Solution 4 – Second Order Statistics of Random Vectors, Gaussian Random Vector

Problem 1:

1. Based on the definition, a positive semi-definite matrix satisfies $\underline{a}^T B \underline{a} \geq 0 \quad \forall \underline{a}$. Let us choose the vector $\underline{a} = [1\ 0 \ldots 0]$ and we get:

   $$\underline{a}^T B \underline{a} = b_{11} \geq 0$$

   It can be proven similarly for the rest of the elements in the diagonal.

2. An invertible matrix has a full rank. Let us prove by contradiction:

   It is given that the matrix B is positive definite. Let us assume that B does not have a full rank, namely that an $a \neq \underline{0}$ exists such that $B\underline{a} = 0$. After multiplying by $\underline{a}^T$ in the left, we get: $\underline{a}^T B \underline{a} = 0$, which is a contradiction to the given $\underline{a}^T B \underline{a} > 0 \quad \forall \underline{a}$.

3. The matrix B satisfies $0 \leq \underline{v}^T B \underline{v} = (*)$.

   We will choose $\underline{v}$ to be the eigenvector of the matrix and, thus, $B\underline{v} = \lambda \underline{v}$, where λ is an eigenvalue of B.
   Therefore:

   $$(*) = \underline{v}^T \lambda \underline{v} = \lambda \|\underline{v}\|^2 \geq 0$$
   $$\Rightarrow \lambda \geq 0$$

<u>Second part</u> –

1. Note that the matrix A can be written as $A = U\Lambda U^T$ with $UU^T = U^T U = I$. Then, U^T is invertible for every $a \in \mathbb{R}^N$ we can find $a = U\tilde{a}$.
   Going back to our calculation we get
   $$a^t A a = \tilde{a}^T U^T U \Lambda U^T U \tilde{a} = \tilde{a}^T \Lambda \tilde{a} = \sum \tilde{a}_i^2 \lambda_i$$
   Because $\lambda_i \geq 0\ \forall i$ we obtain the required result.

2. We write the general expression for an entry of the matrix $D = AXB + C$
   $$d_{ij} = \sum_k \sum_l a_{ik} x_{kl} b_{lj} + c_{ij}$$
   Expectation is taken on each entry separately such that –
   $$E[d_{ij}] = E[\sum_k \sum_l a_{ik} x_{kl} b_{lj} + c_{ij}] = \sum_k \sum_l a_{ik} E[x_{kl}] b_{lj} + c_{ij}$$
   And that will be equal in matrix form to
   $$E[D] = E[AXB + C] = AE[X]B + C$$

Problem 2:

Let us start by finding the characteristic function of a standard Gaussian random variable: $Z \sim N(0,1)$.

$$f_Z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$\phi_Z(\omega) = E\left[e^{i\omega Z}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{i\omega z} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2i\omega z\right)} dz$$

After completing the square of the expression in the exponent, we get:

$$\phi_Z(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2i\omega z\right)} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2i\omega z - \omega^2 + \omega^2\right)} dz = e^{-\frac{1}{2}\omega^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - i\omega)^2} dz}_{=1} = e^{-\frac{1}{2}\omega^2}$$

All in all:

$$\phi_Z(\omega) = E\left[e^{i\omega Z}\right] = e^{-\frac{1}{2}\omega^2}$$

Now, recall that any Gaussian random variable $X \sim N(\mu, \sigma^2)$ may be expressed as a standard Gaussian variable which was "colored":

$$X = \sigma Z + \mu, \qquad Z \sim N(0,1)$$

The characteristic function of X is, therefore:

$$\phi_X(\omega) = E\left[e^{i\omega X}\right] = E\left[e^{i\omega(\mu + \sigma Z)}\right] = e^{i\omega\mu} E\left[e^{i(\omega\sigma)Z}\right] = e^{i\omega\mu - \frac{1}{2}\sigma^2\omega^2}$$

Notice that this expression is true also for a deterministic Gaussian random variable ($\sigma^2 = 0$, meaning $X = \mu$ with a probability of 1).

Problem 3:

$\underline{X}$ is a Gaussian random vector and, therefore, $Y = \underline{a}^T \underline{X}$ is a Gaussian random variable with: $\eta_Y = \underline{a}^T \underline{\eta}$, $\sigma_Y^2 = \underline{a}^T C \underline{a}$.

$$P\left(\underline{a}^T \underline{X} \geq b\right) = P\left(Y \geq b\right) = \int_b^{\infty} f_Y(y) dy$$

$$= \int_b^{\infty} \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\eta_Y)^2}{2\sigma_Y^2}} dy \overset{s = \frac{y-\eta_Y}{\sigma_Y}}{=} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \int_{\frac{b-\eta_Y}{\sigma_Y}}^{\infty} e^{-\frac{s^2}{2}} \sigma_Y ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{b-\eta_Y}{\sigma_Y}}^{\infty} e^{-\frac{s^2}{2}} ds = Q\left(\frac{b - \eta_Y}{\sigma_Y}\right) = Q\left(\frac{b - \underline{a}^T \underline{\eta}}{\sqrt{\underline{a}^T C \underline{a}}}\right)$$

Problem 4:

1. Due to the symmetry of the distribution, we have $B = \begin{cases} 1 & w.p.\ 0.5 \\ -1 & w.p.\ 0.5 \end{cases}$.

   Now, for $w < 0$, we have $F_W(w) = 0$.
   For $w \geq 0$, we have
   $$\begin{aligned} F_W(w) = \Pr(W \leq w) &= \Pr(-w \leq X \leq w) \\ &= \Pr(-w \leq X \leq 0) + \Pr(0 < X \leq w) = 2\Pr(0 < X \leq w) \\ &= 2F_X(w) - 1 \end{aligned}$$
   where we used the symmetry of the distribution again. Thus,
   $$f_W(w) = 2f_X(w)u(w)$$

2. We can prove this in two ways: via pdf or via the characteristic function.

<u>Option 1</u>:

We will start by finding the conditional PDF $f_{W|B}(w|b)$ ($b \in \{-1,1\}$). Clearly,

$$f_{W|B}(w|b) = 0 \qquad w < 0$$

For $w \geq 0$, based on the law of total probability:

$$f_{W|B}(w|b) = \int_{-\infty}^{\infty} f_{W|B,X}(w|b,x) f_{X|B}(x|b) dx$$

Now:

$$f_{X|B}(x|b) = \frac{\mathrm{P}(B=b \mid X=x) f_X(x)}{\mathrm{P}(B=b)} = \begin{cases} 2 \cdot f_X(x) & sign(x) = b \\ 0 & o.w. \end{cases}$$

$$f_{W|B,X}(w|b,x) = \delta(w - |x|)$$

Therefore, if $b = 1$:

$$f_{W|B}(w|1) = \int_0^{\infty} \delta(w - |x|) 2 f_X(x) dx = 2 f_X(w)$$

If $b = -1$:

$$f_{W|B}(w|-1) = \int_{-\infty}^{0} \delta(w - |x|) 2 f_X(x) dx \underset{x' = -x}{=} \int_0^{\infty} \delta(w - |x|) 2 f_X(x) dx = 2 f_X(w)$$

That is to say:

$$f_{W|B}(w|-1) = f_{W|B}(w|1) = f_W(w) \qquad \forall w$$

<u>Option 2</u>: We consider the joint characteristic function of (W, B). Specifically, we use the fact that W and B are independent if and only if:

$$\varphi_{WB}(\omega,\beta)=\varphi_W(\omega)\cdot\varphi_B(\beta)$$

First find $\varphi_B(\beta)$: $\varphi_B(\beta)=E\left[e^{j\beta B}\right]=\frac{1}{2}\cdot e^{-j\beta}+\frac{1}{2}\cdot e^{j\beta}=\cos(\beta)$. Now,

$$\begin{aligned}
&\varphi_{WB}(\omega,\beta)=E\left[\exp\left(j\left(\omega W+\beta B\right)\right)\right]=E\left[\exp\left(j\left(\omega|X|+\beta\text{sign}(X)\right)\right)\right]\\
&=\int_{-\infty}^{0} f_X(x)e^{j\omega(-x)+j\beta(-1)}dx+\int_{0}^{\infty} f_X(x)e^{j\omega(x)+j\beta(+1)}dx\\
&=e^{-j\beta}\int_{-\infty}^{0} f_X(x)e^{-j\omega x}dx+e^{j\beta}\int_{0}^{\infty} f_X(x)e^{j\omega x}dx
\end{aligned}$$

Since $f_X(x)$ is an even function, both integrals are equal and we get:

$$\varphi_{WB}(\omega,\beta)=\left(e^{j\beta}+e^{-j\beta}\right)\int_0^\infty f_X(x)e^{j\omega x}dx=\cos(\beta)\cdot\int_0^\infty 2f_X(x)e^{j\omega x}dx=\varphi_B(\beta)\cdot\varphi_W(\omega)$$

Problem 5:

1. Notice that:

$$W=Y^2+Z^2=\cos^2(X)+\sin^2(X)=1$$

namely, W is deterministic. From here, $E[W]=1$

2. We will find $E[Z^2]$ directly, without finding the distribution of Z. We'll use Euler's formula:

$$\cos(a)=\frac{e^{ja}+e^{-ja}}{2}$$

Let us calculate:

$$\begin{aligned}
&E[Z^2]=E[\sin^2(X)]=E\left[\frac{1-\cos(2X)}{2}\right]=\frac{1}{2}-\frac{1}{2}E[\cos(2X)]=\\
&=\frac{1}{2}-\frac{1}{2}E\left[\frac{e^{2jX}+e^{-2jX}}{2}\right]=\frac{1}{2}-\frac{1}{4}E\left[e^{2jX}\right]-\frac{1}{4}E\left[e^{-2jX}\right]
\end{aligned}$$

Notice the expression: $E[e^{2jX}]$. Recall that the characteristic function of a random variable is defined as:

$$\omega_X(\omega)=E[e^{j\omega X}]$$

In other words:

$$E[e^{2jX}]=\phi_X(\omega)|_{\omega=2}$$

The characteristic function of a standard Gaussian random variable is equal to:

$$\phi_X(\omega) = e^{-\frac{1}{2}\omega^2}$$

And, thus:

$$E[e^{2jX}] = e^{-\frac{1}{2}\cdot 2^2} = e^{-2}$$

Similarly:

$$E[e^{-2jX}] = e^{-\frac{1}{2}\cdot(-2^2)} = e^{-2}$$

All in all:

$$E[Z^2] = \frac{1}{2} - \frac{1}{4}e^{-2} - \frac{1}{4}e^{-2} = \frac{1}{2}(1-e^{-2})$$

3. We'll show first that:

$$E[Z^2 \mid Z \geq 0] = E[Z^2 \mid Z \leq 0]$$

$$\begin{aligned} E[Z^2 \mid Z \geq 0] &= E[\sin^2(X) \mid \sin(X) \geq 0] = \\ &= E\left[(-\sin(X))^2 \mid -\sin(X) \leq 0\right] = \\ &\overset{(a)}{=} E[\sin^2(-X) \mid \sin(-X) \leq 0] = \\ &\overset{(b)}{=} E[\sin^2(W) \mid \sin(W) \leq 0] = \\ &\overset{(c)}{=} E[Z^2 \mid Z \leq 0] \end{aligned}$$

(a) is true since $\sin(x)$ is an **odd** function: $-\sin(x) = \sin(-x)$,

(b) we substituted: $W = -X$

(c) follows since X and $W = -X$ have the same distribution.

Now, we shall use the results from the previous section in the following way:

$$\begin{aligned} E[Z^2] &= E[Z^2 \mid Z \geq 0]\cdot P(Z \geq 0) + E[Z^2 \mid Z \leq 0]\cdot P(Z \leq 0) = \\ &= E[Z^2 \mid Z \geq 0]\cdot P(Z \geq 0) + E[Z^2 \mid Z \geq 0]\cdot P(Z \leq 0) = \\ &= E[Z^2 \mid Z \geq 0]\cdot \underbrace{\left(P(Z \geq 0) + P(Z \leq 0)\right)}_{=1} = E[Z^2 \mid Z \geq 0] \end{aligned}$$

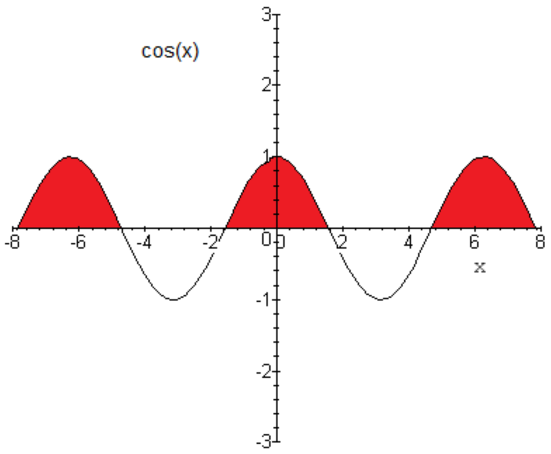
Thus, based on the previous section:

$$E[Z^2 \mid Z \geq 0] = E[Z^2] = \frac{1-e^{-2}}{2}$$

4. Let us check:

$$E[Y^2 \mid Y \geq 0] = E[\cos^2(X) \mid \cos(X) \geq 0]$$

Notice that the cosine function is **not** an odd function, and, for this reason, we cannot use the claim from the last section. In other words, $E[\cos^2(X)\,|\,\cos(X)\geq 0]\neq E[\cos^2(X)\,|\,\cos(X)\leq 0]$. This can be seen below:



The red regions are regions where $\cos(X)\geq 0$. One can see that the integral on the expectation of a Gaussian distribution, which is symmetrical around zero, in the red regions, will be **different** than the same integral in the white regions.

In order to calculate $E[Y^2\,|\,Y\geq 0]$, we would need to calculate it explicitly.

<u>Problem 1:</u>

1. The vector consists of N independent Gaussian random variables thus, they are jointly Gaussian. From here we conclude that it is indeed a Gaussian random vector.

2.

$$
\begin{aligned}
&X_0 = X_0\\
&X_1 = \beta X_0 + V_0\\
&X_2 = \beta X_1 + V_1 = \beta^2 X_0 + \beta V_0 + V_1\\
&X_3 = \beta X_2 + V_2 = \beta^3 X_0 + \beta^2 V_0 + \beta V_1 + V_2\\
&\vdots\\
&X_N = \beta X_{N-1} + V_{N-1} = \beta^N X_0 + \beta^{N-1}V_0 + \beta^{N-2}V_1 + \cdots + V_{N-1}
\end{aligned}
$$

Let us write the system of equations as matrices:

$$
\underline{X} = \begin{bmatrix} X_0\\ X_1\\ X_2\\ \vdots\\ X_N \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & \beta\\ \beta & 1 & 0 & \cdots & \beta^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \beta^{N-1} & \beta^{N-2} & \beta^{N-3} & \cdots & \beta^N \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} V_0\\ V_1\\ \vdots\\ V_{N-1}\\ X_0 \end{bmatrix}
$$

The matrix $\mathbf{A}$ is a deterministic matrix, and since the vector $\begin{bmatrix} V_0 & V_1 & \cdots & V_{N-1} & X_0 \end{bmatrix}^T$ is a Gaussian random vector (all its elements are Gaussian and independent), the vector $\begin{bmatrix} X_0 & X_1 & \cdots & X_N \end{bmatrix}^T$ is a Gaussian random vector (linear transformation of a Gaussian random vector is also a Gaussian random vector).

Now, let us check statistical dependence. We can, of course, calculate the covariance matrix, but in order to show dependence on a vector, it is enough to find an example of two elements of the vector that are dependent of each other. (Intuition – there is a recursive relationship between the elements of the vector and, thus, there exists a dependence).

For independent random variables $E\{X \mid Y\} = E\{X\}$ holds. Let us check the case $X = X_1, Y = X_0$ and we will see that X and Y are dependent:

$$E\{X_1\} = E\{\beta X_0 + V_0\} = \beta E\{X_0\} + E\{V_0\} = 0$$
$$E\{X_1 \mid X_0\} = E\{\beta X_0 + V_0 \mid X_0\} = \beta E\{X_0 \mid X_0\} + E\{V_0 \mid X_0\} = \beta X_0 + E\{V_0\} = \beta X_0$$

Meaning that $E\{X_1\} \neq E\{X_1 \mid X_0\}$, and, therefore, X_0 and X_1 are dependent, and one cannot claim that the vector's elements are independent.

3. Let us calculate it directly from the recursion formula:

$$E\{X_i\} = \beta E\{X_{i-1}\} + \underbrace{E\{V_{i-1}\}}_{0} = \beta^2 E\{X_{i-2}\} + \beta \underbrace{E\{V_{i-2}\}}_{0} = \cdots = \beta^i \underbrace{E\{X_0\}}_{0} = 0$$
$$Var\{X_i\} = E\{X_i^2\} = E\{(\beta X_{i-1} + V_{i-1})^2\} = \beta^2 E\{X_{i-1}^2\} + 2\beta E\{X_{i-1} V_{i-1}\} + E\{V_{i-1}^2\}$$

Take a look at the expectation $E\{X_{i-1} V_{i-1}\}$: X_{i-1} is a linear function of $\{X_0, V_0, ..., V_{i-2}\}$ and, thus, is independent of V_{i-1}. Therefore, $E\{X_{i-1} V_{i-1}\} = E\{X_{i-1}\} E\{V_{i-1}\} = 0$. That is to say, we got:

$$E\{X_i^2\} = \beta^2 E\{X_{i-1}^2\} + E\{V_{i-1}^2\}$$

We'll start by examining the first few elements in order to find a general expression for the recursion formula:

$$E\{X_0^2\} = \sigma^2$$
$$E\{X_1^2\} = \beta^2 \sigma^2 + 1$$
$$E\{X_2^2\} = \beta^2 E\{X_1^2\} + 1 = \beta^4 \sigma^2 + \beta^2 + 1$$
$$\vdots$$
$$E\{X_i^2\} = \beta^{2i} \sigma^2 + \sum_{j=0}^{i-1} \beta^{2j} = \beta^{2i} \sigma^2 + \frac{1 - \beta^{2i}}{1 - \beta^2} = \frac{1}{1 - \beta^2} + \beta^{2i} \left(\sigma^2 - \frac{1}{1 - \beta^2} \right)$$

And, since the expected value is zero, the variance is the second moment.