PARALLEL ALGORITHMS FOR NORMALIZATION

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ABSTRACT. Given a reduced affine algebra A over a perfect field K, we present parallel algorithms to compute the normalization $\overline{A}$ of A. Our starting point is the algorithm of [Greuel et al. 2010], which is an improvement of de Jong's algorithm, see [de Jong 1998, Decker et al. 1999]. First, we propose to stratify the singular locus $\mathrm{Sing}(A)$ in a way which is compatible with normalization, apply a local version of the normalization algorithm at each stratum, and find $\overline{A}$ by putting the local results together. Second, in the case where $K = \mathbb{Q}$ is the field of rationals, we propose modular versions of the global and local algorithms. We have implemented our algorithms in the computer algebra system SINGULAR and compare their performance with that of other algorithms. In the case where $K = \mathbb{Q}$, we also discuss the use of modular computations of Gröbner bases, radicals and primary decompositions. We point out that in most examples, the new algorithms outperform the algorithm of [Greuel et al. 2010] by far, even if we do not run them in parallel.

1. INTRODUCTION

Normalization is an important concept in commutative algebra, with applications in algebraic geometry and singularity theory. We are interested in computing the normalization $\overline{A}$ of a reduced affine K–algebra A, where K is a perfect field. For this, a number of algorithms have been proposed, but not all of them are of practical interest (see the historical account in [Greuel et al. 2010]). A milestone is de Jong's algorithm, see [de Jong 1998, Decker et al. 1999], which is based on the normality criterion of [Grauert and Remmert 1971], and which has been implemented in SINGULAR, MACAULAY2, and MAGMA, see [Bosma et al. 1997]. The algorithm of [Greuel et al. 2010] (*GLS normalization algorithm* for short), which is alsobased on the Grauert and Remmert criterion, is an improvement of de Jong's algorithm. It is implemented in SINGULAR. The algorithm proposed by [Leonard and Pellikaan 2003] respectively [Singh and Swanson 2009] is designed for the characteristic p case. It is implemented in SINGULAR and MACAULAY2, and works well for small p.

In view of modern multi-core computers, the parallelization of fundamental algorithms becomes increasingly important. Our objective in this paper is to present parallel versions of the GLS normalization algorithm in that we reduce the general problem to computational problems which are easier and do not depend on each other. It turns out that in most cases, the new algorithms outperform the GLS algorithm by far, even if we do not run them in parallel.

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We start in Section 2 by reviewing the basic ideas of the algorithm. In particular, we recall the normality criterion of Grauert and Remmert.

In Section 3, we present a local version of the normality criterion which applies to a stratification of the singular locus $\mathrm{Sing}(A)$ of A, and show how to put the individual results together to get $\overline{A}$.

Section 4 contains a discussion of modular methods for the GLS algorithm and its local version.

Timings are presented in Section 5.

2. The GLS Normalization Algorithm

Referring to [Greuel et al. 2010] and [Greuel and Pfister 2007] for details and proofs, we sketch the GLS normalization algorithm. We begin with some general remarks. For these, A may be any reduced Noetherian ring.

Definition 2.1. Let A be a reduced Noetherian ring. The *normalization* of A, written $\overline{A}$, is the integral closure of A in its total ring of fractions $\mathrm{Q}(A)$. We call A *normal* if $A = \overline{A}$.

We write

$$\mathrm{Spec}(A) = \{P \subseteq A \mid P \text{ prime ideal}\}$$

for the *spectrum* of A and $V(J) = \{P \in \mathrm{Spec}(A) \mid P \supseteq J\}$ for the *vanishing locus* of an ideal J of A. Taking into account that normality is a local property, we call

$$N(A) = \{P \in \mathrm{Spec}(A) \mid A_P \text{ is not normal}\}$$

the *non–normal locus* of A. Furthermore, we write

$$\mathrm{Sing}(A) = \{P \in \mathrm{Spec}(A) \mid A_P \text{ is not regular}\}$$

for the *singular locus* of A. Then $N(A) \subseteq \mathrm{Sing}(A)$, with equality holding if A is the coordinate ring of a curve (see [de Jong and Pfister 2000], Theorem 4.4.9).

Definition 2.2. Let A be a reduced Noetherian ring. The *conductor* of A in $\overline{A}$ is the ideal

$$\mathcal{C}_A = \mathrm{Ann}_A(\overline{A}/A) = \{a \in A \mid a\overline{A} \subseteq A\}.$$

Lemma 2.3. *Let A be a reduced Noetherian ring. Then $N(A) \subseteq V(\mathcal{C}_A)$. Furthermore, $\overline{A}$ is module–finite over A if and only if $\mathcal{C}_A$ contains a non–zerodivisor of A. In this case, $N(A) = V(\mathcal{C}_A)$.*

To state the aforementioned Grauert and Remmert criterion, we need:

Lemma 2.4. *Let A be a reduced Noetherian ring, and let $J \subseteq A$ be an ideal containing a non–zerodivisor g of A. Then the following hold:*

1. *If $\varphi \in \mathrm{Hom}_A(J, J)$, then the fraction $\varphi(g)/g \in \overline{A}$ is independent of the choice of g.*
2. *There are natural inclusions of rings*

$$A \subseteq \mathrm{Hom}_A(J, J) \cong \frac{1}{g}\,(gJ :_A J) \subseteq \overline{A} \subseteq \mathrm{Q}(A),\ a \mapsto \varphi_a,\ \varphi \mapsto \frac{\varphi(g)}{g},$$

where $\varphi_a : J \to J$ denotes the multiplication by $a \in A$.

Proposition 2.5 ([Grauert and Remmert 1971])**.** *Let A be a reduced Noetherian ring, and let $J \subseteq A$ be an ideal satisfying the following conditions:*

(1) *J contains a non–zerodivisor g of A,*
(2) *J is a radical ideal,*
(3) *$V(\mathcal{C}_A) \subseteq V(J)$.*

Then A is normal iff $A \cong \mathrm{Hom}_A(J,J)$ via the map which sends a to φ_a.

Definition 2.6. A pair (J, g) as in Proposition 2.5 is called a *test pair* for A, and J is called a *test ideal* for A.

By Lemma 2.3, test pairs exist iff $\overline{A}$ is module–finite over A. Given such a pair (J, g), the idea of finding $\overline{A}$ is to successively enlarge A until the normality criterion allows us to stop (since A is Noetherian, this will eventually happen in the module–finite case). Starting from $A_0 = A$, we get a chain of extensions of reduced Noetherian rings

$$A = A_0 \subseteq \cdots \subseteq A_{i-1} \subseteq A_i \subseteq \cdots \subseteq A_m = \overline{A}.$$

Here, $A_i = \mathrm{Hom}_{A_{i-1}}(J_i, J_i) \cong \frac{1}{g}(gJ_i :_A J_i)$, where J_i is the radical of the extended ideal JA_i, for $i \geq 1$. Note that (J_i, g) is indeed a test pair for A_i:

Remark 2.7. Let A be a reduced Noetherian ring such that $\overline{A}$ is module–finite over A, and let $A \subseteq A' \subseteq \overline{A}$ be an intermediate ring. Clearly, every non–zerodivisor $g \in A$ of A is a non–zerodivisor of $\mathrm{Q}(A)$. In particular, it is a non–zerodivisor of A'. Furthermore, if $\mathcal{C}_{A'}$ is the conductor of A' in $\overline{A'} = \overline{A}$, then $\mathcal{C}_{A'} \supseteq \mathcal{C}_A$. It follows that every prime ideal $Q \in N(A') = V(\mathcal{C}_{A'})$ contracts to a prime ideal $P \in N(A) = V(\mathcal{C}_A)$. Hence, if (J, g) is a test pair for A, then $P \supseteq J$, which implies that $Q \supseteq \sqrt{JA'} =: J'$. We conclude that (J', g) is a test pair for A'.

Explicit computations rely on explicit representations of the A_i as A–algebras. These will be obtained as an application of Lemma 2.8 below. To formulate the lemma, we use the following notation. Let $J \subseteq A$ be an ideal containing a non–zerodivisor g of A, and let A–module generators $u_0 = g, u_1, \ldots, u_s$ for $gJ :_A J$ be given. Choose variables $T_1, \ldots, T_s$, and consider the epimorphism

$$\Phi : A[T_1, \ldots, T_s] \to \frac{1}{g}\,(gJ :_A J),\ T_i \mapsto \frac{u_i}{g}.$$

The kernel of Φ describes the A–algebra relations on the u_i. We single out two types of relations:

- Each A–module syzygy
$$\alpha_0 u_0 + \alpha_1 u_1 + \ldots + \alpha_s u_s = 0, \quad \alpha_i \in A,$$
gives an element $\alpha_0 + \alpha_1 T_1 + \ldots + \alpha_s T_s \in \mathrm{Ker}\,\Phi$, which we call a *linear relation.*
- Developing each product $u_i u_j$, $1 \leq i \leq j \leq s$, as a sum $u_i u_j = \sum_k \beta_{ijk} u_k$, we get elements $T_i T_j - \sum_k \beta_{ijk} T_k$ in $\mathrm{Ker}\,\Phi$, which we call *quadratic relations.*

It is easy to see that these linear and quadratic relations already generate $\mathrm{Ker}\,\Phi$. We thus have:

Lemma 2.8. *Let A be a reduced Noetherian ring, and let $J \subseteq A$ be an ideal containing a non–zerodivisor g of A. Then, given A–module generators $u_0 = g, u_1, \ldots, u_s$ for $gJ :_A J$, we have an isomorphism of A–algebras*

$$A[T_1, \ldots, T_s]/R \cong \frac{1}{g}\,(gJ :_A J),\ T_i \mapsto \frac{u_i}{g},$$

where R is the ideal generated by the linear and quadratic relations described above.

The following result from [Greuel et al. 2010] will allow us to find the normalization such that all calculations except the computation of the radicals $\sqrt{J_i}$ can be carried through in the original ring A:

Theorem 2.9. *Let A be a reduced Noetherian ring, let $J \subseteq A$ be an ideal containing a non–zerodivisor g of A, let $A \subseteq A' \subseteq \mathrm{Q}(A)$ be an intermediate ring such that A' is module–finite over A, and let $J' = \sqrt{JA'}$. Let U and H be ideals of A such that $A' = \frac{1}{d}U$ and $J' = \frac{1}{d}H$, respectively. Then*

$$(gJ' :_{A'} J') = \frac{1}{d}\,(dgH :_A H) \subseteq \mathrm{Q}(A).$$

Remark 2.10. In the case where $A = K[x_1 \ldots, x_n]/I$ is a reduced affine algebra over a field K, let $P_1, \ldots, P_r$ be the associated primes of the radical ideal I. Then

$$\overline{A} \cong \overline{K[X_1, \ldots, X_n]/P_1} \times \cdots \times \overline{K[X_1, \ldots, X_n]/P_r},$$

and $\overline{A}$ is module–finite over A by Emmy Noether's finiteness theorem, see [Swanson and Huneke 2006]. Thus, using techniques for primary decomposition as in [Greuel et al. 2010, Remark 4.6], the computation of normalization can be reduced to the case where A is an affine domain (that is, I is a prime ideal). When writing our algorithms in pseudocode, we will always start from a domain A. Talking about a non–zerodivisor then just means to talk about a non–zero element.

Remark 2.11. If A is an affine domain over a perfect field K, we can apply the Jacobian criterion, see [Eisenbud 1995]: If M is the Jacobian ideal[1] of A, then M is non-zero and contained in the conductor $\mathcal{C}_A$, see [Greuel et al. 2010, Lemma 4.1], so that we may choose $\sqrt{M}$ together with any non–zero element g of M as an initial test pair. Implementing all this, the GLS normalization algorithm will find an ideal $U \subseteq A$ and a denominator $d \in M$ such that

$$\overline{A} = \frac{1}{d}\,U \subseteq \mathrm{Q}(A).$$

Since M is contained in $\mathcal{C}_A$, any non–zero element of M is valid as a denominator: If $0 \neq c \in M$, then $c \cdot \frac{1}{d}U =: W$ is an ideal in A, so that $\frac{1}{d}U = \frac{1}{c}W$.

For the purpose of comparison with the local approach of the next section, we illustrate the algorithm by an example:

Example 2.12. For

$$A = K[x, y] = K[X, Y]/\langle X^4 + Y^2(Y-1)^3\rangle,$$

the radical of the Jacobian ideal is

$$J := \langle x, y\,(y-1)\rangle_A\,,$$

and we can take $g := x \in J$ as a non–zerodivisor of A. In its first step, starting with the initial test pair (J, x), the normalization algorithm produces the following data:

$$U^{(1)} := xJ \ :_A \ J = \left\langle x, y(y-1)^2 \right\rangle_A \text{ and } A_1 := A[t_1] := A[T_1]/I_1 \cong \frac{1}{x}U^{(1)},$$

[1]The *Jacobian ideal* of A is generated by the images of the $(c \times c)$–minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$, where c is the codimension, and $f_1, \ldots, f_r$ are polynomial generators for I.

with relations and isomorphism given by

$$I_1 = \left\langle -T_1x + y(y-1)^2, T_1y(y-1) + x^3, T_1^2 + x^2(y-1) \right\rangle_{A[T_1]}$$

and

$$t_1 \mapsto \frac{y(y-1)^2}{x},$$

respectively. In the next step we find

$$\begin{aligned} J_1 &:= \sqrt{\langle x, y(y-1) \rangle_{A_1}} = \langle x, y(y-1), t_1 \rangle_{A_1} \\ &= \frac{1}{x} \left\langle x^2, xy(y-1), y(y-1)^2 \right\rangle_A =: \frac{1}{x} H_1. \end{aligned}$$

Using the test pair (J_1, x) and applying Theorem 2.9 and Lemma 2.8, we get

$$\begin{aligned} \frac{1}{x}(xJ_1 \ :_{A_1} \ J_1) &= \frac{1}{x^2}(x^2 H_1 \ :_A \ H_1) \\ &= \frac{1}{x^2} \left\langle x^2, xy(y-1), y(y-1)^2 \right\rangle_A =: \frac{1}{x^2} U^{(2)} \end{aligned}$$

and

$$A_2 := A[t_2, t_3] := A[T_2, T_3]/I_2 \cong \frac{1}{x^2} U^{(2)},$$

with relations and isomorphism given by

$$\begin{aligned} I_2 = \big\langle & T_2x - T_3(y-1), -T_3x + y(y-1), T_2y(y-1) + x^2, T_2y^2(y-1)^2 + T_3x^3 \\ & T_2^2 + (y-1), T_2T_3 + x, T_3^2 - T_2y \big\rangle \end{aligned}$$

and

$$t_2 \mapsto \frac{y(y-1)^2}{x^2}, \ t_3 \mapsto \frac{y(y-1)}{x},$$

respectively. In the final step, we find that A_2 is normal, so that $\overline{A} = A_2$.

3. Normalization via Localization

In this section, we discuss a local–global approach to computing the normalization. Our starting point is the following result:

Proposition 3.1. *Let A be a reduced Noetherian ring with a finite singular locus $\mathrm{Sing}(A) = \{P_1, \dots, P_s\}$. For $i = 1, \dots, s$, let $S_i = A \setminus P_i$ and suppose that an intermediate ring $A \subset A^{(i)} \subset \overline{A}$ is given such that $S_i^{-1} A^{(i)} = \overline{S_i^{-1} A}$. Then*

$$\sum_{i=1}^{s} A^{(i)} = \overline{A}.$$

Proof. We will show a more general result in Proposition 3.2 below. □

Proposition 3.1 applies, for example, to the coordinate ring of a curve. Whenever $\mathrm{Sing}(A) = \{P_1, \dots, P_s\}$ is finite, the proposition allows us to find $\overline{A}$ by normalizing locally at each P_i using Proposition 3.3 below, and putting the local results together. In the case where $\mathrm{Sing}(A)$ is not finite, replacing $P_1, \dots, P_s$ above by the minimal associated primes will not give the correct result. However, it is still possible to obtain $\overline{A}$ as a finite sum of local results: The idea is to stratify $\mathrm{Sing}(A)$ in a way which is compatible with normalization. For this, if $P \in \mathrm{Sing}(A)$, set

$$L_P = \bigcap_{P \supseteq \widetilde{P} \in \mathrm{Sing}(A)} \widetilde{P}.$$

We stratify the singular locus by the value of L_P ranging over all $P \in \mathrm{Sing}(A)$. So if we consider the set

$$\mathcal{L} = \{L_P \mid P \in \mathrm{Sing}(A)\}$$

of all possible values of L_P, the strata are the sets

$$V_L = \{P \in \mathrm{Sing}(A) \mid L_P = L\}$$

for $L \in \mathcal{L}$. We write $\mathrm{Strata}(A) = \{V_L \mid L \in \mathcal{L}\}$ for the set of all different strata. If $M_1, ..., M_r$ denote the minimal associated primes of the singular locus, we have

$$\mathcal{L} \subseteq \left\{\bigcap_{i\in\Gamma} M_i \mid \Gamma \subseteq \{1, ..., r\}\right\}$$

and hence the set of strata is finite. By construction the singular locus is the disjoint union of all strata. For $V \in \mathrm{Strata}(A)$ write L_V for the constant value of L_P for $P \in V$.

As a generalization of Proposition 3.1 we obtain:

Proposition 3.2. *Let A be a reduced Noetherian ring with stratification of the singular locus $\mathrm{Strata}(A) = \{V_1, ..., V_s\}$. For $i = 1, \ldots, s$, suppose that an intermediate ring $A \subseteq A^{(i)} \subseteq \overline{A}$ is given such that $S^{-1}A^{(i)} = \overline{S^{-1}A}$ for all $P \in V_i$ and $S = A \setminus P$. Then*

$$\sum_{i=1}^{s} A^{(i)} = \overline{A}.$$

Proof. By construction, $B := \sum_{i=1}^{s} A^{(i)} \subseteq \overline{A}$. We wish to show equality. It suffices to show that if $P \in \mathrm{Spec}(A)$ is a prime ideal and $S = A \setminus P$, then $S^{-1}B = S^{-1}\overline{A}$. If $P \in \mathrm{Sing}(A)$, then $P \in V_i$ for some i, and the local equality is obtained from the chain of inclusions

$$S^{-1}A^{(i)} \subseteq S^{-1}B \subseteq S^{-1}\overline{A} = \overline{S^{-1}A}.$$

Indeed, $S^{-1}A^{(i)} = \overline{S^{-1}A}$ by assumption. If $P \notin \mathrm{Sing}(A)$, then $S^{-1}A$ is normal, and the local equality follows likewise from the chain of inclusions

$$S^{-1}A \subseteq S^{-1}B \subseteq S^{-1}\overline{A} = \overline{S^{-1}A}.$$

□

The modification of the Grauert and Remmert criterion below will allow us to find a local contribution to $\overline{A}$ on V along the lines of the previous section:

Proposition 3.3. *Let A be a reduced Noetherian ring, and let $A \subseteq A'$ be a module–finite ring extension. Let $V \in \mathrm{Strata}(A)$, let $J' = \sqrt{L_V A'}$. Suppose that L_V contains a non–zerodivisor g of A. If*

$$A' \cong \mathrm{Hom}_{A'}(J', J')$$

via the map which sends a' to $\varphi_{a'}$, then the localization $S^{-1}A'$ with $S = A \setminus P$ is normal for all $P \in V$.

Proof. The assumption and Proposition 2.10 in [Eisenbud 1995] give

$$S^{-1}A' \cong S^{-1}(\mathrm{Hom}_{A'}(J', J')) \cong \mathrm{Hom}_{S^{-1}A'}(S^{-1}J', S^{-1}J').$$

Hence, the result will follow from the Grauert and Remmert criterion (Proposition 2.5) applied to $S^{-1}A'$ once we show that the localized ideal $S^{-1}J'$ satisfies the three conditions of the criterion. First, since forming radicals commutes with localization, $S^{-1}J'$ is a radical ideal. Second, the image of g in $S^{-1}A'$ is a non–zerodivisor of $S^{-1}A'$ contained in $S^{-1}J'$. Third, we show that $V(\mathcal{C}_{S^{-1}A'}) = N(S^{-1}A') \subseteq V(S^{-1}J'_P)$. For this, we first note that

$$V(\mathcal{C}_{S^{-1}A}) = N(S^{-1}A) = \{S^{-1}\widetilde{P} \mid \widetilde{P} \in \operatorname{Sing}(A),\ \widetilde{P} \subseteq P\}$$

since prime ideals in $S^{-1}A$ correspond to prime ideals in A contained in P. Let now $Q \in N(S^{-1}A')$. Then, as shown in Remark 2.7, Q contracts in A to some $\widetilde{P} \in \operatorname{Sing}(A)$, $\widetilde{P} \subseteq P$. This implies that

$$Q \supseteq \sqrt{(S^{-1}\widetilde{P})(S^{-1}A')} = \sqrt{S^{-1}(\widetilde{P}A')} = S^{-1}(\sqrt{\widetilde{P}A'}) \supseteq S^{-1}J',$$

as desired. □

In the situation of Proposition 3.3, let a non–zerodivisor $g \in L_V$ of A be known. Then, using (L_V, g) instead of a test pair as in Definition 2.6, and proceeding as in the previous section, we get a chain of rings

$$A \subseteq A_1 \subseteq \cdots \subseteq A_m \subseteq \overline{A}$$

such that $S^{-1}(A_m)$ is normal and, hence, equal to $S^{-1}\overline{A} = \overline{S^{-1}A}$ for all $P \in V$, $S = A \setminus P$.

Summing up, we are lead to Algorithms 1 and 2 below.

Algorithm 1 Normalizing the localizations

Input: An affine domain $A = K[X_1, \dots, X_n]/I$ over a perfect field K, a stratum $V \in \operatorname{Strata}(A)$, and $0 \neq g \in L_V$.

Output: $U \subseteq A$ ideal and $d \in A$ with $\frac{1}{d}U \subseteq \overline{A}$ and $S^{-1}(\frac{1}{d}U) = \overline{S^{-1}A}$ for all $P \in V$, $S = A \setminus P$.

return the result of the GLS normalization algorithm applied to (L_V, g);

Algorithm 2 Normalization via localization

Input: An affine domain $A = K[X_1, \dots, X_n]/I$ over a perfect field K.

Output: $U \subseteq A$ ideal and $d \in A$ such that $\overline{A} = \frac{1}{d}U \subseteq \mathrm{Q}(A)$.

$J := \sqrt{M}$, where M is the Jacobian ideal of I;
choose $0 \neq g \in J$;
compute the strata of the singular locus $\operatorname{Strata}(A) = \{V_1, ..., V_s\}$;
for all i **do**
  apply Algorithm 1 to (V_i, g) to find ideals $U_i \subseteq A$ and powers $d_i = g^{m_i}$ with $A \subseteq \frac{1}{d_i}U_i \subseteq \overline{A}$ and $S^{-1}(\frac{1}{d_i}U_i) = \overline{S^{-1}A}$ for all $P \in V_i$, $S = A \setminus P$;
$m := \max\{m_1, \dots, m_s\},\ d := g^m,\ U := \sum_i g^{m-m_i}U_i$;
return (U, d);

Remark 3.4. In Algorithm 2, it may be more efficient to choose possibly different non–zero elements $g_i \in L_{V_i}$. The algorithm computes, then, pairs (U'_i, d_i) with ideals $U'_i \subseteq A$ and powers $d_i = g_i^{m_i}$. As explained in [Greuel et al. 2010, Remark 4.3], starting from the (U'_i, d_i), we may always find a denominator $d \in M$ and ideals $U_i \subseteq A$ such that $\frac{1}{d}U_i = \frac{1}{d_i}U'_i$ for all i. Hence, the desired result is $(\sum_i U_i, d)$.

Remark 3.5. In Algorithm 2, it is sufficient to consider the minimal strata V, that is, those with minimal values L_V with respect to inclusion. Denote, as above, the minimal associated primes of the singular locus of A by $M_1, ..., M_r$. We can obtain the minimal L_V as all possible intersections $\bigcap_{i\in\Gamma} M_i$ such that $\Gamma \subseteq \{1, ..., r\}$ is maximal with the property that $\sum_{i\in\Gamma} M_i \neq \langle 1\rangle$.

Example 3.6. We come back to the coordinate ring A of the curve C with defining polynomial $f(X,Y) = X^4 + Y^2(Y-1)^3$ from Example 2.12 to discuss normalization via localization. The curve C has a double point of type A_3 at $(0,0)$ and a triple point of type E_6 at $(0,1)$. We illustrate Algorithm 2 using for both singular points the non–zerodivisor $g = x$:

For the A_3-singularity, consider

$$P_1 = \langle x, y\rangle_A \quad \text{and} \quad S_1 = A \setminus P_1.$$

The local normalization algorithm yields $\overline{S_1^{-1}A} = S_1^{-1}(\frac{1}{d_1}U_1)$, where

$$d_1 = x^2 \quad \text{and} \quad U_1 = \left\langle x^2,\ y(y-1)^3\right\rangle_A.$$

For the E_6 singularity, considering

$$P_2 = \langle x, y-1\rangle_A \quad \text{and} \quad S_2 = A \setminus P_2,$$

we get $\overline{S_2^{-1}A} = S_2^{-1}(\frac{1}{d_2}U_2)$, where

$$d_2 = x^2 \quad \text{and} \quad U_2 = \left\langle x^2,\ xy^2\,(y-1)\,,\ y^2\,(y-1)^2\right\rangle_A.$$

Combining the local contributions, we get

$$\frac{1}{d}U = \frac{1}{d_1}U_1 + \frac{1}{d_2}U_2$$

with $d = x^2$ and

$$U = \left\langle x^2,\ xy^2\,(y-1)\,,\ y(y-1)^3,\ y^2\,(y-1)^2\right\rangle_A.$$

A moments thought shows that U coincides with the ideal $U^{(2)}$ computed in Example 2.12.

4. Modular Methods

Algorithm 2 from Section 3 is parallel in nature since the computations of the local normalizations do not depend on each other. In this section, we describe a modular way of parallelizing both the normalization algorithm from Section 2 and that from Section 3 in the case where $K = \mathbb{Q}$ is the field of rationals. One possible approach is to just replace *all* involved Gröbner basis respectively radical computations by their modular variants as introduced by [Idrees et al. 2011]. These algorithms are either probabilistic or require rather expensive tests to verify the results at the end. In order to reduce the number of verification tests, it is reasonable to also try direct modularization and parallelization for the normalization algorithms: The approach we propose requires, in principle, only one verification. Let us point out, however, that in the local setup (in the case where $K = \mathbb{Q}$) the Gröbner basis computation for the Jacobian ideal, the subsequent primary decomposition, and the recombination of the local results can be handled in an efficient way by modular techniques. We exemplarily describe the modularization of the GLS normalization algorithm as outlined in Section 2. Each of the local normalizations in Algorithm 2 from Section 3 can be modularized similarly.

We consider the polynomial ring $\mathbb{Q}[X]$, where $X = \{X_1, \ldots, X_n\}$ is a set of variables, fix a global monomial ordering $>$ on $\mathbb{Q}[X]$, and use the following notation: If $S \subseteq \mathbb{Q}[X]$ is a set of polynomials, then $\mathrm{LM}(S) := \{\mathrm{LM}(f) \mid f \in S\}$ denotes the set of leading monomials of S. If $f \in \mathbb{Q}[X]$ is a polynomial, $I = \langle f_1, \ldots, f_r\rangle \subseteq \mathbb{Q}[X]$ is an ideal, and p is a prime number which does not divide any denominator of the coefficients of $f, f_1, \ldots, f_r$, then we write $f_p := f \mod p \in \mathbb{F}_p[X]$ and $I_p := \langle (f_1)_p, \ldots, (f_r)_p\rangle \subseteq \mathbb{F}_p[X]$. If $J \subseteq \mathbb{Q}[X]$ is an ideal satisfying $I \subseteq J$, then we denote the ideal induced by J in $\mathbb{Q}[X]/I$ also by J.

In the following, $I = \langle f_1, \ldots, f_r\rangle \subseteq \mathbb{Q}[X]$ will be a prime ideal. The GLS normalization algorithm applied to $A = \mathbb{Q}[X]/I$ returns a reduced Gröbner basis $G = \{g_1, \ldots, g_s\}$ of an ideal $U \subseteq \mathbb{Q}[X]$ and a polynomial d such that $\frac{1}{d}U \subseteq \mathrm{Q}(A)$ is the normalization of A. In addition to this, we also obtain the ideal $R \subseteq A[T_1, \ldots, T_s]$ of linear and quadratic relations representing $\overline{A}$ as an A–algebra as specified in Lemma 2.8. Recall from Remark 2.11 that the Jacobian ideal M of I is contained in the conductor $\mathcal{C}_A$ and contains a non–zero element, and that any such element c may be chosen as a denominator: With notation as above, there exists an ideal $W \subseteq \mathbb{Q}[X]$ such that $\frac{1}{d}U = \frac{1}{c}W \subseteq \mathrm{Q}(A)$. If we fix a denominator d, then the reduced Gröbner basis G is uniquely determined. With respect to modularization, we then have:

Lemma 4.1. *With notation as above, let p be a sufficiently general prime number. Then A_p is a domain, d_p is a non–zero element in the conductor of A_p, and $\frac{1}{d_p}U_p \subseteq \mathrm{Q}(A_p)$ is the normalization of $A_p = \mathbb{F}_p[X]/I_p$. In particular, $G_p = \{(g_1)_p, \ldots, (g_s)_p\}$ is the reduced Gröbner basis of I_p.*

Proof. The first two statements are clear. Testing the Grauert Remmert criterion amounts to a Gröbner basis computation. Furthermore, reducing a Gröbner basis modulo a sufficiently general p gives a Gröbner basis of the reduced ideal, see [Idrees et al. 2011]. Hence, if the Grauert and Remmert criterion is satisfied for $\frac{1}{d}U$, then it also holds for $\frac{1}{d_p}U_p$. □

Relying on the lemma, the basic idea of the modular normalization algorithm is as follows. First, compute the Jacobian ideal M and choose a non–zero "universal denominator" $d \in M$ representing the denominator for all normalization computations in positive characteristic (apply again Remark 2.11). Second, choose a set $\mathcal{P}$ of prime numbers at random, and compute reduced Gröbner bases $G_p \subseteq \mathbb{F}_p[X]$, $p \in \mathcal{P}$, such that $\frac{1}{d_p}\langle G_p\rangle \subseteq \mathrm{Q}(A_p)$ is the normalization of A_p. Third, lift the modular Gröbner bases to a Gröbner basis $G \subseteq \mathbb{Q}[X]$ and define $U := \langle G\rangle$.

The lifting process consists of two steps. First, the set $\mathcal{GP} := \{G_p \mid p \in \mathcal{P}\}$ is lifted to $G_N \subseteq (\mathbb{Z}/N\mathbb{Z})[X]$ with $N := \prod_{p\in\mathcal{P}} p$ by applying the Chinese remainder algorithm to the coefficients of the polynomials occurring in $\mathcal{GP}$. Second, we obtain $G \subseteq \mathbb{Q}[X]$ by lifting the modular coefficients occurring in G_N to rational coefficients via the Farey rational map[2]. This map is guaranteed to be bijective provided that

[2] *Farey fractions* refer to rational reconstruction. A definition of *Farey fractions*, the *Farey rational map*, and remarks on the required bound on the coefficients can be found in [Kornerup and Gregory 1983].

$\sqrt{N/2}$ is larger than the moduli of all coefficients of elements in G. In the same way, we proceed for the set of modular relations $\mathcal{RP} := \{R_p \mid p \in \mathcal{P}\}$.

We now define a property of the set of primes $\mathcal{P}$ which guarantees that the modular normalizations of the A_p, $p \in \mathcal{P}$, can be computed via the GLS-algorithm using the denominators d_p, that the modular results can be lifted, and that this lifting is indeed the normalization of A. This property is essential for the algorithm.

Definition 4.2. Let $G \subseteq \mathbb{Q}[X]$ be a Gröbner basis, and let $d \in \mathbb{Q}[X]$ such that $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is the normalization of A. Moreover, let $R \subseteq A[T_1, \ldots, T_s]$ be the ideal of relations of $\frac{1}{d}\langle G\rangle$ over A.

(1) A prime number p is called *lucky* for A if the following hold:
  (a) A_p is a domain.
  (b) d_p is a non–zero element in the conductor of A_p.
  (c) $\mathrm{LM}(G) = \mathrm{LM}(G_p)$.
  (d) $\mathrm{LM}(R) = \mathrm{LM}(R_p)$.

  Otherwise p is called *unlucky* for A.
(2) A set $\mathcal{P}$ of lucky primes for A is called *sufficiently large* for A if
$$\prod_{p\in\mathcal{P}} p \geq \max\{2 \cdot |c|^2 \mid c \text{ coefficient occuring in } G \text{ or } R\}.$$

From a theoretical point of view, the idea of the algorithm is now as follows: Consider a sufficiently large set $\mathcal{P}$ of lucky primes for A, compute the normalizations of the A_p, $p \in \mathcal{P}$, and lift the results to the normalization of A as aforementioned.

From a practical point of view, we face the problem that the conditions from Definition 4.2 can only be tested a posteriori.

To handle this problem in a randomized way, we fix a natural number t and an arbitrary set of primes $\mathcal{P}$ of cardinality t. For every $p \in \mathcal{P}$, we check whether A_p is a domain respectively whether d_p is a non–zero element. Otherwise, p is unlucky and we delete p from $\mathcal{P}$. Subsequently, having computed $\mathcal{GP}$ and $\mathcal{RP}$, we use the following test to modify $\mathcal{P}$ such that all primes in $\mathcal{P}$ are lucky with high probability:

deleteUnluckyPrimesNormal: *We define an equivalence relation on* $(\mathcal{GP}, \mathcal{RP}, \mathcal{P})$ *by* $(G_p, R_p, p) \sim (G_q, R_q, q) :\Longleftrightarrow \big(\mathrm{LM}(G_p) = \mathrm{LM}(G_q)$ *and* $\mathrm{LM}(R_p) = \mathrm{LM}(R_q)\big)$. *Then the equivalence class of largest cardinality is stored in* $(\mathcal{GP}, \mathcal{RP}, \mathcal{P})$*, the others are deleted.*

Since we do not know a priori whether the equivalence class chosen is indeed lucky and whether it is sufficiently large for A, we proceed in the following way: First, lift the set $\mathcal{GP}$ to $G \subseteq \mathbb{Q}[X]$ (and $\mathcal{RP}$ to R) as described earlier. Since the final verification that $\frac{1}{d}\langle G\rangle \subseteq \mathbb{Q}[X]$ is normal and integral over A can be very expensive, especially if $\frac{1}{d}\langle G\rangle$ is *not* the normalization of A, we prefix a test in positive characteristic:

pTestNormal: *We randomly choose a prime number* $p \notin \mathcal{P}$ *such that* A_p *is a domain,* d_p *is a non–zero element in* A_p*, and* p *does not divide the numerator and denominator of any coefficient occuring in a polynomial in* G*,* R *or* $\{f_1, \ldots, f_r\}$*. The test returns true if* $\frac{1}{d_p}\langle G_p\rangle$ *is the normalization of* A_p *and satisfies the corresponding relation* R_p*, and false otherwise.*

If pTestNormal returns false, then $\mathcal{P}$ is not sufficiently large for A or the equivalence class of prime numbers chosen was unlucky, and hence $\frac{1}{d}\langle G\rangle$ is not the normalization of A. In this case, we enlarge the set $\mathcal{P}$ by t new primes and repeat the whole process. On the other hand, if pTestNormal returns true, then $\frac{1}{d}\langle G\rangle$ is most likely the normalization of A. In this case, we verify the result over the rationals using the following lemma:

Lemma 4.3. *The ring $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is the normalization of A if and only if*

(1) *the elements of $\frac{1}{d}\langle G\rangle$ satisfy the ideal of relations $R \subseteq A[T_1,\dots,T_s]$ obtained by lifting the relations $R_p \subseteq A_p[T_1,\dots,T_s]$ in characteristic p for $p \in \mathcal{P}$, and hence are integral over A, and*
(2) *the Grauert and Remmert criterion is satisfied, that is, $\frac{1}{d}\langle G\rangle$ is normal.*

Proof. If $\frac{1}{d}\langle G\rangle$ is integral over A, then $\frac{1}{d}\langle G\rangle \subseteq \overline{A}$; if it is also normal, then equality holds. $\square$

We summarize modular normalization in Algorithm 3.[3]

Algorithm 3 Modular normalization

Input: $I \subseteq \mathbb{Q}[X]$ a prime ideal.
Output: $G \subseteq \mathbb{Q}[X]$ and $d \in \mathbb{Q}[X]$ such that $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is the normalization of $A = \mathbb{Q}[X]/I$.

compute J, the Jacobian ideal of I;
choose $d \in J$, a non–zero element in A;
choose $\mathcal{P}$, a list of random primes;
$\mathcal{GP} = \emptyset$, $\mathcal{RP} = \emptyset$;
loop
  for $p \in \mathcal{P}$ **do**
    if A_p is not a domain or d_p is zero **then**
      delete p;
    else
      compute G_p, the reduced Gröbner basis such that $\frac{1}{d_p}\langle G_p\rangle \subseteq \mathrm{Q}(A_p)$ is the normalization of A_p, and R_p, the ideal of relations (via the GLS normalization algorithm);
      $\mathcal{GP} = \mathcal{GP} \cup \{G_p\}$, $\mathcal{RP} = \mathcal{RP} \cup \{R_p\}$;
  $(\mathcal{GP}, \mathcal{RP}, \mathcal{P}) =$ deleteUnluckyPrimesNormal$(\mathcal{GP}, \mathcal{RP}, \mathcal{P})$;
  lift $(\mathcal{GP}, \mathcal{RP}, \mathcal{P})$ to $G \subseteq \mathbb{Q}[X]$ and $R \subseteq A[T_1,\dots,T_s]$ via Chinese remaindering and the Farey rational map;
  if pTestNormal(I, d, G, R, P) **then**
    if $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is integral over A **then**
      if $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is normal **then**
        return (G, d);
  enlarge $\mathcal{P}$;

Remark 4.4. In Algorithm 3, the normalizations $\frac{1}{d_p}\langle G_p\rangle$ can be computed in parallel. Furthermore, we can parallelize the final verifications of integrality and normality.

[3]The corresponding procedures are implemented in Singular in the library `modnormal.lib`.

Remark 4.5. Algorithm 3 is also applicable without the final tests (that is, without the verification that $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is integral over A and normal). In this case, the algorithm is probabilistic, that is, the output $\frac{1}{d}\langle G\rangle \subseteq \mathrm{Q}(A)$ is the normalization of A only with high probability. This usually accelerates the algorithm considerably.

Remark 4.6. The computation of the ring structure R of the normalization via lifting of the relations R_p may require a large number of primes. Hence, if the number of cores available is limited, a better choice is to compute the relations R over the rationals. For this approach, the initial ideals of the relations need not be tested in pTestNormal.

5. Timings

We compare the GLS normalization algorithm[4] (denoted in the tables by `normal`) with Algorithm 2 from Section 3 (`locNormal`) and Algorithm 3 from Section 4 (`modNormal`). For all modular computations, we use the simplified algorithm as specified in Remark 4.6. Note that at this writing, modularized versions of `locNormal` have not yet been implemented.

In many cases, it turns out that the final verification that the result is indeed the normalization is a time consuming step of `modNormal`. To quantify the improvement of computation times by omitting the verification, we give timings for the resulting, now probabilistic, version of Algorithm 3 (denoted by `modNormal`* in the tables). In all examples computed so far, the result of the probabilistic algorithm is indeed correct.

All implementations are available in the Singular libraries `modnormal.lib` and `locnormal.lib`. All timings are in seconds on an AMD Opteron 6174 machine with 48 cores, 2.2GHz, and 128GB of RAM running a Linux operating system. Computations which did not finish within 2000 seconds are marked by a dash. The maximum number of cores used is written in square brackets. For the single core version of `modNormal`, we indicate the number of primes used by the algorithm in brackets.

Fast modular computation of associated primes has only been implemented in Singular so far for the zero dimensional case. For this reason, as computation of the associated primes is required by the local approach, we first focus on the case of curves. The plane curves defined by the equations

$$f_{1,k} = \left(x^{k+1} + y^{k+1} + z^{k+1}\right)^2 - 4\left(x^{k+1}y^{k+1} + y^{k+1}z^{k+1} + z^{k+1}x^{k+1}\right)$$

were constructed in [Hirano 1992], and have $3\,(k+1)$ singularities of type A_k. The timings for $z = 3x - 2y + 1$ (to have all singularities of the projective curve in the affine chart) and $k = 2, ..., 5$ are:

	$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$
`normal[1]`	.34	14	–	–
`locNormal[1]`	.57	2.0	2.1	38
`locNormal[20]`	.42	1.3	1.4	11
`modNormal[1]`	4.4 (3)	73 (4)	250 (5)	–
`modNormal[10]`	4.1	68	240	–
`modNormal[1]`*	.57 (3)	7.4 (4)	11 (5)	–
`modNormal[10]`*	.31	2.1	2.5	–

[4]We use the implementation available in Singular in the library `normal.lib`.

Both the local and the probabilistic modular approach have a better performance than the GLS algorithm, and they improve further in their parallel versions. The modular algorithm with final verification is slower, but can still handle much bigger examples than GLS.

Timings for the plane curves defined by

$$
\begin{aligned}
f_{2,k} &= ((x-1)^k - y^3)((x+1)^k - y^3)(x^k - y^3)((x-2)^k - y^3)((x+2)^k - y^3) + y^{15},\\
f_3 &= x^{10} + y^{10} + (x-2y+1)^{10} + 2(x^5(x-2y+1)^5 - x^5y^5 + y^5(x-2y+1)^5),\\
f_4 &= (y^5 + 2x^8)(y^3 + 7(x-1)^4)((y+5)^7 + 2x^{12}) + y^{11},\\
f_5 &= 9127158539954x^{10} + 3212722859346x^8y^2 + 228715574724x^6y^4\\
&\quad - 34263110700x^4y^6 - 5431439286x^2y^8 - 201803238y^{10} - 134266087241x^8\\
&\quad - 15052058268x^6y^2 + 12024807786x^4y^4 + 506101284x^2y^6 - 202172841y^8\\
&\quad + 761328152x^6 - 128361096x^4y^2 + 47970216x^2y^4 - 6697080y^6\\
&\quad - 2042158x^4 + 660492x^2y^2 - 84366y^4 + 2494x^2 - 474y^2 - 1,
\end{aligned}
$$

are presented in the following table:

	$f_{2,7}$	$f_{2,8}$	$f_{2,9}$	f_3	f_4	f_5
`normal[1]`	7.7	12	383	–	474	1620
`locNormal[1]`	4.4	116	118	1.9	19	1.2
`locNormal[20]`	1.4	30	31	1.4	18	.93
`modNormal[1]`	38 (3)	69 (3)	146 (3)	142 (3)	–	50 (8)
`modNormal[10]`	38	69	146	84	–	43
`modNormal[1]`*	.70 (3)	1.2 (3)	1.2 (3)	88 (3)	9.8 (3)	7.0 (8)
`modNormal[10]`*	.47	.70	.74	30	4.7	.98

As described in [Böhm et al. 2011], in the case of curves, the performance of the local algorithm will be improved significantly by computing the local normalizations via an algorithm using Hensel lifting and Puiseux series.

In the next table, we consider surfaces in $\mathbb{A}^3$ cut out by

$$
\begin{aligned}
f_{6,k} &= xy(x-y)(x+y)(y-1)z + (x^k - y^2)(x^{10} - (y-1)^2),\\
f_{7,k} &= z^2 - (y^2 - 1234x^3)^k(15791x^2 - y^3)(1231y^2 - x^2(x+158))(1357y^5 - 3x^{11}),\\
f_8 &= z^5 - ((13x-17y)(5x^2-7y^3)(3x^3-2y^2)(19y^2-23x^2(x+29)))^2.
\end{aligned}
$$

We omit the verification of the modular algorithm, as this step is too time consuming.

	$f_{6,11}$	$f_{6,12}$	$f_{6,13}$	$f_{7,2}$	$f_{7,3}$	f_8
`normal[1]`	2.6	11	6.4	–	–	–
`locNormal[1]`	.25	.26	.29	80	113	70
`locNormal[20]`	.21	.22	.24	80	113	70
`modNormal[1]`*	2.2 (2)	.60 (2)	.78 (2)	12 (5)	17 (5)	2.3 (2)
`modNormal[10]`*	1.5	.52	.67	3.5	4.7	1.7

Modular primary decomposition in general, which will improve the performance of the local approach, will be available in SINGULAR in the near future.

Timings for the curves in $\mathbb{A}^3$ defined by the ideals

$$I_{9,k} = \left\langle z^3 - (19y^2 - 23x^2(x+29))^2, x^3 - (11y^2 - 13z^2(z+1))^k \right\rangle$$

and the surface in $\mathbb{A}^4$ defined by

$$I_{10} = \left\langle z^2 - (y^3 - 123456w^2)(15791x^2 - y^3)^2, wz - (1231y^2 - x(111x+158)) \right\rangle$$

are given in the following table:

	$I_{9,1}$	$I_{9,2}$	I_{10}
`normal[1]`	3.2	–	150
`locNormal[1]`	4.2	36	83
`locNormal[20]`	4.1	35	82
`modNormal[1]`	–	–	28 (4)
`modNormal[10]`	–	–	14
`modNormal[1]`*	8.9 (5)	–	8.4 (4)
`modNormal[10]`*	2.1	–	2.5

To summarize, both the local and the probabilistic modular approach provide a significant improvement over the GLS algorithm in computation times and size of the examples covered. The probabilistic method is very stable in the sense that it produces the correct result. As usual, the verification step in the modular setup is the most time consuming task, and a refinement of this step will be the focus of further research. The modular technique parallelizes completely, the local approach parallelizes best if the complexity distributes evenly over the minimal strata of the singular locus. In general, the localization technique, even when not run in parallel, is a major improvement to the GLS algorithm, especially in the case of curves, where the local contributions can also be obtained by other normalization techniques as described in [Böhm et al. 2011].

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