Problem Set 3 – Random Vectors, Conditional Expectation

Problem 1:

Consider two random variables X, Y with joint PDF as follows:

$$f_{XY}(x,y)=\begin{cases} A\cdot x\cdot y & \begin{pmatrix} -1\le x<0, & -1\le y<0 \\ 0\le x\le 1, & 0\le y\le 1 \end{pmatrix} \\ 0 & elsewhere \end{cases}$$

1. Find A.
2. Find $f_X(x)$.

Problem 2:

Consider two random variables X, Y with joint CDF as follows:

$$F_{XY}(x,y)=(1-e^{-x})(1-e^{-y})u(x)u(y)$$

1. What is the joint PDF, $f_{XY}(x,y)$?
2. What are the marginal PDFs, $f_X(x), f_Y(y)$?
3. Are X, Y independent?

Now, let us define the random variables W,Z as follows:

$$W = X + Y$$
$$Z = X - Y$$

4. Find $f_{WZ}(w,z)$.
5. Are W,Z independent?

Problem 3:

The point A is drawn uniformly from the section [0,1]. Afterwards, the point B is drawn uniformly in the section [A,1]. Find the expected value and the variance of the distance between A and B.

Problem 4:

A sequence of N **continuous** random variables is given, $X_1,..,X_N$, which all have the same PDF, $f_X(x)$. The random variables are mutually independent. The following two random variables are defined:

$$W = \underset{i}{Min}\{X_i\} \quad ; \quad Z = \underset{i}{Max}\{X_i\}$$

Find the PDF of W and of Z.

Problem 5:

Let $X_1, X_2, \cdots X_n$ be a sequence of i.i.d random variables with expected value 0 and variance 1, and let M be a random variables that takes the values 1,2,…n , each with probability $\frac{1}{n}$, and independent of $X_1, X_2, \cdots X_n$. Let us define:

$$Y = \sum_{i=1}^{M} X_i$$

1. Find the expectation of Y given X_1.
2. Find the variance of Y given X_1.
3. Find the variance of Y.

Problem 6:

Consider a continuous random variable with uniform distribution:

$$X \sim U[-\pi, \pi)$$

Let us define the following random variables:

$$Y = \cos(X)\ ,\ Z = \sin(X)$$

Moreover, we define $W = Y^2 + Z^2$.

1. Find $E(W)$.
2. Find $E(Y^2)$.
3. Find $E(Y^2 \mid Y \geq 0)$.

Problem 7:

In this problem, we will deal with a general linear transformation of random vectors.

1. Given is a linear transformation $g : R^n \to R^n$, which may be expressed by means of $n \times n$ matrix T, such that $\vec{Y} = g\left(\vec{X}\right) = T\vec{X}$. In addition, it is given that the inverse transformation is known (namely, the inverse of the matrix is known). Prove that the PDF of the random vector $\underline{Y}$, as defined by this transformation, is given by:

$$f_{\vec{Y}}\left(\vec{y}\right) = \frac{1}{|\det T|} f_{\vec{X}}\left(T^{-1}\vec{y}\right)$$

2. Now assume that a specific transformation is known: $\begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \end{cases}$

   Find the joint PDF of the random variables Y_1 and Y_2.
   Use this result to prove that for two independent random variables, the PDF of their sum is the convolution of the separate PDFs.

   (**<u>Hint:</u>** You may and should use Y_2, and you must assume that X_1 and X_2 are independent.)