

Problem Set 5 – Estimation of Random Variables

Problem 1:

We would like to estimate the random variable X from Y with distortion measure $d(e) = |e| = |X - \hat{X}|$.

1. Suppose X and Y have the joint PDF $f_{XY}(x, y)$. Show that the expectation of the distortion measure can be expressed as follows:

$$E[|X - \hat{X}|] = \int_{-\infty}^{\infty} dy f_Y(y) \left[\int_{-\infty}^{g(y)} (g(y) - x) f_{X|Y}(x|y) dx + \int_{g(y)}^{\infty} (x - g(y)) f_{X|Y}(x|y) dx \right]$$

2. Find a condition for the optimal estimator in this case (Hint: by deriving with respect to $g(y)$).

Problem 2:

Let X be a uniform random variable in the section $[0, a]$, $a > 0$. Let us define the random variable Y , which, given $X = x$, distributes uniformly in $[0, x]$.

1. Find the MMSE estimator of Y from X .
2. Find the MMSE estimator of X from Y .
3. Find the linear MMSE estimator of Y from X .
4. Find the linear MMSE estimator of X from Y .

Problem 3:

Given is the following system: $X \rightarrow Q(\cdot) \rightarrow Y$

X is an exponentially distributed random variable with parameter $\lambda > 0$, namely:

$$f_X(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The function $Q(\cdot)$ takes discrete values such that:

$$y = Q(x) = \lfloor x \rfloor$$

1. Calculate $P(Y = n)$, $n \in \mathbb{Z}$.
2. Calculate $f_{X|Y}(x|y)$.

- Find the optimal estimator in the sense of MSE and in the sense of minimum probability of error of Y given X .
- Find the MMSE estimator of X given Y . You may use the identity:

$$\int x e^{-\lambda x} dx = -\frac{e^{-\lambda x}(\lambda x + 1)}{\lambda^2}$$

- Find the linear MMSE estimator of X given Y .

Define $Z = X - Y$.

- Find $f_{Z|Y}(z|y)$.
- Are Z and Y uncorrelated? Independent?
- Find the MMSE and the linear MMSE estimators of Z given Y .

Problem 4 (Exam 2019):

Given is a random matrix $A = \begin{pmatrix} X & \rho Y \\ \rho Y & X \end{pmatrix}$ where $X, Y \sim N(0,1)$ are independent of each other, and ρ is a deterministic constant, $|\rho| \leq 1$.

- Let $v = (0 \ 1)^T$. Is $v^T A v$ a Gaussian random variable for any $|\rho| \leq 1$?
- Is $v^T A$ a Gaussian random vector, for **any** vector $v \in \mathbb{R}^2$ and any $|\rho| \leq 1$?

From now on assume that $\rho = 1$.

In the following sections we define several linear transformations, such that in each section we get a vector in \mathbb{R}^2 , which we denote as $Z = (Z_1 \ Z_2)^T$. In each section, find the MMSE estimator of Z_2 given Z_1 and the corresponding MSE.

- $Z = (1 \ -1) \cdot A$
- $Z = (0 \ 1) \cdot A \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
- $Z = (1 \ 2) \cdot A$

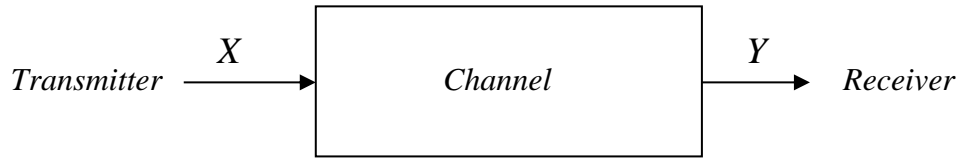
Problem 5:

Prove that if the random variables X, Y, Z are jointly Gaussian with expected value zero, then $E[XYZ] = 0$.

Problem 6 – (channel with interferences):

In this question we will deal with information transference through a channel that has interferences: the transmitter wants to transfer a message represented by the random variable X , and does this through a transmission medium (the channel), which connects the transmitter to the receiver. Since the medium might be non-ideal, there exists a possibility of the information sent to be distorted by the channel, and, thus, in

the end of the medium we get the random variable Y (which is statistically dependent of X). In this question, we will deal with two typical channels.



Binary Symmetric Channel (BSC):

Given:

$$X \sim \text{Ber}(p)$$

$$N \sim \text{Ber}(\varepsilon) \quad Y = X \oplus N \quad \oplus = \text{XOR}$$

X, N are independent and $0 < p, \varepsilon < 1$.

1. Find the following probabilities (for $x, y \in \{0,1\}$):

$$\Pr\{Y = y \mid X = x\}, \quad \Pr\{Y = y\}, \quad \Pr\{X = x \mid Y = y\}$$

2. Find the probability of error in the output of the channel, i.e.:

$$\Pr\{X \neq Y\}$$

Binary Channel with Additive Gaussian Noise (BIAGN):

Given:

$$I \sim \text{Ber}(p)$$

$$X = (-1)^{(I+1)}$$

$$N \sim N(0, \sigma^2) \quad Y = X + N$$

I, N are independent and $0 < p < 1$.

3. Find the distribution of X and the distribution of Y . Draw the PDF of each one of the distributions.

4. At the output of the channel, an attempt at guessing the transmitted message is made as follows:

$$Z = \text{sign}(Y) = \begin{cases} -1 & Y \leq 0 \\ 1 & Y > 0 \end{cases}$$

Calculate the probability of error for the decision of X , that is, $\Pr\{Z \neq X\}$

Make use of the Q-function in your final answer.
