Fuzzy Logic

What we learned so far!

Logic (Boolean)

Knowledge Representation (Rules as Boolean expression)

Inferencing: forward or backward chaining

Boolean Logic is the only possibility?

Limitations of Boolean Logic

Variable function value is either 0 or 1.

Vehicle speed measurement

- IF Speed > 60Kmph THEN speed is HIGH
- What IF Speed 60 or 59.9 KMph?
- Uncertainty inherent in speed measurement!

Need a mechanism to apply inferencing in the presence of uncertainty

Uncertainty can arise due to

- Lack of precise measurement
- Incomplete information
- Linguistic variables

Lack of precise measurement

Consider the following scenario

- You are driving a vehicle
- Another vehicle in front of you, moving slowly
- You want to overtake
- Check if lane on right is free
- If yes then turn into right lane and accelerate

An optimal execution of the above would require information about

- speed of the vehicle ahead of you
- distance between the two vehicles

Do we make such precise measurements while driving?

Can we make such precise measurements while driving?

Incomplete information

Consider a hot summer afternoon

You are on the street and very thirsty

Dying for a cold drink

You notice a shop which has the follow signboard

Is it likely that the shop sells cold drinks?

Notice that a part of the signboard is occluded by another poster

Linguistic Variables

Consider that you are learning to cook a new dish from a video posted on YouTube

The chef says “Add salt to taste”

How much salt will you add?

- Half teaspoon, One teaspoon …?

Humans deal with these statements quite easily

Can we write computer programs for dealing with these statements?

Algorithms have to be precise with no ambiguity!

Introducing fuzzy sets

Analogous to crisp sets, fuzzy sets are defined through a characteristic function

These functions are also called **membership functions**

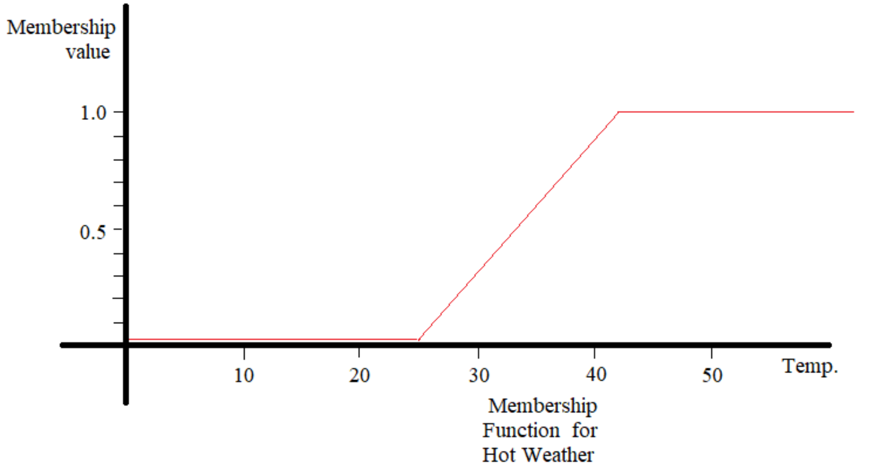
Unlike crisp sets, the membership function can have value in [0, 1]

- any real number in the range 0 to 1

Caution: Do not confuse membership values with probability

Membership values denote a ***degree of belongingness***

Membership function for Hot Weather



Consider interviewing a large number of people

Ask them when do they consider the weather to be hot

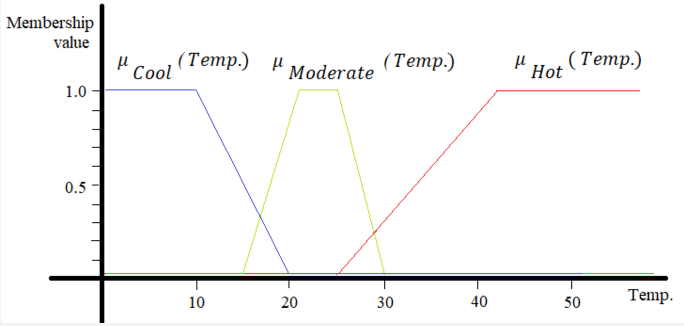
- All agree that T≤25 ℃ is not hot
- A few people say that 26 ℃ is hot
- Some more people say that 27 ℃ is hot
- …
- Everybody says that 42 ℃ is hot

$\mu_{Hot}(Temp. \leq 25) = 0$

$\mu_{Hot}(Temp. \geq 42) = 1$

Other membership values are in [0, 1]

Monotonically increasing from 0 to 1 in this case



Extending the idea

As with Hot Weather, we can create membership functions for

- Cool Weather
- Moderate Weather

The granularity is the choice of the modeler

- We can have further sub-divisions

Notice that the independent variable is Temperature for all

We will generalize it later to have more independent variables

Using piece-wise linear functions is not essential

We can use any function that is logically correct.

Interpreting the membership values

The membership value defines the ***degree of belongingness***

- if $\mu_{Hot}(Temp.) = 0$ then Temp. does not belong the Hot set
- if $\mu_{Hot}(Temp.) = 1$ then Temp. definitely belongs to the Hot set
- if $0 < \mu_{Hot}(Temp.) < 1$ then Temp. belongs to Hot partially

Effectively, the set Hot, is defined by its membership function

Similarly for other sets like Moderate, Cool etc.

Note:

$$\int_{-\infty}^{\infty} \mu_{Hot}(Temp.)\, dTemp = \infty$$

The above shows why membership functions should not be interpreted as probabilities

Operations on fuzzy sets

We can define Union, Intersection, Complement of fuzzy sets in terms of membership functions

- Actually, infinitely many definitions are possible
- We will use the most common definitions which are analogous to those of crisp sets

Given three fuzzy sets A, B and C with $C = A \cup B$

We can define the membership function of C as

$$\mu_C(x) = \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Similarly, given three sets A, B and C with $C = A \cap B$

We can define the membership function of C as

$$\mu_C(x) = \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Given a set A with membership function $\mu_A(x)$ we can define the membership function of complement of A as

$$\mu_{!A}(x) = 1 - \mu_A(x)$$

!A is the symbol for complement of A

Defining set inclusion (subset) and equality

The concepts of set inclusion and set equality for fuzzy sets can be defined using membership functions

Given two fuzzy sets, A and B, with membership functions $\mu_A(x)$ & $\mu_B(x)$

We can define the set inclusion relation as

$$A \subset B \quad iff \quad \forall(x) \;\; \mu_A(x) \leq \; \mu_B(x)$$

Similarly, equality of two sets, A and B, can be defined as

$$A = B \quad iff \quad \forall(x) \;\; \mu_A(x) = \mu_B(x)$$

Theorems for fuzzy sets

In classical (crisp) set theory we learn about the following theorems

Associativity of Union: $A \cup (B \cup C) = (A \cup B) \cup C$

Commutativity of Union: $A \cup B = B \cup A$

Associativity of Intersection: $A \cap (B \cap C) = (A \cap B) \cap C$

Commutativity of Intersection: $A \cap B = B \cap A$

Distributivity of Union with respect to Intersection: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributivity of Intersection with respect to Union: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Existence of a null set Φ such that for any set A: $A \cup \Phi = A$ and $A \cap \Phi = \Phi$

Reflexity of Complementation: $(A^c)^c = A$

De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Law of excluded middle: $A \cup A^c = U$ and $A \cap A^c = \Phi$ where U is the Universal set

Theorems for fuzzy sets contd.

How many of the theorems of crisp sets also hold for fuzzy sets?

Each theorem needs to be proved

Don’t try to use Venn diagrams, they are not applicable to fuzzy sets!

Proving Associativity of Intersection: $A \cap (B \cap C) = (A \cap B) \cap C$

LHS = min($\mu_A(x)$, min ($\mu_B(x)$, $\mu_C(x)$))

RHS = min(min ($\mu_A(x)$, $\mu_B(x)$), $\mu_C(x)$))

Take specific cases like $\mu_A(x) \leq \mu_B(x) \leq \mu_C(x)$

- show that LHS = RHS

Try it for DeMorgan’s law

Try it for Law of Excluded Middle