Solution of Problem Set 3

Problem 1:

1. We will demand that the double integral of $f_{XY}(x,y)$ throughout R^2 result in 1:

$$1=\iint_{R^2} f_{XY}(x,y)dxdy = A\left[\int_{-1}^{0}\int_{-1}^{0} xydxdy+\int_{0}^{1}\int_{0}^{1} xydxdy\right]=2A\left(\int_0^1 xdx\right)\left(\int_0^1 ydy\right)=\frac{A}{2}$$
$$\Rightarrow A=2$$

2.

$$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)dy=\begin{cases}\int_{-1}^{0} 2xydy & -1<x<0\\ \int_{0}^{1} 2xydy & 0<x<1\\ 0 & \text{o.w}\end{cases}$$

$$=\begin{cases}-x & -1<x<0\\ x & 0<x<1\\ 0 & \text{o.w}\end{cases}$$

$$=\begin{cases}|x| & |x|<1\\ 0 & \text{o.w}\end{cases}$$

Problem 2:

1. $$f_{XY}(x,y)=\frac{\partial^2}{\partial x\partial y}F_{XY}(x,y)=e^{-x}e^{-y}u(x)u(y)$$

   The CDF is continuous throughout all R^2 and, thus, the PDF does not include delta functions.

2. It can be seen that the PDF may be separated: $f_{XY}(x,y)=f_X(x)f_Y(y)$

$$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)dy=\int_{-\infty}^{\infty} f_X(x)f_Y(y)dy=f_X(x)\underbrace{\int_{-\infty}^{\infty} f_Y(y)dy}_{=1}=f_X(x)=e^{-x}u(x)$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)dx=e^{-y}u(y)$$

3. Since $f_{XY}(x,y)=f_X(x)f_Y(y)$, X and Y are independent.
4. The inverse transformation is:

$$X=\tfrac{1}{2}(W+Z)$$
$$Y=\tfrac{1}{2}(W-Z)$$

   And the Jacobian is:

$$|J| = \left|\frac{\partial(W,Z)}{\partial(X,Y)}\right| = \left|\det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right| = 2$$

Therefore:

$$f_{WZ}(w,z) = \frac{f_{XY}\left(\frac{1}{2}(w+z),\frac{1}{2}(w-z)\right)}{|J|} = \tfrac{1}{2}e^{-w}u(w+z)u(w-z)$$

$$= \begin{cases} \frac{1}{2}\mathrm{e}^{-w} & w>0, \quad -w<z<w \\ 0 & \text{o.w} \end{cases}$$

Notice that:

$$u(w+z) = u\left(\tfrac{1}{2}(w+z)\right), u(w-z) = u\left(\tfrac{1}{2}(w-z)\right)$$

5. We will start by finding the marginal PDFs and then we will check whether they are independent:

$$f_W(w) = \int_{-\infty}^{\infty} f_{WZ}(w,z)dz = \int_{-w}^{w} \tfrac{1}{2}e^{-w}dz = we^{-w}u(w)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{WZ}(w,z)dw = \tfrac{1}{2}\int_{|z|}^{\infty} e^{-w}dw = \tfrac{1}{2}e^{-|z|}$$

$$\Rightarrow f_W(w)f_Z(z) = \tfrac{1}{2}we^{-w}e^{-|z|}u(w) \neq f_{WZ}(w,z)$$

From here, W and Z are not independent.

Problem 3:

We will start by finding the expectation of the distance given A, and then we will reach the expectation on A using the smoothing theorem:

$$E(B-A\,|\,A) = E(B\,|\,A) - A = \frac{A+1}{2} - A = \frac{1-A}{2}$$

$$Var(B-A\,|\,A) = Var(B\,|\,A) - 0 = \frac{(1-A)^2}{12}$$

$$E((B-A)^2\,|\,A) = Var(B-A\,|\,A) + \left(E(B-A\,|\,A)\right)^2 = \frac{(1-A)^2}{3}$$

To get rid of the conditioning, let us use the smoothing theorem:

$$E(B-A) = E(E(B-A\,|\,A)) = E\left(\frac{1-A}{2}\right) = \frac{1-E(A)}{2} = \frac{1-\frac{1}{2}}{2} = \frac{1}{4}$$

$$E((B-A)^2) = E(E((B-A)^2\,|\,A)) = E\left(\frac{(1-A)^2}{3}\right) = \frac{E(A^2)-2E(A)+1}{3} = \frac{\frac{1}{3}-1+1}{3} = \frac{1}{9}$$

$$\Rightarrow Var(B-A) = \frac{1}{9} - \left(\frac{1}{4}\right)^2 = \frac{7}{144}$$

Consider this – what's not right with the following solution?

$$Var(B-A)=E\left[Var(B-A|A)\right]=E\left[\frac{(1-A)^2}{12}\right]=\frac{1}{4}E\left(\frac{(1-A)^2}{3}\right)=\frac{1}{4}\frac{1}{9}=\frac{1}{36}$$

<u>Problem 4:</u>

1. First, let us find the distribution function – and from it, we will derivate the density function:

$$F_Z(z)=\Pr(Z\le z)=\Pr\left(\max_i X_i\le z\right)=\Pr(X_1\le z,X_2\le z,\dots,X_N\le z)$$
$$=\Pr(X_1\le z)\cdot\Pr(X_2\le z)\cdots\Pr(X_N\le z)=\left(F_X(z)\right)^N$$

$$f_Z(z)=\frac{dF_Z(z)}{dz}=N\cdot\left(F_X(z)\right)^{N-1}f_X(z)$$

$$F_W(w)=\Pr(W\le w)=\Pr\left(\min_i X_i\le w\right)=1-\Pr\left(\min_i X_i>w\right)$$
$$=1-\Pr(X_1>w,X_2>w,\dots,X_N>w)$$
$$=1-\Pr(X_1>w)\cdots\Pr(X_N>w)=1-\left(1-F_X(w)\right)^N$$

$$f_Z(z)=\frac{dF_W(w)}{dw}=N\cdot\left(1-F_X(w)\right)^{N-1}f_X(w)$$

<u>Problem 5:</u>

1. We will start by calculating given M:

$$E(Y\mid X_1=x_1,M=m)=E(x_1+X_2+\cdots+X_m\mid X_1=x_1,M=m)=$$
$$x_1+E(X_2+\cdots+X_m\mid X_1=x_1,M=m)\overset{(a)}{=}x_1+E\left(X_2\right)+\cdots+E\left(X_2\right)=x_1$$
$$\Rightarrow E(Y\mid X_1)=E(E(Y\mid X_1,M)\mid X_1)=E(X_1\mid X_1)=X_1$$

where in the equality (a) we based on the fact that the variables $X_2,\dots,X_m$ are i.i.d.

2. Again, let us first calculate given M:

$$Var(Y\mid X_1=x_1,M=m)=Var(x_1+X_2+\cdots+X_m\mid X_1=x_1,M=m)$$
$$\overset{(a)}{=}Var(X_2+\cdots+X_m\mid X_1=x_1,M=m)\overset{(b)}{=}Var\left(X_2\right)+\cdots+Var\left(X_m\right)=m-1$$

where (a) results from the fact that adding a constant has no effect on the variance, and (b) since the variables $X_2,\dots,X_m$ are i.i.d.

Now, let us use:

$$Var\left(Y \mid X_1, M\right) = E\left(Y^2 \mid X_1, M\right) + E^2\left(Y \mid X_1, M\right)$$

and the result from the last section so as to obtain:

$$E(Y^2 \mid X_1, M) = M - 1 + X_1^{\ 2}$$

$$\Rightarrow E(Y^2 \mid X_1) = E(M - 1 + X_1^{\ 2} \mid X_1) = \tfrac{n+1}{2} - 1 + X_1^{\ 2} = \tfrac{n-1}{2} + X_1^{\ 2}$$

$$\Rightarrow Var(Y \mid X_1) = E(Y^2 \mid X_1) - E^2(Y \mid X_1) = \tfrac{n-1}{2} + X_1^{\ 2} - X_1^{\ 2} = \tfrac{n-1}{2}$$

3. We will start by using the smoothing theorem to calculate the first two moments of Y:

$$E(Y \mid X_1) = X_1$$

$$\Rightarrow E(Y) = E(X_1) = 0$$

$$E(Y^2 \mid X_1) = \tfrac{n-1}{2} + X_1^{\ 2}$$

$$\Rightarrow E(Y^2) = E\left(\tfrac{n-1}{2} + X_1^{\ 2}\right) = \tfrac{n-1}{2} + 1 = \tfrac{n+1}{2}$$

$$\Rightarrow Var(Y) = \tfrac{n+1}{2} - 0^2 = \tfrac{n+1}{2}$$

Problem 6:

1. Let us find according to the definition:

$$E\{W\} = E\{Y^2 + Z^2\} = E\{\cos^2(X) + \sin^2(X)\} =_{(a)} E\{1\} =_{(b)} 1$$

where (a) results from a basic trigonometric identity, and (b) is true since the expectation of a deterministic number is the number itself.

2. The integral of $\cos^2(X)$ throughout a whole period is equal to the integral of $\cos^2(Y)$:

$$\int_{-\pi}^{\pi} \cos^2(x)dx \overset{y=x+\pi/2}{=} \int_{-\pi+\pi/2}^{\pi+\pi/2} \cos^2(y-\pi/2)dy = \int_{-\pi+\pi/2}^{\pi+\pi/2} \sin^2(y)dy = \int_{-\pi}^{\pi} \sin^2(y)dy$$

And, thus:

$$E\{Y^2\} = E\{\cos^2(X)\} = \int_{-\infty}^{+\infty} \cos^2(x) f_X(x)dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos^2(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sin^2(x)dx = E\{\sin^2(X)\} = E\{Z^2\}$$

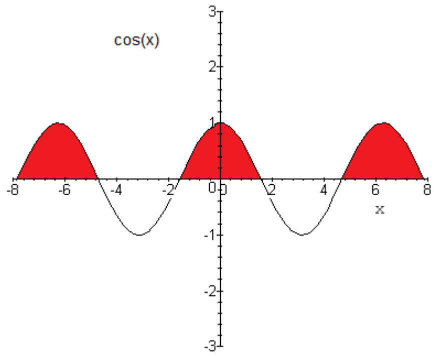
From here we get:

$$E\{Y^2\} = \frac{E\{Y^2\} + E\{Z^2\}}{2} = \frac{E\{Y^2 + Z^2\}}{2} = \frac{E\{1\}}{2} = \frac{1}{2}$$

3. Let us find:

$$E\left\{Y^2 \middle| Y \ge 0\right\} = E\left\{\cos^2(X) \middle| \cos(X) \ge 0\right\}$$

We will make use of the symmetry and periodicity of the cosine function:



$$E\{\cos^2(X)\,|\,\cos(X)\ge 0\} = E\{\cos^2(\underbrace{X+\pi}_{X'})\,|\,\cos(X+\pi)\le 0\} = E\{\cos^2(X')\,|\,\cos(X')\le 0\}$$

Since both X and X' distribute uniformly throughout a whole period, $\cos\left(X'\right)$ distributes like $\cos\left(X\right)$:

$$E\{\cos^2(X)\,|\,\cos(X)\ge 0\} = E\{\cos^2(X)\,|\,\cos(X)\le 0\}$$

Now we will use the law of total probability:

$$E\{\cos^2(X)\} = P(\cos(X)\ge 0)\cdot E\{\cos^2(X)\,|\,\cos(X)\ge 0\} + P(\cos(X)\le 0)\cdot E\{\cos^2(X)\,|\,\cos(X)\le 0\} =$$
$$= P(\cos(X)\ge 0)E\{\cos^2(X)\,|\,\cos(X)\ge 0\} + P(\cos(X)\le 0)E\{\cos^2(X)\,|\,\cos(X)\le 0\}$$
$$= E\{\cos^2(X)\,|\,\cos(X)\ge 0\}$$

Thus, all in all:

$$E\{Y^2\,|\,Y\ge 0\} = E\{Y^2\} = \frac{1}{2}$$

Problem 7:

1. We will use the proposition of transformation of random vectors for a linear transformation. Clearly, the Jacobian of this transformation is simply the determinant of the matrix $J\left(g^{-1}(\underline{y})\right) = T^{-1}$.

   This can easily be seen, since for a specific x_i:

   $$x_i = T_{i1}^{-1}y_1 + T_{i2}^{-1}y_2 + T_{i3}^{-1}y_3 + T_{i4}^{-1}y_4 + \dots$$

   Thus, the derivative of x_i by y_j will yield the element T_{ij}^{-1} in the matrix. Thus, if we proposition this in the above formula, when $\det\left(T^{-1}\right) = 1/\det(T)$, we get the following result:

   $$f_{\vec{Y}}(\vec{y}) = \frac{1}{|\det T|} f_{\vec{X}}\left(T^{-1}\vec{y}\right)$$

2. Let us use the proposition on the given transformation. It is obvious that the inverse matrix of A is:

$$A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\rightarrow A^{-1}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$$

when the inverse may be found by simple inversion or by inverting the transformation itself:

$$\begin{cases}X_1=Y_1\\ X_2=Y_2-Y_1\end{cases}$$

Therefore, we get the joint distribution $f_{Y_1,Y_2}\left(y_1,y_2\right)=\frac{1}{|1|}f_{X_1,X_2}\left(y_1,y_2-y_1\right)$.

Let us observe the distribution of Y_2, which is of course X_1+X_2. Assume that X_1 and X_2 distribute independently; we get the following marginal distribution:

$$f_{X_1+X_2}\left(y_2\right)=f_{Y_2}\left(y_2\right)=\int_{-\infty}^{\infty}f_{Y_1,Y_2}\left(y_1,y_2\right)dy_1$$
$$=\int_{-\infty}^{\infty}f_{X_1,X_2}\left(y_1,y_2-y_1\right)dy_1=\int_{-\infty}^{\infty}f_{X_1}\left(y_1\right)f_{X_2}\left(y_2-y_1\right)dy_1$$

We can see that we got a distribution equal to $f_{X_1}*f_{X_2}$, as required.