Lipmans Desingularization Algorithm

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Abstract

1 Introduction

2 Lipmans approach

The strategy of separating a singular point of a curve into several points and even resolve the singularity by blowing it up successively leads to the desingularization of the curve. It is the statement of the following proposition.

Proposition 1 (Liu06, Prop. 8.1.26) *Let $X \to spec(k)$ be an integral projective curve over a field k. Then the sequence*

$$\cdots X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

is finite whereat $X_i \to X_{i-1}$ is the blow up of X_{i-1} along the singular locus $Sing(X_{i-1})$ which is a closed subset of X_{i-1}.

If we have a surface over $spec(k)$ we can modify the process of successfully blowing up points by normalizing the surface first. Normal rings are regular in codimension one, so again we only have points as singularities. We can also apply the same process to an arithmetic surface $\pi : X \to S$, at which S is a Dedekindscheme of characteristic zero. Arithmetic surfaces are models of algebraic curves over number fields. X is excellent and then $Sing(X)$ is a closed subset of X. Normalizations and blow ups of excellent schemes are locally of finite type and therefore excellent as well. Hence the sequence

$$\cdots X_{i+1} \to X_i' \to X_i \to \cdots \to X_1 \to X_0 = X$$

with normalizations $X_i' \to X_i$ and blow ups $X_{i+1} \to X_i'$ along $Sing(X_i')$ is defined.

Theorem 2 (Lip78) *Let X be an excellent, Noetherian and reduced 2-dimensional scheme. Then the above sequence is finite and X has a strong desingularization.*

The proof of this theorem is complex and would go beyond the scope of this article. But we want to refer to the proof in [CS86,Ch.XI], which is more geometric than the original proof of Lipman.

An arithmetic surface $\pi : X \to S$ is of finite type over S since the structure morphism π is projective. Consequently X is an excellent, Noetherian and reduced scheme of dimension 2. The map π is a birational morphism of integral schemes and it particularly induces an isomorphism on open subsets. Therefore the generic fibres are isomorphic.

Corollary 3 *Let $\pi : X \to S$ be an arithmetic surface over a Dedekind-scheme S of characteristic zero. Then there exists a strong desingularization $X' \to X$ of X. In particular X' is irreducible and reduced and the generic fibres X'_η and X_η are isomorphic.*

So we can construct an algorithm to desingularize excellent Noetherian surfaces over fields and over the integers. There are three things we need for the construction in particular: The computation of a blow up, the singular locus and the normalization of a reduced ring. Blow ups have been studied and implemented in SINGULAR by Anne Frühbis-Krüger in the library resolve.lib. The computation of the singular locus over the integers asks for a different approach than we know from computing the singular locus over a field. The tightest bottleneck of the algorithm though are the normalizations. There are three various algorithms implemented for reduced rings over fields in the library normal.lib, modnormal.lib and locnormal.lib in SINGULAR by Böhm, Decker, Greuel, Pfister et al. Two of them are implemented by us for reduced rings over the integers in the library normalint.lib. We will disclose the basic concepts of the necessary computations in the following sections.

2.1 Singular Locus

As we are working with affine varieties and schemes we will define the singular locus as follows:

Definition 4 (Liu06,Def. 4.2.8) *Let X be a scheme, $x \in X$ and let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and $k := \mathcal{O}_{X,x}/\mathfrak{m} = \kappa(x)$. Then X is called regular in x, if and only if $dim_k(\mathfrak{m}/\mathfrak{m}^2) = dim_k(\mathcal{O}_{X,x})$. The scheme X is called regular if it is regular in all its points. We call a point $x \in X$ singular if X is not regular in x.*

For a Noetherian ring R this means algebraically spoken, that R is called regular if all its localizations R_P, $P \in \mathrm{spec}(R)$ are regular. This definition alone is not enough for the actual computation of the singular locus of a scheme. If we have an ideal I of height h over a field k, the well known jacobian-criterion tells us that its singular locus is defined by the h-minors of its jacobian ideal J_I and the ideal I itself, so $\mathrm{Sing}(I) = \mathrm{V}(\langle J_I, I \rangle)$. If

I is an ideal over the integers it is not as easy to compute the singular locus. But we know, that if we take the radical of the above defined ideal $\langle J_I, I \rangle$ we get a so called test ideal J of I. It can be shown that $\mathrm{Sing}(I) \subseteq \mathrm{V}(J)$. So there are only finitely many points which we have to check for regularity. (We will go into other useful properties of the ideal J in the section about the normalizations.)

The following lemma states a criterium to decide if a point of a scheme is regular or singular.

Lemma 5 (Liu06, Cor. 4.2.12) *Let $(A, \mathfrak{m})$ be a regular, local ring and $f \in \mathfrak{m} \setminus \{0\}$. Then $A/\langle f \rangle$ is regular if and only if $f \notin \mathfrak{m}^2$.*

Proof: A is an integral ring, so f is a non-zero-divisor on A and therefore $\dim(A/\langle f \rangle) = \dim(A) - 1$. Let us set $\mathfrak{n} := \mathfrak{m}/\langle f \rangle$ and $k = A/\mathfrak{m}$. Then we know that $(A/\langle f \rangle)$ is regular if and only if $dim_k(\mathfrak{n}/\mathfrak{n}^2) = dim_k(\mathfrak{m}/\mathfrak{m}^2) - 1$. Because of $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/(\mathfrak{m}^2 + \langle f \rangle)$ it follows that this is equivalent to $f \notin \mathfrak{m}^2$.

<div align="right">

q.e.d.

</div>

We are working with schemes X of the form $X = \mathrm{spec}(R/I)$ where R is a Noetherian ring and $I \subset R$ is a radical ideal. So we can write $I = P_1 \cap \cdots \cap P_n$ with its minimal associated primes $P_i = \langle f_{i1}, \cdots, f_{ir_i} \rangle$. (The necessary procedures are implemented in the librabry primdecint.lib in SINGULAR.) Then $\mathrm{spec}(R/I) = \mathrm{spec}(R/P_1) \cup \cdots \cup \mathrm{spec}(R/P_n)$ where $\mathrm{spec}(R/P_i) := X_i$. So we can compute the singular locus of each component X_i of X as follows:

First we compute the corresponding test ideal J_i which we decompose into its minimal associated primes and then we can apply the above lemma. For if we have a localization $R_{P_i} = \{\frac{f}{g} \mid f \in R, g \notin P_i\}$, then P_i is the only maximal ideal of R_{P_i}.

> **Algorithm (**slocusZ(I)**)**
>
> INPUT: An ideal $I = \langle f_1, \ldots, f_k \rangle \subset \mathbb{Z}[x_1, \ldots, x_n]$.
> OUTPUT: A list of: [1], the ideal describing the singular locus of $\mathrm{spec}(\mathbb{Z}[x_1, \ldots, x_n]/I)$ and [2], the components of [1]
>
> - Compute the minimal associated primes $I = P_1 \cap \cdots \cap P_n$.
> - For each $i = 1, \ldots, n$ compute the testideal J_i of P_i.
> - Compute the minimal associated primes of each test ideal $J_i = p_{i1} \cap \cdots \cap p_{is_i}$.
> - If $f_{ij} \in p_{is}^2$ for $j = 1, \ldots, r_i$ and $s = 1, \ldots, s_i$, then p_{is} is singular in $\mathbb{Z}[x_1, \ldots, x_n]/I$, else, p_{is} is regular.
> - Intersect all singular prime ideals p_{is}.

2.2 Blow Ups

2.3 Normalizations

The normalization $\overline{A}$ of a reduced Noetherian ring A is defined as its integral closure in the total ring of fractions $Q(A)$, which is the localization

of A with respect to the non-zerodivisors on A. A is called normal if $A = \overline{A}$. One basic tool we need for the three normalization algorithms we present here is the notion of a test ideal. For this we recall the Grauert and Remmert criterion of normality:

Proposition 6 *Let A be a Noetherian reduced ring and $J \subset A$ an ideal satisfying the following conditions:*

(1) J contains a non-zerodivisor on A,

(2) J is a radical ideal,

(3) $N(A) \subset V(J)$, where

$$N(A) = \{P \subset A, \text{ prime ideal} \mid A_P \text{ is not normal}\}$$

is the non-normal locus of A.

Then A is normal if and only if $A \cong \text{Hom}_A(J, J)$, via the canonical map which maps $a \in A$ to the multiplication by a.

A pair (J, x) with a test ideal J and x a non-zerodivisor on A is called test pair. Such a test pair exists if and only if $\overline{A}$ is module-finite over A. We can choose any radical ideal J such that $N(A) \subset V(J)$ and any non-zerodivisor $x \in J$, which means any element $x \in J$ if $A = R/I$ with a prime ideal I. That is one reason why we usually do a primary decomposition of a radical ideal I and then we apply the normalization algorithms to each associated prime of I. (Another reason is that we need the property of module-finiteness of the ring extensions in section 2.3.1, such that the chain of rings becomes stationary.) One option for the choice of J is the radical ideal J we described in section 2.

Now we will disclose the distinctive features of the normalization algorithms we implemented for reduced rings over the integers.

2.3.1 Grauert-Remmert-de Jong: Computation through an increasing chain of rings

The theoretical background of this algorithm comes from the inclusions

$$A \subseteq \text{Hom}_A(J, J) \cong \frac{1}{x}(xJ, J) \subseteq \overline{A} \subseteq Q(A)$$

where A is a reduced Noetherian ring and $J \subset A$ an ideal containing a nonzero-divisor x. The first inclusion is defined by the map $a \mapsto m_a$. For the other inclusions we have the following Lemma:

Lemma 7 *Let A be a reduced (not necessarily Noetherian) ring, $Q(A)$ its total ring of fractions, and I, J two A-submodules of $Q(A)$. Assume that I contains a non-zerodivisor x on A.*

(1) The map

$$\phi : Hom_A(I, J) \to \frac{1}{x}(xJ :_{Q(A)} I) = (J :_{Q(A)} I), \ \varphi \mapsto \frac{\varphi(x)}{x},$$

is independent of the choice of x and an isomorphism of A-modules.

(2) If $J \in A$ then $(xJ :_{Q(A)} I) = (xJ :_A I)$.

4

Proof:

(1) Let $q \in I$ be another non-zerodivisor on A. Write $x = x_1/x_0$ and $q = q_1/q_0$, with $x_0, q_0 \in A$ non-zerodivisors and $x_1, q_1 \in A$. Then $c := x_0 q_0 \in A$ is a non-zerodivisor and $cx, cq \in A$ with $cxq \in I$. Since $\varphi \in \mathrm{Hom}_A(I, J)$ is A-linear, we can write

$$cx\varphi(q) = \varphi(cxq) = cq\varphi(x),$$

so $\varphi(x)/x = \varphi(q)/q$ in $Q(A)$, showing that ϕ is independent of x. Moreover, for any $f \in I$ we have

$$\frac{\varphi(x)}{x} f = \frac{\varphi(cx)}{cx} f = \frac{\varphi(cxf)}{cx} = \frac{cx\varphi(f)}{cx} = \varphi(f) \in J$$

in particular $\varphi(x)f \in xJ$. This shows that the image $\phi(\varphi)$ is in $1/x\,(xJ :_{Q(A)} I)$ and $\varphi(x) = 0 \Leftrightarrow \forall f \in I \; \varphi(f) = 0 \Leftrightarrow \varphi = 0$, therefore ϕ is injective.

To see that ϕ is surjective, let $q \in Q(A)$ satisfy $qI \subset J$. Denote by $m_q \in \mathrm{Hom}_A(I, J)$ the multiplication by q. Then $\phi(m_q) = qx/x = q$ showing that ϕ is surjective.

(2) During the proof of (1) we have seen that

$$(xJ :_{Q(A)} I) = \{\varphi(x) \mid \varphi \in \mathrm{Hom}_A(I, J)\}.$$

Hence, the claimed equality holds if and only if $\varphi(x) \in A$ for all $\varphi \in \mathrm{Hom}_A(I, J)$, which is clearly true if $J \subset A$.

q.e.d.

The inclusion $1/x\,(xJ, J) \subseteq \overline{A}$ follows from the fact that any $b \in (xJ : J)$ satisfies an integral relation $b^p + a_{p-1}b^{p-1} + \cdots + a_0 = 0$, $a_i \in \langle x^i \rangle$, hence b/x is integral over A.

If A is not normal, we get a proper ring extension $A \subsetneq (\mathrm{Hom}_A(J, J) =: A_1$. Then we check the normality of A_1 by applying Proposition 6. If A_1 is not normal we obtain a new ring A_2 by that same proposition, which then has to be tested for normality, and so on. That means, we get a chain of inclusions of rings

$$A \subset A_1 \subset A_2 \subset \cdots$$

(with rings $A_i = A[t_1, \ldots, t_{s_i}]/I_i$, where I_i is an ideal in A_i, and natural maps $\psi_i : A \hookrightarrow A_i$). If we get a normal ring A_N, then $A_N \cong \overline{A}$, because $\overline{A}$ is a finitely generated A-module. The fact which makes this algorithm practicable, is the isomorphism $\mathrm{Hom}_A(J, J) \cong \frac{1}{x}(xJ :_A J)$ allowing us to compute $\mathrm{Hom}_A(J, J)$ over A.

Lemma 8 *Let A be a reduced Noetherian ring, and (J, x) a test pair for A. Let $\{u_0 = x, u_1, \ldots, u_s\}$ be a system of generators for the A-module $(xJ :_A J)$. If $t_1, \ldots, t_s$ denote new variables, then $t_j \mapsto u_j/x$, $1 < j < s$, defines an isomorphism of A-algebras*

$$A_1 := A[t_1, \ldots, t_s]/I_1 \longrightarrow \frac{1}{x}(xJ :_A J),$$

with I_1 the kernel of the map $t_j \mapsto u_j/x$ from $A[t_1, \ldots, t_s] \to \frac{1}{x}(xJ :_A J)$.

See Greuel and Pfister (2008, Lemma 3.6.7) for the computational aspects of this lemma.

If we are computing in a ring over the integers, we are talking about modules instead of algebras. But the algorithm works there as well, if we replace the procedures in the computation through integer-compatible ones. The computation of the test ideal J and the increasing number of variables in the computation of $\text{Hom}_A(J, J)$ can become sooner unpractical than over a field, coefficient explosions may even interrupt the computations. This algorithm is implemented in the library normalint.lib in SINGULAR for reduced rings over the integers.

2.3.2 Greuel-Laplagne-Seelisch: Computation through an increasing chain of ideals

This algorithm [Greuel et al. 2010] (GLS normalization algorithm for short), which is also based on the Grauert and Remmert criterion, is an improvement of de Jongs algorithm. Most of its computations are over the original ring, so we can avoid many computations with additional variables. The idea is to compute ideals $U_1, \ldots, U_N \subset A$ and non-zerodivisors $d_1, \ldots, d_N$ on A, such that

$$A \subset \frac{1}{d_1} U_1 \subset \cdots \subset \frac{1}{d_N} U_N = \overline{A} \subset Q(A)$$

The GLS normalization algorithm is general more effective than de Jong's algorithm. The only computation in $\text{Hom}_A(J, J)$ is the radical of the test ideal (GLS paper prop. 3.2). When the characteristic of the base field is $q > 0$, it is possible to compute also this radical over the original ring. For this, the Frobenius map is used, as described in Matsumoto (2001). From the construction we know that $\frac{1}{d_i} U_i$ is a finitely generated R-module and hence there is a surjection

$$R_i := R[t_1, t_2, \ldots, t_{s_i}] \to \frac{1}{d_i} U_i, \ t_j \mapsto u_j,$$

where $\{d_i, u_1, \ldots, u_s\}$ is a set of R-module generators of U_i. If I_i denotes the kernel of this map, we get a ring isomorphism

$$\varphi_i : A_i := R_i / I_i \to \frac{1}{d_i} U_i \subset Q(A).$$

(Note that the definition of I_i is now slightly different from the one given in 3.2.1). To obtain A-module generators of $\varphi_i(J_i)$ it is not enough to compute the images of the generators of J_i. But there is a simple algorithm to get these generators (Algorithm 2, GLS paper).

Once we have computed a test pair (J_i, x_i) in A_i, the next step is to compute the quotient $(pJ_i :_{A_i} Ji)$. The following theorem shows that this computation can be carried out in the original ring A.

Theorem 9 *Let $A = R/I$, $A' = A[t_1, \ldots, t_s]/I'$ be a finite ring extension and maps $\psi : A \hookrightarrow A'$, $\varphi : A' \hookrightarrow Q(A)$. Let (J, x) be a test pair for A and*

(J', x') *a test pair for* A', *with* $x' = \psi(x)$. *Let* U, H *be ideals of* A *and* $d \in A$ *such that* $\varphi(A') = \frac{1}{d}U$ *and* $\varphi(J') = \frac{1}{d}H$. *Then*

$$(x'J' :_{A'} J') = \frac{1}{d}(dxH :_A H).$$

Proof: The proof is an easy consequence of Lemma 7. Omitting ψ and φ in the following notations and applying Lemma 7 to $x \in J \subset A$ we get

$$(xJ' :_A J') = (p'J' :_{Q(A)} J') = (pH :_{Q(A)} H),$$

since $Q(A') = Q(A)$ and $J' = \frac{1}{d}H$. On the other hand, we can apply Lemma 7 to $dp \in H \subset A$ and get

$$\frac{1}{d}(dpH :_A H) = \frac{1}{d}(dpH :_{Q(A)} H) = (pH :_{Q(A)} H).$$

<div align="right">

q.e.d.

</div>

So now we know how to compute the ideals $U_i \subset A$. This algorithm works whenever Gröbner bases, radicals and ideal quotients can be computed in rings of the form $R[t_1, \cdots, t_s]$, so the implementation for reduced rings over the integers requested only the customary changes like taking care not to divide trough any coefficients and replacing particular procedures through their relative integer-suitable ones. This algorithm is also implemented in the library normalint.lib in SINGULAR for reduced rings over the integers.

In 2011 Böhm-Decker-Pfister-Laplagne-Steenpass-Steidel implemented a third algorithm for the normalization of a reduced Noetherian ring A over a field. In view of modern multi-core computers it uses parallelization of the GLS normalization via stratifying $\mathrm{Sing}(A)$.

If $\mathrm{Sing}(A) = \{P_1, \ldots, P_s\}$ is finite, we set $S_i = A \setminus P_i$ for $i = 1, \ldots, s$. Then for any intermediate ring $A \subset A^{(i)} \subset \overline{A}$ with $S_i^{-1}A^{(i)} = \overline{S_i^{-1}A}$ the equation $\overline{A} = \sum_{i=1}^{s} A^{(i)}$ holds. So in the case of a reduced ring A over the integers we can compute $\mathrm{Sing}(A)$ and normalize locally at each P_i to get $\overline{A}$.

All corresponding implementations for reduced rings over fields are available in the SINGULAR libraries modnormal.lib and locnormal.lib. This computation also uses modular methods and it is in general faster than the pure GLS-normalization. It is planned to be implemented for polynomial rings over the integers.

3   Example

We want to desingularize the surface $x^2 + y^3z^3 + y^2z^5 \subset \mathbb{Q}[x, y, z]$ with Lipmans algorithm. The output is a list consisting of two lists of rings. The first component gives the final charts, and the second component is the list of all charts.

```
> LIB "lipmanresolve.lib";              // load the library lipmanresolve
> ring r=0,(x,y,z),dp;                  // ring of char 0 with
                                        // variables x,y and z

> ideal i=x^2+y^3*z^3 +y^2*z^5;

> list l=lipmanresolve(i);              // compute the desingularization

> size(l);                              // the output is a list
2                                       // of 2 components

> l[1];                                 // list of final charts
                                        // every chart is given by
                                        // its defining ring
                                        // there are three final charts

[1]:
// characteristic :  0
// number of vars :  2
//        block 1 :  ordering dp
//                :  names y(2) y(3)
//        block 2 :  ordering C

// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names x(2) y(1) y(3)
//        block 2 :  ordering C

// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names x(3) y(1) y(2)
//        block 2 :  ordering C
```

```
> l[2];                                        // list of all charts
[1]:
// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names x y z
//        block 2 :  ordering C
[2]:
// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names T(1) T(2) T(3)
//        block 2 :  ordering C
// characteristic :  0
// number of vars :  2
//        block 1 :  ordering dp
//                :  names y(2) y(3)
//        block 2 :  ordering C

// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names x(2) y(1) y(3)
//        block 2 :  ordering C

// characteristic :  0
// number of vars :  3
//        block 1 :  ordering dp
//                :  names x(3) y(1) y(2)
//        block 2 :  ordering C
```

To see the information which is inside a concrete ring or chart it is enough to
set the selected ring and proceed as follows:

```
> def R22=l[2][2];                        // define a new ring
> setring R22;                            // set this ring
==== intermediate scheme:
> i;                                      // ideal is called i in an intermediate ring
_[1]=T(3)^3+T(1)^2+T(2)*T(3)              // R22/i is the surface after a normalization
==== Images of variables of original ring:   // morphism defining
> liph;                                   // the normalization
_[1]=-T(1)*T(2)*T(3)
_[2]= T(2)
_[3]= T(3)
> print(Path);                            // path to this chart
 0,1,
 0,1,
 1,1,
// from [0,0,1] ([initial ring, initial vertex,first branch])
// to [1,1,1] ([normalization, vertex, branch of vertex above])
```

```
> def R13=l[1][3];                        // define a new ring
> setring R13;                            // set this ring
==== First final chart:
> lip;                                    //
_[1]=x(3)*y(1)^2+y(1)*y(2)+1              // morphism defining
==== Images of variables of original ring:   // morphism defining
> liph;                                   // the normalization and the blow up
_[1]=-x(3)^5*y(1)^2*y(2)
 -2*x(3)^4*y(1)*y(2)^2 -x(3)^3*y(2)^3
_[2]=-x(3)^2*y(1)*y(2) -x(3)*y(2)^2
_[2]= x(3)
==== The exceptional divisor:
> eD;                                     // the blow up
_[1]=x(3)
_[2]=y(1)*y(2)+1
> print(Path);                            // path to this chart
 0,1,-1
 0,1,4
 1,1,3
// from [0,0,1] ([initial ring, initial vertex,first branch])
// over [1,1,1] ([normalization, vertex, branch of vertex above]]
// to [-1,4,3] ([blow up, vertex, branch of vertex above])
```

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