# EL 625 Lecture 10

## Pole Placement and Observer Design

### Pole Placement

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

The solution to this system is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) \tag{2}$$

If the eigenvalues of $A$ all lie in the open left half plane, $\mathbf{x}(t)$ asymptotically approaches the origin as time goes to infinity (the system is *asymptotically stable*). If the eigenvalues of $A$ lie in the closed left half plane (with some eigenvalues possibly on the imaginary axis), the state $\mathbf{x}(t)$ remains bounded (the system is *stable*). If some eigenvalues of $A$ lie in the right half plane, the system is *unstable*.

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{3}$$

The state $\mathbf{x}$ is measured and can be used to calculate a suitable value of $\mathbf{u}$ to make the system stable. The simplest feedback is a static linear state feedback, $\mathbf{u} = K\mathbf{x}$ (where $K$ is a constant matrix). With this feedback,

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ &= A\mathbf{x} + BK\mathbf{x} \\ &= (A + BK)\mathbf{x} \qquad (4) \\ &= A_0\mathbf{x} \qquad (5) \end{aligned}$$

$$A_0 = A + BK \qquad (6)$$

If it is possible to make the eigenvalues of $A_0$ lie in the open left half plane by using the gains $K$, then, choosing $u = K\mathbf{x}$ would make the resulting closed loop system stable. It can be proved that the eigenvalues of $(A + BK)$ can be arbitrarily assigned through a suitable choice of $K$ if the pair $(A, B)$ is controllable.

If(A,B) is a controllable pair, a similarity transformation, $T$ can be found such that the transformed matrices, $\hat{A} = T^{-1}AT$ and $\hat{B} =$

$T^{-1}B$ are in the *controller canonical form*, shown below.

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & \ldots & -a_{n-1} \end{bmatrix} \tag{7}$$

$$\hat{B} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

$a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are the coefficients of the characteristic polynomial (the characteristic polynomial is $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$). The similarity transformation to achieve this 'controller canonical' structure for the single-input case is described below.

$$\text{Characteristic polynomial}, p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \tag{9}$$

$$
\begin{aligned}
&= \lambda \underbrace{\left(\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_1\right)}_{p_1(\lambda)} \\
&\quad + a_0 \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
p_1(\lambda) &= \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_1 \qquad (11) \\
&= \lambda \underbrace{\left(\lambda^{n-2} + a_{n-1}\lambda^{n-3} + \ldots + a_2\right)}_{p_2(\lambda)} \\
&\quad + a_1 \qquad (12) \\
&\vdots \\
p_{n-1}(\lambda) &= \lambda + a_{n-1} \qquad (13) \\
p_n(\lambda) &= 1 \qquad (14)
\end{aligned}
$$

Define the set of vectors $(v_1, \ldots, v_n)$ as

$$
\begin{aligned}
v_1 &= p_1(A)B \\
v_2 &= p_2(A)B \\
&\vdots \\
v_n &= p_n(A)B \qquad (15)
\end{aligned}
$$

$p_i(A)$ is the polynomial $p_i(\lambda)$ evaluated at $\lambda = A$.

To prove that this coordinate transformation achieves the desired transformed form, we have to compute the effect of $A$ on the coordinate vectors, $v_1, v_2, \ldots, v_n$.

$$
Av_1 = A(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2 A + a_1 I)B
$$

$$
\begin{aligned}
&= (A^n + a_{n-1}A^{n-1} + \ldots + a_2A^2 + a_1A) \\
&= (A^n + a_{n-1}A^{n-1} + \ldots + a_2A^2 + a_1A + a_0I - a_0I)B \\
&= -a_0IB \\
&= -a_0B \\
&= -a_0v_n \qquad (16)
\end{aligned}
$$

(By Cayley Hamilton theorem, $p(A) = A^n + a_{n-1}A^{n-1} + \ldots + a_2A^2 + a_1A + a_0I = 0$).

$$
\begin{aligned}
Av_2 &= A(A^{n-2} + a_{n-1}A^{n-3} + \ldots + a_2I)B \\
&= (A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2A)B \\
&= (A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2A + a_1I - a_1I)B \\
&= v_1 - a_1v_n \qquad (17) \\
&\vdots \\
Av_n &= AB \\
&= (A + a_{n-1}I - a_{n-1}I)B \\
&= (A + a_{n-1}I)B - a_{n-1}B \\
&= v_{n-1} - a_{n-1}B \\
&= v_{n-1} - a_{n-1}v_n \qquad (18)
\end{aligned}
$$

Also, $B = v_n$. Hence, in the new coordinates, $\mathbf{z} = T^{-1}\mathbf{x}$ where

$T = [v_1, v_2, \ldots, v_n]$, the transformed matrices would have the form shown in (7) and (8).

Once the matrices, $A$ and $B$ have been transformed to the controller canonical form, designing the state-feedback controller is easy.

Consider

$$\begin{aligned}
\hat{A} + \hat{B}\hat{K} &= \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & \ldots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{k}_0 & \hat{k}_1 & \ldots & \ldots & \hat{k}_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & 0 & 1 \\ -a_0 + \hat{k}_0 & -a_1 + \hat{k}_1 & \ldots & \ldots & \ldots & -a_{n-1} + \hat{k}_{n-1} \end{bmatrix} \qquad (19)
\end{aligned}$$

The characteristic polynomial of $(\hat{A}+\hat{B}\hat{K})$ is $\lambda^n + (a_{n-1} - \hat{k}_{n-1})\lambda^{n-1} + \ldots + (a_0 - \hat{k}_0)$. Thus, by choosing the gains $(\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_{n-1})$ appropriately, the coefficients of the characteristic polynomial can be changed arbitrarily which implies that the eigenvalues can be assigned arbitrarily. Thus, the closed loop poles of the system can be arbitrarily placed using a linear static state feedback, $u = \hat{K}\mathbf{z} = \hat{K}T^{-1}\mathbf{x} = K\mathbf{x}$.

Example:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{20}$$

where

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{21}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{22}$$

The characteristic polynomial of A is

$$p(\lambda) = \lambda^3 - \lambda^2 + \lambda - 1 \tag{23}$$

The polynomials $p_1(\lambda), p_2(\lambda)$ and $p_3(\lambda)$ are

$$p_1(\lambda) = \lambda^2 - \lambda + 1 \tag{24}$$

$$p_2(\lambda) = \lambda - 1 \tag{25}$$

$$p_3(\lambda) = 1 \tag{26}$$

Hence, the vectors $v_1, v_2$ and $v_3$ are

$$\begin{aligned} v_1 &= p_1(A)B \\ &= (A^2 - A + I)B \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \tag{27}$$

$$\begin{aligned} v_2 &= p_2(A)B \\ &= (A - I)B \\ &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{aligned} \tag{28}$$

$$\begin{aligned} v_3 &= p_3(A)B \\ &= IB \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \tag{29}$$

The transformation matrix, $T$ is

$$T = \; = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \tag{30}$$

In the transformed coordinates, $\hat{A}$ and $\hat{B}$ are

$$\begin{aligned} \hat{A} &= T^{-1}AT \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \end{aligned} \tag{31}$$

$$\begin{aligned} \hat{B} &= T^{-1}B \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{32}$$

$\hat{A}$ and $\hat{B}$ are now in the controller canonical form.

The characteristic polynomial of $(\hat{A}+\hat{B}\hat{K})$ is $\lambda^3+(-1-k_2)\lambda^2+(1-k_1)\lambda+(-1-k_0)$. If we want the closed loop poles to be $-5, -1+j$ and $-1-j$, the closed loop characteristic polynomial must be $\lambda^3+7\lambda^2+12\lambda+10$. Hence, choose the gains $\hat{k}_0, \hat{k}_1$ and $\hat{k}_2$ as

$$\hat{k}_0 = -11 \tag{33}$$

$$\hat{k}_1 = -11 \tag{34}$$

$$\hat{k}_2 = -8 \tag{35}$$

Hence, the state feedback, $u = \hat{K}\mathbf{z}$ achieves the required pole placement.

$$\begin{aligned} u &= \hat{K}\mathbf{z} && (36) \\ &= \hat{K}T^{-1}\mathbf{x} = K\mathbf{x} \\ \Longrightarrow K &= \hat{K}T^{-1} && (37) \\ &= \begin{bmatrix} -11 & -11 & -8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -16 & 8 \end{bmatrix} && (38) \end{aligned}$$

To verify, the eigenvalues of $(A + BK)$ can be computed.

$$\begin{aligned} (A + BK) &= \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & -16 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & -2 \\ -3 & -15 & 8 \\ -2 & -17 & 9 \end{bmatrix} && (39) \end{aligned}$$

The characteristic polynomial of $(A + BK)$ is $\lambda^3 + 7\lambda^2 + 12\lambda + 10$, as desired.

## State Reconstruction: Asymptotic Observers

In the previous lecture, static full state feedback was employed to modify the eigenvalues and hence the dynamic response of a linear time-invariant system. This was possible when the realization {A,B,C} was controllable and all state variables were available for feedback purposes. However, measurements of all the states may not be feasible or impossible. In some cases, the estimate of the state variables may be desirable rather than their direct measurement since the sensor/transducer required to measure that state may be susceptible to excessive noise or it is hard if not impossible to physically measure some particular state variables. In these cases, we are interested in reconstructing the state variables. It will be shown that the states may be obtained from the knowledge of the system input and the system output. Furthermore, in a later section it will be shown that the problem of state estimation is dual to the state feedback problem and states can be attained if the system { A,B,C} is observable.

The states may also be recovered through differentiation of the output; however, this is undesirable. We will construct a dynamical system called an asymptotic observer to estimate the states of a system. The observer is named asymptotic since the estimates converge to the true states after a transient period. Having attained an asymptotic observer, it will be utilized to implement state feedback. It will be shown that the overall closed-loop system will be internally stable when an observer is utilized in the feedback loop to reconstruct the states for feedback purposes. However, it should be noted that utilization of an observer will deteriorate the transient response of a system. This should be expected since the observer will provide a "good" estimate of the states after a transient period.

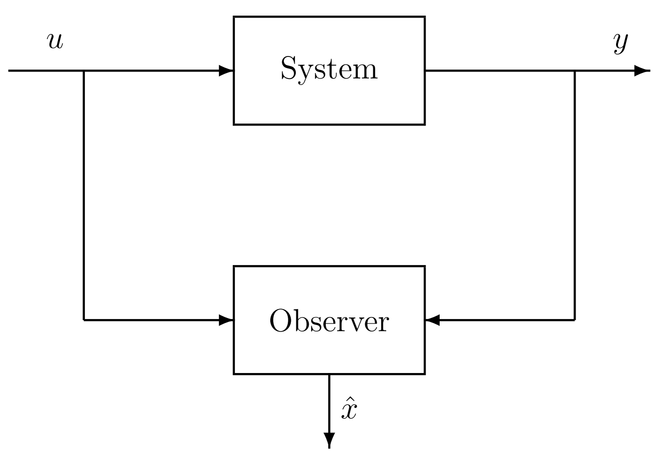


Figure 1: An observer.

# 1 Full Order Observers

Consider a linear time-invariant system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu && (40) \\ y &= Cx \quad . && (41) \end{aligned}$$

Differentiating the output map, we obtain

$$\begin{aligned} \dot{y} &= C\dot{x} = CAx + CBu && (42) \\ \ddot{y} &= CA\dot{x} + CB\dot{u} = CA^2x + CABu + CB\dot{u} \quad . && (43) \end{aligned}$$

Therefore, repeated differentiation of the output will yield

$$\mathcal{Y} = \Gamma_n x + \mathcal{T}\mathcal{U} \quad . \qquad (44)$$

where $\Gamma_n$ is the observability matrix and

$$\mathcal{Y} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}, \qquad \mathcal{U} = \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}, \qquad (45)$$

$$\mathcal{T} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ CB & 0 & 0 & \cdots & & 0 \\ CAB & CB & 0 & \cdots & & 0 \\ CA^2B & CAB & CB & 0 & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ CA^{n-2}B & CA^{n-3}B & & \cdots & CB & 0 \end{bmatrix}. \tag{46}$$

The matrix $\mathcal{T}$ is called a lower triangular Toeplitz matrix.

Given the realization of a system, i.e., {A,B,C}, and its input and output signals, the states may be recovered through the differentiation of the input and output signals. Of course, this is possible if the system is observable. This can be easily verified from (44). The set of equations derived in (44) is linear and has a solution if the vector $\mathcal{Y} - \mathcal{T}\mathcal{U}$ is contained in the range space of the linear transformation $\Gamma_n$. Therefore, if $\Gamma_n$ has full rank, then this system of equations may be solved for the states. As shown before, $\Gamma_n$ having full rank corresponds to the observability of the system. This process for determining states is undesirable since derivative of input and output signals are necessary. In the next sections, dynamical systems, namely observers, are introduced for state estimation.

### 1.1 Open-Loop Observers

Consider the following dynamical system

$$\dot{\hat{x}} = A\hat{x} + Bu \ ; \qquad \hat{x}(0) = \hat{x}_0 \tag{47}$$

$$\hat{y} = C\hat{x}. \tag{48}$$

The above system is an observer for the system given by (40)-(41). This observer is a replica of the original system (Figure (2)) and utilizes the knowledge of the system dynamics (i.e., the triple {A,B,C}), and the input to the system. However, the initial conditions for the observer are the crucial information needed. If the initial conditions of the original system, namely $x(0) = x_0$, are exactly known, then the observer will produce the state variables exactly. However, due to disturbances and in many cases only a partial knowledge of the initial conditions and dynamics of the system at hand, there exists a discrepancy between the initial conditions of the original system and the observer. Therefore, it is important to study the effect of an error in the knowledge of the initial conditions.

Define the error between the exact value of the state and its estimate at time $t$ as $e(t)$, i.e.,

$$e(t) = \hat{x}(t) - x(t). \tag{49}$$

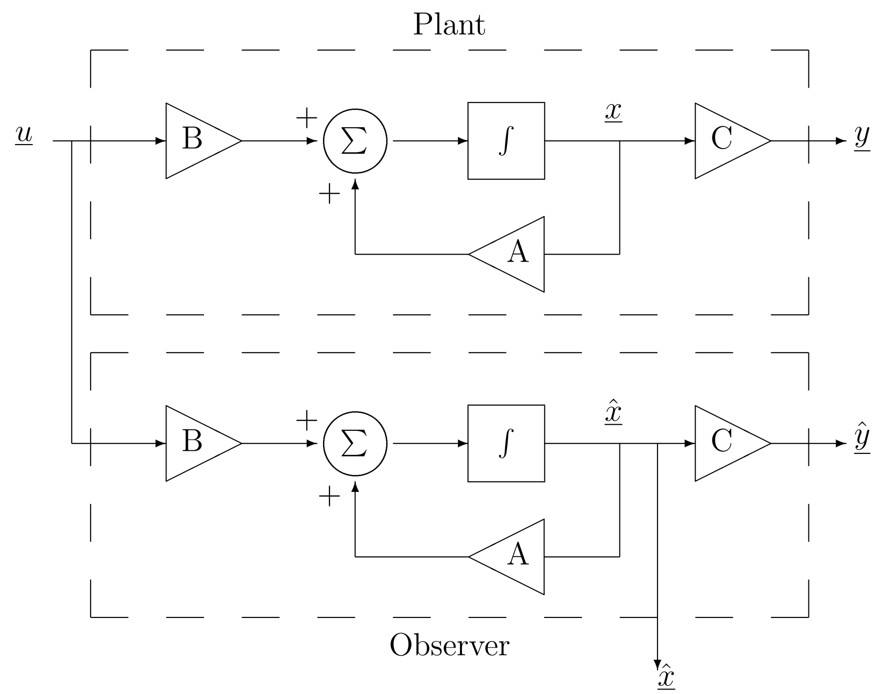


Figure 2: Open-loop Observer.

The dynamics of the error are

$$\dot{e} = \dot{\hat{x}} - \dot{x} = A(\hat{x} - x) = Ae \quad ; \qquad e(0) = \hat{x}_0 - x_0 \ . \tag{50}$$

From (50), it is easy to see that if the initial conditions are exactly known, then the error is identically zero. However, if the initial condition on the error is nonzero, then the behavior of the error is governed by (50). Therefore, if the open-loop system is unstable, the error will grow unbounded and the estimate of the states will not converge to the true state variables. Even if the open-loop system is asymptotically stable, the behavior of the error dynamics will be sluggish if the open-loop system has eigenvalues close to the $j\omega$-axis. Since we are interested in using the estimates of the states in a full state feedback design, if the error dynamics are sluggish and the error persists for a long time, then the overall controlled system performance will be unsatisfactory. In the next section, we will introduce a scheme to alleviate this problem.

### 1.2 Closed-Loop Observers

Recall that the original problem was to estimate the state variables of a linear time-invariant system knowing its dynamics, and its input and output signals. In the proposed observer scheme in the previous section, we utilized all the given information excluding the knowledge of the output measurements. This is the reason why the above scheme is called an open-loop observer. Therefore, we propose to use a modified version of the open-loop observer that makes use of the output measurements.

Consider the following dynamical system

$$\dot{\hat{x}} = F\hat{x} + G_1 u + G_2 y \quad . \tag{51}$$

Define the error as before (i.e., given as in (49)). Then, the error dynamics in this case are

$$\begin{aligned}
\dot{e} &= \dot{\hat{x}} - \dot{x} = F\hat{x} + G_1 u + G_2 y - Ax - Bu \\
&= F(x+e) + (G_1 - B)u + G_2 C x - Ax \\
&= Fe + (F + G_2 C - A)x + (G_1 - B)u \quad . \qquad (52)
\end{aligned}$$

Now, the matrices $F$ and $G_1$ may be chosen as

$$F = A - G_2 C \qquad (53)$$

$$G_1 = B \quad . \qquad (54)$$

Substituting (53) and (54) in (51) and (52), we obtain

$$\dot{\hat{x}} = (A - G_2 C)\hat{x} + Bu + G_2 y \qquad (55)$$

and

$$\dot{e} = Fe = (A - G_2 C)e \quad . \qquad (56)$$

Equations (55) and (56) describe the dynamics of the observer and the error respectively. First we will examine the error dynamics and then we will discuss the observer topology. As derived, the error dynamics depend on the system dynamics through the pair {A,C} and the unknown gain $G_2$. Since the unknown gain $G_2$ is chosen by the designer, there is a chance to modify the dynamics of the error. Therefore, the question is: given a pair {A,C}, is it possible to choose a matrix $G_2$ such that the eigenvalues of the matrix $A - G_2C$ lies in the open left-half plane of the complex plane?

The answer to the above question is yes if the pair {A,C} is an observable pair (or the original system is observable). The above question is similar to the pole placement by full state feedback encountered in the previous chapter. There we considered placing the eigenvalues of $A + BK$ when the pair {A,B} were given and the matrix $K$ needed to be chosen. The pole placement was indeed solvable if and only if the system was controllable. The current problem turns out to be a dual problem of the pole placement, and as we should expect the controllability condition in the case of pole placement should be changed to the observability in the observer problem.

Knowing that the eigenvalues of a matrix and its transpose are the same, we may consider pole placement for the transpose of the error dynamics matrix (i.e., $F = A - G_2C$) instead of pole placement for $A - G_2C$, i.e.,

$$det(\lambda I - F) = det[\lambda I - (A - G_2C)] = det[\lambda I - (A^T - C^T G_2^T)] = det(\lambda I - F^T) \quad .$$

From the above observation, ensuring that the eigenvalues of $A - G_2C$ are in the open left-half plane is equivalent to pole placement for the pair $\{A^T, C^T\}$, i.e., find a gain $K$ such that the eigenvalues of $A^T + C^T K$ are arbitrarily placed in the open-left half plane. After finding this gain through the techniques described in the previous chapter for pole placement through state feedback, the observer gain is given by

$$G_2 = -K^T \quad . \tag{57}$$

To further study the structure of the observer, rewrite the observer dynamics given by (55) as

$$\dot{\hat{x}} = A\hat{x} + Bu + G_2 y - G_2 C\hat{x} \quad . \tag{58}$$

Now define an output for the observer of the form $\hat{y} = C\hat{x}$. Therefore, (58) may be rewritten as

$$\dot{\hat{x}} = A\hat{x} + Bu + G_2(y - \hat{y}) \tag{59}$$

$$\hat{y} = C\hat{x} \quad . \tag{60}$$

It is interesting to note that the above observer is similar to the open-loop observer (or dynamics of the original plant) with the difference that the observer dynamics contain a term corresponding to the output error. For this reason the observer has been named a closed-loop observer since the dynamics of the observer is being driven by the error between the actual output of the system and the estimated output by the observer. This term corresponding to the output error is also known as the innovation process (especially in the Kalman Filtering problem). Henceforth, through the proper choice of the error gain, the error dynamics may be stabilized.

The choice of the observer poles is an important one since the observer should generate the estimates of the state variables in real-time for control purposes. Therefore, the observer poles should be selected to be faster than the closed-loop poles of the overall system. A figure of merit is that the observer time constants should be three to five times faster than the closed-loop system time constants. The reasons for not choosing the observer poles far in the left-half complex plane are: 1) larger gains in the observer and hence saturation, and 2) increased bandwidth which makes the observer more susceptible to sensor noise occurring at high frequencies. Another important issue is the robustness of the observer and hence the closed-loop system to parameter variations in the plant dynamics.

**Example .1** Consider a voltage driven DC motor where its shaft position is being measured and we are interested in designing an observer to estimate the shaft velocity of the motor. Ignoring the electrical time-constant of the motor, the dynamics are:

$$J_m\ddot{\theta} + (B_m + \frac{K_m K_b}{R})\dot{\theta} = \frac{K_m}{R}V. \tag{61}$$

Choosing the shaft position and velocity as the state variables, i.e., $x_1 = \theta$ and $x_2 = \dot{\theta}$, the state variable representation of the dynamics is

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & -\tau \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{62}$$

where $\tau = (B_m + \frac{K_m K_b}{R})/J_m$ and $u = \frac{K_m}{RJ_m}V$.

In this case, the output is the shaft position, i.e.,

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Therefore, the full-order observer is

$$\begin{aligned} \dot{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}} &= \begin{bmatrix} 0 & 1 \\ 0 & -\tau \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} (y - \hat{y}) \qquad (63) \\ \hat{y} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}. \end{aligned}$$

The gain vector $G = [g_1 \;\; g_2]^T$ needs to be chosen such that the error dynamics are asymptotically stable. Let's choose the eigenvalues of the error dynamics at $-5\tau$ and $-6\tau$. Then, the gain matrix may be found. Consider

$$\{A^T, C^T\} = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & -\tau \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Then, pole placement techniques may be utilized to place the poles at $-5\tau$ and $-6\tau$ and obtain the gain as $G = -K^T$. Since the system has only two states, it is rather easy to proceed directly to place the poles of $A - GC$, i.e.,

$$A - GC = \begin{bmatrix} 0 & 0 \\ 1 & -\tau \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -g_1 & 1 \\ -g_2 & -\tau \end{bmatrix}$$

$$\longrightarrow \sigma(A-GC) = \{\lambda | \lambda^2 + (g_1+\tau)\lambda + g_1\tau + g_2 = 0\} = \{-5\tau, -6\tau\}.$$

Therefore,

$$\begin{cases} g_1 + \tau = 11\tau \\ g_1\tau + g_2 = 30\tau^2 \end{cases} \longrightarrow \begin{cases} g_1 = 10\tau \\ g_2 = 20\tau^2 \end{cases}$$

The observer is given as follows:

$$\dot{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & -\tau \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 10\tau \\ 20\tau^2 \end{bmatrix} (y - \hat{y}) \qquad (64)$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}.$$

## 2 Reduced Order Observers

The state space dimension of the observers mentioned in the previous section is the same as the original plant. For this reason, these observers are named full order observers. However, the output measurements may contain some of the state variables directly, if not linear combinations of the states. Therefore, it is natural to wonder if the dimension of the observer may be reduced by an intelligent use of the output vector. The answer to this question is yes! To show this, we first assume that through an appropriate change of coordinates the output matrix is rewritten in the following form:

$$C = [I_{n_1} \;\; | \;\; 0] \tag{65}$$

where $I_{n_1}$ represents an $n_1$ by $n_1$ identity matrix and $n_1$ is the row rank of the original output matrix.

Partition the original plant dynamics as follows:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \ . \tag{66}$$

where $x_1 \in \mathcal{R}^{n_1 \times n_1}$.

The choice of the output equation guarantees that the first $n_1$ states are measured directly. Therefore, the states $x_2$ need to be estimated. Hence, a reduced order observer of dimension $n - n_1$ should be sufficient where $n$ is the dimension of the state space of the plant. From the previous analysis on the full order observers, the following choice for the dynamics of the reduced order observer is taken

$$\dot{\hat{x}}_2 = F\hat{x}_2 + G_1 u + G_2 y + G_3 \dot{y} \quad . \tag{67}$$

From the above dynamics, it seems that the derivative of the output is also an input to the reduced order observer and as discussed before this is not desirable. However, the term containing the derivative of the output may be eliminated by an appropriate change of coordinates. These coordinates are

$$z = \hat{x}_2 - G_3 y \quad . \tag{68}$$

Utilizing this new choice of state variables, the observer dynamics may be rewritten as

$$\dot{z} = Fz + G_1 u + (G_2 + FG_3)y \quad . \tag{69}$$

At this point, the error dynamics need to be examined. Define the error as before, i.e.,

$$e = \hat{x}_2 - x_2 \quad .$$

Then

$$\begin{aligned}\dot{e} &= F\hat{x}_2 + G_1 u + G_2 y + G_3 \dot{y} - (A_{21}x_1 + A_{22}x_2 + B_2 u) \\ &= Fe + (G_1 - B_2 + G_3 B_1)u + (G_2 + G_3 A_{11} - A_{21})x_1 \\ &\qquad +(F + G_3 A_{12} - A_{22})x_2. \end{aligned} \tag{70}$$

Therefore, the following choice of the gain matrices will make the error dynamics independent of the other signals in the system (i.e., $\dot{e} = Fe$)

$$G_1 = B_2 - G_3 B_1 \tag{71}$$

$$G_2 = A_{21} - G_3 A_{11} \tag{72}$$

$$F = A_{22} - G_3 A_{12} \quad . \tag{73}$$

The observer design is reduced to finding a gain, namely $G_3$, such that the error dynamics are asymptotically stable, i.e., $\sigma(F) \subset \mathcal{C}^-$. It is worthwhile to note that this problem is very similar to the finding the gains for the full order observer which in turn is dual to the pole placement problem. In the case of full order observer design, it was proved that the observability of the original system guaranteed that the eigenvalues of the error dynamics may be placed arbitrarily. For the problem at hand, it seems that the pair $\{A_{22}, A_{12}\}$ need to be an observable pair. However, it can be proved that the original plant dynamics are observable if and only if the pair $\{A_{22}, A_{12}\}$ is observable. Therefore, the observability of the original system is necessary and sufficient to attain an asymptotic reduced order observer.

## 3 Observer-Based State Feedback: Separation Principle

In the previous sections, observers were studied to reconstruct the states of a dynamical system. Having attained the full states, state feedback may be utilized to shape the dynamic response of the system (Figure (3)). However, the overall stability of the closed-loop system need to be examined when an observer (full-order or reduced-order) is injected in the feedback loop. In this section, we will show that the observer dynamics may be realized independent of the gains for the state feedback. Therefore, the state feedback gains may be calculated as if the full states were accessible. Furthermore, the observer gain may be chosen independent of the selection of controller gains. This result is known as the *separation theorem* since the gains for the observer and the controller may be calculated independently. We will prove this result in the analysis to follow.

Consider the state feedback resulting from the use of the estimates

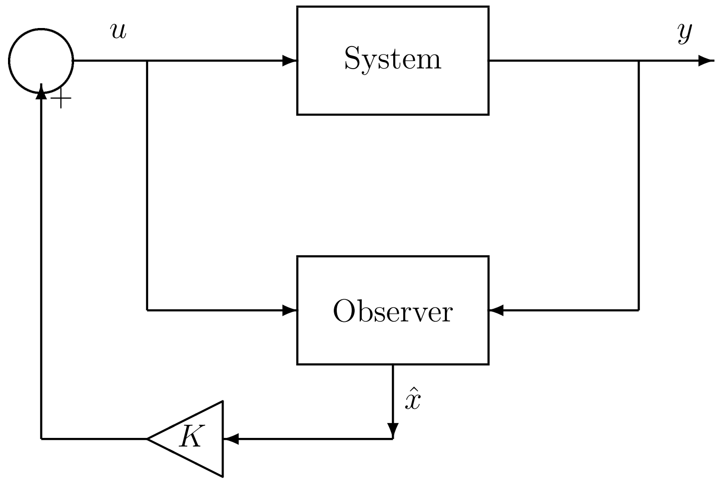


Figure 3: An observer-based state feedback.

of the states generated by the observer, i.e.,

$$u = K\hat{x} \tag{74}$$

where $\hat{x}$ is the states of the observer.

The closed-loop dynamics are as follows:

$$\begin{aligned}\dot{x} &= Ax + BK\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + G(y - \hat{y}) = (A + BK - GC)\hat{x} + GCx\end{aligned}$$

or

$$\dot{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}} = \begin{bmatrix} A & BK \\ A + BK - GC & GC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{75}$$

The above system can be rewritten in a triangular structure by utilizing the following similarity transformation:

$$T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}. \tag{76}$$

Transform the system given by (75) into its new coordinates, namely $Z = TX$ where $X = [x^T \ \ \hat{x}^T]^T$, by the the transformation given in (76). Then,

$$\dot{Z} = \hat{A}Z \tag{77}$$

where

$$\begin{aligned} \hat{A} &= TAT^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} A & BK \\ GC & A + BK - GC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \\ &= \begin{bmatrix} A + BK & BK \\ 0 & A - GC \end{bmatrix}. \end{aligned}$$

Since $\hat{A}$ has a triangular structure, its eigenvalues are

$$\sigma(\hat{A}) = \sigma(A + BK) \cup \sigma(A - GC). \tag{78}$$

From (78), we observe that the eigenvalues of the overall closed-loop system is composed of the eigenvalues of the open-loop system with a state feedback (as if the true states are available) and the eigenvalues of the error dynamics for the observer. This is a very nice result since the implication is that the pole-placement and the observer design can be performed independent of each other. This result is known as the *separation principle* or *separation property*.