Introduction Robotics

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WTB Dynamics and Control

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Outline

- Recapitulation
- Velocity kinematics
- Manipulator Jacobian
- Kinematic singularities
- Inverse velocity kinematics

Recapitulation

The general IK problem (1/2)

- Given a homogenous transformation matrix $H \in SE(3)$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

find (multiple) solution(s) $q_1,\ldots,q_n$ to equation

$$T_n^0(q_1, q_2, \ldots, q_n) = H.$$

- Here, H represents the desired position and orientation of the tip coordinate frame $o_n x_n y_n z_n$ relative to coordinate frame $o_0 x_0 y_0 z_0$ of the base; $T^0{}_n$ is product of homogenous transformation matrices relating successive coordinate frames:

$$T_n^0(q_1, q_2, \ldots, q_n) = A_1(q_1) A_2(q_2) \cdots A_n(q_n)$$

The general IK problem (2/2)

- Since the bottom rows of both T^0_n and H are equal to [0 0 0 1], equation

$$T_n^0(q_1, q_2, \ldots, q_n) = H$$

gives rise to 4 trivial equations and 12 equations in n unknowns $q_1,\ldots,q_n$:

$$T_{ij}(q_1, q_2, \ldots, q_n) = H_{ij} \qquad i = 1, 2, 3 \quad j = 1, 2, 3, 4$$

Here, T_{ij} and H_{ij} are nontrivial elements of T^0_n and H.

Kinematic decoupling (1/3)

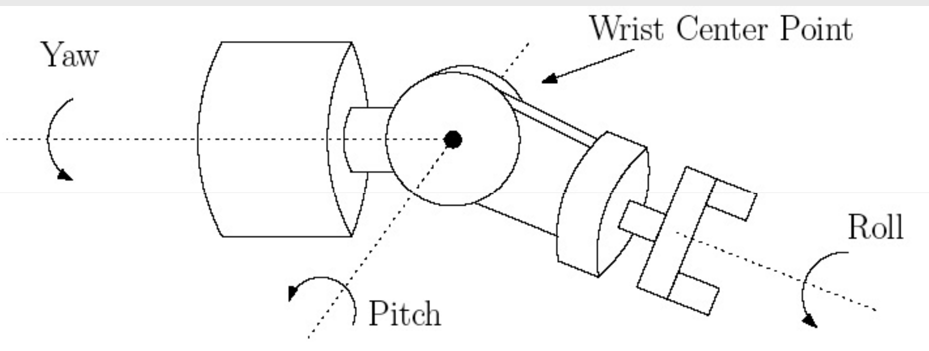
- General IK problem is difficult BUT for manipulators having 6 joints with the last 3 joint axes intersecting at one point, it is possible to decouple the general IK problem into two simpler problems: inverse position kinematics and inverse orientation kinematics.
- IK problem: for given R and o solve 9 rotational and 3 positional equations:

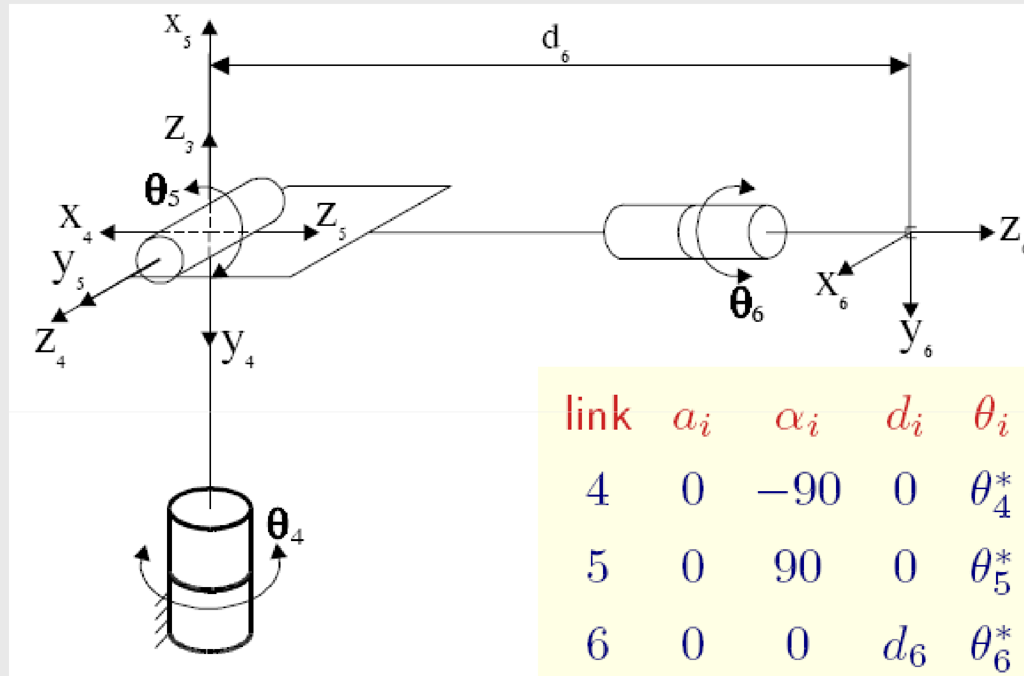
$$R_6^0(q_1, q_2, \ldots, q_6) = R$$

$$o_6^0(q_1, q_2, \ldots, q_6) = o$$

Kinematic decoupling (2/3)

- Spherical wrist as paradigm.

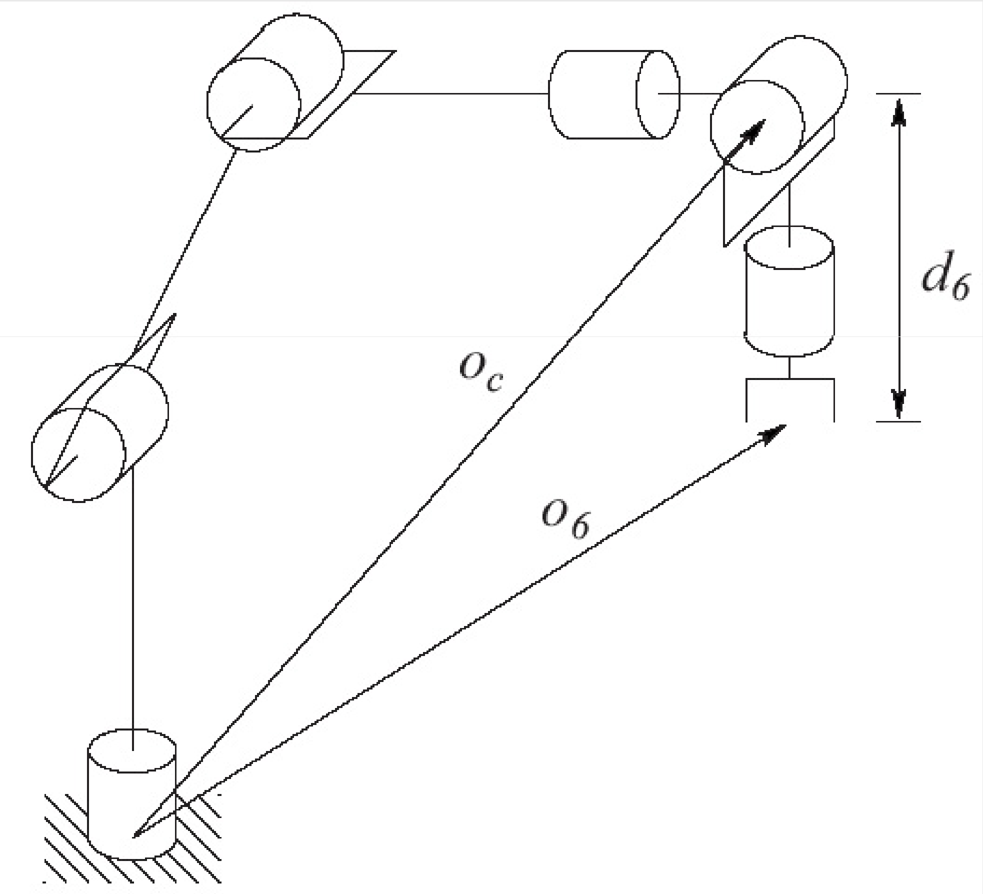




link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

- Let o_c be the intersection of the last 3 joint axes; as z_3, z_4, and z_5 intersect at o_c, the origins o_4 and o_5 will always be at o_c;
  the motion of joints 4, 5 and 6 will not change the position of o_c;
  only motions of joints 1, 2 and 3 can influence position of o_c.

Kinematic decoupling (3/3)



$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} \quad o_c^0 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix} \Rightarrow q_1, q_2, q_3$$

$$R = R_3^0 R_6^3 \quad \Rightarrow$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R \Rightarrow q_4, q_5, q_6$$

Velocity Kinematics

Scope

- Mathematically, forward kinematics defines a function between the space of joint positions and the space of Cartesian positions and orientations of a robot tip; the velocity kinematics are then determined by the Jacobian of this function.
- Jacobian is encountered in many aspects of robotic manipulation: in the planning and execution of robot trajectories, in the derivation of the dynamic equations of motion, etc.

Angular velocity: the fixed axis case

- When a rigid body moves in a pure rotation about a fixed axis, every point of the body moves in a circle; the centers of all these circles lie on the axis of rotation.
- Let θ be the angle swept out by the perpendicular from a point to the axis of rotation; if k is a unit vector in the direction of the axis of rotation, then the angular velocity is given by

$$\omega = \dot{\theta} k.$$

- Given the angular velocity ω, the linear velocity of any point is

$$v = \omega \times r$$

where r is a vector from the origin (laying on the axis of rotation) to the point.

Skew symmetric matrices

- An $n \times n$ matrix S is skew symmetric if and only if

$$S + S^T = 0.$$

- The set of all such matrices is denoted by $so(n)$.

- From this definition, we see that the diagonal elements of S are zero, i.e. $s_{ii} = 0$; also, we see that $S \in so(3)$ contains only 3 independent entries and has the form

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Properties of skew symmetric matrices

- For a vector $a=[a_x, a_y, a_z]^T$ we define

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

1) $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$ for all $a, b \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$

2) $S(a)p = a \times p$ for all $a, p \in \mathbb{R}^3$

3) $RS(a)R^T = S(Ra)$ for all $R \in SO(3)$ and $a \in \mathbb{R}^3$

4) $x^T S x = 0$ for all $S \in so(n)$ and $x \in \mathbb{R}^n$

The derivative of a rotation matrix

- If $R(\theta) \in SO(3)$, then $R(\theta)R^T(\theta) = I$. Differentiating both sides w.r.t. θ yields

$$\underbrace{[\frac{d}{d\theta}R(\theta)]R^T(\theta)}_{S} + \underbrace{R(\theta)[\frac{d}{d\theta}R^T(\theta)]}_{S^T} = 0.$$

- Multiplying both sides on the right by R and using the fact that $S^T = -S$, we get

$$\frac{d}{d\theta}R(\theta)\left[R^T(\theta)R(\theta)\right] - SR(\theta) = 0.$$

- Since $R(\theta)R^T(\theta) = I$, we obtain:

$$\frac{d}{d\theta}R(\theta) = SR(\theta)$$

Derivative of $R_{x,\theta}$ as an example

$$S = [\frac{d}{d\theta}R(\theta)]R^T(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s_\theta & -c_\theta \\ 0 & c_\theta & -s_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i)$$

Hence: $$\frac{d}{d\theta}R_{x,\theta} = S(i)R_{x,\theta}.$$

Similarly we can get

$$\frac{d}{d\theta}R_{y,\theta} = S(j)R_{y,\theta} \qquad \frac{d}{d\theta}R_{z,\theta} = S(k)R_{z,\theta}$$

Derivative of $R_{l,\theta}$

- Let $R_{l,\theta}$ be a rotation matrix about the axis defined by unit vector l.

  Then

$$\frac{d}{d\theta}R_{\ell,\theta} = S(\ell)R_{\ell,\theta}$$

Angular velocity: general case

- Consider angular velocity ω about an arbitrary, possibly moving, axis. Suppose that $R(t) \in SO(3)$ is a time-dependent rotation matrix. Then

$$\dot{R}(t) = S(\omega(t))R(t)$$

where $\omega(t)$ is the angular velocity of the rotating frame with respect to the fixed frame at time t.

Proof that ω is the angular velocity vector

- If p is a point rigidly attached to a moving frame, then

$$p^0 = R_1^0 p^1.$$

Differentiating, we obtain

$$\begin{aligned} \frac{d}{dt} p^0 &= \dot{R}_1^0 p^1 \\ &= S(\omega) R_1^0 p_1 \\ &= \omega \times R_1^0 p_1 \\ &= \omega \times p^0. \end{aligned}$$

Addition of angular velocities (1/3)

- Let $o_0x_0y_0z_0$, $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$ be three frames such that
  - $o_0x_0y_0z_0$ is fixed,
  - all three share a common origin,
  - $R^0{}_1(t)$ and $R^1{}_2(t)$ represent time-varying relative orientations of frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$.
- Also let $\omega^k{}_{i,j}$ denotes the angular velocity vector corresponding to the derivative of $R^i{}_j$, expressed relative to the frame $o_kx_ky_kz_k$.

Addition of angular velocities (2/3)

$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

$$\dot{R}_2^0 = S(\omega_{0,1}^0)R_1^0 R_2^1 + R_1^0 S(\omega_{1,2}^1)R_2^1 \quad \Leftarrow \quad \dot{R} = S(\omega)R$$

$$\dot{R}_2^0 = S(\omega_{0,1}^0)R_1^0 R_2^1 + R_1^0 S(\omega_{1,2}^1)(R_1^0)^T R_1^0 R_2^1 \quad \Leftarrow \quad R^T R = I$$

$$\dot{R}_2^0 = S(\omega_{0,1}^0)R_1^0 R_2^1 + S(R_1^0 \omega_{1,2}^1)R_1^0 R_2^1 \quad \Leftarrow \quad RS(\omega)R^T = S(R\omega)$$

$$S(\omega_{0,2}^0)R_2^0 = [S(\omega_{0,1}^0) + S(R_1^0 \omega_{1,2}^1)]R_2^0 \quad \Leftarrow \quad \dot{R} = S(\omega)R$$

$$\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 \quad \Leftarrow \quad S \text{ is linear}$$

Addition of angular velocities (3/3)

- For an arbitrary number of coordinate systems:

$$R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$$

$$\omega_{0,n}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1}$$

$$\omega_{0,n}^0 = \omega_{0,1}^0 + \omega_{1,2}^0 + \cdots + \omega_{n-1,n}^0 .$$

Linear velocity of a point attached to a moving frame (1/2)

- Suppose that p is rigidly attached to the frame $o_1x_1y_1z_1$ and that $o_1x_1y_1z_1$ is rotating relative to the frame $o_0x_0y_0z_0$.
- Then, we have

$$
\begin{aligned}
p^0 &= R_1^0(t)p^1 \\
\dot{p}^0 &= \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 \\
&= S(\omega^0)R_1^0(t)p^1 \qquad \qquad \dot{p}^1 = 0 \\
&= S(\omega^0)p^0 \\
&= \omega^0 \times p^0.
\end{aligned}
$$

Linear velocity of a point attached to a moving frame (2/2)

- Suppose that the motion of $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ is given by a homogeneous transformation matrix

$$H(t) = \begin{bmatrix} R_1^0(t) & o_1(t) \\ 0 & 1 \end{bmatrix}.$$

- Dropping the argument t, subscripts and superscripts, we get

$$p^0 = Rp^1 + o$$

$$\dot{p}^0 = \dot{R}p^1 + \dot{o} = S(\omega)Rp^1 + \dot{o} = \omega \times r + v$$

where $r = Rp^1$ (vector from o_1 to p expressed in the orientation of $o_0x_0y_0z_0$) and v is the velocity at which the origin o_1 is moving.

Manipulator Jacobian

Derivation of the Jacobian

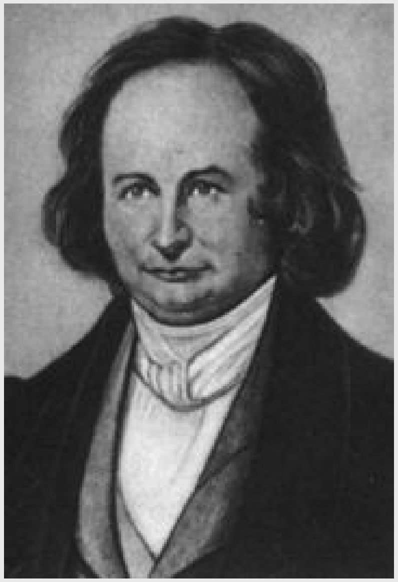
- Consider an n-link manipulator with joint variables $q_1, q_2, \ldots, q_n$.
- Let $q = [q_1, q_2, \ldots, q_n]^T$.
- Let the transformation from the end-effector to the base frame be:

$$T_n^0 = \begin{bmatrix} R_n^0(q) & o(q) \\ 0 & 1 \end{bmatrix}.$$

- Let the angular velocity of the end-effector ω^0_n be

$$S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T.$$

- Linear velocity of the end-effector is $v_n^0 = \dot{o}_n^0$.
- We seek expressions $v_n^0 = J_v \dot{q}$ and $\omega_n^0 = J_\omega \dot{q}$.



Karl Gustav Jacob Jacobi (1804-1851)

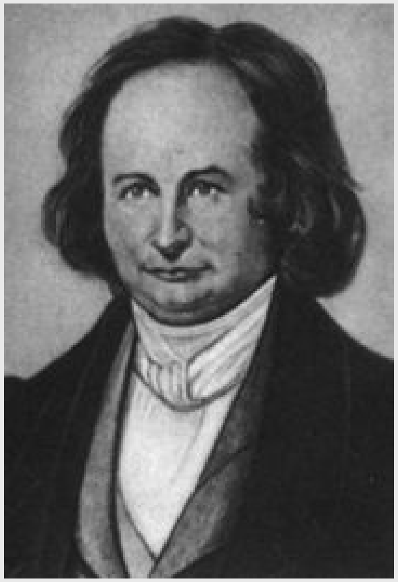
The manipulator Jacobian

- The manipulator Jacobian:

$$J := \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$$\xi = J\dot{q}$$

$$\xi := \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} \qquad \text{body velocity}$$



Karl Gustav Jacob Jacobi (1804-1851)

Angular velocity

- If the i^{th} joint is revolute: the axis of rotation is given by z_{i-1}; let $\omega^{i-1}_{i-1,i}$ represent the angular velocity of the link i w.r.t. the frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. Then, we have

$$\omega^{i-1}_{i-1,i} = \dot{q}_i z^{i-1}_{i-1}.$$

- If the i^{th} joint is prismatic: the motion of frame i relative to frame i-1 is a translation and

$$\omega^{i-1}_{i-1,i} = 0.$$

Overall angular velocity

- By using already derived formula

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + \cdots + R_{n-1}^{0} \omega_{n-1,n}^{n-1},$$

we get

$$\omega_{0,n}^{0} = \rho_1 \dot{q}_1 z_0^0 + \rho_2 \dot{q}_2 R_1^0 z_1^1 + \ldots + \rho_n \dot{q}_n R_{n-1}^{0} z_{n-1}^{n-1} =$$

$$= \rho_1 \dot{q}_1 z_0^0 + \rho_2 \dot{q}_2 z_1^0 + \ldots + \rho_n \dot{q}_n z_{n-1}^{0},$$

where

$$\rho_i = \begin{cases} 1 & \text{if joint } i \text{ is revolute} \\ 0 & \text{if joint } i \text{ is prismatic} \end{cases}.$$

Angular velocity Jacobian

- The complete Jacobian:

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

- Jacobian for angular velocities:

$$J_\omega = \begin{bmatrix} \rho_1 z_0^0 & \rho_2 z_1^0 & \cdots & \rho_n z_{n-1}^0 \end{bmatrix}$$

Linear velocity Jacobian

- The linear velocity of the end effector is just $\dot{o}_n^0$.

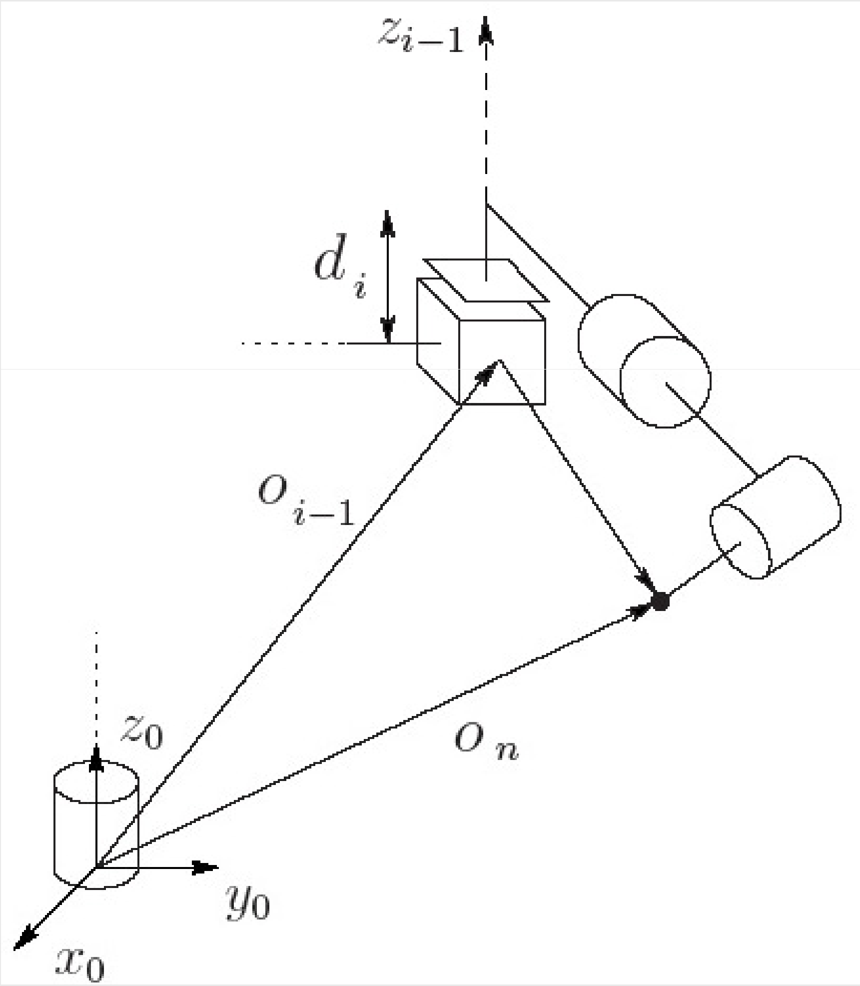
- By the chain rule for differentiation

$$\dot{o}_n^0 = \frac{\partial o_n^0}{\partial q_1}\dot{q}_1 + \frac{\partial o_n^0}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial o_n^0}{\partial q_n}\dot{q}_n,$$

we find Jacobian for linear velocities

$$J_v = \begin{bmatrix} \frac{\partial o_n^0}{\partial q_1} & \frac{\partial o_n^0}{\partial q_2} & \cdots & \frac{\partial o_n^0}{\partial q_n} \end{bmatrix}$$

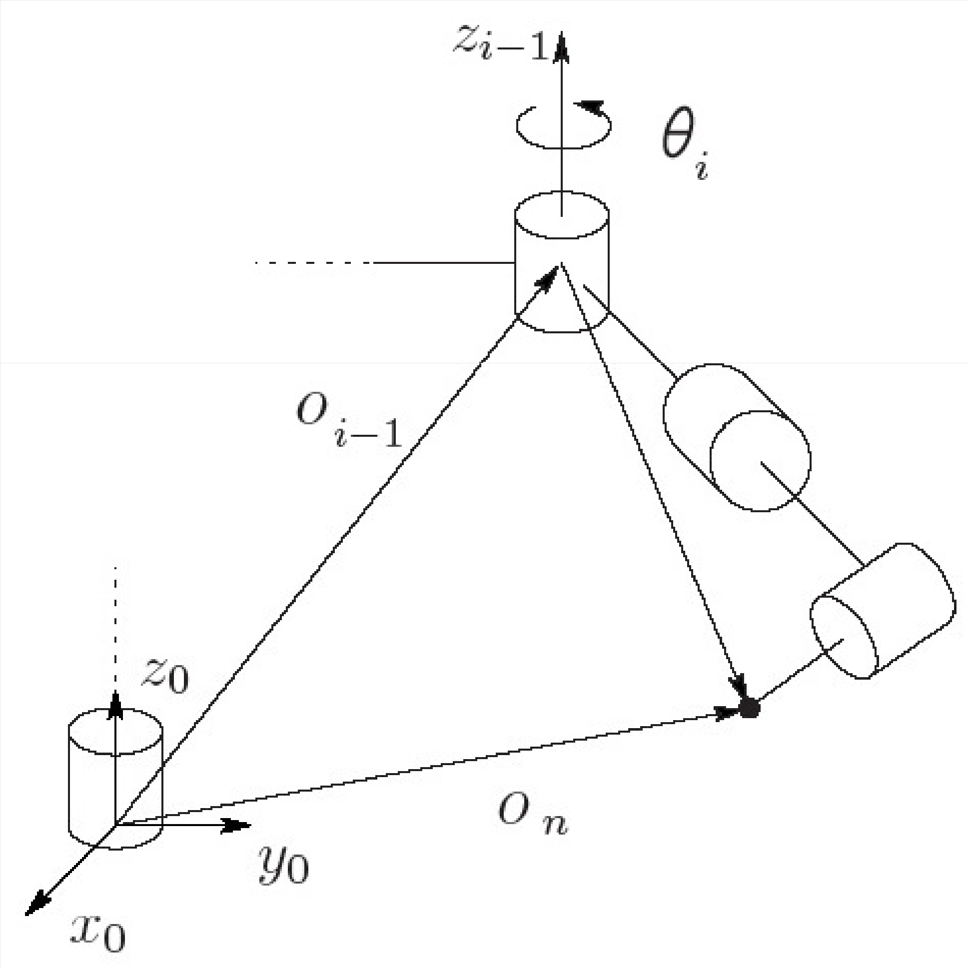
Case 1: prismatic joints



$$\dot{o}_n^0 = \dot{d}_i z_{i-1}^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial o_n^0}{\partial q_i} = z_{i-1}^0.$$

Case 2: revolute joints



- The linear velocity of the end-effector is of form

$$\omega \times r$$

where

$$\omega = \dot{\theta}_i z_{i-1}^0$$
$$r = o_n^0 - o_{i-1}^0$$

- Hence we get

$$\frac{\partial o_n^0}{\partial q_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0).$$

Combining the linear and angular velocity Jacobians

- The Jacobian is given by

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

where

$$J_v = \begin{bmatrix} J_{v_1} & J_{v_2} & \cdots & J_{v_n} \end{bmatrix}$$

$$J_{v_i} = \begin{cases} z_{i-1}^0 \times (o_n^0 - o_{i-1}^0) & \text{for revolute joint } i \\ z_{i-1}^0 & \text{for prismatic joint } i \end{cases}$$

and

$$J_\omega = \begin{bmatrix} J_{\omega_1} & J_{\omega_2} & \cdots & J_{\omega_n} \end{bmatrix}$$

$$J_{\omega_i} = \begin{cases} z_{i-1}^0 & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

Computation of the Jacobian

- We need to compute

$$z_i^0 \qquad \text{and} \qquad o_i^0 .$$

- The former is equal to the first three elements of the 3rd column of matrix $T^0{}_i$, whereas the latter is equal to the first three elements of the 4th column of the same matrix.
- Conclusion: it is straightforward to compute the Jacobian once the forward kinematics is worked out.

Kinematic singularities

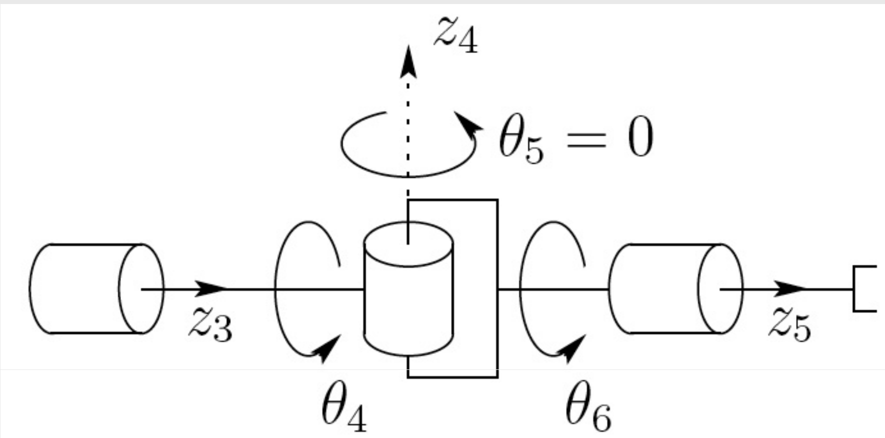
Kinematic singularities

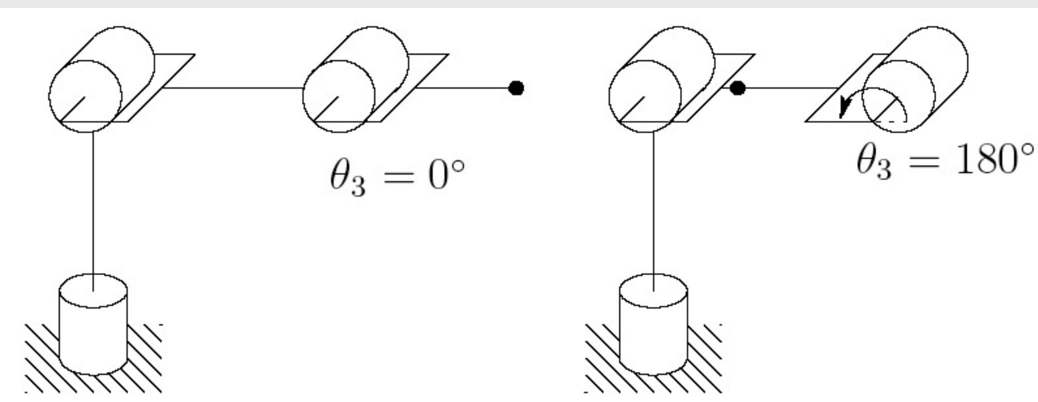
- The $6 \times n$ manipulator Jacobian $J(q)$ defines mapping
$$\xi = J(q)\dot{q}$$
- All possible end-effector velocities are linear combinations of the columns J_i of the Jacobian
$$\xi = J_1\dot{q}_1 + J_2\dot{q}_2 + \ldots + J_n\dot{q}_n$$
- The rank of a matrix is the number of linearly independent columns (or rows) in the matrix; for $J \in \mathbb{R}^{6\times n}$:
$$\operatorname{rank} J \leq \min(6, n)$$
- The rank of Jacobian depends on the configuration q; at singular configurations, $\operatorname{rank} J(q)$ is less than its maximum value.

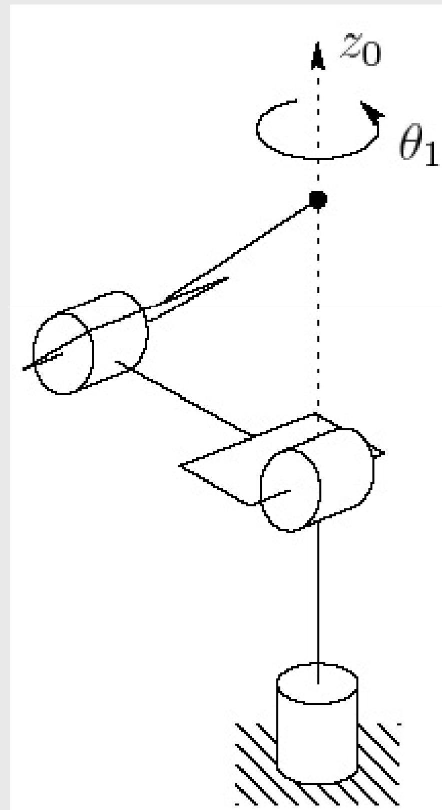
Properties of kinematic singularities

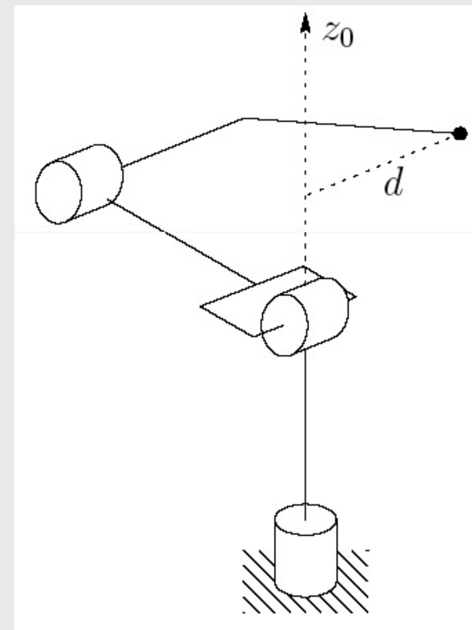
- At singular configurations:
  - certain directions of end-effector motion may be unattainable,
  - bounded end-effector velocities may correspond to unbounded joint velocities,
  - bounded joint torques may correspond to unbounded end-effector forces and torques.
- Singularities correspond to points:
  - on the boundary of the manipulator workspace,
  - within the manipulator workspace that may be unreachable under small perturbations of the link parameters (e.g. length, offset, etc.).

Examples of kinematic singularities (1/2)

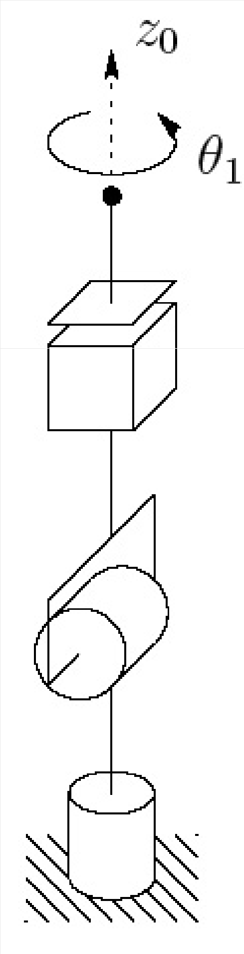


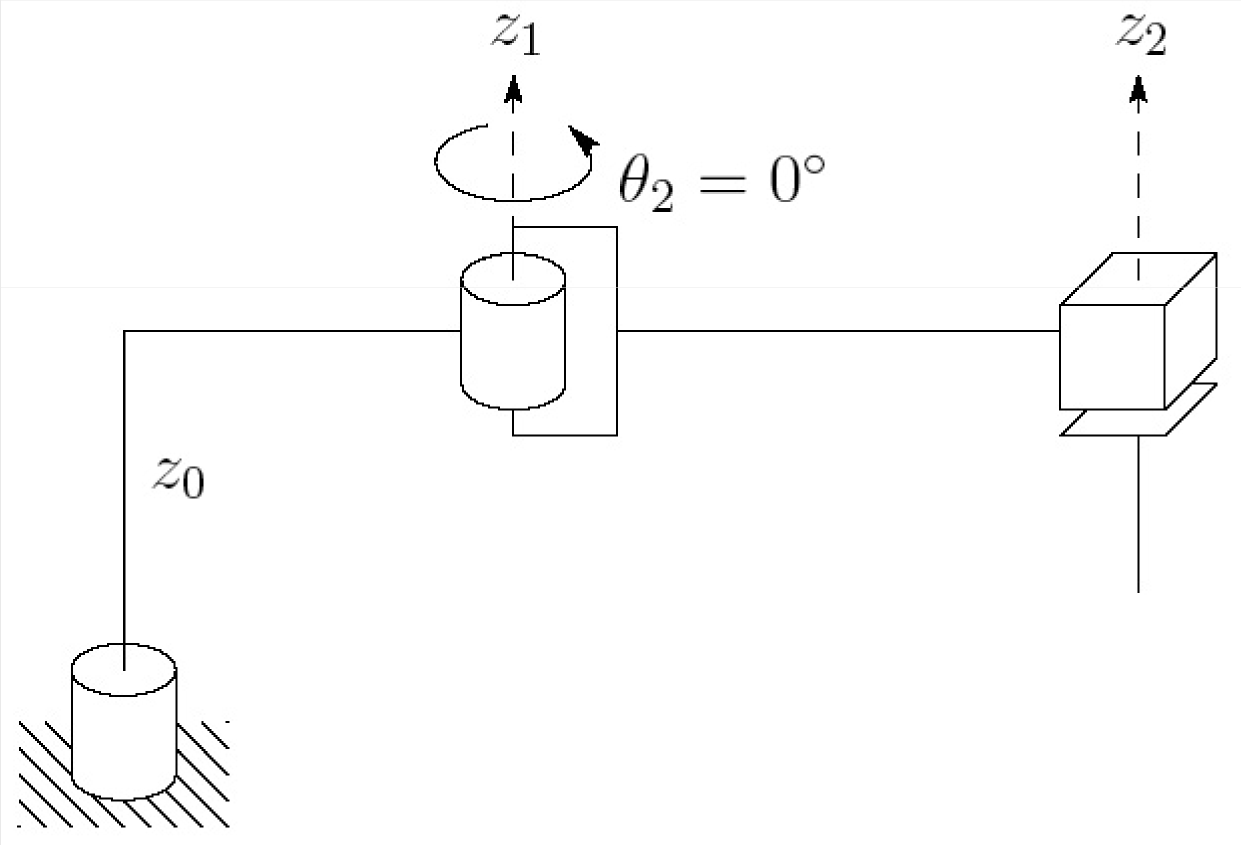






Examples of kinematic singularities (2/2)





Inverse velocity kinematics

Inverse velocity problem

- The Jacobian kinematic relationship:

$$\xi = J\dot{q}.$$

- The inverse velocity problem is to find joint velocities that produce the desired end-effector velocity.
- When Jacobian is square (manipulator has 6 joints) and nonsingular, one gets:

$$\dot{q} = J^{-1}\xi.$$

- If the number of joints is not exactly 6, J cannot be inverted; then the inverse velocity problem has a solution (obtained using e.g. Gaussian elimination) if and only if

$$\operatorname{rank} J = \operatorname{rank} \begin{bmatrix} J & \xi \end{bmatrix}.$$

Pseudoinverse of Jacobian

- When number of joints n is above 6, the manipulator is kinematically redundant; then, the inverse velocity problem can be solved using the pseudoinverse of J.
- Suppose that $\operatorname{rank} J = m$ and $m<n$. Then, the right pseudoinverse of J is given by

$$J^{+} = J^{T}(JJ^{T})^{-1}.$$

- Note that

$$JJ^{+} = I.$$

- It holds

$$\dot{q} = J^{+}\xi + (I - J^{+}J)b$$

where $b \in \mathbb{R}^{n}$ is an arbitrary vector.

Computation of pseudoinverse

- Take the singular value decomposition of J as

$$J = U \Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are both orthogonal matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is given by

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix}.$$

Formula for pseudoinverse

- The right pseudoinverse of J is

$$J^{+} = V\Sigma^{+}U^{T}$$

where

$$\Sigma^{+} = \begin{bmatrix} \sigma_1^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^{-1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_m^{-1} & 0 & \cdots & 0 \end{bmatrix}^{T}.$$

Measures of kinematic manipulability

- Indicate how close is manipulator to a singular configuration.
- In terms of singular values σ_i of the manipulator Jacobian J, kinematic manipulability is defined by:

$$\mu = \sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_m$$

- In terms of eigenvalues λ_i of J or determinant of J, μ is given by:

$$\mu = \sqrt{\det JJ^T} = \left|\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_m\right|$$

- Condition number of J is another manipulability measure:

$$\operatorname{cond} J = \frac{\max \sigma_i}{\min \sigma_i}; \qquad i = 1, \cdots, m.$$