Solution to Problem Set 8 – Stationarity, Power Spectral Density

Problem 1:

1. A direct calculation yields:

$$S_X(\omega)=\int_{-\infty}^{\infty}R_X(\tau)e^{-i\omega\tau}d\tau=\underbrace{\int_{-\infty}^{\infty}R_X(\tau)\cos(\omega\tau)d\tau}_{\text{an even function}}=2\int_{0}^{\infty}R_X(\tau)\cos(\omega\tau)d\tau$$

$$=2\int_{0}^{\infty}e^{-\alpha\tau}\cos(\omega\tau)d\tau=2\operatorname{Re}\left\{\int_{0}^{\infty}e^{-(\alpha+i\omega)\tau}d\tau\right\}=2\operatorname{Re}\left\{\frac{1}{\alpha+i\omega}\right\}$$

$$=2\operatorname{Re}\left\{\frac{\alpha-i\omega}{\alpha^2+\omega^2}\right\}=\frac{2\alpha}{\alpha^2+\omega^2}$$

2. Again by direct calculation:

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)e^{i\omega\tau}d\omega=\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}S_X(\omega)\cos(\omega\tau)d\omega}_{\text{an even function}}=\frac{1}{\pi}\int_{0}^{\infty}S_X(\omega)\cos(\omega\tau)d\omega$$

$$=\frac{1}{\pi}\int_{0}^{B}\cos(\omega\tau)d\omega=\begin{cases}\dfrac{\sin(B\tau)}{\pi\tau} & \tau\neq 0\\[2ex] \dfrac{B}{\pi} & \tau=0\end{cases}$$

Problem 2:

1. Using direct calculation and Fourier transform:

$$R_Y(\tau)=E\{(X(t+\tau+a)-X(t+\tau-a))(X(t+a)-X(t-a))\}=$$
$$=R_X(\tau)-R_X(\tau+2a)-R_X(\tau-2a)+R_X(\tau)=2R_X(\tau)-(R_X(\tau+2a)+R_X(\tau-2a))$$
$$\Leftrightarrow S_Y(\omega)=2S_X(\omega)-\left(S_X(\omega)e^{j\omega 2a}+S_X(\omega)e^{-j\omega 2a}\right)=2S_X(\omega)[1-\cos(2a\omega)]=4S_X(\omega)\sin^2(a\omega)$$

2. From the problem conditions $X(t)$ is WSS.

$$R_Y(t,t+\tau)=E\{X(t)X(t+\tau)\}E\{G(t)G(t+\tau)\}=R_X(\tau)R_G(\tau)$$
$$E\{Y(t)\}=E\{X(t)G(t)\}=E\{X(t)\}E\{G(t)\}=0$$

We got that $Y(t)$ is WSS as well. The cross-correlation is hence:

$$R_{XY}(t,t+\tau)=E\{X(t)X(t+\tau)G(t+\tau)\}=E\{X(t)X(t+\tau)\}E\{G(t+\tau)\}=R_X(\tau)$$

which means $X(t)$ and $Y(t)$ are JWSS.

Problem 3:

1. Notice that $X_{n+5}=\alpha^5X_n+\sum_{i=0}^{4}\alpha^iW_{n+5-i}$ where the second element is a deterministic (linear) function of the random variables $\{W_l\}_{l=n+1}^{n+5}$. Since X_n is a

deterministic (linear) function of $\{W_k\}_{k=-\infty}^{n}$ and since the process W_n is i.i.d, it follows that X_n and $\sum_{i=0}^{4}\alpha^i W_{n+5-i}$ are independent. From here it follows that:

$$\hat{X}_{n+5}(X_n) = E\{X_{n+5} \mid X_n\} = E\left\{\alpha^5 X_n + \sum_{i=0}^{4}\alpha^i W_{n+5-i} \mid X_n\right\}$$
$$= \alpha^5 X_n + E\left\{\sum_{i=0}^{4}\alpha^i W_{n+5-i} \mid X_n\right\} = \alpha^5 X_n + E\left\{\sum_{i=0}^{4}\alpha^i W_{n+5-i}\right\} = \alpha^5 X_n$$

2. Since the samples $\{X_k\}_{k=-\infty}^{n}$ are a deterministic (linear) function of $\{W_l\}_{l=-\infty}^{n}$, it holds that:

$$\hat{X}_{n+5}\left(\{X_k\}_{k=-\infty}^{n}\right) = E\left\{X_{n+5} \mid \{X_k\}_{k=-\infty}^{n}\right\} = E\left\{\alpha^5 X_n + \sum_{i=0}^{4}\alpha^i W_{n+5-i} \mid \{X_k\}_{k=-\infty}^{n}\right\}$$
$$= \alpha^5 X_n + E\left\{\sum_{i=0}^{4}\alpha^i W_{n+5-i} \mid \{X_k\}_{k=-\infty}^{n}\right\} = \alpha^5 X_n + E\left\{\sum_{i=0}^{4}\alpha^i W_{n+5-i} \mid \{W_k\}_{k=-\infty}^{n}\right\}$$
$$= \alpha^5 X_n + E\left\{\sum_{i=0}^{4}\alpha^i W_{n+5-i}\right\} = \alpha^5 X_n$$

Namely, we got that the optimal estimator of the future given all the past only makes use of the last sample from the past (X_n).

3. Since the expectations of Y_{n+5} and Y_n are zero, the LMMSE estimator is given by the formula:

$$\hat{Y}_{n+5}(Y_n) = \frac{E\{Y_{n+5}Y_n\}}{E\{Y_{n+5}^2\}} \cdot Y_n = \frac{R_Y[5]}{R_Y[0]} \cdot Y_n$$

$$R_Y[k] = E\{Y_{n+k}Y_n\} = E\{(X_{n+k}+Z_{n+k})(X_n+Z_n)\} = E\{X_{n+k}X_n\} + E\{Z_{n+k}Z_n\} =$$
$$= E\{X_{n+k}X_n\} + \delta[k]$$

$$E\{X_{n+k}X_n\} \overset{k>0}{=} E\left\{\left(\alpha^k X_n + \sum_{i=0}^{k-1}\alpha^i W_{n+k-i}\right)X_n\right\} = \alpha^k E\{X_n^2\}, \qquad k>0$$

$$\Rightarrow \qquad E\{X_{n+k}X_n\} = \alpha^{|k|}E\{X_n^2\}, \qquad \forall k$$

$$\Rightarrow \qquad R_Y[k] = \alpha^{|k|}E\{X_n^2\} + \delta[k]$$

$$E\{X_n^2\} = E\{(\alpha X_{n-1}+W_n)(\alpha X_{n-1}+W_n)\} = \alpha^2 E\{X_{n-1}^2\} + E\{W_n^2\} \overset{\{X_n\}\ WSS}{=} \alpha^2 E\{X_n^2\} + 1$$

$$\Rightarrow \qquad E\{X_n^2\}(1-\alpha^2) = 1 \qquad \Rightarrow \qquad E\{X_n^2\} = \frac{1}{1-\alpha^2}$$

$$\Rightarrow \qquad R_Y[k] = \alpha^{|k|}E\{X_n^2\} + \delta[k] = \frac{\alpha^{|k|}}{1-\alpha^2} + \delta[k]$$

Let us substitute in the expression of the estimator and we get:

$$\hat{Y}_{n+5}(Y_n)=\frac{R_Y[5]}{R_Y[0]}\cdot Y_n=\frac{\frac{\alpha^{|5|}}{1-\alpha^2}+\delta[5]}{\frac{\alpha^{|0|}}{1-\alpha^2}+\delta[0]}\cdot Y_n=\frac{\frac{\alpha^5}{1-\alpha^2}}{\frac{1}{1-\alpha^2}+1}\cdot Y_n\overset{\alpha=0.5}{=}\frac{1}{56}\cdot Y_n$$

4. Let us define the vector $\underline{Y}=\left[Y_n\,Y_{n-1}\right]^T$ and use the formula for the optimal linear estimator of a scalar from a vector:

$$\hat{Y}_{n+5}\left(\underline{Y}\right)=E\left\{Y_{n+5}\underline{Y}^T\right\}E\left\{\underline{Y}\,\underline{Y}^T\right\}^{-1}\underline{Y}=$$

$$=\begin{bmatrix} E\left\{Y_{n+5}Y_n\right\} & E\left\{Y_{n+5}Y_{n-1}\right\}\end{bmatrix}\begin{bmatrix} E\left\{Y_nY_n\right\} & E\left\{Y_nY_{n-1}\right\} \\ E\left\{Y_{n-1}Y_n\right\} & E\left\{Y_{n-1}Y_{n-1}\right\}\end{bmatrix}^{-1}\underline{Y}=$$

$$=\begin{bmatrix} R_Y[5] & R_Y[6]\end{bmatrix}\begin{bmatrix} R_Y[0] & R_Y[1] \\ R_Y[1] & R_Y[0]\end{bmatrix}^{-1}\underline{Y}=$$

$$=\frac{1}{3\cdot 8}\begin{bmatrix} 1 & \frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix} 7/3 & 2/3 \\ 2/3 & 7/3\end{bmatrix}^{-1}\underline{Y}=\frac{1}{3\cdot 8}\begin{bmatrix} 1 & \frac{1}{2}\end{bmatrix}\cdot\frac{1}{15}\cdot\begin{bmatrix} 7 & -2 \\ -2 & 7\end{bmatrix}\cdot\underline{Y}=$$

$$=\begin{bmatrix} \frac{1}{60} & \frac{1}{240}\end{bmatrix}\cdot\underline{Y}=\frac{1}{60}Y_n+\frac{1}{240}Y_{n-1}$$

Notice that, in contrast to the estimator from section 2, which uses only the last sample of X_n to estimate the future, the estimator we got here uses the two samples Y_n, Y_{n-1} in order to estimate the future.

Problem 4:

1. The initial condition is $X_0=0$, this means that it is deterministically known and, thus, $f_{X_0}(x)=\delta(x)$. Let us calculate for the following samples:

$$X_1=\frac{1}{2}\cdot X_0+W_1=\frac{1}{2}\cdot 0+W_1=W_1=\begin{cases}1 & w.p.\quad 1/2\\ 0 & w.p.\quad 1/2\end{cases}$$

$$\Rightarrow\qquad f_{X_1}(x)=\frac{1}{2}\cdot\delta(x)+\frac{1}{2}\cdot\delta(x-1)$$

$$X_2=\frac{1}{2}\cdot X_1+W_2=\begin{cases}1.5 & w.p.\quad 1/4\\ 1 & w.p.\quad 1/4\\ 0.5 & w.p.\quad 1/4\\ 0 & w.p.\quad 1/4\end{cases}$$

$$\Rightarrow\qquad f_{X_2}(x)=\frac{1}{4}\left[\delta(x)+\delta(x-0.5)+\delta(x-1)+\delta(x-1.5)\right]$$

Similarly, for a general n we get:

$$f_{X_n}(x)=\frac{1}{2^n}\left[\sum_{k=0}^{2^{n-1}-1}\delta\left(x-k\cdot 2^{-(n-1)}\right) \quad + \quad \sum_{k=0}^{2^{n-1}-1}\delta\left(x-1-k\cdot 2^{-(n-1)}\right)\right]$$

We got that X_n distributes uniformly through 2^n points ordered uniformly in the interval between 0 and $2-2^{-(n-1)}$. It can be seen that, at the limit $n\to\infty$, the distribution of the sample X_n approaches uniform distribution $Unif\left(0,2\right)$.

2.

$$X_n=\frac{1}{2}X_{n-1}+W_n=\frac{1}{2}\left(\frac{1}{2}X_{n-2}+W_{n-1}\right)+W_n=\left(\frac{1}{2}\right)^2X_{n-2}+\frac{1}{2}W_{n-1}+W_n=$$

$$=\ldots=\left(\frac{1}{2}\right)^nX_0+\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{n-k}W_k$$

For initial condition $X_0=0$ we get:

$$X_n=\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{n-k}W_k$$

Notice that this form suits a binary expansion of a number from $[0,2]$:

$W_n\,.W_{n-1}\,W_{n-2}\,\ldots W_1$ where each W_k takes the value 0 or 1. When $n\to\infty$ approaches an infinite binary expansion of any number in the interval $[0,2]$.

3. We examine the distribution $Unif\left(0,2\right)$ as a stationary distribution. Assume that $X_{n-1}\sim Unif\left(0,2\right)$ and check what is the distribution of X_n in this case:

$$f_{X_n}(x)=f_{\frac{1}{2}X_{n-1}}(x)*f_{W_n}(x)=\left(2f_{X_{n-1}}(2x)\right)*f_{W_n}(x)=$$

$$=\left(2\cdot\begin{cases}0.5 & 0\le 2x\le 2\\ 0 & else\end{cases}\right)*\left(\frac{1}{2}\delta(x)+\frac{1}{2}\delta(x-1)\right)=$$

$$=\left(\begin{cases}1 & 0\le x\le 1\\ 0 & else\end{cases}\right)*\left(\frac{1}{2}\delta(x)+\frac{1}{2}\delta(x-1)\right)=$$

$$=\frac{1}{2}\left(\begin{cases}1 & 0\le x\le 1\\ 0 & else\end{cases}\right)*\delta(x) \quad + \quad \frac{1}{2}\left(\begin{cases}1 & 0\le x\le 1\\ 0 & else\end{cases}\right)*\delta(x-1)=$$

$$=\frac{1}{2}\left(\begin{cases}1 & 0\le x\le 1\\ 0 & else\end{cases}\right) \quad + \quad \frac{1}{2}\left(\begin{cases}1 & 1\le x\le 2\\ 0 & else\end{cases}\right)=$$

$$=\frac{1}{2}\left(\begin{cases}1 & 0\le x\le 2\\ 0 & else\end{cases}\right)=\left(\begin{cases}0.5 & 0\le x\le 2\\ 0 & else\end{cases}\right)=Unif\,(0,2)$$

We got that $f_{X_n}(x)=f_{X_{n-1}}(x)$ and, thus, $Unif\left(0,2\right)$ is a stationary distribution. In other words, we get an S.S.S process if we choose:

$$X_0\sim Unif\left(0,2\right)$$