Problem Set 2 – Moments and Moment-generating functions

Problem 1:

Find the moment-generating function $M_X(s) = E\left(e^{sX}\right)$ and use it to calculate the expected value and variance of the following random variables:

1. Binomial random variable:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0,1,\ldots,n$$

2. Exponential random variable: $f_X(x) = \lambda e^{-\lambda x} u(x)$

Problem 2:

Prove the following preposition (expected value of a function of a random variable):
If X is a random variable and g is a continuous and increasing function, then

$$E\big(g(X)\big) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Problem 3:

X is a random variable with Moment-generating function $M_X(s)$. Let $X_1, X_2, \dots, X_n$ be a sequence of n i.i.d. random variables with the same distribution as X. Define $Y = \sum_{i=1}^{n} X_i$. Find $M_Y(s)$, the moment generating function of Y.

Problem 4:

Definition: A random variable Y distributes log-normal with parameters μ and σ^2 (marked as $Y \sim LN(\mu,\sigma^2)$) iff its PDF satisfies:

$$f_Y(y) = \frac{1}{y \cdot \sqrt{2\pi\sigma^2}} e^{-\frac{\left(\ln(Y)-\mu\right)^2}{2\sigma^2}} \cdot u(y)$$

If $\mu = 0$ and $\sigma^2 = 1$, Y is called standard log-normal.

1.
   a. Show that if $X \sim N(\mu,\sigma^2)$, then $e^X \sim LN(\mu,\sigma^2)$.
   b. Show that if $Y \sim LN(\mu,\sigma^2)$, then $Y^r \sim LN(r\mu, r^2\sigma^2)$.

2. Show that the sequence of moments of the random variable $Y_1 \sim LN(0,1)$ satisfies:

$$a_k = E\left\{Y_1^k\right\} = e^{\frac{k^2}{2}} \qquad\qquad k = 1,2....$$

(Hint: Use the moment-generating function of a standard normal random variable)