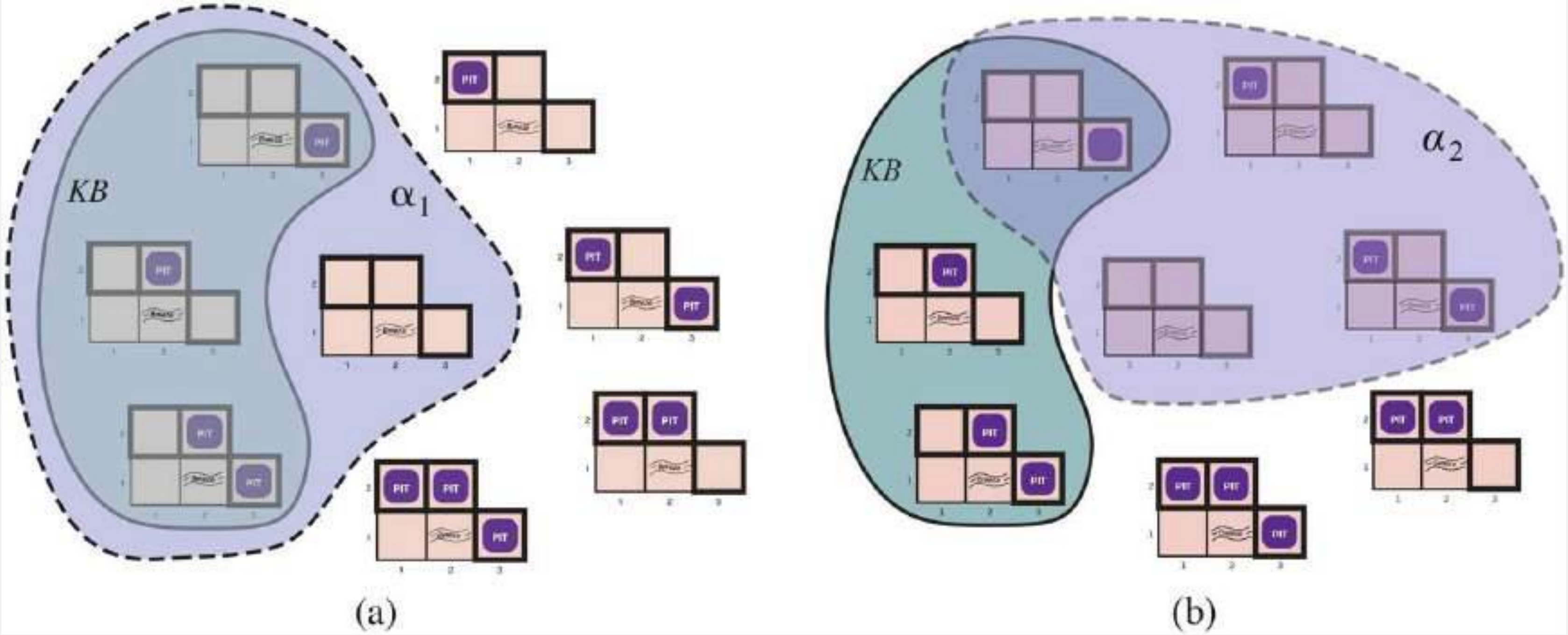
Logic

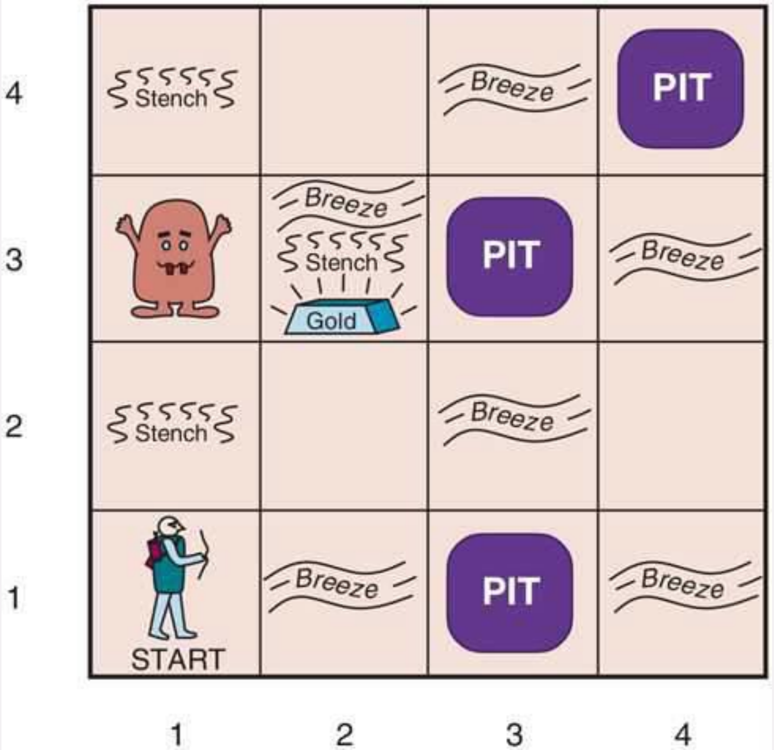
Logical Inference

- α_1 = “There is no pit in [1,2].”
- α_2 = “There is no pit in [2,2].”



(a) (b)

- By inspection:
- In every model in which KB is true, α1 is also true. Hence $KB \models \alpha_1$
- Conclusion: There is no pit in [1,2].
- In some models in which KB is true, α2 is false. Hence, KB does not entail α2
- Agent cannot conclude that there is no pit in [2,2].



Propositional Logic

$$
\begin{array}{rcl}
\textit{Sentence} & \rightarrow & \textit{AtomicSentence} \mid \textit{ComplexSentence} \\
\textit{AtomicSentence} & \rightarrow & \textit{True} \mid \textit{False} \mid P \mid Q \mid R \mid \ldots \\
\textit{ComplexSentence} & \rightarrow & (\,\textit{Sentence}\,) \\
 & \mid & \neg\,\textit{Sentence} \\
 & \mid & \textit{Sentence} \land \textit{Sentence} \\
 & \mid & \textit{Sentence} \lor \textit{Sentence} \\
 & \mid & \textit{Sentence} \;\Rightarrow\; \textit{Sentence} \\
 & \mid & \textit{Sentence} \;\Leftrightarrow\; \textit{Sentence} \\
 & & \\
\textsc{Operator Precedence} & : & \neg, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

Semantics

- The semantics defines the rules for determining the truth of a sentence with respect to a particular model.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

- $\neg P$ is true iff P is false in m.
- $P \wedge Q$ is true iff both P and Q are true in m.
- $P \vee Q$ is true iff either P or Q is true in m.
- $P \Rightarrow Q$ is true unless P is true and Q is false in m.
- $P \Leftrightarrow Q$ is true iff P and Q are both true or both false in m.

Propositional Theorem Proving

- Applying rules of inference directly to the sentences in our knowledge base to construct a proof of the desired sentence without consulting models.
- If the number of models is large but the length of the proof is short, then theorem proving can be more efficient than model checking.

- Logical Equivalence
- Validity
- Satisfiability

Logical Equivalence

- Two sentences α and β are logically equivalent if they are true in the same set of models.

$$\alpha \equiv \beta$$

- Any two sentences α and β are equivalent if and only if each of them entails the other:

$\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$

$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha) \equiv \alpha$	double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$	De Morgan
$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$	De Morgan
$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

Validity

- A sentence is valid if it is true in all models.

$$P \lor \neg P$$

- Valid sentences are also known as tautologies—they are necessarily true.

- Deduction theorem

For any sentences α and β, $(\alpha \models \beta)$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

$$R_1 : \quad \neg P_{1,1} .$$

$$R_2 : \quad B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1}) .$$
$$R_3 : \quad B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1}) .$$

$$R_4 : \quad \neg B_{1,1} .$$
$$R_5 : \quad B_{2,1} .$$

Satisfiability

- A sentence is satisfiable if it is true in, or satisfied by, some model.

- The knowledge base given earlier, ($R_1 \wedge R_2 \wedge R_3 \wedge R_4 \wedge R_5$), is satisfiable because there are three models in which it is true.

- Validity and satisfiability are of course connected: α is valid iff ¬α is unsatisfiable;
- contrapositively, α is satisfiable iff ¬α is not valid.

$\alpha \models \beta$ if and only if the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable

Inference and proofs

- **Modus Ponens rule:** Whenever any sentences of the form α ⇒ β and α are given, then the sentence can be inferred.

$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

- If (WumpusAhead ∧WumpusAlive) ⇒ Shoot and (WumpusAhead ∧WumpusAlive) are given, then Shoot can be inferred.

- **And-Elimination:** from a conjunction, any of the conjuncts can be inferred

$$\frac{\alpha \wedge \beta}{\alpha}$$

- (WumpusAhead ∧WumpusAlive), WumpusAlive can be inferred.

how to prove ¬P1,2, that is, there is no pit in [1,2]?

$$R_1 : \quad \neg P_{1,1} .$$

$$R_2 : \quad B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1}) .$$
$$R_3 : \quad B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1}) .$$

$$R_4 : \quad \neg B_{1,1} .$$
$$R_5 : \quad B_{2,1} .$$

Finding a proof can be more efficient because the proof can ignore irrelevant propositions, no matter how many of them there are!

1. Apply biconditional elimination to R_2 to obtain

$$R_6 : \quad (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}) .$$

2. Apply And-Elimination to R_6 to obtain

$$R_7 : \quad ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}) .$$

3. Logical equivalence for contrapositives gives

$$R_8 : \quad (\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1})) .$$

4. Apply Modus Ponens with R_8 and the percept R_4 (i.e., $\neg B_{1,1}$), to obtain

$$R_9 : \quad \neg(P_{1,2} \vee P_{2,1}) .$$

5. Apply De Morgan's rule, giving the conclusion

$$R_{10} : \quad \neg P_{1,2} \wedge \neg P_{2,1} .$$

That is, neither [1,2] nor [2,1] contains a pit.

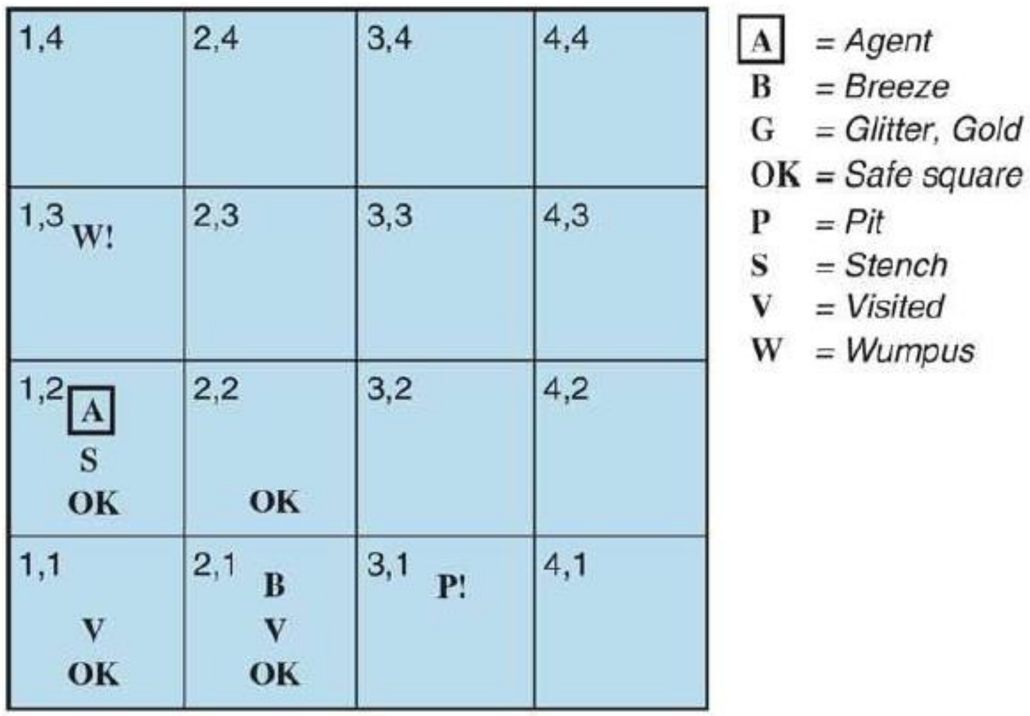
Monotonicity

- The set of entailed sentences can only increase as information is added to the knowledge base.
- For any sentences α and β,

$$\text{If } KB \models \alpha \text{ then } KB \land \beta \models \alpha$$

Proof by Resolution

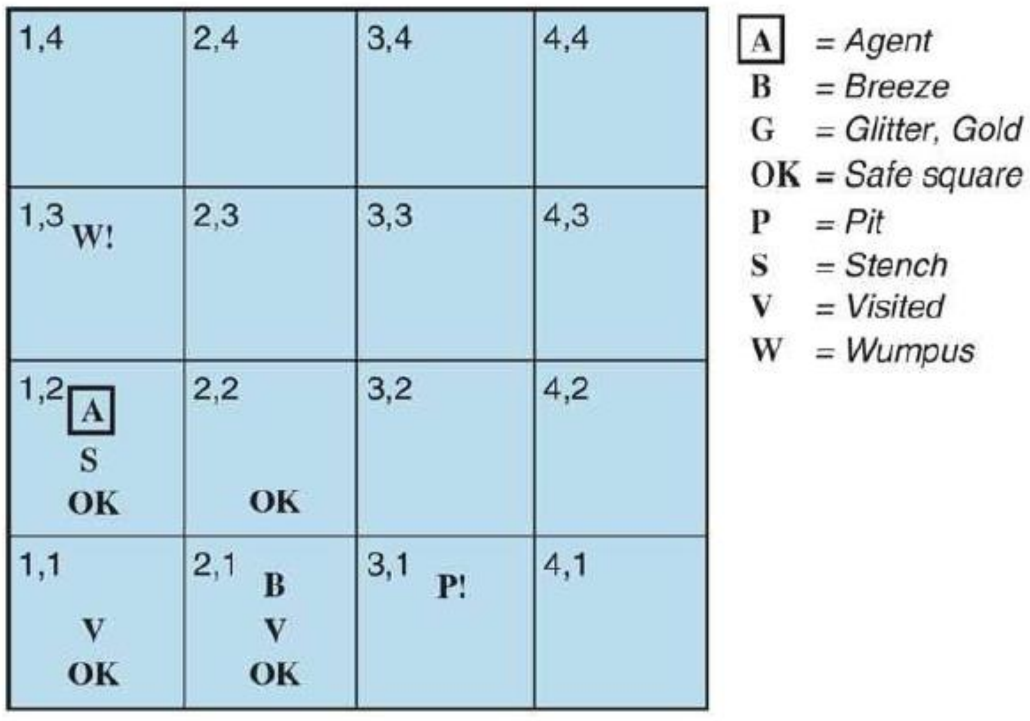
- A single inference rule, **resolution**, that yields a complete inference algorithm when coupled with any complete search algorithm.
- The agent returns from [2,1] to [1,1] and then goes to [1,2], where it perceives a stench, but no breeze.



(a)

Proof by Resolution

- The agent returns from [2,1] to [1,1] and then goes to [1,2], where it perceives a stench, but no breeze. We add the following facts to the knowledge base:
- $R_{11} : \neg B_{1,2}$
- $R_{12} : B_{1,2} \Leftrightarrow (P_{1,1} \lor P_{2,2} \lor P_{1,3})$



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Proof by Resolution

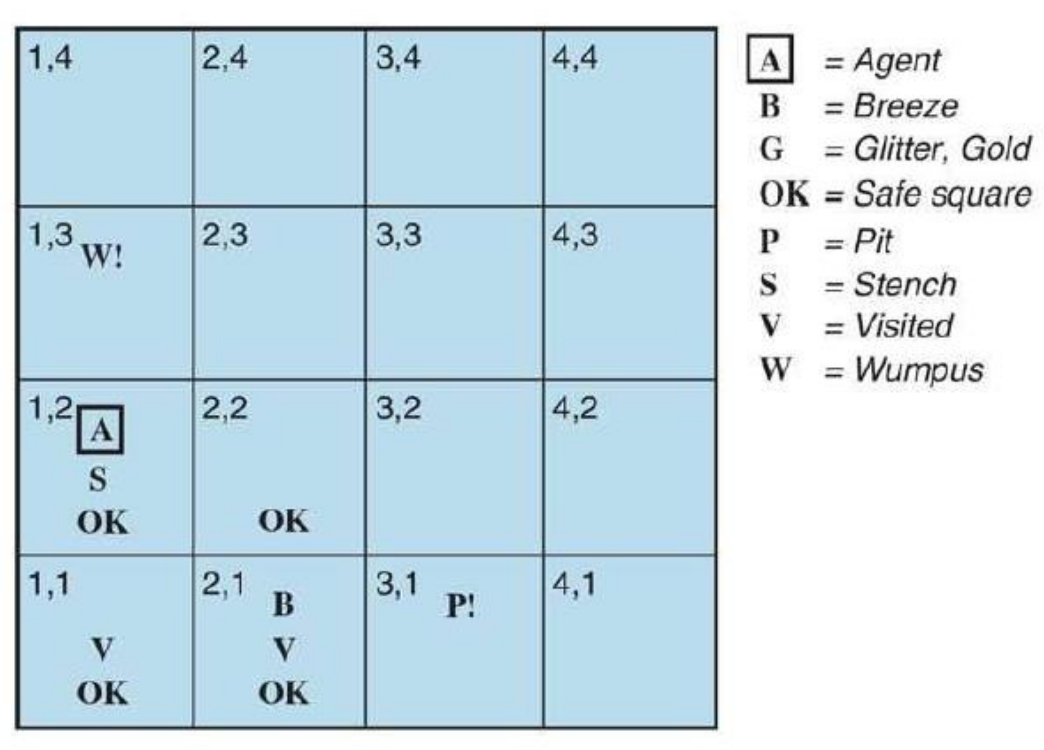
- By the same process that led to R_{10} earlier, we can now derive the absence of pits in [2,2] and [1,3].

$$R_{13} : \neg P_{2,2}$$

$$R_{14} : \neg P_{1,3}$$

- We can also apply biconditional elimination to R_3, followed by Modus Ponens with R_5, to obtain the fact that there is a pit in [1,1], [2,2], or [3,1]

$$R_{15} : P_{1,1} \vee P_{2,2} \vee P_{3,1}$$



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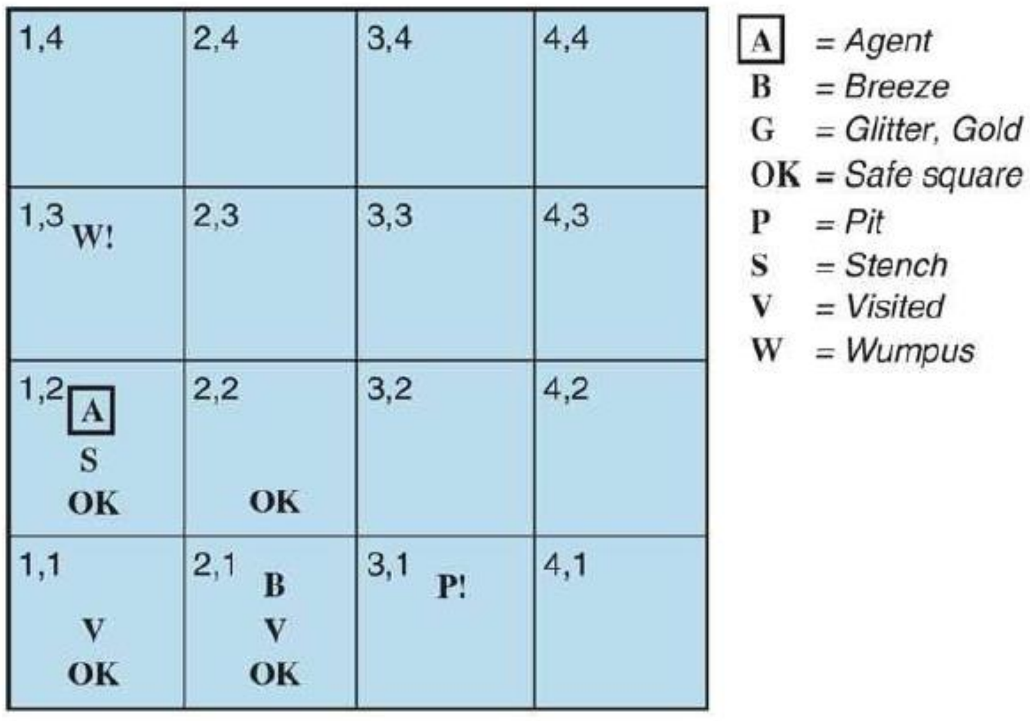
Proof by Resolution

- Now comes the first application of the resolution rule: the literal $\neg P_{2,2}$ in R_{13} resolves with the literal $P_{2,2}$ in R_{15} to give the resolvent

$$R_{16} : P_{1,1} \vee P_{3,1}$$

- Similarly, the literal $\neg P_{1,1}$ in R_1 resolves with the literal $P_{1,1}$ in R_{16} to give

$$R_{17} : P_{3,1}$$



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