Dynamics and Control of Formation Flying Satellites In Earth Orbit

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Outline

Lecture 1- Dynamics of relative motion

- What is formation flying?
- Clohessy-Wiltshire (CW) or Hill's equations
  - History
  - Periodic Solutions
  - Assumptions/limitations
- Curvilinear coordinates
- Design with differential orbital elements
- Elliptic reference orbit
- J2 effects
- Classification of relative motion orbits
- Comparison of theories
- Utilizing natural motion

Outline

Lecture 2- Control concepts and navigation error effect

- Example of the Error Analysis of a LEO satellite
- Relative Navigation Error Effect
- Constellation Control

Texas A&M Team

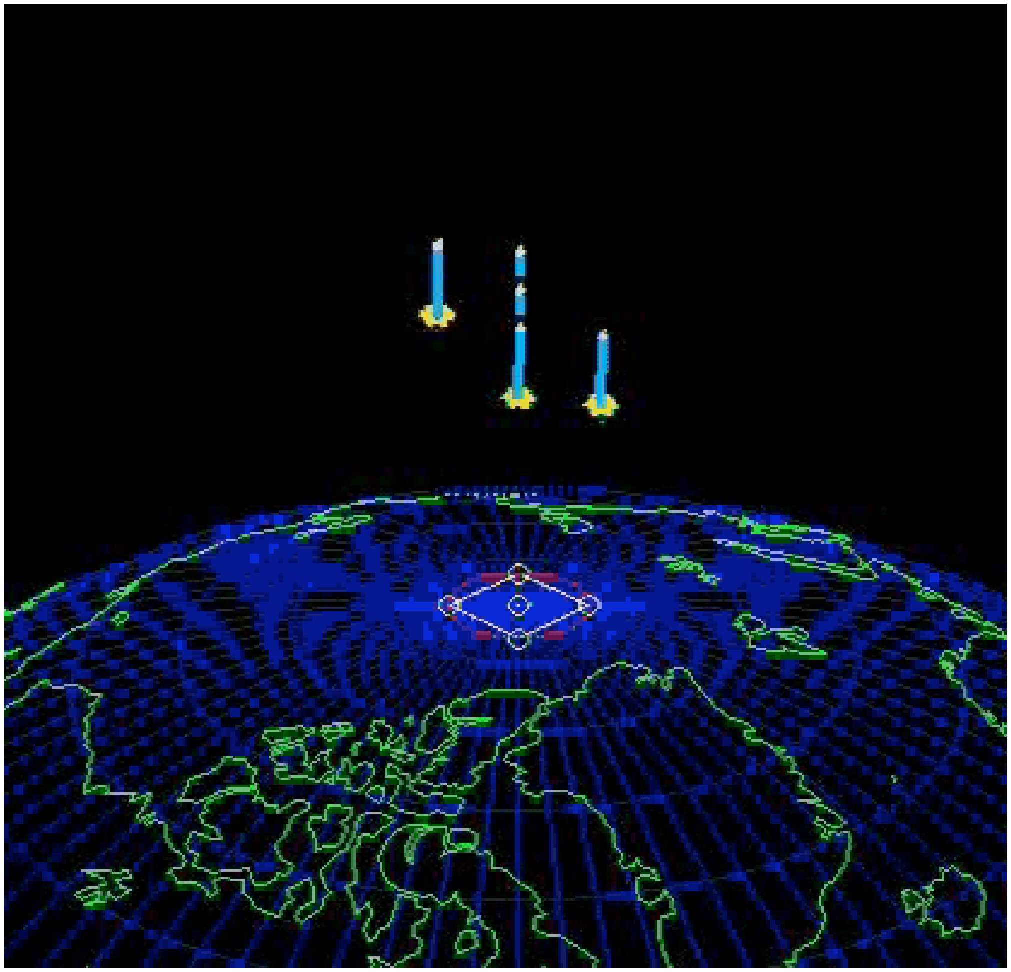
- Faculty
  - Alfriend
  - Vadali
  - Junkins
- Postdocs
  - Schaub (former)
- Graduate Students
  - Gim
  - Vaddi
  - Yan
  - Wilkins
  - Naik
  - Subbarao (former)
- Contracts
  - AFOSR
  - NASA Goddard (3)
  - AFRL (TechSat 21)
  - AFRL (Power Sail)

What is Formation Flying?

Formation flying is multiple satellites orbiting in close proximity in a cooperative manner.

Autonomous operation will be necessary.

Example



Background

- Interest in the US in the use of swarms of small satellites flying in precise formation
  - Mission enabling and mission enhancing technology
  - Long baseline interferometers, Virtual antennas
- ARFL TechSat 21 program - cancelled
- NASA/GSFC
  - Landsat 7 - EO1
  - ORION
  - MMS

Formation flying satellites is not a difficult concept. Flying autonomously in precise formation is a new challenge.

Distributed Spacecraft Missions

Representative List of Satellite Missions Utilizing Distributed Spacecraft Techniques

Projected Launch Year	Mission Name	Mission Type
00	New Millennium Program (NMP) Earth Observing-1	Earth Science
01	Gravity Recovery and Climate Recovery (GRACE)	Earth Science
03	University Nanosats/Air Force Research Laboratory Nanosat 1	Technology Demonstrator
03	University Nanosats/Air Force Research Laboratory Nanosat 2	Technology Demonstrator
03	NMP ST-5 Nanosat Constellation Trailblazer	Space Science
04	Techsat-21/AFRL	Technology Demo
04	Auroral Multiscale Mission (AMM)/APL	Space Science/SEC
04	ESSP-3-Cena (w/ Aqua)	Earth Science
05	Starlight (ST-3)	Space Science/ASO
05	Magnetospheric Multiscale (MMS)	Space Science/SEC
05	Space Interferometry Mission (SIM)	Space Science/ASO
06	MAGnetic Imaging Constellation (MAGIC)	Space Science
06	COACH	Earth Science
07	Global Precipitation Mission (EOS-9)	Earth Science
07	Geospace Electrodynamic Connections (GEC)	Space Science/SEC
08	Constellation-X	Space Science/SEU
08	Magnetospheric Constellation (DRACO)	Space Science/SEC
08	Laser Interferometer Space Antenna (LISA)	Space Science/SEU
09	DARWIN Space Infrared Interferometer/European Space Agency	Space Science
10	Leonardo (GSFC)	Earth Science
15	Stellar Imager (SI)	Space Science/ASO
11	Terrestrial Planet Finder (TPF)	Space Science/ASO
	Astronomical Low Frequency Array (ALFA)/Explorers	Space Science
05+	MAXIM Pathfinder	Space Science/SEU
05+	Living with a Star (LWS)	Space Science
05+	Soil Moisture and Ocean Salinity Observing Mission (EX-4)	Earth Science
05+	Time-Dependent Gravity Field Mapping Mission (EX-5)	Earth Science
05+	Vegetation Recovery Mission (EX-6)	Earth Science
05+	Cold Land Processes Research Mission (EX-7)	Earth Science
05+	Hercules	Space Science/SEC
05+	Orion Constellation Mission	Space Science/SEC
15	Submillimeter Probe of the Evolution of Cosmic Structure (SPECS)	Space Science/SEU
15+	Planet Imager (PI)	Space Science/ASO
15+	MAXIM X-ray Interferometry Mission	Space Science/SEU
15+	Solar Flotilla, IHC, OHRM, OHRI, ITM, IMC, DSB Con	Space Science/SEC
15+	NASA Goddard Space Flight Center Earth Sciences Vision	Earth Science
15+	NASA Institute of Advanced Concepts/Very Large Optics for the Study of Extrasolar Terrestrial Planets	Space Science
15+	NASA Institute of Advanced Concepts /Ultra-high Throughput X-Ray Astronomy Observatory with a New Mission Architecture	Space Science
15+	NASA Institute of Advanced Concepts /Structureless Extremely Large Yet Very Lightweight Swarm Array Space Telescope	Space Science

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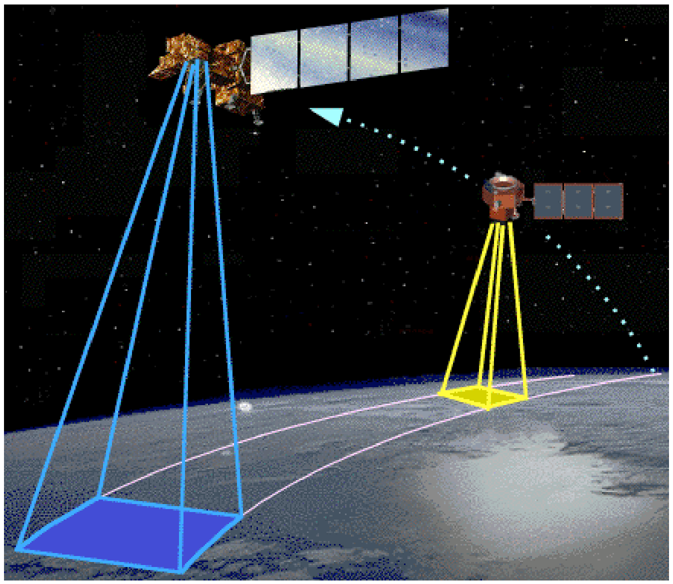
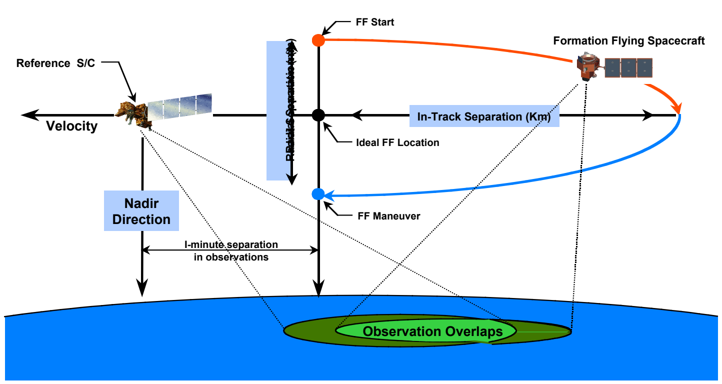
Notes: ASO-Astronomical Search for Origins, SEC-Sun Earth Connections, SSE-Solar System Exploration, SEU- structure and Evolution of the Universe

TechSat21

- Air Force Technology Mission
- 3-170 kg Satellites
- Orbit:550 km altitude, 35.4 deg inclination
- Relative Navigation: DCGPS
  - 1σ Rqmt: 10 cm position, 0.5 mm/s velocity
- Formations
  - Leader Follower: 30 m-5 km
  - In-Plane Elliptic: 60 m-5 km
- Autonomous Orbit Maintenance and Reconfiguration
- Challenges/Constraints
  - 1 thruster (PPT)-rotate S/C to fire thruster
  - Differential drag
  - Collision avoidance

NMP EO-1 Enhanced Formation Flying (EFF) Experiment





Level - I:

Demonstrate the Capability to Fly Over the Same Ground Track As LandSat-7 Within 3 Km at a Nodal Separation Interval of Nominally One Minute During Which Time an Image Is Collected.

Level-II:

***EFF*- Shall Provide the Autonomous Capability of Flying Over the Same Groundtrack of Another S/C at Fixed Separation Times.**

***Autonomy* - Shall Provide On-Board Autonomous Relative Navigation and Formation Flying Control for EO-1 and LandSat-7.**

***AutoCon Flight Control System* - Shall Provide Autonomous Formation Flying Control Via AutoCon (to provide future reusability).**

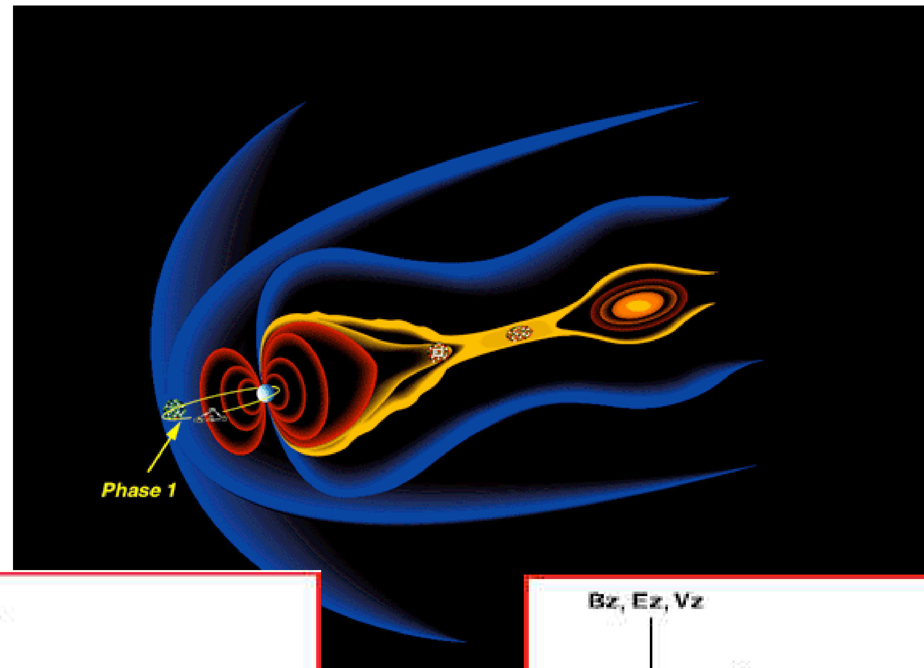
***Ground Track* - EO-1 Shall Fly the Same Ground Track As LandSat-7.**

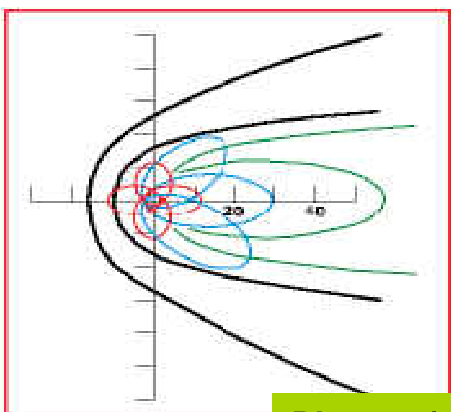
***Separation* - EO-1 Shall Remain Within a 1-Minute In-Track Separation from LandSat-7.**

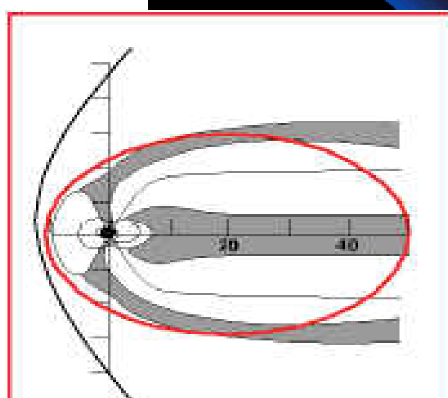
Magnetospheric Multi-scale

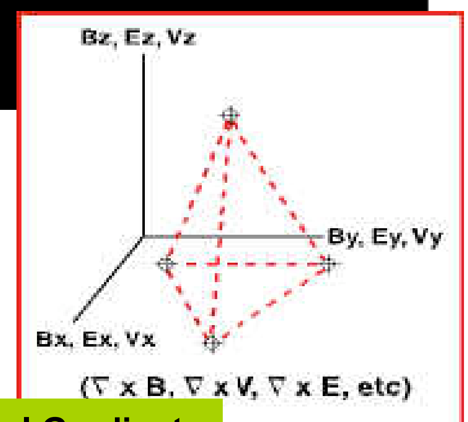
How do small-scale processes control large-scale phenomenology, such as magnetotail dynamics, plasma entry into the magnetosphere, and substorm initiation?

•5 identical spacecraft in a variably spaced tetrahedron (1 km to several *earth radii*)
•4 orbit phases, orbit adjust
•2 year in-orbit (minimum) mission life
•Interspacecraft ranging and communication
•Advanced instrumentation, integrated payload
•Attitude knowledge < 0.1°, spin rate 20 rpm



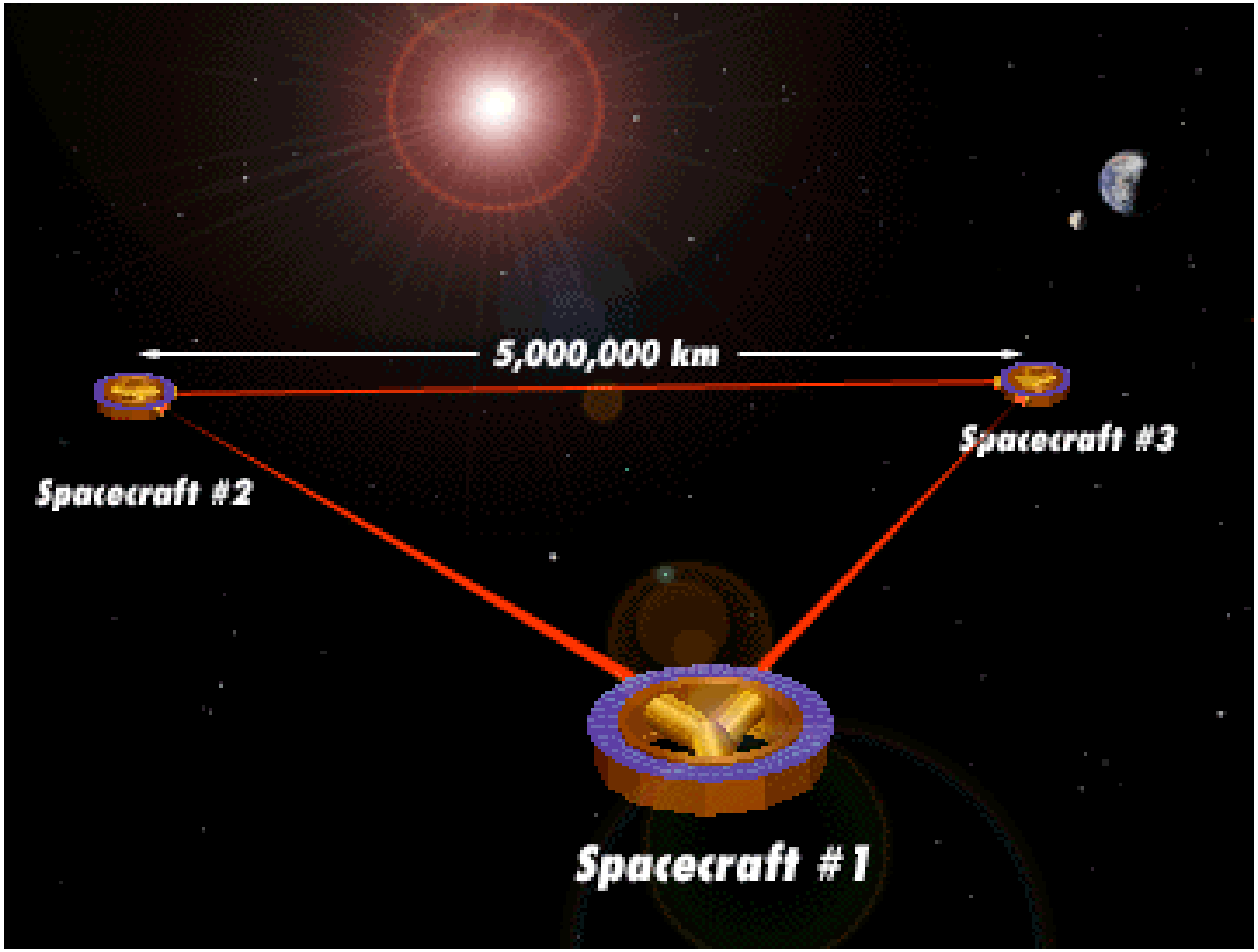






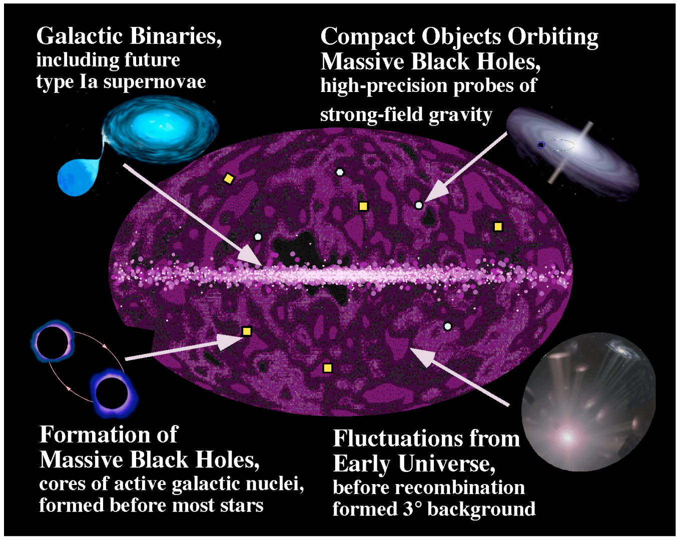
Phases 1-3, Equatorial - Phase 4, Polar - Determination of Spatial Gradients

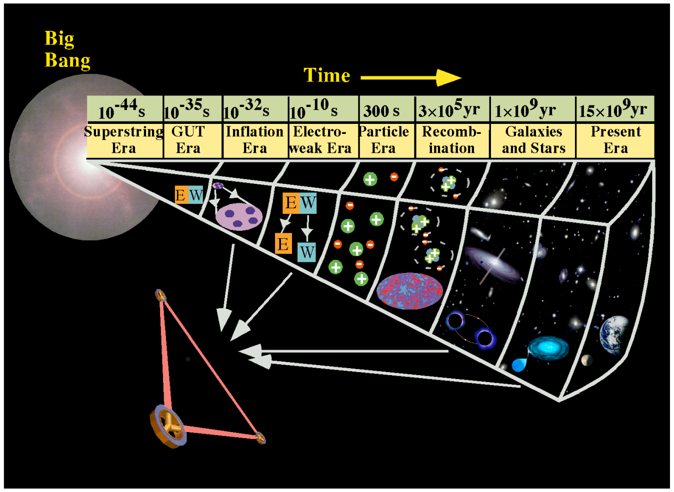
The LISA Configuration



LISA Science Overview

- **LISA will observe gravitational waves from compact galactic binaries and from extragalactic supermassive black holes.**
- **LISA will the address the following questions;**
  - What is the structure of space and time near massive black holes?
  - How did the massive black holes at the centers of galaxies form?
  - What happened in the Universe before light could propagate?
  - What are the precursors to Type Ia supernovae?





Relative Motion Equations

$$\ddot{\vec{\rho}} + 2\vec{\omega}\times\dot{\vec{\rho}} + \dot{\vec{\omega}}\times\vec{\rho} + \vec{\omega}\times\left(\vec{\omega}\times\vec{\rho}\right) + \ddot{\vec{r}}_c = -\left(\frac{\mu}{r^3}\vec{r} - \frac{\mu}{R^3}\vec{r}_c\right)$$

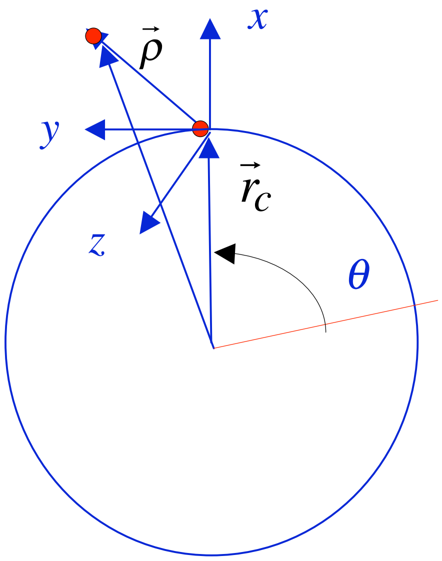
$$\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}y - \dot{\theta}^2 x = -\frac{\mu(r_c + x)}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} + \frac{\mu}{r_c^2} + J_2 f_x\left(\mathrm{e,x}\right)$$

$$\ddot{y} + 2\dot{\theta}\dot{x} + \ddot{\theta}x - \dot{\theta}^2 y = -\frac{\mu y}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} + J_2 f_y\left(\mathrm{e,x}\right)$$

$$\ddot{z} = -\frac{\mu z}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} + J_2 f_z\left(\mathrm{e,x}\right)$$

$$\ddot{r}_c = r_c\dot{\theta}^2 - \frac{\mu}{r_c^2}\left[1 + J_2 f_r\left(\mathbf{e}\right)\right]$$

$$\ddot{\theta} = -\frac{2\dot{r}_c\dot{\theta}}{r_c}\left[1 + J_2 f_\theta\left(\mathbf{e}\right)\right]$$



Clohessy-Wiltshire (Hill's) Equations

Assumptions

- Spherical Earth
- Circular reference orbit
- Equations can be linearized

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 0$$

$$\ddot{y} + 2n\dot{x} = 0$$

$$\ddot{z} + n^2 z = 0$$

$$x = 2\left(2x_0 + \dot{y}_0 / n\right) - \left(3x_0 + 2\dot{y}_0 / n\right)\cos\psi + \left(\dot{x}_0 / n\right)\sin\psi$$

$$y = \left(y_0 - 2\dot{x}_0 / n\right) - 3\left(2x_0 + \dot{y}_0 / n\right)\psi + \left(2\dot{x}_0 / n\right)\cos\psi + 2\left(3x_0 + 2\dot{y}_0 / n\right)\sin\psi$$

$$z = z_0 \cos\psi + \left(\dot{z}_0 / n\right)\sin\psi$$

$$\psi = nt$$

With θ = argument of latitude and initial conditions at the equator

$$x = \delta a - a\delta e \cos M$$

$$y = a\left(\delta\theta_0 + \delta\Omega\cos i + 2\delta e \sin\omega\right) - 1.5\delta a\theta + 2a\delta e \sin M$$

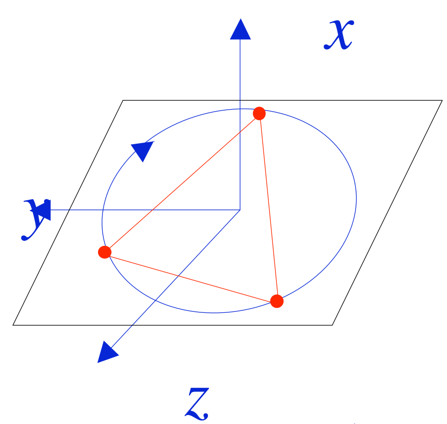
$$z = a\left(\delta i \sin\theta - \delta\Omega \sin i \cos\theta\right)$$

Periodic Solutions

Leader-Follower, x=z=0, y=const

$$x = A\sin(\psi+\beta), y = 2A\cos(\psi+\beta), z = B\sin(\psi+\alpha)$$

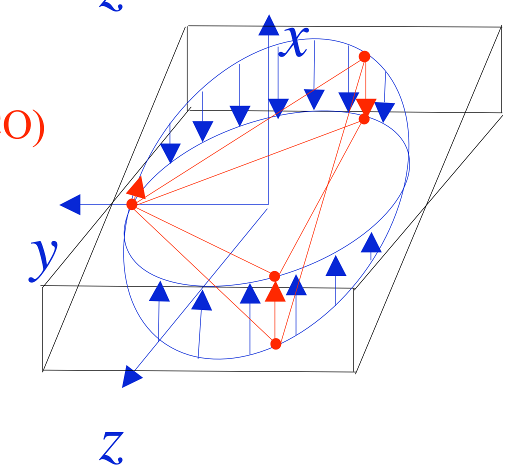
$x^2 + (y/2)^2 = A^2$; 2-1 ellipse



Circular relative orbit

B=$\sqrt{3}A, \alpha = \beta$

$x^2 + y^2 + z^2 = 4A^2$



Horizontal Plane Circular Orbit or Projected Circular Orbit (PCO)

A=B/2 , α=β

$y^2 + z^2 = B^2$

CW Equations (cont)

$$\ddot{x} - 2n\dot{y} - 3n^2 x = u_x + O(\rho / R) + O(e) + O(J_2)$$
$$\ddot{y} + 2n\dot{x} = u_y + O(\rho / R) + O(e) + O(J_2)$$
$$\ddot{z} + n^2 z = u_z + O(\rho / R) + O(e) + O(J_2)$$

(u_x, u_y, u_z) are the control forces

Due to $J_2 : e \geq O(J_2)$

For formations of size 1-30 km the neglected nonlinear terms are of $O(10^{-3})$. Therefore, all the neglected terms in general are of the same order of magnitude, so for a theory or model to be valid for all types of relative motion orbits it must include all these effects. If curvilinear coordinates are used instead of Cartesian coordinates then the nonlinear terms are generally not needed for orbits less that 30-40 km. As will be seen later the differential J_2 effects are small for some relative motion orbits and can be neglected.

The Problem

- Precise relative position knowledge is important, precision control often is not important. Position maintenance within bounds is sufficient.
- Minimum fuel control is essential.
- Maintenance of nominal relative motion orbits results in different fuel requirements for each satellite.
- System lifetime is defined by the first satellite becoming non-operational.
- Equal fuel consumption/satellite over a long period of time.
- During thruster firings many satellites are not operational and the relative navigation could be disturbed. Maximize time between thruster firings.

The Problem (cont)

- Mathematical model is accurate.
- System can usually be described by linear equations.
- Control is easy, minimum fuel control for the real problem with equal fuel consumption for each satellite over a long period of time is not easy

This is not a standard Linear-Quadratic-Gaussian control problem.

Philosophy/Principles/Approach

- Minimum fuel orbits will result from finding the initial conditions that yield the orbits that require minimum fuel to maintain. ⇨ Nonlinear analysis
- Minimum fuel control will result from the most accurate model.
- Use, do not fight Kepler.
- Must take into account gravitational perturbations.
- Build on our extensive knowledge of orbital mechanics and optimal orbit maneuvers.
- To maximize system lifetime all satellites should have near equal fuel consumption.
- The mathematical model should be consistent with the needs of the mission and the accuracy of the relative navigation system.

Know and use the physics

Elliptic Reference Orbit

$u = x / r_c . v = y / r_c , w = z / r_c$

change independent variable; use true anomaly f

$$\frac{\mathrm{d}}{\mathrm{dt}} = \frac{df}{dt}\frac{d}{df} = \frac{h}{r_c^2}\frac{d}{df}$$

$$\frac{d^2}{dt^2} = \left(\frac{h}{r_c^2}\right)^2 \frac{d^2}{df^2} + \frac{2h^3}{r_c^5}\frac{e\sin f}{p}\frac{d}{df}$$

$$u'' - 2v' - 3u / \left(1 + e\cos f\right) = 0$$

$$v'' + 2u' = 0$$

$$z'' + z = 0$$

1 . Lawden, D. F., *Optimal Trajectories forSpace Navigation*, Butterworths, London, 1963.

2 . Tschauner, J. and Hempel, P., “Optimale Beschleunigeungsprogramme fur das Rendezvous-Manouever,” *Astronautica Acta*, Vol. 10, 1964, pp. 296-307.

3 . Carter, T. E., “New Form for the Optimal Rendezvous Equations near a Keplerian Orbit,” *AIAA J. of Guidance, Control and Dynamics*, Vol. 13, No. 1, Jan.-Feb. 1990.

Nonlinear Effects

λ=out-of-plane angle

$\tan\lambda = \dfrac{z}{r}$

The linearized equations of motion are

$\delta r'' - 2r_c\phi' - 3n^2\delta r = 0$

$r_c\phi'' + 2\delta r' = 0$

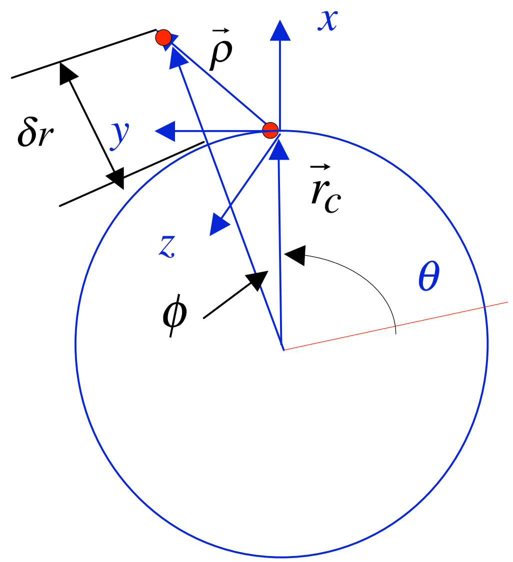
$\lambda'' + n^2\lambda = 0$

Identical to CW equations with $x \to \delta r, y \to r_c\phi, z \to r_c\lambda$

$r = r_c + \delta r$

rather than

$r = \left[\left(r_c + x\right)^2 + y^2 + z^2\right]^{1/2}$



Use of curvilinear coordinates reduces the nonlinearity effects.

Nonlinear Effect References

1. Karlgard, C. D. and Lutze, F. H., “Second Order Relative Motion Equations,” AAS 01-464, *AAS/AIAA Astrodynamics Specialist Conference*, Quebec City, Quebec, July 2001.
2. Mitchell, J. W., and Richardson, D. L., “A Third Order Analytical Solution for Relative Motion with a Circular Reference Orbit,” Paper AAS 02-147, *AAS/AIAA 2002 Space Flight Mechanics Conference*, San Antonio, TX, January 2002.
3. Vaddi, S. S., Vadali, S. R. and Alfriend, K. T., “Formation Flying: Accommodating Nonlinearity and Eccentricity Perturbations,” *AIAA J. of Guidance, Control and Dynamics*, Vol. 26, No. 2, Mar.-Apr. 2002, pp. 214-223.
4. Alfriend, K. T. and Yan, H., “An Orbital Elements Approach to the Nonlinear Formation Flying Problem,” International Formation Flying Conference: Missions and Technologies, Toulouse, France, October 2002.

Refs. 1-3 treat the adjustment of the initial conditions for the CW equations in Cartesian coordinates to obtain periodic orbits. Ref. 4 is a nonlinear theory using differential orbital elements that includes J2 effects.

Design With Differential Orbital Elements

Why?

For a circular reference orbit

1. The semi-major axis of the in-plane 2x1 ellipse is $0.5a\delta e$.
2. The maximum out-of-plane distance is $a[(\delta i)^2+(\delta\Omega sin)^2]^{1/2.}$
3. The argument of perigee and initial mean anomaly determine the phasing.

These basic concepts continue to hold with non-circular reference orbits. Thus, it is easier to design the relative motion orbit with differential orbital elements rather than trying to find the initial conditions in the relative Cartesian space.

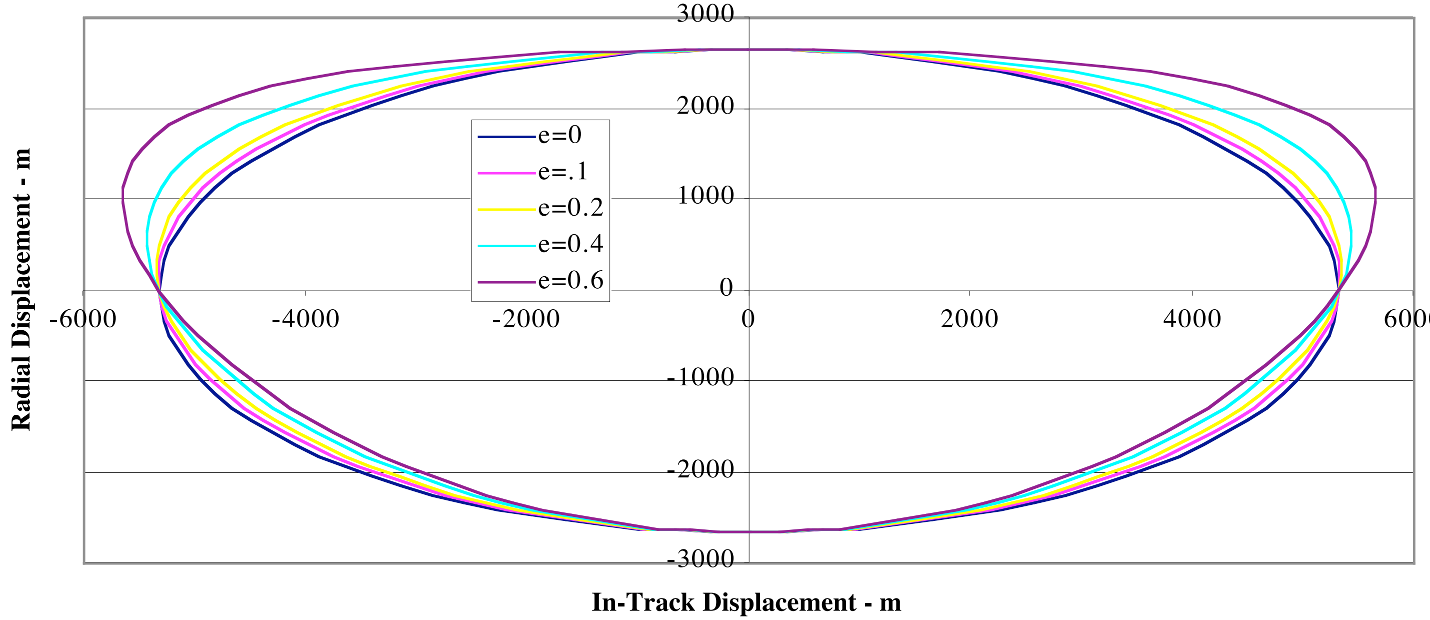
$$x = \delta a - a\delta e \cos M$$

$$y = a\left(\delta\theta_0 + \delta\Omega\cos i + 2\delta e\sin\omega\right) - 1.5\delta a\theta + 2a\delta e\sin M$$

$$z = a\left(\delta i\sin\theta - \delta\Omega\sin i\cos\theta\right)$$

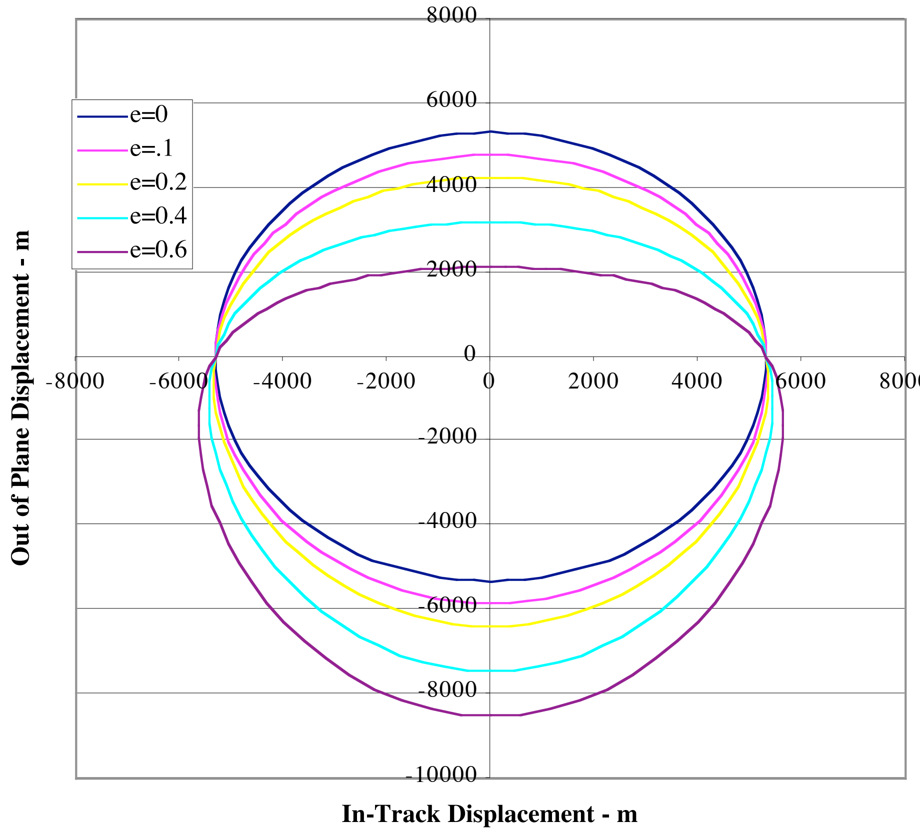
Orbital Element Design -In-Plane

a=26,608 km, δe=0.0001, vary e



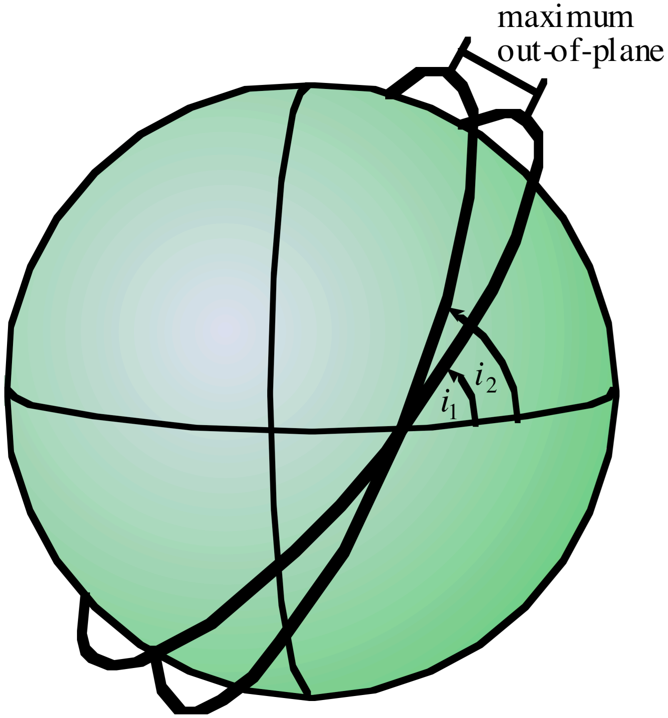
Orbital Element Design-Out-of-Plane

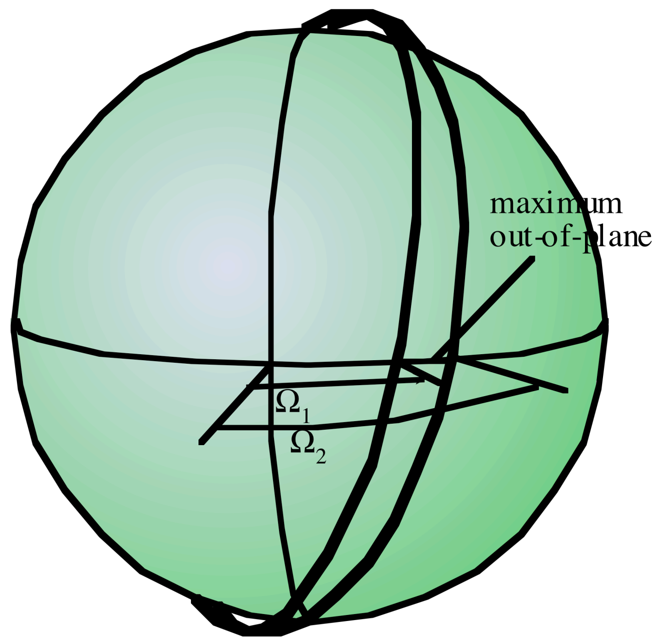
a=26,608 km, δe=0.0001, δi=0.0002, vary e



Gravitational Perturbation Effects

Out of Plane Motion





Gravitational Perturbation Effects

- J_2 is the dominant effect
  - Nodal precession $\dot{\Omega} = -1.5 J_2 (R_e / p)^2 n \cos i$
  - Perigee drift $\dot{\omega} = 0.75 J_2 (R_e / p)^2 n (5\cos^2 i - 1)$
  - Mean motion $\delta n = -0.75 J_2 (R_e / p)^2 \eta (1 - 3\cos^2 i) n$

$$\delta\dot{\Omega} = 1.5 J_2 \left(\frac{R_e}{p}\right)^2 n \sin i \,\delta i - 3 J_2 \left(\frac{R_e}{p}\right)^2 \frac{n \cos i}{(1-e^2)} \delta(e^2) + 3 J_2 \left(\frac{R_e}{p}\right)^2 n \cos i \frac{\delta a}{a}$$

Similar effects for perigee rotation and change of period

Gravitational Perturbation Effects

- Both absolute and differential gravitational effects must be considered.
- For two satellites to remain close (quasi-periodic relative motion orbits) the differential mean angle rates must be zero.
- Differential mean rates are created by changes in (a,e,i).
- Design orbits in mean element space, not osculating and transform to osculating for the ICs.

Mean Elements

Delaunay variables

$$l = mean\ \ anomaly, g = arg.of\ perigee, h = rt.ascension$$

$$L = \sqrt{ua}, G = L\eta,\ H = G\cos i,\ \eta = \left(1 - e^2\right)^{1/2}$$

Hamiltonian (normalized variables)

$$M = -\frac{1}{2L^2} + J_2\frac{\mu}{2r}\left(\frac{R_e}{r}\right)^2\left(1 - 3\sin^2 i\sin^2\theta\right)$$

Non-Averaged

$$M = -\frac{1}{2L^2} - J_2\frac{1}{4L^6}\left(\frac{L}{G}\right)^3\left(1 - 3\frac{H^2}{G^2}\right)$$

Averaged

Angular Rates

$$\dot{l} = \frac{1}{L^3} + \varepsilon\frac{3}{4L^7}\left(\frac{L}{G}\right)^3\left(1 - 3\frac{H^2}{G^2}\right) = \frac{1}{L^3} + \varepsilon\frac{3}{4L^7\eta^3}\left(1 - 3\cos^2 i\right)$$

$$\dot{g} = \varepsilon\frac{3}{4L^7}\left(\frac{L}{G}\right)^4\left(1 - 5\frac{H^2}{G^2}\right) = \varepsilon\frac{3}{4L^7\eta^4}\left(1 - 5\cos^2 i\right)$$

$$\dot{h} = \varepsilon\frac{3}{2L^7}\left(\frac{L}{G}\right)^4\left(\frac{H}{G}\right) = \varepsilon\frac{3}{2L^7\eta^4}\cos i$$

Bounded Relative Motion Orbits

For two satellites to not drift apart

$$\delta\bar{\dot{l}} = \delta\bar{\dot{g}} = \delta\bar{\dot{h}} = 0 \text{ or } \delta\bar{\dot{M}} = \delta\bar{\dot{\omega}} = \delta\bar{\dot{\Omega}} = 0$$

$$\delta L = \delta G = \delta H = 0, \delta a = \delta e = \delta i = 0$$

In the circular orbit, spherical Earth problem this means

$$\delta a = 0 \Rightarrow \text{equal energy}$$

Possible Constraints

Period Matching Constraint (In-track drift)

$$\lambda = l + g = M + \omega$$

$$\delta\dot{\lambda} + \delta\dot{\Omega}\cos i = -\frac{3}{L^4}\delta L + \varepsilon\frac{3}{4L^7\eta^5}\left(3\eta + 4\right)\left[\eta\sin 2i\delta i + \left(3\cos^2 i - 1\right)\delta\eta\right] = 0$$

$$\delta a = -\frac{J_2}{2\eta^5}\left(\frac{R_e}{a}\right)^2\left(3\eta + 4\right)\left[\eta\sin 2i\delta i + \left(3\cos^2 i - 1\right)\delta\eta\right]a$$

$$\delta\dot{\Omega} = -\varepsilon\frac{3}{2L^7\eta^5}\left(4\cos i\delta\eta + \eta\sin i\delta i\right) = 0$$ Out-of Plane Drift

$$\delta\dot{\omega} = \varepsilon\frac{3}{4L^7\eta^5}\left[5\eta\sin 2i\delta i + 4\left(5\cos^2 i - 1\right)\delta\eta\right] = 0$$ Radial Drift

Relative Motion Orbit Classification

Type 1 - Three Constraints

$\delta a = \delta e = \delta i = 0$

Equal energy, angular momentum and polar component of angular momentum

Type 1A: Leader Follower $\delta\Omega=0, \delta\lambda, \delta\omega$ *constant*

Type 1B Linear Out of Plane, $\delta\Omega \neq 0$

No fuel for maintenance.

Type 2 - Two Constraints

Period Matching $\delta a = -\frac{J_2}{2\eta^5}\left(\frac{R_e}{a}\right)^2 (3\eta+4)\left[\eta \sin 2i\delta i + \left(3\cos^2 i - 1\right)\delta\eta\right]a$

Type 2A: J_2 Invariant with Perigee Drift

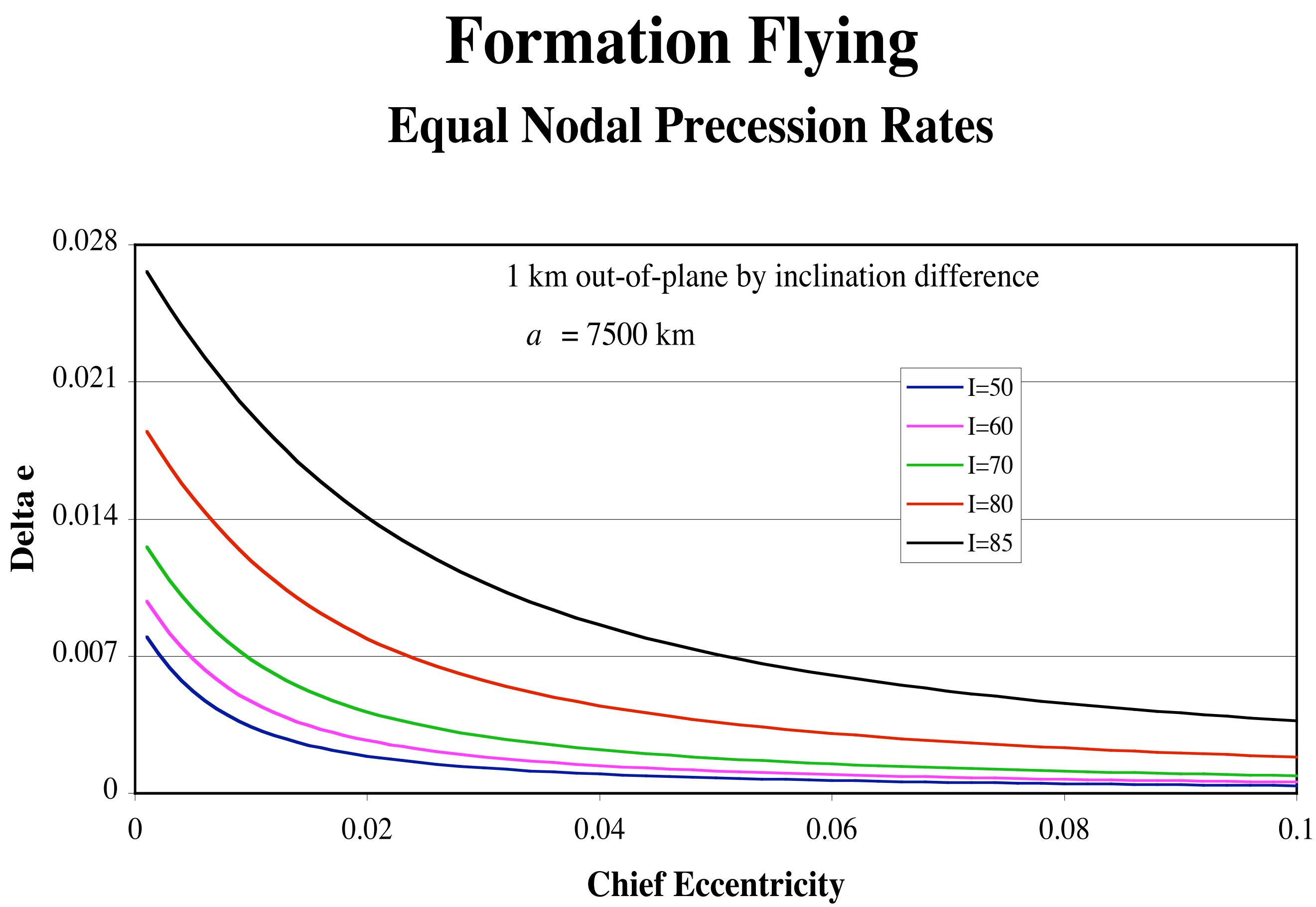
$$\eta \sin i\delta i = -4\cos i\delta\eta, \delta e = -\frac{\eta}{e}\delta\eta$$

Problems with a prescribed $\delta\iota$ and near circular or near polar

$$\Delta v / orbit = 6\varepsilon\pi\left(\frac{R_e}{a}\right)^2 \frac{nae}{\eta^6}\left(1+5\cos^2 i\right)\delta\eta$$

Circular orbit or circular projection orbit with prescribed $\delta\Omega$, $\delta e = \rho/2a$

Class 2A Required δe



Class 2 - Two Constraints

Type 2B: J_2 Invariant with Right Ascension Drift

$$5\eta \sin 2i\delta i = -4\left(5\cos^2 i - 1\right)\delta\eta$$

Same problems as 2A with prescribed δi

$$\Delta v / orbit = 1.2\varepsilon\pi na\left(\frac{R_e}{a}\right)^2\left(1 + 5\cos^2 i\right)\tan i\delta\eta$$

Class 3 One Constraint

Use Period Matching constraint.

Right Ascension Drift

$$\Delta v / orbit = 3\pi J_2 na \sin i \left(\frac{R_e}{a}\right)^2 \left(4\cos i\delta\eta + \eta \sin i\delta i\right)$$

Perigee Drift

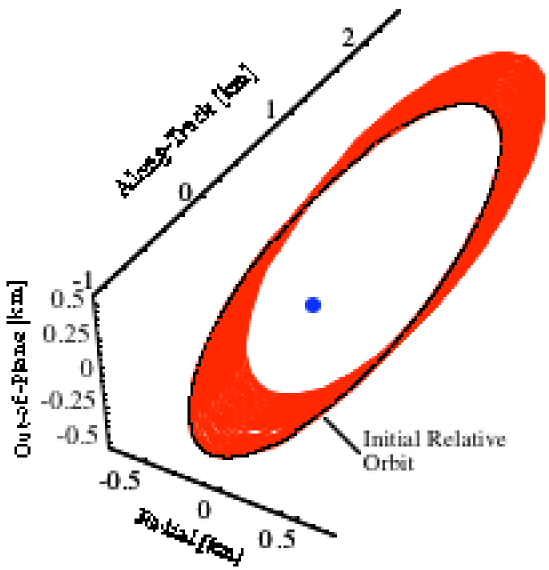
$$\Delta v / orbit = 1.5\pi J_2 nae \left(\frac{R_e}{a}\right)^2 \left[5\eta \sin 2i\delta i + 4\left(5\cos^2 i - 1\right)\delta\eta\right]$$

It may also be necessary to control the absolute perigee drift. To maintain the PCO and GCO this is required because of the different in-plane and out-of-plane frequencies

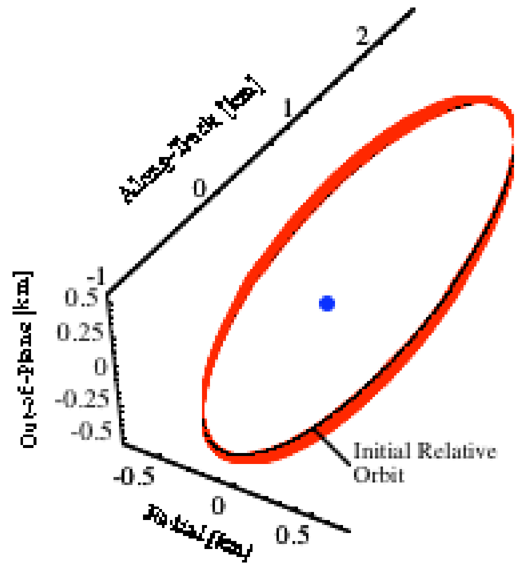
Example 1

$a = 7153$, $e = 0.05$, $i = 48$ deg, $l = h = 0$, $g = 30$ deg

$\delta h = 0.005$, $\delta e = 0.0001$



Osculating Elements



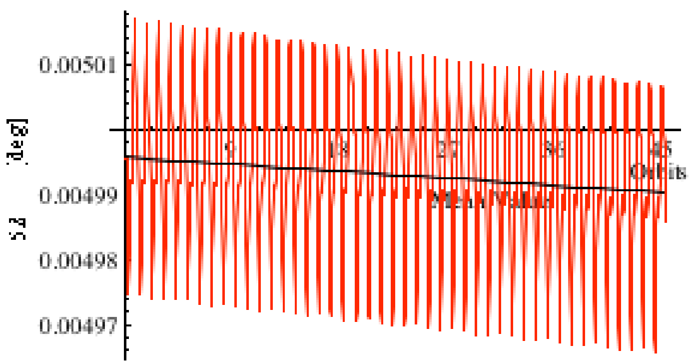
Mean Elements

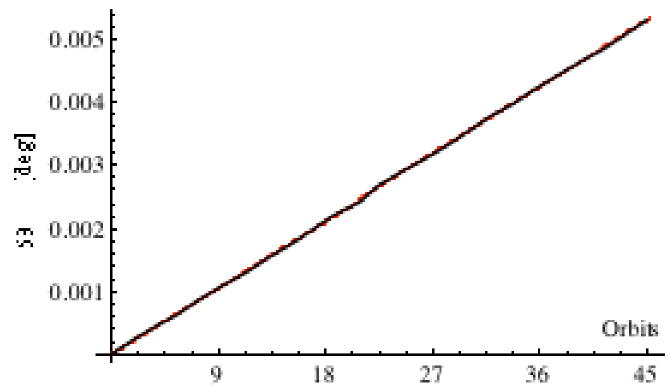
$\delta a = 0.351$ m, $\delta i = 0.001$ deg

Example 1 - (cont)

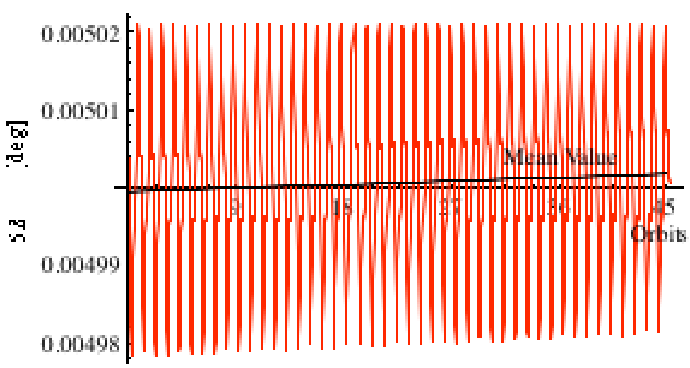
Osculating

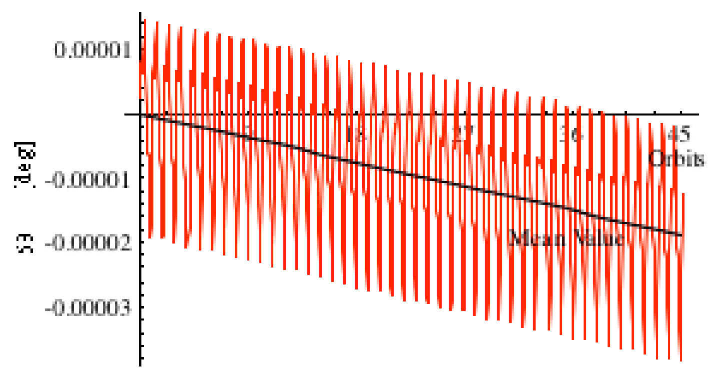
$\delta\Omega$ $\delta\theta$





Mean

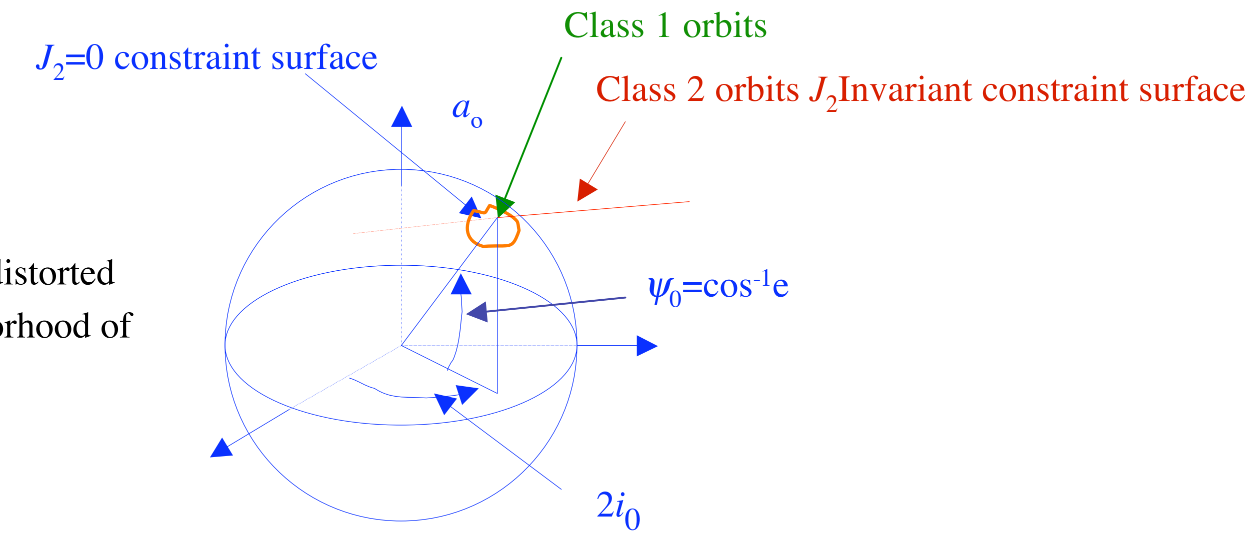




J_2 Changes the Physics

- With no gravitational perturbations the initial conditions that yield bounded relative motion are constrained to a 5-dimensional manifold or in momenta space a 2-dimensional manifold, the sphere of radius a_0, $\delta a=0$.
- The addition of J_2 changes the dimension of the constraint manifold according to the number of constraints

Class 3 orbits are a distorted sphere in the neighborhood of the point



Generalized Relative Motion Solution

Need:

A solution to the relative motion that is valid for all eccentricities and includes at the minimum 1st order absolute and differential gravitation perturbation effects.

Two Possible Approaches

- Relative Orbit Theory - Geometric Method
  - Use existing analytic orbit theories and expand about the Chief orbit to obtain relative orbital elements.
  - Transform from relative orbital element space to relative motion space (x,y,z).
  - Does not consider nonlinear terms
- Generalized Hill's equations
  - Using relative motion variables obtain Hamiltonian including gravitational perturbations, chief orbit ellipticity and nonlinear terms.
  - Transform to action-angle variable space.
  - Use Lie transforms to obtain perturbation solution

Geometric Method

Objective: Obtain a state transition matrix that is valid for all eccentricities and includes the 1st order absolute and differential J2 effects.

$$\mathbf{x}^T = \left(x, \dot{x}, y, \dot{y}, z, \dot{z}\right), \mathbf{e}^T = \left(a, \theta, i, q_1, q_2, \Omega\right), q_1 = e\cos\omega, q_2 = e\sin\omega$$

$$\mathbf{x}\left(t\right) = A\left(\mathbf{e}_c\right)\delta\mathbf{e}_{osc} = A\left(\mathbf{e}_c\right)D\delta\mathbf{e}_m = A\left(\mathbf{e}_c\right)D\Phi_{\mathbf{e}m}\left(t\right)\delta\mathbf{e}_m\left(t_0\right)$$

$$\mathbf{x}\left(t\right) = A\left(\mathbf{e}_c\right)D\Phi_{\mathbf{e}m}\left(t\right)D^{-1}\left(t_0\right)A^{-1}\left(t_0\right)\mathbf{x}\left(t_0\right)$$

$$\mathbf{x}\left(t\right) = \Phi_{\mathbf{x}}\left(t\right)\mathbf{x}_0, \Phi_{\mathbf{x}}\left(t\right) = A\left(\mathbf{e}_c\right)D\Phi_{\mathbf{e}m}\left(t\right)D^{-1}\left(t_0\right)A^{-1}\left(t_0\right)$$

$$D = \frac{\partial\mathbf{e}_{osc}}{\partial\mathbf{e}_m}$$

For a mean element solution $D=I$

Geometric Method - *A* Matrix

$$\vec{R}_d = \vec{R}_c + \vec{\rho} = \left(R + x\right)\vec{e}_{xc} + y\vec{e}_{yc} + z\vec{e}_{zc}$$

$$\vec{R}_d^C = T^{CE}T^{ED}\vec{R}_d^D$$

$$\vec{R}_d^C = T^{CE}\left(T^{EC} + \delta T^{EC}\right)\begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} = \left(I + T^{CE}\delta T^{EC}\right)\begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{R}_d^C = \begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} + R_c T^{CE}\begin{pmatrix} \delta T_{11} \\ \delta T_{12} \\ \delta T_{13} \end{pmatrix} = \begin{pmatrix} R_c + \delta R_c \\ 0 \\ 0 \end{pmatrix} + R_c\begin{pmatrix} 0 \\ \delta\theta + \delta\Omega\cos i_c \\ -\cos\theta_c \sin i_c \delta\Omega + \sin\theta_c \delta i \end{pmatrix}$$

A Matrix

$$\vec{V} = (\dot{R})\hat{e}_x + (R\varpi_n)\hat{e}_y + (-R\varpi_t)\hat{e}_z \equiv (V_{rJ_2})\hat{e}_x + (V_{tJ_2})\hat{e}_y + (V_{nJ_2})\hat{e}_z$$

$$\vec{V}_d = (V_{rJ_2} + \dot{x} - y\varpi_n + z\varpi_t)\,\hat{e}_x + \left(V_{tJ_2} + \dot{y} + x\varpi_n - z\varpi_r\right)\hat{e}_y + \left(V_{nJ_2} + \dot{z} - x\varpi_t + y\varpi_r\right)\hat{e}_z$$

$$\vec{V}_d = \begin{pmatrix} V_{rJ_2} + \delta V_{rJ_2} \\ V_{tJ_2} + \delta V_{tJ_2} \\ V_{nJ_2} + \delta V_{nJ_2} \end{pmatrix} + V_{rJ_2}\begin{pmatrix} 0 \\ \delta\theta + c_i\,\delta\Omega \\ s_\theta\,\delta i - c_\theta\,s_i\,\delta\Omega \end{pmatrix} + V_{tJ_2}\begin{pmatrix} -\delta\theta - c_i\,\delta\Omega \\ 0 \\ c_\theta\,\delta i + s_\theta\,s_i\,\delta\Omega \end{pmatrix} + V_{nJ_2}\begin{pmatrix} -s_\theta\,\delta i + c_\theta\,s_i\,\delta\Omega \\ -c_\theta\,\delta i - s_\theta\,s_i\,\delta\Omega \\ 0 \end{pmatrix}$$

The angular velocity $\vec{\varpi} = \left(\varpi_r, \varpi_t, \varpi_n\right)$ is different for mean and osculating elements.

Example

Circular horizontal plane orbit with 500 m radius

Chief Elements

$$a_c = 7100 km$$
$$e_c = 0.005$$
$$i_c = 70 \deg$$
$$\Omega_c = \omega = f = 0$$
$$\theta_c = 0$$
$$q_{1c} = 0.005 \; q_{2c} = 0$$
$$n = \sqrt{\frac{\mu}{a^3}} = 1.05531 \times 10^{-3} r/s$$

Deputy Elements

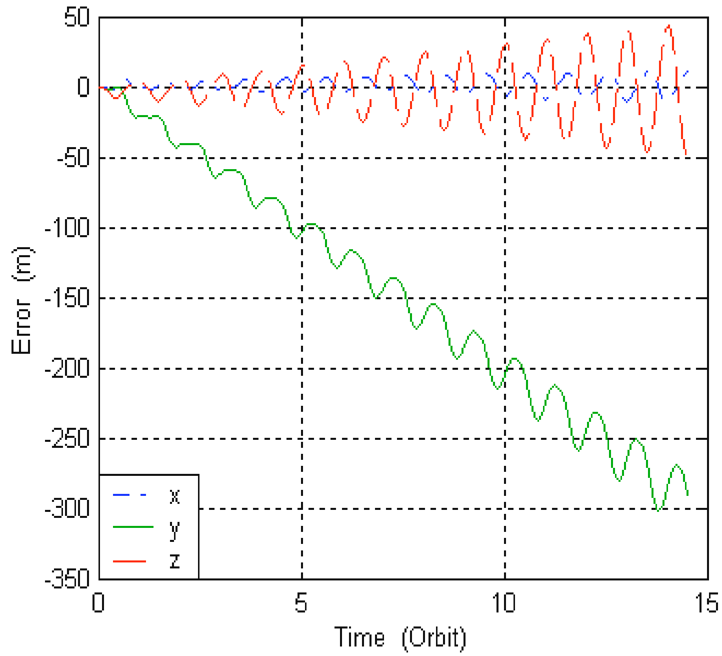
$$x = 0$$
$$\dot{x} = n_c / 4$$
$$y = 0.5 km$$
$$\dot{y} = 0$$
$$z = 0$$
$$\dot{z} = n / 2$$

$$\delta a = 0$$
$$\delta\theta = 0.004055 \deg$$
$$\delta i = 0.0040148 \deg$$
$$\delta q_1 = 0$$
$$\delta q_2 = -3.556 \times 10^{-5}$$
$$\delta\Omega = 0$$

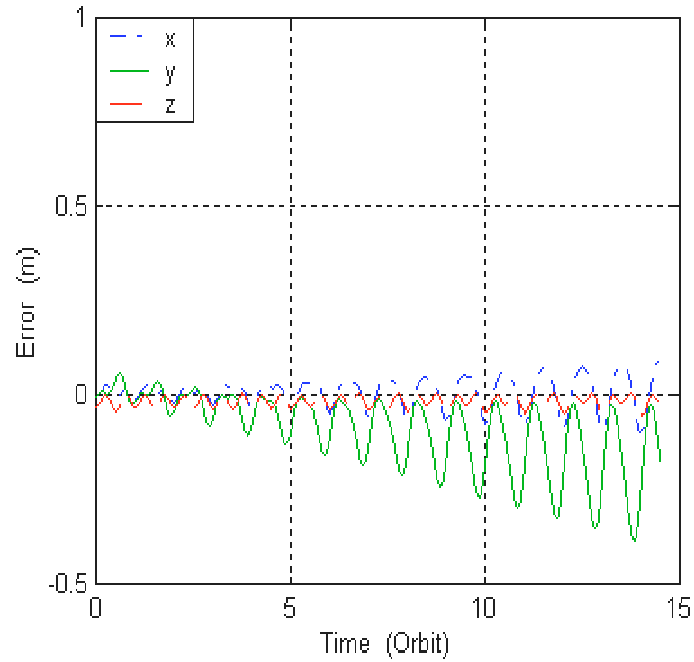
Results

a=7100 km, e=0.005, i=70 deg

Circular projected orbit, ρ=500 m



CW equations



Geometric Method

References

1. Gim, D-W and Alfriend, K.T., "The State Transition Matrix of Relative Motion for the Perturbed Non-Circular Reference Orbit," AIAA J. of Guidance, Control and Dynamics, Vol. 26, No. 6 Nov-Dec 2003, pp 956-971.
2. Gim, D-W and Alfriend, K.T., "Satellite Relative Motion Using Differential Equinoctial Elements," Paper No. AAS 02-186,2002 AAS/AIAA Space Flight Mechanics Conference, San Antonio, January 26-29, 2002, to appear in Int. J. of Celestial Mechanics and Dynamical Astronomy. Same as 1 but with differential equinoctial elements to avoid all element singularities.

Theory Comparison

or

Which relative motion theory do you use?

Problem

- For a given problem how do you select which theory or model to use?
- How do you compare accuracy?
- What is acceptable accuracy?

Modeling Error Index

Given the system

$\dot{x} = f(x,t), x(0) = x_0$

The initial conditions $x_{io}, i = 1, N$ are initial conditions on the desired relative motion orbit, e.g., projected circular orbit.

Let $\bar{x}_i(t)$ be the exact solution for the initial condition $\bar{x}_i(0)$ and let $x_i(t)$ be the solution for the model. Non-dimensionalize the variables by $y = Wx$.

The modeling error index is defined to be

$$\nu(t) = \sup_{i=1,N} \left| \frac{\bar{y}_i^T(t)\bar{y}_i(t)}{y_i^T(t)y_i(t)} - 1 \right|$$

For a one dimensional system $y = \bar{y}(1+\varepsilon)$

$\nu = 2\varepsilon$

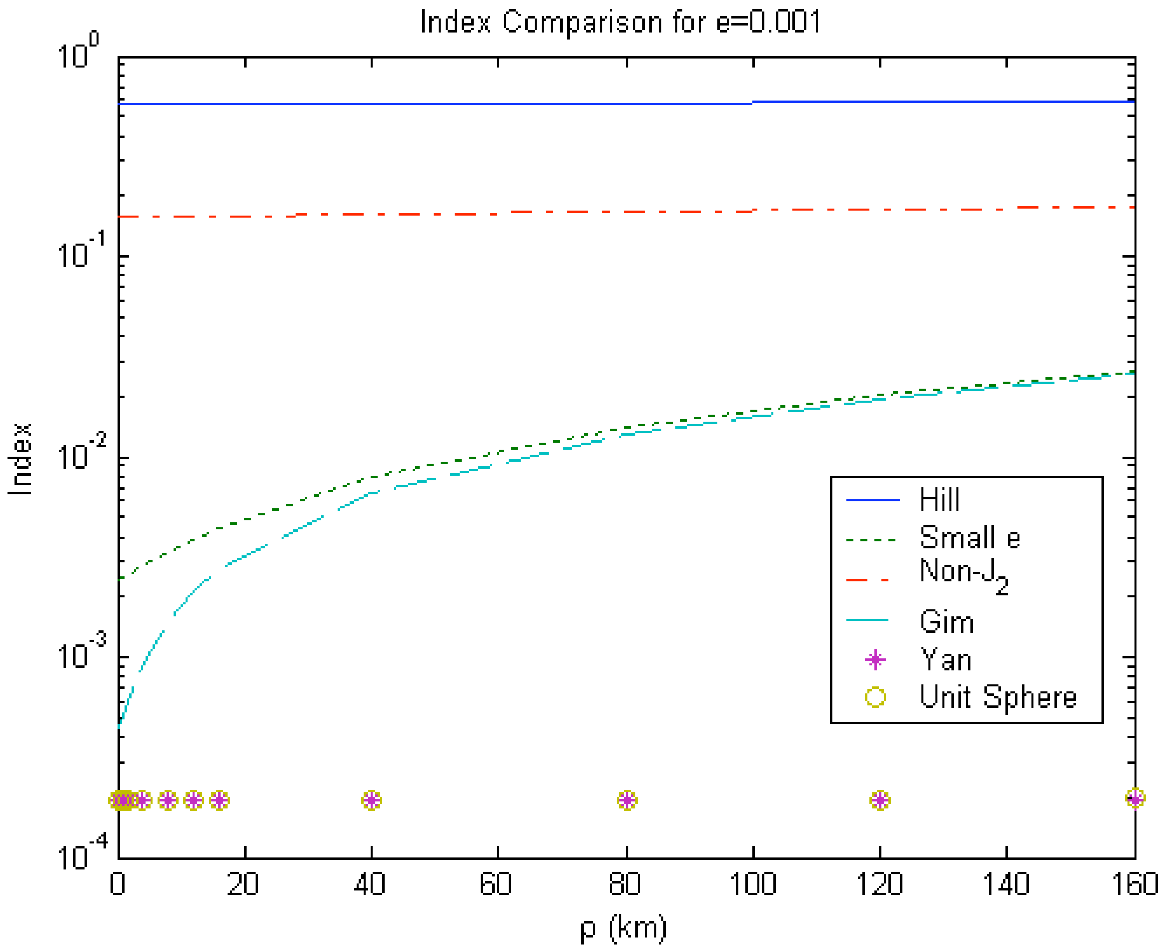
Theories Compared

- Hill, C-W
- Linear, spherical Earth (non-J_2)
  - Lawden, Tschauner-Hempel, Carter, Garrison
  - Obtained by setting $J_2=0$ in Gim-Alfriend
- Small eccentricity.
  - Contains $O(e)$ for non-J_2 terms and $e=0$ J_2 terms
- Gim-Alfriend
- Vadali Unit Sphere
- Yan-Alfriend nonlinear theory

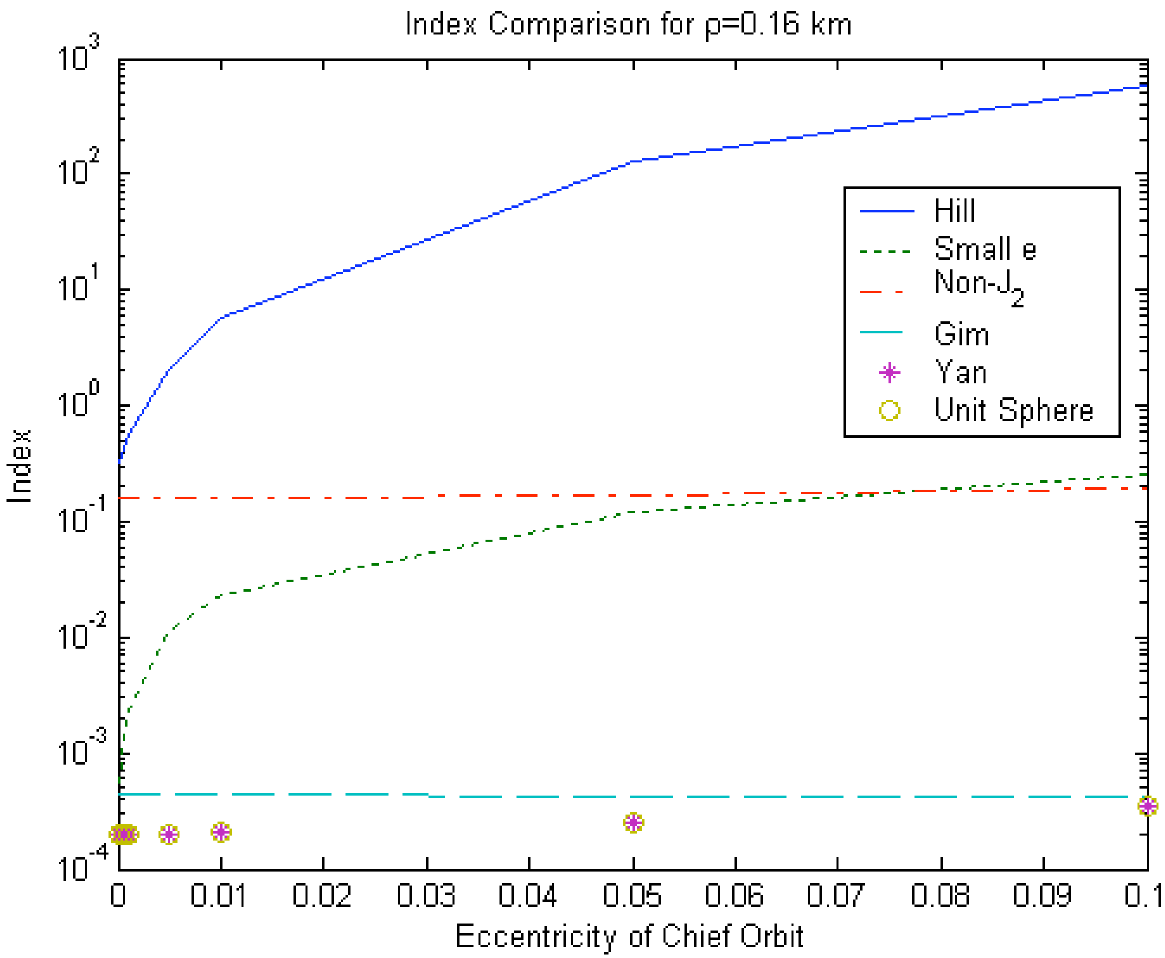
Comparison

- a=8000 km, i=50 deg
- Projected circular orbit (PCO)
- Vary e and ρ (size of PCO)
- 100 equally spaced points on the PCO, calculate error index after 1 day.

Results - e=0.001



Results ρ=160 m



Comparison Summary

- Need to include chief eccentricity in the model.
- Need to include J_2 in the model if the relative motion orbit involves a differential inclination.
- Nonlinear effects are small for relative motion orbits up to 40 km in size if curvilinear coordinates are used.
- Small e theory with J_2 is good for $e<0.005$

Natural Dynamics

Purpose: Investigate utilizing the natural dynamics of the relative motion as best as possible.

Example: For leader-follower in an elliptic orbit one can maintain a constant angular separation with a differential argument of perigee but not a constant distance.

Natural Dynamics

Constant Dimensionless Radius for PCO

$$u_y = u_z = 0$$

$$u_x = 1.5 R\dot{\theta}^2 \left(\frac{e\cos f}{1 + e\cos f} \right) \left(\frac{\rho}{a} \right) \sin(\theta + \alpha) = 1.5 \left(\frac{\rho}{a} \right) \left(\frac{R}{p} \right) R\dot{\theta}^2 e\cos f \sin(\theta + \alpha)$$

Constant Radius

$$u_x = \left(\frac{\rho e}{p} \right) R\dot{\theta}^2 \sin(2f + \beta)$$

$$u_y = \left(\frac{\rho e}{2p} \right) R\dot{\theta}^2 \left[\cos\beta + \cos(2f + \beta) \right]$$

$$u_z = -\left(\frac{\rho e}{2p} \right) R\dot{\theta}^2 \left[3\sin(2f + \beta) - \sin\beta \right]$$

Natural Dynamics

$$J = \frac{1}{T}\int_0^T \left(u_x^2 + u_y^2 + u_z^2\right)dt$$

$$J_{ND} = \frac{9e^2}{64\eta^5}\rho^2 n^4\left[\left(4+3e^2\right) - 2\left(1+e^2\right)\cos 2\beta\right]$$

$$J_D = \frac{n^4\rho^2 e^2}{4\eta^9}\left[\left(8+21e^2-\frac{1}{8}e^4\right) + e^2\left(6+\frac{5}{8}e^2\right)\cos 2\beta\right]$$

$$\frac{J_{ND}}{J_D} = \frac{9}{16}\frac{\left[\left(4+3e^2\right) - 2\left(1+e^2\right)\cos 2\beta\right]}{\left[\left(8+21e^2-\frac{1}{8}e^4\right) + e^2\left(6+\frac{5}{8}e^2\right)\cos 2\beta\right]}$$

$$\frac{J_{ND}}{J_D} \approx \frac{9}{64}\left(2-\cos 2\beta\right)$$

Natural Dynamics

Leader-Follower

$$J_{ND} = 0$$

$$u_y = -\left(R\dot{\theta}^2\right)\frac{\rho}{p}e\cos f$$

$$u_x = 2\left(R\dot{\theta}^2\right)\frac{\rho}{p}e\sin f$$

$$J_D = \frac{\rho^2 e^2 n^4}{32}\left(40 + 42e^2 + 9e^4\right)$$

Circular Relative Orbit

$$u_y = u_z = 0$$

$$u_x = 1.5\left(\frac{\rho}{a}\right)\left(\frac{R}{p}\right)R\dot{\theta}^2 e\cos f\sin\left(\theta + \alpha\right)$$

Constant Radius

$$u_x = \left(\frac{\rho e}{p}\right)R\dot{\theta}^2\sin\left(2f + \beta\right)$$

$$u_y = \left(\frac{\rho e}{2p}\right)R\dot{\theta}^2\left[\cos\beta + \cos\left(2f + \beta\right)\right]$$

$$u_z = -\left(\frac{\sqrt{3}\rho e}{4p}\right)R\dot{\theta}^2\left[3\sin\left(2f + \beta\right) - \sin\beta\right]$$

$$\frac{J_{ND}}{J_D} = \frac{9\eta^4}{4}\frac{\left[\left(4 + 3e^2\right) - 2\left(1 + e^2\right)\cos 2\beta\right]}{\left[\left(51 + 147e^2 + \frac{71}{4}e^4\right) + \left(1 + 42e^2 + \frac{175}{16}e^4\right)\cos 2\beta\right]} \backslash$$

$$\frac{J_{ND}}{J_D} = \frac{9}{2}\frac{\left(2 - \cos 2\beta\right)}{\left(51 + \cos 2\beta\right)}$$

Dynamics Summary

- Include perturbation effects in defining initial conditions for relative motion
- Define relative motions in mean elements, transform to osculating.
  - (a,e,i) or (L,G,H) define relative drift rates
- Conditions derived for (δa, δe, δi) for J_2 invariant relative orbits
  - Can only constrain perigee drift or differential nodal precession
  - Near circular Master orbit and near polar Master orbit require large, possibly undesirable d e for J_2 relative motion invariant orbits
- Look for periodic solutions and determine their stability
  - Establish attractive/stable reference motions
  - Establish/validate optimal navigation and control concepts to maintain these relative motions
- For more precision higher order gravity terms and transformation terms can be included.
- Utilize the natural dynamics to the maximum extent possible