COMPUTATIONAL TOOLS FOR MODELS OF CURVES: NORMALIZATION

Our Project: Algorithmic methods for arithmetic surfaces and regular, minimal models.

$2-$dimensional, irreducible, reduced schemes $\pi : X \implies S$ are arithmetic surfaces if S is a Dedekindscheme and π is projective and flat. They are models of algebraic curves over number fields.

One of our main topics is Lipmans desingularization algorithm:
Let X be an excellent, Noetherian, reduced and $2-$dimensional scheme. Then the following sequence

$$\cdots X_{i+1} \to X_i' \to X_i \to \cdots \to X_1 \to X_0 = X$$

with normalizations $X_i' \to X_i$ and blow-ups $X_{i+1} \to X_i'$ along $\mathrm{Sing}(X_i')$ is finite and X has a strong desingularization.

Lipman also works for arithmetic surfaces because they are of finite type over S and hence Noetherian and excellent. The bottleneck of the algorithm are the normalizations. Let us look at three different normalization algorithms:

Let I be a radical ideal in a Noetherian ring R and $A = R/I$ (reduced Noetherian ring). We want to compute the normalization $\overline{A}$ of A.

1. Grauert-Remmert-de Jong: Computation through an increasing chain of rings. The theoretical background comes from the inclusions

   $$A \subseteq \mathrm{Hom}_A(J, J) \cong \frac{1}{x}(xJ, J) \subseteq \overline{A} \subseteq Q(A)$$

   where (J, x) is a so called test pair for A. That means $A = \overline{A} \iff A = \mathrm{Hom}_A(J, J)$. The computation of the radical J (test ideal) and the increasing number of variables in the computation of $\mathrm{Hom}_A(J, J)$ can become unpractical.

   Implemented in Singular for reduced rings over the integers.

2. Greuel-Laplagne-Seelisch: Computation through an increasing chain of ideals. We compute ideals $U_1, \ldots, U_N \subset A$ and non-zerodivisors $d_1, \ldots, d_N$ on A, such that

$$A \subset \frac{1}{d_1} U_1 \subset \cdots \subset \frac{1}{d_N} U_N = \overline{A} \subset Q(A)$$

   In general more effective than algorithm 1, the only computation in $\mathrm{Hom}_A(J, J)$ is the radical of the test ideal.
   Works whenever Gröbner bases, radicals and ideal quotients can be computed in rings of the form $R[t_1, \cdots, t_s]$.

   Also implemented in Singular for reduced rings over the integers.

3. Böhm-Decker-Pfister-Laplagne-Steenpass-Steidel: Parallelization by stratifying $\mathrm{Sing}(A)$. (Non-normal-locus $N(A) \subset \mathrm{Sing}(A)$.)
   Used techniques: Normalization via localization and modular methods.

   In general even faster than algorithm 2, next thing to look at for polynomial rings over the integers!