4.1. Observability of Nonlinear Systems

Nonlinear systems of the following form are considered:

$$\Sigma : \begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x} \in \mathcal{M} \subseteq R^n \\ \boldsymbol{y} & = & \boldsymbol{h}(\boldsymbol{x}), \quad \boldsymbol{y} \in R^m \end{array}$$

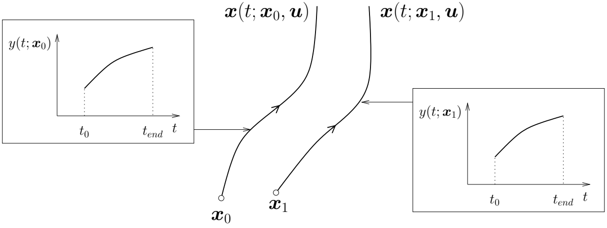
The discussion follows Hermann and Krener (1977)

4.1.1. Observability Definitions

Definition: Indistinguishable states

*Two states $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{M}$ are said to be **indistinguishable**, if for every admissible input $\boldsymbol{u}(t)$, $t_0 \leq t \leq t_{end}$, identical outputs result:*

$$\boldsymbol{y}(t; \boldsymbol{x}_0) \equiv \boldsymbol{y}(t; \boldsymbol{x}_1) \text{ for } t_0 \leq t \leq t_{end}$$

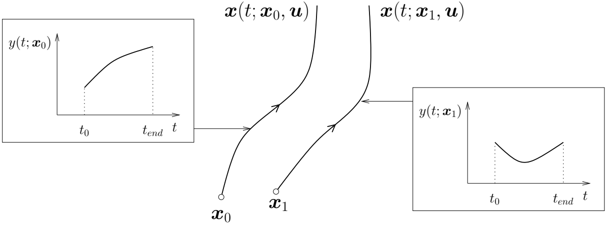


Notation: $I(\boldsymbol{x}_0)$ = set of all points that are indistinguishable from $\boldsymbol{x}_0$.

4.1.1. Observability Definitions

Definition: Observability

*System Σ is **observable at x_0**, if $I(\boldsymbol{x}_0) = \boldsymbol{x}_0$.*
*System Σ is **observable**, if $I(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathcal{M}$.*



Remark:

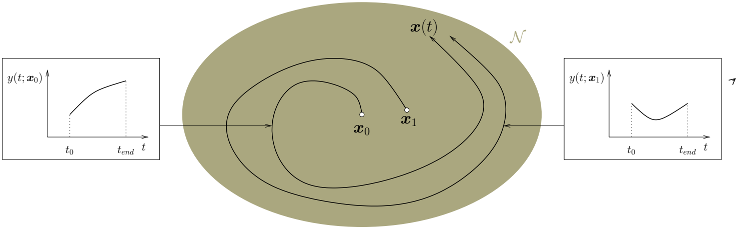
- Observability is sometimes called “global observability”.
- Reconstruction of $\boldsymbol{x}$ from measurement data may be possible for certain inputs $\boldsymbol{u}(t)$, $t_0 \leq t \leq t_{end}$ only.

4.1.1. Observability Definitions

Definition: Local observability

System Σ *is* ***locally observable at*** $\boldsymbol{x}_0$, *if* for every open neighbourhood $\mathcal{N}$ *of* $\boldsymbol{x}_0$ *and* for every solution $\boldsymbol{x}(t)$ completely in $\mathcal{N}$ $I_{\mathcal{N}}(\boldsymbol{x}_0) = \boldsymbol{x}_0$.

System Σ *is* ***locally observable***, *if* $I_{\mathcal{N}}(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathcal{M}$.

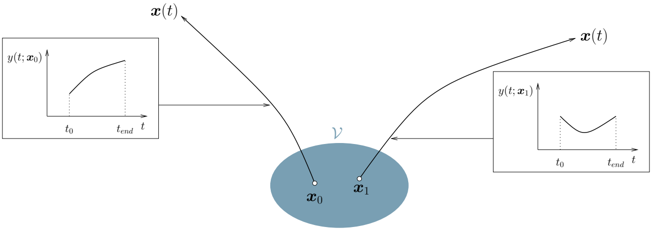


Note that local observability is a stronger property than observability.

4.1.1. Observability Definitions

Definition: Weak observability

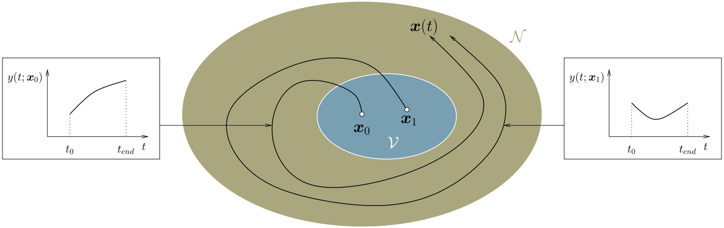
*System Σ is **weakly observable at $\mathbf{x}_0$**, if there is* some *neighbourhood $\mathcal{V}$ of $\mathbf{x}_0$, where $I(\mathbf{x}_0) \cap \mathcal{V} = \mathbf{x}_0$.*
*System Σ is **weakly observable**, if such a neighbourhood $\mathcal{V}$ exists for all $\mathbf{x} \in \mathcal{M}$.*



4.1.1. Observability Definitions

Definition: Local weak observability

*System Σ is **locally weakly observable at x_0**, if there is* some *neighbourhood $\mathcal{V}$ of $\boldsymbol{x}_0$, where $I_{\mathcal{N}}(\boldsymbol{x}_0) \cap \mathcal{V} = \boldsymbol{x}_0$, for all solutions $\boldsymbol{x}(t)$ completely in* any *neighbourhood $\mathcal{N}$ of $\boldsymbol{x}_0$.*
*System Σ is **locally weakly observable**, if this property holds for all $\boldsymbol{x} \in \mathcal{M}$.*



4.1.1. Observability Definitions

Summary:

$$\begin{array}{ccc} \Sigma \text{ locally observable} & \Longrightarrow & \Sigma \text{ observable} \\ \Downarrow & & \Downarrow \\ \Sigma \text{ locally weakly observable} & \Longrightarrow & \Sigma \text{ weakly observable} \end{array}$$

For LTI systems:

observable $\Longleftrightarrow$ locally observable $\Longleftrightarrow$ weakly observable $\Longleftrightarrow$ locally weakly observable

4.1.2. Observability Mapping

Lie Derivative

Consider autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$:

Time derivatives of the measurements:

$$\frac{d\boldsymbol{y}}{dt} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x})$$

Higher derivatives of $\boldsymbol{y}$ can be written compactly by introducing the operator $L_f[.]$ (**Lie derivative**):

$$L_f[\boldsymbol{h}] := \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) = \text{ time derivative of } \boldsymbol{h} \text{ along the system trajectory } \boldsymbol{x}$$

$$\begin{aligned} \Rightarrow \frac{d^2\boldsymbol{y}}{dt^2} &= \frac{\partial}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) \right) \boldsymbol{f}(\boldsymbol{x}) = L_f\left[L_f\left[\boldsymbol{h}\right]\right] = L_f^2\left[\boldsymbol{h}\right] \\ \frac{d^k\boldsymbol{y}}{dt^k} &= L_f^k[\boldsymbol{h}] \end{aligned}$$

4.1.2. Observability Mapping

Observability mapping $\boldsymbol{Q}(\boldsymbol{x})$ for an autonomous system

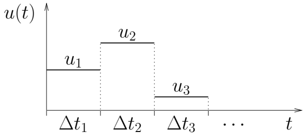
$$\begin{pmatrix} \boldsymbol{y} \\ \dot{\boldsymbol{y}} \\ \ddot{\boldsymbol{y}} \\ \vdots \end{pmatrix} = \begin{pmatrix} L_f^0[\boldsymbol{h}(\boldsymbol{x})] \\ L_f^1[\boldsymbol{h}(\boldsymbol{x})] \\ L_f^2[\boldsymbol{h}(\boldsymbol{x})] \\ \vdots \end{pmatrix} =: \boldsymbol{Q}(\boldsymbol{x})$$

- $\boldsymbol{Q}(\boldsymbol{x})$ defines a set of nonlinear equations for determining $\boldsymbol{x}$ from $\boldsymbol{y}$, $\dot{\boldsymbol{y}}$, $\ddot{\boldsymbol{y}}$,...
- For a general nonlinear system, the number of time derivatives is not fixed (Cayley Hamilton theorem is not applicable!)
- For LTI systems: $\boldsymbol{Q}(\boldsymbol{x}) = \mathcal{O}\boldsymbol{x}$ with Kalman observability matrix $\mathcal{O}$.

4.1.2. Observability Mapping

Construction of the observability mapping $Q(x)$ for a non-autonomous system

- Assume input signals as piecewise constant.
  Abbreviation: $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_i) =: \boldsymbol{f}_i(\boldsymbol{x})$.



- Form derivatives of $\boldsymbol{y}$ with respect to $\Delta t_1, \Delta t_2, \ldots, \Delta t_k$.
  It can be shown (Hermann and Krener, 1977) that

$$\frac{\partial}{\partial \Delta t_1}\left(\frac{\partial}{\partial \Delta t_2}\left(\cdots \frac{\partial \boldsymbol{y}}{\partial \Delta t_k}\right)\right)\Bigg|_{\Delta t_1 = \cdots = \Delta t_k = 0} = L_{\boldsymbol{f}_1}\left[L_{\boldsymbol{f}_2}\left[\cdots L_{\boldsymbol{f}_k}[\boldsymbol{h}]\right]\right]$$

- The rows of $\boldsymbol{Q}(\boldsymbol{x})$ consist of all possible Lie derivatives of the above type.

4.1.3. Observability Rank Condition

Definition:

System Σ satisfies the ***observability rank condition*** *at $\boldsymbol{x}_0$, if $\left.\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0}$ contains n linear independent row vectors.*

Observability indices:

System Σ is said to have observability indices $\kappa_1, \kappa_2, \ldots, \kappa_m$ at $\boldsymbol{x}_0$, if observability rank condition is satisfied after forming κ_i derivatives of y_i.

Observability indices of a given system are not unique.

Observability condition:

The observability rank condition is **sufficient, but not necessary** for **local weak observability**.

4.1.4. Linearised System

Consider a steady state solution $\boldsymbol{x}_s$ of an autonomous system with $\boldsymbol{f}(\boldsymbol{x}_s) = \boldsymbol{0}$ and the linearised system

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{A}\Delta\boldsymbol{x} \text{ with } \boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_s} \\ \Delta\boldsymbol{y} &= \boldsymbol{C}\Delta\boldsymbol{x} \text{ with } \boldsymbol{C} = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_s} \end{aligned}$$

Observability of the linearised system is equivalent to the observability rank condition at $\boldsymbol{x}_s$.

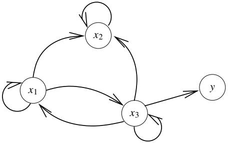
4.1.5. Structural Observability

- Full structural rank of $\partial \boldsymbol{Q}(\boldsymbol{x})/\partial \boldsymbol{x}$ is necessary to satisfy the observability rank condition.
- Output connectivity of all states is necessary for observability of the nonlinear system.

Example:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= f_3(x_1, x_3) \\ y &= h(x_3) \end{aligned}$$

$$\left[\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right] = \begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ * & 0 & * \\ \vdots & \vdots & \vdots \end{pmatrix}$$



$\Rightarrow$ observability rank condition not satisfied

$\Rightarrow$ x_2 not output connected.

4.1.6. Nonlinear Canonical Forms

Observability canonical form:

Consider an autonomous system with a scalar measurement:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad y = h(\boldsymbol{x})$$

Idea for nonlinear state transformation:

Use y and its time derivatives as new state vector $\bar{\boldsymbol{x}}$:

$$\bar{\boldsymbol{x}} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix} = \begin{pmatrix} h(\boldsymbol{x}) \\ L_f h(\boldsymbol{x}) \\ \vdots \\ L_f^{n-1} h(\boldsymbol{x}) \end{pmatrix} =: \bar{\Phi}(\boldsymbol{x})$$

$\bar{\boldsymbol{x}} = \bar{\Phi}(\boldsymbol{x})$ defines a nonlinear coordinate transformation

4.1.6. Nonlinear Canonical Forms

Observability canonical form:

Resulting transformed system equations:

$$
\begin{aligned}
\bar{S}:\ \dot{\bar{\boldsymbol{x}}} &= \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \\ \vdots \\ \bar{x}_n \\ \bar{f}_n(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \bar{\boldsymbol{x}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{f}_n(\bar{x}_1, \ldots, \bar{x}_n) \end{pmatrix} \\
y &= \bar{x}_1
\end{aligned}
$$

Condition for existence of the transformation $\boldsymbol{x} \leftrightarrow \bar{\boldsymbol{x}}$:

- $\bar{\Phi}^{-1}$ exists ($\Rightarrow$ system is observable)

Or:

- $\dfrac{\partial \bar{\Phi}}{\partial \boldsymbol{x}}$ is invertible ($\Rightarrow$ system is locally weakly observable)

4.1.6. Nonlinear Canonical Forms

Nonlinear observer canonical form

$$S^*: \qquad \dot{\boldsymbol{x}}^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \boldsymbol{x}^* - \begin{pmatrix} a_{n-1}(y) \\ a_{n-2}(y) \\ \vdots \\ a_1(y) \\ a_0(y) \end{pmatrix}$$

$$y = h^*(\boldsymbol{C}\boldsymbol{x}^*) \Longrightarrow \boldsymbol{C}\boldsymbol{x}^* = h^{*-1}(y) =: \gamma(y)$$

- all nonlinearities depend on y only $\Rightarrow$ simple observer design.
- transformation to a system in observer canonical form is very difficult in most cases.