Reachability/Observability Gramians and Balanced Realization of Periodic Descriptor Systems

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Abstract

In this paper necessary and sufficient conditions are derived for complete reachability and observability of periodic time-varying descriptor systems. Applying these conditions, the symmetric positive semi-definite reachability/observability Gramians are defined and can be shown to satisfy some projected generalized discrete-time periodic Lyapunov equations. We propose a numerical method for solving these projected Lyapunov equations, and an illustrative numerical example is given. As an application of our results, the balanced realization of periodic descriptor systems is discussed.

Key words. periodic systems, descriptor systems, reachability and observability Gramians, Hankel singular values, balanced realization, numerical method.

1 Introduction

In the second-half of the last century, the development of systems and control theory, together with the achievements of digital control and signal processing, has set the stage for renewed interests in the study of periodic systems, both in continuous and discrete time; see, e.g., [28, 51, 42, 10, 15, 12] and the survey papers [3, 4]. This has been amplified by specific application demands in the aerospace realm [20, 29, 19], computer control of industrial processes [5] and communication systems [41, 10, 40, 50]. The number of contributions on linear time-varying discrete-time periodic systems has been increasing in recent times; see, e.g., [13, 18, 21, 43, 45, 47] and the references therein. This increasing interest in periodic systems has also been motivated by the large variety of processes that can be modelled through linear discrete-time periodic systems (e.g., multirate sampled-data systems, chemical processes, periodically time-varying filters and networks, and seasonal phenomena [2, 3, 6, 14, 27, 32, 49]).

We consider here periodic time-varying descriptor systems of the form

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbb{Z}, \tag{1.1}$$

where the matrices E_k, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $C_k \in \mathbb{R}^{p\times n}$ are periodic with period $K \geq 1$, i.e., $E_k = E_{k+K}$, $A_k = A_{k+K}$, $B_k = B_{k+K}$, $C_k = C_{k+K}$, and the matrices E_k are allowed to be singular for all k. Recently, this class of periodic descriptor systems (1.1) is discussed and studied extensively in the problem of solvability and conditionability [34], the computation of H_∞-norm and system zeros [33, 48], and the compensating and regularization problems for periodic descriptor systems [8, 22].

It is well known that the dynamics of the discrete-time periodic descriptor system (1.1) depend critically on the regularity and the eigenstructure of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ which satisfy the homogeneous systems of (1.1):

$$E_k x_{k+1} = A_k x_k, \quad k \in \mathbb{Z}. \tag{1.2}$$

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The matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are said to be regular when $\det[C((\alpha_k, \beta_k)_{k=0}^{K-1})] \not\equiv 0$, where

$$C((\alpha_k, \beta_k)_{k=0}^{K-1}) \equiv \begin{bmatrix} \alpha_0 E_0 & 0 & \cdots & 0 & -\beta_0 A_0 \\ -\beta_1 A_1 & \alpha_1 E_1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & 0 & -\beta_{K-1} A_{K-1} & \alpha_{K-1} E_{K-1} \end{bmatrix}, \tag{1.3}$$

in which α_k, β_k are complex variables for $k = 0, \ldots, K-1$.

Definition 1.1. *[25] Let $\{(E_k, A_k)\}_{k=0}^{K-1}$ be $n \times n$ regular matrix pairs. If there exist complex numbers $\alpha_0, \cdots, \alpha_{K-1}$, $\beta_0, \cdots, \beta_{K-1}$ which satisfy*

$$\det[C((\alpha_k, \beta_k)_{k=0}^{K-1})] = 0 \ , \quad \left(\prod_{k=0}^{K-1} \alpha_k, \prod_{k=0}^{K-1} \beta_k \right) \equiv (\pi_\alpha, \pi_\beta) \neq (0, 0) \tag{1.4}$$

then (π_α, π_β) is an eigenvalue pair of $\{(E_k, A_k)\}_{k=0}^{K-1}$.

Note that if (π_α, π_β) is an eigenvalue pair of $\{(E_k, A_k)\}_{k=0}^{K-1}$, then (π_α, π_β) and $(\tau\pi_\alpha, \tau\pi_\beta)$ represent the same eigenvalue for any nonzero τ. If $\pi_\beta \neq 0$, then $\lambda = \pi_\alpha / \pi_\beta$ is a finite eigenvalue; otherwise $(\pi_\alpha, 0)$ represents an infinite eigenvalue. The spectrum, or the set of all eigenvalue pairs, of $\{(E_k, A_k)\}_{k=0}^{K-1}$ is denoted by $\sigma\left(\{(E_k, A_k)\}_{k=0}^{K-1}\right)$. We shall assume throughout the paper that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are regular, and also use the notation $\sigma(M)$ to denote the spectrum of a square matrix M.

It is easily seen that the determinant of $C((\alpha_k, \beta_k)_{k=0}^{K-1})$ is a homogeneous polynomial in π_α and π_β of degree n of the form

$$\sum_{k=0}^{n} c_k \pi_\alpha^k \pi_\beta^{n-k}, \tag{1.5}$$

where $c_0, \cdots, c_n$ are complex numbers uniquely determined by $\{(E_k, A_k)\}_{k=0}^{K-1}$. For the regular matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$, at least one of the c_k's is nonzero, and hence we see from Definition 1.1 that there are exact n eigenvalue pairs (counting multiplicity) for $\{(E_k, A_k)\}_{k=0}^{K-1}$.

It was shown in [34] that the solvability of (1.2) is equivalent to the condition that the pencil

$$\alpha\mathcal{E} - \beta\mathcal{A} := \begin{bmatrix} \alpha E_0 & 0 & \cdots & 0 & -\beta A_0 \\ -\beta A_1 & \alpha E_1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & 0 & -\beta A_{K-1} & \alpha E_{K-1} \end{bmatrix} \tag{1.6}$$

is regular i.e. $\det(\alpha\mathcal{E} - \beta\mathcal{A}) \not\equiv 0$. From (1.5) it is easy to check that

$$\sigma\left(\{(E_k, A_k)\}_{k=0}^{K-1}\right) = \left\{(\alpha^K, \beta^K) \,|\, \det(\alpha\mathcal{E} - \beta\mathcal{A}) = 0\right\}. \tag{1.7}$$

Hence, from (1.7), the solvability of (1.2) is equivalent to the regularity of $\{(E_k, A_k)\}_{k=0}^{K-1}$.

For discrete-time descriptor systems, the concepts of reachability and observability Gramians, causal and noncausal Hankel singular values, and balanced realization are well-established [1, 39]. Moreover, numerical methods are proposed in [35] to solve the projected generalized Lyapunov equations for continuous-time descriptor systems. However, to our best knowledge, similar results have not been developed for periodic descriptor systems.

In summary, there are three main contributions from this paper. First, in Section 3, we give a set of necessary and sufficient conditions of complete reachability and observability for the periodic time-varying descriptor system (1.1). Second, with the aid of the fundamental matrices $\varphi_{i,j}$ defined as in (2.6), the reachability/observability Gramians and their corresponding projected generalized discrete-time periodic Lyapunov equations (GDPLE) are derived in terms of the original system matrices E_k, A_k, B_k and C_k, $k = 0, 1, \ldots, K-1$, respectively. These fundamental matrices play an important role here and are not natural extension of those defined for the descriptor system with period $K = 1$ [35, 39]. Third, in Sections 6 and 7, Hankel singular values and balanced realization are discussed, for the first time, for completely reachable and observable periodic descriptor systems. These concepts are likely to be crucial in the model reduction problem of periodic descriptor systems.

This paper is organized as follows. Section 2 contains some notations and definitions, as well as some preliminary results. In Section 3 the necessary and sufficient conditions are derived for complete reachability and observability of periodic descriptor systems, respectively. With these equivalent conditions, the periodic reachability and observability Gramians, which satisfy some generalized periodic Lyapunov equations, are developed in Section 4. In Section 5 we propose a numerical method for solving these equations under the assumption of pd-stability. A numerical example is given to illustrate its feasibility and reliability. The concept of Hankel singular values is generalized for periodic descriptor systems in Section 6. The problem of balanced realization for the completely reachable and completely observable periodic descriptor systems is discussed in Section 7.

2 Preliminaries

For period $K = 1$ and a regular matrix pair (E, A), it is well known that the discrete-time descriptor system (E, A, B, C) is asymptotically stable if and only if all finite eigenvalues of (E, A) lie inside the unit circle [11, 36, 37]. Similarly, the asymptotic stability of the periodic descriptor system (1.1) can be characterized in terms of the spectrum of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$.

Definition 2.1. *Let $\{(E_k, A_k)\}_{k=0}^{K-1}$ be $n \times n$ regular matrix pairs. The periodic descriptor system (1.1) is asymptotically stable if and only if all finite eigenvalues of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ lie inside unit circle. The periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are called pd-stable if the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are regular and all their finite eigenvalues lie inside the unit circle.*

In a similar fashion to the Kronecker canonical form for a regular matrix pair, we can transform regular periodic matrix pairs into periodic Kronecker canonical forms [22].

Lemma 2.1. *Suppose that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ in systems (1.1) are regular. Then for $k = 0, \ldots, K-1$, there exist nonsingular matrices X_k and Y_k such that*

$$X_k E_k Y_{k+1} = \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix}, \quad X_k A_k Y_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix}, \tag{2.1}$$

where $Y_K \equiv Y_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f \equiv J_k$ is an $n_1 \times n_1$ Jordan matrix corresponding to the finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b \equiv N_k$ is an $n_2 \times n_2$ nilpotent Jordan matrix corresponding to the infinite eigenvalues, and $n = n_1 + n_2$.

Remark. If ν_k is the nilpotency of the nilpotent matrix N_k for $k = 0, 1, \ldots, K-1$, then these K values are defined as the indices [22] of regular periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$. Hence we define the index of the periodic descriptor system (1.1) as $\nu \equiv \max\{\nu_0, \nu_1, \cdots, \nu_{K-1}\}$. We say that the periodic descriptor system (1.1) is of index at most 1 if $\nu \leq 1$, i.e., E_k are all nonsingular or $N_k = 0$ for all k.

For each $k \in \mathbb{Z}$, we let

$$x_k = Y_k \begin{bmatrix} x_k^f \\ x_k^b \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad X_k B_k = \begin{bmatrix} B_k^f \\ B_k^b \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad C_k Y_k = \underset{\quad n_1 \quad\ n_2}{\begin{bmatrix} C_k^f & C_k^b \end{bmatrix}}, \tag{2.2}$$

and by using Lemma 2.1 we can decompose the original system (1.1) into forward and backward periodic subsystems, respectively:

$$x_{k+1}^f = A_k^f x_k^f + B_k^f u_k, \quad y_k^f = C_k^f x_k^f, \tag{2.3}$$
$$E_k^b x_{k+1}^b = x_k^b + B_k^b u_k, \quad y_k^b = C_k^b x_k^b, \tag{2.4}$$

with $y_k = y_k^f + y_k^b, \quad k \in \mathbb{Z}$.

Notice that the state transition matrix of the forward subsystem (2.3) equals $\Phi_f(i,j) = A_{i-1}^f A_{i-1}^f \cdots A_j^f$ when $i > j$ with $\Phi_f(i,i) := I_{n_1}$. The state transition matrix of the backward subsystem (2.4) is $\Phi_b(i,j) = E_i^b E_{i+1}^b \cdots E_{j-1}^b$ when $i < j$ with $\Phi_b(i,i) := I_{n_2}$. The state transition matrix over one period $\Phi_f(\tau+K,\tau) \in \mathbb{R}^{n_1 \times n_1}$ is called the monodromy matrix of the forward subsystem (2.3) at time τ. It is well known that its eigenvalues, called the characteristic multipliers, are independent of τ [44, 26].

For $k = 0, 1, \ldots, K-1$, the $n \times n$ matrices

$$P_r(k) = Y_k \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1}, \quad P_l(k) = X_k^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} X_k, \tag{2.5}$$

are respectively the spectral projections onto the kth right and left deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues. Moreover, the fundamental matrices $\varphi_{i,j}$ $(i, j \in \mathbb{Z})$ of the periodic descriptor system (1.1) are defined by

$$\varphi_{i,j} = \begin{cases} Y_i \begin{bmatrix} \Phi_f(i,j+1) & 0 \\ 0 & 0 \end{bmatrix} X_j, & \text{if } i > j, \\ Y_i \begin{bmatrix} 0 & 0 \\ 0 & -\Phi_b(i,j) \end{bmatrix} X_j, & \text{if } i \leq j. \end{cases} \tag{2.6}$$

These matrices play an essential role for the periodic discrete-time descriptor system (1.1). For the discrete-time descriptor system with period $K = 1$, these fundamental matrices coincide with the coefficient matrices of the Laurent expansion of the generalized resolvent $(\lambda E - A)^{-1}$ at infinity [24, 39].

3 Complete reachability and observability

In this Section we shall give a characterization of complete reachability and observability for the periodic discrete-time descriptor systems (1.1).

Definition 3.1. (i) *The periodic descriptor system (1.1) is reachable at time t if for any state $\bar{x} \in \mathbb{R}^n$, there exist two integers s, ℓ with $s < t < \ell$ and a set of control inputs $\{u_i\}_{i=s}^{\ell}$ which carry $x_s = 0$ into $x_t = \bar{x}$. The periodic descriptor system (1.1) is called completely reachable if it is reachable at all time t.*

(ii) *The forward subsystem (2.3) is reachable at time t if for any state $\bar{\xi}_1 \in \mathbb{R}^{n_1}$, there exists an integer s with $s < t$ and a set of control inputs $\{u_i\}_{i=s}^{t-1}$ which carry $x_s^f = 0$ into $x_t^f = \bar{\xi}_1$. The periodic subsystem (2.3) is called completely reachable if it is reachable at all time t.*

(iii) *The backward subsystem (2.4) is reachable at time t if for any state $\bar{\xi}_2 \in \mathbb{R}^{n_2}$, there exists an integer ℓ with $\ell > t$ and a set of control inputs $\{u_i\}_{i=t}^{\ell}$ such that $x_t^b = \bar{\xi}_2$. The periodic subsystem (2.4) is completely reachable if it is reachable at all time t.*

Remark. It is easily seen from Definition 3.1 that the periodic discrete-time descriptor system (1.1) is completely reachable if and only if both its forward and backward subsystems are completely reachable.

Theorem 3.1 (Forward Reachability). *The following statements are equivalent.*

(a) *The forward subsystem (2.3) is completely reachable.*

(b) *For* $t = 0, 1, 2, \cdots, K-1$, *the matrices*

$$\mathcal{R}^f(t) \equiv \left[B^f_{t-1},\ A^f_{t-1}B^f_{t-2},\ \cdots,\ \Phi_f(t, t-n_1K+1)B^f_{t-n_1K}\right]$$

have full row rank .

(c) *For* $t = 0, 1, 2, \cdots, K-1$, *and*

$$\mathcal{B}^f_t \equiv \left[B^f_{t-1},\ A^f_{t-1}B^f_{t-2},\ A^f_{t-1}A^f_{t-2}B^f_{t-3},\ \cdots,\ \Phi_f(t, t-K+1)B^f_{t-K}\right] ,$$

the matrices

$$\left[\mathcal{B}^f_t,\ \Phi_f(t, t-K)\mathcal{B}^f_t,\ (\Phi_f(t, t-K))^2\mathcal{B}^f_t,\ \cdots,\ (\Phi_f(t, t-K))^{n_1-1}\mathcal{B}^f_t\right]$$

have full row rank.

(d) *For* $\prod_{i=0}^{K-1} \alpha_i \in \sigma(\Phi_f(K, 0))$, *the matrix*

$$U^f(\alpha_0, \cdots, \alpha_{K-1}) \equiv \left[\begin{array}{ccccc|cccc} \alpha_0 I & 0 & \cdots & 0 & -A^f_0 & B^f_0 & & & \\ -A^f_1 & \alpha_1 I & \ddots & & 0 & & B^f_1 & & \\ 0 & -A^f_2 & \ddots & \ddots & \vdots & & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 & & & & \ddots \\ 0 & \cdots & 0 & -A^f_{K-1} & \alpha_{K-1} I & & & & B^f_{K-1} \end{array}\right]$$

has full row rank.

(e) *For* $t = 0, 1, 2, \cdots, K-1$,

$$y^T\Phi_f(t+K, t) = \lambda y^T \ \text{ and } \ y^T\Phi_f(t, j)B^f_{j-1} = 0 \ \text{ for } \ j = t-K+1, \cdots, t-1, t$$

imply $y = 0$.

Proof. (a) $\Rightarrow$ (e): Suppose the statement (a) is true. For any $t \in \{0, 1, \cdots, K-1\}$, assume that

$$y^T\Phi_f(t+K, t) = \lambda y^T \ \text{ and } \ y^T\Phi_f(t, j)B^f_{j-1} = 0 \ \text{ for } \ j = t-K+1, \cdots, t-1, t. \tag{3.1}$$

Since the forward subsystem (2.3) is reachable at time t, there exist an integer s with $s < t$ and control inputs u_i, $s \le i \le t-1$, which carry $x^f_s = 0$ into $x^f_t = y$. Thus, we have

$$y = x^f_t = \sum_{i=s}^{t-1} \Phi_f(t, i+1)B^f_i u_i.$$

Moreover, from the assumptions (3.1), it follows that

$$y^T y = y^T \sum_{i=s}^{t-1} \Phi_f(t, i+1)B^f_i u_i = 0.$$

Therefore, $y = 0$ and hence the condition (e) holds.

(e) $\Rightarrow$ (d): Assume that the condition (e) holds, and let vectors $y_0, y_1, \cdots, y_{K-1} \in \mathbb{R}^{n_1}$ satisfy

$$(y^T_0, y^T_1, \cdots, y^T_{K-1})U^f(\alpha_0, \alpha_1, \cdots, \alpha_{K-1}) = 0,$$

or

$$\begin{cases} \alpha_0 y^T_0 = y^T_1 A^f_1 \\ \alpha_1 y^T_1 = y^T_2 A^f_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{K-2} y^T_{K-2} = y^T_{K-1} A^f_{K-1} \\ \alpha_{K-1} y^T_{K-1} = y^T_0 A^f_0 \end{cases} \quad \text{and} \quad \begin{cases} y^T_0 B^f_0 = 0 \\ y^T_1 B^f_1 = 0 \\ \cdots\cdots\cdots \\ y^T_{K-2} B^f_{K-2} = 0 \\ y^T_{K-1} B^f_{K-1} = 0. \end{cases} \tag{3.2}$$

Notice that for the vector y_{K-1}, it can be easily checked from (3.2), for $j = -K+1, \cdots, -2, -1, 0$, that

$$\begin{aligned} y_{K-1}^T \Phi_f(K,0) &= y_{K-1}^T A_{K-1}^f A_{K-2}^f \cdots A_0^f \\ &= (\alpha_{K-2}\alpha_{K-3} \cdots \alpha_0 \alpha_{K-1}) y_{K-1}^T, \end{aligned}$$

and

$$y_{K-1}^T \Phi_f(0,j) B_{j-1}^f = 0 \ .$$

By condition (e), if the product $\alpha_{K-2}\alpha_{K-3}\cdots\alpha_0\alpha_{K-1} \in \sigma(\{(E_k, A_k)\}_{k=0}^{K-1})$, we then have $y_{K-1} = 0$. Similarly, it can be shown that $y_{K-2} = y_{K-3} = \cdots = y_0 = 0$. Therefore, the matrix $U^f(\alpha_0, \cdots, \alpha_{K-1})$ has full row rank and condition (d) is proved.

(d) $\Rightarrow$ (c): Suppose that (d) holds. It suffices to prove condition (c) for time instant $t = 0$. Since condition (d) holds, it follows that

$$U^f(1, \cdots, 1, \lambda) = \left[\begin{array}{ccccc|cccc} I & 0 & \cdots & 0 & -A_0^f & B_0^f & & & \\ -A_1^f & I & \ddots & & 0 & & B_1^f & & \\ 0 & -A_2^f & \ddots & \ddots & \vdots & & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 & & & & \ddots \\ 0 & \cdots & 0 & -A_{K-1}^f & \lambda I & & & & B_{K-1}^f \end{array} \right] \tag{3.3}$$

has full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$. By elementary row operations, the matrix $U^f(1, \cdots, 1, \lambda)$ can be transformed to

$$\tilde{U}^f \equiv \left[\begin{array}{ccccc|ccccc} I & * & \cdots & \cdots & * & * & & & & \\ 0 & I & \ddots & & \vdots & \vdots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \ddots & & \\ \vdots & & \ddots & I & * & * & \cdots & \cdots & * & \\ 0 & \cdots & \cdots & 0 & \lambda I - \Phi_f(K,0) & \Phi_f(K,1)B_0^f & \cdots & \cdots & A_{K-1}^f B_{K-2}^f & B_{K-1}^f \end{array} \right],$$

which is of full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$. Equivalently, the last row blocks of $\tilde{U}^f$ has full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$, i.e.,

$$\text{rank} \left[\begin{array}{c|c} \lambda I - \Phi_f(K,0) & \mathcal{B}_0^f \end{array} \right] = n_1 \quad \text{for } \ \lambda \in \sigma(\Phi_f(K,0)),$$

where the matrix $\mathcal{B}_0^f$ is defined in the (c). This proves condition (c) for $t = 0$. Similar arguments apply for $1 \le t \le K-1$.

(c) $\Rightarrow$ (b): This follows from the periodicity of the matrices A_k^f and B_k^f, i.e., $A_{k+K}^f = A_k^f$ and $B_{k+K}^f = B_k^f$ for all k.

(b) $\Rightarrow$ (a): Assume that condition (b) holds. For any time $t \in \mathbb{Z}$ (mod K) and any given state $\bar{\xi}_1 \in \mathbb{R}^{n_1}$, there exist an integer $s \equiv t - n_1 K$ and a set of control inputs u_i, $s \le i \le t-1$, which satisfy

$$\sum_{i=s}^{t-1} \Phi_f(t, i+1) B_i^f u_i = \bar{\xi}_1,$$

since $\mathcal{R}^f(t)$ has full row rank. With these control inputs, the given state $\bar{\xi}_1$ can be reached at time t from the initial state $x_s^f = 0$, and hence the complete reachability of the forward subsystem (2.3) is proved. $\square$

Theorem 3.2 (Backward Reachability). *The following statements are equivalent.*

(a) *The backward subsystem (2.4) is completely reachable.*

(b) *For* $t = 0, 1, 2, \cdots, K-1$, *the matrices*

$$\mathcal{R}^b(t) \equiv \left[B_t^b,\, E_t^b B_{t+1}^b,\, \cdots,\, \Phi_b(t, t+\nu K-1) B_{t+\nu K-1}^b\right]$$

have full row rank.

(c) *For* $t = 0, 1, 2, \cdots, K-1$, *and*

$$\mathcal{B}_t^b \equiv \left[B_t^b,\, E_t^b B_{t+1}^b,\, \cdots,\, E_t^b E_{t+1}^b \cdots E_{t+K-2}^b B_{t+K-1}^b\right] \ ,$$

the matrices $\left[N_t,\, \mathcal{B}_t^b\right]$ *have full row rank.*

(d) *The pair* $(\mathcal{E}_b, \mathcal{B}_b)$ *is reachable, where*

$$\mathcal{E}_b \equiv \begin{bmatrix} 0 & E_0^b & & & \\ 0 & 0 & E_1^b & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & & \ddots & E_{K-2}^b \\ E_{K-1}^b & 0 & \cdots & \cdots & 0 \end{bmatrix} \ \text{and}\ \mathcal{B}_b \equiv \begin{bmatrix} B_0^b & & & \\ & B_1^b & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & B_{K-1}^b \end{bmatrix}. \tag{3.4}$$

Proof. (a) $\Leftrightarrow$ (b): For any time $t \in \mathbb{Z}$, it can be easily seen that

$$x_t^b = -\sum_{i=t}^{t+\nu K-1} \Phi_b(t, i) B_i^b u_i = -\mathcal{R}^b(t) \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+\nu K-1} \end{bmatrix}.$$

Therefore, any given state $\bar{\xi}_2 \in \mathbb{R}^{n_2}$ can be reached at time t, i.e., $x_t^b = \bar{\xi}_2$, through a set of control inputs $\{u_i\}_{i=t}^{t+\nu K-1}$ if and only if the matrix $\mathcal{R}^b(t)$ is of full row rank.

(b) $\Leftrightarrow$ (c): Since $N_t = \Phi_b(t, t+K) = E_t^b E_{t+1}^b \cdots E_{t+K-1}^b$ and $N_t^\nu = 0$ for $t = 0, 1, \ldots, K-1$, it follows that

$$\mathcal{R}^b(t) = \left[\mathcal{B}_t^b,\, N_t \mathcal{B}_t^b,\, \cdots,\, N_t^{\nu-1} \mathcal{B}_t^b\right].$$

Thus, $\text{rank}(\mathcal{R}^b(t)) = n_2$ if and only if $\text{rank}\left[\lambda I - N_t,\, \mathcal{B}_t^b\right] = n_2$ for any $\lambda \in \sigma(N_t)$. Since the matrix N_t is nilpotent, $\sigma(N_t) = \{0\}$, and hence $\text{rank}(\mathcal{R}^b(t)) = n_2$ if and only if $\text{rank}\left[-N_t,\, \mathcal{B}_t^b\right] = n_2$. Equivalently, $\text{rank}\left[N_t,\, \mathcal{B}_t^b\right] = n_2$.

(b) $\Leftrightarrow$ (d): Notice that the matrix $\mathcal{E}_b$ is nilpotent with the property that $\mathcal{E}_b^\nu = 0$ and $\mathcal{E}_b^{\nu-1} \neq 0$. It is well known that the pair $(\mathcal{E}_b, \mathcal{B}_b)$ is reachable if and only if $\mathbb{B}_b \equiv \left[\mathcal{B}_b,\, \mathcal{E}_b \mathcal{B}_b,\, \cdots,\, \mathcal{E}_b^{\nu-1} \mathcal{B}_b\right]$ has full row rank. Furthermore, it can be checked that the row blocks of the matrix $\mathbb{B}_b$ are just $\mathcal{R}^b(t)$ with different t. Therefore, statements (b) and (d) are equivalent. $\square$

Definition 3.2. (i) *The periodic descriptor system (1.1) is observable at time* t *if there exist two integers* s, ℓ *with* $s < t < \ell$ *such that any state at time* t *can be determined from knowledge of* $\{y_i\}_{i=s}^{\ell}$ *and* $\{u_i\}_{i=s}^{\ell}$. *The periodic descriptor system (1.1) is called completely observable if it is observable at all time* t.

(ii) *The forward subsystem (2.3) is observable at time* t *if there exists an integer* ℓ *with* $\ell > t$ *such that any state at time* t *can be determined from knowledge of* $\{y_i\}_{i=t}^{\ell}$ *and* $\{u_i\}_{i=t}^{\ell}$. *The periodic subsystem (2.3) is called completely observable if it is observable at all time* t.

(iii) *The backward subsystem (2.4) is observable at time* t *if there exists an integer* s *with* $s < t$ *such that any state at time* t *can be determined from knowledge of* $\{y_i\}_{i=s}^{t}$ *and* $\{u_i\}_{i=s}^{t}$. *The periodic subsystem (2.4) is completely observable if it is observable at all time* t.

Remark. It is easily seen from Definition 3.2 that the periodic discrete-time descriptor system (1.1) is completely observable if and only if both its forward and backward subsystems are completely observable.

We shall state the following Theorems without proofs, which are similar to those of Theorems 3.1 and 3.2.

Theorem 3.3 (Forward Observability). *The following statements are equivalent.*

(a) *The forward subsystem (2.3) is completely observable.*

(b) *For $t = 0, 1, 2, \cdots, K-1$, the matrices*

$$\mathcal{O}^f(t) \equiv \begin{bmatrix} C_t^f \\ C_{t+1}^f A_t^f \\ C_{t+2}^f A_{t+1}^f A_t^f \\ \vdots \\ C_{t+n_1K-1}^f \Phi_f(t+n_1K-1, t) \end{bmatrix}$$

have full column rank.

(c) *For $t = 0, 1, 2, \cdots, K-1$, and*

$$\mathcal{C}_t^f \equiv \left[(C_t^f)^T,\ (A_t^f)^T (C_{t+1}^f)^T,\ \cdots,\ \Phi_f(t+K-1, t)^T (C_{t+K-1}^f)^T \right]^T,$$

the matrices

$$\begin{bmatrix} \mathcal{C}_t^f \\ \mathcal{C}_t^f \Phi_f(t+K, t) \\ \mathcal{C}_t^f (\Phi_f(t+K, t))^2 \\ \vdots \\ \mathcal{C}_t^f (\Phi_f(t+K, t))^{n_1-1} \end{bmatrix}$$

have full row rank.

(d) *For $\prod_{i=0}^{K-1} \alpha_i \in \sigma(\Phi_f(K, 0))$, the matrix*

$$V^f(\alpha_0, \cdots, \alpha_{K-1}) \equiv \left[\begin{array}{ccccc} \alpha_0 I & 0 & \cdots & 0 & -A_{K-1}^f \\ -A_0^f & \alpha_1 I & \ddots & & 0 \\ 0 & -A_1^f & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A_{K-2}^f & \alpha_{K-1} I \\ \hline C_0^f & & & & \\ & C_1^f & & & \\ & & \ddots & & \\ & & & C_{K-2}^f & \\ & & & & C_{K-1}^f \end{array} \right]$$

has full column rank.

(e) *For $t = 0, 1, 2, \cdots, K-1$,*

$$\Phi_f(t+K, t)x = \lambda x \ \ \textit{and} \ \ C_i^f \Phi_f(i, t)x = 0 \ \ \textit{for} \ \ i = t, t+1, \cdots, t+K-1$$

imply $x = 0$.

Theorem 3.4 (Backward Observability). *The following statements are equivalent.*

(a) *The backward subsystem (2.4) is completely observable.*

(b) *For $t = 0, 1, 2, \cdots, K-1$, the matrices*

$$\mathcal{O}^b(t) \equiv \begin{bmatrix} C_t^b \\ C_{t-1}^b E_{t-1}^b \\ C_{t-2}^b E_{t-2}^b E_{t-1}^b \\ \vdots \\ C_{t-\nu K+1}^b \Phi_b(t-\nu K+1, t) \end{bmatrix}$$

have full column rank .

(c) *For $t = 0, 1, 2, \cdots, K-1$, and*

$$\mathcal{C}_t^b \equiv \left[(C_t^b)^T,\, (E_{t-1}^b)^T (C_{t-1}^b)^T,\, \cdots,\, \Phi_b(t-K+1, t)^T (C_{t-K+1}^b)^T\right]^T \ ,$$

the matrices

$$\begin{bmatrix} \mathcal{C}_t^b \\ \mathcal{C}_t^b N_t \\ \mathcal{C}_t^b N_t^2 \\ \vdots \\ \mathcal{C}_t^b N_t^{\nu-1} \end{bmatrix}$$

have full column rank

(d) *The pair $(\mathcal{E}_b, \mathcal{C}_b)$ is observable, where $\mathcal{E}_b$ is defined in (3.4) and $\mathcal{C}_b \equiv \operatorname{diag}(C_0^b, C_1^b, \cdots, C_{K-1}^b)$.*

4 Periodic reachability and observability Gramians

It is well known that Gramians play an important role in many applications, such as the model reduction problem [16, 30, 52]. In this Section, the concepts of reachability and observability Gramians are generalized for periodic discrete-time descriptor systems (1.1).

Consider the causal and noncausal reachability matrices given by

$$\mathcal{R}_+(t) \equiv \left[\varphi_{t,t-1} B_{t-1},\, \varphi_{t,t-2} B_{t-2}, \cdots, \varphi_{t,i} B_i, \cdots\right] \quad (t = 0, 1, \cdots, K-1)$$

and

$$\mathcal{R}_-(t) \equiv \left[\varphi_{t,t} B_t,\, \varphi_{t,t+1} B_{t+1}, \cdots, \varphi_{t,t+\nu K-1} B_{t+\nu K-1}\right] \quad (t = 0, 1, \cdots, K-1),$$

respectively, with $\varphi_{i,j}$ $(i, j \in \mathbb{Z})$ as defined in (2.6).

Definition 4.1 (Reachability Gramians). *Suppose that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable.*

(i) *The causal reachability Gramians of the periodic descriptor system (1.1) are defined by*

$$G_k^{cr} \equiv \mathcal{R}_+(k) \mathcal{R}_+(k)^T = \sum_{i=-\infty}^{k-1} \varphi_{k,i} B_i B_i^T \varphi_{k,i}^T, \quad k = 0, 1, \ldots, K-1.$$

(ii) *The noncausal reachability Gramians of the periodic descriptor system (1.1) are defined by*

$$G_k^{nr} \equiv \mathcal{R}_-(k) \mathcal{R}_-(k)^T = \sum_{i=k}^{k+\nu K-1} \varphi_{k,i} B_i B_i^T \varphi_{k,i}^T, \quad k = 0, 1, \ldots, K-1.$$

(iii) *The reachability Gramians of the periodic descriptor system (1.1) are defined via*

$$G_k^r \equiv G_k^{cr} + G_k^{nr}, \quad k = 0, 1, \ldots, K-1.$$

The causal and noncausal observability matrices are respectively defined by

$$\mathcal{O}_{+}(t) \equiv \left[\varphi_{t,t-1}^{T} C_{t}^{T}, \varphi_{t+1,t-1}^{T} C_{t+1}^{T}, \cdots, \varphi_{i,t-1} C_{i}^{T}, \cdots\right]^{T} \quad (t = 0, 1, \cdots, K-1)$$

and

$$\mathcal{O}_{-}(t) \equiv \left[\varphi_{t-\nu K,t-1}^{T} C_{t-\nu K}^{T}, \varphi_{t-\nu K+1,t-1}^{T} C_{t-\nu K+1}^{T}, \cdots, \varphi_{t-1,t-1}^{T} C_{t-1}^{T}\right] \quad (t = 0, 1, \cdots, K-1).$$

Definition 4.2 (Observability Gramians). *Suppose that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable.*

(i) *The causal observability Gramians of the periodic descriptor system (1.1) are defined by*

$$G_k^{co} \equiv \mathcal{O}_{+}(k)^T \mathcal{O}_{+}(k) = \sum_{i=k}^{\infty} \varphi_{i,k-1}^T C_i^T C_i \varphi_{i,k-1}, \quad k = 0, 1, \ldots, K-1.$$

(ii) *The noncausal observability Gramians of the periodic descriptor system (1.1) are defined by*

$$G_k^{no} \equiv \mathcal{O}_{-}(k)^T \mathcal{O}_{-}(k) = \sum_{i=k-\nu K}^{k-1} \varphi_{i,k-1}^T C_i^T C_i \varphi_{i,k-1}, \quad k = 0, 1, \ldots, K-1.$$

(iii) *The observability Gramians of the periodic descriptor system (1.1) are defined by*

$$G_k^{o} \equiv G_k^{co} + G_k^{no}, \quad k = 0, 1, \ldots, K-1.$$

Remarks. (i) The infinite series appeared in the definition of Gramians G_k^{cr} and G_k^{co} converge because of the pd-stability of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$.

(ii) The Gramians G_k^{cr}, G_k^{nr}, G_k^{co} and G_k^{no} are $n \times n$ symmetric positive semi-definite matrices for all k.

(iii) Definitions 4.1 and 4.2 are natural generalizations of the Gramians defined for descriptor systems with period $K = 1$; see, e.g., [1, 39].

The following theorem indicates that these Gramians of the periodic descriptor system (1.1) satisfy some projected generalized discrete-time periodic Lyapunov equations with special right-hand sides.

Theorem 4.1. *Consider the periodic discrete-time descriptor system (1.1), where the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable.*

(i) *The causal and noncausal reachability Gramians $\{G_k^{cr}\}_{k=0}^{K-1}$ and $\{G_k^{nr}\}_{k=0}^{K-1}$ are the unique symmetric positive semi-definite solutions of the projected GDPLE*

$$\begin{aligned} &E_k G_{k+1}^{cr} E_k^T - A_k G_k^{cr} A_k^T = P_l(k) B_k B_k^T P_l(k)^T, \\ &G_k^{cr} = P_r(k) G_k^{cr} P_r(k)^T, \quad k = 0, 1, 2, \ldots, K-1, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &E_k G_{k+1}^{nr} E_k^T - A_k G_k^{nr} A_k^T = -(I - P_l(k)) B_k B_k^T (I - P_l(k))^T, \\ &P_r(k) G_k^{nr} = 0, \quad k = 0, 1, 2, \ldots, K-1, \end{aligned} \tag{4.2}$$

respectively, where $G_K^{cr} \equiv G_0^{cr}$ and $G_K^{nr} \equiv G_0^{nr}$.

(ii) *The causal and noncausal observability Gramians $\{G_k^{co}\}_{k=0}^{K-1}$ and $\{G_k^{no}\}_{k=0}^{K-1}$ are the unique symmetric positive semi-definite solutions of the projected GDPLE*

$$\begin{aligned} &E_{k-1}^T G_k^{co} E_{k-1} - A_k^T G_{k+1}^{co} A_k = P_r(k)^T C_k^T C_k P_r(k), \\ &G_k^{co} = P_l(k-1)^T G_k^{co} P_l(k-1), \quad k = 0, 1, \ldots, K-1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &E_{k-1}^T G_k^{no} E_{k-1} - A_k^T G_{k+1}^{no} A_k = -(I - P_r(k))^T C_k^T C_k (I - P_r(k)), \\ &G_k^{no} P_l(k-1) = 0, \quad k = 0, 1, 2, \ldots, K-1, \end{aligned} \tag{4.4}$$

respectively, where $G_K^{co} \equiv G_0^{co}$, $G_K^{no} \equiv G_0^{no}$, $E_{-1} \equiv E_{K-1}$ *and* $P_l(-1) \equiv P_l(K-1)$.

(iii) *The reachability and observability Gramians* $\{G_k^r\}_{k=0}^{K-1}$ *and* $\{G_k^o\}_{k=0}^{K-1}$ *are the unique symmetric positive semi-definite solutions of the projected GDPLE*

$$\begin{aligned} &E_k G_{k+1}^r E_k^T - A_k G_k^r A_k^T = P_l(k) B_k B_k^T P_l(k)^T - (I - P_l(k)) B_k B_k^T (I - P_l(k))^T, \\ &P_r(k) G_k^r = G_k^r P_r(k)^T, \quad k = 0, 1, 2, \ldots, K-1, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &E_{k-1}^T G_k^o E_{k-1} - A_k^T G_{k+1}^o A_k = P_r(k)^T C_k^T C_k P_r(k) - (I - P_r(k))^T C_k^T C_k (I - P_r(k)), \\ &P_l(k-1)^T G_k^o = G_k^o P_l(k-1), \quad k = 0, 1, 2, \ldots, K-1, \end{aligned} \tag{4.6}$$

respectively, where $G_K^r \equiv G_0^r$, $G_K^o \equiv G_0^o$, $E_{-1} \equiv E_{K-1}$ *and* $P_l(-1) \equiv P_l(K-1)$.

Proof. We shall verify only (4.1) here and the other cases can be shown similarly. Rewrite (4.1) into an enlarged Lyapunov equation

$$\mathcal{E}\mathcal{G}\mathcal{E}^T - \mathcal{A}\mathcal{G}\mathcal{A}^T = \mathcal{B}\mathcal{B}^T, \tag{4.7}$$

where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \cdots, E_{K-1}), \quad \mathcal{B} = \operatorname{diag}(P_l(0)B_0, P_l(1)B_1, \cdots, P_l(K-1)B_{K-1}),$$

$$\mathcal{A} = \begin{bmatrix} & & & A_0 \\ A_1 & & & \\ & \ddots & & \\ & & A_{K-1} & \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} G_1^{cr} & & & \\ & G_2^{cr} & & \\ & & \ddots & \\ & & & G_0^{cr} \end{bmatrix}.$$

Since the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable, the matrix pencil $\lambda\mathcal{E} - \mathcal{A}$ is regular and all its generalized eigenvalues lie inside the unit circle. Then the Lyapunov equation (4.7) has a unique solution and hence the uniqueness of solutions of the projected GDPLE (4.1) is guaranteed. On the other hand, it can be shown that the causal reachability Gramians G_k^{cr}, $k = 0, 1, \ldots, K-1$, satisfy the projected GDPLE (4.1). Indeed, simple calculation gives that

$$\begin{aligned} & E_k G_{k+1}^{cr} E_k^T - A_k G_k^{cr} A_k^T \\ = \;& E_k \left(\sum_{i=-\infty}^{k} \varphi_{k+1,i} B_i B_i^T \varphi_{k+1,i}^T \right) E_k^T - A_k \left(\sum_{i=-\infty}^{k-1} \varphi_{k,i} B_i B_i^T \varphi_{k,i}^T \right) A_k^T \\ = \;& E_k Y_{k+1} \left(\sum_{i=-\infty}^{k} \begin{bmatrix} \Phi_f(k+1, i+1) & 0 \\ 0 & 0 \end{bmatrix} X_i B_i B_i^T X_i^T \begin{bmatrix} \Phi_f(k+1, i+1)^T & 0 \\ 0 & 0 \end{bmatrix} \right) Y_{k+1}^T E_k^T \\ & - A_k Y_k \left(\sum_{i=-\infty}^{k-1} \begin{bmatrix} \Phi_f(k, i+1) & 0 \\ 0 & 0 \end{bmatrix} X_i B_i B_i^T X_i^T \begin{bmatrix} \Phi_f(k, i+1)^T & 0 \\ 0 & 0 \end{bmatrix} \right) Y_k^T A_k^T \\ = \;& X_k^{-1} \begin{bmatrix} \sum_{-\infty}^{k} \Phi_f(k+1, i+1) B_i^f (B_i^f)^T \Phi_f(k+1, i+1)^T & 0 \\ 0 & 0 \end{bmatrix} X_k^{-T} \\ & - X_k^{-1} \begin{bmatrix} \sum_{-\infty}^{k-1} \Phi_f(k+1, i+1) B_i^f (B_i^f)^T \Phi_f(k+1, i+1)^T & 0 \\ 0 & 0 \end{bmatrix} X_k^{-T} \\ = \;& X_k^{-1} \begin{bmatrix} B_k^f (B_k^f)^T & 0 \\ 0 & 0 \end{bmatrix} X_k^{-T} = P_l(k) B_k B_k^T P_l(k)^T, \end{aligned}$$

and

$$\begin{aligned} & P_r(k) G_k^{cr} P_r(k)^T \\ = \;& Y_k \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1} \left(\sum_{i=-\infty}^{k-1} \varphi_{k,i} B_i B_i^T \varphi_{k,i}^T \right) Y_k^{-T} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} Y_k^T \end{aligned}$$

$$= \quad Y_k \begin{bmatrix} \sum_{i=-\infty}^{k-1} \Phi_f(k,i+1) B_i^f (B_i^f)^T \Phi_f(k,i+1)^T & 0 \\ 0 & 0 \end{bmatrix} Y_k^T = G_k^{cr},$$

for $k = 0, 1, \ldots, K-1$. Therefore, the causal reachability Gramians $\{G_k^{cr}\}_{k=0}^{K-1}$ are the unique symmetric positive semi-definite solutions of the projected GDPLE (4.1). □

The following theorem shows that complete reachability/observability of the periodic descriptor system (1.1) can be characterized via the reachability/observability Gramians.

Theorem 4.2. *Consider the periodic discrete-time descriptor system (1.1). Assume that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable.*

(i) *The periodic descriptor system (1.1) is completely reachable if and only if the reachability Gramians G_k^r are positive definite for $k = 0, 1, 2, \ldots, K-1$.*

(ii) *The periodic descriptor system (1.1) is completely observable if and only if the observability Gramians G_k^o are positive definite for $k = 0, 1, 2, \ldots, K-1$.*

Proof. Here we shall only prove statement (i) and statement (ii) can be verified similarly. For $k = 0, 1, \ldots, K-1$, pre-multiply (4.5) by X_k and post-multiply (4.5) by X_k^T, it follows that

$$X_k E_k Y_{k+1} \widehat{G}_{k+1}^r Y_{k+1}^T E_k^T X_k^T - X_k A_k Y_k \widehat{G}_k^r Y_k^T A_k^T X_k^T = \begin{bmatrix} B_k^f (B_k^f)^T & 0 \\ 0 & -B_k^b (B_k^b)^T \end{bmatrix}, \tag{4.8}$$

where $\widehat{G}_k^r \equiv Y_k^{-1} G_k^r Y_k^{-T}$.

From Definition 4.1 it is easily seen, for $k = 0, 1, \ldots, K-1$, that

$$\widehat{G}_k^r = Y_k^{-1} G_k^r Y_k^{-T} = \begin{bmatrix} \widehat{G}_{k,1}^{cr} & 0 \\ 0 & \widehat{G}_{k,2}^{nr} \end{bmatrix}, \tag{4.9}$$

with

$$\widehat{G}_{k,1}^{cr} \equiv \sum_{i=-\infty}^{k-1} \Phi_f(k,i+1) B_i^f (B_i^f)^T \Phi_f(k,i+1)^T, \quad \widehat{G}_{k,2}^{nr} \equiv \sum_{i=k}^{k+\nu K-1} \Phi_b(k,i) B_i^b (B_i^b)^T \Phi_b(k,i)^T .$$

Then by (2.1) and (4.9), equations (4.8) are decomposed into two periodic Lyapunov equations, for $k = 0, 1, 2, \ldots, K-1$:

$$\widehat{G}_{k+1,1}^{cr} - A_k^f \widehat{G}_{k,1}^{cr} (A_k^f)^T = B_k^f (B_k^f)^T, \tag{4.10}$$

$$\widehat{G}_{k,2}^{nr} - E_k^b \widehat{G}_{k+1,2}^{nr} (E_k^b)^T = B_k^b (B_k^b)^T. \tag{4.11}$$

Rewrite (4.10) and (4.11) to two enlarged Lyapunov equations:

$$\mathcal{G}_{cr} - \mathcal{A}_f \mathcal{G}_{cr} \mathcal{A}_f^T = \mathcal{B}_f \mathcal{B}_f^T, \tag{4.12}$$

$$\mathcal{G}_{nr} - \mathcal{E}_b \mathcal{G}_{nr} \mathcal{E}_b^T = \mathcal{B}_b \mathcal{B}_b^T, \tag{4.13}$$

where $\mathcal{G}_{cr} = \mathrm{diag}(\widehat{G}_{k,1}^{cr}, \cdots, \widehat{G}_{K-1,1}^{cr}, \widehat{G}_{0,1}^{cr})$, $\mathcal{G}_{nr} = \mathrm{diag}(\widehat{G}_{0,2}^{nr}, \widehat{G}_{1,2}^{nr}, \cdots, \widehat{G}_{K-1,2}^{nr})$, $\mathcal{E}_b$ and $\mathcal{B}_b$ as defined in (3.4), and

$$\mathcal{A}_f = \begin{bmatrix} & & & A_0^f \\ A_1^f & & & \\ & \ddots & & \\ & & A_{K-1}^f & \end{bmatrix}, \quad \mathcal{B}_f = \begin{bmatrix} B_0^f & & & \\ & B_1^f & & \\ & & \ddots & \\ & & & B_{K-1}^f \end{bmatrix}.$$

Since the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable and the matrix $\mathcal{E}_b$ is nilpotent with index ν, the pairs $(\mathcal{A}_f, \mathcal{B}_f)$ and $(\mathcal{E}_b, \mathcal{B}_b)$ are reachable if and only if the solutions $\mathcal{G}_{cr}$ and $\mathcal{G}_{nr}$ of Lyapunov equations (4.12), (4.13) are symmetric positive definite. Equivalently, followed from (4.9), all reachability Gramians G_k^r ($k = 0, 1, \ldots, K-1$) are symmetric positive definite. Moreover, from Theorems 3.1–3.2 and the Remark following Definition 3.1, we know that the periodic descriptor system (1.1) is completely reachable if and only if the pairs $(\mathcal{A}_f, \mathcal{B}_f)$ and $(\mathcal{E}_b, \mathcal{B}_b)$ are reachable. This completes the proof of statement (i). □

5 Numerical solutions of projected GDPLEs

In this Section, a numerical method is proposed for the symmetric positive semi-definite solutions of the projected generalized discrete-time periodic Lyapunov equations (4.1) and (4.3), for pd-stable $\{(E_k, A_k)\}_{k=0}^{K-1}$. We first consider the numerical solutions of the GDPLE (4.3).

GDPLE for observability Gramians G_k^{co}

As $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable, there exist orthogonal matrices V_k and U_k, with $U_K \equiv U_0$ and for $k = 0, 1, \ldots, K-1$, such that

$$V_k^T E_k U_{k+1} = \begin{bmatrix} E_{k,1} & E_{k,3} \\ 0 & E_{k,2} \end{bmatrix}, \quad V_k^T A_k U_k = \begin{bmatrix} A_{k,1} & A_{k,3} \\ 0 & A_{k,2} \end{bmatrix} \tag{5.1}$$

are upper triangular except $V_0^T A_0 U_0$ is qusi-upper triangular [7, 18]. The matrices $E_{k,1}$ and $A_{k,2}$ are nonsingular, and $E_{k,2}E_{k+1,2}\cdots E_{k+K-1,2}$ are nilpotent for $k = 0, 1, \ldots, K-1$. All finite eigenvalues of the periodic matrix pairs $\{(E_{k,1}, A_{k,1})\}_{k=0}^{K-1}$ lie inside the unit circle and the spectrum of the periodic matrix pairs $\{(E_{k,2}, A_{k,2})\}_{k=0}^{K-1}$ contains only infinite eigenvalues, with

$$\sigma\left(\{(E_{k,1}, A_{k,1})\}_{k=0}^{K-1}\right) \cap \sigma\left(\{(E_{k,2}, A_{k,2})\}_{k=0}^{K-1}\right) = \emptyset. \tag{5.2}$$

Computationally, these matrix decompositions can be accomplished via the periodic QZ algorithm (PQZ) with reordering strategies.

Notice that

$$\begin{bmatrix} I & Z_k \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{k,1} & E_{k,3} \\ 0 & E_{k,2} \end{bmatrix} \begin{bmatrix} I & -W_{k+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_{k,1} & 0 \\ 0 & E_{k,2} \end{bmatrix}, \tag{5.3}$$

$$\begin{bmatrix} I & Z_k \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{k,1} & A_{k,3} \\ 0 & A_{k,2} \end{bmatrix} \begin{bmatrix} I & -W_k \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{k,1} & 0 \\ 0 & A_{k,2} \end{bmatrix}, \tag{5.4}$$

if the matrices Z_k and W_k, with $W_K \equiv W_0$ and for $k = 0, 1, \ldots, K-1$, satisfy the generalized periodic Sylvester equations

$$\begin{aligned} E_{k,1}W_{k+1} - Z_k E_{k,2} &= E_{k,3}, \\ A_{k,1}W_k - Z_k A_{k,2} &= A_{k,3}. \end{aligned} \tag{5.5}$$

From condition (5.2), the generalized periodic Sylvester equations (5.5) have unique solutions Z_k and W_k. Therefore, the nonsingular matrices X_k, Y_k in (2.1) satisfy

$$X_k = \begin{bmatrix} I & Z_k \\ 0 & I \end{bmatrix} V_k^T, \quad Y_k = U_k \begin{bmatrix} I & -W_k \\ 0 & I \end{bmatrix},$$

and the right and left spectral projections $P_r(k)$, $P_l(k)$ are given as

$$P_l(k) = V_k \begin{bmatrix} I & Z_k \\ 0 & 0 \end{bmatrix} V_k^T, \quad P_r(k) = U_k \begin{bmatrix} I & W_k \\ 0 & 0 \end{bmatrix} U_k^T. \tag{5.6}$$

Let, for $k = 0, 1, \ldots, K-1$,

$$V_{k-1}^T G_k^{co} V_{k-1} = \begin{bmatrix} G_{k,1}^{co} & G_{k,3}^{co} \\ (G_{k,3}^{co})^T & G_{k,2}^{co} \end{bmatrix}, \quad C_k U_k = \begin{bmatrix} C_{k,1}, & C_{k,2} \end{bmatrix}. \tag{5.7}$$

Substituting (5.1), (5.6) and (5.7) into the projected GDPLE (4.3), for $k = 0, 1, \ldots, K-1$, we have

$$E_{k-1,1}^T G_{k,1}^{co} E_{k-1,1} - A_{k,1}^T G_{k+1,1}^{co} A_{k,1} = C_{k,1}^T C_{k,1}, \tag{5.8}$$

$$E_{k-1,1}^T G_{k,1}^{co} E_{k-1,3} + E_{k-1,1}^T G_{k,3}^{co} E_{k-1,2} - A_{k,1}^T G_{k+1,1}^{co} A_{k,3} - A_{k,1}^T G_{k+1,3}^{co} A_{k,2} = C_{k,1}^T C_{k,1} W_k, \tag{5.9}$$

$$E_{k-1,3}^T G_{k,1}^{co} E_{k-1,3} + E_{k-1,3}^T G_{k,3}^{co} E_{k-1,2} + E_{k-1,2}^T (G_{k,3}^{co})^T E_{k-1,3} + E_{k-1,2}^T G_{k,2}^{co} E_{k-1,2} -$$

$$A_{k,3}^T G_{k+1,1}^{co} A_{k,3} - A_{k,3}^T G_{k+1,3}^{co} A_{k,2} - A_{k,2}^T (G_{k+1,3}^{co})^T A_{k,3} - A_{k,2}^T G_{k+1,2}^{co} A_{k,2} = W_k^T C_{k,1}^T C_{k,1} W_k. \tag{5.10}$$

Again from the pd-stability of $\{(E_{k,1}, A_{k,1})\}_{k=0}^{K-1}$, the generalized discrete-time periodic Lyapunov equations (5.8) have unique symmetric positive semi-definite solutions $G_{k,1}^{co}$. Furthermore, it follows from (5.5) that (5.9) can be rearranged as

$$E_{k-1,1}^T (G_{k,3}^{co} - G_{k,1}^{co} Z_{k-1}) E_{k-1,2} - A_{k,1}^T (G_{k+1,3}^{co} - G_{k+1,1}^{co} Z_k) A_{k,2} = 0. \tag{5.11}$$

Again, from (5.2), we deduce that

$$G_{k,3}^{co} = G_{k,1}^{co} Z_{k-1}, \quad k = 0, 1, \dots, K-1. \tag{5.12}$$

From (5.5), (5.8) and (5.12), (5.10) can be rewritten as

$$E_{k-1,2}^T (G_{k,2}^{co} - Z_{k-1}^T G_{k,1}^{co} Z_{k-1}) E_{k-1,2} - A_{k,2}^T (G_{k+1,2}^{co} - Z_k^T G_{k+1,1}^{co} Z_k) A_{k,2} = 0. \tag{5.13}$$

Now, since the periodic matrix pairs $\{(E_{k,2}, A_{k,2})\}_{k=0}^{K-1}$ have only infinite eigenvalues, we then have

$$G_{k,2}^{co} = Z_{k-1}^T G_{k,1}^{co} Z_{k-1}, \quad k = 0, 1, \dots, K-1. \tag{5.14}$$

Therefore, the solutions of the projected GDPLE (4.3) have the form

$$G_k^{co} = V_{k-1} \begin{bmatrix} G_{k,1}^{co} & G_{k,1}^{co} Z_{k-1} \\ Z_{k-1}^T G_{k,1}^{co} & Z_{k-1}^T G_{k,1}^{co} Z_{k-1} \end{bmatrix} V_{k-1}^T, \quad k = 0, 1, \dots, K-1, \tag{5.15}$$

where the matrices $G_{k,1}^{co}$ are the unique symmetric positive semi-definite solutions of the generalized periodic Lyapunov equations (5.8). Moreover, from (5.6) and (5.15) they also satisfy $P_l(k-1)^T G_k^{co} P_l(k-1) = G_k^{co}$.

In many applications it is necessary to have the Cholesky factors of the solutions of the Lyapunov equations rather the solutions itself [23]. In particular, these full-ranked factors are useful for computing numerically the Hankel singular values (see next Section). If $L_{k,1}$ denotes a Cholesky factor of each matrix $G_{k,1}^{co}$, i.e., $G_{k,1}^{co} = L_{k,1}^T L_{k,1}$, then we compute the QR factorization

$$L_{k,1} = Q_{k,L} \begin{bmatrix} T_{k,L} \\ 0 \end{bmatrix},$$

where $Q_{k,L}$ is orthogonal and $T_{k,L}$ has full row rank, for $k = 0, 1, \dots, K-1$. The full-ranked factorizations of the solutions G_k^{co}, for $k = 0, 1, \dots, K-1$, are given by

$$\begin{aligned} G_k^{co} &= V_{k-1} \begin{bmatrix} L_{k,1}^T \\ Z_{k-1}^T L_{k,1}^T \end{bmatrix} \begin{bmatrix} L_{k,1}, & L_{k,1} Z_{k-1} \end{bmatrix} V_{k-1}^T \\ &= V_{k-1} \begin{bmatrix} T_{k,L}^T \\ Z_{k-1}^T T_{k,L}^T \end{bmatrix} \begin{bmatrix} T_{k,L}, & T_{k,L} Z_{k-1} \end{bmatrix} V_{k-1}^T \\ &\equiv L_k^T L_k, \end{aligned}$$

where $L_k \equiv \begin{bmatrix} T_{k,L}, & T_{k,L} Z_{k-1} \end{bmatrix} V_{k-1}^T$ has full row-rank.

GDPLE for reachability Gramians G_k^{cr}

Similarly for the projected GDPLE (4.1), for $k = 0, 1, \dots, K-1$, we let

$$U_k^T G_k^{cr} U_k = \begin{bmatrix} G_{k,1}^{cr} & G_{k,3}^{cr} \\ (G_{k,3}^{cr})^T & G_{k,2}^{cr} \end{bmatrix}, \quad V_k^T B_k = \begin{bmatrix} B_{k,1} \\ B_{k,2} \end{bmatrix}. \tag{5.16}$$

Substituting (5.1), (5.6) and (5.16) into the projected GDPLE (4.1), we then have

$$\begin{aligned}E_{k,1}G^{cr}_{k+1,1}E^T_{k,1} - A_{k,1}G^{cr}_{k,1}A^T_{k,1} &= -E_{k,1}G^{cr}_{k+1,3}E^T_{k,3} - E_{k,3}(G^{cr}_{k+1,3})^T E^T_{k,1} - E_{k,3}G^{cr}_{k+1,2}E^T_{k,3} \\ &\quad + A_{k,1}G^{cr}_{k,3}A^T_{k,3} + A_{k,3}(G^{cr}_{k,3})^T A^T_{k,1} + A_{k,3}G^{cr}_{k,2}A^T_{k,3} \\ &\quad + (B_{k,1} + Z_k B_{k,2})(B_{k,1} + Z_k B_{k,2})^T, \qquad (5.17)\\ E_{k,1}G^{cr}_{k+1,3}E^T_{k,2} - A_{k,1}G^{cr}_{k,3}A^T_{k,2} &= -E_{k,3}G^{cr}_{k+1,2}E^T_{k,2} + A_{k,3}G^{cr}_{k,2}A^T_{k,2}, \qquad (5.18)\\ E_{k,2}G^{cr}_{k+1,2}E^T_{k,2} - A_{k,2}G^{cr}_{k,2}A^T_{k,2} &= 0, \quad k = 0, 1, \ldots, K-1. \qquad (5.19)\end{aligned}$$

Since the periodic matrix pairs $\{(E_{k,2}, A_{k,2})\}_{k=0}^{K-1}$ have only infinite eigenvalues, it follows from (5.19) that

$$G^{cr}_{k,2} = 0, \quad k = 0, 1, \ldots, K-1. \qquad (5.20)$$

Furthermore, (5.18) can be simplified to

$$E_{k,1}G^{cr}_{k+1,3}E^T_{k,2} - A_{k,1}G^{cr}_{k,3}A^T_{k,2} = 0. \qquad (5.21)$$

Then from (5.2), we have

$$G^{cr}_{k,3} = 0, \quad k = 0, 1, \ldots, K-1. \qquad (5.22)$$

From (5.20) and (5.22), (5.17) can be rewritten as

$$E_{k,1}G^{cr}_{k+1,1}E^T_{k,1} - A_{k,1}G^{cr}_{k,1}A^T_{k,1} = (B_{k,1} + Z_k B_{k,2})(B_{k,1} + Z_k B_{k,2})^T. \qquad (5.23)$$

Therefore, the solutions of the projected GDPLE (4.1) have the form

$$G^{cr}_k = U_k \begin{bmatrix} G^{cr}_{k,1} & 0 \\ 0 & 0 \end{bmatrix} U^T_k, \quad k = 0, 1, \ldots, K-1, \qquad (5.24)$$

where the matrices $G^{cr}_{k,1}$ are the unique symmetric positive semi-definite solutions of the generalized periodic Lyapunov equations (5.23). Moreover, from (5.6) and (5.24) they also satisfy $P_r(k)G^{cr}_k P_r(k)^T = G^{cr}_k$.

If $R_{k,1}$ denotes a Cholesky factor of each matrix $G^{cr}_{k,1}$, i.e., $G^{cr}_{k,1} = R_{k,1}R^T_{k,1}$, then we compute the QR factorization

$$R^T_{k,1} = Q_{k,R} \begin{bmatrix} T^T_{k,R} \\ 0 \end{bmatrix},$$

where $Q_{k,R}$ is orthogonal and $T_{k,R}$ has full column-rank. The full-ranked factorizations of the solutions G^{cr}_k are given by

$$\begin{aligned} G^{cr}_k &= U_k \begin{bmatrix} R_{k,1} \\ 0 \end{bmatrix} \begin{bmatrix} R^T_{k,1}, & 0 \end{bmatrix} U^T_k \\ &= U_k \begin{bmatrix} T_{k,R} \\ 0 \end{bmatrix} \begin{bmatrix} T^T_{k,R}, & 0 \end{bmatrix} U^T_k \\ &\equiv R_k R^T_k, \end{aligned}$$

where $R^T_k \equiv \begin{bmatrix} T^T_{k,R}, & 0 \end{bmatrix} U^T_k$ has full row-rank for $k = 0, 1, \ldots, K-1$.

Algorithm GDPLEs

We now summarize the main steps for computing the full-ranked Cholesky factors of the causal Gramians, via the solution of the GDPLEs (4.1) and (4.3). For simplicity in Algorithm 5.1, we shall ignore the obvious qualification for k, i.e., $k = 0, 1, \cdots, K-1$.

Algorithm 5.1 (GDPLEs)

Input: System matrices (E_k, A_k, B_k, C_k), with $\{(E_k, A_k)\}_{k=0}^{K-1}$ being pd-stable.

Output: Full-ranked Cholesky factors R_k and L_k $(k = 0, 1, \ldots, K-1)$, where $G_k^{cr} = R_k R_k^T$ and $G_k^{co} = L_k^T L_k$.

Step 1. Use the PQZ algorithm [7, 18] to compute orthogonal matrices V_k and U_k, with $U_K \equiv U_0$, such that

$$V_k^T E_k U_{k+1} = \begin{bmatrix} E_{k,1} & E_{k,3} \\ 0 & E_{k,2} \end{bmatrix}, \quad V_k^T A_k U_k = \begin{bmatrix} A_{k,1} & A_{k,3} \\ 0 & A_{k,2} \end{bmatrix}$$

are upper triangular except $V_0^T A_0 U_0$ is quasi-upper triangular. The matrices $E_{k,1}$ and $A_{k,2}$ are nonsingular, and $E_{k,2}E_{k+1,2}\cdots E_{k+K-1,2}$ are nilpotent.

Step 2. Use the Cyclic Schur and Hessenberg-Schur methods [9] to compute the solutions of the generalized periodic Sylvester equations

$$\begin{aligned} E_{k,1}W_{k+1} - Z_k E_{k,2} &= E_{k,3}, \\ A_{k,1}W_k - Z_k A_{k,2} &= A_{k,3}. \end{aligned}$$

Step 3. Compute the matrices

$$V_k^T B_k = \begin{bmatrix} B_{k,1} \\ B_{k,2} \end{bmatrix}, \quad C_k U_k = \begin{bmatrix} C_{k,1}, & C_{k,2} \end{bmatrix}.$$

Step 4. Compute the Cholesky factors $R_{k,1}$ and $L_{k,1}$ of the solutions $G_{k,1}^{cr} = R_{k,1}R_{k,1}^T$ and $G_{k,1}^{co} = L_{k,1}^T L_{k,1}$ of the generalized discrete-time periodic Lyapunov equations

$$\begin{aligned} E_{k,1}G_{k+1,1}^{cr}E_{k,1}^T - A_{k,1}G_{k,1}^{cr}A_{k,1}^T &= (B_{k,1} + Z_k B_{k,2})(B_{k,1} + Z_k B_{k,2})^T, \\ E_{k-1,1}^T G_{k,1}^{co} E_{k-1,1} - A_{k,1}^T G_{k+1,1}^{co} A_{k,1} &= C_{k,1}^T C_{k,1}. \end{aligned}$$

Step 5. Compute the QR factorizations

$$R_{k,1}^T = Q_{k,R} \begin{bmatrix} T_{k,R}^T \\ 0 \end{bmatrix}, \quad L_{k,1} = Q_{k,L} \begin{bmatrix} T_{k,L} \\ 0 \end{bmatrix}.$$

Step 6. Compute the full-ranked Cholesky factors

$$R_k = U_k \begin{bmatrix} T_{k,R} \\ 0 \end{bmatrix}, \quad L_k = \begin{bmatrix} T_{k,L}, & T_{k,L}Z_{k-1} \end{bmatrix} V_{k-1}^T.$$

Remark. One can extend the techniques in [31], for the numerical solution of the generalized Lyapunov equations, to solve the generalized discrete-time periodic Lyapunov equations given in Step 4. A thorough error analysis and practical implementation details for the algorithm extended from [31] are still under investigation.

A numerical example

We shall illustrate the feasibility and reliability of the proposed algorithm with an example. All computations were performed in MATLAB/version 6.5 on a PC with an Intel Pentium-III processor at 866 MHz, with 768 MB RAM, using IEEE double-precision floating-point arithmetic. The machine precision is approximately 2.22×10^{-16}.

For approximate solutions $\widetilde{X}_k$ of the projected generalized discrete-time periodic Lyapunov equations (4.1) and (4.3), we compute the relative residuals defined by

$$\gamma_k^{cr} = \frac{\|E_k\widetilde{X}_{k+1}E_k^T - A_k\widetilde{X}_kA_k^T - P_l(k)B_kB_k^TP_l(k)^T\|_2}{\|\widetilde{X}_k\|_2},$$

$$\gamma_k^{co} = \frac{\|E_{k-1}^T \widetilde{X}_k E_{k-1} - A_k^T \widetilde{X}_{k+1} A_k - P_r(k)^T C_k^T C_k P_r(k)\|_2}{\|\widetilde{X}_k\|_2}.$$

Example 1. We consider a periodic discrete-time descriptor system (1.1) with $n = 10$, $m = 2$, $p = 3$ and period $K = 3$. For $k = 0, 1, 2$, we have

$$E_k^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c_1 & s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -s_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & s_1 & 1 & 0 & c_2 & s_2 & 0 & 0 & 0 \\ 0 & -s_1 & c_1 & 0 & 1 & -s_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & s_2 & 1 & 0 & c_3 & s_3 & 0 \\ 0 & 0 & 0 & -s_2 & c_2 & 0 & 1 & -s_3 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 & s_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s_3 & c_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_k^{(0)} = \mathrm{diag}(1.01, A_{01}, A_{02}, A_{03}, A_{04}, 1.001), \quad \theta_k := 2\pi k/K,$$

$$B_k^T = \begin{bmatrix} 4 & -1 & 3 & 5 & 0 & -2 & 0 & 8 & 1 & 0 \\ 1 & 1 & s_1+1 & -2 & 1 & 0 & 0 & -3 & 0 & 1 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05+c_1 & 0 & 0 \end{bmatrix},$$

where

$$c_1 = \cos(\theta_k), \quad c_2 = 0.2c_1, \quad c_3 = 0.6c_1,$$
$$s_1 = \sin(\theta_k), \quad s_2 = 0.2s_1, \quad s_3 = 0.6s_1,$$
$$A_{01} = \begin{bmatrix} r_1\cos(\pi/3) & r_1\sin(\pi/3) \\ -r_1\sin(\pi/3) & r_1\cos(\pi/3) \end{bmatrix}, \quad A_{02} = \begin{bmatrix} r_2\cos(7\pi/5) & r_2\sin(7\pi/5) \\ -r_2\sin(7\pi/5) & r_2\cos(7\pi/5) \end{bmatrix},$$
$$A_{03} = \begin{bmatrix} r_3\cos(\pi/4) & r_3\sin(\pi/4) \\ -r_3\sin(\pi/4) & r_3\cos(\pi/4) \end{bmatrix}, \quad A_{04} = \begin{bmatrix} r_4\cos(\pi/10) & r_4\sin(\pi/10) \\ -r_4\sin(\pi/10) & r_4\cos(\pi/10) \end{bmatrix},$$

and

$$r_1 = 0.5, \quad r_2 = 0.05, \quad r_3 = -0.02, \quad r_4 = 0.12.$$

We define a Householder transformation $V = I - 2uu^T$ with $u = [1, 1, \cdots, 1, 1]^T/\sqrt{10} \in \mathbb{R}^{10}$, and the K-periodic system matrices (E_k, A_k, B_k, C_k) are given by

$$E_k \equiv V^T E_k^{(0)} V, \quad A_k \equiv V^T A_k^{(0)} V, \quad k = 0, 1, 2.$$

The computed open-loop spectrum of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ consists of two infinite eigenvalues and four pairs of complex conjugate finite eigenvalues lying inside the unit circle. Thus, the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable with $n_1 = 8$ and $n_2 = 2$. Accurate numerical results were produced by the proposed algorithm, as shown in Table 1.

k	$\|G_k^{cr}\|_2$	γ_k^{cr}	$\|G_k^{co}\|_2$	γ_k^{co}
0	8.30×10^4	2.17×10^{-16}	1.14×10^3	1.39×10^{-16}
1	7.11×10^3	3.11×10^{-16}	9.70×10^0	4.17×10^{-15}
2	5.82×10^2	6.73×10^{-16}	9.74×10^1	9.18×10^{-15}

Table 1: Norms and relative residuals of causal Gramians

6 Hankel singular values

Similar to standard state space systems [16] and continuous-time descriptor systems [35, 38], the controllability and observability Gramians can be used to define Hankel singular values for the periodic descriptor systems (1.1), which are of great importance in the model reduction problem via the balanced truncation method.

For the discrete-time descriptor systems, the causal and noncausal Hankel singular values are defined via the nonnegative eigenvalues of the matrices $\mathcal{G}_{dcc}E^T\mathcal{G}_{dco}E$ and $\mathcal{G}_{dnc}A^T\mathcal{G}_{dno}A$. Here $\mathcal{G}_{dcc}$, $\mathcal{G}_{dnc}$, $\mathcal{G}_{dco}$ and $\mathcal{G}_{dno}$ denote the causal/noncausal reachability Gramians and the causal/noncausal observability Gramians, respectively [39].

Lemma 6.1. *Let the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ be pd-stable. Then the matrices $\mathbf{H}_k^c \equiv G_k^{cr}E_{k-1}^TG_k^{co}E_{k-1}$ and $\mathbf{H}_k^{nc} \equiv G_k^{nr}A_k^TG_{k+1}^{no}A_k$, $k = 0, 1, 2, \ldots, K-1$, have real and nonnegative eigenvalues.*

Proof. From Definitions 4.1, 4.2 and (2.6) and for $k = 0, 1, 2, \ldots, K-1$, we have

$$\mathbf{H}_k^c = Y_k \begin{bmatrix} \widehat{G}_{k,1}^{cr}\widehat{G}_{k,1}^{co} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1},$$

where

$$\widehat{G}_{k,1}^{cr} \equiv \sum_{i=-\infty}^{k-1} \Phi_f(k, i+1)B_i^f(B_i^f)^T\Phi_f(k, i+1)^T, \quad \widehat{G}_{k,1}^{co} \equiv \sum_{i=k}^{\infty} \Phi_f(i, k)(C_i^f)^TC_i^f\Phi_f(i, k).$$

Since the $n_1 \times n_1$ matrices $\widehat{G}_{k,1}^{cr}$ and $\widehat{G}_{k,1}^{co}$ are symmetric positive semi-definite, it follows that $\mathbf{H}_k^c$ have real and nonnegative eigenvalues. Similarly, it can be shown that $\mathbf{H}_k^{nc}$ also share the same property. $\square$

Notice that, in the proof of Lemma 6.1, the matrices $\mathbf{H}_k^c$ and $\mathbf{H}_k^{nc}$ have at least n_2 and n_1 zero eigenvalues, respectively. Hence, we have the following definition of Hankel singular values for the periodic descriptor system (1.1).

Definition 6.1. *Suppose that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable and let n_1, n_2 be the dimensions of the periodic deflating subspaces of $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding respectively to the finite and infinite eigenvalues.*

(i) *For $k = 0, 1, \ldots, K-1$, the square roots of the largest n_1 eigenvalues of the matrices $\mathbf{H}_k^c$, denoted by $\zeta_{k,j}$, are called the causal Hankel singular values of the periodic descriptor system (1.1).*

(ii) *For $k = 0, 1, \ldots, K-1$, the square roots of the largest n_2 eigenvalues of the matrices $\mathbf{H}_k^{nc}$, denoted by $\theta_{k,j}$, are called the noncausal Hankel singular values of the periodic descriptor system (1.1).*

Remarks. (i) When $K = 1$, the causal and noncausal Hankel singular values defined in Definition 6.1 coincide with those for discrete-time descriptor systems (see [39] and references therein). For $E_k = I$, the causal Hankel singular values are the classical Hankel singular values of linear periodic discrete-time systems [46].

(ii) As in the case of descriptor systems, the causal and noncausal Hankel singular values of the periodic descriptor system (1.1) are invariant under system equivalence transformations.

From Theorem 4.2 and Lemma 6.1 we obtain the following result.

Corollary 6.2. *Consider the periodic discrete-time descriptor system (1.1), where the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable. The following statements are equivalent.*

(a) *The periodic descriptor system (1.1) is completely reachable and completely observable.*

(b) *For $k = 0, 1, 2, \ldots, K-1$, we have*

$$\begin{aligned} \mathrm{rank}(G_k^{cr}) &= \mathrm{rank}(G_k^{co}) = \mathrm{rank}(\mathbf{H}_k^c) = n_1, \\ \mathrm{rank}(G_k^{nr}) &= \mathrm{rank}(G_k^{no}) = \mathrm{rank}(\mathbf{H}_k^{nc}) = n_2. \end{aligned}$$

(c) *The causal and noncausal Hankel singular values of (1.1) are nonzero.*

For pd-stable $\{(E_k, A_k)\}_{k=0}^{K-1}$, the causal and noncausal reachability and observability Gramians are symmetric and positive semi-definite. Thus, there exist full-ranked factorizations

$$\begin{aligned} G_k^{cr} &= R_k R_k^T, \quad G_k^{co} = L_k^T L_k, \\ G_k^{nr} &= \widetilde{R}_k \widetilde{R}_k^T, \quad G_k^{co} = \widetilde{L}_k^T \widetilde{L}_k, \end{aligned} \tag{6.1}$$

where the matrices R_k, L_k^T, $\tilde{R}_k$ and $\tilde{L}_k^T$ are of full column-rank. The connections between the causal/noncausal Hankel singular values and the singular values of the matrices $L_k E_{k-1} R_k$ and $\widetilde{L}_{k+1} A_k \widetilde{R}_k$ are considered in the following Lemma.

Lemma 6.3. *For the periodic descriptor system (1.1), where the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable. Suppose that the causal and noncausal Gramians of (1.1) have the full-ranked factorizations defined as in (6.1). Then for $k = 0, 1, 2, \ldots, K-1$, the nonzero causal Hankel singular values are the nonzero singular values of the matrices $L_k E_{k-1} R_k$, while the nonzero noncausal Hankel singular values are the nonzero singular values of the matrices $\widetilde{L}_{k+1} A_k \widetilde{R}_k$.*

Proof. Notice that for $k = 0, 1, \ldots, K-1$, we have

$$\begin{aligned} \zeta_{k,j}^2 &= \lambda_j(R_k R_k^T E_{k-1}^T L_k^T L_k E_{k-1}) = \lambda_j(R_k^T E_{k-1}^T L_k^T L_k E_{k-1} R_k) = \sigma_j^2(L_k E_{k-1} R_k), \\ \theta_{k,j}^2 &= \lambda_j(\widetilde{R}_k \widetilde{R}_k^T A_k^T \widetilde{L}_{k+1}^T \widetilde{L}_{k+1} A_k) = \lambda_j(\widetilde{R}_k^T A_k^T \widetilde{L}_{k+1}^T \widetilde{L}_{k+1} A_k \widetilde{R}_k) = \sigma_j^2(\widetilde{L}_{k+1} A_k \widetilde{R}_k), \end{aligned}$$

where $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, the eigenvalues and singular values of the corresponding matrices. □

7 Balanced realization

It is well known [16] that for any minimal realization (A, B, C) of a stable continuous-time or discrete-time system, there exists a transformation such that the controllability and observability Gramians for the transformed realization equal to some diagonal matrix. Such a realization is called a(n) (internally) balanced realization. Recently, the issues of balanced realization and model reduction via the balanced truncation method are discussed for continuous-time descriptor systems [35, 38] and asymptotically stable linear discrete-time periodic systems [45, 46]. In this Section the problem of balanced realization is generalized for periodic descriptor systems. We shall assume that the periodic descriptor system (1.1) is completely reachable/observable with $\{(E_k, A_k)\}_{k=0}^{K-1}$ being pd-stable.

Definition 7.1. *A realization (E_k, A_k, B_k, C_k) of the periodic descriptor system (1.1) is called balanced if*

$$G_k^{cr} = G_k^{co} = \begin{bmatrix} D_{k,1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad G_k^{nr} = G_{k+1}^{no} = \begin{bmatrix} 0 & 0 \\ 0 & D_{k,2} \end{bmatrix},$$

where $D_{k,1}$ and $D_{k,2}$ are diagonal matrices for $k = 0, 1, \ldots, K-1$.

We shall show that for a realization (E_k, A_k, B_k, C_k) of the periodic descriptor system (1.1), there exist nonsingular periodic matrices S_k and T_k ($k = 0, 1, \ldots, K-1$) with $T_K \equiv T_0$, such that the transformed realization

$$(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k) \equiv (S_k^T E_k T_{k+1}, S_k^T A_k T_k, S_k^T B_k, C_k T_k) \tag{7.1}$$

is balanced.

Consider the full-ranked factorizations (6.1) of the causal/noncausal reachability/observability Gramians. For $k = 0, 1, \cdots, K-1$, let

$$L_k E_{k-1} R_k = U_k \Sigma_k V_k^T, \quad \widetilde{L}_{k+1} A_k \widetilde{R}_k = \widetilde{U}_k \Theta_k \widetilde{V}_k^T, \tag{7.2}$$

be the singular value decompositions [17] of $L_k E_{k-1} R_k$ and $\widetilde{L}_{k+1} A_k \widetilde{R}_k$. Here U_k, V_k, $\widetilde{U}_k$, $\widetilde{V}_k$ are orthogonal, and Σ_k and Θ_k are diagonal and nonsingular. From Corollary 6.2 and Lemma 6.3, we have $\Sigma_k = \mathrm{diag}(\zeta_{k,1}, \cdots, \zeta_{k,n_1}) > 0$ and $\Theta_k = \mathrm{diag}(\theta_{k,1}, \cdots, \theta_{k,n_2}) > 0$. Furthermore, it is easily seen from Theorem 4.1 and (2.5) that

$$\begin{aligned} &G_k^{cr} = P_r(k) G_k^{cr} P_r(k)^T, \quad G_k^{co} = P_l(k-1)^T G_k^{co} P_l(k-1), \\ &P_r(k) G_k^{nr} = 0, \quad G_k^{no} P_l(k-1) = 0, \\ &E_{k-1} P_r(k) = P_l(k-1) E_{k-1}, \quad A_k P_r(k) = P_l(k) A_k. \end{aligned}$$

Simple calculations then yield $G_k^{no} E_{k-1} G_k^{cr} = G_k^{co} E_{k-1} G_k^{nr} = G_{k+1}^{no} A_k G_k^{cr} = G_{k+1}^{co} A_k G_k^{nr} = 0$. Hence, for $k = 0, 1, \ldots, K-1$, we have

$$\widetilde{L}_k E_{k-1} R_k = L_k E_{k-1} \widetilde{R}_k = \widetilde{L}_{k+1} A_k R_k = L_{k+1} A_k \widetilde{R}_k = 0. \tag{7.3}$$

Now for $k = 0, 1, \ldots, K-1$, consider the $n \times n$ matrices

$$S_k = \left[\begin{array}{cc} L_{k+1}^T U_{k+1} \Sigma_{k+1}^{-1/2}, & \widetilde{L}_{k+1}^T \widetilde{U}_k \Theta_k^{-1/2} \end{array} \right], \quad \check{S}_k = \left[\begin{array}{cc} E_k R_{k+1} V_{k+1} \Sigma_{k+1}^{-1/2}, & A_k \widetilde{R}_k \widetilde{V}_k \Theta_k^{-1/2} \end{array} \right],$$

It follows from (7.2) and (7.3) that

$$S_k^T \check{S}_k = \left[\begin{array}{cc} \Sigma_{k+1}^{-1/2} U_{k+1}^T L_{k+1} E_k R_{k+1} V_{k+1} \Sigma_{k+1}^{-1/2} & \Sigma_{k+1}^{-1/2} U_{k+1}^T L_{k+1} A_k \widetilde{R}_k \widetilde{V}_k \Theta_k^{-1/2} \\ \Theta_k^{-1/2} \widetilde{U}_k^T \widetilde{L}_{k+1} E_k R_{k+1} V_{k+1} \Sigma_{k+1}^{-1/2} & \Theta_k^{-1/2} \widetilde{U}_k^T \widetilde{L}_{k+1} A_k \widetilde{R}_k \widetilde{V}_k \Theta_k^{-1/2} \end{array} \right] = I_n,$$

i.e., the matrices S_k and $\check{S}_k$ are nonsingular and $S_k^{-1} = \check{S}_k^T$. Similarly, it can be shown that the matrices

$$T_k = \left[\begin{array}{cc} R_k V_k \Sigma_k^{-1/2}, & \widetilde{R}_k \widetilde{V}_k \Theta_k^{-1/2} \end{array} \right], \quad \check{T}_k = \left[\begin{array}{cc} E_{k-1}^T L_k^T U_k \Sigma_k^{-1/2}, & A_k^T \widetilde{L}_{k+1}^T \widetilde{U}_k \Theta_k^{-1/2} \end{array} \right]$$

are also nonsingular and $T_k^{-1} = \check{T}_k^T$. Therefore, with the transformation matrices S_k and T_k defined above and (7.3), the causal reachability and observability Gramians of the transformed periodic descriptor system (7.1) become

$$\begin{aligned} \widehat{G}_k^{cr} &\equiv T_k^{-1} G_k^{cr} T_k^{-T} = \check{T}_k^T G_k^{cr} \check{T}_k \\ &= \left[\begin{array}{cc} \Sigma_k^{-1/2} U_k^T L_k E_{k-1} R_k R_k^T E_{k-1}^T L_k^T U_k \Sigma_k^{-1/2} & \Sigma_k^{-1/2} U_k^T L_k E_{k-1} R_k R_k^T A_k^T \widetilde{L}_{k+1}^T \widetilde{U}_k \Theta_k^{-1/2} \\ \Theta_k^{-1/2} \widetilde{U}_k^T \widetilde{L}_{k+1} A_k R_k R_k^T E_{k-1}^T L_k^T U_k \Sigma_k^{-1/2} & \Theta_k^{-1/2} \widetilde{U}_k^T \widetilde{L}_{k+1} A_k R_k R_k^T A_k^T \widetilde{L}_{k+1}^T \widetilde{U}_k \Theta_k^{-1/2} \end{array} \right] \\ &= \left[\begin{array}{cc} \Sigma_k & 0 \\ 0 & 0 \end{array} \right], \end{aligned}$$

and

$$\begin{aligned} \widehat{G}_k^{co} &\equiv S_{k-1}^{-1} G_k^{co} S_{k-1}^{-T} = \check{S}_{k-1}^T G_k^{cr} \check{S}_{k-1} \\ &= \left[\begin{array}{cc} \Sigma_k^{-1/2} V_k^T R_k^T E_{k-1}^T L_k^T L_k E_{k-1} R_k V_k \Sigma_k^{-1/2} & \Sigma_k^{-1/2} V_k^T R_k^T E_{k-1}^T L_k^T L_k A_{k-1} \widetilde{R}_{k-1} \widetilde{V}_{k-1} \Theta_{k-1}^{-1/2} \\ \Theta_{k-1}^{-1/2} \widetilde{V}_{k-1}^T \widetilde{R}_{k-1}^T A_{k-1}^T L_k^T L_k E_{k-1} R_k V_k \Sigma_k^{-1/2} & \Theta_{k-1}^{-1/2} \widetilde{V}_{k-1}^T \widetilde{R}_{k-1}^T A_{k-1}^T L_k^T L_k A_{k-1} \widetilde{R}_{k-1} \widetilde{V}_{k-1} \Theta_{k-1}^{-1/2} \end{array} \right] \\ &= \left[\begin{array}{cc} \Sigma_k & 0 \\ 0 & 0 \end{array} \right]. \end{aligned}$$

On the other hand, one can also show that the noncausal reachability and observability Gramians of the transformed periodic descriptor system (7.1) satisfy

$$\widehat{G}_k^{nr} \equiv T_k^{-1} G_k^{nr} T_k^{-T} = \left[\begin{array}{cc} 0 & 0 \\ 0 & \Theta_k \end{array} \right] = S_k^{-1} G_{k+1}^{no} S_k^{-T} \equiv \widehat{G}_{k+1}^{no}, \quad k = 0, 1, \ldots, K-1.$$

Consequently, S_k and T_k ($k = 0, 1, \ldots, K-1$) are the desired balancing transformations such that the realization (7.1) is balanced. In summary, we have the following theorem.

Theorem 7.1. *For completely reachable and completely observable periodic discrete-time descriptor system (1.1) with $\{(E_k, A_k)\}_{k=0}^{K-1}$ being pd-stable, there exist nonsingular periodic matrices S_k and T_k $(k = 0, 1, \ldots, K-1)$ with $T_K \equiv T_0$ such that the transformed realization (7.1) is balanced.*

Remark. As in the cases of standard state space systems [16, 30] and descriptor systems [35, 38], the balancing transformation matrices for periodic descriptor system (1.1) are not unique. Indeed, if $\{(S_k, T_k)\}_{k=0}^{K-1}$ denotes a set of balancing transformation pairs for the periodic descriptor system (1.1), then for any diagonal matrix D with diagonal entries ± 1, the set of matrix pairs $\{(S_k D, T_k D)\}_{k=0}^{K-1}$ are also the balancing transformation matrices for the periodic descriptor system (1.1).

8 Concluding remarks

In this paper we have derived the necessary and sufficient conditions for complete reachability and complete observability of periodic time-varying descriptor systems. Furthermore, the important concepts of reachability/observability Gramians, Hankel singular values and balanced realization have been generalized for periodic discrete-time descriptor systems. These are useful in the model reduction problem via the balanced truncation method.

In addition, in Theorem 4.1, the reachability/observability Gramians are shown to satisfy some projected GDPLE which can be computed numerically by applying the PQZ algorithm with reordering strategies. A numerical example is given to illustrate the feasibility and reliability of the proposed algorithm in Section 5.

References

[1] D. J. Bender, *Lyapunov-like equations and reachability/observability gramians for descriptor systems*, **IEEE Trans. Auto. Control**, 32 (1987), pp. 343–348.

[2] M. C. Berg, N. Amit, and J. D. Powell, *Multirate digital control system design*, **IEEE Trans. Auto. Control**, 33 (1988), pp. 1139–1150.

[3] S. Bittanti, *Deterministic and stochastic linear periodic systems*, in **Time Series and Linear Systems**, S. Bittanti, ed., Springer Verlag, New York, 1986, pp. 141–182.

[4] S. Bittanti and P. Colaneri, *Analysis of discrete-time linear periodic systems*, in **Control and Dynamic Systems**, C. T. Leondes, ed., vol. 78, Academic Press, New York, 1996.

[5] S. Bittanti and P. Colaneri, *Periodic control*, in **Wiley Encyclopedia of Electrical and Electronic Engineering**, J. G. Webster, ed., vol. 16, Wiley, New York, 1999, pp. 59–74.

[6] S. Bittanti, P. Colaneri, and G. D. Nicolao, *The difference periodic Riccati equation for the periodic prediction problem*, **IEEE Trans. Auto. Control**, 33 (1988), pp. 706–712.

[7] A. Bojanczyk, G. H. Golub, and P. Van Dooren, *The periodic Schur decomposition. Algorithms and applications*, in **Proc. SPIE Conference**, vol. 1770, San Diego, 1992, pp. 31–42.

[8] R. Bru, C. Coll, and N. Thome, *Compensating periodic descriptor systems*, **Sys. Contr. Lett.**, 43 (2001), pp. 133–139.

[9] R. Byers and N. Rhee, *Cyclic Schur and Hessenberg Schur numerical methods for solving periodic Lyapunov and Sylvester equations*, Technical Report, Dept. of Mathematics, Univ. of Missouri at Kansas City, 1995.

[10] R. E. Crochiere and L. R. Rabiner, **Mutirate Digital Signal Processing**, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[11] L. Dai, **Singualr Control Systems**, Springer-Verlag Berlin, Heidelberg, 1989.

[12] A. Feuer and G. C. Goodwin, **Sampling in Digital Signal Processing and Control**, Birkhauser, New York, 1996.

[13] D. S. Flamm and A. J. Laub, *A new shift-invariant representation of perioidc linear systems*, **Sys. Contr. Lett.**, 17 (1991), pp. 9–14.

[14] B. Francis and T. T. Georgiou, *Stability theory for linear time-invariant plants with periodic digital controllers*, **IEEE Trans. Auto. Control**, 33 (1988), pp. 820–832.

[15] W. A. Gardner, **Cyclostationarity in Communnications and Signal Processing**, IEEE Press, New York, 1994.

[16] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their* L^∞*-errors bounds*, **Int. J. Control**, 39 (1984), pp. 1115–1193.

[17] G. H. Golub and C. F. Van Loan, **Matrix Computations, 3rd ed.**, The Johns Hopkins University Press, 1996.

[18] J. J. Hench and A. J. Laub, *Numerical solution of the discrete-time periodic Riccati equation*, **IEEE Trans. Auto. Control**, 39 (1994), pp. 1197–1210.

[19] R. W. Isniewski and M. Blanke, *Fully magnetic attitude control for spacecraft subject to gravity gradient*, **Automatica**, 35 (1999), pp. 1201–1214.

[20] W. Johnson, **Helicopter Theory**, Princeton University Press, Princeton, NJ, 1996.

[21] M. Kono, *Eigenvalue assignment in linear discrete-time system*, **Int. J. Control**, 32 (1980), pp. 149–158.

[22] Y.-C. Kuo, W.-W. Lin, and S.-F. Xu, *Regularrization of linear discrete-time perioidc descriptor systems by derivative and proportional state feedback.* To appear in **SIAM J. Matrix Analy. Appl.**, 2004.

[23] A. J. Laub, M. T. Heath, C. C. Paige, and R. C. Ward, *Computation of system balancing transformations and other applications of simutaneous diagonalization algorithms*, **IEEE Trans. Auto. Control**, 32 (1987), pp. 115–122.

[24] F. L. Lewis, *Fundamental, reachability, and observability matrices for discrete descriptor systems*, **IEEE Trans. Auto. Control**, 30 (1985), pp. 502–505.

[25] W.-W. Lin and J.-G. Sun, *Perturbation analysis for eigenproblem of periodic matrix pairs*, **Lin. Alg. Appl.**, 337 (2001), pp. 157–187.

[26] W.-W. Lin and J.-G. Sun, *Perturbation analysis of the periodic discrete-time algebraic Riccati equation*, **SIAM J. Matrix Analy. Appl.**, 24 (2002), pp. 411–438.

[27] M.-L. Liou and Y.-L. Kuo, *Exact analysis of switched capacitor circuits with arbitrary inputs*, **IEEE Trans. Circuits and Systems**, 26 (1979), pp. 213–223.

[28] A. Marzollo, **Periodic Optimization**, Springer Verlag, Berlin, 1972.

[29] R. McKillip, *Periodic model following controller for the control-configured helicopter*, **Journal of the American Helicopter Society**, 36 (1991), pp. 4–12.

[30] B. C. Moore, *Principal component analysis in linear systems: controllability, observability, and model reduction*, **IEEE Trans. Auto. Control**, 26 (1981), pp. 17–32.

[31] T. Penzl, *Numerical solution of generalized Lyapunov equations*, **Adv. Comput. Math.**, 8 (1998), pp. 33–48.

[32] J. A. Richards, **Analysis of Periodically Time-Varying Systems**, Springer-Verlag, Berlin, 1983.

[33] J. Sreedhar and P. Van Dooren, *Forward/backward decomposition of periodic descriptor systems and two point boundary value problems*, in **European Control Conf.**, 1997.

[34] J. Sreedhar and P. Van Dooren, *Periodic descriptor systems: solvability and conditionability*, **IEEE Trans. Auto. Control**, 44 (1999), pp. 310–313.

[35] T. Stykel, *Model reduction of descriptor systems*, Technical Report 720-2001, Institut für Mathematik, TU Berlin, D-10263 Berlin, Germany, 2001.

[36] T. Stykel, *Analysis and numerical solution of generalized Lyapunov equations*, PhD Dissertation, Institut für Mathematik, Tecnische Universität Berlin, Berlin, 2002.

[37] T. Stykel, *Stability and inertia theorems for generalized Lyapunov equations*, **Lin. Alg. Appl.**, 355 (2002), pp. 297–314.

[38] T. Stykel, *Balanced truncation model reduction for semidiscretized Stokes equation*, Technical Report 04-2003, Institut für Mathematik, TU Berlin, D-10263 Berlin, Germany, 2003.

[39] T. Stykel, *Input-output invariants for descriptor systems*, preprint PIMS-03-1, Pacific Institute for the Mathematical Sciences, Canada, 2003.

[40] L. Tong, G. Xu, and T. Kailath, *Blind identification and equalization based on second-order statistics: A time domain approach*, **IEEE Trans. Information Theory**, 40 (1994), pp. 340–349.

[41] P. P. Vaidyanathan, *Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial*, in **Proc. IEEE**, vol. 78, 1990, pp. 56–93.

[42] P. P. Vaidyanathan, **Mutirate Systems and Filter-Banks**, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[43] P. Van Dooren and J. Sreedhar, *When is a periodic discrete-time system equivalent to a time invariant one?*, **Lin. Alg. Appl.**, 212/213 (1994), pp. 131–151.

[44] A. Varga, *Periodic Lyapunov equations: some applications and new algorithms*, **Int. J. Control**, 67 (1997), pp. 69–87.

[45] A. Varga, *Balancing related methods for minimal realization of perioidc systems*, **Sys. Contr. Lett.**, 36 (1999), pp. 339–349.

[46] A. Varga, *Balanced truncation model reduction of periodic systems*, in **Proc. of IEEE Conference on Decision and Control, Sydney, Australia**, 2000.

[47] A. Varga, *Robust and minimum norm pole assignment with periodic state feedback*, **IEEE Trans. Auto. Control**, 45 (2000), pp. 1017–1022.

[48] A. Varga and P. Van Dooren, *Computing the zeros of periodic descriptor systems*, **Sys. Contr. Lett.**, 50 (2003), pp. 371–381.

[49] J. Vlach, K. Singhai, and M. Vlach, *Computer oriented formulation of equations and analysis of switched-capacitor networks*, **IEEE Trans. Circuits and Systems**, 31 (1984), pp. 735–765.

[50] J. Xin, H. Kagiwada, A. Sano, H. Tsuj, and S. Yoshimoto, *Regularization approach for detection of cyclostationary signals in antenna array processing*, in **IFAC Symposium on System Identification**, vol. 2, 1997, pp. 529–534.

[51] V. A. Yakubovich and V. M. Starzhinskii, **Linear Differential Equations with Perioidc Coefficients**, Wiley, New York, 1975.

[52] K. Zhou, J. C. Doyle, and K. Glover, **Robust and Optimal Control**, Prentice-Hall, Upper Saddle River, NJ, 1996.