Observability and the Separation Principle

- Observability: Definition, tests, canonical/normal forms
- Duality
- Observer design and detectability
- Separation principle
- Design of stabilizing output-feedback controllers

Related Reading

[KK] 4.1-4.4 [AM]: 7.1-7.3

State Reconstruction

We have designed stabilizing state-feedback controllers for

$$\dot{x} = Ax + Bu.$$

An implementation requires that the full state is measurable.

Assuming availability of all states for control is unrealistic.

Let us hence suppose that we only have the output

$$y = Cx + Du$$

as information available that can be used by the controller.

Two - still rather vague - questions come to mind:

- If we only know u and y, is it possible to reconstruct x?
  This is the problem of state-reconstruction.
- Can one implement a controller with the reconstructed state?

State Reconstruction

Typically y has much fewer components than x. It is hence impossible to extract x from the measurements y instantaneously. However the system is dynamic, and we could try to collect the information in y over time in order to reconstruct x.

Definition 1 *The linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable if, for any $T > 0$, it is possible to uniquely determine $x(t)$ for $t \in [0, T]$ based on knowledge of $u(t)$ and $y(t)$ for $t \in [0, T]$.*

Given $y(t)$ for $t \in [0, T]$, how can we possibly extract more information? One idea is to differentiate $y(t)$. In fact we have at our disposal

$$y(t),\ \dot{y}(t),\ \ddot{y}(t),\ \dots\ , y^{(n-1)}(t).$$

We could use even higher derivatives but they do not provide additional information, as we will see.

State Reconstruction

Now observe that there exists a matrix $\mathcal{D}$ such that

$$Y(t) := \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}}_{W} x(t) + \mathcal{D} \underbrace{\begin{pmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}}_{U(t)}.$$

Definition 2 *W is called the* **observability matrix** *of the system $\dot{x} = Ax + Bu$, $y = Cx + Du$ or of the pair (A, C).*

If W has full column rank the equation $W^+W = I$ has a solution W^+. With any such matrix we can reconstruct $x(t)$ from $Y(t)$ and $U(t)$ as $x(t) = W^+Y(t) - W^+\mathcal{D}U(t)$. This leads us to the following result.

Theorem 3 *The linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable iff its observability matrix W has full column rank.*

Proof

If W has full column rank, reconstruction can be done as seen above.

Suppose that W does not have full column rank. This implies that

$$\begin{pmatrix} C^T & A^TC^T & \cdots & (A^{n-1})^TC^T \end{pmatrix} \text{ does not have full row rank.}$$

Due to the Hautus-test for controllability of (A^T, C^T) we infer that there exist $\lambda \in \mathbb{C}$ and a complex vector $e \neq 0$ with $e^* \begin{pmatrix} A^T - \lambda I & C^T \end{pmatrix} = 0$. Hence we infer $Ae = \bar{\lambda}e$ and $Ce = 0$. Therefore $c(t) = e^{\bar{\lambda}t}e$ satisfies $\dot{c}(t) = Ac(t)$ and $Cc(t) = 0$ for all $t \geq 0$. Hence, with $u(t) = 0$, either $r(t) = \text{Re}[c(t)]$ or $s(t) = \text{Im}[x(t)]$ is a **nonzero** system trajectory for which the output is identically zero. Suppose $r(t) \neq 0$ for $t \in [0, T]$.

This prevents observability: Indeed let $y(t)$ be an output for $x(t)$, $u(t)$. Then both $x(t)$ and $x(t) + r(t)$ are **different** state-trajectories for the very same input/output pair $(u(t), y(t))$. Thus the state-trajectory is certainly not uniquely defined by the input- and output-trajectories.

Duality

Let us recognize, as just exploited, that

$$(A, C) \text{ is observable } \iff (A^T, C^T) \text{ is controllable.}$$

Indeed the transposed observability matrix of (A, C) is just the Kalman matrix of (A^T, C^T). More generally, we can exploit everything that we already learnt about controllability of (A^T, C^T) and translate the related insights into facts about observability of the pair (A, C).

Translating questions about observability of (A, B, C, D) into the corresponding questions on controllability of (A^T, C^T, B^T, D^T) (or vice-versa) is called the **duality principle** in linear control.

This statement is brought to life through the examples considered next. Note that it saves a lot of work since we do not need to re-derive results about observability, but we just need to dualize the results of Lecture 3.

Hautus Test for Observability

As a first example let us formulate the Hautus-test for observability.

Theorem 4 *The pair (A, C) is observable if and only if*

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} \text{ has full column rank for all } \lambda \in \mathbb{C}.$$

Equivalently, there exists no eigenvector $e \neq 0$ of A with $Ce = 0$

Proof. (A, C) is observable iff (A^T, C^T) is controllable iff (Hautus test for controllability) $\begin{pmatrix} A^T - \lambda I & C^T \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$; the row rank of a complex matrix is equal to the column rank of its transpose (without conjugation); hence the latter property is equivalent to $\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}$ having full column rank for all $\lambda \in \mathbb{C}$. ■

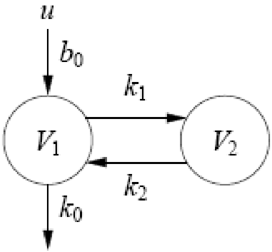
We have now all the means to verify observability of (A, C) in practice.

Example

Consider the two-compartment model (e.g. for drug admin, [AM] pp.85):

$$\begin{aligned} \dot{c} &= \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} c \end{aligned}$$

$$b_0 > 0,\ k_1 > 0,\ k_2 > 0.$$



By measuring the concentration in the first compartment, it is indeed possible to reconstruct the not directly accessible concentration in the second compartment: Since $k_1 \neq 0$,

$$W = \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{pmatrix}$$ has full column rank (Kalman criterion).

Similarly $Ae = \lambda e$ and $Ce = 0$ implies $e_1 = 0$ (due to $Ce = 0$) and then $e_2 = 0$ (first equation and $k_1 \neq 0$) - we can apply the Hautus criterion.

Unobservable Subspace and Modes

If the observability matrix does not have full column rank, it has a non-zero null-space. From slide 4 we infer that non-zero state-trajectories in this space are "swallowed" by W and "cannot be seen" at the output.

Definition 5 $N(W)$ *is called* **unobservable subspace** *of* (A, C).

Based on an eigenvalue λ for which we find an eigenvector e of A such that $Ce = 0$, we actually constructed such a state-trajectory on slide 5. These eigenvalues carry a special name.

Definition 6 *Any* $\lambda \in \mathbb{C}$ *for which* $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ *does not have full column rank is called an* **unobservable mode** *of* (A, C).

By using a duality argument it is now very simple to nicely exhibit the unobservable subspace and the unobservable modes as seen below.

A Geometric Characterization

Theorem 7 *The unobservable subspace of (A, C) is the largest A-invariant subspaces that is contained in the null-space of C.*

Proof 1. $x \in N(W)$ implies $CA^k x = 0$ for $k = 0, \ldots, n-1$. Hence $x \in N(C)$. Moreover $A^n x$ is a linear combination of $x, Ax, \ldots, A^{n-1}x$. Thus $WAx = 0$, i.e., $Ax \in N(W)$. Thus $AN(W) \subset N(W) \subset N(C)$.

Let $\mathcal{V}$ be any other A-invariant subspace contained in $N(C)$. $x \in \mathcal{V}$ then implies $Cx = 0$ and hence $CAx = 0$, $CA^2x = 0$, ..., $CA^{n-1}x = 0$. Thus $x \in N(W)$ and hence $\mathcal{V} \subset N(W)$. That's why $N(W)$ is largest.

Proof 2. Recall that $N(W) = R(W^T)^\perp$. Also $R(W^T)$ is the smallest A^T-invariant subspace containing $R(C^T)$; by standard duality facts from linear algebra its orthogonal complement is the smallest A-invariant subspace and contained in $N(C)$. ■

Observability Normal Form

Theorem 8 *There exists a state-coordinate change* $z = Tx$ *with invertible* T *that transforms* $\dot{x} = Ax + Bu$, $y = Cx + Du$ *into*

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u = \tilde{A}z + \tilde{B}u,$$

$$y = \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du = \tilde{C}z + Du$$

such that (A_{11}, C_1) *is observable.*

Learn to read these equations more explicitly as

$$\dot{z}_1 = A_{11}z_1 + B_1u, \quad \dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u, \quad y = C_1z_1 + Du.$$

We infer that z_1 and hence also y are **not influenced** by z_2. For example a modification of the initial condition $z_2(0)$ cannot be observed in y.

Corollary 9 *The unobservable subspace of* $(\tilde{A}, \tilde{C})$ *is* $\{(0, z_2) : \ z_2 \in \mathbb{R}^{\dim(z_2)}\}$ *and its unobservable modes are just the eigenvalues of* A_{22}*.*

Proofs

First Theorem. Just transform (A^T, C^T, B^T, D^T) into the controllability normal form on slide 3-26 and transpose the matrices. **Note:** Clearly T^{-1} can be chosen as $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$ that is non-singular and such that the columns of S_2 form a basis of the null-space of W.

Corollary. The observability matrix $\tilde{W}$ of $(\tilde{A}, \tilde{C})$ has the rows

$$\begin{pmatrix} C_1 & 0 \end{pmatrix}, \begin{pmatrix} C_1 A_{11} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} C_1 A_{11}^{n-1} & 0 \end{pmatrix}.$$

Now observe that $\tilde{W} z = 0$ if and only if z_2 is arbitrary and z_1 satisfies

$$C_1 z_1 = 0,\ C_1 A_{11} z_1 = 0,\ \ldots,\ C_1 A_{11}^{n-1} z_1 = 0. \qquad (\star)$$

Since (A_{11}, C_1) is observable $(\star)$ holds if and only if $z_1 = 0$.

Since $\begin{pmatrix} A_{11} - \lambda I \\ C_1 \end{pmatrix}$ has full column rank, $\begin{pmatrix} A_{11} - \lambda I & 0 \\ A_{21} & A_{22} - \lambda I \\ C_1 & 0 \end{pmatrix}$ can only loose column rank if $A_{22} - \lambda I$ is not invertible.

Example

Let's consider the example system on p.191 of [F]:

$$A = \begin{pmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & 6 & 4 & 2 \end{pmatrix}.$$

The observability matrix is

$$W = \begin{pmatrix} 7 & 6 & 4 & 2 \\ -10 & -9 & -6 & -3 \\ 16 & 15 & 10 & 5 \\ -28 & -27 & -18 & -9 \end{pmatrix}$$

and can be written as $W = LU$ (LU-factorization with `lu`) where

$$L = \begin{pmatrix} -0.25 & 1 & 0 & 0 \\ 0.36 & -0.86 & 1 & 0 \\ -0.57 & 0.57 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -28 & -27 & -18 & -9 \\ 0 & -0.75 & -0.5 & -0.25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

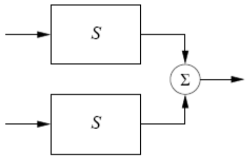
Example

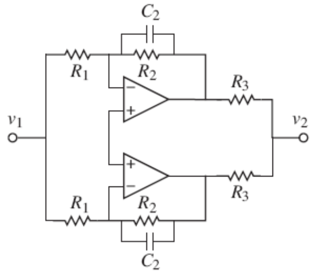
If we define $S = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.67 & -0.33 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ the last columns form a basis of the null space of U and hence of W, and the overall matrix is by construction non-singular. Let's check that it transforms (A, C) into the observability normal form:

$$\left(\begin{array}{c|c} S^{-1}AS & S^{-1}B \\ \hline CS & 0 \end{array}\right) = \left(\begin{array}{rrrr|r} 2 & 3 & \color{red}{0} & \color{red}{0} & 1 \\ -4 & -5 & \color{red}{0} & \color{red}{0} & -1 \\ -2 & -2 & -2.67 & 0.67 & 2 \\ -2 & -2 & -0.67 & -4.33 & -1 \\ \hline 7 & 6 & \color{red}{0} & \color{red}{0} & 0 \end{array}\right).$$

$\left(\begin{pmatrix} 2 & 3 \\ -4 & -5 \end{pmatrix}, \begin{pmatrix} 7 & 6 \end{pmatrix}\right)$ is observable; due to the red zeros this is indeed the desired observability normal form. Hence we infer that (A, C) has the unobservable modes eig $\begin{pmatrix} -2.67 & 0.67 \\ -0.67 & -4.33 \end{pmatrix} = \{-3, -4\}$.

Unobservable Systems





There are many causes for unobservability. For example if we interconnect two identical observable systems (A_S, B_S, C_S, D_S) as in the block-diagram, we get $\dot{x} = Ax + Bu$, $y = Cx + Du$ with

$$A = \begin{pmatrix} A_S & 0 \\ 0 & A_S \end{pmatrix},\ B = \begin{pmatrix} B_S & 0 \\ 0 & B_S \end{pmatrix},\ C_S = \begin{pmatrix} C_S & C_S \end{pmatrix},\ D_S = \begin{pmatrix} D_S & D_S \end{pmatrix}.$$

The observability matrix of (A, C) has two identical block columns and, hence, cannot have full column rank. If A_S has the dimension n, the unobservable subspace of (A, C) actually equals $\left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : \ x \in \mathbb{R}^n \right\}$.

Observability Canonical Form

Let us consider a system with a **single output** only.

Theorem 10 *If $\dot{x} = Ax + Bu$, $y = Cx + Du$ has one output (y scalar) and is observable, there exists a coordinate change $z = Tx$ (T invertible) that transforms it into*

$$\begin{aligned}
\dot{z} \;&=\; \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\alpha_{n-1} & 0 & \cdots & 0 & 1 \\ -\alpha_n & 0 & 0 & \cdots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} u = \tilde{A}z + \tilde{B}u \\
y \;&=\; \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} z + Du = \tilde{C}z + Du.
\end{aligned}$$

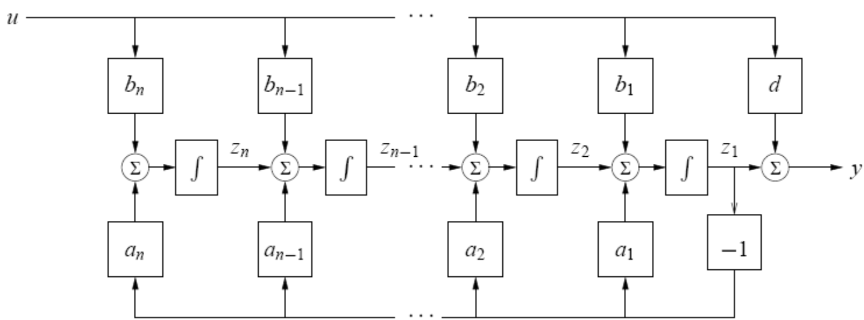
As in Lecture 3, $\alpha_1, \ldots, \alpha_n$ are uniquely determined by the coefficients of the characteristic polynomial of A:

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n.$$

Block-Diagram

The proof is immediate by dualization. Transform (A^T, C^T, B^T, D^T) into the controllable canonical form and transpose both the system matrices and the transformation matrix.

A system in observability normal form can be decomposed into an interconnection of first order system as depicted in



Summary of Duality: $(A, B, C, D) \leftrightarrow (A^T, C^T, B^T, D^T)$

Controllability	**Observability**
$K = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$ full row rk	$W = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ full column rk
Range space K = Controllable subspace	Null space W = Unobservable subspace
$\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ full row rk $\forall \lambda \in \mathbb{C}$	$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ full column rk $\forall \lambda \in \mathbb{C}$
λ with rk drop: Uncontrollable Modes	λ with rk drop: Unobservable Modes
Normal Forms	
$\left(\begin{array}{cc\|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right)$, (A_{11}, B_1) controllable	$\left(\begin{array}{cc\|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right)$, (A_{11}, C_1) observable
SISO: $\left(\begin{array}{cccc\|c} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_n & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \hline \gamma_1 & \gamma_2 & \cdots & \gamma_n & D \end{array}\right)$	SISO: $\left(\begin{array}{cccc\|c} -\alpha_1 & 1 & \cdots & 0 & \beta_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} & \cdots & 0 & 1 & \beta_{n-1} \\ -\alpha_n & \cdots & 0 & 0 & \beta_n \\ \hline 1 & 0 & \cdots & 0 & D \end{array}\right)$

Observers

Instantaneous reconstruction of the state as on slide 4 is not practical, since one should avoid differentiation of signals (measurement noise!) and since the observability matrix might be ill-conditioned.

This motivates to rather try to reconstruct the state asymptotically, by a linear dynamical system that has as its inputs the signals u and y. This leads us to the following fundamental concept.

Definition 11 *An* **observer** *for the linear system* $\dot{x} = Ax + Bu$, $y = Cx + Du$ *is the dynamical system*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x} + Du.$$

that is specified by choosing an **observer gain** $L \in \mathbb{R}^{n \times k}$.

An observer is a **copy** of the original system with a correction term $L(y - \hat{y})$ that servers to drive the **estimated state** $\hat{x}$ towards x in case that the measured output y deviates from the estimated output $\hat{y}$.

Observers

How should we choose L? Actually we would like the **estimation error**

$$\tilde{x} = x - \hat{x}$$

to converge quickly to zero. Let us hence determine its dynamics:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + L(y - \hat{y})] =$$

$$A\tilde{x} - L(Cx + Du - C\hat{x} - Du) = (A - LC)\tilde{x}.$$

The **error dynamics** is described by $\dot{\tilde{x}} = (A - LC)\tilde{x}$.

Hence L should render $A - LC$ Hurwitz such that $\lim_{t\to\infty} \tilde{x}(t) = 0$. The speed and the "character" of the response (such as the overshoot) is determined by the eigenvalues of $A - LC$ and by $e^{(A-LC)t}$.

Theorem 12 *If (A, C) is observable and α a real monic polynomial of degree n, there exists a real matrix L with $\chi_{(A-LC)} = \alpha$.*

Detectability

Indeed if (A, C) is observable then (A^T, C^T) is controllable; hence we can place the eigenvalues of $A^T - C^T F$ to the desired locations; then $A - LC$ has exactly the same eigenvalues for $L = F^T$. Use `place`.

It suffices to just place the eigenvalues into the open left half-plane. This leads to the following concept.

Definition 13 *The system* $\dot{x} = Ax + Bu$, $y = Cx + Du$ *or the pair* (A, C) *is said to be* **detectable** *if there exists some matrix* L *(of compatible dimension) such that* $A - LC$ *is Hurwitz.*

Theorem 14 (Hautus-Test) (A, C) *is detectable iff all its unobservable modes are located in the open left half-plane. Equivalently:*

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \quad \text{has full column rank for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}(\lambda) \geq 0.$$

Detectability

Proof of Hautus test. Since $A-LC$ is Hurwitz iff $A^T-C^TL^T$ is Hurwitz, detectability of (A,C) is equivalent to stabilizability of (A^T,C^T). Now just transpose the Hautus-test for stabilizability.

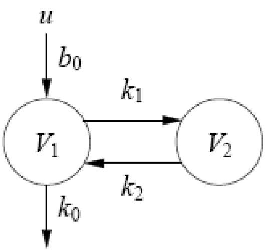
There do exist as well various trajectory-based characterization of detectability (often used as definitions); let us formulate one as follows.

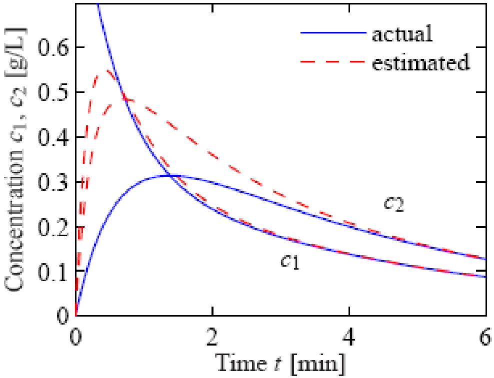
Theorem 15 *The system $\dot{x}=Ax+Bu$, $y=Cx+Du$ is detectable iff $u(t)=0$ and $y(t)=0$ for $t\geq 0$ imply $\lim_{t\to\infty} x(t)=0$.*

Proof. Let the system be transformed into observability normal form on slide 11. With $u(t)=0$ and $y(t)=0$, we actually infer $\dot{z}_1(t)=A_{11}z_1(t)$, $y(t)=C_1z_1(t)=0$ for $t\geq 0$; since (A_{11},C_1) is observable this is equivalent to $z_1(t)=0$ for $t\geq 0$. Hence only $z_2(t)$ is nontrivial and equals $z_2(t)=e^{A_{22}t}z_2(0)$. Therefore, the whole state $z(t)$ converges to zero for $t\to\infty$ iff A_{22} is Hurwitz.

Example

The design of an observer for the compartment model on slide 8 leads to the following simulation results:





Based on measuring $y = c_1$, the observer nicely achieves the task of asymptotically reconstructing the unmeasured concentration c_2.

An Evaluation

Here's a quote from [AM] which emphasizes the relevance of observers:

> "The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no hidden dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs.
>
> As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a virtual sensor that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called sensor fusion."

Separation Principle

We have seen in Lecture 3 how to stabilize a system by static state-feedback, which requires that all states of the system are measured on-line (at all times). In this lecture we learnt how to asymptotically reconstruct the system state from a measured output.

It was a tremendously influential idea to merge these techniques.

For the stabilizable and detectable linear time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

design F and L such that $A - BF$ and $A - LC$ are Hurwitz. Then $u = -Fx$ stabilizes the system, and the observer with gain L generates a state-estimate $\hat{x}$ which asymptotically reconstructs x. The key idea is to replace the unavailable x by the available $\hat{x}$ for control:

$$u = -F\hat{x}.$$

Separation Principle

Definition 16 *For the design parameters* F *and* L, *the linear system*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x} + Du, \quad u = -F\hat{x}$$

is called **observer-based output-feedback controller** *for (1).*

The following two versions are obviously equivalent implementations:

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u + Ly, \quad u = -F\hat{x};$$

$$\dot{\hat{x}} = (A - LC - BF + LDF)\hat{x} + Ly, \quad u = -F\hat{x}.$$

Theorem 17 *The interconnection of the observer-based output-feedback controller with (1) leads to the* **closed-loop** *system*

$$\begin{aligned} \dot{x} &= Ax - BF\hat{x} \\ \dot{\hat{x}} &= (A - LC - BF)\hat{x} + LCx. \end{aligned}$$

This is **asymptotically stable** *iff* $A-BF$ *and* $A-LC$ *are Hurwitz.*

Proof

The closed-loop system as derived above reads as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BF \\ LC & A - LC - BF \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

Let us perform the coordinate change

$$\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = T \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

A very elementary calculation shows

$$T \begin{pmatrix} A & -BF \\ LC & A - LC - BF \end{pmatrix} T^{-1} = \begin{pmatrix} A - BF & BF \\ 0 & A - LC \end{pmatrix}.$$

This leads to the closed-loop system description

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A - BF & BF \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

whose state is x and the estimation error $\tilde{x} = x - \hat{x}$. (This description can as well be obtained by just writing down the differential equations for x and $\tilde{x}$.) Then $\mathcal{A}$ is Hurwitz iff $A - BF$ and $A - LC$ are Hurwitz.

Summary

The resulting design procedure can be described as follows:

- Test whether (A, B) is stabilizable and (A, C) is detectable.
  If **no** one can show that no linear stabilizing controller can exist.
- If **yes** one can determine F and L such that $A - BF$ and $A - LC$ are Hurwitz.
- The observer-based controller leads to a closed-loop system whose eigenvalues are eig$(A - BF) \cup$ eig$(A - LC)$.
- If (A, B) is controllable and (A, C) is observable, one can even place the closed-loop eigenvalues (symmetrically) to arbitrary locations.

In view of the independent construction of F and L for state-feedback stabilization and observer design, the proposed controller is said to be based on the **separation principle**.

How to Choose the Observer Gain?

- The open-loop eigenvalues of the system and the controller, A, $A - LC - BF + LDF$, and the closed-loop eigenvalues of

$$\dot{x} = (A - BF)x + BF\tilde{x}, \quad \dot{\tilde{x}} = (A - LC)\tilde{x}.$$

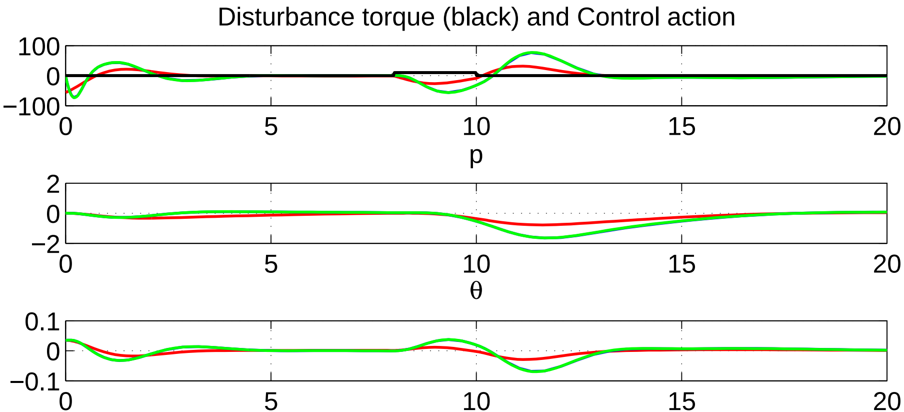
  can all be different. In particular it might happen that the controller itself is unstable. Then care has to be taken in implementing such a controller in practice (start-up, fault-prevention)!

- The structure of the closed-loop system nicely displays the influence of a non-zero initial estimation error $\tilde{x}(0) = x(0) - \hat{x}(0)$ onto the dynamics of x via the term $BF\tilde{x}$; it's visible as a transient response.

- It also illustrates that we wish to use "fast" eigenvalues for the error dynamics $A - LC$. However, they should not be taken too fast since measurement noise (not modeled here) might then be amplified. Moreover large observer gains can adversely influence robustness.

Example: Segway

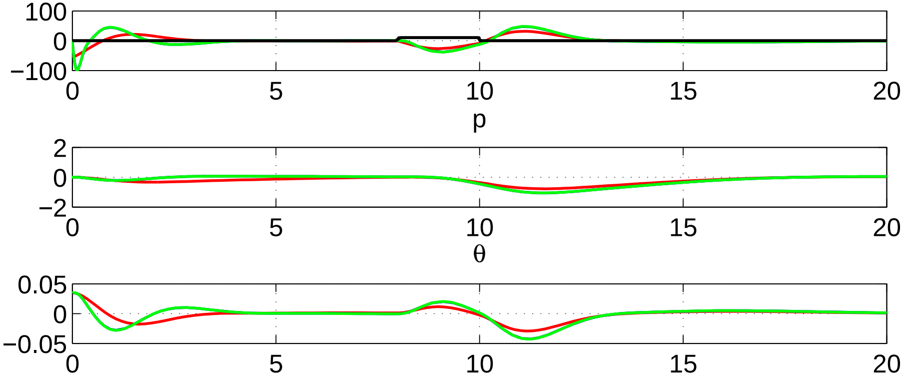
Based on slide 3-46 design an output-feedback controller with observer error-dynamics eigenvalue locations equal to multiples of those used for designing the state-feedback gain. We assume that only p and θ (and not their derivatives) are measured. With the error dynamics eigenvalues $-4 \pm 6.93i, -1.4 \pm 1.43i$ we get



sf (red), of with linear model (blue), of with non-linear model (green)

Example: Segway

"Speeding up" the error dynamics to $-8 \pm 13.86i, -2.8 \pm 2.86i$ leads to output-feedback responses closer to those for state-feedback synthesis:



If "speeding up" the error dynamics, the blue/green responses come closer and closer to the red curves. **However**, this results in high-gain feedback controllers that amplify measurement noise!

How to Analyze Sensivity to Noise?

If the measured output is affected by a noise signal v, the system and controller are described, with $A_c = (A - LC - BF + LDF)$, as

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx + Du + v \end{cases} \quad \text{and} \quad \begin{cases} \dot{\hat{x}} &= A_c\hat{x} + Ly, \\ u &= -F\hat{x}. \end{cases}$$

With the system's transfer matrix $G(s) = C(sI - A)^{-1}B + D$, the controller transfer matrix $K(s) = F(sI - A_c)^{-1}L$ and if assuming zero initial conditions, this reads in the Laplace domain as

$$\hat{y}(s) = G(s)\hat{u}(s) + \hat{v}(s) \quad \text{and} \quad \hat{u}(s) = -K(s)\hat{y}(s)$$

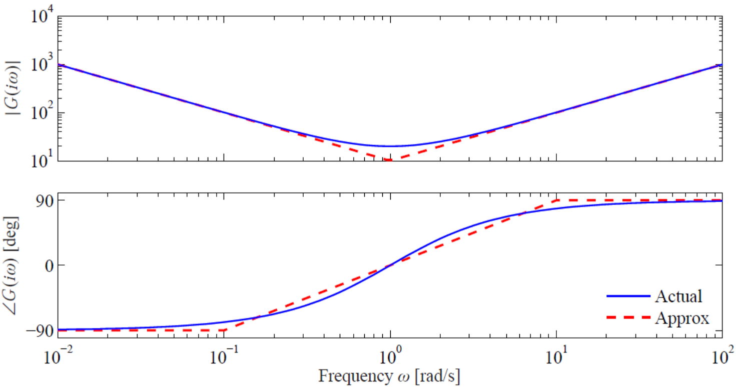
(with apologies for the ambiguous use of the notation $\hat{}$!) We obtain

$$\hat{u}(s) = -K(s)(I + G(s)K(s))^{-1}\hat{v}(s).$$

If this transfer matrix is SISO, we can hence analyze the influence of the measurement noise onto the control input in closed-loop by considering the Bode-magnitude plot of $-K(s)(I + G(s)K(s))^{-1}$.

Bode-Plot

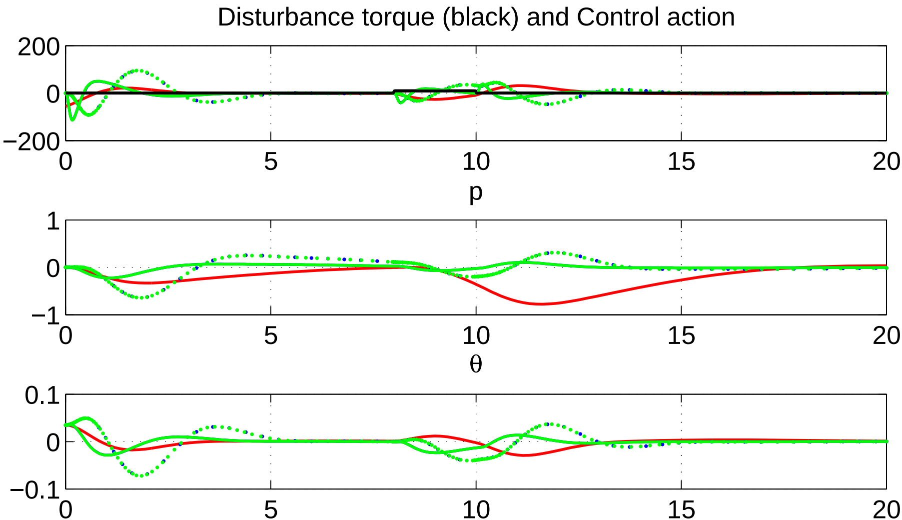
Let $H(s)$ be real-rational. The curve that depicts the absolute value $|H(i\omega)|$ over the frequency $\omega \geq 0$ is called a **Bode magnitude plot** of $H(s)$. The curve of the argument of $H(i\omega)$ over the frequency ω is the **Bode phase plot** of $H(s)$. The whole frequency response $H(i\omega)$ can be represented by two curves, the so-called **Bode plot** of $H(s)$.



Example: Bode plot of $H(s) = 20 + 10/s + 10s$.

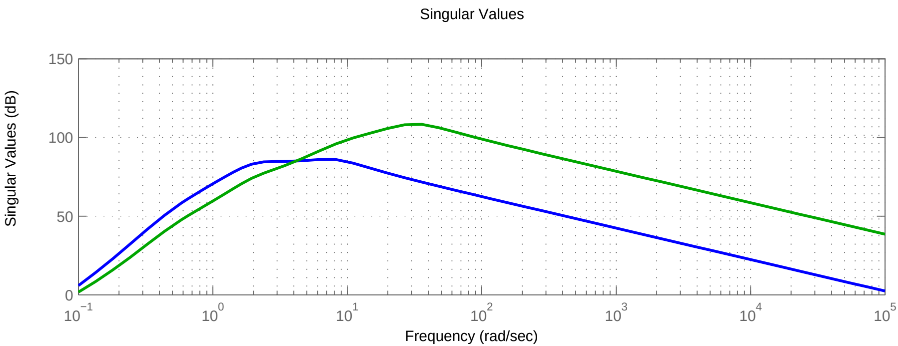
Example

Suppose only p is measured. Compare the designs for error dynamics eigenvalues $-4 \pm 6.9i, -1.4 \pm 1.4i$ (dotted) and $-16 \pm 27.7i, -5.6 \pm 5.7i$. The state-feedback responses are now **not** approximated any more:



Example

Moreover the plot of $\omega \to |K(i\omega)(I + G(i\omega)K(i\omega))^{-1}|$ for the transfer function from v to u in closed-loop shows that measurement noise is considerably amplified at u for faster error dynamics:



By means of this example we have identified an important trade-off in controller synthesis: Improved performance comes at the expense of higher noise amplification. There are many more!

Covered in Lecture 5

- Observability

  observers, observability, detectability, unobservable modes, duality

  observer design

- Separation Principle

  Output feedback synthesis for stability and response shaping