Yufan Xu
yufanxu@g.ucla.edu

Homework 4

AOS 212A
March 10, 2022

2D shallow water equations

Fully nonlinear, single-layer shallow water equation system in rotational form is given by

$$\begin{aligned} \frac{\partial u}{\partial t} - qhv + \frac{\partial B}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + qhu + \frac{\partial B}{\partial y} &= 0 \\ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0, \end{aligned} \tag{1}$$

where h is the total water column height (can later be replaced by a bottom topography $h + z_b(x, y)$); $q = (\zeta + f)/h$ is the barotropic potential vorticity; $\zeta = (\partial v/\partial x - \partial u/\partial y)$ is the vertical vorticity; and $B = gh + (u^2 + v^2)/2$ is the total energy. This system conserves total mass, energy, and potential enstrophy, $(1/2)q^2$.

Numerical Schemes

Spatial discretization: MAC grid (Arakawa C staggered grid)

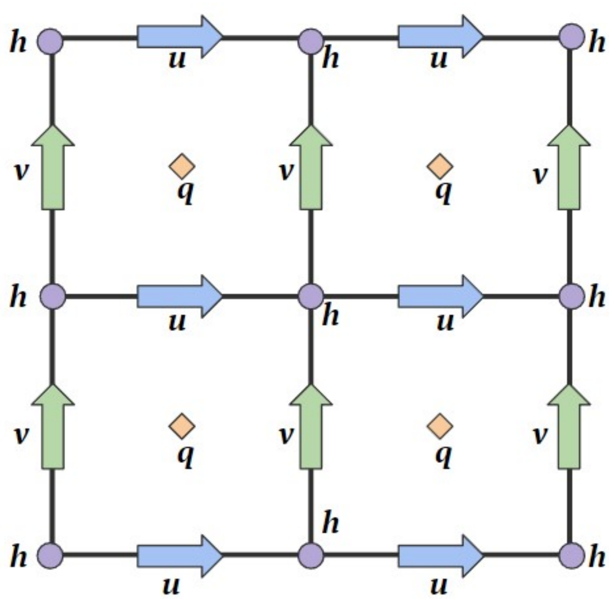

Figure 1: Arakawa C grid, modified from Sadourny 1974.

Time discretization: Euler-Forward

Interpolations:

$$\begin{aligned}\delta_x q(x,y) &= \frac{1}{\Delta x}\left[q(x+\frac{\Delta x}{2},y) - q(x-\frac{\Delta x}{2},y)\right] \\ \delta_y q(x,y) &= \frac{1}{\Delta y}\left[q(x,y+\frac{\Delta y}{2}) - q(x,y-\frac{\Delta y}{2})\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\overline{q}^x(x,y) &= \frac{1}{2}\left[q(x+\frac{\Delta x}{2},y) - q(x-\frac{\Delta x}{2},y)\right] \\ \overline{q}^y(x,y) &= \frac{1}{2}\left[q(x,y+\frac{\Delta y}{2}) - q(x,y-\frac{\Delta y}{2})\right]\end{aligned} \tag{3}$$

Redefinition of variables:

$$\begin{aligned}U &= \overline{h}^x u, \quad V = \overline{h}^y v \\ B &= gh + \frac{1}{2}\left(\overline{u^2}^x + \overline{v^2}^y\right) \\ q &= \left(\delta_x v - \delta_y u + f\right)/\overline{\overline{h}^x}^y\end{aligned} \tag{4}$$

Enstrophy-conserving system

$$\begin{aligned}\frac{\partial u}{\partial t} - \overline{q}^y \overline{\overline{V}^x}^y + \delta_x B = 0 \\ \frac{\partial v}{\partial t} + \overline{q}^x \overline{\overline{U}^y}^x + \delta_y B = 0 \\ \frac{\partial h}{\partial t} + \delta_x U + \delta_y V = 0\end{aligned} \tag{5}$$

Energy

$$E = \frac{1}{2}\sum\left(gh^2 + gh\overline{u^2}^x + gh\overline{v^2}^y\right) = \sum B \tag{6}$$

Enstrophy

$$Z = \frac{1}{2}\sum q^2 \overline{\overline{h}^x}^y \tag{7}$$

Initial condition for the test case, the height h is given as

$$h(x,y,t=0) = H\left[1 + \frac{1}{2}\exp\left(-\frac{(x-x_0)^2 + (y-y_0)^2}{a^2}\right)\right] \tag{8}$$