# Eigenvalues and Eigenfunctions

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***Abstract -*** The article describes the eigenvalue and eigenfunction problems. Basic properties, some applications and examples in system analysis are provided.

***Keywords -*** Eigenvalues, Eigenvectors, Eigenfunctions, Singular Decomposition and Singular Values, State Matrix, Jordan Form, Characteristic Equation and Polynomial, Eigenvalue Localization, Modal Analysis, Participation Factors, Eigenvalue Sensitivity, Observability, QR Transformation, Bifurcations, Small-Signal Stability, Damping of Oscillations.

## 1. DEFINITION OF EIGENVALUE AND EIGENFUNCTION

Many physical system models deal with a square matrix $A = [a_{i,j}]_{n \times n}$ and its eigenvalues and eigenvectors. The eigenvalue problem aims to find a nonzero vector $x=[x_i]_{1 \times n}$ and scalar $\lambda$ such that satisfy the following equation:

$$Ax = \lambda x \tag{1.1}$$

where $\lambda$ is the *eigenvalue* (or characteristic value, or proper value) of matrix $A$, and $x$ is the corresponding right *eigenvector* (or characteristic vector ,or proper vector) of $A$.

The necessary and sufficient condition for the above equation to have a non-trivial solution for vector $x$ is that the matrix $(\lambda I - A)$ is singular. Equivalently, the last requirement can be rewritten as a *characteristic equation* of $A$:

$$det(\lambda I - A) = 0 \tag{1.2}$$

where $I$ is the identity matrix. All $n$ roots of the characteristic equation are all $n$ eigenvalues $[\lambda_1, \lambda_2, .., \lambda_n]$. Expansion of $det(\lambda I - A)$ as a scalar function of $\lambda$ gives the *characteristic polynomial* of $A$:

$$L(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 \tag{1.3}$$

where $\lambda^k$, $k=1,...,n$, are the corresponding $k$-th powers of $\lambda$, and $a_k$, $k=1,...,n$, are the coefficients determined via the elements $a_{ij}$ of $A$.

Each eigenvalue also corresponds to a left eigenvector $y$ which is the right eigenvector of matrix $A^T$ where the superscript $^T$ denotes the transpose of $A$. The left eigenvector satisfies the equation

$$(\lambda I - A^T)l = 0 \tag{1.4}$$

The set of all eigenvalues is called the *spectrum* of $A$.

*Eigenfunction* is defined for an operator in the functional space. For example, oscillations of an elastic object can be described by the equation:

$$\varphi'' = L\varphi \tag{1.5}$$

where $L\varphi$ is some differential expression. If a solution of (1.5) has the form $\varphi = T(t)u(x)$, then with respect to function $u(x)$ the following equation holds

$$L(u) + \lambda u = 0 \tag{1.6}$$

In a restricted region and under some homogenous conditions on its boundary, parameter $\lambda$ is called eigenvalue, and nonzero solutions of (1.6) are called eigenfunctions. More description of eigenfunction is given in the sequel [12,13,16,24,41-43].

Along with the eigenvalues, *singular values* are often used. If a matrix (m×n) can be transformed in the following form:

$$U^*AV = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } S = diag[\sigma_1, \sigma_2, ... , \sigma_r] \tag{1.7}$$

where $U$ and $V$ are (m×m) and (n×n) orthogonal matrix respectively, and all $\sigma_k \geq 0$, then expression (1.7) is called singular value decomposition. The values $\sigma_1, \sigma_2, ... , \sigma_r$ are called singular values of $A$, and r if the rank of $A$. If $A$ is a symmetric matrix, then matrices $U$ and $V$ coincide, and $\sigma_k$ are equal to the absolute values of eigenvalues of $A$. The singular

decomposition (1.7) is often used in the least square method, especially when $A$ is ill-conditioned [12].

## 2. SOME PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Eigenvalues of a real matrix appear as real numbers or complex conjugate pairs.

A symmetric real matrix has all real eigenvalues.

The product of all eigenvalues of $A$ equals to the determinant of $A$, i.e.

$$\lambda_1\lambda_2\cdots\lambda_n = \det A \tag{2.1}$$

Eigenvalues of a triangle or diagonal matrix are the diagonal components of the matrix.

The sum of all eigenvalues of a matrix equals to its *trace*, i.e.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr}A = a_{11} + a_{22} + \cdots + a_{nn} \tag{2.2}$$

Eigenvalues for $A^k$ are $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$, e.g. eigenvalues for $A^{-1}$ are $\lambda_1^{-1}, \lambda_1^{-1}, \cdots, \lambda_1^{-1}$.

A symmetric matrix $A$ can be put in a diagonal form with eigenvalues as the elements alone the diagonal as shown below:

$$A = T\Lambda T^* = T\,\text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]T^* \tag{2.3}$$

where $T = [t_{ij}]_{\text{nxn}}$ is the *transformation matrix* and $T^*$ is its complex conjugate transpose matrix, $T^* = [t_{ji}^*]_{\text{nxn}}$.

However non-semi-simple matrices can not be put into diagonal form, though, they can be put into the so-called *Jordan form*. For a non-semi-simple *multiple eigenvalue* $\lambda$, the eigenvector $u^1$ is dependent on *(m-1)* generalized eigenvectors $u^2, .., u^m$:

$$Au^1 = \lambda u^1$$

$$Au^2 = \lambda u^2 + u^1$$

$$\text{.......} \tag{2.4}$$

$$Au^m = \lambda u^m + u^{m-1}$$

$$Au^{m+1} = \lambda_{m+1}u^{m+1}$$

$$\dots\dots$$

$$Au^n = \lambda_n u^n$$

From these equations, the matrix form can be obtained as below:

$$T^{-1}AT = J \tag{2.5}$$

where $J$ is A matrix containing a *Jordan block,* and $T$ is the *modal matrix* containing the generalized eigenvectors, $T = [u^1 u^2, ..., u^m, ..., u^n]$. For example, when the number of multiple eigenvalues is 3, matrix $J$ takes the form:

$$J = \begin{bmatrix} \lambda & \delta & & & & \\ & \lambda & \delta & & 0 & \\ & & \lambda & & & \\ & & & \lambda_{m+1} & & \\ & 0 & & & \lambda_{m+2} & \\ & & & & & \lambda_n \end{bmatrix} \tag{2.6}$$

where $\delta = 0$ or $1$ [12].

### 3. EIGENVALUE ANALYSIS FOR ORDINARY DIFFERENTIAL EQUATIONS

Eigenvalue approach is applied to *solving the ordinary differential equations* (ODE) given in the following linear form:

$$\frac{dx}{dt} = Ax + Bu \tag{3.1}$$

where $A$ is the *state matrix* and $u$ is the vector of controls. When $A$ is a matrix with all different eigenvalues $\lambda_i$ and (3.1) is homogeneous (that is $u=0$), then a solution of (3.1) can be found in the following general form:

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \tag{3.2}$$

where $c_i$ are coefficients that are determined by the initial conditions $x(0)$. For the case of $m<n$ different eigenvalues, the general solution of (3.1) for $u = 0$ is

$$x(t) = \sum_{i=1}^{m} \sum_{k=0}^{K_m-1} c_{ik} t^k e^{\lambda_i t} \tag{3.3}$$

where $K_m$ is the multiplicity of $\lambda_t$. If the system is inhomogeneous (that is u is nonzero), a solution of (3.1) can be found as a sum of a general solution for the homogeneous system (3.2) or (3.3) and a particular solution of the inhomogeneous system.

The elements of (3.2) and (3.3) corresponding to each real eigenvalue $\lambda_i = \alpha_i$ or to each pair of complex conjugate eigenvalues $\lambda_i = \alpha_i \pm j\omega_i$ are called *aperiodic and oscillatory modes* of the system motion respectively. The eigenvalue real part $\alpha_i$ is called damping of the mode $i$, and the imaginary part $\omega_i$ determines the frequency of oscillations.

When $A$ is a matrix with all different eigenvalues, by substituting

$$x = Tx', \quad u = Tu' \tag{3.4}$$

the original ODE can be transformed into:

$$Tdx'/dt = ATx' + Tu' \tag{3.5}$$

If $T$ is a nonsingular matrix chosen so that

$$T^{-1}AT = \Lambda = \text{diag} \ [\lambda_1, \lambda_2, .., \lambda_n] \tag{3.6}$$

Then we get a modal form of ODE:

$$dx'/dt = T^{-1}ATx' + u' = \Lambda x' + u' \tag{3.7}$$

In the modal form, state variables $x'$ and equations are independent. Diagonal elements of the matrix $\Lambda$ are eigenvalues of $A$, which can be used to solve ODE [12].

For a general $n$-th order differential equation as given below:

$$A_n \frac{d^n x}{dt^n} + A_{n-1} \frac{d^{n-1}x}{dt^{n-1}} \cdots + A_1 \frac{dx}{dt} + A_0 x = 0 \tag{3.8}$$

Besides solving it through transfering it into set of first order differential equations [12, 41], it can Laos be solved using the original coordinate. The matrix polynomial of system (3.8) is given below:

$$L(\lambda) = A_n \lambda^n + A_{n-1} \lambda^{n-1} + ... + A_1 \lambda + A_0 \tag{3.9}$$

The solutions and eigenvalues as well as eigenvectors of the system (3.8) can be obtained by solving the eigenvalue equation:

$$L(\lambda) \, u = 0 \tag{3.10}$$

where $L(\lambda)$ is the matrix (3.9) containing an eigenvalue $\lambda$ having the corresponding eigenvector $u$. If vectors $u^1$, $u^2$, ..., $u^m$, where $m < n$, satisfies the equation:

$$L(\lambda)u^1 = 0$$
$$L(\lambda)u^2 + \frac{1}{1!}\frac{dL(\lambda)}{d\lambda}u^1 = 0$$
$$\dots \qquad\qquad (3.11)$$
$$L(\lambda)u^m + \frac{1}{1!}\frac{dL(\lambda)}{d\lambda}u^{m-1} + \cdots + \frac{1}{(m-1)!}\frac{d^{m-1}L(\lambda)}{d\lambda^{m-1}}u^1 = 0$$

then

$$x(t) = (t^{m-1}u^1/(m-1)! + \dots + t\,u^{m-1}/1! + u^m)\,e^{\lambda t} \qquad\qquad (3.12)$$

is a solution to the ODE system [12]. The set of equations (3.11) defines the *Jordan Chain* of the multiple eigenvalue $\lambda$ and the eigenvector $u^1$.

## 4. EIGENVALUES AND EIGENFUNCTIONS FOR INTEGRAL EQUATIONS

An integral equation takes the following general form [12]:

$$x(t) = f(t) + \lambda \int_0^t k(\zeta, t)x(\zeta)d\zeta \qquad\qquad (4.1)$$

In eigenanalysis, our concentration is paid to the integral equation, which can be rewritten as:

$$x(t) = f(t) + \lambda \int_a^b \sum_{i=1}^{n} r_i(\zeta)q_i(\zeta)x(\zeta)d\zeta \qquad\qquad (4.2)$$

where

$$\int_a^b r_i(\zeta)q_i(\zeta)x(\zeta)d\zeta = const. \qquad\qquad (4.3)$$

So equation (4.2) can be reduced to the following problem:

$$x(t) = f(t) + \lambda \sum_{j=1}^{n} x_j q_j(t) \qquad\qquad (4.4)$$

By substitution, it can be drawn:

$$x_i = \int_a^b r_i(\zeta)[f(t) + \lambda \sum_{j=1}^{n} x_j q_j(t)]d\zeta, i = 1,2,\dots,n \qquad\qquad (4.5)$$

The equation of the system can be obtained as:

$$(I - \lambda A)x = b \qquad\qquad (4.6)$$

where

$$\boldsymbol{x} = [x_1, x_2, ..., x_n]^T,$$

$$A = [a_{i,j}] = [\int_a^b r_i(\zeta)q_j(\zeta)d\zeta]$$

$$b = [b_i] = [\int_a^b r_i(\zeta)q_j(\zeta)d\zeta]$$

The values of $\lambda$ which satisfy

$$det\,[I - \lambda A] = 0 \qquad\qquad (4.7)$$

are the eigenvalues of the integral equation.

To find $x(t)$ by solving an integral equation similar to (4.1) except for the interval which is $[a, b]$ instead of $[0, t]$, the eigenfunction approach can be also used. First, $x(t)$ is rewritten as:

$$x(t) = f(t) + \sum_{n=1}^{\infty} a_n \phi_n(t) \qquad\qquad (4.8)$$

where $\phi_1(t)$, $\phi_2(t)$, ... are eigenfunctions of the system, and satisfy

$$\phi(t) = \lambda_n \int_a^b k(\zeta, t)\phi_n(\zeta)d\zeta \qquad\qquad (4.9)$$

where $\lambda_1, \lambda_2, ...$ are eigenvalues of the integral equation. After substituting the eigenfunction into the integral equation and further simplification, the solution $x(t)$ is obtained as:

$$x(t) = f(t) + \sum_{n=1}^{\infty} \frac{\lambda f_n}{\lambda_n - \lambda}\phi_n(t) \qquad\qquad (4.10)$$

where $f_n = \int_a^b f(\zeta)\phi(\zeta)d\zeta$ .

## 5. LINEAR DYNAMIC MODELS AND EIGENVALUES

*A) State space modeling*

In control systems, where the purpose of control is to make a variable adhere to a particular value, the system can be modeled by using the *state space equation* and *transfer functions*. The state space equation is given below:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

(5.1)

where $x$ is the $(n \times 1)$ vector of *state variables*, $\dot{x}$ is its first order derivative vector, $u$ is the $(p \times 1)$ *control vector*, and $y$ is the $(q \times 1)$ *output vector*. Accordingly, $A$ is the $(n \times n)$ *state matrix*, $B$ is a $(n \times p)$ matrix, $C$ is a $(q \times n)$ matrix, and D is of $(q \times p)$ dimension.

### 0   Model analysis on the base of eigenvalues and eigenvectors

Model analysis is based on the *state space representation* (5.1). It also explores *eigenvalues*, *eigenvectors* and *transfer functions* [15, 28, 33].

Consider a case where matrix $D$ is a zero matrix. Then the state space model can be transformed using *Laplace transformation* in a *transfer function* that maps input into output:

$$G(s) = C(sI - A)^{-1}B$$

(5.2)

where $s$ is the Laplace complex variable, and $G(s)$ is composed of denominator $a(s)$ and a numerator $b(s)$:

$$G(s) = b(s)/a(s) = (b_0 s^n + b_1 s^{n-1} + \dots b_n )/(s^n + a_1 s^{n-1} + \dots a_n)$$

(5.3)

The *closed-loop transfer function* for a *feedback system* is:

$$G_c(s) = [1+G(s)H(s)]^{-1}G(s)$$

(5.4)

where H(s) is a feedback transfer function.

The system model (5.1) can be analyzed using the observability and controllability concepts. *Observability* indicates whether all of the system's modes can be observed by monitoring only the sensed outputs. *Controllability* decides whether the system state can be moved from an initial point to any other point in the state space within infinite time, and if every mode is connected to controlled input. They can be described more precisely as follows [15,28,44]:

1)   For a linear system, if within a infinite time interval, $t_0 < t < t_1$, there exist an piecewise continuous control signal $u(t)$, so that the system states can be moved from any initial mode $x(t_0)$ to any final mode $x(t_1)$, then the system is said to be controllable at the time $t_0$. If every system mode is controllable, then the system is state

controllable. If at least one of the states is not controllable, then the system is not controllable.

2)     For a linear system, if within a infinite time interval, $t_0 < t < t_1$, every initial mode $x(t_0)$ can be observed exclusively by the sensed value $y(t)$, then the system is said to be fully observable.

Matrix transformations are required  to assess the observability and controllability. To study the controllability, it is necessary to introduce the control canonical form as below:

$$A_c = \begin{bmatrix} -a_1 & -a_2 & ... & -a_n \\ 1 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.5}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & ... & b_n \end{bmatrix}, \qquad D_c = 0$$

where the subscript $c$ denotes that the associated matrix is in control canonical form.

For a linear time-invariant system, the necessary and sufficient condition for the system state controllability is the full rank of controllability matrix $Q_c$ . The controllability matrix is:

$$Q_c = [B \vdots AB \vdots ... \vdots A_{n-1}B], \tag{5.6}$$

and the system is controllable if and only if Rank $Q_c = n$.

When the linear time-invariant system has distinct eigenvalues, then after the modal transformation,  the new system becomes as follows:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu \tag{5.7}$$

where $T^{-1}AT$ is diagonal matrix. Under such condition, the sufficient and necessary conditions for state controllability is that there are no rows in the matrix $T^{-1}B$ containing all zero elements.

When matrix $A$ has multiple eigenvalues, and every multiple eigenvalue corresponds to the same eigenvector, then the system can be transformed into the new state space form, which is called the *Jordan canonical form*:

$$\dot{z} = Jz + T^{-1}Bu \tag{5.9}$$

where the matrix $J$ is Jordan canonical matrix. Then the sufficient and necessary condition for state controllability is that not all the elements in the matrix $T^{-1}B$, corresponding to the last row of every Jordan sub-matrix in matrix $J$, are zero.

The *output controllability sufficient and necessary condition* for linear time-invariant system is that the matrix $[CB \vdots CAB \vdots ... \vdots CA^{n-1}B \vdots D]$ is full rank, i.e.

$$rank[CB \vdots CAB \vdots ... \vdots CA^{n-1}B \vdots D] = n \tag{5.10}$$

Similarly, the sufficient and necessary observability condition for linear time-invariant system is that the observability matrix is full rank, i.e.

$$Q_o = [C \vdots CA \vdots ... \vdots CA^{n-1}]^T \tag{5.11}$$

and *rank* $Q_o = n$. When the system has distinct eigenvalues, then after a linear non-singular transformation, the system takes the form (when control vector $u$ is zero):

$$\begin{aligned} \dot{z} &= T^{-1}ATz \\ y &= CTz \end{aligned} \tag{5.12}$$

then the condition for observability is that there are no rows in the matrix $CT$ have only zero elements. While the system has multiple eigenvalues, and every multiple eigenvalue corresponds to the same eigenvector, the system after transformation looks like:

$$\begin{aligned} \dot{z} &= Jz \\ y &= CTz \end{aligned} \tag{5.13}$$

where $J$ is the Jordan matrix. The observability condition is that there are no columns corresponding to the first row of each Jordan sub-matrix having only zero elements.


6. Eigenvalues And Stability

The Lyapunov stability theory forms a basis for stability analysis. There are two approaches to evaluate system stability [5,15,24,28,44]:

(1) The first Lyapunov method, and

(2) The second Lyapunov method.

The *first Lyapunov method* is based on eigenvalue and eigenvector analysis for linearized system and small disturbances. It finds its application in many cases, for example, in the area of power systems engineering.

To study small-signal stability, it is necessary to clarify some basic concepts regarding the following *Differential-Algebraic Equation (DAE)*:

$$\dot{x} = f(x, y, p) \quad f : R^{n+m+q} \to R^n$$
$$0 = g(x, y, p) \quad g : R^{m+n+q} \to R^m$$

(6.1)

where $x \subset R^n$, $y \subset R^m$, $p \subset R^q$; $x$ is the vector of dynamic state variables, $y$ is the vector of static or instantaneous state variables, and $p$ is a selected system parameter affecting the studied system behavior. Variable $y$ usually represents a state variable whose dynamics is instantaneously completed as compared to that of the dynamic state variables $x$. Parameter $p$ belongs to the system parameters which have no dynamics at all at least if modeled by (6.1) [40]. For example, in power system engineering, typical dynamic state variables are chosen from the time dependent variables such as machine angle, machine speed, excitation field voltage, etc. While the static variables are the load flow variables including bus voltages and angles. Parameter $p$ can be selected from static load powers, or control system parameters.

A system is said to be in its *equilibrium condition* when the derivatives of its state variables equal to zero, which means there is no variation of the state variables. For the system modeled in (6.1), this condition is given as the following:

$$0 = f(x, y, p)$$
$$0 = g(x, y, p)$$

(6.2)

Solutions, $(x_0, y_0, p_0)$, of the above system are the system *equilibrium points*. Small-signal stability analysis uses the system represented in the linearized form, which is done by differentiating the original system respect to the system variables and parameters around its equilibrium point $(x_0, y_0, p_0)$. This linearization is necessary for the Lyapunov method and via computing system eigenvalues and eigenvectors.

For the original system (6.1), its linearized form is given in the following:

$$\Delta \dot{x} = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$
$$0 = \frac{\partial g}{\partial x} \Delta x + \frac{\partial g}{\partial y} \Delta y$$

(6.3)

For simplicity, the above system is rewritten as below:

$$\Delta \dot{x} = A \Delta x + B \Delta y$$
$$0 = C \Delta x + D \Delta y$$

(6.4)

where matrices *A*, *B*, *C* and *D* are the partial derivatives' matrices. If the algebraic matrix *D* is not singular, i.e. *detD* ≠ *0*, the state matrix $A_s$ is given as:

$$A_s = A - BD^{-1}C$$

(6.5)

is studied in stability analysis using eigenvalue and eigenvector approach.

The use of the Lyapunov first method involves the following steps [2, 5,26]:

1) linearization of the original system  (6.1) as in (6.3);

2) elimination of the algebraic variables to form the reduced dynamic state matrix $A_s$;

3) computation of the eigenvalues and eigenvectors of the state matrix $A_s$;

4) stability study of the system [21]

   a) If eigenvalues of the state matrix are located in the left hand side of the complex plane, then the system is said small-signal stable at the studied equilibrium point;

   b) If the most right eigenvalue is zero, the system is on the edge of small-signal aperiodic instability;

   c) If the most right complex conjugate pair of eigenvalues has zero real part and nonzero imaginary part, the system on the edge of oscillatory instability depending on the transversality condition [36];

   d) If the system has eigenvalue with positive real parts, the system is not stable;

   e) For the stable case, analyze several characteristics including damping and frequencies for all modes, eigenvalue sensitivities to the system parameters, excitability, observability and controllability of the modes.

More precise definitions for the first Lyapunov method have been addressed in the literature [14,21]. The general time-varying or non-autonomous form is as follows:

$$\dot{x} = f(t, x, u) \tag{6.6}$$

where $t$ represents time, $x$ is the vector of state variables, and $u$ is the vector of system input. In a special case of the system (6.1), $f$ is not explicitly dependent on time $t$, i.e.

$$\dot{x} = f(x) \tag{6.7}$$

and the system is said to be autonomous or time-invariant. Such kind of system does not change its behavior at different time [21].

An equilibrium point $x_0$ of the autonomous system (6.7) is,

(1) stable if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$, $\forall\, t \geq 0$

(2) unstable otherwise

(3) asymptotically stable, if it is stable, and $\delta$ can be chosen such that: $\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(0) =$

$x_0$.

The definition can be represented in the form of eigenvalue approach as given in the Lyapunov first method theorem below:

*Let $x_0$ be an equilibrium point for the autonomous system (6.7), where f: $D \to R^n$ is*

*continuously differentiable and D is a neighborhood of the origin. Let system Jacobian be: A*

$= \dfrac{\partial f}{\partial x}(x)\Big|_{x=x_0}$, *and $\lambda = [\lambda_1, \lambda_2, ..., \lambda_n]$ be the eigenvalues of A, then the origin is*

*asymptotically stable if Re $\lambda_i < 0$ for all eigenvalues of A, or the origin is unstable if Re $\lambda_i$*

*> 0 for one or more eigenvalues of A.*

Take one step further, the stability definition for a time-varying system, where the system behavior depends on the origin at the initial time $t_0$, is as follows. The equilibrium point $x_0$ for the system (6.6) is [21]:

(1) stable, if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$, $\forall t \geq t_0$

$\geq 0$;

(2) uniformly stable, if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$,

$\forall t \geq t_0 \geq 0$;

(3) unstable otherwise;

(4) asymptotically stable, if it is stable and there is $c=c(t_0)>0$ such that, for all $\|x(t_0)\|<c$,

$$\lim_{t\to\infty} x(t) = x_0;$$

(5) uniformly asymptotically stable if it is uniformly stable and there is a time invariant $c > 0$,

such that for all $\|x(t_0)\| < c$, $\lim_{t\to\infty} x(t) = x_0$. This is hold for each $\varepsilon > 0$, there is $T = T(\varepsilon) > 0$,

such that $\|x(t)\| < \varepsilon$, $\forall\, t \geq t_0 + T(\varepsilon)$, $\forall\| x(t_0)\| < c$;

(6) globally uniformly asymptotically stable if it is uniformly stable and for each pair of

positive numbers $\varepsilon$ and $c$, there is $T = T(\varepsilon,c) > 0$, such that $\|x(t)\| < \varepsilon$, $\forall\, t \geq t_0 + T(\varepsilon,c)$, $\forall\|$

$x(t_0)\| < c$ .

The corresponding stability theorem is listed below:

*Let  =0 be an equilibrium point for the nonlinear time varying system (6.6), where f: [0,*

$\infty)\times D \to R^n$ *is continuously differentiable, D = {x$\in R^n$ | ||x||$_2$ < r}, the Jacobian matrix is*

*bounded and Lipschitz on D, uniformly in t. A(t)=$\dfrac{\partial f}{\partial x}(t, x)\Big|_{x=x_0}$ is the Jacobian, then the*

*origin is exponentially stable for the nonlinear system if it is an exponentially stable*

*equilibrium point for the linear system $\dot{x} = A(t)x$.*

The second Lyapunov method is potentially most reliable and powerful method for the

original nonlinear and non-autonomous (or time-varying) systems. But it relays on the

Lyapunov function, which is hard to find for many physical systems.

## 1   EIGENVALUES AND BIFURCATIONS

The bifurcation theory has a rich mathematical description and literature for various areas

of applications. Many physical systems can be modeled by the general form

$$x' = f(x, p) \tag{7.1}$$

where $x$ is vector of the system state variables, and $p$ is the system's parameter which may

vary during system operation in normal as well as contingency conditions.

Bifurcations occur where, by slowly varying certain system parameters in some direction,

the system properties change qualitatively or quantitatively at a certain point [2,39]. Local

bifurcations can be detected by monitoring the behavior of eigenvalues of the systems operation point. In some direction of parameter variation, the system may become unstable due to the singularity of the system dynamic state matrix associated with zero eigenvalue, or because of a pair of complex conjugate eigenvalues crossing the imaginary axes of the complex plane. These two phenomena are saddle node and Hopf bifurcations respectively. Other conditions which may drive the system state into instability may also occur, such as singularity induced bifurcations, cyclic fold, period doubling, and blue sky bifurcations or even chaos [25,26,39].

For the general system (7.1), a point $(x_0, p_0)$ is said to be a *saddle node bifurcation* point if it is a equilibrium point of the system, i.e. $f(x_0, p_0) = 0$ ,the system Jacobian matrix, $f_y(x_0, p_0)$ has a simple zero eigenvalue $\lambda(p_0) = 0$, and the transversal conditions hold [1,36]. More generally [7], the saddle bifurcation satisfy the following conditions:

1) The point is the system's equilibrium point, i.e. $f(x_0, p_0) = 0$

2) The Jacobian matrix, $f_y(x_0, p_0)$ has a simple and unique eigenvalue $\lambda(p_0) = 0$ with the corresponding right and left eigenvectors $l$ and $r$ respectively.

3) Transversality condition of the first order derivative: $l^T f_y(x_0, p_0) \neq 0$

4) Transversality condition of the second order derivative: $l^T[f_{yy}(x_0, p_0)r]r \neq 0$

A Hopf bifurcation occurs when the following conditions are satisfied:

1) The point is a system operation equilibrium point, i.e. $f(x_0, p_0) = 0$

2) The Jacobian matrix, $f_y(x_0, p_0)$ has a simple pair of pure imaginary eigenvalues $\lambda(p_0) = 0 \pm j\omega$, and no other eigenvalues with zero real part

3) Transversality condition: $d[\text{Re}\lambda(p_0)] / dp \neq 0$

The last condition guarantees the transversal crossing of the imaginary axis. The sign of $d[\text{Re}\lambda(p_0)] / dp$ determines whether there is a birth or death of a *limit cycle* at $(x_0, p_0)$. Depending on the direction of transversal crossing the imaginary axis, Hopf bifurcation can be further categorized into supercritical and subcritical ones. The *supercritical Hopf bifurcation* happens when the critical eigenvalue moves from the right half plane to the left

half plane. The *sub-critical Hopf bifurcation* occurs when the eigenvalue moves from the right half plane to the left half plane, and is unstable. The system transients are diverged into an oscillatory style at the vicinity of the sub-critical Hopf bifurcation points.

*Singularity induced bifurcations* occur when the system's equilibrium approaches singularity, and some of the system eigenvalues become unbounded along the real axis, i.e. $\lambda_i \rightarrow \infty$. In case of the DAE model (6.1), the singularity of the algebraic Jacobian $D=g_y$ causes the singularity induced bifurcations. In that case, singular perturbations or noise techniques will have to be used to analysis the system dynamics [23]. When singularity induced bifurcation occurs, the system behavior becomes hardly predictable, and may cause fast claps type instability [23].

A graphical illustration of these three major bifurcations is given in Figure 1.

*Methods of computing bifurcations* can be categorized into direct and indirect approaches. The direct method has been practiced by many researchers in this area [2,3,7-11,20,23,25,26,35-39,40]. For example, the direct method computes the Hopf bifurcation condition by solving directly the set of equations [9,25,26,36]:

$$f(x, p_0 + \tau \Delta p) = 0 \tag{7.2}$$

$$A^t{}_S(x, p_0 + \tau \Delta p)l' + \omega l'' = 0 \tag{7.3}$$

$$A^t{}_S(x, p_0 + \tau \Delta p)l'' - \omega l' = 0 \tag{7.4}$$

$$\|l\| = 1 \tag{7.5}$$

where $A_S = A-BD^{-1}C = f_x - f_y g_y^{-1} g_x$ is the state matrix, $0 + j\omega$ is its eigenvalue, $l = l' + jl''$ is the corresponding left eigenvector, $p = p_0 + \tau \Delta p$ is the system parameter vector varying form the point $p_0$ in direction $\Delta p$. By taking zero $\omega$ and $l''$, saddle node bifurcation can be computer as well.

Indirect methods are mainly Newton-Raphson type method using predictor and corrector to trace the bifurcation diagram. A detailed description of the continuation methods can be found in [3,8-10,34,36].

As an *example of applied bifurcation analysis*, let us consider a task from the area of power system analysis [9,25]. The power system model is composed of two generators and one load bus. The system is shown in Figure 3.

Static and an induction motor load are connected with the load bus in the middle of the network. A capacitor device is also connected with the same bus to provide reactive power supply and control the voltage magnitude; $E$ and $\delta$ are generator terminal voltage and angle respectively; $V$ is load bus voltage, $Y$ is line conductance, and $M$ stands for induction motor load. The system is modeled by the equations below:

$$\delta'_m = \omega$$

$$M\dot{\omega} = -\delta_m\omega + P_m + E_m y_m V\sin(\delta - \delta_m - \theta_m) + E^2_m y_m \sin\theta_m$$

$$K_{qw}\dot{\delta} = -K_{qv2}V^2 - K_{qv}V + E'_0 y'_0 V\cos(\delta + \theta'_0) + E_m y_m V\cos(\delta - \delta_m + \theta_m)-$$

$$- (y'_0\cos\theta_0 + y_m\cos\theta_m V^2 - Q_0 - Q_1$$

$$TK_{qw}K_{pv}\dot{V} = K_{qw}K^2_{qv}V^2 + (K_{pw}K_{qv} - K_{qw}K_{pv})V + \sqrt{(K^2_{qw} + K^2_{pw})}[-E'_0 y'_0 V\cos(\delta + \theta_0 - h)-$$

$$- E_m y_m V\cos(\delta - \delta_m + \theta_m - \eta)+(y'_0\cos(\theta_0 - \eta) + y_m\cos(\theta_m - \eta))V^2]-K_{qw}(P_0 + P_1) + K_{pw}(Q_0 + Q_1)$$

where $\eta = \tan^{-1}(K_{qw}/K_{pw})$. The active and reactive loads are featured by the following equations:

$$P_d = P_0 + P_1 + K_{pw}\delta + K_{pv}(V + TV')$$

$$Q_d = Q_0 + Q_1 + K_{qw}\delta + K_{qv}V + K_{qv2}V^2$$

The system parameter $Q_1$ is selected as the bifurcation parameter to be increased slowly. Voltage $V$ is taken as a dependent parameter for illustration. Figure 4 shows the dynamics for the system in the form of a Q-V curve [39].

The eigenvalue trajectory around a Hopf bifurcation point is given in Figure 5. where both supercritical and subcritical Hopf bifurcations can be seen.

## 7. NUMERICAL METHODS FOR THE EIGENVALUE PROBLEM

*A) Computing eigenvalues and eigenvectors*

Although roots of the characteristics polynomial $L(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ are eigenvalues of the matrix $A$, a direct computing of these roots is not recommended because of the round errors and high sensitivity of the roots to coefficients $a_i$ [12].

We start by introducing the *power method*, which locates the eigenvalue with the largest magnitude. Suppose a matrix $A$ has eigenvalues $\Lambda = [\lambda_1, \lambda_2, ..., \lambda_n]^{\mathrm{T}}$ and the corresponding right eigenvectors $R = [r_1, r_2, ..., r_n]^{\mathrm{T}}$, and $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$. For any vector $x \neq 0$, we have

$$x = \sum_{i=1}^{n} c_i r_i \tag{8.1}$$

By multiplying (8.1) by $A, A^2, ...$, it can be obtained that:

$$x^{(1)} = Ax = \sum_{i=1}^{n} c_i \lambda_i r_i$$

$$x^{(2)} = Ax^{(1)} = \sum_{i=1}^{n} c_i \lambda_i^2 r_i \tag{8.2}$$

$$.........$$

$$x^{(m)} = Ax^{(m-1)} = \sum_{i=1}^{n} c_i \lambda_i^m r_i$$

After a number of iterations, $x^{(m)} \rightarrow \lambda_1^m c_1 r_1$. Therefore $\lambda_1$ can be obtained by dividing the corresponding elements of $x^{(m)}$ and $x^{(m-1)}$ after a sufficient number of iterations, and the eigenvector can be obtained by scaling $x^{(m)}$ directly. Other eigenvalues and eigenvectors can be computed by applying the same method to the new matrix:

$$A_1 = A - \lambda_1 r_1 v^{1*} \tag{8.3}$$

where $v^1$ is the reciprocal vector of the first eigenvector $r_1$. It can be observed that the matrix $A_1$ has the same eigenvalues as $A$ except the first eigenvalue, which is set to zero by the transformation. By applying the method successively, all eigenvalues and corresponding right eigenvectors of matrix $A$ can be located. The applicability of this method is restricted by computational errors. Convergence of the method depends on separation of eigenvalues determined by the ratios $\|\lambda_i/\lambda_1\|$, $\|\lambda_i/\lambda_2\|$, etc. As evident, the method can only compute one eigenvalue and eigenvector at a time.

The *Schur algorithm* can also be used to locate eigenvalues from $\lambda_2$ while knowing $\lambda_1$ by applying power method to $A_1$ after the following transformation [12]:

$$\begin{bmatrix} \lambda_1 & B_1 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} 1 \\ y^{(2)} \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ y^{(2)} \end{bmatrix} \tag{8.4}$$

A general idea of the *inverse power method* is to use the power method determining the minimum eigenvalues. By shifts, any eigenvalue can be made the minimum one. The inverse method can compute eigenvectors accurately even when the eigenvalues are not well separated. The method implies the following. Let $\lambda_i^*$ be an approximation of one of the eigenvalues $\lambda_i$ of $A$. The steps involved is given below:

- Obtain a tridiagonal matrix $T$ by reduction of matrix $A$;

- Find zeros of $(T - \lambda_i^* I)^{-1} y_0$ , i.e. $\{z_1 \mid (T - \lambda_i^* I)z_1 = y_0\}$;

- set $y_1 = z_1 / \| z_1 \|$;

- solve $(T - \lambda_i^* I)^{-1} z_2 = y_1$, etc.

The eigenvector $l_i$ of $\lambda_i$ is approximated by $y_k = z_k / \| z_k \|$ provided $y_0$ contains a non-zero term in $l_i$. If $|\lambda_i^* - \lambda_i|$ is sufficiently small, the inverse iteration method obtains the eigenvector associated with $\lambda_i$ within only several iterations.

The *QR method* is one of the most popular algorithms for computing the eigenvalues and eigenvectors. By using a factorization into the product of a unitary matrix $Q$ and an upper-triangular matrix R, this method involves the iteration process as given below:

$$A_1 = RQ = Q_1 R_1$$

$$\dots \tag{8.5}$$

$$A_i = R_{i-1}Q_{i-1} = Q_i R_i = Q^{-1}_{i-1} A_{i-1} Q_{i-1}$$

where the t-th unitary matrix $Q_t$ is obtained by solving:

$$Q_t^T A = R_t$$

and $Q_t^T$ is determined in a factorized form, such as the product of plane rotations or of elementary Hermitians. Then, the matrix $R_t Q_t$ is obtained by successive post-multiplication of $R_t$ with the transposes of the factors of $Q_t^T$ [41].

After a number of iterations, diagonal elements of $R_m$ approximate eigenvalues for $A$ [12]. To reduce the number of iterations and speed-up computations due to less computational efforts required at each iteration, the studied matrix $A$ is initially reduces to the Hessenberg form which is preserved during iterations [12.41].

A more *general form of the eigenvalue problem* can be modeled as

$$Ax = \lambda Bx \tag{8.6}$$

If $A$ and $B$ are non-singular, the problem can be transformed into the standard form of eigenvalue problem by expressing

$$B^{-1}Ax = \lambda x. \ \text{ or } \ A^{-1}Bx = \lambda^{-1}x \tag{8.7}.$$

Then methods discussed above can be applied to solve the problem.

There are cases when the computation can be simplified [12]. When both $A$ and $B$ are symmetric and $B$ is positive definite, matrix $B$ can be decomposed as $B = C^T C$ where $C$ is a non-singular triangular matrix. Then the problem can be expressed as:

$$Ax = \lambda \, C^T Cx \tag{8.8}$$

If vector $y$ is chosen so that $y = Cx$, the final transformation is obtained as:

$$(C^T)^{-1}AC^{-1}y = Gy = \lambda y \tag{8.9}$$

and the problem is simplified into the eigenvalue problem with matrix $G$. Techniques dealing with other situations of the generalized eigenvalue problem can be found in [31].

In many cases, matrix $A$ is a sparse matrix with many zero elements. Different techniques solving the *sparse matrix eigenvalue problem* are derived. The approaches can be categorized into two major branches: (1) problems where the *LU factorization* is possible, or (2) it is impossible.

In the first case, after transformation of the generalized eigenvalue problem as $B^{-1}Ax = \lambda x$, or $y = L^T x$, so that $L^{-1}AL^{-T}y = \lambda y$, the resulting matrices may not necessarily be sparse. There are several aspects for the problem: first, the matrix should be represented in such a way that it dispenses with zero elements and allows new elements to be inserted as they are generated

by the elimination process during the decomposition; second, pivoting must be performed during the elimination process to preserve sparsity and insure numerical stability [6,37]. Power method is sometimes used for large sparse matrix problem to compute eigenvalues.

When matrices $A$ and $B$ are becoming very large, the difficulties of performing the LU factorization for the general eigenvalue problem becomes more and more difficult. In this case, a function should be constructed so that it reaches its minimum at one or more of the eigenvectors, and the problem is to minimize this function with an appropriate numerical method [37]. For example, the successive search method can be used to minimize this function. Also, other gradient methods can be employed.

Among all these computation methods for eigenvalue problem, there are many factors which influence the efficiency of a particular method. For a large matrix, whether it is dense or sparse, the power method is suitable when only a few large eigenvalues and corresponding eigenvectors are required. The inverse iteration method is the most robust and accurate in calculating eigenvectors. Though, the most popular general method for eigenvalue and eigenvector computations is the QR method. However, in many cases, especially when the matrix is Hermitian or real symmetric, many methods can provide satisfactory results.

## 0 Localization of eigenvalues

Along with the direct method based on computation of eigenvalues, there are several indirect method to determine a domain in the complex plane where the eigenvalues are located. A particular interest for the stability studies is to decide whether all eigenvalues have negative real parts. Some methods can also count the number of stable and unstable modes without solving the general eigenvalue problem. Also, there are methods that determines a bounded region where the eigenvalues are located.

The following algebraic results can help to identify stability of a matrix [24]:

- If the matrix $A \in R_{nxn}$ is stable and $W \in R_{nxn}$ is positive (nonnegative) definite, then there exists a real positive (positive or nonnegative) definite matrix $V$ such that $AV' +$

$$VA = -W.$$

- Let $V \in R_{nxn}$ be positive definite, define the real symmetric matrix $W$ by $A'V + VA = -W$. Then $A$ is stable iff for the right eigenvector $r$ associated with every distinct eigenvalue of $A$, there holds the relation $r^*Wr > 0$ where $r^*$ means conjurigate transpose of eigenvector $r$.

- If $W$ is positive definite, then $A$ is stable iff $A'V + VA = -W$ has a positive definite solution matrix $V$.

Also, the stability problem can be studied by locating the eigenvalues using coefficients of the characteristic polynomial $\det(\lambda I - A) = 0$ rather then the matrix itself. The *Routh-Hurwitz criterion* is one of these approaches. For the monic polynomial with real coefficients,

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n \tag{8.10}$$

the *Hurwitz matrices* are defined as:

$$H_1 = a_1,$$

$$H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},$$

$$\ldots \tag{8.11}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & & 0 \\ a_3 & a_2 & a_1 & 1 & & 0 \\ a_5 & a_4 & a_3 & a_2 & & . \\ . & . & . & & . & . \\ a_{2n-1} & . & . & & a_{n+1} & a_n \end{bmatrix}$$

The criteria says that all the zeros of the polynomial $f(z)$ have negative real parts iff $\det H_i > 0$, for $i = 1, 2, .., n$. This also indicates the eigenvalues $\lambda$ of the matrix associated with the characteristic polynomial $f(\lambda)$, have all negative real parts, so the matrix is stable.

The *Nyquist stability criterion* is another indirect approach of evaluating stability conditions. For a feedback system given in Figure 2, it relates the system open loop frequency response to the number of closed-loop poles in the right half of the complex plane [15].

Stability of the system is analyzed by studying the Nyquist plot (polar plot) of the open loop transfer function $KG(j\omega)$. As it is based on the poles of the closed-loop system which is

decided by *1 + KG(s)=0*, the point −*1* is the critical point for study of the curve *KG(jω)* in the polar plot. The following steps are involved. First, draw the magnitude and angle of KG(*jω*). Second, count the number of clockwise encirclements of −*1* as *N*. Third, find the number of unstable poles of *G(s)*, which is *P*. The system is stable if the number of unstable closed-loop roots *Z* = *N*−*P* is zero, which means there are no closed-loop poles in the right half plane. There are other methods exploiting godographs of the system transfer function in as functions of *ω*.

The *Gershgorin's theorem* is also used in eigenvalue localization. It states that any of the eigenvalues of a matrix $A = [a_{i,j}]_{nxn}$ lies in at least one of the circular discs with centers $a_{i,i}$ and radii as sum of $\| a_{i,j} \|$ for all $i \neq j$. If there are *s* such circular discs forming a connected domain isolated from other discs, then *A* has exactly *s* eigenvalues within this domain [18,41]. This theorem finds its application in perturbation analysis of eigenvalues.


## 9. SOME PRACTICAL APPLICATIONS OF EIGENVALUES AND EIGENVECTORS

*A) Some useful comments*

In the area of stability and control, eigenvalues give such important information as *damping*, *phase* and *magnitude* of oscillations [22,25,26]. For example, for the system dynamic state matrix $A_s$ critical eigenvalue $\lambda_i = \alpha_i \pm j\omega_i$, which is the eigenvalue with the largest real part $\alpha_i$, the damping constant is $\sigma = -\alpha_i$, and frequency of oscillation is $\omega_i$ in radius per second unit, or $\omega_i/2\pi$ in Hz unit.

The *eigenvalue sensitivity analysis* is often needed to assess the influence of certain system parameters *p* on damping and enhance system stability [13, 26]:

$$\frac{\partial \alpha_j}{\partial p_i} = \text{Re} \left\{ \frac{l_j^T \frac{\partial A_s}{\partial p_i} r_j}{l_j^T r_j} \right\} \tag{9.1}$$

$l_j$ and $r_j$ are the corresponding left and right eigenvectors for the *j*-th eigenvalue $\alpha_j$, and $\partial A_s/\partial p_i$ is the sensitivity of the dynamic state matrix to the *i*-th parameter $p_i$.

The left and right vectors are also associated with important features of the system dynamics.

The left eigenvector is a normal vector to the equal damping surfaces, and the right eigenvector shows an initial dynamics of the system at a disturbance [26]. Also, they provide an efficient mathematical approach to locate these equal damping surfaces in the parameter spaces [25,26].

### B) A power system example

In order to reveal some power system stability characteristics a General Method is proposed by the authors in [26]. For the DAE power system model, the method gives the following important small-signal stability characteristic points:

- load flow feasibility points, beyond where there exists no solution for the system load flow equations;

- aperiodic and oscillatory stability points;

- min/max damping points;

The method employs the following constrained optimization problem:

$$a^2 \Rightarrow min/max \qquad (9.2)$$

Subject to

$$f(x, p_0 + \tau \Delta p) = 0 \qquad (9.3)$$
$$As(x, p_0 + \tau \Delta p)l' - al' + \omega l'' = 0 \qquad (9.4)$$
$$As(x, p_0 + \tau \Delta p)l'' - al'' - \omega l' = 0 \qquad (9.5)$$
$$l_i' - 1 = 0 \qquad (9.6)$$
$$l_i'' = 0 \qquad (9.7)$$

where $a$ is the real part of system eigenvalue of interest, $\omega$ is the imaginary part, $l'$ and $l''$ are real and imaginary parts of the corresponding left eigenvector $l$; $l'_i + jl''_i$ is the $i$-th element of the left eigenvector $l$; $p_0 + \tau \Delta p$ specifies a ray in the space of $p$; $A_S$ stands for the state matrix. In the above set, (9.3) is the load flow equation and conditions (9.4)-(9.7) provide an eigenvalue with real part of $a$ and the corresponding left eigenvector.

The problem may have a number of solutions, and all of them presents different aspects of the small-signal stability problem as shown in Figure 6.

The minimum and maximum damping points correspond to zero derivative $da/d\tau$. The constraint set (9.3)-(9.7) gives all unknown variables at these points. The minimum and maximum damping, determined for all oscillatory modes of interest, provides an essential information about damping variations caused by a directed change of power system parameters.

The saddle node or Hopf bifurcations correspond to $a = 0$. They indicate the small-signal stability limits along the specified loading trajectory $p_0 + \tau\Delta p$. Besides revealing the type of instability (aperiodic for $\omega = 0$ or oscillatory for $\omega \neq 0$), the constraint set (9.3)-(9.7) gives the frequency of critical oscillatory mode. The left eigenvector $l = l' + jl''$ (together with the right eigenvector $r = r' + jr''$ which can be easily computed in its turn) determine such essential factors as sensitivity of $a$ with respect to $p$, the mode, shape, participation factors, observability and excitability of the critical oscillatory mode [20,17,22].

The load flow feasibility boundary points (9.3) reflect the maximal power transfer capabilities of the power system. Those conditions play a decisive role when the system is stable everywhere on the ray $p_0 + \tau\Delta p$ up to the load flow feasibility boundary. The optimization procedure stops at these points as the constraint (9.3) can not be satisfied anymore.

The problem (9.2)-(9.7) takes into account only one eigenvalue each time. The procedure must be repeated for all eigenvalues of interest. The choice of eigenvalues depends upon the concrete task to be solved. The eigenvalue sensitivity, observability , excitability and controllability factors [19,20,22] can help to determine the eigenvalues of interest, and trace them during optimization.

The result of optimization depends on the initial guesses for all variables in (9.2)-(9.7). To get all characteristic points for a selected eigenvalue, different initial points may be computed for different values of $\tau$.

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Saddle Node Bifurcation

Hopf Bifurcation

Superitical and Subcritical
Hopf Bifurcations

Singularity Induced Bifurcations

Supercritical Bifurcations

Subcritical bifurcations
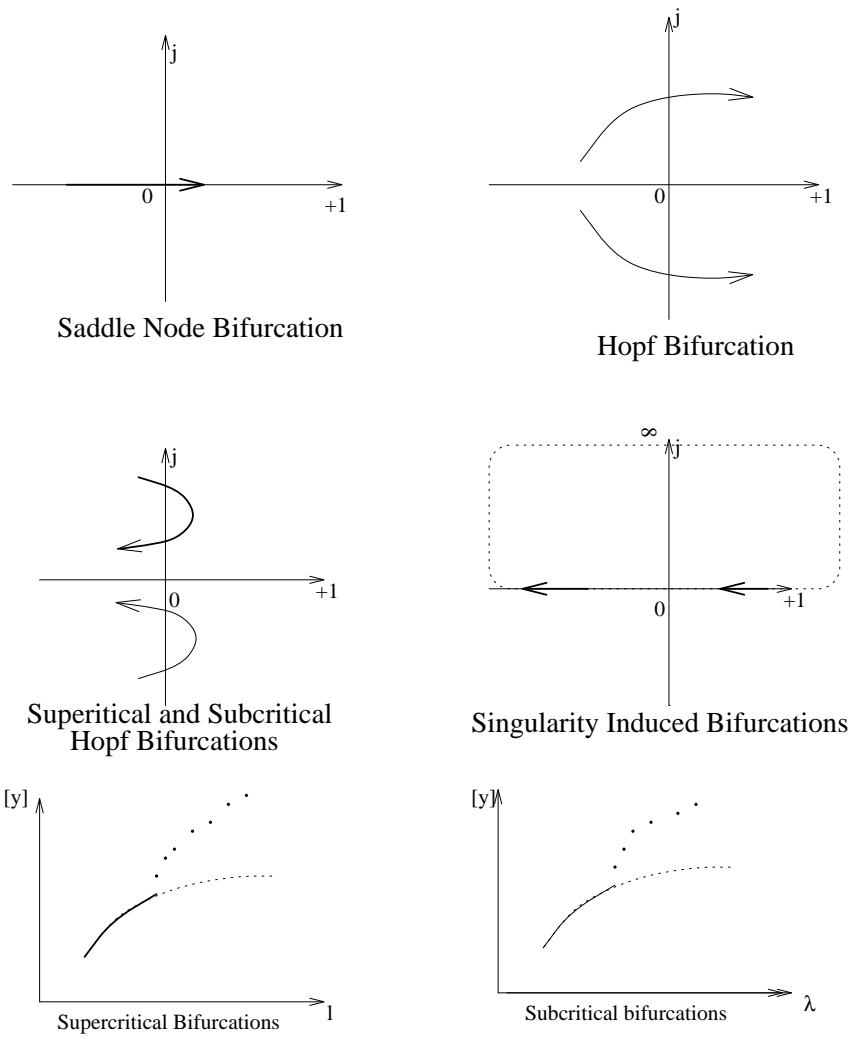
Figure 1. Bifurcation diagrams for different bifurcations.
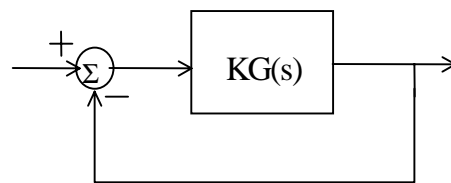


Figure 2. Block diagram for the feedback system:
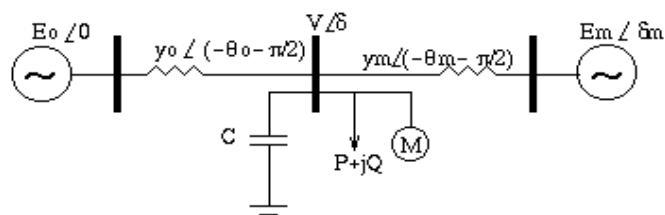$Y(s)/R(s) = H(s) = KG(s)/[1+KG(s)]$.

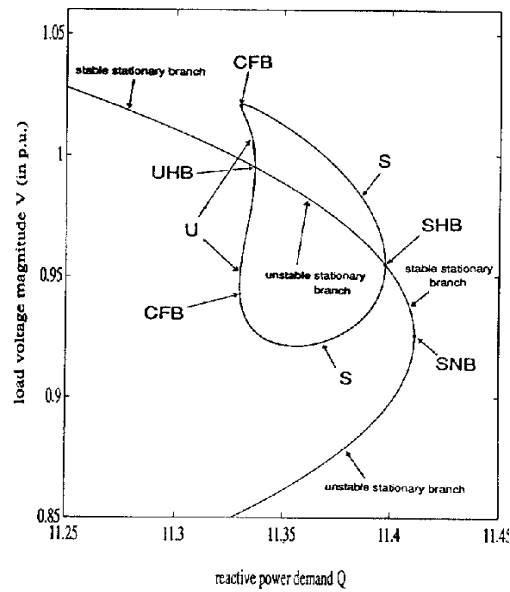Figure 3. A Simple Power System Model



Figure 4. The Q-V Curve Branch diagrams.

S - Stable periodic branch, U - Unstable periodic branch.
SNB - Saddle Node Bifurcation,
SHB - Stable (Supercritical) Hopf Bifurcation,
UHB - Unstable (Subcritical) Hopf Bifurcation,
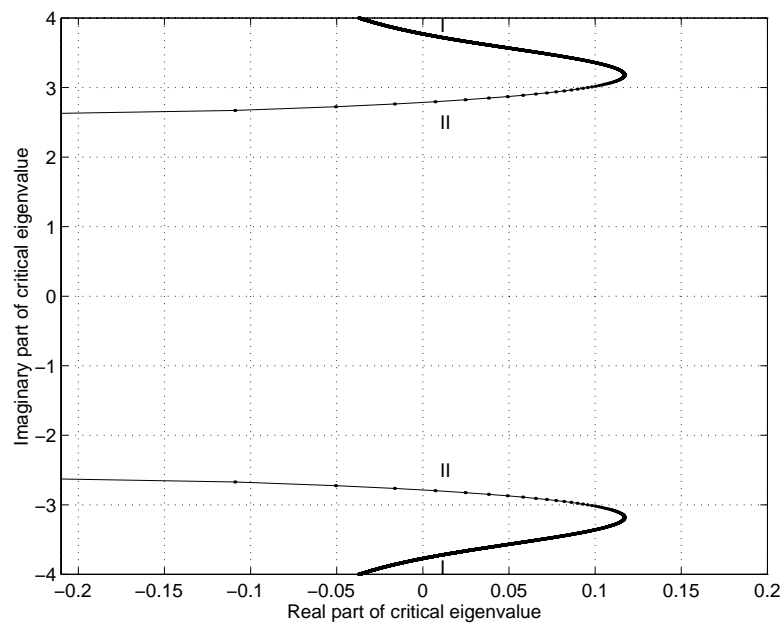CFB - Cyclic Fold Bifurcation.



Figure 5. An illustration of subcritical (I) and supercritical (II) Hopf bifurcations
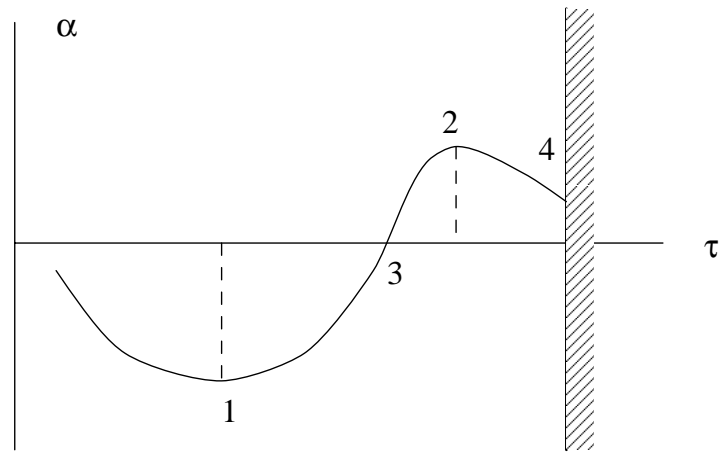
Figure 6. Different solutions of the problem (9.2)-(9.7):

1, 2 - minimum and maximum damping
3 - saddle ($\omega = 0$) or Hopf ($\omega \neq 0$) bifurcations
4 - load flow feasibility boundary