On the Dynamics of Floating-base Robots: Linking the Recursive Formulation to the Reduced Euler-Lagrange Equations

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***Abstract*—In this paper, we link the matrix-based recursive dynamics with the *Reduced Euler-Lagrange* (REL) equations from geometric mechanics for a Floating-base Robotic System (FRS). Specifically, a FRS is characterized by a non-flat *mechanical connection* arising from the conservation of the momentum map. The aforestated link is firstly motivated to reveal this non-flatness (curvature), which is hidden in the Coriolis/Centrifugal (CC) terms of the commonly used recursive form of the Hamel's equations. Secondly, although inertia diagonalization transformation of Hamel's equations have been addressed before, the velocity dependencies in the transformed CC matrix have not been clarified. By linking with REL equations, the recursive form acquires its rich structure in form of a diagonalized inertia and, importantly, separated velocity dependencies in the CC matrix. To this end, we firstly separate velocity dependencies in the recursive forms to reveal exact closed-form expressions. Secondly, as main result, we provide novel matrix-based dynamics expressions that relate directly to the REL equations. As a consequence of ensuing simplifications, we obtain novel recursive expressions for the interaction and curvature terms of the REL equation. Finally, we discuss practical applications, like observer design, which are expected to benefit from the proposed structure.**

I. Introduction

Floating-base Robotic Systems (FRS) have emerged as a key solution in domains such as On-Orbit Servicing (OOS) [24, 10, 4], humanoids [7, 8] and, aerial [16] and underwater [23] robotics, see Fig. 1. The dynamics of a FRS are defined by a system of Hamel's equations [22, 14], which consist of an Euler-Poincaré equation and an Euler-Lagrange equation corresponding to motion of FRS-base and the shape (joints), respectively. Hamel's equations are particularly attractive in FRS applications due to the computational efficiency of its recursive form [5, §9.4]. This dynamic description possesses the passivity (or skew-symmetric) structure, which is common in mechanical systems, and has been utilized for control purposes in the past [4]. Specifically for an underactuated scenario like OOS, in which the FRS-base actuation is precluded in the so-called *free-floating* control [24], a simplification of Hamel's form is Lagrangian reduction. This yields reduced shape-space dynamics on a momentum map level-set and is an outcome of the commonly-known property of momentum map conservation. In geometric mechanics, this property is modeled using a *mechanical connection* (see [12, Ex. 4]), which is distinctly non-flat (curvature) [22] for a FRS. The effect of curvature is clearly observed from the net displacement of the FRS-base along closed-path periodic motions in shape-space. In fact, a key feature of locomotion approaches from geometric mechanics is to approximate this displacement over cyclic shape changes with area integrals of the curvature [19, 15]. For this class of mechanical systems, *Reduced Euler Lagrange*

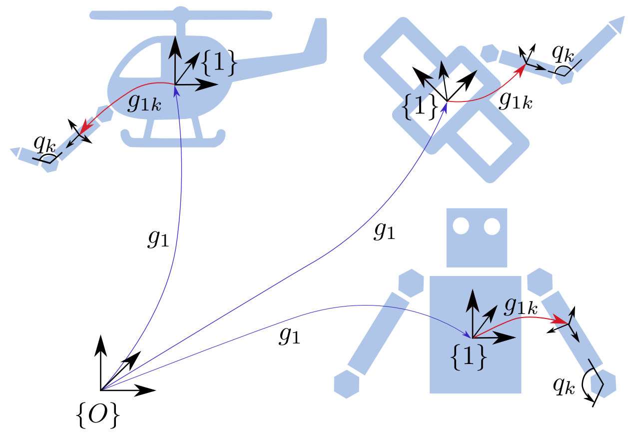

Fig. 1. Diagram of Floating-base Robotic Systems (FRS) whose configuration space is $\{g_1, q\} \in \mathcal{Q}$, where $g_1 \in \mathrm{SE}(3)$ is the pose of the FRS-base frame $\{1\}$ relative to inertial frame $\{O\}$, and $q \in \mathbb{R}^n$ are the n-joint positions.

(REL) equations [2, §5.3][14, pp. 141] are alternatively used to describe the dynamics. In particular, REL equations consist of locked and shape-space variations as an Euler-Poincaré equation and an Euler-Lagrange equation, respectively. The main advantages in the structure of REL equations are block-diagonal inertia, separation of velocity dependencies in the Coriolis/Centrifugal (CC) terms and apparentness of the curvature form. Thus, a deeper insight into FRS dynamics can be gained from REL equations.

A first step that unintentionally brought the robotics community close to the REL equations was a coordinate transformation of the recursive form of Hamel's equations [8, 10]. It is important to acknowledge that, although an explicit reference to REL equations was not made in these works, the resulting dynamics are actually a spatial-momentum formulation [19, eq. 8] of these equations, which lead to a block-diagonal inertia. An outcome of this was that the transformed CC matrix could be decomposed into two matrices of block-diagonal and off-diagonal terms, which satisfied the passivity and skew-symmetric properties, respectively. A similar result was obtained in the domain of multi-agent systems using nonholonomic passive decomposition [13].

However, a link between recursive dynamics and REL equations has not been established before. One outcome of this was that the aforestated works in [8, 10] only exposed the block-diagonal structure of inertia, whereas velocity dependencies in the transformed Coriolis/Centrifugal (CC) were neither separated nor in closed-form. Secondly, the curvature terms are clubbed together with the CC terms and are therefore concealed. In fact, the curvature computation used in locomotion approaches for FRS is largely restricted to symbolic computations and, therefore, applicable only to limited degrees of freedom (DoF) [22].

The contributions of this paper are the following. We derive a novel closed-form computation of the CC matrix, which clearly shows the separation of velocity dependencies for the REL equations of a FRS. In particular, the computation is decomposed into two CC matrices, one of which depends on shape-space velocity, while the other depends on locked velocity. This decomposition of the CC term leads to three new results. Firstly, we prove that the CC matrix of block-diagonal terms satisfying the passivity property depends only on shape-space velocities while the CC matrix of off-diagonal terms satisfying the skew-symmetric property depends only on the locked velocity. This immediately provides more intuition in Lyapunov-based control design leading to improved controllers, and we demonstrate this using a simulation example. Secondly, from the two proposed CC matrices, we prove commutativity properties for the matrices of inertia derivative and inertia velocity. This is a very useful property for simplification in specific cases, e.g. velocity observer design, where a difference of CC forces due to the actual and reference velocities appear. We illustrate this idea with an example. Thirdly, we derive closed-form expressions for the curvature form by isolating the terms that depend on both locked and shape-space velocities in the shape-space part of REL equations.

The paper is organized as follows. In Sec. II, the notation of motion on SE(3) used in this paper is summarized. The FRS is modeled using differential geometry in Sec. III, and the *mechanical connection* is introduced, which leads towards the REL equations. The main result is stated and and derived as a recursive form in Sec. IV. Herein, the skew-symmetric/passivity and commutativity properties are proved. Furthermore, the closed-form expression of curvature is also derived. In Sec. V, a discussion of applications that might benefit from the proposed method are provided. Concluding remarks are given in Sec. VI.

II. Modeling motion on $\mathrm{SE}(3)$ group

In this section, relevant details about motion on $\mathrm{SE}(3)$ group are provided. The reader is referred to App. 1 for the matrix descriptions of introduced quantities. The pose of a rigid body is a matrix group representation of $\mathrm{SE}(3)$, which is written as $g = (R, p)$, where $R \in \mathrm{SO}(3)$ is the rotation matrix and $p \in \mathbb{R}^3$ is the position. The identity of the $\mathrm{SE}(3)$ group is $\mathbb{I}_{4,4}$, where $\mathbb{I}_{k,k}$ is an identity matrix of dimension k. The tangent space at $\mathbb{I}_{4,4}$ is the $\mathfrak{se}(3)$ algebra, which may be referenced in either the body or a spatial frame. Analogously, the cotangent space at $\mathbb{I}_{4,4}$ is the dual space of wrenches, denoted as $\mathfrak{se}(3)^*$. The $\mathfrak{se}(3)$ algebra and its dual $\mathfrak{se}(3)^*$ are isomorphic to the space of velocity twists and wrenches on $\mathbb{R}^6$ using $(\bullet)^\wedge : \mathbb{R}^6 \to \mathfrak{se}(3), \mathfrak{se}(3)^*$ and $(\bullet)^\vee : \mathfrak{se}(3), \mathfrak{se}(3)^* \to \mathbb{R}^6$, e.g., given a twist, $\mathcal{V} \in \mathbb{R}^6$, $\mathcal{V}^\wedge \in \mathfrak{se}(3)$. A pose between two frames is subscripted with symbols of both frames, e.g., in Fig. 1, the pose of $\{k\}$ relative to $\{1\}$ is g_{1k}. Poses and velocities that are subscripted once are referenced relative to an inertial frame, $\{O\}$, e.g. $\{g_1,\ V_1\}$ are pose and velocity of $\{1\}$, which is the FRS-base, relative to $\{O\}$. The Adjoint action, $\mathrm{Ad} : \mathfrak{se}(3) \to \mathfrak{se}(3)$, of a pose g, transforms elements of $\mathfrak{se}(3)$ algebra between spatial and body frames as $\mathcal{V}^s = \mathrm{Ad}_g \mathcal{V}$, see [3]. In the following text, for simplicity of notation, we indicate the pose argument in Ad by its ordered frames. e.g., $\mathrm{Ad}_{g_{1k}} = \mathrm{Ad}_{1k}$. The adjoint map of the $\mathfrak{se}(3)$ algebra onto itself is, $\mathrm{ad} : \mathfrak{se}(3) \to \mathfrak{se}(3)$, which is the differential of the Ad map. This is denoted as $\mathrm{ad}_{\mathcal{V}}$ and its coadjoint map is, $\mathrm{ad}_{\mathcal{V}}^\top : \mathfrak{se}(3)^* \to \mathfrak{se}(3)^*$.

A. Kinematics/Dynamics of a Rigid body

Given a rigid body (k^{th} link in Fig. 1) with pose $g_k \in \mathrm{SE}(3)$, the $\mathrm{SE}(3)$ reconstruction formula is,

$$\text{Kinematics}\left\{\ \dot{g}_k = g_k V_k^\wedge,\ g_k \equiv (R_k, p_k)\right. \tag{1}$$

where $V_k^\wedge \in \mathfrak{se}(3)$ is the body-referenced Lie algebra. The *Euler-Poincaré* equation is written using $\mathfrak{se}(3) \cong \mathbb{R}^6$ as,

$$\text{Dynamics}\left\{\ M_k \dot{V}_k - \mathrm{ad}_{V_k}^\top M_k V_k = F_k\right. , \tag{2}$$

where $F_k \in \mathbb{R}^6 \cong \mathfrak{se}(3)^*$ is the body-referenced wrench, $M_k \in \mathbb{R}^{6\times 6}$ is the inertia matrix (see App. 1) depending on $m_k \in \mathbb{R}_+$ and $I_k \in \mathbb{R}^{3\times 3} \succ 0$, which denote the body's mass and inertia tensor respectively. In (2), the velocity-dependent terms arise due to non-abelian nature of $\mathrm{SE}(3)$ and encapsulate its structure coefficients, which can be generalized using a non-unique operator, $C_k : \mathfrak{se}(3) \to \mathfrak{se}(3)^*$, as,

$$C_k\, V_k = -\mathrm{ad}_{V_k}^\top\, M_k V_k\ . \tag{3}$$

Different factorizations for C_k are possible using the properties of the adjoint map. Notable ones are those leading to a skew-symmetric C_k, since this implies the passivity property of the robot, see [7, 21, 17], which is key to nonlinear Lyapunov-based analysis. The choice of C_k used in this paper, is based on the left-invariant Riemannian connection map, $\nabla : \mathfrak{se}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$, used as $\nabla_V W$ for $V, W \in \mathbb{R}^6 \cong \mathfrak{se}(3)$, [3, eq. 8]. The skew-symmetric operator, $\mathrm{ad}_\bullet^\sim : \mathfrak{se}(3) \to \mathfrak{se}(3)^*$, is defined as,

$$\mathrm{ad}_V^\top M_k W = \mathrm{ad}_{M_k W}^\sim V,\ \text{(see App. 1)} \tag{4}$$

and can be used to rewrite the connection map as,

$$\nabla_V W = \frac{1}{2}\big(\mathrm{ad}_V - M_k^{-1}(\mathrm{ad}_V^\top M_k + \mathrm{ad}_{M_k V}^\sim)\big) W\ . \tag{5}$$

From (5), $C_k(V) = M_k \nabla_V$ is skew-symmetric and therefore used as the factorization in this paper due to its interpretation as the inertia-scaled covariant derivative operator and its usefulness in obtaining the main result. In this text, the velocity dependency is highlighted with a superscript as $C_k^V,\ V \in \mathbb{R}^6 \cong \mathfrak{se}(3)$, while the subscript denotes the corresponding body's (k) inertia matrix which is used to scale ∇_V. The following properties are used in the paper.

Property 1: Corresponding to a group action of $g \in \mathrm{SE}(3)$, the following properties hold,

- $\mathrm{Ad}_g \, \mathrm{ad}_{\mathcal{V}} \, \mathrm{Ad}_g^{-1} = \mathrm{ad}_{\mathrm{Ad}_g \mathcal{V}}$ (6a)
- $\mathrm{Ad}_g^{\top} \, \mathrm{ad}^{\sim}_{M\mathcal{V}} \, \mathrm{Ad}_g = \mathrm{ad}^{\sim}_{\mathrm{Ad}_g^{\top} M\mathcal{V}}$ (6b)

Property 2: Torsion-free connection: For the k^{th} body, given $X, Y \in \mathbb{R}^6 \cong \mathfrak{se}(3)$,

- $C_k^X \, Y = C_k^Y \, X - M_k \, \mathrm{ad}_Y \, X$ (7)

Property 3: For the k^{th} body, given $X \in \mathbb{R}^6 \cong \mathfrak{se}(3)$,

- $\mathrm{ad}_X^{\top} M_k X = \mathrm{ad}^{\sim}_{M_k X} X = -C_k^X X$ (8)

III. Geometric Modeling of the FRS

In this section, we first define the FRS using concepts of differential geometry. Formally, it can be stated that[1],

Def. 1: A FRS is a multibody system of $n+1$ rigid links (see Fig. 1), which comprises of the FRS-base and n joints. The configuration space of the FRS, as seen in Fig. 1, is $\mathcal{Q} \equiv \mathrm{SE}(3) \times \mathbb{T}^n$ with coordinates $r = \{g_1, q\} \in \mathcal{Q}$, where $g_1 \in \mathrm{SE}(3)$ is the pose of the FRS-base and $q \in \mathbb{T}^n$ denotes the n joint positions of the robotic manipulator. The tangent space at r is denoted by $\{\dot{g}_1, \dot{q}\} \in T_r\mathcal{Q}$ and the tangent bundle is given by $\{(g_1, q), (\dot{g}_1, \dot{q})\} \in T\mathcal{Q}$.

The presence of SE(3) coordinate, g_1, implies a SE(3) action, which drops g_1 to yield a reduced Lagrangian [2, §5], $l(q, V)$, where $V = \begin{bmatrix} V_1^{\top} & \dot{q}^{\top} \end{bmatrix}^{\top}$ is the system velocity and $V_1^{\wedge} = g_1^{-1}\dot{g}_1 \in \mathfrak{se}(3)$ corresponding to the FRS-base pose. Using $l(q, V)$, the dynamics from the Hamel's equations [14, §. 6] for a FRS [22, eq. 4,5] are obtained. The forced dynamics result from recursive computations [6, 7, 21], which are compactly written as,

$$M(q)\dot{V} + C(q, V)V = F \tag{9}$$

where $F = \begin{bmatrix} \mathcal{F}_1^{\top} & \tau^{\top} \end{bmatrix}^{\top}, \mathcal{F}_1 \in \mathbb{R}^6 \cong \mathfrak{se}(3)^*, \tau \in \mathbb{R}^n$ contains the actuation and external torques acting on the FRS-base and joints. $M, C \in \mathbb{R}^{(6+n)\times(6+n)}$ are the joint-space inertia and the CC dynamic matrices, respectively. A brief outline of the recursive method is given in App. 2.

A. The Mechanical Connnection $\mathcal{A}$

Here, we introduce the *mechanical connection*, which is the key element in REL equations for the FRS.

Def. 2: A Momentum map [22, App.] is a mapping, $\mathcal{J} : T\mathcal{Q} \to \mathfrak{se}(3)^*$, which can be defined using $\mathfrak{se}(3)$ algebra notation as $\mathcal{J} = \mathrm{Ad}_1^{-\top}\left(\Lambda_b(q)V_1 + M_{bq}(q)\dot{q}\right)$, where Λ_b is the body-referenced locked-inertia (derived later) of the FRS and M_{bq} is given from (44) in App. 2.

Note that in robotics, $\mathcal{J}$ is called the generalized momentum [9, 8]. The invariance of $l(q, V)$ to g_1, implies a SE(3) symmetry, which leads to a conservation law on $\mathcal{J}$ according to Noether's theorem [2, §4.1]. This law is modeled as a *mechanical connection* [12, §3.2] and is defined below.

[1]Configuration space of the manipulator is a Riemannian manifold, $\mathbb{T}^n$, with the manipulator inertia as the metric tensor and locally, $\mathbb{T}^n \cong \mathbb{R}^n$.

Def. 3: [1, def. 3.1] The *mechanical connection* is a SE(3)-equivariant map $\mathcal{A} : T\mathcal{Q} \to \mathfrak{se}(3)$ and written as, $\mathcal{A} = (\mathrm{Ad}_{g_1^{-1}}^{\top} \Lambda_b \, \mathrm{Ad}_{g_1^{-1}})^{-1}\mathcal{J}$ [2, eq. 5.3.1].

Using Def. 2 above to define $\mathcal{A}$ in def. 3,

$$(\mathrm{Ad}_1^{-\top} \Lambda_b \, \mathrm{Ad}_1^{-1})^{-1}\mathcal{J} = \underbrace{\mathrm{Ad}_1 \begin{bmatrix} \mathbb{I}_{6,6} & \mathcal{A}_l(q) \end{bmatrix} \begin{bmatrix} V_1 \\ \dot{q} \end{bmatrix}}_{\mathcal{A}(q,V)} \tag{10}$$

where $\mathcal{A}_l\dot{q}$ is the local *mechanical connection* and $\mathcal{A}_l = \Lambda_b^{-1}(q)M_{bq}(q)$. Note that $\mathcal{A}_l$ is exactly the *dynamic-coupling* factor as proposed by [24]. The *mechanical connection* splits $T\mathcal{Q}$ into mutually orthogonal complements.

Firstly, the horizontal $\mathfrak{se}(3)$-algebra [2, §3.3] of $\mathcal{A}$ is defined as, $\mathrm{hor}_r = \{V | \mathcal{A} = 0\}$. Using R.H.S of (10), we get, $\mathrm{hor}_r = (-\mathcal{A}_l\dot{q}, \; \dot{q})$. Secondly, we get the vertical part as $\mathrm{ver}_r = V - \mathrm{hor}_r = (V_1 + \mathcal{A}_l(q)\dot{q}, \; 0_n) = (\mu, 0_n)$, where $\mu \in \mathfrak{se}(3)$ is,

$$\mu = \mathrm{Ad}_1^{-1} \mathcal{A} = V_1 + \mathcal{A}_l\dot{q} = \Lambda_b^{-1}p = \mathrm{Ad}_1 \Lambda_b^{-1} \mathrm{Ad}_1^{\top} \mathcal{J} \tag{11}$$

and is a non-integrable quasi-velocity (locked velocity), which is a *velocity shift* [14, pp. 141] of the FRS-base velocity, V_1. It can also be noted in (11) that $\mathcal{A}$ is simply the spatial variant of locked body velocity μ.

$\mathcal{A}$ plays a fundamental role in theory of geometric phases which requires holonomy of the *mechanical connection* i.e. the extent to which parallel transport around a closed-path in $\mathbb{T}^n$ fails to preserve $\mathcal{A}$, and is related to its curvature [12]. This phenomenon is commonly known as the falling-cat problem [14]. $\mathcal{A}$ is also used in separating internal and external motion of a mechanical system.

B. REL equations: The motivation

In this section, we recall the REL equations from differential geometry and write them in a form consistent with robotics literature. The ver_r and hor_r forms in Sec. III-A can be minimally represented with μ and $\dot{q}$, respectively. Hence, the reduced Lagrangian of FRS can be written with new bundle coordinates as $l(q, \xi)$, instead of $l(q, V)$, where $\xi = \begin{bmatrix} \mu^T & \dot{q}^T \end{bmatrix}$. From here on, given $X, Y \in \mathbb{R}^t$, and $X \in \mathbb{R}^{t\times t}$, $\langle X, Y\rangle_Z = X^{\top}ZY$.

Lemma 1: The motion of an unforced FRS with a reduced Lagrangian, $l(q, \xi) = \frac{1}{2}\langle \xi, \xi\rangle_{\Lambda}$, in which the *mechanical connection* defines bundle coordinates, (q, ξ), is modeled using REL equations [2, §5.3] and are written as,

$$\Lambda_b(q)\dot{\mu} + \frac{d\Lambda_b(q)}{dt}\mu = \mathrm{ad}_{\mu}^{\top}\Lambda_b(q)\mu - \mathrm{ad}_{\mathcal{A}_l(q)\dot{q}}^{\top}\Lambda_b(q)\mu \tag{12}$$

$$\Lambda_q(q)\ddot{q} + \frac{d\Lambda_q(q)}{dt}\dot{q} - \frac{\partial}{2\partial q}\langle \dot{q}, \dot{q}\rangle_{\Lambda_q} = \tilde{N}(q, \mu, \dot{q}) \tag{13}$$

$$\tilde{N} = \left(\mathcal{B}_l(\dot{q})^{\top}\Lambda_b(q)\mu + \frac{\partial}{2\partial q}\langle \mu, \mu\rangle_{\Lambda_b} + \mathcal{A}_l^{\top}\mathrm{ad}_{\mu}\Lambda_b\mu\right) \tag{14}$$

where $\mathcal{B}_l(\dot{q})\bullet = (d\mathcal{A}_l(\dot{q}) \bullet - [(\mathcal{A}_l\dot{q})^{\wedge}, (\mathcal{A}_l\bullet)^{\wedge}])$ is the local curvature of the *mechanical connection*, C_b^{μ} is defined in Prop. 6, $\Lambda_q \in \mathbb{R}^{n\times n}$ is the reduced shape-space inertia and Λ is block-diagonal in Λ_b, Λ_q. Note that, (12) and (13) correspond to momentum and shape-space (reduced) dynamics respectively.

Proof: For (12), ee App. 3. In (13), $\tilde{N}$ collects the μ-dependent dynamic terms that affect the shape-space dynamics and its expanded form is proved in App. 4. ■

It is clear in (12) and (13) that the velocity-dependent couplings are separated in terms of $(\dot{q}, \dot{q})$ on L.H.S and $(\dot{q}, \mu), (\mu, \mu)$ on R.H.S. Hence, in this paper, we start by simplifying the closed-form expressions from previous works in [7, 8] and separate these couplings. We note that, instead of the exact form of REL equations in (12) and (13), we obtain a reformulation of the REL equations in the next section, which preserves the the skew-symmetric property unlike (12) and (13).

IV. Proposed form of Reduced Euler-Lagrange equations

In this section, we state and prove the main result of this paper. To this end, we recall from Sec. III-B that, instead of using (q, V) as bundle coordinates, we use (q, ξ) to define the dynamics. A consequence of this is that, in contrast to Sec. 2, the link velocity is determined as, $V_k = \tilde{T}_k(q)\xi$, such that,

$$\tilde{T}_k(q) = \begin{bmatrix} \mathrm{Ad}_{1k}^{-1}(q) & \tilde{J}_k(q) \end{bmatrix} \tag{15}$$

where $\tilde{J}_k = J_k - \mathrm{Ad}_{1k}^{-1}\,\mathcal{A}_l$, is the generalized Jacobian for the k^{th} link. We define the following two fundamental matrices [18] which aid in linking the main result with (12) and (13).

Def. 4: Inertia velocity matrix: Given an arbitrary shape-space velocity $x \in \mathbb{R}^n$, a matrix, $P(q, x)$, which can be interpreted as $P(\dot{q}) = \frac{d}{dt}\Lambda_b(q)$ when $x = \dot{q}$.

Def. 5: Inertia derivative matrix: Given locked velocity $x \in \mathbb{R}^6 \cong \mathfrak{se}(3)$, a matrix, $S(q, x)$, which arises in the partial derivative of the locked kinetic energy relative to the shape-space positions, $\frac{\partial \langle x, x\rangle_{\Lambda_b}}{2\partial q} = S(q, x)^\top x$.
Note that Def.5 defines the first term on R.H.S of (13) and that $\tilde{S}^\top x = S^\top x$.

Theorem 1: Main result: For the FRS in Def. 1, the matrix-based recursive form of proposed *Reduced Euler-Lagrange* equations are written as,

$$\underbrace{\begin{bmatrix} \Lambda_b(q) & 0 \\ 0 & \Lambda_q(q) \end{bmatrix}}_{\Lambda(q)} \begin{bmatrix} \dot{\mu} \\ \ddot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}P(\dot{q}) & 0 \\ 0 & \Gamma_q'(\dot{q}) \end{bmatrix}}_{\mathcal{D}_{\dot{q}}(q,\xi)} \underbrace{\begin{bmatrix} \mu \\ \dot{q} \end{bmatrix}}_{\xi}$$
$$= \underbrace{\begin{bmatrix} C_b^{\mu} & S(\mu) - \mathrm{ad}^{\sim}_{\Lambda_b\mu}\mathcal{A}_l \\ -S(\mu)^\top - \mathcal{A}_l^\top\,\mathrm{ad}^{\sim}_{\Lambda_b\mu} & -2B_2(q,\mu) \end{bmatrix}}_{\mathcal{D}_{\mu}(q,\mu)} \begin{bmatrix} \mu \\ \dot{q} \end{bmatrix} \tag{16}$$
$$+ \underbrace{\begin{bmatrix} \mathcal{F}_1 \\ \tau - \mathcal{A}_l^\top \mathcal{F}_1 \end{bmatrix}}_{\mathcal{F}}$$

where Λ is the block-diagonal inertia matrix, $\mathcal{D}_{\dot{q}}, \mathcal{D}_\mu$ are the two proposed CC matrices which depend only on $\dot{q}$ and μ, respectively. B_2 is directly related to the curvature of the *mechanical connection* affecting shape-space dynamics and $\mathcal{F}$ contains external forces acting on the FRS. In particular,

$$\Lambda_b(q) = \sum_k \mathrm{Ad}_{1k}^{-\top}\, M_k\, \mathrm{Ad}_{1k}^{-1} = M_b \tag{17a}$$
$$M_{bq}(q) = \sum_k \mathrm{Ad}_{1k}^{-\top}\, M_k\, J_k,\;\; \mathcal{A}_l = \Lambda_b^{-1}(q) M_{bq}(q), \tag{17b}$$
$$\Lambda_q(q) = \sum_k J_k^\top\, M_k\, J_k, -\mathcal{A}_l^\top\, \Lambda_b\, \mathcal{A}_l \tag{17c}$$

- $$P(x) = \tilde{P} + \sum_k \mathrm{ad}^{\sim}_{\mathrm{Ad}_{1k}^{-\top} M_k J_k x}$$ $$\tilde{P}(x) = 2\sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k^{J_k x} - M_k\,\mathrm{ad}_{J_k x}\right)\mathrm{Ad}_{1k}^{-1} \tag{18a}$$
- $$S(x)^\top = \tilde{S}^\top(x) + \sum_{k=b} \frac{J_k^\top}{2} M_k \mathrm{ad}_{\mathrm{Ad}_{1k}^{-1} x} \mathrm{Ad}_{1k}^{-1}$$ $$\tilde{S}(x)^\top = -\sum_{k=b} J_k^\top C_k^{\mathrm{Ad}_{1k}^{-1} x} \mathrm{Ad}_{1k}^{-1} \tag{18b}$$
- $$\Gamma_q'(\dot{q}) = \sum_k \tilde{J}_k^\top\left(C_k^{\tilde{J}_k \dot{q}}\,\tilde{J}_k + M_k\,\dot{\tilde{J}}_k\right), \tag{18c}$$
- $$B_2 = \sum_k J_k^\top\, C_k^{\mathrm{Ad}_{1k}^{-1}\mu} J_k + S(\mu)^T \mathcal{A}_l - \mathcal{A}_l^\top\, S(\mu) + \mathcal{A}_l^\top\,\mathrm{ad}^{\sim}_{\Lambda_b\mu}\,\mathcal{A}_l\;. \tag{18d}$$

The closed-form expressions for all matrices in (16) are provided in the next two subsections.

Proof: The proof of Theorem 1 is split into parts 1-4. Firstly, in part. 1, we derive the inertia matrix $\Lambda(q)$, which is followed by closed-form expressions for the matrices of inertia velocity, $P(\dot{q})$, and inertia derivative, $S(\mu)$ in part. 2. In part. 3, we separate the velocity dependencies from the recursive formulation and in part. 4, we prove and reveal the form in the main result in (16).

Throughout this section, the following identities will be used for simplifying the expressions.

$$\sum_k \mathrm{Ad}_{1k}^{-\top}\, M_k\, \tilde{J}_k = 0 \tag{19a}$$
$$\sum_k \mathrm{Ad}_{1k}^{-\top}\, M_k\, \dot{\tilde{J}}_k = \sum_k \mathrm{Ad}_{1k}^{-\top}\,\mathrm{ad}_{J_k\dot{q}}^\top\, M_k\, \tilde{J}_k \tag{19b}$$
$$\sum_k \mathrm{Ad}_{1k}^{-\top}\,\mathrm{ad}^{\sim}_{M_k \tilde{J}_k \dot{q}}\,\mathrm{Ad}_{1k}^{-1} = 0 \tag{19c}$$
$$\sum_k \mathrm{Ad}_{1k}^{-\top}\,\mathrm{ad}^\top_{\mathrm{Ad}_{1k}^{-1}\mu}\, M_k \tilde{J}_k = 0 \tag{19d}$$
$$\sum_k \mathrm{Ad}_{1k}^{-\top}\left(\mathrm{ad}^\top_{J_k\dot{q}} - \mathrm{ad}^\top_{\tilde{J}_k\dot{q}}\right) M_k \tilde{J}_k = 0 \tag{19e}$$

To prove (19a), one has to replace the expression of $\tilde{J}_k$ and use (44), while all the other identities are a direct consequence of (19a), (6a), (6b) and (4).

The proof starts by replacing $T_k(q)$ with $\tilde{T}_k(q)$ in the recursive computations to obtain an equation, which is analogous to (20) as,

$$\Lambda(q)\dot{\xi} + \Gamma(q, \xi)\xi = \mathcal{F} \tag{20}$$

where Γ is a transformed CC matrix. Note that, (20) results in the same form as [8, eq. 15]. A notable difference is that a coordinate transformation was used in [8], but in (20), a recursive method has been used. However, we note that both methods are equivalent. As a consequence of using (q,ξ) as bundle coordinates, a corresponding transformation in matrices of inertia, Λ, and CC matrix, Γ occurs. The corresponding partitioning of the dynamic matrices is,

$$\Lambda = \begin{bmatrix} \Lambda_b & \Lambda_{bq} \\ \Lambda_{bq}^\top & \Lambda_q \end{bmatrix} \qquad \Gamma = \begin{bmatrix} \Gamma_b & \Gamma_{bq} \\ \Gamma_{qb} & \Gamma_q \end{bmatrix} . \tag{21}$$

Proof part 1: (Inertia matrix): Λ is obtained straightforwardly by using the partitioning of (15), invoking (19a) and taking $\mathcal{A}_l$ outside the summation to yield (17a)-(17c). Hence, Λ is already in the same form as it appears in (16).

Proof part 2: (Inertia velocity/derivative matrices): Firstly, a closed-form expression of the matrix in Def. 4, $P(q,\dot{q})$, is simply obtained by computing $\frac{d}{dt}\Lambda_b(q)$ which yields,

$$P(\dot{q}) = -\sum_k \mathrm{Ad}_{1k}^{-\top}\left(\mathrm{ad}_{J_k\dot{q}}^\top M_k + M_k \mathrm{ad}_{J_k\dot{q}}\right)\mathrm{Ad}_{1k}^{-1} \tag{22}$$

Using the definition of C_k in Sec. II, we obtain (18a). Secondly, for the derivation of the closed-form expression of the matrix in Def. 5, the proof is provided in App. 5.

Proof part 3: (Separation of velocity dependencies): The main idea of the proof is to start from the closed-form expressions of the partitioned CC matrices in (20) and separate the terms according to their velocity dependencies for each of the four blocks. In fact, the key feature of the structure in (16) is the isolation of different terms according to the dependency on shape-space and locked velocities. For the CC matrix, Γ, using instead (19b) and $C_k = -C_k^\top$ yields

$$\Gamma_b = \sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k - M_k\,\mathrm{ad}_{J_k\dot{q}}\right)\mathrm{Ad}_{1k}^{-1} = C_b \tag{23a}$$

$$\begin{aligned} \Gamma_{bq} &= \sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k \tilde{J}_k + M_k\,\dot{\tilde{J}}_k\right) \\ &= \sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k + \mathrm{ad}_{J_k\dot{q}}^\top M_k\right)\tilde{J}_k \end{aligned} \tag{23b}$$

$$\Gamma_{qb} = \sum_k \tilde{J}_k^\top\left(C_k - M_k\,\mathrm{ad}_{J_k\dot{q}}\right)\mathrm{Ad}_{1k}^{-1} = -\Gamma_{bq}^\top \tag{23c}$$

$$\Gamma_q = \sum_k \tilde{J}_k^\top\left(C_k\tilde{J}_k + M_k\,\dot{\tilde{J}}_k\right) . \tag{23d}$$

To derive the form in (16), Γ must also be factorized into $(\dot{q},\dot{q})$, $(\dot{q},\mu)$ and (μ,μ) couplings. The key idea, is to use (15) to write $V_k = \mathrm{Ad}_{1k}^{-1}\mu + \tilde{J}_k\dot{q}$ and split C_k as the sum of contributions depending on $\dot{q}$ and μ, for example,

$$C_k = M_k(\nabla_{\mathrm{Ad}_{1k}^{-1}\mu} + \nabla_{\tilde{J}_k\dot{q}}) = C_k^{\mathrm{Ad}_{1k}^{-1}\mu} + C_k^{\tilde{J}_k\dot{q}} . \tag{24}$$

Firstly, we simplify Γ_b by splitting C_k and using $P(\dot{q})$ matrix in (18a) as,

$$\begin{aligned} \Gamma_b &= \sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k^{\mathrm{Ad}_{1k}^{-1}(\mu-\mathcal{A}_l\dot{q}+J_k\dot{q})} - M_k\,\mathrm{ad}_{J_k\dot{q}}\right)\mathrm{Ad}_{1k}^{-1} \\ &= C_b^{(\mu-\mathcal{A}_l\dot{q})} + \frac{1}{2}P(\dot{q}) - \frac{1}{2}\sum_k \mathrm{ad}^{\sim}_{\mathrm{Ad}_{1k}^{-\top}M_k J_k\dot{q}} \end{aligned} \tag{25}$$

Next, for Γ_{bq}, we use (45) from Prop.6 after expanding the generalized Jacobian matrix $\tilde{J}_k$ and identity (19e), to get

$$\begin{aligned} \Gamma_{bq} &= \sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k^{\mathrm{Ad}_{1k}^{-1}\mu} + C_k^{\tilde{J}_k\dot{q}} + \mathrm{ad}_{J_k\dot{q}}^\top M_k\right)\tilde{J}_k \\ &= \underbrace{\sum_k \mathrm{Ad}_{1k}^{-\top}\,C_k^{\mathrm{Ad}_{1k}^{-1}\mu}\,J_k - C_b^\mu\,\mathcal{A}_l}_{-\tilde{S}(q,\mu)} \\ &\quad + \underbrace{\sum_k \mathrm{Ad}_{1k}^{-\top}\left(C_k^{\tilde{J}_k\dot{q}} + \mathrm{ad}_{\tilde{J}_k\dot{q}}^\top M_k\right)\tilde{J}_k}_{B_1(q,\dot{q})^\top} . \end{aligned} \tag{26}$$

where $\tilde{S}(\mu)$ derived from Def. 5 is used. The last term in Γ to consider is Γ_q which can be expanded using (24) as,

$$\Gamma_q = \underbrace{\sum_k \tilde{J}_k^\top\,C_k^{\mathrm{Ad}_{1k}^{-1}\mu}\,\tilde{J}_k}_{B_2(q,\mu)} + \underbrace{\sum_k \tilde{J}_k^\top\left(C_k^{\tilde{J}_k\dot{q}}\,\tilde{J}_k + M_k\,\dot{\tilde{J}}_k\right)}_{\Gamma_q'(q,\dot{q})} , \tag{27}$$

where, using (45) from Prop.6 after expanding $\tilde{J}_k$,

$$\begin{aligned} B_2 &= \sum_k J_k^\top\,C_k^{\mathrm{Ad}_{1k}^{-1}\mu} J_k + \tilde{S}(\mu)^T\mathcal{A}_l - \mathcal{A}_l^\top\,\tilde{S}(\mu) \\ &\quad + \mathcal{A}_l^\top\,C_b^\mu\,\mathcal{A}_l . \end{aligned} \tag{28}$$

Note that $\Gamma_q'(\dot{q})$ is already in the form of (18c). This concludes the simplification obtained through separation of dependencies in the recursive formulation.

Proof part 4: (Matching the proposed REL equations): In L.H.S of (16), we see that the $\mathcal{D}_\mu$ matrix only has μ dependencies whereas Γ_{bq} and $-\Gamma_{bq}^\top$ in Γ have $\dot{q}$ dependencies too. So, we invoke the the following identities,

$$B_1(q,\dot{q})^\top\dot{q} = \frac{1}{2}\sum_k \mathrm{Ad}_{1k}^{-\top}\,M_k\,\mathrm{ad}_{\tilde{J}_k\dot{q}}\,\tilde{J}_k\dot{q} = 0 \tag{29}$$

$$-B_1(q,\dot{q})\mu = \sum_k \tilde{J}_k^\top\,C_k^{\mathrm{Ad}_{1k}^{-1}\mu}\,\tilde{J}_k\dot{q} = B_2(q,\mu)\dot{q} . \tag{30}$$

Furthermore, B_2 is simplified using (19d) on the last three terms in (28) to reveal (18d).

Using (29) with block matrices in (26), we get,

$$\begin{aligned} \Gamma_b\,\mu + \Gamma_{bq}\,\dot{q} &= \big(C_b^{(\mu-\mathcal{A}_l\dot{q})} + \frac{1}{2}(P(\dot{q}) \\ &\quad - \sum_k \mathrm{ad}^{\sim}_{\mathrm{Ad}_{1k}^{-\top}M_k J_k\dot{q}})\big)\mu - (S + C_b^\mu\,\mathcal{A}_l)\dot{q}, \end{aligned} \tag{31}$$

Lemma 2: In (31),

$$C_b^{\mathcal{A}_l\dot{q}}\mu + C_b^\mu\mathcal{A}_l\dot{q} + \mathrm{ad}^{\sim}_{\mathrm{Ad}_{1k}^{-T}M_k J_k\dot{q}} = -\mathrm{ad}^{\sim}_{\Lambda_b\mu}\mathcal{A}_l\dot{q} \tag{32}$$

Proof: See App. 6. ■

Invoking Lemma 2 to simplify (31), we get,

$$\Gamma_b\,\mu + \Gamma_{bq}\,\dot{q} = \big(C_b^{\mu} + \frac{1}{2}P\big)\mu - (S - \mathrm{ad}^{\sim}_{\Lambda_b\mu}\mathcal{A}_l)\dot{q}, \tag{33}$$

Similarly using (30) with block matrices in (27), we get,

$$\begin{aligned} -\Gamma_{bq}^{\top}\mu + \Gamma_q\dot{q} &= (S + C_b^{\mu}\,\mathcal{A}_l)^{\top}\mu + (2B_2 + \Gamma'_q)\dot{q} \\ &= (S^{\top} + \mathcal{A}_l^{\top}\mathrm{ad}^{\sim}_{\Lambda_b\mu})\mu + (2B_2 + \Gamma'_q)\dot{q} \end{aligned} \tag{34}$$

where equivalence in Prop. 2 (3) has been used. Finally, separating dependencies in (33) and (34) as $(\dot{q},\dot{q})$ on L.H.S and all $(\dot{q},\mu),(\mu,\mu)$ on R.H.S, the result in (16) follows. ■

We point out, that the Γ matrix in (20) is analogous to the result in [8, eq. 18] where block matrices were functionally dependent on V and not the chosen velocity ξ. In contrast, by further simplification of Γ, block matrices in (16) are explicitly defined in terms of the system velocity ξ. For the FRS defined in Def. 1, using the split of Γ into two CC matrices, namely $\mathcal{D}_{\dot{q}}, \mathcal{D}_{\dot{q}}$, the following two properties are established.

Property 4: The skew-symmetric property, $z^{\top}(\dot{\Lambda} - 2\Gamma)z = 0$ for $z = \begin{bmatrix} x^{\top} & y^{\top}\end{bmatrix}^{\top}, x \in \mathbb{R}^6 \cong \mathfrak{se}(3), y \in \mathbb{R}^n$, can be viewed in (16), as satisfying the following,

$$\begin{gathered} x^{\top}(\frac{d\Lambda_b}{dt} - P(\dot{q}))x = 0,\ y^{\top}(\frac{d\Lambda_q}{dt} - 2\Gamma'_q)y = 0 \\ x^{T}C_b^{\mu}x = 0,\ y^{\top}B_2(q,\mu)y = 0,\ z^{\top}\mathcal{D}_{\mu}z = 0 \end{gathered} \tag{35}$$

Proof: The first follows straightforwardly. The second follows by using the corresponding matrix expansions in (17c) and (18c). For the third, fourth and fifth, C_b^{μ} in Lem. 6, B_2 in (28) and $\mathcal{D}_{\mu}$ in (16), respectively, are skew-symmetric. Note that, in (16), the the CC matrix of block-diagonal terms, $\mathcal{D}_{\dot{q}}$, satisfying the passivity property depends only on $\dot{q}$ while the CC matrix of off-diagonal terms, $\mathcal{D}_{\mu}$ satisfying the skew-symmetric property depends only on μ. ■

Property 5: Given locked velocities, $x, y \in \mathbb{R}^6$, and shape-space velocity, $z \in \mathbb{R}^n$, the following commutativity properties for the fundamental matrices from Def. 4 and Def. 5 hold.

$$S(q,x)^{\top}y = S(q,y)^{\top}x,\ \frac{1}{2}P(q,z)y = -S(q,y)z \tag{36}$$

which extends the result for generalized coordinates in [18, Prop. 5,6] to the locked-shape velocity formulation.

Proof: For the first, invoking Prop. 2 (7), and for the second, using (4) and Prop. 2 (3), leads to the result. ■

Since, (16) preserves the skew-symmetric property through Prop. 4, which is important in robotics, we treat (16) as the main result. However, a direct comparison with the REL structure in (12) and (13) has not been established yet.

Theorem 2: For the FRS in Def. 1, the matrix-based recursive form in (16) is equivalent to (12) and (13).

Proof: Applying second of (36), we can simplify $S(\mu)\dot{q} = -\frac{1}{2}P(\dot{q})\mu$. Using this simplification and rearranging first row of (16), we obtain,

$$\Lambda_b\dot{\mu} + \frac{d\Lambda_b}{dt}\mu = -C_b^{\mu}\mu - \mathrm{ad}^{\sim}_{\Lambda_b\mu}\mathcal{A}_l\dot{q} \tag{37}$$

Note that (37) can be simplified by using Properties 2 and 3 to derive the momentum equation in (12).

Secondly, for the shape-space equation, by matching $(\dot{q},\dot{q})$ couplings in (13) and second row of (16), we see that $\frac{d\Lambda_q\dot{q}}{dt} - \frac{\partial\langle\dot{q},\dot{q}\rangle_{\Lambda_q}}{2\partial q} = \Gamma'_q(q,\dot{q})$. We recall the R.H.S of (13), which is expanded in (14). In particular, revealing terms which are analogous in (14) yields the closed-form expressions for μ-dependent terms. We state the following Lemma to provide a closed-form expression for the curvature using recursive method.

Lemma 3: The local curvature [2, Def. 3.5] is a $\mathfrak{se}(3)$-valued two-form given by the exterior covariant derivative of the local *mechanical connection* $\mathcal{A}_l\dot{q}$ and its closed-form is,

$$\mathcal{B}_l(q,\dot{q})X = \Lambda_b(q)^{-1}\mathcal{B}^{\top}(q,\dot{q})X,\ X \in \mathbb{R}^n \tag{38}$$

where,

$$\begin{aligned} \mathcal{B}(\dot{q}) = -2\Big(\sum_k J_k^{\top}\,\big(C_k^{J_k\dot{q}} - M_k\mathrm{ad}_{J_k\dot{q}}\big)\mathrm{Ad}^{-1}_{1k} \\ S(\mathcal{A}_l\dot{q})^{\top} + \frac{1}{2}\mathcal{A}_l^{\top}P(\dot{q}) + \mathcal{A}_l\mathrm{ad}^{\top}_{\mathcal{A}_l\dot{q}}\Lambda_b\Big) \end{aligned} \tag{39}$$

Proof: First, (14) is equated to the μ-dependent terms in bottom row of (16). Lemma (5) in (14) is applied to eliminate the $\frac{\partial}{2\partial q}\langle\mu,\mu\rangle_{\Lambda_b}$ term. Using Prop. 3, $\mathcal{A}_l^{\top}\mathrm{ad}_{\mu}\Lambda_b\mu = \mathcal{A}_l^{\top}\mathrm{ad}^{\sim}_{\Lambda_b\mu}$, which eliminates the interaction term. Therefore, the only remaining term,

$$\mathcal{B}_l(\dot{q})^{\top}\Lambda_b(q)\mu = -2B_2(\mu)\dot{q} \Rightarrow \mathcal{B}_l^{\top}\Lambda_b(q)\mu = \mathcal{B}(\dot{q})\,\mu \tag{40}$$

where a factorization $-2B_2\dot{q} = \mathcal{B}\mu$ is obtained from (18d) using both of Properties 36 and 2. Rearrangement yields the result[2]. ■

Applying Lem. 3, the result in Th. (2) follows, which proves the equivalence between the recursive method and the Lagrangian formulation in differential geometry. Finally, the main features in (16) are i) block-diagonalized inertia, ii) separated velocity-dependencies in CC matrix, iii) skew-symmetric/passivity property with evident velocity-dependencies iv) commutativity properties of two sub-matrices of CC terms, and v) explicit form of the curvature. ■

V. Discussion and Future Work

In this section, we provide possible applications that motivated the proposed method and briefly outline our future work.

1) Control design: : The Prop. 4, provides a more detailed view of the CC matrix than previous works and aids Lyapunov-based control design by clearly showing the power-preserving terms. To illustrate this idea, let us consider an underactuated FRS with $\mathcal{J} \neq 0_6$, which makes it an *affine* nonholonomic system. This results in additional shape-space coupling torques, which are given on R.H.S in (16). For shape regulation, these torques should be compensated to decouple

[2]The term, $-\mathrm{ad}_{\mathcal{A}_l\dot{q}}\mathcal{A}_l X$, appearing in *both* L.H.S of (38) and $\mathcal{B}_l$ of (14) is the Lie bracket, which corresponds to the Christoffel symbols that capture the extrinsic changes in the connection as the space of allowable velocities rotates with the body frame. The first three terms are $d\mathcal{A}_l$, which measures the intrinsic change in the connection across the shape-space [11, §4.1].

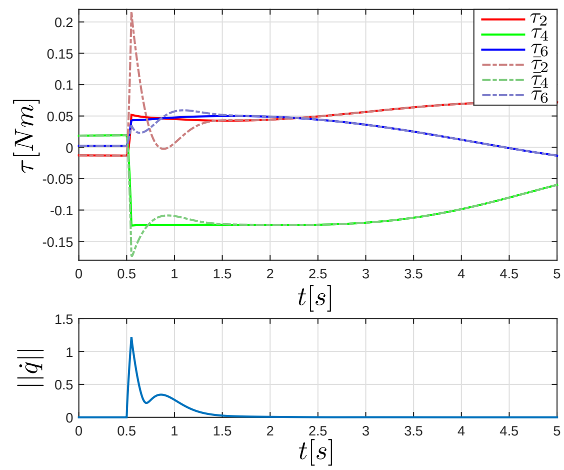

Fig. 2. Comparison of decoupling shape-space control torques (top) and shape velocities (bottom).

the controller from μ [9]. In the following simulation, we consider a gravity-free FRS with the same 7-DoF manipulator from [9] and a FRS-base with $m_1 = 3.50$[kg] and $I_1 =$ blkdiag $\begin{bmatrix} .12 & .14 & .12 \end{bmatrix}^T$ [Kg.m^2]. In the simulation scenario, the initial angular locked velocity was, $\mu_{\omega y} = 12$[°/s], while all other components are zero. Between $[0.51, 0.55][s]$, only a disturbance torque $\mathcal{F}_{1\omega} = \begin{bmatrix} 5 & 5 & 5 \end{bmatrix}^T$ [Nm] was applied. The controller in [9] was used for the shape-space task in the aforementioned scenario. The same steps from [9] were followed using a coordinate transformation of the Hamel's form that yielded (20). To decouple the shape-space controller from μ, torques are found to be $\bar{\tau} = -\Gamma_{bq}(\xi)^T\mu$. Alternatively, a second controller was implemented using (16) and decoupling torques were found to be $\tau = (\tilde{S}^\top + \mathcal{A}_l^\top \text{ad}^{\sim}_{\Lambda_b\mu})\mu$. The recursive formulation for (16) is particularly useful for this purpose. In contrast to [9], an advantage in structure of (16) is that it directly indicates the coupling torques with only (μ, μ)-dependency. This observation is not immediately evident in (20). In fact, using (26), we see that $\bar{\tau} = \tau - B_1(\dot{q})\mu = \tau + B_2(\mu)\dot{q}$. The fourth of Prop. 4 implies that the second term can be neglected without compromising asymptotic stability. Although, $B_2(\mu)\dot{q}$ is power-preserving, it still affects the dynamic equations in bottom of (16). This effect can be seen in Fig 2 (top), $\bar{\tau}$ (dashed) has peaking torques upon impact, as opposed to the proposed controller. For clarity of display, we chose joints with three highest torques, namely $2, 4, 6$. From the plot at bottom, the transient effect of $||\dot{q}||$ on $\bar{\tau}$ can be observed between $[0.5, 1.5][s]$, while the controller based on (16) is unaffected by $\dot{q}$. We remark that both controllers were able to achieve asymptotic stability, and it can be seen that the steady state torques before and after disturbance are actually the same. This example emphasizes the additional insight gained through Prop. 4 about FRS dynamics. As future work, the proposed controller will be extended to task-space.

2) Stability analysis: In Lyapunov analysis of observer and tracking control problems, a difference of CC forces due to actual and reference velocities appear. In such cases, closed-form expressions and commutativity properties of CC matrices are useful [20]. To illustrate this, let us take a function $\mathcal{W} = \langle \mu_e, \mu_e \rangle_{\Lambda_b}$, where $\mu_e = \mu - \hat{\mu}$, $\mu, \hat{\mu}$ are actual and reference locked velocities. In $\dot{\mathcal{W}}$, the mixed term, $\langle \mu_e, \big(S(\mu) - S(\hat{\mu})\big)\dot{q} \rangle$, is encountered using (16). This term can be simplified into a quadratic form as $-\frac{1}{2}\langle \mu_e, \mu_e \rangle_{P(\dot{q})}$ using Prop. 36 to analyze sign-definiteness of $\dot{\mathcal{W}}$. Note that such a simplification is not evident in using transformations in [8, eq. 18] and [9, eq. 13]. As future work, (16) will be used in observer design for μ given $\{q, \dot{q}, g_1\}$ measurements.

3) Curvature of $\mathcal{A}$: : The FRS-base pose g_1 is reconstructed using, $\dot{g}_1 = g_1\big(\mu - \mathcal{A}_l(q)\dot{q}\big)^\wedge$. For $\mu = 0_6$, the area integral of the curvature form provides the deviation from $g_1(0)$ for a period of cyclical shape-space motion (gait) [15, eq. 2.23]. The exact closed-form expression of the curvature in (38) might be useful for gaining intuition about gaits using curvature function plots [11, Fig. 3] and is a future topic for investigation.

VI. Conclusion

Hence, in this paper, we established a congruence between REL equations from geometric mechanics and the recursive forms from FRS literature as the overarching objective. In this pursuit, we derived a novel closed-form computation of the CC matrix, which clearly shows the separation of locked and shape-space velocity dependencies. From the proposed dynamics in (16), we obtained three new results. Firstly, we revealed a more elaborate form of the skew-symmetric/passivity property in Prop. 4, which is not only a reformulation, but also has practical significance in control design. Secondly, we established commutativity properties between two submatrices of the CC matrix in Prop. 36 and demonstrated its utility in aiding Lyapunov-based observer design. Finally, we also derived a closed-form expression for the curvature form of the FRS in Lem. 3, which is a key quantity in locomotion.

Appendix

1) Matrix Representation: Given $g \equiv (R, p) \in \text{SE}(3)$ with body velocity twist, $\mathcal{V} = \begin{bmatrix} \omega^\top & v^\top \end{bmatrix}^\top$ (declared in Sec. II), the following quantities are detailed,

$$g = \begin{bmatrix} R & p \\ 0_{1,3} & 1 \end{bmatrix}, \; \mathcal{V}^\wedge = \begin{bmatrix} \omega_\times & v \\ 0_{1,3} & 0 \end{bmatrix}$$
$$\text{Ad}_g = \begin{bmatrix} R & 0_{3,3} \\ p_\times R & R \end{bmatrix}, \; \text{ad}_{\mathcal{V}} = \begin{bmatrix} \omega_\times & 0_{3,3} \\ v_\times & \omega_\times \end{bmatrix} \tag{41}$$

where $(\bullet)_\times$ is a skew-symmetric matrix for the vector and, ω (v) is the angular (linear, respectively) velocity. For k^{th}-link (see Sec. II-A) with momentum $h_k = \begin{bmatrix} h_\omega^T & h_v^T \end{bmatrix}^T = M_k V_k$,

$$M_k = \begin{bmatrix} I_k & 0_{3,3} \\ 0_{3,3} & m_k \mathbb{I}_{3,3} \end{bmatrix}, \; \text{ad}^{\sim}_{M_k V_k} = \begin{bmatrix} h_{\omega\times} & h_{v\times} \\ h_{v\times} & 0_{3,3} \end{bmatrix} \tag{42}$$

2) Recursive Dynamics on $\mathbb{R}^6 \times \mathbb{R}^n$: In this part, a recursive method to compute dynamic quantities is briefly outlined. Before describing the dynamic terms, firstly, we note that the link velocity, $V_k = T_k(q)V$, $T_k = \begin{bmatrix} \text{Ad}_{1k}^{-1} & J_k \end{bmatrix}$, with a link Jacobian $J_k(q)$ relative to the FRS-base. The closed-form expression of the dynamic matrices can be obtained using T_k, e.g. as shown by [7]. In particular,

$$M = \sum_k T_k^\top M_k T_k, \; C = \sum_k T_k^\top \left(C_k T_k + M_k \dot{T}_k \right) \quad (43)$$

The partitioning in T_k can be used to further add detail to the dynamic matrices, e.g., in $M = \begin{bmatrix} M_b & M_{bq} \\ M_{bq}^\top & M_q \end{bmatrix}$, we get,

$$M_b = \sum_k \mathrm{Ad}_{1k}^{-\top} M_k \mathrm{Ad}_{1k}^{-1}, \; M_{bq} = \sum_k \mathrm{Ad}_{1k}^{-\top} M_k J_k \quad (44)$$

Property 6: Given the FRS (see Fig. 1) defined in Def. 1, for $\mathcal{V} \in \mathbb{R}^6 \cong \mathfrak{se}(3)$,

$$\sum_k \mathrm{Ad}_{1k}^{-\top} C_k^{\mathrm{Ad}_{1k}^{-1}\mathcal{V}} \mathrm{Ad}_{1k}^{-1} = C_b^{\mathcal{V}} = \Lambda_b \nabla_{\mathcal{V}} \quad (45)$$

Proof: Using Prop. 1 for simplification on L.H.S, we get the result. ∎

3) Proof of Lem. 1 (12)*:* The momentum equation [19, eq. 6] is,

$$\langle \frac{dp}{dt}, \eta \rangle = \langle p, [(-\mathcal{A}_l \dot{q} + \Lambda_b^{-1} p)^\wedge, \eta^\wedge]^\vee \rangle, \; \eta \in \mathbb{R}^6 \quad (46)$$

Moving η to the left on L.H.S and R.H.S after using SE(3) Lie-bracket isomorphism, $[X^\wedge, \bullet]^\vee = \mathrm{ad}_X, \; X \in \mathbb{R}^6$, substituting $p = \Lambda_b \mu$ and, eliminating η yields the result.

4) Proof of Lem. 1 (13)*:* To obtain closed-form expressions for $\tilde{N}$ in (13), we recall [19, eq. 7][3], which provides a scalar product form in terms of body momentum, $p = \Lambda_b \mu$, as,

$$\begin{aligned} \langle \tilde{N}, \delta q \rangle = &\Big\langle p, d\mathcal{A}_l(q, \dot{q}, \delta q) - [(\mathcal{A}_l \dot{q})^\wedge, (\mathcal{A}_l \delta q)^\wedge]^\vee \\ &+ \frac{1}{2} \frac{\partial (\Lambda_b^{-1} p)}{\partial q} \delta q + [(\Lambda_b^{-1} p)^\wedge, (\mathcal{A}_l \delta q)^\wedge]^\vee \Big\rangle \end{aligned} \quad (47)$$

In (47), the property, $[X^\wedge, \bullet]^\vee = \mathrm{ad}_X$ as in Lem. 3 is used. Also, in vector notation, $d\mathcal{A}_l(q, \dot{q}, \delta q) = d\mathcal{A}_l(q, \dot{q})\delta q$. Moving all δq terms towards the left in both L.H.S and R.H.S of (47), and substituting p, we get,

$$\begin{aligned} \langle \delta q, \tilde{N} \rangle = &\Big\langle \delta q, \big((d\mathcal{A}_l(q, \dot{q}))^\top - \mathcal{A}_l^\top \mathrm{ad}_{\mathcal{A}_l \dot{q}}^\top) \Lambda_b(q) \mu \\ &+ \frac{1}{2} \frac{\partial \mu^\top \Lambda_b(q) \mu}{\partial q} + \mathcal{A}_l^\top \mathrm{ad}_\mu^\top \Lambda_b(q) \mu \big) \Big\rangle \end{aligned} \quad (48)$$

Removing δq variations and invoking equivalence in Prop. 2 (3) to substitute $C_b^\mu = \Lambda_b \nabla_\mu$ from Prop. 6, we get (14).

5) Proof in Def. 4:

$$\begin{aligned} \text{L.H.S} &= \frac{\partial}{\partial q} \big(\sum_{k=b} x^\top \mathrm{Ad}_{1k}^{-\top}(q) M_k \mathrm{Ad}_{1k}^{-1}(q) x \big) \\ &= \frac{1}{2} \sum_{k=b} \left(\Pi_k^\top x + x^T \Pi_k \right), \; \Pi_k = \mathrm{Ad}_{1k}^{-\top} M_k \frac{\partial \mathrm{Ad}_{1k}^{-1} x}{\partial q} \end{aligned} \quad (49)$$

[3] In [19, eq. 7], the μ-dependent terms are on the L.H.S.

The following leads to the result in (18b). Given a column-wise detail of Jacobian as, $J_k = \begin{bmatrix} J_k^1 & \dots & J_k^n \end{bmatrix}$ for the k^{th} link, for the j^{th} joint, using [7, Prop. 4], we have,

$$\begin{aligned} \frac{\partial \mathrm{Ad}_{1k}^{-1} x}{q_j} &= -\mathrm{ad}_{J_k^j} \mathrm{Ad}_{1k}^{-1} x \\ \Rightarrow \frac{\partial \mathrm{Ad}_{1k}^{-1} x}{\partial q} &= \begin{bmatrix} -\mathrm{ad}_{J_k^1} \mathrm{Ad}_{1k}^{-1} x & \dots \end{bmatrix} \end{aligned} \quad (50)$$

Lemma 4:

$$\sum_{k=b} \Pi_k^\top x = \sum_{k=b} J_k^\top \mathrm{ad}_{\mathrm{Ad}_{1k}^{-1} x}^\top M_k \mathrm{Ad}_{1k}^{-1} x \quad (51)$$

Proof: Using the property, $\mathrm{ad}_X Y = -\mathrm{ad}_Y X$ in the result of (50), where $X, Y \in \mathbb{R}^6 \cong \mathfrak{se}(3)$, and isolating, J_k^j to obtain J_k, we first obtain $\frac{\partial \mathrm{Ad}_{1k}^{-1} x}{\partial q} = \mathrm{ad}_{\mathrm{Ad}_{1k}^{-1} x} J_k$. Substituting this in the first term in the L.H.S yields the result. ∎

Lemma 5:

$$\sum_{k=b} x^T \Pi_k = \sum_{k=b} J_k^\top \mathrm{ad}_{M_k \mathrm{Ad}_{1k}^{-1} x}^\sim \mathrm{Ad}_{1k}^{-1} x \quad (52)$$

Proof: We exploit the property in (4) as follows,

$$\begin{aligned} \text{L.H.S} &= \sum_{k=b} x^T \mathrm{Ad}_{1k}^{-\top} M_k \mathrm{Ad}_{1k}^{-1} \begin{bmatrix} -\mathrm{ad}_{J_k^1} \mathrm{Ad}_{1k}^{-1} x & \dots \end{bmatrix} \\ &= -\sum_{k=b} \begin{bmatrix} (\mathrm{ad}_{J_k^1}^\top M_k \mathrm{Ad}_{1k}^{-1} x)^\top \mathrm{Ad}_{1k}^{-1} x & \dots \end{bmatrix} \\ &= -\sum_{k=b} \begin{bmatrix} (\mathrm{ad}_{M_k \mathrm{Ad}_{1k}^{-1} x}^\sim J_k^1)^\top \mathrm{Ad}_{1k}^{-1} x & \dots \end{bmatrix} \\ &= \sum_{k=b} J_k^\top \mathrm{ad}_{M_k \mathrm{Ad}_{1k}^{-1} x}^\sim \mathrm{Ad}_{1k}^{-1} x \end{aligned} \quad (53)$$

where J_k^j is collected to get J_k, leading to the result. ∎

Applying Lemmas 4 and 5 to last of (49), invoking definition of $C_k^{\mathrm{Ad}_{1k}^{-1} x}$, we obtain the result in Def. 5.

6) Proof of Lem. 2: Using Prop. 2 (7) on second of L.H.S,

$$\begin{aligned} \text{L.H.S} =& 2 C_b^{\mathcal{A}_l \dot{q}} \mu - \Lambda_b \mathrm{ad}_{\mathcal{A}_l \dot{q}} \mu + \mathrm{ad}_{\mathrm{Ad}_{1k}^{-\top} M_k J_k \dot{q}}^\sim \\ =& -\mathrm{ad}_{\mathcal{A}_l \dot{q}}^\top \Lambda_b \mu + \mathrm{ad}_{\mathrm{Ad}_{1k}^{-\top} M_k (J_k - \mathrm{Ad}_{1k}^{-1} \mathcal{A}_l) \dot{q}}^\sim \end{aligned} \quad (54)$$

Using the definition of $\tilde{J}_k$ and (19c), the second term in (54) vanishes. Rearranging the first term using equivalence in (4), the result follows.

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