**AOS 212A – Homework 4**

Shallow water equations in 2D

Due date: Friday, March 4 (end of Week 9)

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## Problem description

We are interested in the solution of the fully nonlinear, single-layer shallow water equation system in rotational form given by

$$\frac{\partial u}{\partial t} - qhv + \frac{\partial B}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + qhu + \frac{\partial B}{\partial y} = 0 \tag{2}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{3}$$

where $h$ is the total water column height, $q = (\zeta + f)/h$ is the barotropic potential vorticity, $\zeta = (\partial v/\partial x - \partial u/\partial y)$ is the vertical vorticity, and $B = gh + (u^2 + v^2)/2$ is the total energy in the water column. This system conserves total mass, energy, and potential enstrophy (i.e. $(1/2)q^2$), even though it is hard to produce a simple discrete set that also conserves all three quantities at the same time. Note that if you want to simulate flow over topography, all you need to do is to replace $h$ by $h + z_b$ (where $z_b(x, y)$ is the height of the bottom surface) in the potential energy term appearing in $B$.

## Numerical approach

You can choose your discretization scheme, but I suggest you use a simple scheme for the time discretization (Euler-Forward works well enough, but you could also use leapfrog or AB3). The best approach for spatial discretization is to use the MAC grid (Arakawa C staggered grid). You will need potential vorticity, and it is best to stagger it with respect to all other variables (see paper mentioned below for more details). As you know by now, how you handle interpolations and other aspects of the spatial discretization is critical to ensure conservation properties on the discrete grid and to keep your code from blowing up due to nonlinear instability. There are two simple schemes proposed by Sadourny (1975) that are widely used (one conserves energy and the other conserves enstrophy). I uploaded the paper to CCLE, and I suggest you choose one of them (conserving potential enstrophy is usually preferred in this system). Before you start, please read at least the first two sections of the paper.

Depending on your choices, you may need a small viscous term to help keep the nonlinear instability in check. If you do so, I suggest you do not worry much about being too careful with the discretization of the viscous term. Finally, code your boundary conditions in a flexible way so that you can shift between free-slip walls and periodic boundary conditions.

Note: this problem set is all about paying attention to details. It is really important that you understand all the interpolations in Sadourny's paper, and it is even more important that you organize your staggered grid carefully and determine your index numbers for each variable. To give you extra motivation to do so, I will require you to include a sketch of your domain with the details of your choices of index numbers for each variable in your report!

### Test case

For a test case, we will study the geostrophic adjustment problem on the domain $0 \le x \le L_x$ and $0 \le y \le L_y$. For initial condition we will prescribe a quiescent state (with $u(t=0) = v(t=0) = 0$ everywhere) and a Gaussian unbalanced height given by

$$h(x, y, t = 0) = H \left[ 1 + \frac{1}{2} \exp \left( - \frac{(x - x_0)^2 + (y - y_0)^2}{a^2} \right) \right]. \tag{4}$$

Here $a$ is the radius of the vortex centered at $(x_0, y_0)$. I suggest you use periodic boundary conditions on all four boundaries, as this is the easiest approach to use here. If you want to use a radiation boundary condition to allow your waves to propagate out of the domain, take a look at Chapter 9 in Durran's book. Note that to implement periodic boundary conditions you can either use ghost nodes or you can simply code you indices in a way that they wrap around the domain (which is clean, but less flexible).

To keep things simple, we will use a test case cooked up by my good friend Andrew, in which a reduced gravity $g = 10^{-3}\,\text{m/s}^2$ is used. The flow depth is $H = 1000\,\text{m}$, and we use $f = 10^{-4}\,\text{s}^{-1}$. The appropriate velocity scale is the phase speed $c = \sqrt{gH} = 1\,\text{m/s}$ and the horizontal length scale is the Rossby radius of deformation $\lambda_R = c/f = 10^4\,\text{m}$. The depth $H$ is the appropriate vertical length scale. You can set $L_x = L_y = 10\lambda_R$, center the disturbance in your domain by setting $x_0 = y_0 = 5\lambda_R$, and set the radius of the disturbance to be $a = 1.25\lambda_R$). Please run your simulation long enough to document the propagation of the original wave and the result of its interaction with the boundaries.

### Choose one more flow (optional)

Here are many things you can actually do without much extra work:

- You can compare energy conserving and enstrophy conserving schemes or you can improve your time discretization;
- You can add $\beta$ effect and/or bottom topography and document what happens;
- You can use radiation boundary conditions to let the waves out of the domain and see if your solution reaches geostrophic balance;
- You can also simulate vortices or jets over topographic features;
- You can add a solid wall and simulate Kelvin waves (If you run the geostrophic adjustment case with free-slip solid walls for a very long time you will see the Kelvin waves – there are other setups that can get you there much faster, though);
- You can probably come up with ideas better than the ones I outlined above...

**Report**

I want you to draft about a short text indicating your choices of discretization, treatment of boundary conditions, etc. Did you include a viscous term? Did you choose to conserve energy or enstrophy? How did energy and enstrophy varied in the simulation? Make sure you document properly your choices and justify them whenever you fell it is appropriate. Please also include a few figures or animations that show you have accomplished the goal of running the test case. Finally, if you do additional simulations, add a few details and one figure or movie documenting that. Do not forget to include the sketch of your domain with your staggered grid organization and the index numbers for all variables. Upload everything as a single .zip or .tgz file to Bruin Learn. PLEASE: spend your time writing your code, studying your choices, running simulations, and having fun. Do not spend too much time on your report. If you use any additional references, please list them as well.