Contollability & Observability

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Outline

- Controllability.
- Observability.
- Stabilizability.
- Detectability.
- Identical tests for CT and DT systems.

Controllability

Definition 8.4

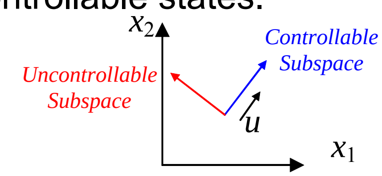
- An LTI system is controllable if for any initial state $\boldsymbol{x}(k_0)$ there exists a control sequence $\boldsymbol{u}(k), k_0 = 1,2,\ldots,k_f - 1$, such that an arbitrary final state $\boldsymbol{x}(k_f)$ can be reached in finite time.

Uncontrollable State

- $\boldsymbol{x}_{uc}$ is uncontrollable if it is orthogonal to the zero state response for all k and **all inputs** $\boldsymbol{u}(k)$.

$$\boldsymbol{x}_{uc}^T \boldsymbol{x}_{ZS}(k) = \sum_{i=k_0}^{k} \boldsymbol{x}_{uc}^T A_d^{k-i-1} B_d \boldsymbol{u}(i) = 0, \forall k, \forall \boldsymbol{u}(k)$$

- Inputs can only drive the system in directions **orthogonal** to the uncontrollable states.
- Sum is identically zero
$\boldsymbol{x}_{uc}^T A_d^k B_d = \boldsymbol{0}^T, \forall k$
$\boldsymbol{x}_{uc}^T = \boldsymbol{w}_i^T$
$= left\ eigenvector$



Left Eigenvector

- Eigenvectors on the left side of A
$$\boldsymbol{w}_i^T A = \lambda_i \boldsymbol{w}_i^T, i = 1,2,\ldots,n$$
- Transpose: can use MATLAB **eig**
$$A^T \boldsymbol{w}_i = \lambda_i \boldsymbol{w}_i, i = 1,2,\ldots,n$$
- Uncontrollable state $\boldsymbol{x}_{uc}^T = \boldsymbol{w}_i^T$
$$\boldsymbol{x}_{uc}^T A_d^k B_d = \boldsymbol{w}_i^T A_d^k B_d = \lambda_i^k \boldsymbol{w}_i^T B_d = \mathbf{0}^T, \forall k$$
  - Controllable: no uncontrollable states

Theorem 8.4: Controllability

- An LTI system is controllable if and only if the products
$$\boldsymbol{w}_i^T B_d \neq \mathbf{0}^T, i = 1,2,\ldots,n$$

$\boldsymbol{w}_i^T = i^{th}$ left eigenvector, $B_d =$ input matrix.

$\boldsymbol{w}_i^T B_d = \mathbf{0}^T$: i^{th} mode is uncontrollable

Proof: Necessity

- Zero-input response
$$\boldsymbol{x}_{ZI}(k) = \sum_{i=1}^{n} Z_i \lambda_i^k \, \boldsymbol{x}(k_0)$$
- Can only decay to zero asymptotically, not in finite time.
- Each mode must be influenced by the input to go to zero in finite time.
- We need $Z_i B_d \neq [0], i = 1,2,\ldots,n$

Proof: Necessity (Cont.)

$$\boldsymbol{x}_{ZS}(k) = \sum_{i=k_0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i), \qquad A_d^k = \sum_{i=1}^{n} Z_i B_d \lambda_i^k$$

$$Z_i B_d = \boldsymbol{v}_i \boldsymbol{w}_i^T B_d \neq [0], i = 1,2,\ldots,n$$

only if

$$\boldsymbol{w}_i^T B_d \neq \mathbf{0}^T, i = 1,2,\ldots,n$$

- If $\boldsymbol{w}_j^T B_d = \mathbf{0}^T$, the the j^{th} mode is uncontrollable.

Cayley-Hamilton Theorem

- Every matrix satisfies its own characteristic equation.

$$\det[\lambda I_n - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$
$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = [\mathbf{0}]$$
$$A^n = -\left[a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n\right]$$

- By induction, all higher powers $A^j, j \geq n$ can be written in terms of $I_n, A, \ldots, A^{n-1}$
- $\forall k, A^k$ can be expressed in terms of $I_n, A, \ldots, A^{n-1}$

Proof: Sufficiency

$$\mathbf{x} = \boldsymbol{x}(k) - \boldsymbol{x}_{ZI}(k) = \sum_{i=k_0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i)$$
$$= \sum_{i=0}^{n-1} \sum_{j=1}^{n} Z_j \lambda_j^{k-i-1} B_d \boldsymbol{u}(i)$$
$$\mathbf{x} = \left[\sum_{j=1}^{n} Z_j B_d \lambda_j^{n-1} \,|\ldots|\, \sum_{j=1}^{n} Z_j B_d\right] \left[\begin{array}{c} \boldsymbol{u}(0) \\ \hline \boldsymbol{u}(1) \\ \hline \vdots \\ \hline \boldsymbol{u}(n-1) \end{array}\right] = L\mathbf{u}$$

Proof: Sufficiency (Cont.)

- $L = \left[\sum_{j=1}^{n} Z_j B_d \lambda_j^{n-1} \,|\ldots|\, \sum_{j=1}^{n} Z_j B_d\right]$ is full rank given that B_d is full rank and $Z_j B_d$ are nonzero.
- $Z_j B_d$ are all rank 1 and are linearly independent provided if $\boldsymbol{w}_i^T B_d \neq \mathbf{0}^T, i = 1,2, \ldots, n$

$$rank\{L|\boldsymbol{x}\} = rank\{L\} = n$$

- Can solve the following equation for the input sequence that drives the system to a specified final state, but nonuniquely

$$\mathbf{x} = \boldsymbol{x}(k) - \boldsymbol{x}_{ZI}(k) = L\mathbf{u}$$

Controllability Rank Condition

Theorem 8.5: A LTI system is completely controllable if and only if the $n \times m.n$ controllability matrix has rank n.

$$\mathcal{C} = \left[B_d | A_d B_d | \ldots | A_d^{n-2} B_d | A_d^{n-1} B_d\right]$$

- If not full rank, there is a vector $\boldsymbol{x}_{uc} = \boldsymbol{w}_i^T$ such that $\boldsymbol{w}_i^T \mathcal{C} = \mathbf{0}^T$
- Rank deficit=number of uncontrollable modes

Proof

$$\mathbf{x} = \boldsymbol{x}(k) - \boldsymbol{x}_{ZI}(k) = \sum_{i=k_0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i)$$

- Use the Cayley-Hamilton Theorem to write

$$\mathbf{x} = [B_d | A_d B_d | \ldots | A_d^{n-1} B_d | A_d^n B_d] \begin{bmatrix} \boldsymbol{u}(n-1) \\ \hline \boldsymbol{u}(n-2) \\ \hline \vdots \\ \hline \boldsymbol{u}(0) \end{bmatrix} = \boldsymbol{\mathcal{C}}\mathbf{u}$$

Solution for $\mathbf{u}$ exists if and only if $rank(\boldsymbol{\mathcal{C}}) = n$

Example

Determine the controllability of the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -0.4 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u(k)$$

Solution

- The controllability matrix of the system is

$$\boldsymbol{\mathcal{C}} = [B_d | A_d B_d | A_d^2 B_d]$$
$$= \left[\begin{array}{cc|cc|cc} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & -0.5 & -0.9 \\ 1 & 1 & -0.5 & -0.9 & -0.15 & 0.05 \end{array}\right]$$
$$rank(\boldsymbol{\mathcal{C}}) = 3$$

- Controllability matrix has rank 3: controllable.
- First 3 columns of matrix linearly independent: sufficient to conclude controllability.
- In general, compute more columns until n linearly independent columns are obtained.

Theorem 8.6: Controllability of Systems in Normal Form

A system in normal form is controllable if and only if its input matrix has no zero rows. Furthermore, if the input matrix has a zero row then the corresponding mode in uncontrollable.

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1^T \\ \vdots \\ \boldsymbol{b}_n^T \end{bmatrix} \boldsymbol{u}(k)$$

Proof:Necessity

- The diagonal form is equivalent to:
  $x_i(k+1) = \lambda_i x_i(k) + \boldsymbol{b}_i^T \boldsymbol{u}(k), i = 1,2,\dots,n$

Necessity:

- If $\boldsymbol{b}_i^T = \mathbf{0}^T$, then the system can only converge to zero asymptotically.
- For controllability we must have convergence in finite time.
- If $\boldsymbol{b}_i^T = \mathbf{0}^T$, the i^{th} mode is not controllable.

Proof: Sufficiency

$$x_i(k+1) = \lambda_i x_i(k) + \boldsymbol{b}_i^T \boldsymbol{u}(k), i = 1,2,\dots,n$$

$$x_i(n) = \lambda_i^n x_i(0) + \sum_{j=0}^{n-1} \lambda_i^{n-j-1} \boldsymbol{b}_i^T \boldsymbol{u}(j)$$

$$\mathbf{x} = \boldsymbol{x}(n) - \Lambda^n \boldsymbol{x}(0) = L\mathbf{u}$$

$$= \begin{bmatrix} \lambda_1^{n-1}\boldsymbol{b}_1^T & \lambda_1^{n-2}\boldsymbol{b}_1^T & \cdots & \lambda_1 \boldsymbol{b}_1^T \\ \lambda_2^{n-1}\boldsymbol{b}_2^T & \lambda_2^{n-2}\boldsymbol{b}_2^T & \cdots & \lambda_2 \boldsymbol{b}_2^T \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{n-1}\boldsymbol{b}_n^T & \lambda_n^{n-2}\boldsymbol{b}_n^T & \cdots & \lambda_n \boldsymbol{b}_n^T \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(0) \\ \hline \boldsymbol{u}(1) \\ \hline \vdots \\ \hline \boldsymbol{u}(n-1) \end{bmatrix}$$

- If B_d has no zero tows the matrix L is full rank
- We can find a control $\mathbf{u}$ to go to any $\mathbf{x}$

MATLAB

- Controllability matrix: same for CT and DT

```
>> A=[[0;0],eye(2);-6,-11,-6]; B=[1;-1;1];
>> C=ctrb(A,B)
>> rank(C)
      ans =
          1
```

(2 uncontrollable modes)

- For diagonal form use ss2ss
- For the eigenvectors use eig with A' (rows)

Transfer Function (not reduced)

```
>> g=zpk(ss(A,B,C,0))
Zero/pole/gain from input to output...
      12 (s+3) (s+2)
 #1:  ----------------
     (s+3) (s+2) (s+1)
        (s+3) (s+2)
 #2:  ----------------
     (s+3) (s+2) (s+1)
```

Cancel Poles and Zeros

```
>> minreal(g)
  Transfer function from input to
    output...
       12
  #1:  -----
      s + 1
        1
  #2:  -----
      s + 1
```


Stabilizability

A system is stabilizable if all its uncontrollable modes decay to zero asymptotically.

- **Stabilizable:** all unstable modes are controllable.
- Stability and controllability: independent properties.
- Physical systems are often stabilizable but not controllable: not a problem if the uncontrollable dynamics decay to zero sufficiently fast.

Observability

A system is said to be observable if any initial state $\boldsymbol{x}(k_0)$ can be estimated from the control sequence

$$\boldsymbol{u}(k), k = k_0, k_0 + 1, \dots, k_f - 1,$$

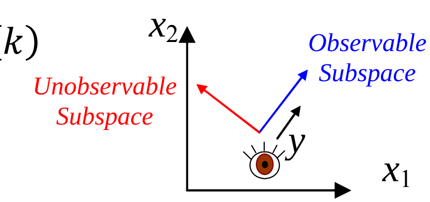
and the measurements

$$\boldsymbol{y}(k), k = k_0, k_0 + 1, \dots, k_f$$

Unobservable States

$$\boldsymbol{y}_{ZI}(k) = C\boldsymbol{x}(k) = CA_d^k \boldsymbol{x}_{uo} = \mathbf{0}, \forall k$$

$$\boldsymbol{y}_{ZI}(k) = C\boldsymbol{x}(k) = \begin{bmatrix} \boldsymbol{c}_1^T \\ \vdots \\ \boldsymbol{c}_l^T \end{bmatrix} \boldsymbol{x}(k)$$



$\boldsymbol{c}_i^T \boldsymbol{x}(k) = \mathbf{0}, i = 1,2, \dots, l$ **(orthogonal)**

- Unobservable state $\boldsymbol{x}_{uo} \neq \mathbf{0}$
- All vectors $\alpha\, \boldsymbol{x}_{uo}$ are indistinguishable.

Right Eigenvector

$$A_d \boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

$$\boldsymbol{x}_{uo} = \boldsymbol{v}_i, C A_d^k \boldsymbol{x}_{uo} = C A_d^k \boldsymbol{v}_i = C \lambda_i^k \boldsymbol{v}_i = \boldsymbol{0}, \forall k$$

$$C \boldsymbol{v}_i = \boldsymbol{0} \Leftrightarrow \boldsymbol{x}_{uo} = \boldsymbol{v}_i$$

- Unobservable state $\boldsymbol{x}_{uo}$ eigenvector
- $\forall k$ the response remains zero (along $\boldsymbol{x}_{uo}$)
- Observable system has no unobservable states

Observability Condition

Theorem 8.7: A system is observable if and only if $C\boldsymbol{v}_i$ is nonzero for $i = 1, 2, \dots, n$, where $\boldsymbol{v}_i$ is the i^{th} eigenvector of the state matrix. Furthermore, if the product $C\boldsymbol{v}_i$ is zero then the i^{th} mode is unobservable.

- Recall that the i^{th} column of the output matrix of the diagonal form is given by $C\boldsymbol{v}_i, i = 1, \dots, n$

Observability Rank Test

Theorem 8.5: A LTI system is completely observable iff the $l.n \times n$ observability matrix has rank n.

$$\mathcal{O} = \begin{bmatrix} C \\ C A_d \\ \vdots \\ C A_d^{n-1} \end{bmatrix}$$

- If not full rank, there is a vector $\boldsymbol{x}_{uo} = \boldsymbol{v}_i$ such that $\mathcal{O}\boldsymbol{v}_i = \boldsymbol{0}$
- Rank deficit=number of unobservable modes

Theorem 8.9: Observability of Systems in Normal Form

A system in normal form is observable if and only if its output matrix has no zero columns. Furthermore, if the output matrix has a zero row column the corresponding mode in unobservable.

Recall: For normal form each state variable associated with a different mode.

$$\boldsymbol{y}(k) = [\boldsymbol{c}_{z1} \quad \cdots \quad \boldsymbol{c}_{zn}]\boldsymbol{x}(k)$$

Proof of Necessity

- Assume observable with rank deficient matrix gives a contradiction. For rank deficient $\mathcal{O}$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \boldsymbol{x}_{uo} = \mathbf{0} \Leftrightarrow CA^i \boldsymbol{x}_{uo} = \mathbf{0}, i = 0, \dots, n-1, \boldsymbol{x}_{uo} \neq \mathbf{0}$$

$$\Rightarrow \boldsymbol{y}_{ZI}(k) = C\boldsymbol{x}(k) = CA^k \boldsymbol{x}_{uo} = \sum_{i-0}^{n-1} \alpha_i(t) CA^i \boldsymbol{x}_{uo} = \mathbf{0}, \forall t$$

- $\boldsymbol{x}_{uo}$ unobservable state and system cannot be observable: contradicts observability assumption.
- Rank deficit= number of eigenvectors s.t.$\boldsymbol{y}_{ZI}(k) = \mathbf{0}$

Proof of Sufficiency

- Assume full rank observability matrix then

$$CA^i \boldsymbol{x}(0) = \mathbf{0}, i = 0,1, \dots, n-1 \Longrightarrow \boldsymbol{x}(0) = \mathbf{0}$$

$$CA^k \boldsymbol{x}(0) = C \left[\sum_{i=0}^{n-1} \alpha_i(t) A^i \right] \boldsymbol{x}(0)$$

$$= [\alpha_0(t) I_n \quad \dots \quad \alpha_{n-1}(t) I_n] \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} \boldsymbol{x}(0)$$

$$\Longrightarrow \boldsymbol{x}(0) = \mathbf{0}$$

- No unobservable states hence observable.

Example 8.10

Determine the observability of the system using two different tests.

If the system is not completely observable, determine the unobservable modes.

$$A_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix}, C = [0 \quad 0 \quad 1]$$

Solution

- State matrix in companion form.
- Characteristic equation and modal matrix

$$\lambda^3 - 4\lambda^2 + 3\lambda = \lambda(\lambda - 1)(\lambda - 3) = 0$$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 9 \end{bmatrix}$$

$$CV = [0 \quad 0 \quad 1] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 9 \end{bmatrix} = [0 \quad 1 \quad 9]$$

$C\boldsymbol{v}_1 = \mathbf{0}$: output-decoupling zero at zero, i.e. one unobservable mode. The unobservable mode is stable (inside the unit circle): detectable system

Rank Test

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \\ CA_d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & 4 \\ 0 & -12 & -13 \end{bmatrix}$$

- Rank $= 2$
- Rank deficit $= 3 - 2 = 1$
- One unobservable mode.

MATLAB Commands

% Calculate observability matrix
>>o = obsv(A, C)
» rank(o) % Find the rank of the matrix.

Example: Eigenvector Test

- Check the controllability and observability of the CT system

$$A = \begin{bmatrix} -1 & 0 & 5 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & -11 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 10 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example Continued: Controllability

$$A = \begin{bmatrix} -1 & 0 & 5 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & -11 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 & 1.25 & -0.25 \\ 0 & 1 & 4 & -1.1429 \\ 0 & 0 & 4.3084 & -1.4361 \\ 0 & 0 & 0 & 1.0841 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad WB = \begin{bmatrix} 0.75 \\ -0.1429 \\ 1.4361 \\ 1.0841 \end{bmatrix}$$

controllable

Example Continued: Observability

$$A=\begin{bmatrix}-1 & 0 & 5 & 0\\ 0 & -4 & 4 & 0\\ 0 & 0 & -5 & -2\\ 0 & 0 & 0 & -11\end{bmatrix}, V=\left[\begin{array}{c|c|c|c}1 & 0 & -0.2901 & -0.1537\\ 0 & 1 & -0.9284 & -0.1757\\ 0 & 0 & 0.2321 & 0.3075\\ 0 & 0 & 0 & 0.9225\end{array}\right]$$

$$C=\begin{bmatrix}10 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad CV=\left[\begin{array}{c|c|c|c}10 & 0 & -2.4371 & -0.9225\\ 0 & 0 & 0 & 0.9225\end{array}\right]$$

e^{-4t} *unobservable*

Example: Rank Test

$$\mathcal{C}=\begin{bmatrix}B & AB & A^2B & A^3B\end{bmatrix}$$
$$=\begin{bmatrix}1 & -1 & -9 & 169\\ 1 & -4 & 8 & 96\\ 0 & -2 & 32 & -402\\ 1 & -11 & 121 & -1331\end{bmatrix}$$

$rank[\mathcal{C}]=4$

controllable

Detectability

- A system is detectable if all its unobservable modes decay to zero asymptotically.
- Detectable: all unstable modes are observable.
- Observability and stability: independent properties.
- Physical systems are typically detectable but not observable: not a problem if the unobservable modes decay to zero sufficiently fast.

Example: Normal Form

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\\ x_3(k+1)\end{bmatrix}=\left[\begin{array}{ccc}-0.4 & 0 & 0\\ 0 & 3 & 0\\ \hline 0 & 0 & -2\end{array}\right]\begin{bmatrix}x_1(k)\\ x_2(k)\\ x_3(k)\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}\mathbf{u}(k)$$

$$y(k)=\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\\ x_3(k)\end{bmatrix}$$

- Zero rows in B_d, zero column in C
- 3^kuncontrollable, $(-2)^k$unobservable.
- $|3|>1$ not stabilizable.
- $|-2|>1$ not detectable.

Example: Diagonal Form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 100 & 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$e^{-5t} \; uncontrolable, \; e^{-11t} \; unobservable$$

Important Relations

- Internally stable systems are stabilizable and detectable: no unstable modes.
- Observable systems are detectable: no unobservable modes.
- Controllable systems are stabilizable: no uncontrollable modes.
- For minimal realizations, BIBO stability and internal stability are equivalent.

Kalman Decomposition

- Any system can be decomposed into four subsystems as shown in the figure: