but, by acting as a measurement signal of the state, conditioning on the input may still affect the probability of the state. This loose notion is the subject of Solo's work on generalizing Granger causality [Sol07, Sol08, Gra01]. We canvas Solo's research and show that it is a necessary condition for a linear system to be reachable in our sense.

1.3 Contributions

The contributions of this dissertation are summarized as follows:

1. In Chapter 2, we extend deterministic linear definitions for observability and reconstructibility, first to stochastic linear systems, and then to stochastic nonlinear systems.

   – The deterministic matrix conditions are recovered from stochastic definitions for linear systems;

   – The close connection between reconstructibility and estimability – a property which specifies the minimality of the Kalman filter – is demonstrated.

   – Linear stochastic definitions of observability and reconstructibility are recovered from nonlinear versions, which are formalized in the language of information theory.

2. In Chapter 3, we demonstrate computability in finite time of complete observability and reconstructibility in finite-state hidden Markov models (HMM).

   – In fact, for HMMs, a rank condition reminiscent of that for linear systems is shown to guarantee complete observability.

   – Connections between observability/reconstructibility and the performance of the HMM filter and smoother are explored.

3. In Chapter 4, we show the necessity of reconstructibility for improvement of the optimal feedback-control strategy over the optimal open-loop inputs and investigate the dual purposes of control via simulations of a hidden Markov model.

4. In Chapter 5, we demonstrate connections between our observability and that of van Handel [vH09a], which describes stability of the optimal filter (forgetting of the initial condition).

5. In Chapter 6, we extend deterministic linear definitions for controllability and reachability to stochastic linear systems and stochastic nonlinear systems.

   – The deterministic matrix conditions are recovered from stochastic definitions for linear systems.

   – Some backwards compatibility of the nonlinear to linear definitions is shown.

   – A connection between controllability/reachability and Granger causality [Gra88, Sol08] is shown.

Chapter 2

Stochastic Observability and Reconstructibility

In deterministic systems, observability describes the ability to reconstruct the state of a system using the input and output (measurement) signals, and arises as a key property for state-estimator-controller design. The generalization to stochastic systems is not obvious as, in the presence of noise, one generally cannot reconstruct the state exactly. We explore the extension problem.

Our formulation has two objectives. First, we would like an unobservable stochastic system to be one in which the output signal contains no useful information for estimating some part of the state. Second, we require that the definition of stochastic observability in linear systems reduce, under mild assumptions, to the well known deterministic rank condition.

We begin with the analysis of linear systems to ensure the second aim is achieved and to take advantage of the machinery provided by Kalman filtering theory both for quantifying the information content of the output in terms of the estimation-error covariance, and for connecting our definition to related formulations by Chen [Che77, Che79, Che80] and Baram and Kailath [BK88]. Our definition of stochastic observability describes the ability to reduce the uncertainty of the initial state in all directions in the state space via use of the outputs. From this we distinguish the concept of reconstructibility, which describes the ability to reduce the uncertainty of the *current* state.

In nonlinear systems, we borrow the concepts of entropy and mutual information to generalize from the conditional covariance. Analogous to our discussion of observable directions in the state space for linear systems, we permit measurable functions of the state of a nonlinear system to be unobservable. This, in turn, motivates the definition of complete observability of a nonlinear system, where every measurable function of the state is observable. We show that a linear system which is completely stochastic-observable in the linear sense must also be completely observable in the nonlinear sense.

2.1 Introduction

Consider the time-invariant, discrete-time linear system,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \quad x_k \in \mathbb{R}^n, \\ y_k &= Cx_k + Du_k + v_k. \end{aligned} \tag{2.1}$$

We make the following standing assumptions.

Assumption 1. *The process noise sequence, $\{w_k\}_{k=0}^{\infty}$, and measurement noise sequence, $\{v_k\}_{k=0}^{\infty}$, are white and satisfy $E[w_k] = E[v_k] = 0$, $E[w_k w_j^T] = Q\delta_{kj}$, $E[v_k v_j^T] = R\delta_{k=j}$, and $E[w_k v_j^T] = 0$, where $\delta_{k=j} = 1$ if $k = j$ and $\delta_{k=j} = 0$ otherwise.*

The absence of noise is defined as $w_k = 0$ and $v_k = 0$.

2.1.1 Deterministic Linear Systems

Definition 1 (Linear Deterministic Observability [Kai96]). *The system given by* (2.1) *with $w_k = 0$ and $v_k = 0$ is linear deterministically observable if the state x_0 can be reconstructed exactly given the sequence of inputs and outputs,*

$$\begin{aligned} \mathcal{U}_n &= \begin{pmatrix} u_0^T & u_1^T & \dots & u_{n-1}^T \end{pmatrix}^T \quad \textit{and} \\ \mathcal{Y}_n &= \begin{pmatrix} y_0^T & y_1^T & \dots & y_{n-1}^T \end{pmatrix}^T. \end{aligned}$$

Equivalently, (2.1) *is linear deterministically observable if any two distinct initial states,* $x_0^1 \neq x_0^2$*, yield distinct corresponding output sequences* $\mathcal{Y}_n^1 \neq \mathcal{Y}_n^2$ *for the same control sequence* $\mathcal{U}_n$*.*

In the deterministic nonlinear case, this definition may be extended, as is done in various works [HK77, NvdS90, BBBB03, LAM05]. Because the system (2.1) is linear, the deterministic observability depends only on the matrices C and A and not on the input sequence. In nonlinear, deterministic systems, this control independence does not hold in general [NvdS90]. We include the control input in our development from the outset in order to make observations about how the interaction between control and stochastic observability compares between systems of different type.

We next introduce the concept of linear deterministic reconstructibility.

Definition 2 (Linear Deterministic Reconstructibility [Kai96])**.** *The system given by* (2.1) *with* $w_k = 0$ *and* $v_k = 0$ *is linear deterministically reconstructible if the state* x_n *can be reconstructed exactly given the sequence of inputs* $\mathcal{U}_n$ *and outputs* $\mathcal{Y}_n$*.*

In deterministic systems, reconstructibility and observability are only equivalent for continuous time systems with time invariant system matrices. In general, reconstructibility is necessary but not sufficient for observability in discrete-time systems of the form (2.1). Consider such a system with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C = [1 \; 0],$$

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } D = 0.$$

The state x_k for any $k > 0$ is evidently zero so the system is reconstructible, but the system is not observable.

We remark that these definitions of linear deterministic observability and reconstructibility do not extend simply to the consideration of stochastic systems such as (2.1) which have non-zero driving noises. The state in general cannot be exactly recovered from the measurements. We further note that in stochastic systems, while the state at time zero might be known exactly, it does not follow that this information suffices for future state exact reconstructibility. This issue of dealing with state distributions is central to these concepts in the stochastic case.

2.2 Gaussian Linear Systems

In order to handle stochastic systems, we commence with the consideration of Gaussian linear systems. We will denote that z is normally distributed with mean $\bar{z}$ and covariance Σ_z by $z \sim N(\bar{z}, \Sigma_z)$. Consider the uncontrolled, time-invariant, discrete-time, linear system (2.1). If the initial state, process noise, and measurement noise are Gaussian, $x_0 \sim N(\bar{x}_0, \Sigma_0)$, $w_k \sim N(0, Q)$, $v_k \sim N(0, R)$, then x_k and y_ℓ are jointly Gaussian for all $k, \ell \geq 0$. In this case, the minimum-mean-square state estimator of x_0 given inputs $\mathcal{U}_n$ and output measurements $\mathcal{Y}_n$ is the conditional mean and is provided by the Kalman estimator. As usual, we denote the conditional mean of the state at time k given output measurements $\{y_0, y_1, \ldots, y_\ell\}$ by

$$\mathrm{E}[x_k | y_0, \ldots, y_\ell] = \hat{x}_{k|\ell},$$

and we denote the conditional covariance of this estimate by $\Sigma_{k|\ell}$, where ℓ may be greater than, less than, or equal to k.

2.2.1 Stochastic Observability

A variety of definitions of stochastic-linear observability in different contexts is given, but for our purpose of extending known definitions to the network state estimation problem, we will adopt the following meaning.

Definition 3 (Complete Stochastic-Linear Observability). *Consider the linear system (2.1) with initial state mean $E[x_0] = \bar{x}_0$ and covariance $\Sigma_{0|-1}$. If for every n-vector ξ we have either;*

$$\xi^T \Sigma_{0|-1} \xi = 0, \ \ \textit{or} \ \xi^T \Sigma_{0|-1} \xi > 0 \ \textit{and} \ \xi^T \Sigma_{0|N} \xi < \xi^T \Sigma_{0|-1} \xi \tag{2.2}$$

for a known control sequence $\mathcal{U}_N$ *and every* $N \geq N_0$ *for some finite* N_0*, then we call the system completely stochastic-linear-observable for the given initial state distribution.* [1] *Clearly, this would follow if* $\Sigma_{0|N} < \Sigma_{0|-1}$*, where this inequality is interpreted in the usual sense of partial ordering of non-negative-definite matrices.*

Definition 3 is closely related to the formulation of stochastic observability due to Chen [Che77, Che79, Che80, Che85], who also defines observability in terms of the strict reduction of the conditional state covariance given the outputs. Chen's definition addresses a more specific estimation problem, in which a finite state-estimator covariance may be recovered when there is no prior knowledge of the initial state, namely, when $\Sigma_{0|-1} = \lim_{\alpha\to\infty} \alpha I$, I being the $n \times n$ identity matrix. Our reason for adapting his definition is that the action of setting the initial state covariance to an arbitrarily large value with specific matrix structure may be unsuitable for generalization to nonlinear systems.

Theorem 1. *Consider the linear stochastic system* (2.1) *with Gaussian initial state,* x_0*, and Gaussian noise processes,* $\{w_k\}_0^\infty$ *and* $\{v_k\}_0^\infty$*, subject to the Assumption 1, with the measurement noise covariance,* $R > 0$*, and a known control sequence* $\{u_k\}_0^{n-1}$*. Then the system* (2.1) *is completely stochastic-linear-observable if the following conditions hold.*

(i) *R is finite.*

(ii) *Q is finite if $n > 1$.*

(iii)

$$\mathit{rank}\begin{pmatrix} C^T & A^TC^T & A^{2^T}C^T & \dots & A^{(n-1)^T}C^T \end{pmatrix} = n.$$

[1]The strict inequalities in (2.2) are needed since, for any $\xi \in \mathbb{R}^n$, $\xi^T\Sigma_{0|N}\xi \leq \xi^T\Sigma_{0|-1}\xi$ is always true.

Proof. We have the following relationship.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0 + \begin{pmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & \dots & D \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & \dots & 0 \\ C & 0 & \dots & 0 \\ CA & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2} & CA^{n-3} & \dots & C \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-2} \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}. \quad (2.3)$$

With obvious notation, we denote these matrices as,

$$\mathcal{Y} = \mathcal{O}x_0 + \mathcal{B}\mathcal{U} + \mathcal{H}\mathcal{W} + \mathcal{V}.$$

Further denote $\mathcal{Q} = I_{n-1} \otimes Q$ and $\mathcal{R} = I_n \otimes R$. Clearly since x_0 is Gaussian, as are the $\{w_k\}_0^{n-1}$ and $\{v_k\}_0^{n-1}$ sequences, and the system (2.1) is linear, x_0 and $\mathcal{Y}$ are jointly Gaussian;

$$\begin{bmatrix} x_0 \\ \mathcal{Y} \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_0 \\ \mathcal{O}\bar{x}_0 + \mathcal{B}\mathcal{U} \end{bmatrix}, \begin{bmatrix} \Sigma_0 & \Sigma_0\mathcal{O}^T \\ \mathcal{O}\Sigma_0 & \mathcal{O}\Sigma_0\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R} \end{bmatrix} \right).$$

Whence,

$$\mathrm{E}\left[x_0 \,|\mathcal{Y}\right] = \bar{x}_0 + \Sigma_0\mathcal{O}^T \left(\mathcal{O}\Sigma_0\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R}\right)^{-1} \left(\mathcal{Y} - \mathcal{O}\bar{x}_0 - \mathcal{B}\mathcal{U}\right), \quad (2.4)$$
$$\sim N\left(\bar{x}_0, \Sigma_0 - \Sigma_0\mathcal{O}^T \left(\mathcal{O}\Sigma_0\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R}\right)^{-1} \mathcal{O}\Sigma_0\right). \quad (2.5)$$

Establishing complete observability requires proving that

$$\begin{aligned} \xi^T\mathcal{M}\xi &\triangleq \xi^T\Sigma_0\mathcal{O}^T \left(\mathcal{O}\Sigma_0\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R}\right)^{-1} \mathcal{O}\Sigma_0\xi \\ &> 0, \end{aligned} \quad (2.6)$$

for all vectors ξ such that $\xi^T\Sigma_{0|-1}\xi > 0$.

Since by assumption $R > 0$, $\mathcal{R}$ is positive definite and the matrix

$$\mathcal{O}\Sigma_0\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R}$$

is invertible. The Conditions (i) and (ii) that R and Q be finite guarantees that this inverse is nonzero and is, indeed, positive definite. Define $\zeta = \Sigma_0\xi$, which we assume is non-zero, since otherwise we have nothing else to prove. The full rank Condition (iii) on $\mathcal{O}$ ensures that $\xi^T\mathcal{M}\xi > 0$ and so that $\xi^T\Sigma_{0|N}\xi < \xi^T\Sigma_{0|-1}\xi$ establishing complete stochastic-linear observability. $\square$

The importance of Theorem 1 is that Condition (iii) on the rank of the observability matrix coincides with the deterministic linear observability condition as found in most texts on Linear Control Theory. In the special case where the noise terms tend to zero, $w_k, v_k \to 0$, the deterministic and stochastic definitions of observability, Definitions 1 and 3, coincide. The probabilistic structure required of Definition 3 follows from allowing x_0 to have non-zero covariance.

The consideration of infinite covariances follows from the methods for handling missing data in Signal Processing, where the measurement noise covariance is temporarily set to $R = \infty I$ for those missing samples. This parallels Chen's approach to the complete absence of prior knowledge about the initial state by defining an infinite initial covariance $\Sigma_{0|-1} = \infty I_n$.

As with linear deterministic observability, complete stochastic-linear observability likewise does not depend on the given control sequence although the reconstruction of the initial state clearly uses it. A focus of our study of stochastic observability in nonlinear systems will be on the effects of control on observability and vice versa; the above analysis does not extend in an obvious way.

2.2.2 Stochastic Reconstructibility

Expanding on the ideas of Kalman filtering, we instead define a version of reconstructibility for stochastic systems in terms of the reduction of the present estimator uncertainty given the preceding outputs.

Definition 4. *Consider the stochastic linear system* (2.1). *Denote the state mean and variance based on the initial data only as*

$$\bar{x}_k = E[x_k], \tag{2.7}$$

$$\Sigma_{k|-1} = E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T], \tag{2.8}$$

and the state estimator and covariance from the Kalman Filter as

$$\hat{x}_{k|k} = E[x_k | y_0, \dots y_k],$$
$$\Sigma_{k|k} = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T | y_0, \dots y_k].$$

If for every n-vector ξ and some $N > 0$ we have either;

$$\xi^T \Sigma_{k|-1} \xi = 0, \ \textit{or}$$
$$\xi^T \Sigma_{k|-1} \xi > 0 \ \textit{and} \ \xi^T \Sigma_{k|k} \xi < \xi^T \Sigma_{k|-1} \xi, \ \textit{for all } k > N,$$

then we call the system linear stochastically completely reconstructible for an initial state distribution $x_0 \sim N(\bar{x}_0, \Sigma_{0|-1})$. [2] *This would follow if $\Sigma_{k|k} < \Sigma_{k|-1}$, where this inequality is interpreted in the usual sense of partial ordering of non-negative definite matrices.*

As with observability, we show a connection between the deterministic and stochastic definitions for reconstructibility. For this purpose, we quote the following result [AM79].

Theorem 2. *The system given by* (2.1) *is deterministic-reconstructible if any one of the following equivalent conditions holds:*

(i) *Range* $\left[C^T, A^T C^T \dots A^{T^{n-1}} C^T\right] \supseteq$ *range*(A^{T^n}).

(ii) *If* $CA^i w = 0$ *for* $i = 0, 1, \dots n-1$, *then* $A^n w = 0$.

(iii) *If* $Cw = 0$ *and* $Aw = \lambda w$, *then* $\lambda = 0$ *or* $w = 0$.

Henceforth, we shall refer to linear stochastic complete reconstructibility simply as reconstructibility and distinguish it from deterministic reconstructibility as necessary.

As with stochastic observability, we next develop a linear algebraic condition for stochastic reconstructilibity. We go on to show that, as the noise approaches zero, we show that stochastic reconstructibility implies deterministic reconstructibility. To this end, for the system (2.1), recall the standing assumptions (1):

[2] Analogous to the stochastic observability case, $\xi^T \Sigma_{k|k} \xi \geq \xi^T \Sigma_{k|-1} \xi$ is true for any $\xi \in \mathbb{R}^n$.

> The process noise sequence, $\{w_k\}$, and measurement noise sequence, $\{v_k\}$, are Gaussian white and satisfy $\mathrm{E}[w_k] = \mathrm{E}[v_k] = 0$, $\mathrm{E}[w_k w_j^T] = Q\delta_{kj}$, $\mathrm{E}[v_k v_j^T] = R\delta_{kj}$, and $\mathrm{E}[w_k v_j^T] = 0$. The absence of noise is defined as $w_k = 0$ and $v_k = 0$.

The following result reveals the technical difficulties in analyzing reconstructibility.

Theorem 3. *The system given by* (2.1) *subject to standing assumptions, with positive definite measurement noise covariance R, is reconstructible if and only if*

$$\left\{ range \begin{bmatrix} \Sigma_{0|-1} A^{T^n} \\ Q A^{T^{n-1}} \\ \vdots \\ Q \end{bmatrix} \cap null \begin{bmatrix} C & 0 & \dots & 0 \\ CA & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^n & CA^{n-1} & \dots & C \end{bmatrix} \right\} = 0 \tag{2.9}$$

Proof. From (2.1), the propagated state equation can be written as

$$x_n = A^n x_0 + \begin{bmatrix} A^{n-1} & \dots & I_n \end{bmatrix} \begin{bmatrix} w_0 \\ \vdots \\ w_{n-1} \end{bmatrix}.$$

With obvious notation, we denote these matrices as

$$x_n = A^n x_0 + \mathcal{L}\mathcal{W}.$$

Similarly, arrange the sequence of outputs in a matrix equation,

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} x_0 + \begin{pmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & \dots & D \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & \dots & 0 \\ C & 0 & \dots & 0 \\ CA & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1} & CA^{n-2} & \dots & C \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{bmatrix} + \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

We denote these matrices as

$$\mathcal{Y}_n = \mathcal{O}x_0 + \mathcal{B}\mathcal{U} + \mathcal{H}\mathcal{W} + \mathcal{V}.$$

Further, denote $\mathcal{Q} = I_n \otimes Q$ and $\mathcal{R} = I_{n+1} \otimes R$.

Since x_0 and sequences $\{w_j\}$ and $\{v_j\}$ are Gaussian and the system (2.1) is linear, x_n and $\mathcal{Y}_n$ are jointly Gaussian;

$$\begin{pmatrix} x_n \\ \mathcal{Y}_n \end{pmatrix} \sim N\left(\begin{bmatrix} A^n\bar{x}_0 \\ \mathcal{O}\bar{x}_0 + \mathcal{B}\mathcal{U} \end{bmatrix}, \begin{bmatrix} A^n\Sigma_{0|-1}A^{T^n} + \mathcal{L}\mathcal{Q}\mathcal{L}^T & A^n\Sigma_{0|-1}\mathcal{O}^T + \mathcal{L}\mathcal{Q}\mathcal{H}^T \\ \mathcal{H}\mathcal{Q}\mathcal{L}^T + \mathcal{O}\Sigma_{0|-1}A^{T^n} & \mathcal{O}\Sigma_{0|-1}\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R} \end{bmatrix}\right) \quad (2.10)$$

The state covariance defined in (2.7) can be written as

$$\Sigma_{k|-1} = A^k\Sigma_{0|-1}A^{T^k} + \mathcal{L}\mathcal{Q}\mathcal{L}^T. \quad (2.11)$$

The optimal (Kalman filter) estimator $\hat{x}_{k|k}$ is given by the conditional mean, $E[x_k|\mathcal{Y}_n]$, which from (2.10) is unbiased and has covariance

$$\Sigma_{k|k} = \Sigma_{k|-1} - (A^k\Sigma_{0|-1}\mathcal{O}^T + \mathcal{L}\mathcal{Q}\mathcal{H}^T) (\mathcal{O}\Sigma_{0|-1}\mathcal{O}^T + \mathcal{H}\mathcal{Q}\mathcal{H}^T + \mathcal{R})^{-1}(\mathcal{O}\Sigma_{0|-1}A^{T^k} + \mathcal{H}\mathcal{Q}\mathcal{L}^T). \quad (2.12)$$

Proving linear stochastic complete reconstructibility requires showing

$$\xi^T\Sigma_{n|n}\xi < \xi^T\Sigma_{n|-1}\xi \text{ for all } \xi \text{ such that } \xi^T\Sigma_{n|-1}\xi \neq 0.$$

Equivalently, rearranging (2.12), we need to show that

$$\mathcal{O}\Sigma_{0|-1}A^{T^n}\xi + \mathcal{H}\mathcal{Q}\mathcal{L}^T\xi \neq 0 \text{ for all } \xi \text{ such that} \quad (2.13)$$

$$\begin{bmatrix} \Sigma_{0|-1}A^{T^n} \\ \mathcal{Q}\mathcal{L}^T \end{bmatrix}\xi \neq 0 \quad (2.14)$$

Rearranging (2.13), we equivalently require

$$[\mathcal{O}\ \mathcal{H}]\begin{bmatrix} \Sigma_{0|-1}A^{T^n} \\ \mathcal{Q}\mathcal{L}^T \end{bmatrix}\xi \neq 0 \text{ for all } \xi \text{ such that}$$

$$\begin{bmatrix} \Sigma_{0|-1}A^{T^n} \\ \mathcal{Q}\mathcal{L}^T \end{bmatrix}\xi \neq 0. \quad (2.15)$$

Note that due to the Cayley-Hamilton Theorem, the matrix in (2.15) achieves full rank after n time steps.

This is in turn equivalent to

$$\left\{\text{range}\begin{bmatrix} \Sigma_{0|-1}A^{T^n} \\ \mathcal{Q}\mathcal{L}^T \end{bmatrix} \cap \text{null}[\mathcal{O}\ \mathcal{H}]\right\} = 0.$$

□

The proof of Theorem 3 is analogous to that in Theorem 1, where we recovered the deterministic observability rank condition from linear stochastic complete observability.

We continue the analysis of reconstructibility by demonstrating how to recover the deterministic reconstructibility conditions given in Theorem 2.

Theorem 4. *For the system given in* (2.1) *with* $\Sigma_{0|-1} = \alpha I_n$ *for any* $\alpha \in \mathbb{R}$, $\alpha > 0$, *let* $Q = 0$ *and* $R = rI_p$, *where* $r \in \mathbb{R}$, $r > 0$. *In the limit as* $r \to 0$, *system* (2.1) *is completely (deterministic) reconstructible per Definition 2 if and only if for some* $N > 0$ *and for all* ξ, $\lim_{r\to 0} \xi^T \Sigma_{k|k}\xi = 0$, *for all* $k > N$.

Proof. We will use the reconstructibility condition,

$$\text{range}\left[C^T,\ A^T C^T\ \ldots\ A^{T^{n-1}}C^T\right] \supseteq \text{range}(A^{T^n}).$$

We first prove that $\text{range}(A^{T^n}) \subseteq \text{range}(\mathcal{O}^T) \Rightarrow \xi^T\Sigma_{n|n}\xi \to 0$ as $r \to 0$. Equivalently, show

$$\begin{aligned}\xi^T A^n \Sigma_{0|-1} A^{T^n}\xi &= \xi^T A^n \Sigma_{0|-1}\mathcal{O}^T\left(\alpha\mathcal{O}\Sigma_{0|-1}\mathcal{O}^T + \bar{R}\right)^{-1}\mathcal{O}\Sigma_{0|-1}A^{T^n}\xi \\ &= \alpha\xi^T A^n \lim_{r\to 0}\mathcal{O}^T(\mathcal{O}\mathcal{O}^T + \frac{r}{\alpha}I_p)^{-1}\mathcal{O}A^{T^n}\xi.\end{aligned}$$

Setting $w \triangleq A^{T^n}\xi$ and $\rho \triangleq \left(\frac{r}{\alpha}\right)^{1/2}$, we have

$$\alpha w^T w = \alpha w^T \lim_{\rho\to 0}\mathcal{O}^T(\mathcal{O}\mathcal{O}^T + \rho^2 I_p)^{-1}\mathcal{O}w. \tag{2.16}$$

Equation (2.16) contains the limit form of the Moore-Penrose pseudo-inverse (see [Alb72]),

$$\lim_{\rho\to 0}\mathcal{O}^T(\mathcal{O}\mathcal{O}^T + \rho^2 I)^{-1} = \mathcal{O}^+,$$

where $\mathcal{O}^+$ is the pseudo-inverse of $\mathcal{O}$. By assumption, $w \in \text{range}(\mathcal{O}^T) = \text{range}(\mathcal{O}^+)$, hence $\exists\ v$ such that $w = \mathcal{O}^+ v$. Thus, from (2.16) we have

$$\begin{aligned} \alpha w^T w &= \alpha w^T \mathcal{O}^+ \mathcal{O} w \\ &= \alpha w^T \mathcal{O}^+ v \\ &= \alpha w^T w. \end{aligned}$$

Conversely, we show that $\lim_{r\to 0} \xi^T \Sigma_{n|n} \xi = 0 \Rightarrow \text{range}(A^{T^n}) \subseteq \text{range}(\mathcal{O}^T)$. Make the contradiction assumption, $\text{range}(A^{T^n}) \nsubseteq \text{range}(\mathcal{O}^T)$. Then $\exists\, \xi$ such that $w \triangleq A^{T^n}\xi \notin \text{range}(\mathcal{O}^T)$.

$$\begin{aligned} \Rightarrow\ \xi^T \Sigma_{n|n}\xi =& \alpha \xi^T A^n A^{T^n} \xi \\ & - \alpha^2 \xi^T A^n \mathcal{O}^T (\alpha \mathcal{O}\mathcal{O}^T + rI_p)^{-1} \mathcal{O} A^{T^n} \xi \\ =& \alpha w^T w - \alpha^2 w^T \mathcal{O}^T z \end{aligned}$$

where $w = A^{T^n}\xi$ and $z = (\alpha \mathcal{O}\mathcal{O}^T + rI)^{-1}\mathcal{O}w$.

Since $w \notin \text{range}(\mathcal{O}^T)$, $\mathcal{O}^T z \neq \beta w$, for all $\beta \in \mathbb{R}$. We then have $\xi^T \Sigma_{n|n} \xi \neq 0$, which is a contradiction. Thus, as $r \to 0$, $\xi^T \Sigma_{n|n}\xi = 0 \Rightarrow \text{range}(A^{T^n}) \subseteq \text{range}(\mathcal{O}^T)$.

$\square$

In Theorem 1 the derivation of the observability rank condition from completely stochastic-linear observability assumes $R > 0$. We remark that the same method for having $R = \lim_{r\to 0} rI_p$ with $\Sigma_{0|-1} = \alpha I_n$, $Q = 0$ as in Theorem 4 will show that deterministic observability is a special case of linear stochastic complete observability in system (2.1) with zero noise.

2.2.3 Estimability

There are parallels to estimability defined by Baram and Kailath as a linear stochastic system property which is related to the minimality of the Kalman Filter [BK88].

Definition 5. *Consider the stochastic linear system* (2.1)*. As before, denote the propagated state covariance by* $\Sigma_{k|-1}$*, the conditional mean estimator of* x_k *by* $\hat{x}_{k|k}$

and its covariance by $\Sigma_{k|k}$*. This system is estimable if*

$$\Sigma_{k|k} < \Sigma_{k|-1} \text{ for all } k \geq n-1.$$

That is, for any non-zero n-vector, ξ, $\xi^T \Sigma_{k|k} \xi < \xi^T \Sigma_{k|-1} \xi$ *for all* $k \geq n-1$.

Baram and Kailath interpret estimability as an extension of observability to stochastic systems. Additionally, Davis and Lasdas, in their study of bearings only tracking and of linear diffusions [DL92], generalize estimability to define stochastic observability for (not necessarily linear) Markov processes. Estimability is distinct from deterministic observability, however, in that neither implies the other. Additionally, comparing Definitions 4 and 5, estimability is evidently a stronger version of reconstructibility.

Example 1. *The main difference between estimability and reconstructibility is that a system of the form* (2.1) *with deterministically known components of the state, i.e.* $\xi^T \Sigma_k \xi = 0$*, can be reconstructible, but such a system is never estimable. Consider a linear stochastic system of the form* (2.1)*, with*

$$A = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = [1\ 1],$$
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ R = 1,\ \Sigma_{0|-1} = \begin{bmatrix} 1.8 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\Sigma_{k|-1}$ *is never positive definite for any* $k \geq 0$*, the system is not estimable; however, because* $\xi^T(\Sigma_{k|-1} - \Sigma_{k|k})\xi > 0$*, for all* $\xi \neq \alpha[1,0]^T$*,* $\alpha \in \mathbb{R}$*, the system is reconstructible.*

For brevity we will in the remainder of the paper use the term “observability” to mean both observability and reconstructibility unless otherwise specified.

2.2.4 Minimality of the Kalman Filter

Baram and Kailath show that a reduced-order Kalman Filter may be defined for a stochastic linear system (2.1) from which the estimate of the original full-order Kalman Filter may be recovered if the system is not estimable. We expand this analysis by examining the connection between the minimality of the Kalman Filter and reconstructibility.

Consider the system (2.1) with standing assumptions, and for clarity, let it be time-invariant and asymptotically stable;

$$\lim_{k\to\infty} x_k \sim \mathrm{N}(0,\Sigma),$$

where Σ satisfies

$$\Sigma = A\Sigma A^T + Q. \tag{2.17}$$

Initialize the state distribution as $x_0 \sim \mathrm{N}(0,\Pi)$. Denote the error covariances for the stationary Kalman Filter as,

$$\begin{aligned} \Sigma_f &= \Sigma_p - \Sigma_p C^T (C\Sigma_p C^T + R)^{-1} C\Sigma_p, \\ \Sigma_p &= A\Sigma_f A^T + Q. \end{aligned}$$

When this system is not estimable, a reduced order filter can be defined using a change of variables on the state x_k;

$$\begin{bmatrix} T^T \\ U^T \end{bmatrix} x_k = \begin{bmatrix} z_k \\ \eta_k \end{bmatrix}, \tag{2.18}$$

where the columns of T are the eigenvectors of $\Sigma - \Sigma_f$ corresponding to all positive eigenvalues, and the columns of U are the remaining eigenvectors;

$$\begin{bmatrix} T^T \\ U^T \end{bmatrix} (\Pi - \Sigma_f)\, [T\ U] = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

If the system is reconstructible, for any vector ξ such that $\xi^T(\Sigma - \Sigma_f)\xi = 0$, we have $\xi^T \Sigma \xi = 0$; thus, $U^T \Sigma U = 0$, and from (2.17), $U^T Q U = 0$. The reduced order system follows;

$$\begin{aligned} z_{k+1} &= T^T A T z_k + T^T w_k, \\ y_k &= C T z_k + v_k. \end{aligned} \tag{2.19}$$

For an estimate generated by the Kalman Filter, $\hat{z}_{k|k}$, the Kalman Filter estimate for $\hat{x}_{k|k}$ can be recovered as $\hat{x}_{k|k} = T\hat{z}_{k|k}$. The discarded states under the change of variables, η_k, are zero mean with variance $U^T \Sigma U = 0$, so they are deterministically known.

On the other hand, if the system is not reconstructible, then there exist vectors ξ such that $\xi^T \Sigma \xi \neq 0$, but $\xi^T(\Sigma - \Sigma_f)\xi = 0$. With T and U defined as before, the variance of η_k is evidently $U^T \Sigma U \neq 0$. Nevertheless, the reduced order filter is defined in the same way. To see this, recall from the proof of Theorem 3,

$$\begin{aligned}\Sigma_f = \Sigma - (&A^k \Sigma \mathcal{O}^T + \mathcal{L}\bar{Q}\mathcal{H}^T) \\ &(\mathcal{O}\Sigma\mathcal{O}^T + \mathcal{H}\bar{Q}\mathcal{H}^T + \bar{R})^{-1}(\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T).\end{aligned} \tag{2.20}$$

Since $U^T(\Sigma - \Sigma_f)U = 0$, from (2.20) we have

$$(\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T)U = 0. \tag{2.21}$$

The Kalman Filter estimate is given by

$$\hat{x}_{k|k} = A^n \bar{x}_0 + (\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T)(\mathcal{O}\Sigma\mathcal{O}^T + \mathcal{H}\bar{Q}\mathcal{H}^T + \bar{R})^{-1}(\mathcal{Y}_n - \mathcal{O}\bar{x}_0). \tag{2.22}$$

Since $\bar{x}_0 = 0$, we have

$$\hat{x}_{k|k} = (\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T)(\mathcal{O}\Sigma\mathcal{O}^T + \mathcal{H}\bar{Q}\mathcal{H}^T + \bar{R})^{-1}\mathcal{Y}_n.$$

Applying the change of variables (2.18) and noting (2.21), we arrive at the following formulation for the reduced-order estimator,

$$\hat{z}_{k|k} = T^T(\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T)(\mathcal{O}\Sigma\mathcal{O}^T + \mathcal{H}\bar{Q}\mathcal{H}^T + \bar{R})^{-1}\mathcal{Y}_n.$$

To show that $\hat{\eta}_{k|k} \equiv 0$ for all k, observe that

$$\hat{\eta}_{k|k} = U^T(\mathcal{O}\Sigma A^{T^k} + \mathcal{H}\bar{Q}\mathcal{L}^T)(\mathcal{O}\Sigma\mathcal{O}^T + \mathcal{H}\bar{Q}\mathcal{H}^T + \bar{R})^{-1}\mathcal{Y}_n.$$

Hence, from (2.21) we have $\hat{\eta}_{k|k} = 0$. Thus, we are again able to recover the full-order estimate, $\hat{x}_{k|k} = T\hat{z}_{k|k}$.

2.3 Nonlinear Systems

For linear systems (and notably for Gaussian ones), complete observability or reconstructibility of an n-vector state from m-vector output measurements is tied to the strict ordering of covariances of estimators. As we move to consider nonlinear systems or systems with states on manifolds or discrete sets, covariances will no longer provide a suitable or suitably computable measure of performance. Further, we shall need to distinguish the observability of n-vector processes and the observability of scalars; this is already made apparent in the definitions of linear stochastic observability and reconstructibility and the consideration of scalar functions $\xi^T x$. We also turn attention from estimate performance to estimator performance.

As a working problem, consider the observability of the scalar process $x \in \mathbb{R}$ from measurement $y = |x|$. Clearly, the distribution of x plays a role; x equilikely distributed between the values $\{-1, 1\}$ is completely unobservable, while x equilikely distributed between values $\{-2, -1\}$ is completely observable from $|x|$. On the other hand, for x symmetrically distributed between $\{-2, -1, 1, 2\}$, while the output y reduces uncertainty of the state x, $\operatorname{sgn}(x)$ is, nevertheless, unobservable. Furthermore, when the distribution of x is asymmetric on $\{-2, -1, 1, 2\}$, $\operatorname{sgn}(x)$ is generally observable. An extension of stochastic complete observability to nonlinear systems must account for the situation in which outputs may provide no information about a function of the state or a subset of the state space, even as it reduces the overall state uncertainty. From this we see that, in nonlinear systems, the trade-off for addressing the observability of functions of the state is evidently a dependence of observability on the state distribution.

These concepts will be used to formulate extensions of stochastic observability. In doing this, we shall be motivated by our later application in finite-state hidden Markov model filtering (HMM). In the HMM, the state x_k and output y_k are finitely denumerable, and the evolution of the probability laws may be written as a set of linear equations,

$$\Pi_{k+1} = A\Pi_k, \quad x_0 \sim \Pi_0$$
$$\Psi_k = C\Pi_k,$$

where $x_k \sim \Pi_k$ and $y_k \sim \Psi_k$. Because the conditional state probabilities may be obtained via finite computations of Bayes rule, the HMM is especially suitable for illustrating key ideas. It is germane to refer to the bearings-only tracking example of Davis and Lasdas [DL92], where the angle of a target is measured but not its range. This is a converse problem to the $\operatorname{sgn}(x)$ problem with clear counterparts in terms of observability.

The nonlinear contribution of the control in the HMM convolves the problem of estimation with the control strategy. Thus, although the HMM has the appearance of a linear system, the observability analysis of the controlled HMM differs fundamentally from the linear systems theory, in which the control does not affect observability.

Denote the *entropy* of the random variable, x, by

$$H(x) = -\sum_{x_i \in \mathcal{X}} P(x = x_i) \ln P(x = x_i), \tag{2.23}$$

if x is a discrete random variable with support in $\mathcal{X}$, and

$$H(x) = -\int_{\mathbb{R}} \ln P(x)\, dP(x),$$

if x is a continuous random variable.

For convenience, we shall use the discrete form of the calculation in the following. We define the conditional entropy between two random variables, x and y, in the normal fashion.

Definition 6 (Conditional Entropy)**.**

$$\begin{aligned} H(x|y) &= -\sum_{y_i \in \mathcal{Y}_n} \sum_{x_j \in \mathcal{X}} P(y = y_i, x = x_j) \ln P(x = x_j | y = y_i), \\ &= -\sum_{y_i \in \mathcal{Y}_n} P(y_i) \sum_{x_j \in \mathcal{X}} P(x_j|y_i) \ln P(x_j|y_i), \qquad (2.24) \\ &= H(x,y) - H(y). \end{aligned}$$

Since this definition involves averaging over both the 'state' process and the 'measurement' process distributions, its interpretation is in terms of estimator properties as opposed to evaluation of estimates in the same manner as the linear observability definition involved the estimator covariance. We have the following immediate properties of conditional entropy, [CT06].

$$\begin{aligned}
H(x) &\geq 0, \\
H\left(g(x)\right) &\leq H(x), \text{ with equality if } g(\cdot) \text{ is injective.} && (2.25) \\
H(x|y) &\leq H(x), \\
H(x|y) &= H(x), \Leftrightarrow x \text{ and } y \text{ are independent,} && (2.26) \\
H(x) &= \operatorname{trace}(\Sigma), \text{ if } x \text{ is Gaussian } N(\bar{x}, \Sigma), && (2.27)
\end{aligned}$$

We will also use the following quantity for its intuitive appeal.

Definition 7 (Mutual Information)**.** *The mutual information between random variables x and y is the difference between the entropy of x and the conditional entropy of x given y,*

$$I(x;y) = H(x) - H(x|y) = H(x) + H(y) - H(x,y).$$

Due to (2.26), the mutual information between x and y is non-negative, $I(x;y) \geq 0$, and equals zero if and only if x and y are independent. Conditional entropy captures the linear system properties so far examined using covariances because of the connection (2.27) between entropy and covariance. For the moment, we limit the discussion to the observability of random variables and vectors without reference to a system property of observability.

Definition 8 (Observability)**.** *Random vector x is unobservable from random vector y if $H(x) > 0$ and x is independent from y. It is observable if it is not unobservable. Equivalently, x is observable from y if either $H(x) = 0$ or $H(x|y) < H(x)$.*

Definition 8 has clear connections to the degree of observability articulated by Mohler and Hwang as the mutual information $I(x;y) = H(x) - H(x|y)$. Their work in turn explores relations to another notion of stochastic observability [CJ97], expressed in terms of the Fisher Information Matrix. The mutual information $I(x;y)$ gives us a quantification of observability. We add to this approach an idea which is analogous to the completeness of observability for linear systems, as this addition captures the capacity of the measurement to improve knowledge of all uncertain aspects of the state.

Definition 9 (Complete Observability)**.** *Random vector $x \in \mathcal{X}$ is completely observable from random vector y if, for every scalar measurable function $g : \mathcal{X} \to \mathbb{R}$, $g(x)$ is observable from y. That is, either $H(g(x)) = 0$ or $H(g(x)|y) < H(g(x))$.*

Definition 9 is our point of departure from Mohler and Hwang. Returning to the working problem of $x \in \{-2, -1, 1, 2\}$ uniformly distributed and $y = |x|$, stochastic observability is apparent in either sense, but additionally, the incomplete observability of $g(x) = \mathrm{sgn}(x)$ is established. There is a clear concordance of these definitions with their linear counterparts from the entropy-covariance relationship (2.27) and the class of functions $g(x) = \xi^T x$.

Nonlinear System Properties

Consider the discrete-time, stochastic, finite-dimensional, nonlinear system

$$x_{k+1} = f(x_k, u_k, w_k), \quad x_k \in \mathbb{R}^n, \tag{2.28}$$

$$y_k = h(x_k, u_k, v_k), \tag{2.29}$$

where

— functions $f(\cdot)$ and $h(\cdot)$ are absolutely continuous in each variable except at a finite number of points,

— $\{w_k\}$ and $\{v_k\}$ are independent martingale difference processes of known distribution, and

— $\{u_k\}$ is a deterministic known control signal.

Definition 10 (Observable Stochastic System)**.** *Consider the system (2.28-2.29) with initial state x_0 and initial state distribution Π_0, possessing entropy $H(x_0)$. If x_0 is completely observable from $\{y_0, \ldots, y_m\}$ given particular control sequence $\{u_0, \ldots, u_m\}$ for some finite m, then we say that the system is completely stochastically observable from this initial distribution and for this input sequence.*

That is, the system is completely stochastically observable if the measurement sequence over a sufficiently long interval permits the reduction of the entropy of any non-trivial function of its initial state when the control sequence is known. As variously recognized [HK77, NvdS90], the control input may result in loss of observability, motivating its inclusion in this definition.

Definition 11 (Reconstructible Stochastic System)**.** *Consider the system (2.28-2.29) with initial state x_0 and initial state distribution Π_0, possessing entropy $H(x_0)$. From this state and entropy and using* (2.28)*, compute the entropy of the state at future time m, $H(x_m)$ given particular control sequence $\{u_0, \ldots, u_m\}$. If x_m is completely observable from $\{y_0, \ldots, y_m\}$ for some finite m, then we say that the system is completely stochastically reconstructible from this initial distribution and for this input sequence.*

It is of interest to ask whether linear stochastic complete observability implies complete stochastic observability in the not-necessarily-linear context. We address this in the next result, since this in part validates the use of the nonlinear definition.

Theorem 5. *Consider the linear stochastic system* (2.1) *with Gaussian initial state, x_0, and Gaussian noise processes, $\{w_k\}$ and $\{v_k\}$, subject to Assumption 1, with the measurement noise covariance, $R > 0$, and a known control sequence $\{u_k\}_0^{n-1}$. If the system* (2.1) *is linear stochastically completely observable (respectively reconstructible), then it is [nonlinear] completely stochastically observable (respectively reconstructible) from initial state, x_0.*

Proof. Denote the state being estimated by x and the sequence of output measurements by $\mathcal{Y}$, which as shown in the proof of Theorem 1 are jointly Gaussian. With obvious notation, denote the covariance or cross-covariance matrices Σ_x, $\Sigma_{\mathcal{Y}}$, and $\Sigma_{x\mathcal{Y}}$. Define the gain $K = \Sigma_{x\mathcal{Y}}\Sigma_{\mathcal{Y}}^{-1}$ and the conditional covariance $W = \Sigma_x - \Sigma_{x\mathcal{Y}}\Sigma_{\mathcal{Y}}^{-1}\Sigma_{x\mathcal{Y}}^T$. Since for $W = 0$ we have $H(g(x)) = H(x) = 0$, we assume without loss of generality that $W > 0$. The semi-definite case can be included with some extra detail. Nonzero W further implies that $K \neq 0$, because of the linear stochastic complete observability of (2.1).

For a measurable function $g(\cdot)$ and constant vectors γ and β, define the following two sets

$$I_\gamma = \{x : g(x) < \gamma\},$$
$$I_{\gamma,\beta} = \{x : g(x + K\beta) < \gamma\},$$

and consider the conditional probability

$$\begin{aligned}
&P(g(x) < \gamma | \mathcal{Y} = \beta) \\
&\quad = \int_{I_\gamma} \frac{1}{\sqrt{(2\pi)^n |W|}} \exp\left[-\frac{1}{2}(\zeta - K\beta)^T W^{-1} (\zeta - K\beta)\right] d\zeta, \\
&\quad = \int_{I_{\gamma,\beta}} \frac{1}{\sqrt{(2\pi)^n |W|}} \exp\left[-\frac{1}{2}\zeta^T W^{-1} \zeta\right] d\zeta.
\end{aligned}$$

Since $W > 0$, the (Gaussian) measure under the integral is absolutely continuous with respect to Lebesgue measure. The independence of this integral from the value β, which would be required to contradict complete observability in the nonlinear sense of Definition 10[3], ensures that either $I_{\gamma,\beta}$ has zero Lebesgue measure or that the function $g(\cdot)$ is almost surely a constant. In either case this would require $H(g(x)) = 0$.

Thus we have established either that $H(g(x)) = 0$ or that $g(x)$ and $\mathcal{Y}$ are not independent and thus $H(g(x)|\mathcal{Y}) < H(g(x))$. □

As seen from Theorem 5 and the entropy-covariance relationship (2.27), the conditional entropy formulation unifies the observability definitions for linear and nonlinear systems.

2.4 Chapter Summary

We defined observability and reconstructibility for the Gauss-Markov system, and then extended these definitions using information theory to a large class of nonlinear systems. The linear deterministic matrix conditions for observability and reconstructibility were recovered from the definitions for stochastic systems. A connection between estimability and reconstructibility in linear systems was shown;

[3]While intuitively obvious, this step entails a fair amount of additional mathematical detail. We have omitted it as the additional steps do not add significantly to the discussion.

this also revealed a link between minimality of the Kalman filter and stochastic reconstructibility. Finally, we showed that the linear and nonlinear versions of stochastic observability and reconstructibility are compatible.

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Andrew R. Liu and Robert R. Bitmead - *Stochastic Observability in Network State Estimation and Control*, Automatica, Vol 47, pp 65-78, 2011;

Andrew R. Liu and Robert R. Bitmead - *Reconstructibility and Estimability*, Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, December, 2009.

Chapter 3

Finite-State Hidden Markov Models

Systems with states and outputs which take values in finite sets are especially suitable for the observability analysis developed in Chapter 2. We adopt the framework of the controlled, finite-state, hidden Markov model (HMM) with state $x_k \in X$, measurement $y_k \in Y$, and control input $r_k \in U$, where X, Y, and U are finite alphabets. For convenience, let $X = \{1, 2, \dots n\}$ and $Y = \{1, 2, \dots m\}$. One may define the probability transitions of this HMM as follows.

$$\begin{aligned} \Pi_{k+1} &= A(r_k)\Pi_k, \\ \Psi_k &= C(r_k)\Pi_k, \\ x_k &\sim \Pi_k, \\ y_k &\sim \Psi_k, \\ A(r_k)[i,j] &= P(x_{k+1} = i | x_k = j, r_k), \\ C(r_k)[i,j] &= P(y_k = i | x_k = j, r_k). \end{aligned} \tag{3.1}$$

One may equivalently write state-space equations for the HMM (see [EMA95]). The state-space representation has a linear structure, though the noise terms are anticipatory in the state and output calculations. The state space is defined as the set of elementary unit vectors.

There is a well-known filter for generating the state distribution conditioned on the input and output sequences from the initial distribution $x_0 \sim \Pi_0$ [SAD98, EMA95]. For a particular sequence of inputs $\{r_k\} \triangleq \{r_0, r_1, \dots\}$ and an output sequence $\{y_k\} \triangleq \{y_0, y_1, \dots\}$, the associated filter for generating state distribution estimates is given by the following equations,

$$\begin{aligned} \Pi_{k+1|k} &= A(r_k)\Pi_{k|k}, \\ \Pi_{k+1|k+1} &= \frac{D_{y_{k+1}}\Pi_{k+1|k}}{[1\ 1\ \dots\ 1]\, D_{y_{k+1}}\Pi_{k+1|k}}, \\ D_{y_{k+1}} &= \begin{bmatrix} C(r_k)[\ell,1] & 0 & \dots & 0 \\ 0 & C(r_k)[\ell,2] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C(r_k)[\ell,n] \end{bmatrix}, \end{aligned} \tag{3.2}$$

for a realization of the output, $y_{k+1} = \ell$. The filter is initialized in the obvious manner;

$$\Pi_{0|-1} \triangleq \Pi_0.$$

The given filter has the added advantage that the estimates are exactly the state distributions conditioned on the output sequence, i.e.

$$\Pi_{k|k}(i) = P(x_k = i | \{y_t\}_0^k, \{u_t\}_0^k) \text{ and } \Pi_{k+1|k}(i) = P(x_{k+1} = i | \{y_t\}_0^k, \{u_t\}_0^k).$$

This can be verified by a straightforward inductive argument using Bayes' Theorem. A generalized form of the proof, which applies to wider classes of HMMs, can be found in [EMA95].

3.1 Observability and Reconstructibility

Because the filter generates the conditional probabilities, the analysis of observability is especially transparent for the HMM. Our first result concerns the verifiability of Definition 10 for the HMM.

Theorem 6. *Complete stochastic observability (reconstructibility) of a system where the state and output take values in finite sets is equivalent to complete stochastic observability (reconstructibility) of a finite collection of scalar measurable functions of the state.*

The key property necessary for the proof is (2.25),

$$H\left(g(x)\right) \leq H(x), \text{ with equality if } g(\cdot) \text{ is injective},$$

from which we can show that the set of entropies and conditional entropies of all measurable functions takes on a finite set of values, since the set of measurable functions of state x splits into a finite number of equivalence classes, to each of which a specific measurable function has an injective mapping. When the sets of state and output values are finite, this equivalence class is likewise finite.

To define the equivalence class of entropies (of measurable functions), we construct a set of their corresponding functions.

Lemma 1. *Consider a set $\mathbb{S}$ of all simple functions of the form $s(x) = \sum_i \gamma_i 1_{G_i}(x)$ (1_{G_i} is the indicator function for set G_i), where $x \in X$ and G_i are disjoint subsets such that $\bigcup_i G_i = X$. The system* (3.1) *is completely observable (respectively reconstructible) if and only if the set of all simple functions of x_0 (respectively x_n) in $\mathbb{S}$ is observable from output sequence $\mathcal{Y} \triangleq \{y_k\}_0^n$ for any $n > m$ and some finite m.*

Proof. The proofs for complete observability and complete reconstructibility are interchangeable; hence, denote the state being estimated by x. In testing complete observability, set $x \equiv x_0$, and in testing complete reconstructibility, set $x \equiv x_n$. Note that the system (3.1) is completely observable (reconstructible) if and only for every measurable g, $H(g(x)|\mathcal{Y}) < H(g(x))$, i.e. every measurable function g is observable in the sense of Definition 8.

If part: Since any simple function $s(x)$ is a measurable function of x, $H(s(x)|\mathcal{Y}) < H(s(x))$.

Only if part: Consider any measurable function $g(x)$. We will construct a simple function such that $s'(x) = g(x)$. We will then show that this simple function is observable in the sense of Definition 8 from the outputs. For any value γ_i in the range of $g(x)$, let $G_i = g^{-1}(\gamma_i)$ be the inverse image of γ_i. The inverse image sets are necessarily disjoint since g is a function. Note that since $x \in X$ and

X is a finite set, the set of possible γ_i is likewise finite. Define $s'(x)$ as

$$s'(x) \triangleq \sum_i \gamma_i 1_{G_i}(x).$$

Since $\{G_i\}$ are disjoint, $s(\xi) = \gamma_\ell = g(\xi)$, where $\xi \in G_\ell \Rightarrow g(\xi) = \gamma_\ell$. To show that $s'(x)$ is observable from the output sequence, note from (2.25) that $H(s'(x)|\mathcal{Y}) = H(s(x)|\mathcal{Y})$ and $H(s'(x)) = H(s(x))$. Hence, given, $H(s(x)|\mathcal{Y}) < H(s(x))$, we have $H(g(x)|\mathcal{Y}) = H(s'(x)|\mathcal{Y}) < H(s'(x)) = H(g(x))$, and the system (3.1) is completely observable. $\square$

We remark that we have avoided the analytical tedium of addressing all points in the co-domain of g since we are only concerned with the values taken by $g(x_0)$ (respectively $g(x_n)$ for reconstructibility) in the entropy computations.

Note that when the HMM dynamics are written in state-space form, an equivalence class of measurable functions of state x_k is comprised of functions of the form $\xi^T x_k$, where ξ is any vector of appropriate dimension. This parallels earlier discussion of complete observability for the linear system (2.1), which likewise requires observability of every $\xi^T x_k$.

Proof of Theorem 6. Because X is finite, the class of $\{G_i\}$ is finite, and so $\mathbb{S}$ is likewise finite; therefore, Lemma 1 states that verification of observability requires only a finite number of computations. $\square$

3.1.1 Rank Condition

Due to the explicit dependence of the state and output transition matrices on the control input, we specify the uncontrolled system as a system of the form (3.1) with a constant, nominal control input $r^{(0)}$, and, for notational clarity, write it as follows.

$$\begin{aligned} \Pi_{k+1} &= A\Pi_k, \\ \Psi_k &= C\Pi_k, \end{aligned} \tag{3.3}$$

We introduce the following suggestive notation.

$$\mathcal{O}_{n-1} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{3.4}$$

where A and C are defined as in (3.3)

The following result provides a method for verifying observability of the (uncontrolled) HMM.

Theorem 7. *If the matrix $\mathcal{O}_{n-1}$ defined in* (3.4) *is full rank, then the HMM* (3.3) *is completely stochastic-observable.*

Proof. Make the contradiction assumption: the HMM (3.3) is not completely stochastic-observable. Hence, by Lemma 1, there exists some set $G \subset X$ such that the indicator function $1_G(\cdot)$ is unobservable, i.e.

$$H(1_G(x_0)) = H(1_G(x_0)|y_0 \dots y_N) > 0$$

for some finite N. Note that

$$H(1_G(x_0)) \geq H(1_G(x_0)|y_k) \geq H(1_G(x_0)|y_0 \dots y_N)$$

for $k \in \{0, \dots N\}$, thus, $H(1_G(x_0)) = H(1_G(x_0)|y_k)$, or equivalently, y_k is independent of $1_G(x_0)$ for any $k \in \{0, \dots N\}$. As a result, $P(y_k|x_0 \in G) = P(y_k)$ for any value of y_k. Denote the corresponding distributions $\Psi_{k|G}$ such that $\Psi_{k|G}[i] = P(y_k = i|x_0 \in G)$ and Ψ_k is as in (3.3). We have that

$$\Psi_{k|G} = \Psi_k. \tag{3.5}$$

Due to the Markov property, $P(x_k = i|x_0 = j)$ is given by $A^k[i, j]$. Additionally,

$$\begin{aligned} P(x_k = j, x_0 \in G) &= P(x_0 \in G | x_k = j) P(x_k = j), \\ &= \sum_{i \in G} P(x_0 = i | x_k = j) P(x_k = j), \\ &= \sum_{i \in G} P(x_0 = i, x_k = j), \\ &= \sum_{i \in G} P(x_k = j | x_0 = i) P(x_0 = i), \\ &= A^k \Pi_0^\star(G). \end{aligned}$$

where $\Pi_0^\star(G)[i] = 1_G(i) P(x_0 = i)$. We have that $P(x_0 \in G) < 1$, because $H(1_G(x_0)) > 0$; hence, $\Pi_0^\star(G) \neq \Sigma_{0|-1}$. Note that the conditional probabilities of y_k obey $P(y_k | x_k, x_0 \in G') = P(y_k | x_k)$ for any set $G' \subseteq X$, which follows from the definition of the HMM. As a result, the distributions Ψ_k and $\Psi_{k|G}$ are

$$\begin{aligned} \Psi_k &= CA^k \Sigma_{0|-1} \\ \Psi_{k|G} &= CA^k \Pi_0^\star(G). \end{aligned} \tag{3.6}$$

Substitution into (3.5) yields

$$\begin{aligned} CA^k \Sigma_{0|-1} &= CA^k \Pi_0^\star(G), \\ \Rightarrow \; CA^k(\Sigma_{0|-1} - \Pi_0^\star(G)) &= 0 \end{aligned}$$

for any $k \in \{0, \dots n-1\}$. Since $\mathcal{O}_N$ attains its maximum rank for $N \geq n-1$ due to the Cayley Hamilton Theorem, $\mathcal{O}_{n-1}(\Sigma_{0|-1} - \Pi_0^\star(G)) = 0 \Leftrightarrow \mathcal{O}_N(\Sigma_{0|-1} - \Pi_0^\star(G)) = 0$. Because $\Sigma_{0|-1} \neq \Pi_0^\star(G)$, this is a contradiction. □

Note that, while complete stochastic observability of a system will, in general, depend on the initial distribution $x_0 \sim \Pi_0$, Theorem 7 provides a sufficient condition which is independent of Π_0. A HMM may be observable without satisfying the rank condition for particular choices of $\Sigma_{0|-1}$. For example, any HMM with initial distribution $\Sigma_{0|-1} = 1_\alpha(x_0)$, for some $\alpha \in X$, is completely stochastic observable. Such an initial distribution reflects the modeler's belief that the initial state is perfectly known (in an almost-sure sense) without the need for taking measurements. The admission of almost surely certain events into the class of observable quantities is a consequence of our definition of stochastic observability.

3.2 An Example

We study the network congestion control problem over Transmission Control Protocol (TCP) networks [ÅM08]. A source node segments data into packets, which it transmits through a sequence of routers to a destination node. In order to guarantee delivery of all packets, the destination node returns an acknowledgment packet (ACK) for each successfully delivered source packet. This is illustrated in Figure 3.1. Packet loss is signified by time-outs in awaiting an ACK arrival, or when an ACK is received out of order at the source.

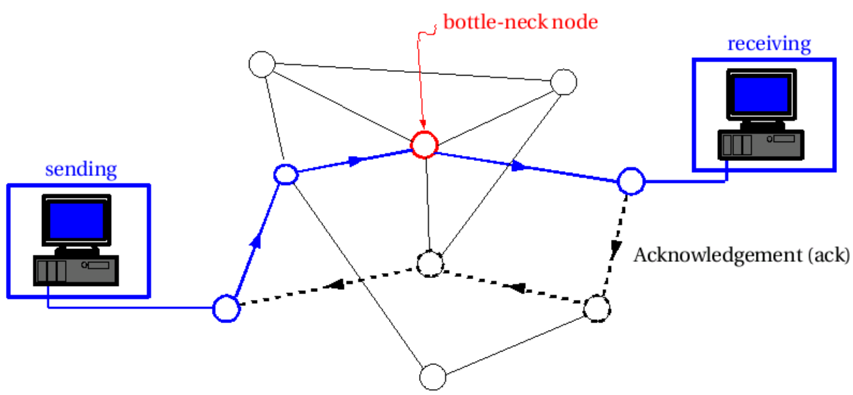


Figure 3.1: A schematic of a typical computer network. Routers are represented by circles.

Various algorithms for controlling the source transmission rate in order to prevent congestion at the intermediate routers have been studied; see for example [LLS07, BP95a, Kel03, JWL$^+$05, LBS06]. These algorithms are typically dependent on the round trip time (RTT), which is the total amount of time between transmission of a packet and receipt of its corresponding ACK at the source, and the window size, which upper-bounds the total number of unacknowledged packets in the network. For example, Additive Increase Multiplicative Decrease (AIMD), a current standard for the internet (see [ÅM08]), reduces the window size by a multiplicative factor, typically a half, whenever packet loss is detected, and increases window size additively, typically by a single packet upon receipt of ACKs for all packets in a given window. An enhancement of AIMD, TCP-Illinois, additionally measures the RTT, using the information to adjust the additive factor [LBS06]. Other variations often include additional information about the router in the packet header files [KS03].

The literature on congestion control also studies controls at the intervening routers. Router control laws intentionally drop packets under certain conditions. The Drop-tail Algorithm [Com06], for example, drops all incoming packets once the router buffer exceeds a limit. The Random Early Detection (RED) Algorithm [FJ93] will preemptively drop an incoming packet with a probability proportional to the fraction of the buffer occupied.

3.2.1 Model

We pose a HMM representation for the interaction between a single TCP source, which controls its transmission rate r_k, and the intermediate routers. We make several simplifying assumptions for clarity. A schematic of the model is given in Figure 3.2. As is commonly done in the analysis of TCP networks [LPW+03],

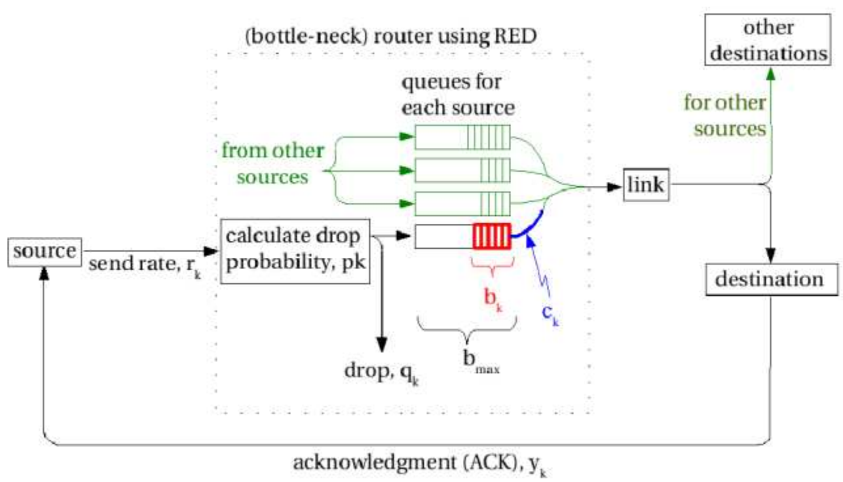


Figure 3.2: A schematic of our TCP/RED model.

we aggregate the dynamics of all links into the behavior of the single bottleneck router constraining throughput. We assume that each source connected to the router receives a dedicated segment of the total buffer, which is pre-allocated when the source establishes communications via the link. The state of the model describes two aspects of the router, the number of packets from the modeled source queued in the buffer (queue length, b_k), and the router service rate for the modeled source (capacity, c_k). The dynamics of other sources are accounted for as variations of the capacity c_k. For example, when traffic from other sources is high, c_k decreases. In the most general case, sources other than the modeled source may have arbitrary behavior. It is then reasonable to model c_k as a Markov process, $c_k \sim \text{Markov}(\Pi_c, P_c)$ [Nor97], where Π_c is the probability distribution of c_0 and P_c is a matrix whose $(i,j)^{\text{th}}$ element contains the transition probability $P(c_{k+1} = i | c_k = j)$. This model (fixed buffer segment and variable capacity) is used to capture the dynamics of a shared buffer and varying arrivals. Because each component, b_k and c_k, has finite cardinality, we can lexicographically assign pairs of (b_k, c_k) values to single values of a state

$$x_k = (b_k, c_k).$$

We assume the router uses a simplified form of RED where b_k is the instantaneous, rather than averaged, queue length, and the drop probability is given by

$$p_k = \frac{b_k}{b_{max}},$$

with b_{max} equal to the maximum queue length.

Packets that pass the router generate ACKs which, for simplicity, we assume return within one time step. The measurement y_k is a counting process of the number of ACKs, i.e. the number of packets which leave the router at time k. With these conditions, the full system with state x_k, send rate r_k, and measurement y_k can now be described as a Hidden Markov Model (HMM) [And99, SAD98, Rab89].

In mapping the network model to the problem of estimating the router state from the source, we view the source send rate r_k as the known control input and the count of ACK arrivals y_k as the measured output of the HMM system (3.3). We go on to show how the state estimate is generated using the input-output data and a priori information.

Since our focus is on exposing fundamental observability ideas, we use unrealistically small values for the maximum queue length and capacity to elucidate our examples and mitigate the computational burden. Consider a source sending packets through a bottleneck router with a capacity $c_k \in \{0, 1, 2\}$ and queue length $b_k \in \{0, 1, 2, 3\}$.

Let the transition matrix for the capacity distribution be

$$P_c = \begin{bmatrix} .2 & .25 & .1 \\ .6 & .5 & .7 \\ .2 & .25 & .2 \end{bmatrix}.$$

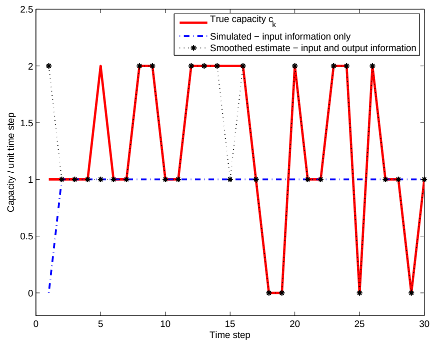
Normally, the capacity transition probabilities would be chosen to capture a certain dynamic in the network, for example the ritual 9 am mass e-mail sorting. For the purposes of this example, detailed consideration in choosing these values is unnecessary. With this transition matrix and from the preceding calculations, we can generate the full state and output transition matrices, $A(r_k)$ and $C(r_k)$ for each r_k.

Compare in Figure 3.3 the estimate generated by the three-step, fixed-lag smoother, which uses the ACK sequence $\{y_k\}$ and the input sequence $\{r_k\}$ due to an AIMD algorithm, to the simulated state estimate which uses only the inputs. Returning to the working definition of observability, one goal is to distinguish the case where, for example, the output data improves the queue length estimate, but not the capacity estimate, from the case where both capacity and queue length estimates are improved, as is suggested by Figures 3.3(a) and 3.3(b). Quantifying this improvement in all aspects of state estimation is a primary goal of defining observability.

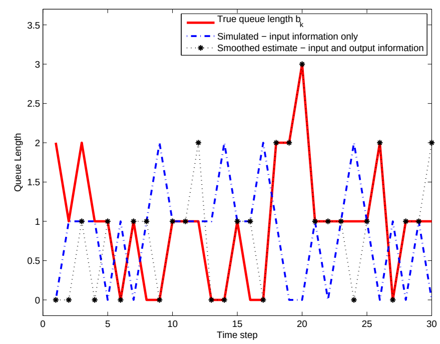
3.2.2 Estimator Improvement due to Additional Information

We modify our HMM system so that the output y_k encapsulates some information about the bottleneck queue length in addition to the count of ACKs. This frames the limitations of purely loss-based congestion control schemes in the developed observability language and encourages study of other architectures.

Practically speaking, the additional information could be appended to the ACK packet header file [KS03], though inclusion of queue length data in the packet



(a)



(b)

Figure 3.3: The inputs r_k are determined by AIMD. Plot (a) compares the actual capacity c_k (solid) with the simulated state estimate (dot-dash), which is generated by propagating the state dynamics (3.3) with the sequence of inputs $\{r_k\}$, and the estimate generated by the three-step smoother (black disks), which uses both input and output data $\{r_k, y_k\}$. The simulated state predicts a constant capacity while the smoothed estimate tracks the actual state with qualitatively obvious precision. Plot (b) compares the actual queue length b_k (solid) with the simulated state estimate (dot-dash), which uses only $\{r_k\}$, and the three-step smoothed state estimate (black disks), which uses $\{r_k, y_k\}$. The variations in the simulated state show that the oscillatory nature of the AIMD control policy is reflected in the state equation as a highly variable queue length. The smoother is able to track the actual queue length with noticeable precision in spite of this.

header file would in general require significant changes in the network to ensure fairness with existing TCP-controlled sources. We introduce randomness to the queue length measurement to account for the possibility of quantization and interference from other, intermediate buffers.

As can be seen from Figure 3.4, the additional queue measurements improve the queue estimate in spite of the quantization. We compare the mutual information between the two estimators:

$$I(b_0; \{y_k\}) = .13751,$$

using only the acknowledgments,

$$I(b_0; \{y_k, \beta_k\}) = .29066,$$

using the acknowledgments in addition to the queue measurements β_k. The capacity estimate already had near perfect performance, so qualitative comparison is

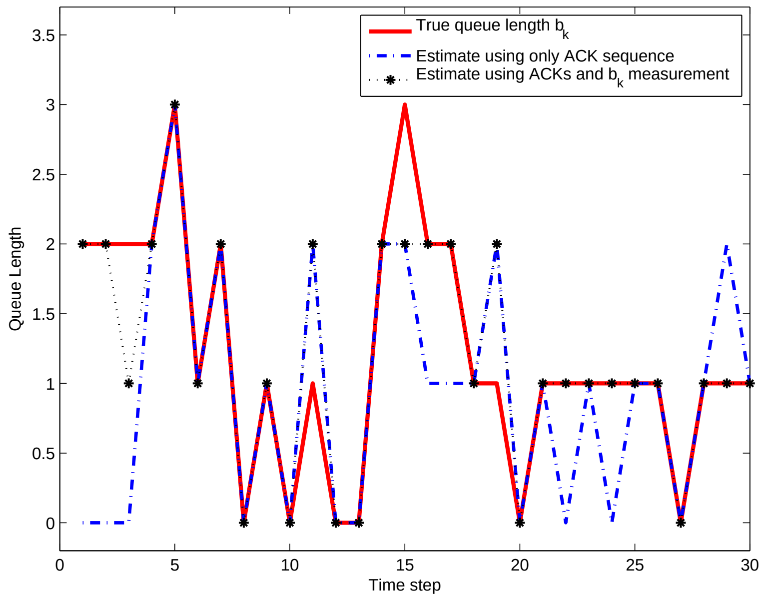


Figure 3.4: The true queue length (solid) is compared to the estimate generated by the HMM filter using only the ACK sequence and send rate (dot-dash) and the estimate generated by the filter using the ACK sequence, the input, and a b_k measurement (black disks). The ACK sequence alone often does not provide enough information to distinguish between an empty and a near empty queue, as seen in the time interval $20 \leq k \leq 25$.

uninformative, and we omit the graphs. For comparison, the mutual informations for the capacity estimates are:

$$I(c_0; \{y_k\}) = .74243,$$

using only the acknowledgments,

$$I(c_0; \{y_k, \beta_k\}) = .76165,$$

using the acknowledgments in addition to the queue measurements β_k.

The apparent connection between qualitative estimator performance and higher mutual information is the premise behind Mohler and Hwang's measure of stochastic observability [MH88], as well as several studies on linear stochastic observability [Ugr03], and is further reason to examine entropy concepts in analyzing stochastic observability.

3.3 Chapter Summary

The HMM was used to study, through example, the characteristics of our definitions of nonlinear stochastic observability and reconstructibility. The similarity between the state evolution of linear systems and the state-probability-law evolution of the HMM suggested that a rank condition on the probability transition matrices should provide a condition for complete stochastic observability of the HMM; we demonstrated this. The effects of observability and reconstructibility on estimator performance were explored via simulations of a HMM representing the network-congestion-control system. Numerical results invite further research on relating the degree of observability as described by Mohler and Hwang [MH88] to estimator performance in a wider class of models.

This chapter is in part a reprint of the materials as it appears in the following:

Andrew R. Liu and Robert R. Bitmead - *Stochastic Observability in Network State Estimation and Control*, Automatica, Vol 47, pp 65-78, 2011;

Andrew R. Liu and Robert R. Bitmead - *Observability and Reconstructibility of Hidden Markov Models: Implications for Control and Network Congestion Control*, 49th IEEE Conference on Decision and Control, December, 2010.

Chapter 4

Implications of Reconstructibility for Optimal Control

Stochastic-unobservable systems lack architectural features necessary for control design. We demonstrate that observability is necessary for feedback control to provide any advantage over open loop control.

Our observability definition is binary (yes/no) in nature. On the other hand, information theory provides the possibility of quantifying a degree of observability, and, thus, the interaction of the control input. We address this avenue of research through further simulations of the HMM for network traffic. In particular, the probing nature of current additive increase, multiplicative decrease (AIMD) protocols for throttling network traffic demonstrate the dual role of control in both achieving a control objective and ensuring that the controller has sufficient information to maintain performance. This reiterates the theme of Mohler and Hwang [MH88].

4.1 Observability and Feedback Control

Stochastic reconstructibility has a direct effect on the problem of optimal feedback control of the HMM system. Consider the following system:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k), \\ y_k &= h(x_k, u_k, v_k). \end{aligned} \tag{4.1}$$

As usual, denote,

$$\{y_t\}_k^N \triangleq \{y_k, y_{k+1}, \ldots, y_N\} \text{ and } \{u_t\}_k^N \triangleq \{u_k, u_{k+1}, \ldots, u_N\}$$

for specified values k and N. Also, denote the probability law for x_0 as $x_0 \sim \Pi_0$. Define the reward function,

$$\begin{aligned} \ell(i, r_k) = {} & \text{reward for being in state } i \text{ at time } k \text{ and} \\ & \text{taking control action } r_k \text{ at time } k. \end{aligned}$$

Denote the state space $x_k \in X$ and output space $y_k \in Y$. The input $r_k \in U$ at time k is a measurable function on $\sigma\{y_0, \ldots, y_{k-1}\}$ for $k \geq 1$. The initial value $r_0 \in U$ is a function of Π_0 (thus, trivially measurable on $\sigma\{x_0\}$). In the following, we will include equivalent expressions for the case where X has finite cardinality, as solutions to the optimal control problem are tractable in this case. Define the value-to-go function as

$$V(\Pi_0, \{y_t\}_0^{k-1}, \{r_t\}_0^{N-1}) = \mathrm{E}_{x_0, \{y_t\}_k^{N-1}} \left[\sum_{t=k}^{N} \ell(x_t, r_t) \middle| \{y_t\}_0^{k-1}, \{r_t\}_0^{N-1} \right],$$

where the subscript on the expectation signifies that the expected value is taken over the probability space generated by the random variables in the subscript, $\{x_0, y_k, \ldots, y_{N-1}\}$. At time $k = 0$, the value function is

$$V(\Pi_0, \{r_t\}_0^{N-1}) = E_{x_0, \{y_t\}_0^{N-1}} \left[\sum_{t=0}^{N} \ell(x_t, r_t) \middle| \{r_t\}_0^{N-1} \right]. \tag{4.2}$$

The optimal control sequence $\mathcal{U}_{N-1} \triangleq \{r_t^*\}_0^{N-1}$, which maximizes (4.2) over all $r_t \in U$ for $t \in \{0, \ldots, N-1\}$, can be generated by the Dynamic Programming algorithm [Ber87]. Following Bertsekas, the optimal value-to-go is given by the Bellman equation:

$$\begin{aligned} V^\dagger(\Pi_0, \{y_t\}_0^{k-1}, \{r_t\}_0^{k-1}) &\triangleq \max_{\{r_t\}_k^{N-1} \in \{U \times U \times \ldots U\}} V(\Pi_0, \{y_t\}_0^{k-1}, \{r_t\}_0^{N-1}), \\ &= \max_{r_k \in U} E_{x_0, y_k} \Big[\ell(x_k, r_k) \\ &\qquad + V^*(\Pi_0, \{y_t\}_0^k, \{r_t\}_0^k) \Big| \{y_t\}_0^{k-1}, \{r_t\}_0^{k-1} \Big]. \end{aligned} \tag{4.3}$$

In the following, we drop the subscript on the expectation when the meaning is clear. As shown in [Ber87], this may be written in terms of the conditional state probability law, which we denote as $\Pi_{k|k-1}(\cdot) = P(x_k \in \cdot | \{r_t, y_t\}_0^{k-1})$ given by the HMM filter (3.2):

$$\begin{aligned} V^\dagger(\Pi_0, \{y_t\}_0^{k-1}, \{r_t\}_0^{k-1}) &= V^*(\Pi_{k|k-1}), \\ &= \max_{r_k \in U_k} \big(E[\ell(x_k, r_k) | \{y_t, r_t\}_0^{k-1}] \\ &\qquad + E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k)) | \{y_t, r_t\}_0^{k-1}] \big) \end{aligned} \tag{4.4}$$

The superscript * on a conditional state probability as in the expected value of (4.4) signifies that the probability is a function of its arguments y_k and r_k (i.e. for fixed r_k and $\omega \in \Omega$, $\Pi^\star_{k+1|k}(y_k(\omega), r_k) \equiv \Pi_{k+1|k}(y_k(\omega), r_k)$). The recursion is initialized as

$$V^*(\Pi_{N|N-1}) = \max_{r_N \in U_N} E[\ell(x_N, r_N) | \{r_t, y_t\}_0^{N-1}], \tag{4.5}$$

and the optimal control value is denoted

$$r_k^* \triangleq \begin{cases} \arg\max_{r_k \in U_k} \big(E[\ell(x_k, r_k) | \{y_t, r_t\}_0^{k-1}] + E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k)) | \{y_t, r_t\}_0^{k-1}] \big), \\ \qquad \text{when } k < N, \\ \arg\max_{r_N \in U_N} \big(E[\ell(x_N, r_N) | \{r_t, y_t\}_0^{N-1}] \big), \\ \qquad \text{when } k = N. \end{cases} \tag{4.6}$$

When X has finite cardinality, we may express (4.4-4.6) as

$$
\begin{aligned}
V^*(\Pi_{k|k-1}) &= \max_{r_k \in U_k} \Bigg(\sum_{i\in X} \ell(i, r_k)\Pi_{k|k-1}(i) \\
&\qquad + \sum_{y_k \in Y} V^*(\Pi^\star_{k+1|k}(y_k, r_k))P(y_k|\Pi_{k|k-1}, r_k) \Bigg), \\
V^*(\Pi_{N|N-1}) &= \sum_{i\in X} \ell(i, r_N)\Pi_{N|N-1}(i), \\
r_k^* &= \arg\max_{r_k\in U} \Bigg(\sum_{i\in X} \ell(i, r_k)\Pi_{k|k-1}(i) \\
&\qquad + \sum_{y_k \in Y} V(\Pi^\star_{k+1|k}(y_k, r_k))P(y_k|\Pi_{k|k-1}, r_k) \Bigg), \quad k < N,
\end{aligned}
$$

and as before, $\Pi_{k|k-1}$ is generated from the HMM filter.

We compare the system using open-loop control to the system using closed-loop control; the difference between the two is the form of the state estimate distribution $\Pi_{k|k-1}$ in (4.4)-(4.6). With closed-loop control the estimate $\Pi_{k|k-1}$ is constructed using both the input sequence $\{r_t\}_0^{k-1}$ and output sequence $\{y_t\}_0^{k-1}$. In contrast, by open-loop we mean that the state probability is derived from using only the known sequence of inputs $\{r_t\}_0^{k-1}$ and Π_0. Denote this open-loop, simulated state estimate distribution by $\mathcal{P}_{k|k-1}$. Define the optimal open-loop control solution constructively through the following lemma.

Lemma 2. *The optimal open-loop control is derived from the following recursion.*

$$
V^o(\mathcal{P}_{k|k-1}) = \max_{r_k\in U} \left(E[\ell(x_k, r_k)|\{r_t\}_0^{k-1}] + V^o(\mathcal{P}^\star_{k+1|k}(r_k)) \right), \tag{4.7}
$$

$$
V^o(\mathcal{P}_{N|N-1}) = \max_{r_N\in U} E[\ell(x_N, r_N)|\{r_t\}_0^{N-1}], \tag{4.8}
$$

$$
r_k^o = \begin{cases} \arg\max_{r_k\in U} \left(E[\ell(x_k, r_k)|\{r_t\}_0^{k-1}] + V^o(\mathcal{P}^\star_{k+1|k}(r_k)) \right), & \\ \qquad\qquad\qquad\qquad \textit{when } k < N, & \\ \arg\max_{r_N\in U} E[\ell(x_N, r_N)|\{r_t\}_0^{N-1}], & \\ \qquad\qquad\qquad\qquad \textit{when } k = N. & \end{cases} \tag{4.9}
$$

$$
\mathcal{U}_k^o \triangleq \{r_t^o\}_0^k,
$$

where $\mathcal{P}_{k|k-1}$ is the conditional probability law for x_k given the filtration generated by U^o_{k-1}.

When X has finite cardinality, the above expressions have simpler forms.

$$\begin{aligned} V^o(\mathcal{P}_{k|k-1}) &= \max_{r_k \in U} \left(\sum_{i \in X} \ell(i, r_k) \mathcal{P}_{k|k-1}(i) + V^o(A(r_k)\mathcal{P}_{k|k-1}) \right), \\ V^o(\mathcal{P}_{N|N-1}) &= \sum_{i \in X} \ell(i, r_N) \mathcal{P}_{N|N-1}(i), \\ r^o_k &= \arg\max_{r_k \in U} \left(\sum_{i \in X} \ell(i, r_k) \mathcal{P}_{k|k-1}(i) + V^o(A(r_k)\mathcal{P}_{k|k-1}) \right), \\ \mathcal{U}^o_k &\triangleq \{r^o_t\}^k_0 \end{aligned} \tag{4.10}$$

where $\mathcal{P}_{k|k-1}$ satisfies $\mathcal{P}_{k|k-1}(i) = P(x_k = i | \mathcal{U}^o_{k-1})$ and is generated from

$$\mathcal{P}_{k|k-1} = A(r^o_{k-1}) \dots A(r^o_1) A(r^o_0) \mathcal{P}_0, \tag{4.11}$$

and $\mathcal{P}_0(i) = \Pi_0(i) = P(x_0 = i)$.

Theorem 8. *Suppose, for $k = 0, \dots, N$, the state x_k of (3.1) is not reconstructible from the preceding outputs $\{y_t\}^k_0$ no matter which sequence of control inputs $\{r_t\}^k_0$ is chosen. Then for any choice of value function $\ell(\cdot, \cdot)$, $V^o(\Pi_0) = V^*(\Pi_0)$, i.e. the cost using the optimal open-loop control is the same as the cost using the optimal closed-loop control, which incorporates the measurement sequence. Furthermore, the optimal control sequences for the open-loop and closed-loop programs are the same.*

Proof. Since x_k is not reconstructible from the preceding outputs $\{y_t\}^k_0$ no matter the sequence of control inputs $\{r_t\}^k_0$, we have $\mathcal{P}_{k|k-1} = \Pi_{k|k-1}$ given the optimal feedback control sequence $\{r^*_t\}^{k-1}_0$ for $k = 0 \dots N$. Then, the Bellman equation (4.4) with outputs $\{y_t\}^{k-1}_0$ equals its open-loop counterpart (4.7). Likewise, the initializations for the Bellman equation in closed loop (4.5) and open loop (4.8) are the same. The result is immediate. □

Additionally, consider the class of control-independent reward functions, $\ell(i, r_k) = \ell(i)$. In this special case, while the state x_k of (3.1) may be reconstructible from the outputs $\{y_t\}^k_0$, feedback control may still have no benefit over open-loop control for some choices of $\ell(\cdot)$ if the system (3.1) is not completely reconstructible. This is captured in the following theorem.

Theorem 9. *Suppose the conditional entropy of the reward function given the inputs and outputs equals the conditional entropy given only the inputs, i.e.*

$$H(\ell(x_k)|\{r_t\}_0^k, \{y_t\}_0^k) = H(\ell(x_k)|\{r_t\}_0^k)$$

no matter which control sequence $\{r_t\}_0^k$ is chosen.[1] *Then $V^o(\Pi_0) = V^*(\Pi_0)$, i.e. the cost using the optimal open-loop control is the same as the cost using the optimal closed-loop control, which incorporates the measurement sequence. Furthermore, the optimal control sequences for the open-loop and closed-loop programs are the same.*

The conditioning on $\{r_t\}_0^k$ requires some explanation. When the sequence $\{r_t\}_0^k$ is fixed, as in the open-loop case, conditioning on $\{r_t\}_0^k$ is equivalent to conditioning on a trivial σ-field almost surely. Compare this to the optimal closed-loop case, $\{r_t\}_0^k$ is a measurable function on the filtration generated by the outputs up to time k[2].

Proof. Recall the following notation,

$$\begin{aligned}
\mathcal{U}_k &\triangleq \{r_t^*\}_0^k, \text{ the optimal closed-loop control sequence,}\\
\mathcal{U}_k^o &\triangleq \{r_t^o\}_0^k, \text{ the optimal open-loop control sequence,}\\
\mathcal{Y}_k &\triangleq \{y_t\}_0^k, \text{ the sequence of outputs.}
\end{aligned}$$

We show $V^o(\Pi_0) = V^*(\Pi_0)$ by induction.

Make the inductive assumption:

For $k < N-1$, if the sequence of inputs satisfies $\mathcal{U}_k = \mathcal{U}_k^o$ almost surely, then $V^*(\Pi_{k+1|k}) = V^o(\mathcal{P}_{k+1|k})$, and $r_{k+1}^* = r_{k+1}^o$.

We are required to prove that:

$$\mathcal{U}_{k-1} = \mathcal{U}_{k-1}^o \Rightarrow V^*(\Pi_{k|k-1}) = V^o(\mathcal{P}_{k|k-1}) \text{ and } r_k^* = r_k^o. \tag{4.12}$$

[1] This is a very strong assumption, as it applies not only to fixed (deterministic sequences) but to random sequences which may be defined on the information filtration.

[2] By induction, the past inputs form a sub-filtration of the filtration generated by the outputs.

From (4.7), showing that $V^*(\Pi_{k|k-1}) = V^o(\mathcal{P}_{k|k-1})$ is equivalent to showing

$$V^*(\Pi_{k|k-1}) = \begin{cases} \max_{r_0 \in U} \left(E[\ell(x_0)] + V^o(\mathcal{P}^\star_{1|0}(r_0)) \right), & \text{when } k = 0, \\ \max_{r_k \in U} \left(E[\ell(x_k) | \mathcal{U}^o_{k-1}] + V^o(\mathcal{P}^\star_{k+1|k}(r_k)) \right), & \text{otherwise.} \end{cases} \tag{4.13}$$

Thus, begin by assuming $\mathcal{U}_{k-1} = \mathcal{U}^o_{k-1}$. We have from (4.4),

$$\begin{aligned} V^*(\Pi_{k|k-1}) = \max_{r_k \in U} \Bigg(& E[\ell(x_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}] \\ & + E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}] \Bigg), \\ = \max_{r_k \in U} & E[\ell(x_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}] \\ & + \max_{r_k \in U} E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k)) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}]. \end{aligned} \tag{4.14}$$

Note that, because the reward function ℓ does not depend on the control,

$$\max_{r_k \in U} E[\ell(x_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}] = E[\ell(x_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}]; \tag{4.15}$$

thus,

$$r^*_k = \arg\max_{r_k \in U} E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}]. \tag{4.16}$$

Since $\mathcal{U}_{k-1}$ is a function of a fixed initial distribution and $\mathcal{Y}_{k-1}$, we may view $E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k) | \mathcal{Y}_{k-1}, \mathcal{U}_{k-1}]$ as a function of $\mathcal{Y}_{k-1}$. For a fixed sequence $\mathcal{Y}_N \equiv \bar{\mathcal{Y}}_N$, we have a corresponding input sequence, $\mathcal{U}_N \equiv \bar{\mathcal{U}}_N$. Thus,

$$\begin{aligned} & E_{y_k}[V^*(\Pi^\star_{k+1|k}(y_k, r_k) | \bar{y}_{k-1}, \bar{u}_{k-1}] \\ & \quad = E\left[\sum_{t=k+1}^{N} E[\ell(x_t) | \bar{\mathcal{Y}}_{t-1}, \bar{\mathcal{U}}_{t-1}] \middle| \bar{\mathcal{Y}}_{k-1}, \bar{\mathcal{U}}_{k-1} \right], \\ & \quad = E\left[\sum_{t=k+1}^{N} E[\ell(x_t) | \bar{\mathcal{U}}_{t-1}] \middle| \bar{\mathcal{U}}_{k-1} \right], \end{aligned}$$

because $\ell(x_k)$ is not reconstructible for any input sequence. Substituting into (4.16) and invoking optimality, we have $r^*_k = r^o_k$.

We have shown that $r^*_k = r^o_k$ for every $k \in \{0, \ldots, N\}$. This establishes

$$V^*(\Pi_{k|k-1}) = \begin{cases} \max_{r_0 \in U} \left(E[\ell(x_0)] + V^o(\mathcal{P}^\star_{1|0}(r_0)) \right), & \text{when } k = 0 \\ \max_{r_k \in U} \left(E[\ell(x_k) | \mathcal{U}_{k-1}, \mathcal{Y}_{k-1}] + V^o(\mathcal{P}^\star_{k+1|k}) \right), & \text{otherwise.} \end{cases} \tag{4.17}$$

To establish (4.13), we must show

$$\max_{r_k \in U} E[\ell(x_k)|\mathcal{U}_{k-1}, \mathcal{Y}_{k-1}] = \max_{r_k \in U} E[\ell(x_k)|\mathcal{U}_{k-1}] \text{ for } k \in \{1, \ldots, N\}. \tag{4.18}$$

This follows immediately from the independence of $\ell(x_k)$ and $\mathcal{Y}_{k-1}$ for any sequence of inputs $\mathcal{U}_{k-1}$. Substitution of (4.18) into (4.17) yields (4.13).

Now prove that the first step of the induction is true, namely that

$$\mathcal{U}_{N-1} = \mathcal{U}^o_{N-1} \Rightarrow V^o(\mathcal{P}_{N|N-1}) = V^*(\Pi_{N|N-1}) \text{ and } r^*_N = r^o_N.$$

This was shown above (4.18) for the general k^{th} time step case with $k \in \{1, \ldots N\}$.

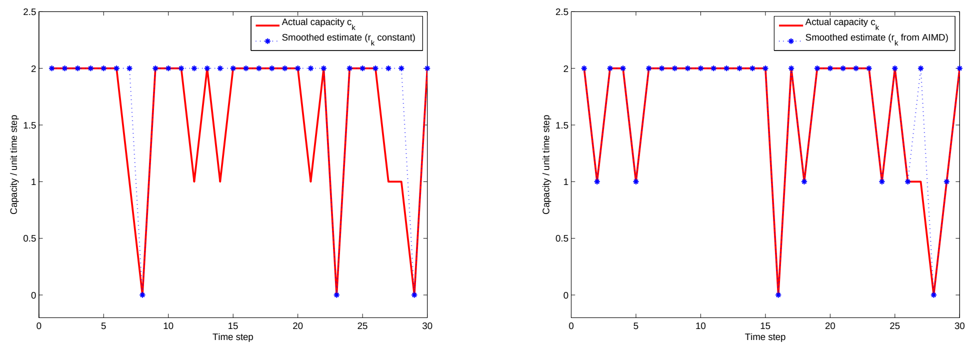
To complete the proof, note that $\mathcal{P}_{0|-1} = \Pi_{0|-1}$ so without loss of generality define $\mathcal{U}_{-1} = \mathcal{U}^o_{-1}$, although there is no physical significance in doing so. □

Theorem 8 addresses the situation where the state is completely unobservable, that is, every measurable function of the state is unobservable from the output sequence regardless of the sequence of inputs. Theorem 9 concerns the observability of the optimal-control reward function, which is a scalar measurable function of the state.

4.1.1 Examples Redux

Returning to the question of how the control input affects estimator quality, we again compare the source using the additive increase, multiplicative decrease algorithm[3] (AIMD) to the source with a constant send rate $r_k = 1$. By exhaustive (but, by Theorem 6, finite) computation we have verified the observability (of the initial state x_0) of both systems. As shown in Figures 4.1(a) and 4.1(b), the capacity estimator for the AIMD-controlled source performs noticeably better. We compare the mutual informations, i.e. the reduction of the entropy with inclusion of the output sequence. With AIMD, $I(c_0, \{y_k\}) = .96951$, and with $r_k = 1$, $I(c_0, \{y_k\}) = .52143$.

[3]See Section 3.2.



(a) (b)

Figure 4.1: Plot (a) compares, for constant send rate r_k, the three-step smoothed capacity estimate (disks) compared to the actual capacity (solid) when the send rate r_k is constant. Note that the smoother is inaccurate when the actual capacity is $c_k = 1$. Plot (b) compares, for send rate r_k governed by an AIMD policy, the three-step smoothed capacity estimate (disks) with the actual capacity (solid). In contrast to Figure 4.1(a), here, the smoother has little difficulty tracking the capacity state even when the actual capacity is $c_k = 1$.

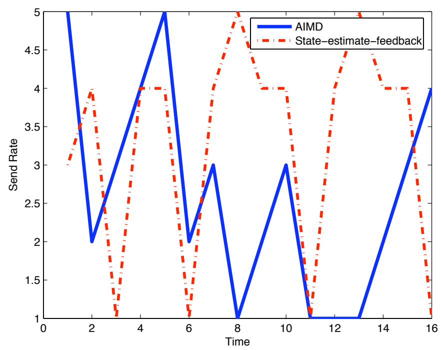
Comparison to state-estimate feedback

In the simulations that follow, we will use a sub-optimal, output-feedback controller, as our purpose is to investigate further connections between the control input and the HMM system observability, rather than to design optimal controllers. To generate the sub-optimal, output-feedback controller, we first solve the optimal, infinite-time horizon *state*-feedback problem, which maximizes,

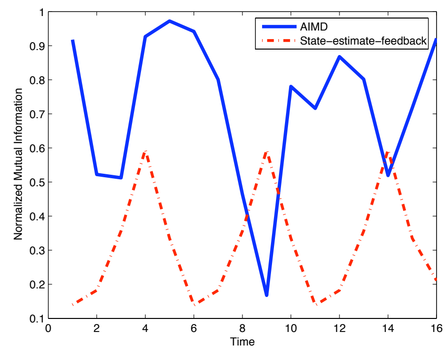
$$V(\Pi_0, \mu(\cdot)) = E\left[\sum_{t=0}^{\infty} \gamma^t \ell(x_t, \mu(x_t))\right]. \tag{4.19}$$

The solution to this problem is well known; see [Ber87]. Note that the policy μ takes as its argument the realization of the state evolution, $x_k \sim \Pi_k$. In the sub-optimal, output-feedback controller, we use a state-estimate from the HMM Filter [And99] in place of the true state realization. Because this control-design algorithm is easy to compute and to implement, it is suitable for our initial computational experiments.

In fact, the oscillatory nature of AIMD is often detrimental to its goal of increasing total data transmission rate. Given the HMM system, we may con-



(a)



(b)

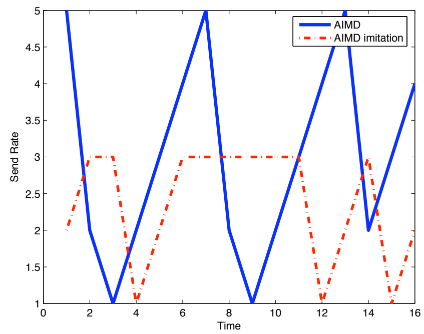
Figure 4.2: We compare AIMD to our sub-optimal state-estimate-feedback policy. Plot (a) shows the send rate of AIMD (solid) and the state-estimate-feedback policy (dot-dash). Plot (b) shows the mutual information (normalized by the state entropy, $H(x_k)$) of AIMD (solid) and the state-estimate-feedback policy (dot-dash) over the same time interval as in Plot (a).

struct state-estimate-feedback policies with higher throughputs and comparable data loss using the process described above with an appropriate reward function ℓ. A comparison of the throughput is given in Figure 4.2(a).

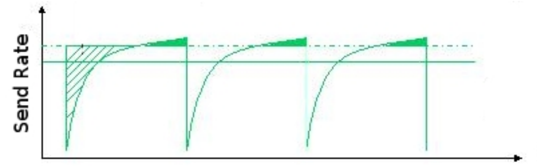
On the other hand, AIMD is comparably successful in maintaining the mutual information between the state and output. Figure 4.2(b) shows that the mutual information between the state and output for the state-estimate-feedback policy, is frequently lower than for AIMD, despite the throughput being higher.

Additionally, we attempt to mimic the behavior of AIMD by formulating an objective function with a quadratic penalty on packet loss and linear reward function for throughput. In simulations (Figure 4.3(a)), the resulting control signal has additive increase and multiplicative decrease qualities, but unlike the common form of AIMD, this controller bounds the send rate. The control-input graph resembles that of simulations of the recently developed TCP Illinois algorithm, designed for high-speed networks (Figure 4.3(b)).

The success of AIMD may partly be due to its role in maintaining congestion estimation accuracy (where the estimation process is implicitly performed by the AIMD algorithm). This behavior may be less beneficial when additional measurements of network congestion are available. Not surprisingly, newer algo-



(a)



(b)

Figure 4.3: We attempt to imitate the behavior of AIMD by defining an appropriate objective function and designing a state-estimate-feedback policy with this function. Plot (a) shows the send rate of AIMD (solid) compared to the send rate from our designed policy (dot-dash). We note a similarity to TCP Illinois; a typical send-rate graph of TCP Illinois is shown in Plot (b).

rithms which rely on additional measurements, for instance packet propagation delay (TCP Vegas [BP95b], TCP Illinois [LBS06]) or explicit signals from the network routers (ECN [Flo94]), often use variations of AIMD. Our simulations suggest that a reconstructibility analysis of the control signal may offer insights for systematic design of these modifications.

4.2 Chapter Summary

We demonstrate the necessity of stochastic reconstructibility for control-performance improvement using output feedback over the optimal open-loop control solution. We also studied the interaction between control and observability/reconstructibility from another direction, with further simulations of the HMM for the router system. Simulations compared the effectiveness of control algorithms in improving the source computer's information about the router state. In the process, we showed that the oscillatory nature of current congestion-control algorithms may be beneficial as they excite the system and result in more informative measurements of the router state at the source computer.

This chapter is in part a reprint of the materials as it appears in the following:

Andrew R. Liu and Robert R. Bitmead - *Stochastic Observability in Network State Estimation and Control*, Automatica, Vol 47, pp 65-78, 2011.

Chapter 5

Nonlinear Filter Stability

When the controller is a function of the state estimator, one may ask whether the state estimate converges to a unique value regardless of the initial distribution. That is, for a fixed system and controller, we would like the controller to converge to a desired controller in spite of poor choices for the initial conditions. Roughly speaking, this is the question of stability of the nonlinear filter, and it is directly related to our observation that stochastic observability and reconstructibility may depend on the initial distribution.

Here, our work intersects the investigations of van Handel [vH09a]. We devote some effort to exploring van Handel's research as he formulates a definition of stochastic observability from a – so far as we know – unique perspective. Whereas in all definitions previously addressed, the state and output vectors have been the fundamental objects of interest, van Handel instead defines observability in terms of the state and output *distributions*. His primary results are that this version of observability guarantees stability of the nonlinear filter. The connection to our work lies in recalling that the rank conditions for the Kalman filter and the hidden Markov model have the appearance of conditions for filter stability, and in fact that these conditions guarantee our version of stochastic observability for *any* initial condition. Here, we provide the general result, showing that a system is observable according to our definition for any initial condition if and only if it is observable according van Handel's definition.

5.1 Background and Definitions

As the following is quite technical, for clarity, we limit our investigation to systems of the following form.

$$\begin{aligned} x_{k+1} &= f_k(x_k, w_k), \\ y_k &= h_k(x_k, v_k). \end{aligned} \tag{5.1}$$

Assume the following:

- x_0 and w_k are independent for each $k \in \{0, 1, \dots\}$;
- x_0 and v_k are independent for each $k \in \{0, 1, \dots\}$;
- w_j and v_k are independent for each $k, j \in \{0, 1, \dots\}$.

We allow that the state space may be a subset of $\mathbb{R}^n$; specifically, we permit that the state space may be compact.

Definition 12 (VH Observability)**.** *Consider two probability laws for the initial state x_0:*

$$x_0 \sim \Pi_0 \text{ and } x_0 \sim \Pi_0'.$$

We say that (5.1) *is VH observable if* $\Pi_0 \neq \Pi_0'$ *implies different corresponding laws for* $\{y_k\}_0^T$ *for some* T*. Denoting by* $P(\{y_k\}_0^T \in \cdot)$ *the probability for the filtration generated by* $\{y_k\}_0^T$ *for initial measure* Π_0 *and* $P'(\{y_k\}_0^T \in \cdot)$ *similarly for* Π_0', (5.1) *is VH observable if*

$$\Pi_0 \neq \Pi_0' \quad \Rightarrow P(\{y_k\}_0^T \in \cdot) \neq P'(\{y_k\}_0^T \in \cdot). \tag{5.2}$$

We quote Van Handel's result for continuous-time systems on a compact state space.

Theorem 10 ([vH09a])**.** *Let* $\Pi_0 << \Pi_0'$ *and suppose that a continuous-time stochastic process* X_t *has right continuous paths with finite left limits. Then for all continuous, bounded functions* f*,*

$$\left| E^{\Pi_0}[f(X_T)|\sigma(\{y_t : t \in [0,T]\})] - E^{\Pi_0'}[f(X_T)|\sigma(\{y_t : t \in [0,T]\})] \right| \to 0, \; P^{\Pi_0} a.s.$$

where $E^{\Pi_0}[\cdot]$ *denotes the expectation taken with respect to the probability consistent with initial measure* Π_0*.*

See [vH09a] for the proof. As we are mainly interested in the result to motivate Van Handel's formulation, we do not engage in developing the discrete-time counterpart. Instead, we consider the connections between VH observability and our version of observability. To avoid ambiguity, in the following we refer to our definition of complete stochastic observability as "CS observability".

5.2 Main Result

Theorem 11. *The system (5.1) is CS observable for every initial measure* $x_0 \sim \Pi_0$ *if and only if it is VH observable.*

Proof. We first show that VH observability implies CS observability irrespective of initial measure Π_0.

Assume g is non-trivial, there is a set G such that $P(g(x_0) \in G) \in (0,1)$. Denote the inverse image of G as $B \triangleq g^{-1}(G)$.

Define a probability P' on $(\Omega, \mathcal{F})$ as

$$P'(\omega \in A) = \frac{P(\omega \in \{A \cap x_0^{-1}(B)\})}{P(\omega \in x_0^{-1}(B))}. \tag{5.3}$$

It is straightforward to verify that P' is a probability.

Step 1: We claim that $P'(y_{[0:T]} \in \cdot) = P(y_{[0:T]} \in \cdot | 1_B(x_0))$ almost surely.[1]

$$\begin{aligned} P'(y_{[0:T]} \in \cdot) &= \frac{P(\omega \in \{\cdot \cap x_0^{-1}(B)\})}{P(\omega \in x_0^{-1}(B))}, \\ &= \frac{E[1_{\{\omega: y_{[0:T]} \in \cdot\}} 1_{\{\omega: x_0 \in B\}}]}{P(x_0 \in B)}, \\ &= \frac{\int \{\omega : x_0 \in B\} 1_{\{\omega: y_{[0:T]} \in \cdot\}} dP}{P(x_0 \in B)}, \\ &= E[1_{\{\omega: y_{[0:T]} \in \cdot\}} | 1_B(x_0)], \\ &= P(y_{[0:T]} \in \cdot | 1_B(x_0)). \end{aligned}$$

Step 2: Assume (5.2) and for some Π_0, set $\Pi_0^1 = \Pi_0$ and $\Pi_0^2 = \Pi_0'$. For the contradiction argument, assume $g(x_0)$ is independent of $y_{[0:T]}$ under the law whose

[1] As usual, $1_B(x_0)$ is the indicator function which equals one when $x_0 \in B$ and zero otherwise.

restriction to x_0 is Π_0. Then

$$P(y_{[0:T]} \in \cdot | x_0 \in B) = P(y_{[0:T]} \in \cdot) = P'(y_{[0:T]} \in \cdot).$$

Since $\Pi_0 \neq \Pi_0'$, we have from (5.2) that $P'(y_{[0:T]} \in \cdot) \neq P(y_{[0:T]} \in \cdot)$, which is a contradiction.

We postpone the proof of the converse to address some technical issues. $\square$

For brevity, define the output map for (5.1) as

$$\Phi_k(x_0, \{w_t\}_0^{k-1}, \{v_t\}_0^{k-1}) \triangleq \begin{bmatrix} h_0(x_0, v_0) \\ h_1(f_0(x_0, w_0), v_1) \\ \vdots \end{bmatrix} = \{y_t\}_0^k. \tag{5.4}$$

The continuation of our proof relies on formalizing the idea that the noises are exogenous. Specifically, the noise $\{w_k, v_k\}$ are defined on a probability space $(\Omega^*, \mathcal{F}^*, P^*)$, while the initial state lies on a probability space $(\Omega^0, \mathcal{F}^0, P^0)$, where Ω^* and Ω^0 are disjoint. The full probability space in the domain of the stochastic process described by (5.1) is the product space $(\Omega^* \times \Omega^0, \mathcal{F}^* \times \mathcal{F}^0, P)$, where the product measure P is defined via the usual transition kernel (see, for instance, [Res99]). In particular, we will require that the product measure satisfy

$$P(\{\omega^* \in A^* \subset \Omega^*\}) = P^*(\{\omega^* \in A^* \subset \Omega^*\}). \tag{5.5}$$

Using this structure, we have the following intermediate result.

Lemma 3. *Let A be an event such that $P(x_0 \in A) \in (0,1)$. Then, for a Borel set B,*

$$P(\{y_t\}_0^k \in B | x_0 \in A) = P(A^*) = P^*(A^*),$$

where

$$A^* \triangleq \{\omega^* \in \Omega^* : \Phi_k(x_0, \{w_t(\omega^*)\}_0^{k-1}, \{w_t(\omega^*)\}_0^k) \in B \;\forall\, x_0 \in A\}, \tag{5.6}$$

and P^* does not change with the initial distribution $x_0 \sim \Pi_0$.

Proof of Lemma 3. By definition of conditional probability,

$$\begin{aligned} E[1_A(x_0)P(\{y_t\}_0^k \in B | 1_A(x_0)] &= E[1_A(x_0)1_B(\{y_t\}_0^k)], \\ &= E[\varphi(x_0, \{w_t\}_0^{k-1}, \{v_t\}_0^k)], \end{aligned}$$

where

$$\varphi(x_0, \{w_t\}_0^{k-1}, \{v_t\}_0^k) = \begin{cases} 1, & \text{if } x_0 \in A \text{ and } \Phi_k(x_0, \{w_t\}_0^{k-1}, \{v_t\}_0^k) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Continuing,

$$\begin{aligned} E[\varphi(x_0, \{w_t\}_0^{k-1}, \{v_t\}_0^k)] &= P(\{\omega^* \in \Omega^* : \Phi_k(x_0, \{w_t(\omega^*)\}_0^{k-1}, \\ &\qquad \{v_t(\omega^*)\}_0^k) \in B \ \forall \ x_0 \in A\}), \\ &= P(\omega^* \in A^*). \end{aligned} \tag{5.7}$$

Finally due to (5.5) and (5.7),

$$P(\omega^* \in A^*) = P^*(\omega^* \in A^*).$$

□

Continuation of the proof of Theorem 11. We show that if (5.1) is not VH observable, then it is not CS observable for some $x_0 \sim \Pi_0$. Let B be any open set in the range of $\{y_t\}_0^k$. Since (5.1) is not VH observable, there are two distinct laws for x_0, Π_0 and Π_0' such that $P(\{y_t\}_0^k \in B) = P'(\{y_t\}_0^k \in B)$, where as before, P is consistent with Π_0, and P' with Π_0'. Because Π_0 and Π_0' are distinct, there exists a set A satisfying

$$P(x_0 \in A) \in (0, 1) \text{ and } P(x_0 \in A) \neq P'(x_0 \in A).$$

Define A^c as the complement of set A on the range of x_0. We have

$$\begin{aligned}
P(\{y_t\}_0^k \in B) &= P(\{y_t\}_0^k \in B | 1_A(x_0) = 1)P(x_0 \in A) \\
&\quad + P(\{y_t\}_0^k | 1_A(x_0) = 0)P(x_0 \in A^c), \\
&= P(\{y_t\}_0^k \in B | 1_A(x_0) = 1)P(x_0 \in A) \\
&\quad + P(\{y_t\}_0^k \in B | 1_A(x_0) = 0)(1 - P(x_0 \in A)), \\
&= P(\{y_t\}_0^k \in B | 1_A(x_0) = 0) \\
&\quad + P(x_0 \in A)(P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) \\
&\quad - P(\{y_t\}_0^k \in B | 1_A(x_0) = 0).
\end{aligned} \tag{5.8}$$

Likewise,

$$\begin{aligned}
P'(\{y_t\}_0^k \in B) = P'(\{y_t\}_0^k \in B | 1_A(x_0) = 0) \\
+ P'(x_0 \in A)\big(P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) \\
- P'(\{y_t\}_0^k | 1_A(x_0) = 0)\big).
\end{aligned} \tag{5.9}$$

Due to Lemma 3,

$$P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) = P'(\{y_t\}_0^k \in B | 1_A(x_0) = 1). \tag{5.10}$$

Combining (5.8-5.10), we have

$$\begin{aligned}
&P(\{y_t\}_0^k \in B | 1_A(x_0) = 0) \\
&\quad + P(x_0 \in A)\left(P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) - P(\{y_t\}_0^k \in B | 1_A(x_0) = 0)\right), \\
&= P(\{y_t\}_0^k \in B | 1_A(x_0) = 0) \\
&\quad + P'(x_0 \in A)\left(P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) - P(\{y_t\}_0^k \in B | 1_A(x_0) = 0)\right).
\end{aligned}$$

Since $P(x_0 \in A) \neq P(x_0 \in A^c)$, we have that

$$P(\{y_t\}_0^k \in B | 1_A(x_0) = 1) = P(\{y_t\}_0^k \in B | 1_A(x_0) = 0);$$

thus,

$$\begin{aligned}
P(\{y_t\}_0^k \in B) &= P(\{y_t\}_0^k \in B | 1_A(x_0) = 1)P(x_0 \in A) \\
&\quad + P(\{y_t\}_0^k \in B | 1_A(x_0) = 0)P(x_0 \in A^c), \\
&= P(\{y_t\}_0^k \in B | 1_A(x_0) = 1)\left(P(x_0 \in A) + P(x_0 \in A^c)\right), \\
&= P(\{y_t\}_0^k \in B | 1_A(x_0) = 1).
\end{aligned}$$

We have verified that

$$P(\{y_t\}_0^k) = P(\{y_t\}_0^k \in B | 1_A(x_0)) \text{ almost surely.} \tag{5.11}$$

Since this holds for any open set B, (5.11) suffices to show independence of $1_A(x_0)$ and $\{y_t\}_0^k$. Thus, (5.1) is not CS observable. □

We remark that van Handel has also defined a concept of uniform observability for state-space models, which guarantees filter stability when the state space is not compact. Notably, well known results from Kalman filtering theory have been shown to follow from uniform observability [vH09b]. Future work may pursue the relations between this uniform observability and our analysis.

5.3 Chapter Summary

Van Handel's definition of stochastic observability implies filter stability when the state space is compact; a stronger condition of uniform observability implies filter stability in the general case. We show equivalence between van Handel's idea and our own definition of stochastic observability.

Chapter 6

Stochastic Controllability and Reachability

In deterministic systems the dual properties of observability and reconstructibility are reachability and controllability. As with observability and reconstructibility, the extensions to stochastic systems are not obvious, since process disturbances in general prevent the state of the system from being steered to an exact point. To approach the problem, we again begin with linear systems where the error covariance provides a convenient quantity for analysis. Compared to the analysis of estimators, here the error is measured against a desired point to which the control attempts to move the state. While there is freedom in formulating the definitions from probabilistic intuitions, our formulation again aims to recover the deterministic rank and matrix conditions.

While in linear deterministic systems, there is a duality between controllability and observability[1], the connections between control and estimation are less apparent for stochastic systems. The reason is that the conditional probability distribution is not appropriate for understanding the effect of the control on the state; the control may have no causal effect on the state, but, by acting as a measurement signal of the state, the control nevertheless affects the conditional probability. Signaling solutions of the Witsenhausen counterexample [Wit68, MHL11] are such an

[1]More precisely, the dualities are between reachability and observability and between controllability and reconstructibility.

instance where the control signal takes on the role of a measurement.

Another distinguishing feature of the control analysis is the necessity of defining admissible controls. In particular, feedback is intuitively necessary to remove disturbances in the stochastic state evolution. On the other hand, we may wish to consider control policies which do not use any feedback. These include open-loop control, vibrational control [KMP98], where an unstable system is stabilized by a periodic control input determined off-line, and parameter-noise control [ACW83], where an unstable system is stabilized via a stochastic parametric input sequence which is independent of the state. In between pure-feedback and pure-noise inputs are mixed policy algorithms which arise frequently as equilibrium solutions for game-theoretic problems. To account for each of these cases, we define an admissible control at each time k for a system with state x_k to be a function $u_k = g_k(\{x_t\}_0^k, r_k, \Pi_0)$, where r_k is independent of the present and past states $\{x_t\}_0^k$ and g_k is measurable on the product σ-algebra generated by $\{x_t\}_0^k$ and r_k. The initialization of the initial state probability law $x_0 \sim \Pi_0$ is user-specified and fixed, so it is trivially measurable.

6.1 Linear Systems

The linear system is

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{6.1}$$

with state $x_k \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$, and disturbance $w_k \in \mathbb{R}^n$. The initial state is normally distributed $x_0 \sim N(\bar{x}_0, \Sigma_0)$ and the disturbances w_k form a white noise, normally distributed sequence $w_k \sim N(0, Q_k)$. Additionally, w_k and x_0 are independent for every non-negative k. Denote by $\mathcal{F}_k$ the filtration generated by $\sigma(\{x_t\}_0^k)$, that is, the σ-algebra generated by the sequence of states from time 0 to k.

For convenience, we write

$$x_n = A^n x_0 + \mathcal{C}_{n-1}\mathcal{U}_{n-1} + \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \tag{6.2}$$

where for times $k = 0, 1, \ldots, n$, $\mathcal{C}_{n-1}$ is the controllability matrix, $\mathcal{U}_{n-1}$ is the vector of control inputs, $\mathcal{A}_{n-1} = [I_n, A, A^2, \ldots, A^{n-1}]$ with I_n being the n-dimensional identity matrix, and $\mathcal{W}_{n-1}$ is the vector of disturbances.

Given an admissible control sequence $\mathcal{U}_{n-1}$, denote by $\Sigma_n(\mathcal{U}_{n-1})$ the covariance of state x_n at time n. Denote by $\Sigma_n(0)$ the covariance of the state when the control input at each time is zero.

6.1.1 Controllability

We define the stochastic version of controllability in terms of the state covariance.

Definition 13. *The system* (6.1) *is said to be completely linear-stochastic controllable if for every n-vector ξ, either*

$$\xi^T x_n \text{ and } x_0 \text{ are independent when } \mathcal{U}_{n-1} \equiv 0 \text{ or} \tag{6.3}$$

$$\xi^T \Sigma_n(\mathcal{U}_{n-1})\xi < \xi^T \Sigma_n(0)\xi, \tag{6.4}$$

for some $\mathcal{U}_{n-1}$.

For convenience, we assume in the following that $\bar{x}_0 = 0$. The linear system (6.1) is completely linear-stochastic controllable if the matrix pair (A, B) satisfies the deterministic conditions for controllability.

Theorem 12. *The system* (6.1) *is completely linear-stochastic controllable for every $\bar{x}_0$, Σ_0, and $\{Q_k\}_0^{n-1}$ if and only if*

$$range(A^n) \subseteq range(\mathcal{C}_{n-1}). \tag{6.5}$$

Proof. ***If part:*** Denote by $\mathcal{C}_{n-1}^+$ the Moore-Penrose pseudo-inverse of $\mathcal{C}_{n-1}$. Fix an n-vector ξ. Because the range of A^n is contained in the range of C_{n-1}, there exists a ζ such that

$$A^n x_0 = \mathcal{C}_{n-1}\zeta.$$

Define $\mathcal{U}_{n-1} \triangleq -\mathcal{C}_{n-1}^{+}\mathcal{C}_{n-1}\zeta$. The expression (6.2) reduces to

$$\begin{aligned} x_n &= A^n x_0 - \mathcal{C}_{n-1}\mathcal{C}_{n-1}^{+}\mathcal{C}_{n-1}\zeta + \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \\ &= A^n x_0 - \mathcal{C}_{n-1}\zeta + \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \\ &= A^n x_0 - A^n x_0 + \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \\ &= \mathcal{A}_{n-1}\mathcal{W}_{n-1}. \end{aligned} \tag{6.6}$$

Due to (6.6), for an n-vector ξ, if $\xi^T A^n x_0 \neq 0$ almost surely, then $\xi^T \Sigma_n(\mathcal{U}_{n-1})\xi < \xi^T \Sigma_n(0)\xi$. On the other hand, if $\xi^T A^n x_0 = 0$ almost surely, then the unforced response of (6.2) is

$$\xi^T x_n = \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}.$$

Since the disturbances $\mathcal{W}_{n-1}$ and the initial state are independent, it follows that $\xi^T x_n$ and x_0 are independent in the unforced system.

Only if part: In the first case, if $\xi^T x_n$ and x_0 are independent in the unforced system, then $E[\xi^T x_n x_0^T] = 0$ so

$$\begin{aligned} 0 &= \xi^T A^n E[x_0 x_0^T] + \xi^T \mathcal{A}_{n-1} E[\mathcal{W}_{n-1} x_0^T], \\ &= \xi^T A^n \Sigma_0. \end{aligned}$$

In particular, let $\Sigma_0 > 0$; then $\xi^T A^n = 0$, i.e. ξ is not in the range of A^n.

On the other hand, when $\xi^T x_n$ and x_0 are not independent in the unforced system, we have $\xi^T A^n \neq 0$. To see this, note that if $\xi^T A^n = 0$ – as argued above – $\xi^T x_n$ and x_0 are independent because x_0 and $\mathcal{W}_{n-1}$ are independent. Combining $\xi^T A^n \neq 0$ with the condition that there is an admissible control $\mathcal{U}_{n-1}$ for which $\xi^T \Sigma_n(\mathcal{U}_{n-1})\xi < \xi^T \Sigma_n(0)\xi$ we are required to show $\xi^T \mathcal{C}_{n-1} \neq 0$.

For the contradiction argument, fix ξ so that $\xi^T A^n \neq 0$ but $\xi^T \mathcal{C}_{n-1} = 0$. Then for any admissible control $\mathcal{U}_{n-1}$, we have by substitution into (6.2),

$$\begin{aligned} &\xi^T x_n = \xi^T A^n x_0 + \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \\ &\Rightarrow \xi^T \Sigma_n(\mathcal{U}_{n-1})\xi = \xi^T A^n \Sigma_0 A^n \xi + \xi^T \mathcal{A}_{n-1}\bar{Q}_{n-1}\mathcal{A}_{n-1}^T \xi = \xi^T \Sigma_n(0)\xi, \end{aligned} \tag{6.7}$$

where $\bar{Q}_{n-1}$ is the block diagonal matrix containing Q_k, $k = 0, 1, \ldots, n-1$. The equality in (6.7) is a contradiction; therefore, $\xi^T \mathcal{C}_{n-1} \neq 0$. $\square$

The condition,

$$\xi^T x_n \text{ and } x_0 \text{ are independent when } \mathcal{U}_{n-1} \equiv 0, \tag{6.8}$$

is not obvious. One interpretation is that stable directions in the state-space forget the initial condition in finite time; additionally, an important case where $\xi x_n \perp \{x_k\}_0^{n-1}$ – when $A \equiv 0$ – is contained in (6.8). Nevertheless, it may be possible to relax (6.8) and derive a result like Theorem 12 with appropriate modifications to the assumptions in the theorem.

6.1.2 Reachability

The definition of stochastic controllability corresponded to the ability to reduce the state covariance with control; in other words, controllability relates to bounding the distance of the state from the origin. On the other hand, we expect reachability to coincide with the ability to use control to bound the distance of the state from an arbitrary point. Thus, for a fixed selection of x_r, define the state error at time n as $\tilde{x}_n \triangleq x_n - x_r$. Denote by $\tilde{\Sigma}_n(\mathcal{U}_{n-1})$ the second moment of $\tilde{x}_n$,

$$\Sigma_n(\mathcal{U}_{n-1}) \triangleq E[\tilde{x}_n \tilde{x}_n^T],$$

subject to control sequence $\mathcal{U}_{n-1}$. Similarly, denote by $\tilde{\Sigma}_n(0)$ the second moment of $\tilde{x}_n$ arising from the unforced system.

Definition 14. *The system* (6.1) *is said to be completely linear-stochastic reachable if for every n-vector ξ and any selection of x_r, either*

$$\xi^T x_r = \xi^T \left(A^n x_0 + \mathcal{A}_{n-1} \mathcal{W}_{n-1} \right) \quad a.s. \ or \tag{6.9}$$

$$\xi^T \tilde{\Sigma}_n(\mathcal{U}_{n-1}) \xi < \xi^T \tilde{\Sigma}_n(0) \xi, \tag{6.10}$$

for some $\mathcal{U}_{n-1}$.

Again, for clarity, assume $\bar{x}_0 = 0$.

Theorem 13. *The system* (6.1) *is completely linear-stochastic reachable for every $\bar{x}_0$, Σ_0, and $\{Q_k\}_0^{n-1}$ if and only if the controllability matrix $\mathcal{C}_{n-1}$ is full rank.*

Proof. We adopt the notation used in the proof of Theorem 12. ***If part:*** Select $\mathcal{U}_{n-1} = \mathcal{C}_{n-1}^{+}(x_r - A^n x_0)$. Since $\mathcal{C}_{n-1}$ is full rank by assumption, (6.2) yields

$$\tilde{x}_n = \mathcal{A}_{n-1}\mathcal{W}_{n-1}.$$

The unforced-system response is

$$\xi^T \tilde{x}_n = \xi^T A^n x_0 - \xi^T x_r + \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}.$$

Suppose $\xi^T A^n x_0 \neq \xi^T x_r$ on a set of non-zero probability. Then, for some $\delta > 0$,

$$\begin{aligned}
\xi^T \tilde{\Sigma}_n(0)\xi &= E[\xi^T (A^n x_0 - x_r)(A^n x_0 - x_r)^T \xi] + \xi^T \mathcal{A}_{n-1}\bar{Q}_{n-1}\mathcal{A}_{n-1}\xi, \\
&\geq \delta^2 P\left((\xi^T (A^n x_0 - x_r))^2 > \delta\right) + \xi^T \mathcal{A}_{n-1}\bar{Q}_{n-1}\mathcal{A}_{n-1}\xi, \qquad (6.11) \\
&> \xi^T \mathcal{A}_{n-1}\bar{Q}_{n-1}\mathcal{A}_{n-1}\xi = \xi^T \tilde{\Sigma}_n(\mathcal{U}_{n-1})\xi,
\end{aligned}$$

since we may choose $\delta > 0$ sufficiently small so that the probability in (6.11) is positive. Thus, either $\xi^T A^n x_0 = \xi^T x_r$ almost surely, or $\xi^T \tilde{\Sigma}_n(0)\xi > \xi^T \tilde{\Sigma}_n(\mathcal{U}_{n-1})\xi$.

Only if part: We show the contrapositive, i.e. if $\mathcal{C}_{n-1}$ is not full rank, then there is at least one ξ and x_r for which $\xi^T x_r \neq \xi^T A^n x_0 + \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}$ almost surely and $\xi^T \tilde{\Sigma}_n(\mathcal{U}_{n-1})\xi = \xi^T \tilde{\Sigma}_n(0)\xi$, for some $\bar{x}_0$, Σ_0, $\{Q_k\}_0^{n-1}$.

Since $\mathcal{C}_{n-1}$ is not full rank, there exists ξ such that $\xi^T \mathcal{C}_{n-1} = 0$. Fix $\bar{x}_0$, Σ_0, $\{Q_k\}_0^{n-1}$ and choose x_r so $\xi^T x_r \neq \xi^T\left(A^n x_0 + \mathcal{A}_{n-1}\mathcal{W}_{n-1}\right)$ almost surely. The error version of (6.2) has the form

$$\begin{aligned}
\xi^T \tilde{x}_n &= \xi^T A^n x_0 - \xi^T x_r + \xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} + \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \\
&= \xi^T A^n x_0 - \xi^T x_r + \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}, \text{ with covariance} \\
\xi^T \tilde{\Sigma}_n(\mathcal{U}_{n-1})\xi &= \xi^T A^n \Sigma_0 A^{nT}\xi + \xi^T x_r x_r^T \xi + \xi^T \mathcal{A}_{n-1}\bar{Q}_{n-1}\mathcal{A}_{n-1}^T \xi,
\end{aligned}$$

for any admissible control $\mathcal{U}_{n-1}$. We have $\xi^T \Sigma_n(\mathcal{U}_{n-1})\xi = \xi^T \Sigma_n(0)\xi$ for every admissible control $\mathcal{U}_{n-1}$, which concludes the proof.

□

As before, there is some ambiguity in the conditions (6.9-6.10). Analogous to the stochastic controllability case, the choice of condition (6.9) was to ensure sufficiency of the rank condition regardless of the initial condition and disturbance distributions. In contrast to the controllability condition (6.3), we need more than independence in order that the rank condition be a necessary condition for complete stochastic reachability.

6.2 Nonlinear Systems

In similar fashion to the observability and reconstructibility analysis, complete controllability/reachability requires the examination of functions of the state. For instance, let $x_0 \sim \text{unif}(-1, 1)$ and

$$x_1 = g(u)x_0, \tag{6.12}$$

where $g(u) = 1$ if $u \geq 0$ and $g(u) = -1$ otherwise. While, intuitively, the sign of the state x_1 is controllable, the magnitude of x_1 is not. Note that although this example is similar to its counterpart in the observability analysis,

$$\begin{aligned} x \sim\ & \text{unif}\{-2, -1, 1, 2\}, \\ y =\ & |x|, \end{aligned}$$

the control question is distinct as it does not concern independence of the uncontrollable function. In particular, if u is any bijective function of x_0, for instance the identity mapping, then $|x_1|$ in (6.12) is exactly determined from u. This reiterates the point that conditional expectation, and thus, the mutual information, are not appropriate tools for reachability/controllability analysis. Thus, we introduce the relative entropy (also known as the Kullback-Leibler divergence) from information theory. Given two probability mass (density) functions, p and $\hat{p}$, for random variable X, the relative entropy between them is

$$D(p||\hat{p}) = E_p\left[\log\frac{p(X)}{\hat{p}(X)}\right], \tag{6.13}$$

where the subscript on E_p denotes that the expectation is taken using the measure (density) p. By convention, $0\log(0/0) \equiv 0$, $0\log(0/q) = 0$ and $0\log(p/0) = \infty$. More generally, if probability measure P is absolutely continuous with respect to probability measure $\hat{P}$, the relative entropy may be defined in terms of the Radon-Nikodym derivative,

$$D(P||\hat{P}) = E_P \left[\log \frac{dP}{d\hat{P}} \right]. \tag{6.14}$$

Recall that $D(P||\hat{P}) \geq 0$ with equality if and only if $P = \hat{P}$.

Because the admissible control inputs include feedback and mixture policies, we fix the system before offering definitions. Let the system be

$$x_{k+1} = f(x_k, u_k, w_k), \tag{6.15}$$

where x_0 and w_k are independent for every k, and w_k is an independently distributed sequence. Consider a measurable function $\ell(x_k)$. Denote by P_ℓ the probability law for $\ell(x_k)$ subject to an admissible control sequence $\{u_t\}_0^{k-1}$. Denote by $x_{k:0}$ the state x_k subject to a fixed control sequence $u_t \equiv 0$, $t = 1, 2, \ldots, k-1$ and let P_ℓ^0 be its probability law.

Definition 15. *Call a measurable function $\ell(x_k)$ (nonlinear-stochastic) controllable if there exists an admissible control sequence $\{u_t\}_0^{k-1}$ such that*

$$D(P_\ell||P_\ell^0) > 0, \tag{6.16}$$

or $\ell(x_{k:0})$ and x_0 are independent.

Rationale for Definition 15

The independence condition between $\ell(x_{k:0})$ and x_0 warrants some discussion. Recall that for a controllable state in a deterministic linear system, either a component of the state may be steered to the origin via the control input, or the component moves to the origin in the nominal system. The condition (6.3) preserves the idea that initial-condition effects may be removed due to the uncontrolled dynamics for linear stochastic systems.

The finite-state Markov chain highlights considerations for nonlinear systems. The concept of the origin is generally not meaningful because the state space has finite cardinality. On the other hand there is a common definition for stability;

a Markov chain is said to be stable if it is irreducible. Irreducibility implies the existence of a unique stationary distribution; that is, the terminal state distribution does not depend on the initial condition. Thus, if an uncontrolled Markov chain is stable in some sense, i.e. if there is a unique stationary distribution for a function of the terminal state, $\ell(x_\infty)$,[2] then the independence of $\ell(x_\infty)$ with x_0 follows. Definition 15 adds that the uncontrolled state dynamics remove dependencies of x_0 in some component of a terminal state $\ell(x_{k:0})$ *in finite time*.

The next result demonstrates the connection between Definitions 15 and 13; we remark on possible shortcomings of Definition 15 after the result.

Theorem 14. *If the system (6.1) is completely linear-stochastic-controllable for every $\bar{x}_0$, Σ_0, and $\{Q_k\}_0^{n-1}$, then it is completely nonlinear-stochastic-controllable.*

Proof. Due to Theorem 12,

$$\mathrm{range}(A^n) \subset \mathrm{range}(\mathcal{C}_{n-1}), \tag{6.17}$$

where, as before, $\mathcal{C}_{n-1}$ denotes the controllability matrix. Since $D(P_\ell || P_\ell^0) = 0$ if and only if $P_\ell = P_\ell^0$, it suffices to show that if (6.17) holds, then there is some Borel set S in the range of ℓ such that $P(\ell(x_n) \in S) \neq P(\ell(x_{n:0}) \in S)$ for an appropriately chosen control signal.

For the contradiction argument, suppose that $\ell(x_n)$ and x_0 are not independent and $P(\ell(x_n) \in S) = P(\ell(x_{n:0}) \in S)$ given any Borel set S in the range of ℓ and control input. Let $S^{\leftarrow} \triangleq \{x_n : \ell(x_n) \in S\}$. Adopting the notation of (6.2), we have

$$\begin{aligned} P(\ell(x_n) \in S) &= P(A^n x_0 + \mathcal{C}_{n-1}\mathcal{U}_{n-1} + \mathcal{A}_{n-1}\mathcal{W}_{n-1} \in S^{\leftarrow}), \\ &= P(A^n x_0 + \mathcal{A}_{n-1}\mathcal{W}_{n-1} \in \{s : s + \mathcal{C}_{n-1}\mathcal{U}_{n-1} \in S^{\leftarrow}\}), \\ &= P(x_{n:0} \in S'), \end{aligned}$$

where $S' \triangleq \{s : s + \mathcal{C}_{n-1}\mathcal{U}_{n-1} \in S^{\leftarrow}\}$. Let Ξ form a basis for $\mathbb{R}^n$. Then

$$P(x_{n:0} \in S') = P\left(\left\{x_{n:0} : \sum_{\xi \in \Xi} \xi^T x_{n:0} = \xi^T s, s \in S'\right\}\right).$$

[2] with some abuse of notation

We suggest without proof[3] that

$$P(x_{n:0} \in S') = P(x_{n:0} \in S^{\leftarrow}) \Rightarrow S' = S^{\leftarrow} \text{ a.s.}$$

If $S' = S^{\leftarrow}$ almost surely, then for each $\xi \in \Xi$, $P(x_{n:0} \in S') = P(x_{n:0} \in S^{\leftarrow})$ implies either $\xi^T \mathcal{C}_{n-1} = 0$ or $\{\xi^T s : s \in S'\} = (-\infty, \infty)$ almost everywhere. For ξ in the former case, $\xi^T A^n = 0$ by (6.17), so

$$\xi^T x_{n:0} = \xi^T \mathcal{A}_{n-1} \mathcal{W}_{n-1}. \tag{6.18}$$

For ξ in the latter case,

$$\{x_{n:0} : \xi^T x_{n:0} = \xi^T s, s \in S'\} = (-\infty, \infty), \quad \text{a.e.} \tag{6.19}$$

Since this holds for any Borel set S in the range of ℓ, (6.18-6.19) imply that $\ell(x_{n:0})$ and x_0 are independent, which is a contradiction. $\square$

One may recover the linear definition from the nonlinear version by considering the class of functions $\ell(x_k) = \xi^T x_k$. This also demonstrates the significance of (6.19).

Definition 15 does not meet some intuitions. Viewing the entropy as the nonlinear counterpart of the covariance, one might expect that a condition for controllability is the capacity for the control input to reduce the entropy of the state. That is, denoting by P_ℓ the probability mass (density) function for $\ell(x_n)$ and P_ℓ^0 the probability mass (density) function for $\ell(x_{n:0})$ with ℓ measurable, we would expect

$$\begin{aligned} &H(\ell(x_n)) < H(\ell(x_{n:0})), \\ &\Leftrightarrow E_{P_\ell}[-\log P_\ell] < E_{P_\ell^0}[-\log P_\ell^0]. \end{aligned} \tag{6.20}$$

The relative entropy requirement (6.16) almost achieves this condition since

$$\begin{aligned} 0 &< D(\ell(x_n)||\ell(x_{n:0})), \\ &= E_{P_\ell}\left[\log \frac{P_\ell}{P_\ell^0}\right], \\ &= E_{P_\ell}[-\log P_\ell^0] - H(\ell(x_n)), \end{aligned}$$

[3] As in the proof of Theorem 5, the additional steps do not contribute much to the discussion.

except that the physical meaning of $E_{P_\ell}[-\log P_\ell^0]$ is unclear. On the other hand, it is potentially problematic to demonstrate a version of Theorem 14 using (6.20) since the differential entropy lacks some key properties of the discrete entropy; for instance, the differential entropy can be negative and unbounded from below, and it is not necessarily invariant with respect to bijective transformations. These technical hurdles are compounded by the freedom in choosing ℓ. It is worth remarking that we did not encounter such difficulties in our observability analysis because the conditions for stochastic observability could be expressed in terms of the mutual information, which retains its key characteristics for continuous random variables (see Chapter 8 of [CT06] for additional details). The relative entropy plays an analogous role in the controllability analysis.

Our formulation of nonlinear reachability follows a similar approach. Let ℓ_d be a point in the range of a measurable function ℓ and denote a sequence of measures $\{P_{d,m}\}_{m=1}^{\infty}$ with mean ℓ_d which converge to $P_d(\cdot) = 1_{\ell_d}(\cdot)$. The terms P_ℓ, P_ℓ^0, and $x_{k:0}$ remain as previously defined. In the following analysis, it will be convenient to have that P_ℓ is absolutely continuous with respect to $P_{d,m}$ and $P_{d,m}$ is absolutely continuous with respect to P_ℓ^0 for every finite m (i.e. $P_\ell << P_{d,m} << P_\ell^0$); we assume this whenever a sequence $\{P_{d,m}\}_1^{\infty}$ is specified.

Definition 16. *Call the point ℓ_d in the range of a measurable function $\ell(x_k)$ (nonlinear-stochastic) reachable if there is an admissible control sequence $\{u_t\}_0^{k-1}$ such that for some sequence $\{P_{d,m}\}_{m=1}^{\infty}$ with first moment ℓ_d and which converges to $P_d(\cdot) = 1_{\ell_d}(\cdot)$, and for every $m > M$ where $M < \infty$,*

$$D(P_{d,m}||P_\ell) < D(P_{d,m}||P_\ell^0) \tag{6.21}$$

[4] *or*

$$\ell(x_{k:0}) = \ell_d \quad \text{almost surely.} \tag{6.22}$$

As with the controllability definition, an entropy condition – e.g.

$$H(\ell(x_k)) < H(\ell(x_{k:0}))$$

[4]Note that there is no loss in generality for using the probabilities rather than the random variables as the arguments of relative entropy, since relative entropy depends only on the probability law of the random variables.

in place of (6.21) – is insufficient. This is well illustrated by example. Consider the following single step problem,

$$x_1 = x_0^{1_{u\neq 0}+1}, \quad x_0 \sim\text{unif}\{-1,1\}.$$

In the nominal ($u \equiv 0$) system, $x_1 \sim$unif$\{-1,1\}$ so $H(x_{1:0}) = 1$ bit. On the other hand, for $u \neq 0$, $x_1 = 1$; thus, $H(x_{1:0}) = 0$ bit. Nevertheless, no control can bring x_1 "closer" to -1.

The form of (6.21) is similar to (6.16) since

$$\begin{aligned}
D(P_{d,m}||P_\ell) &< D(P_{d,m}||P_\ell^0), \\
\Leftrightarrow 0 &< D(P_{d,m}||P_\ell^0) - D(P_{d,m}||P_\ell), \\
&= E_{P_{d,m}}\left[\log\frac{P_{d,m}}{P_\ell^0} - \log\frac{P_{d,m}}{P_\ell}\right], \\
&= E_{P_{d,m}}\left[\log\frac{P_\ell}{P_\ell^0}\right].
\end{aligned}$$

The same derivation holds using Radon-Nikodym derivatives due to absolute continuity assumptions. As before, the physical interpretation is not obvious since the integral is evaluated with respect to the measure $P_{d,m}$.

We demote the result relating Definitions 14 and 16 to conjecture, as the justification is rough.

Conjecture 1. *If (6.1) is completely stochastic-linear-reachable for every $\bar{x}_0$, Σ_0, and $\{Q_k\}_0^n$, then it is completely stochastic-nonlinear-reachable.*

Sketch of argument. We continue using the notation of (6.2). By Theorem 13, $\mathcal{C}_{n-1}$ is full rank. Fix ϵ and let B_ϵ denote an open ball of radius ϵ around ℓ_d. The idea is, for a sufficiently small neighborhood around ℓ_d, we will have

$$\frac{dP_\ell}{dP_\ell^0} = 1.$$

This should hold approximately since

$$E_{P_{d,m}}\left[\log\frac{dP_\ell}{dP_\ell^0}\right] = E_{P_{d,m}}\left[1_{B_\epsilon}\log\frac{dP_\ell}{dP_\ell^0}\right] + E_{P_{d,m}}\left[1_{\mathbb{R}^n\setminus B_\epsilon}\log\frac{dP_\ell}{dP_\ell^0}\right] = 0.$$

Since this holds for each $m > M$, we may select $m' > m$ such that more mass is concentrated around B_ϵ, in which case

$$E_{P_{d,m}}\left[\log \frac{dP_\ell}{dP_\ell^0}\right] \neq 0,$$

unless $dP_\ell/dP_\ell^0 = 1$.

Now using the definition of the Radon-Nikodym derivative, conclude that $P(\ell(x_n) \in B_\epsilon) = P(\ell(x_{n:0}) \in B_\epsilon)$. Since $\mathcal{C}_{n-1}$ is full rank, this leads to a contradiction by the same arguments as in Theorem 14. □

6.3 Statistical Causality

A variety of concepts of causality exist over a range of research fields. Our discussion focuses on the statistical causality introduced by Granger for econometrics. Loosely speaking, given two time series, say X and Y, this version of causality seeks to identify whether X contains information to improve the prediction of future behavior of Y which is not contained in the past observations of Y.

Solo has recently developed an information theoretic basis for extending Granger's definition of causality to nonlinear systems [Gra88, Sol07, Sol08]. His theory provides a procedure to measure the strength of causality as well as a framework which may be used in system identification. We focus on Solo's formulation since the tools of information theory provide a convenient language for expressing our ideas about information.

Our interest in causality is that, in order for a control with an exogenous component r_k to exert a causal effect on the state, (6.1) must possess structural properties consistent with reachability. This provides an option for empirically studying reachability as a causality problem.

In addition to the previously defined terms, the conditional mutual information will be needed. Given random quantities X, Y, and Z, the conditional mutual information between X and Y given Z is

$$I(X;Y|Z) = E\left[\log \frac{P(X,Y|Z)}{P(X|Z)P(Y|Z)}\right]. \tag{6.23}$$

As usual, in the case of continuous random vectors for which densities exist, the densities are used in place of the probabilities.

Definition 17. *[Sol08] We say that u_k **does not cause** x_k if*

$$I(x_{N+1}; \{u_k\}_0^N | \{x_k\}_0^N) = 0, \tag{6.24}$$

for all $N \geq 0$. Equivalently,

$$I(\{x_k\}_{N+1}^{N+m}; u_N | \{x_k\}_0^N, \{u_k\}_0^{N-1}) = 0, \tag{6.25}$$

for all $N \geq 0$ and $m \geq 0$.

The equivalence of (6.24) and (6.25) is shown in [Sol08]. Additionally, these are equivalent to

$$P(x_{N+1} | \{x_k\}_0^N, \{u_k\}_0^N) = P(x_{N+1} | \{x_k\}_0^N), \tag{6.26}$$

since

$$\begin{aligned} &P(x_{N+1} | \{x_k\}_0^N, \{u_k\}_0^N) = P(x_{N+1} | \{x_k\}_0^N) P(\{u_k\}_0^N | \{x_k\}_0^N), \quad \text{but also} \\ &P(x_{N+1} | \{x_k\}_0^N, \{u_k\}_0^N) = P(x_{N+1} | \{u_k\}_0^N, \{x_k\}_0^N) P(\{u_k\}_0^N | \{x_k\}_0^N). \end{aligned}$$

Thus,

$$P(x_{N+1} | \{u_k\}_0^N, \{x_k\}_0^N) = P(x_{N+1} | \{x_k\}_0^N). \tag{6.27}$$

The class of state-feedback policies – where the control is solely a function of the state, $u_k \triangleq \mu_k(\{x_t\}_0^k$ – is insufficient for causality analysis since by (6.24) in conjunction with the information processing inequality,

$$I(x_{N+1}; \{\mu_k\}(\{x_t\}_{t=0}^k)\}_{k=0}^N | \{x_k\}_0^N) = 0.$$

Allow u_k to be a function of the state and an exogenous signal, r_k; that is, $u_k = g(\{x_i\}_0^k, r_k)$. The signal r_k takes values on Ω_r which is disjoint from the sample space Ω. The full probability space is, with obvious notation, the usual product space, $(\Omega \times \Omega_r, \mathcal{F} \times \sigma(\{r_k\}_0^\infty), P \times P_r)$. Define P_r and the corresponding kernel function from which the probability on $\mathcal{F} \times \sigma(\{r_k\}_0^\infty)$ is computed so that $\mathcal{F}$ and $\sigma(\{r_k\}_0^\infty)$ are independent. Naturally, we require that g is measurable on this space. For notational convenience, in the following we will denote the product measure using P and specify by $\bar{P}$ the measure on $(\Omega, \mathcal{F})$.

In linear systems, causality is directly related to stochastic reachability.

Theorem 15. *Let w_k have bounded variance for all k. For every $\xi \neq 0$ there exists g and r_k, $k = 0, 1, \ldots, n-1$ such that*

$$I(\xi^T x_n; \{u_k\}_0^{n-1} | \{x_k\}_0^{n-1}) > 0 \tag{6.28}$$

if and only if

$$rank\left(\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}\right) = n.$$

We adopt the earlier notation (6.2). It suffices to show that there exists $\xi^T x_n$ for which

$$I(\xi^T x_n; \{u_k\}_0^{n-1} | \{x_k\}_0^{n-1}) = 0 \tag{6.29}$$

if and only if $\text{rank}(\mathcal{C}_{n-1}) < n$. Note that due to the Cayley-Hamilton Theorem,

$$\text{rank}(\mathcal{C}_{n-1}) < n \;\Rightarrow\; \text{rank}(\mathcal{C}_{m-1}) < n$$

for every m.

Before engaging in the proof, we quote some properties of conditional expectation for ease of reference.

1. If X is $\mathcal{G}$-measurable for σ-algebra $\mathcal{G}$, then $E[X|\mathcal{G}] = X$. This is immediate from the Radon-Nikodym definition of conditional expectation.

2. If σ-algebras

   $$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{B},$$

   then for an integrable $X \in B$,

   $$E[E[X|\mathcal{G}_1]|\mathcal{G}_2] \;\text{ almost surely.}$$

   See [Res99] for the proof.

3. Let $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a bounded Borel function. Suppose also that $X : \Omega \to \mathbb{R}^n$, $W : \Omega \to \mathbb{R}^m$, $X \in \mathcal{G}$, and W is independent of $\mathcal{G}$. Define

   $$f_\phi(x) = E[\phi(x, W)]. \tag{6.30}$$

Then

$$E[\phi(X,W)|\mathcal{G}] = f_\phi(X) \text{ almost surely.} \tag{6.31}$$

See [Res99] for the proof. Note that in (6.31), $f_\phi(X)$ is itself a $\sigma(X)$-measurable random element; roughly speaking, $f_\phi(X)$ is obtained by integrating away W. Evidently, ϕ may be vectorized to obtain the same relation for vector functions $\bar{\phi}$.

4. If X and Y are conditionally independent given Z. This follows immediately from the equivalences between various formulations for statistical causality, namely (6.26).

Proof. First assume that $\xi^T x_n$ satisfies (6.29). By Property 4,

$$E[\xi^T x_n|\mathcal{U}_{n-1}, \{x_k\}_0^{n-1}] = E[\xi^T x_n|\{x_k\}_0^{n-1}] \text{ a.s.} \tag{6.32}$$

Using (6.2),

$$\begin{aligned} E[\xi^T x_n|\mathcal{U}_{n-1}, \{x_k\}_0^{n-1}] &= E[\xi^T(A^n x_0 + \mathcal{C}_{n-1}\mathcal{U}_{n-1} + \mathcal{A}_{n-1}\mathcal{W}_{n-1})|\mathcal{U}_{n-1}, \{x_k\}_0^{n-1}], \\ &= \xi^T A^n x_0 + \xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} + E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}|\mathcal{U}_{n-1}, \{x_k\}_0^{n-1}], \end{aligned} \tag{6.33}$$

due to Property 1. By construction of $\mathcal{U}_{n-1}$,

$$E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}|\mathcal{U}_{n-1}, \{x_k\}_0^{n-1}] = E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}|\{g(x_k, r_k)\}_0^{n-1}, \{x_k\}_0^{n-1}].$$

Using the definition of conditional expectation,

$$\int 1_S E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1}|\{g(x_k, r_k\}_0^{n-1}, \{x_k\}_0^{n-1}]dP = \int 1_S \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} dP, \tag{6.34}$$

for any $S \in \sigma(\{g(x_k, r_k)\}_0^{n-1})$. Define $S_r \triangleq (S \cap \Omega_r) \cap \Omega$ and $S_x \triangleq (S \cap \Omega) \cup \Omega_r$

(since Ω and Ω_r are disjoint, S_r and S_x are cylinder sets). Then

$$\begin{aligned}
\int 1_S \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} dP &= \int 1_{S_r \cap S_x} \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} dP, \\
&= \int 1_{S_r} 1_{S_x} \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} dP, \\
&= \int 1_{S_r} dP_r \int 1_{S_x} \xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} dP, && (6.35) \\
&= \int 1_{S_r} dP_r \int 1_{S_x} E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}] dP, \quad \text{a.s.} && (6.36) \\
&= \int 1_{S_r} 1_{S_x} E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}] dP, && (6.37) \\
&= \int 1_S E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}] dP, && (6.38)
\end{aligned}$$

Steps (6.35) and (6.37) follow from independence built into the product space. Step (6.36) uses the definition of conditional expectation.

Substituting (6.38) into (6.33),

$$E[\xi^T x_n | \mathcal{U}_{n-1}, \{x_k\}_0^{n-1}] = \xi^T A^n x_0 + \xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} + E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}].$$

Using (6.32)

$$\begin{aligned}
\xi^T A^n x_0 + &\xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} + E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}] \\
&= \xi^T A^n x_0 + \xi^T \mathcal{C}_{n-1} E[\mathcal{U}_{n-1}] + E[\xi^T \mathcal{A}_{n-1}\mathcal{W}_{n-1} | \{x_k\}_0^{n-1}], \\
&\Rightarrow \xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} = \xi^T \mathcal{C}_{n-1} E[\mathcal{U}_{n-1}], \quad \text{almost surely.}
\end{aligned}$$

If we set $u_k = g(x_k, r_k) = r_k$ where r_k is a normally distributed random vector with positive definite covariance, then we have that $\mathcal{U}_{n-1} \neq 0$ but $\xi^T \mathcal{C}_{n-1}\mathcal{U}_{n-1} = 0$ (both relations hold almost everywhere). Thus, $\text{rank}(\mathcal{C}_{n-1}) < n$.

Conversely, assume $\text{rank}(\mathcal{C}_{n-1}) < n$. Then there is some $\xi \in \mathbb{R}^n \ \{0\}$ such that $\xi^T \mathcal{C}_{n-1} = 0$. As a consequence of the Cayley-Hamilton Theorem, $\xi^T \mathcal{C}_m = 0$ for $m > n$. Evidently, $\xi^T \mathcal{C}_m = 0$ for $m \leq n$ as well, and in particular, $\xi^T B = 0$. Thus,

$$\begin{aligned}
P(\xi^T x_m < a | \mathcal{U}_{m-1}, \{x_k\}_0^{m-1}) &= P(\xi^T (A x_{m-1} + w_{m-1}) < a | \mathcal{U}_{m-1}, \{x_k\}_0^{m-1}), \\
&= E[1_{\{\xi^T (A x_{m-1} + w_{m-1}) < a\}} | \mathcal{U}_{m-1}, \{x_k\}_0^{m-1}], \\
&\triangleq E[\phi(x_{m-1}, w_{m-1}) | \mathcal{U}_{m-1}, \{x_k\}_0^{m-1}].
\end{aligned}$$

Since w_{m-1} and $(\mathcal{U}_{m-1}, \{x_k\}_0^{m-1})$ are independent (i.e. their generated σ-algebras are independent), by applying Property 3 we obtain

$$E[\phi(x_{m-1}, w_{m-1})|\mathcal{U}_{m-1}, \{x_k\}_0^{m-1}] = f_\phi(x_{m-1}),$$

hence, $P(\xi^T x_m < a|\mathcal{U}_{m-1}, \{x_k\}_0^{m-1}) = f_\phi(x_{m-1})$, where $f_\phi(\cdot)$ is characterized by (6.30-6.31). For any $S \in \sigma(\{x_k\}_0^{m-1})$,

$$\int 1_S f_\phi(x_{m-1}) dP = \int 1_S E[f_\phi(x_{m-1})|\{x_k\}_0^{m-1} dP, \tag{6.39}$$

$$= \int 1_S E[E[1_{\{\xi^T x_m < a\}}|\mathcal{U}_{m-1}, \{x_k\}_0^{m-1}]|\{x_k\}_0^{m-1}] dP, \tag{6.40}$$

$$= \int 1_S E[1_{\{\xi^T x_m < a\}}|\{x_k\}_0^{m-1}] dP. \tag{6.41}$$

Step (6.39) is from the definition of conditional expectation. Step (6.40) is a direct substitution for $f_\phi(x_{m-1})$ using (6.31). Step (6.41) is due to Property 2.

Finally, (6.41) shows that

$$P(\xi^T x_m < a|\{x_k\}_0^{m-1}) = P(\xi^T x_m < a|\mathcal{U}_{m-1}, \{x_k\}_0^{m-1}).$$

Since a was arbitrary, this suffices to verify that $\xi^T x_m$ and $\mathcal{U}_{m-1}$ are conditionally independent given $\{x_k\}_0^{m-1}$ for every m. □

6.4 Chapter Summary

Definitions for stochastic controllability and reachability were developed, beginning with the Gauss-Markov model and extending to nonlinear systems. Whereas the mutual information was the key tool for extending stochastic observability and reconstructibility from linear to nonlinear systems, the relative entropy played the part in the controllability and reachability analyses. We recovered the deterministic matrix conditions from our stochastic linear definitions. We also showed that the nonlinear definition of complete controllability was compatible with its linear counterpart. An avenue was suggested for demonstrating the same link between the linear and nonlinear versions of stochastic reachability. Finally, we commented on a connection between our ideas and Granger causality from econometrics.

Chapter 7

Conclusion

7.1 Summary

We have formulated and analyzed versions of observability, reconstructibility, controllability, and reachability in discrete-time stochastic systems. Our development was guided by progressively extending definitions starting from linear deterministic systems to stochastic, linear systems, and finally to stochastic nonlinear systems. We furthermore show the role of information theory as an interpretative framework which allows the generalization of ideas from stochastic linear systems theory to stochastic nonlinear systems.

Chapter 2 focused on the formulation of stochastic observability and reconstructibility. A novelty of our linear systems analysis is our emphasis on the property of reconstructibility, and we discuss the connection between reconstructibility and estimability due to Baram and Kailath [BK88]. The deterministic matrix conditions for both observability and reconstructibility are recovered from the stochastic definitions under reasonable assumptions on the stochastic disturbances. The concepts of entropy and mutual information from information theory allow us to generalize the definitions from linear systems.

In Chapter 3 We applied the concepts of Chapter 2 to the finite-state hidden Markov model and, in particular, derive a rank condition analogous to that in the linear systems analysis. Simulations of a hidden Markov model representation of a computer connected to the Internet supplement our results.

Chapter 4 studies some consequences of stochastic observability and reconstructibility. We demonstrate that reconstructibility is needed in order to achieve control performance improvement via feedback. These results show that fundamental limitations on control design coincide with our formulation. The problem is also approached from another angle; we study through simulation the benefit to control performance from improved estimation.

Chapter 5 deals with the relation between van Handel's definition of stochastic observability [vH09a] and our own. Van Handel's observability provides a condition for the stability of the nonlinear filter. Filter stability characterizes, roughly, the robustness of estimates against errors in the initial condition; thus, there is likely a connection between our definitions and the sensitivity of estimators to initial condition mismatches.

Chapter 6 addresses stochastic controllability and reachability. Here, our development is guided by the earlier work on stochastic observability and reconstructibility; thus, we focus on recovering the deterministic matrix conditions for controllability and reachability from properties of the covariance of the stochastic linear systems. The extension to nonlinear systems again uses information theory to extend the conceptual interpretations of covariance reduction for linear systems. Whereas the observability and reconstructibility analysis used the conditional entropy and mutual information, the controllability and reachability analysis rely on the relative entropy. We go on to demonstrate a link between our definitions and statistical causality.

7.2 Future work

The goal of this research is to develop guidelines for control design in systems where the controller has limited information about the state. Information theory provides tools in the form of various entropy concepts which potentially allow us to quantify trade-offs between increasing state information with decreases to control performance. Although our theoretical and numerical results suggest that the level of entropy measures estimator/controller performance, we have not established the connection in more complex systems, for instance the dual adaptive control system [Fel61, ÅW94]. The specific problems are to link the mutual information (between state and output) to the estimator performance, to link the estimator performance or mutual information to controller performance, and to link the relative entropy between nominal and controlled systems to the control performance.

Recent work by Abarbanel, et al. [Aba09, ACFK09, AKW10, QA10] has made some headway in associating the mutual information to the estimator performance. Abarbanel, et al. developed computational methods for estimating the conditional state probability given the outputs – that is, the output of the nonlinear filter – by formulating the problem as a discrete-time path integral and using methods from statistical physics to approximate the integral. Importantly, the mutual information appears as a term in the (exact) estimator. By analogy to the Kalman filter, where the covariance, which is a measure of estimator performance, appears in the estimator, there is reason to suspect that the mutual information plays a similar role in the nonlinear filter.

Our network example relies on a number of simplifying examples which limit its use to conceptual illustrations. Another area of future work is to apply the definitions presented herein to more realistic models. By nature of the information theoretic tools, there is reason to believe that the law of large numbers may be invoked so that our definitions remain useful. On the other hand, the application to more complex systems may require adjustments to the definitions. As a specific example, it is known that chaotic systems generate entropy (see, for example, [BTV96]) at a rate proportional to the sum of the averaged Lyapunov exponents. Thus, rather than requiring non-zero mutual information, the definition of observability in chaotic systems may require mutual information strictly greater than a function of the system's Lyapunov exponents in order to be physically meaningful. Abarbanel's research [Aba09, ACFK09, AKW10, QA10] again provides an avenue for investigation, as his estimators are tested against chaotic systems of high order.

Two directions for applications to control algorithm design come to mind.

One might derive a trade-off between the excitation and regulation roles of the control and solve a separated problem. This approach has the flavor of robust control problems, and there is the possibility of introducing game theory in its study. Game theory in turn often suggests the use of mixed control policies, which is in line with our observations that control excitation plays an important role in general output feedback problems. A second approach is to focus on deriving observability conditions for systems based on the underlying structure of their dynamics. These may in turn be used as constraints in optimal control problems in similar fashion to the persistence of excitation constraint in certain forms of model predictive control.

Appendix A

Odds and ends

For completeness, we include the derivation of the hidden Markov model for the Internet network router example.

A.1 State dynamics

Note that

$$\begin{aligned} P(b_{k+1}, c_{k+1}|b_k, c_k, r_k) &= P(b_{k+1}|b_k, c_k, r_k)P(c_{k+1}|b_k, c_k, r_k), \\ &= P(b_{k+1}|b_k, c_k, r_k)P(c_{k+1}|c_k), \\ &= P(b_{k+1}|b_k, c_k, r_k)P_c[i, j]. \end{aligned}$$

To determine the queue dynamics, which specify $P(b_{k+1}|b_k, c_k, r_k)$, consider the k^{th} time step. Define a time index $t \in \{0, 1, \ldots, t_f\}$ such that $t = 0$ at time k and $t = t_f \triangleq \max(r_k, c_k) - 1$ at some time less than $k+1$. Each $t \in \{0, 1, \ldots, t_f\}$ corresponds to a queue length change due to the arrival of a packet and/or departure of a packet. Define a new variable ζ_t, the intermediate queue length for each t, which is governed by the following recursion.

$$\begin{aligned} \zeta_{t+1} &= \zeta_t + q_t - 1(c_k \geq t), \quad q_t \in \{0, 1\} \\ \zeta_0 &= b_k, \end{aligned}$$

where $q_t = 1$ corresponds to the successful arrival of a packet at time t, and

$$1(c_k \leq t) = \begin{cases} 1, & c_k \leq t, \\ 0, & \text{else.} \end{cases}$$

The successful arrival of a packet is a random variable, which has the following distribution due to the RED algorithm.

$$P(q_t = 0 | \zeta_t, r_k) = \zeta_t / b_{\max}. \tag{A.1}$$

Subject to these dynamics, ζ_t evolves as a Markov process. To simplify notation, define $\mathcal{F}$ as the filtration generated by $\{b_k, c_k, r_k, t\}$. Define the matrix of transition probabilities A_ζ as follows;

$$\begin{aligned} A_{\zeta_t}[i,j] &= P(\zeta_{t+1} = i | \zeta_t = j, \mathcal{F}), \\ &= P(\zeta + q_t - 1(c_k \leq t) = i | \zeta_t = j, \mathcal{F}), \\ &= P(q_t = 1(c_k \leq t) + i - j). \end{aligned}$$

These values are summarized by Table A.1.

Table A.1: State Probability Transitions

Given:			$A_\zeta(t)[i,j] =$		
$r_k > t$	$c_k > t$	$j > 0$	if $i - j + 1 = 0$	if $i - j = 0$	if $i - j - 1 = 0$
true	true	true	$\zeta_t/b_{\max}$	$1 - \zeta_t/b_{\max}$	0
false	true	true	1	0	0
true	false	true	0	$\zeta_t/b_{\max}$	1-$\zeta_t/b_{\max}$
false	false	true	0	1	0
true	true	false	$\zeta_t/b_{\max}$	$1 - \zeta_t/b_{\max}$	0
false	true	false	0	1	0
true	false	false	0	$\zeta_t/b_{\max}$	$1 - \zeta_t/b_{\max}$
false	false	false	0	1	0

Set $\Pi_\zeta(t)[i] \triangleq P(\zeta_t = i | \mathcal{F})$. At time t_f we have,

$$\Pi_\zeta(t_f) = A_\zeta(t_f) A_\zeta(t_f - 1) \ldots A_\zeta(0) \Pi_\zeta(0),$$

where $t_f = \max(c_k, r_k) - 1$. Finally,

$$P(b_{k+1} = i | r_k, c_k, b_k) = \Pi_\zeta(t_f)[i].$$

For (b_k, c_k) pairs assigned to x_k, we can arrange the transition probabilities in a stochastic matrix $A(r_k)$, where, given r_k, $A(r_k)[i,j] = P(x_{k+1} = i | x_k = j, r_k)$. For x_0 with an initial distribution Π_x, it is then apparent that $x_k \sim \text{Markov}(\Pi_x, A(r_k))$ [Nor97].

A.2 Output dynamics

Define random variable z_t as the total number of packets sent through time t, which obeys the following recursion:

$$z_{t+1} = z_t + \eta_t,$$

where η_t corresponds to a packet being sent at time t. Note that so long as capacity is available and packets are waiting to be sent from the router, then $\eta_t = 1$. Hence,

$$\eta_t = 1(c_k > t) * 1(r_k > t \ \cup \ \zeta_t > 0).$$

Again, let $\mathcal{F}$ denote the filtration generated by $\{b_k, c_k, r_k, t\}$. We can write the transition probabilities $P(z_{t+1} = i | z_t = j, \mathcal{F})$ as follows. When $j \geq c_{\max} + 1$,

$$P(z_{t+1} = i | z_t = j, \mathcal{F}) = \begin{cases} 1, & i = j, \\ 0, & \text{else.} \end{cases}$$

Otherwise,

$$\begin{aligned} P(z_{t+1} = i | z_t = j, \mathcal{F}) &= P(i = j + \eta_t(i,j) | \mathcal{F}), \\ &= \sum_{\bar{\zeta}_t} P(i = j + \eta_t(i,j) | \bar{\zeta}_t, \mathcal{F}) P(\bar{\zeta}_t | \mathcal{F}), \\ &\triangleq A_z(t)[i,j], \end{aligned}$$

where $P(\bar{\zeta}_t | \mathcal{F})$ can be recovered from $\Pi_\zeta(t)[\bar{\zeta}_t]$, and with $\eta_t(i,j)$ given by Table A.2.

Table A.2: Output Probability Transitions

Given:			$\eta_t(i,j) =$	
$r_k > t$	$c_k > t$	$\bar{\zeta}_t > 0$	if $i-j=0$	if $i-j-1=0$
true	true	true	0	1
false	true	true	0	1
true	false	true	1	0
false	false	true	1	0
true	true	false	0	1
false	true	false	1	0
true	false	false	1	0
false	false	false	1	0

We have that

$$\Pi_z(t_f) = A_z(t_f)A_z(t_f-1)\dots A_z(0)\Pi_z(0).$$

$$\Pi_z(0)[i] = \begin{cases} 1, & i=1, \\ 0, & \text{else.} \end{cases}$$

As before, $t_f = \max(c_k, r_k) - 1$ and $\Pi_z(t)[i] = P(z_t = i|\mathcal{F})$. Thus,

$$P(y_k = i|r_k, c_k, b_k) = \Pi_z(t_f)[i].$$

Recalling that $(b_k, c_k) \to x_k$, the conditional probability distributions for y_k given x_k can then be arranged in an output transition matrix, $C(r_k)$, where $C(r_k)[i,j] = P(y_k = i|x_k = j, r_k)$.

Combining the state and output transitions, and initializing $x_0 \sim \Pi_x(0)$, we arrive at the HMM description for (x_k, y_k):

$$\begin{aligned} \Pi_x(k+1) &= A(r_k)\Pi_x(k), \quad x_0 \sim \Pi_x(0) \\ \Pi_y(k) &= C(r_k)\Pi_x(k). \end{aligned} \tag{A.2}$$

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