Problem Set 4 – Second Order Statistics of Random Vectors, Gaussian Random Vector

Problem 1:

In the course lecture you learned that the correlation and covariance matrices are positive semi-definite. In this problem, you are required to prove a number of simple characteristics relating positive semi-definite matrices.

Let B be a positive semi-definite matrix. Prove the following characteristics:

1. The elements on the diagonal of B are non-negative.
2. If B is positive definite, then B is invertible.
3. The eigenvalues of B are non-negative.

Problem 2:

Given $X \sim N(\mu,\sigma^2)$. Find the characteristic function of X.

Hint: Start by calculating the characteristic function of Z, a standard Gaussian random variable, $Z \sim N(0,1)$.

Problem 3:

$\underline{X}$ is a Gaussian random vector with expectation $\underline{\eta}$ and covariance matrix C.

For constants $\underline{a}$ and b, prove that:

$$P\left(\underline{a}^T \underline{X} \geq b\right) = Q\left(\frac{b-\underline{a}^T \underline{\eta}}{\sqrt{\underline{a}^T C \underline{a}}}\right)$$

$$where\ Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-S^2/2}\, dS$$

Problem 4:

Consider $X \sim N(0,\sigma^2)$, and let us denote $W = |X|$, $B = sign(X) = \begin{cases} 1 & X \geq 0 \\ -1 & X < 0 \end{cases}$.

1. Find the marginal distributions of W and B.
2. Prove that W and B are independent of each other.

Problem 5:

Given a Gaussian random vector with expected value 0 and variance 1, $X \sim N(0,1)$

The following random variables are defined:

$$Y = \cos(X),\ Z = \sin(X)$$

And also: $W = Y^2 + Z^2$.

1. Find $E(W)$.
2. Find $E(Z^2)$.
3. Find $E(Z^2 \mid Z \geq 0)$. Hint: use symmetry.
4. Can one use symmetry in order to calculate $E(Y^2 \mid Y \geq 0)$? Explain.

Problem 6:

Given is a sequence of N i.i.d standard Gaussian random variables $\{V_0, V_1, ..., V_{N-1}\}$.

Let us denote a new sequence of random variables in the following way: $X_{i+1} = \beta X_i + V_i \quad i = 0,1,2,3,..,N-1$ (N in this question is a deterministic constant).

Given: $X_0 \sim N\left(0,\sigma^2\right)$ independent of the sequence and $\beta \neq 0$.

1. Is the vector $\left[V_0 \quad V_1 \quad \cdots \quad V_{N-1}\right]^T$ a Gaussian random vector?

2. Is the vector $\underline{X} \equiv \left[X_0 \quad X_1 \quad \cdots \quad X_N\right]^T$ a Gaussian random vector? Are the elements of the vector independent?

3. Find the expected value and variance of X_i.