# Solution to Problem Set 9

## Problem 1:

1. $X[n], Z[n]$ are independent and each is WSS so there are JWSS.
   $Y[n]$ is the sum of two independent WSS processes hence is also WSS.

$$R_{XY}[n+k,n] = E(X[n+k]Y[n]) = E(X[n+k]\cdot(X[n]+Z[n]))$$
$$= E(X[n+k]X[n]) = R_X[k]$$

   We have that the cross correlation between $X[n], Y[n]$ depends only on the time difference, hence they are JWSS.

2. Since $X[n], Y[n]$ are jointly Gaussian, JWSS with expectation zero, the optimal estimator of $X[n]$ from $Y[n]$ is also the optimal linear estimator, meaning the Weiner filter:

$$H(\omega) = \frac{S_{XY}(\omega)}{S_Y(\omega)} = \frac{S_X(\omega)}{S_X(\omega)+S_Z(\omega)} = S_X(\omega) = \frac{|\omega|}{\pi}, \qquad |\omega| \le \pi$$

3. Estimation error spectrum is:

$$S_\varepsilon(\omega) = S_X(\omega) - |H(\omega)|^2 S_Y(\omega) = S_X(\omega) - S_X(\omega)^2 = \frac{|\omega|}{\pi} - \frac{\omega^2}{\pi^2}, \qquad |\omega| \le \pi$$

   So the MSE is given via:

$$MSE = \frac{1}{\pi}\int_0^\pi S_\varepsilon(\omega)d\omega = \frac{1}{\pi^2}\int_0^\pi \omega d\omega - \frac{1}{\pi^3}\int_0^\pi \omega^2 d\omega = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

4. $Y''(n) = X(n) + (-1)^n Z(n)$

   We will first show that the process $(-1)^n Z(n)$ is WSS:

$$E\left[(-1)^n Z(n)\right] = (-1)^n E\left[Z(n)\right] = 0$$
$$E\left[(-1)^n Z(n)(-1)^m Z(m)\right] = (-1)^{n-m} R_Z(n-m)$$

   $X[n]$ and $(-1)^n Z(n)$ are WSS and independent, so they are JWSS, and from this the processes $X[n], Y''[n]$ are also JWSS.

5. Since the processes are JWSS and jointly Gaussian, the optimal estimator is again the Weiner filter. From Fourier transform properties, multiplication by $e^{-j\omega n}$ in the time domain is equivalent to shifting by $\omega$ in the frequency domain, so the spectrum of the process $(-1)^n Z(n) = e^{-j\pi n} Z(n)$ is the spectrum of $Z(n)$ shifted by $\pi$, which is identical to the spectrum of $X[n]$, thus:

$$S_{Y''}(\omega) = S_X(\omega) + S_X(\omega) = 2S_X(\omega)$$

   And the filter is :

$$H(\omega) = \frac{S_{XY''}(\omega)}{S_{Y''}(\omega)} = \frac{S_X(\omega)}{2S_X(\omega)} = \frac{1}{2}$$

Meaning the optimal estimator is: $\hat{X}(n)=\frac{1}{2}Y''(n)$

6. The processes $X[n],Y^{'}[n]$ are not JWSS:
$E[X(n)Y'(m)]=E[(-1)^m X(n)X(m)]=(-1)^m R_X(n-m)$

7. The function between $Y^{'}[n]$ and $Y^{''}[n]$ is one-to-one and onto, so the optimal estimator of $X[n]$ from $Y^{'}[n]$ is **identical** to the optimal estimator of $X[n]$ from $Y^{''}[n]$: $\hat{X}(n)=\frac{1}{2}Y''(n)=\frac{1}{2}(-1)^n Y'(n)$
Note that this estimator is linear but it is not LTI.

**Problem 2:**

1. All process are with expectation zero and it is given that $X(t)$ is WSS.
We first show that the autocorrelations depend only on time difference:
$$R_{Y_1}(\tau)=E\left[Y_1(t+\tau)Y_1(t)\right]=E[P^2\cdot X(t+\tau)X\left(t\right)]+E[N_1(t+\tau)N_1\left(t\right)]$$
$$=E[P^2]\cdot E\left[X(t+\tau)X(t)\right]+E[N_1(t+\tau)N_1\left(t\right)]=pR_X(\tau)+R_{N_1}(\tau)$$
$$R_{Y_2}(\tau)=E\left[Y_2(t+\tau)Y_2(t)\right]=E[(1-P)^2\cdot X(t+\tau)X\left(t\right)]+E[N_2(t+\tau)N_2\left(t\right)]$$
$$=E[(1-P)^2]\cdot E\left[X(t+\tau)X(t)\right]+E[N_2(t+\tau)N_2\left(t\right)]=(1-p)R_X(\tau)+R_{N_2}(\tau)$$
We now show that the cross-correlation depends only on time difference:
$$R_{XY_1}(\tau)=E\left[X(t+\tau)Y_1(t)\right]=E[P\cdot X(t+\tau)X\left(t\right)+X(t+\tau)N_1\left(t\right)]$$
$$=E[P]\cdot E\left[X(t+\tau)X(t)\right]=pR_X(\tau)$$
$$R_{XY_2}(\tau)=E\left[X(t+\tau)Y_2(t)\right]=E[(1-P)\cdot X(t+\tau)X\left(t\right)+X(t+\tau)N_2\left(t\right)]$$
$$=E[1-P]\cdot E\left[X(t+\tau)X(t)\right]=(1-p)R_X(\tau)$$
$$R_{Y_1Y_2}(\tau)=E\left[Y_1(t+\tau)Y_2(t)\right]=E[\underbrace{(1-P)\cdot P}_{0}\cdot X(t+\tau)X\left(t\right)+P\cdot X(t+\tau)N_2\left(t\right)$$
$$+(1-P)\cdot X(t)N_1\left(t+\tau\right)+N_1(t+\tau)N_2(t+\tau)]=0$$

2. The optimal linear estimator is given through the Weiner filter:
$$H_1(\omega)=\frac{S_{XY_1}(\omega)}{S_{Y_1}(\omega)}=\frac{pS_X(\omega)}{pS_X(\omega)+S_1(\omega)}$$

3. The optimal linear estimator is given through the Weiner filter:
$$H_2(\omega)=\frac{S_{XY_2}(\omega)}{S_{Y_2}(\omega)}=\frac{(1-p)S_X(\omega)}{(1-p)S_X(\omega)+S_1(\omega)}$$

4. To show this, we will use the fact that the estimation error of the optimal linear estimator is orthogonal to the measurements, $Y_1(t)$ and $Y_2(t)$ are orthogonal and $\hat{X}_i(t)$ is a linear function of $Y_i(t)$:

$$R_{\varepsilon Y_1}(\tau) = R_{XY_1}(\tau) - R_{\hat{X}_1 Y_1}(\tau) - \underbrace{R_{\hat{X}_2 Y_1}(\tau)}_{0} = 0$$

$$R_{\varepsilon Y_2}(\tau) = R_{XY_2}(\tau) - \underbrace{R_{\hat{X}_1 Y_2}(\tau)}_{0} - R_{\hat{X}_2 Y_2}(\tau) = 0$$

We have that $R_{\hat{X}_2 Y_1}(\tau) = R_{\hat{X}_1 Y_2}(\tau) = 0$ since $\hat{X}_2(t)$ is a linear function of $Y_2(t)$ (the result of passing $Y_2(t)$ through a Weiner filter) and $R_{Y_1 Y_2}(\tau) = 0$ thus it is orthogonal to $Y_1(t)$. Similarly, we have $R_{\hat{X}_1 Y_2}(\tau) = 0$.

5. The estimation error is orthogonal to all the measurements, and all process have zero expectation so the estimator is the optimal linear estimator.

**Problem 3:**

1. The optimal linear estimator is Wiener filter sampled at time $t = t_0$. The frequency response of the Wiener filter:

$$H(\omega) = \frac{S_{N_1 N_2}(\omega)}{S_{N_2}(\omega)}$$

2. The processes are JWSS. This can be concluded by the fact that the two processes are WSS (passing of a WSS process through an LTI system) and to check that their cross-correlation is dependent of time difference only:

$$\begin{aligned}
R_{N_1,N_2}(t, t-\tau) &= E\{N_1(t) N_2(t-\tau)\} = E\{[V * g_1](t) \cdot [V * g_2](t-\tau)\} \\
&= E\left\{ \int_{-\infty}^{\infty} V(t-s) g_1(s) ds \cdot \int_{-\infty}^{\infty} V(t-\tau-r) g_2(r) dr \right\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{E\{V(t-s) V(t-\tau-r)\}}_{R_V(\tau+r-s)} g_1(s) g_2(r) ds dr \\
&= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_V(\tau+r-s) g_1(s) ds \right] g_2(r) dr \\
&= \int_{-\infty}^{\infty} [R_V * g_1](\tau+r) g_2(r) dr = \int_{-\infty}^{\infty} [R_V * g_1](\tau-r) g_2(-r) dr \\
&= R_V(\tau) * g_1(\tau) * g_2(-\tau)
\end{aligned}$$

From here we can also find the cross-spectrum:

$$S_{N_1 N_2}(\omega) = S_V(\omega) G_1(\omega) G_2^*(\omega) = \begin{cases} a & \omega_2 < |\omega| < \min(\omega_1, 2) \\ 0 & \text{else} \end{cases}$$

The spectrum and the expectation are found directly:

$$E\{N_1(t)\} = E\{V(t)\} G_1(0) = 0$$

$$E\{N_2(t)\} = E\{V(t)\} G_2(0) = 0$$

$$S_{N_1}(\omega) = S_V(\omega) |G_1(\omega)|^2 = \begin{cases} a & |\omega| \le \min(\omega_1, 2) \\ 0 & \text{else} \end{cases}$$

$$S_{N_2}(\omega)=S_V(\omega)\left|G_2(\omega)\right|^2=\begin{cases} a & \omega_2\le|\omega|\le 2 \\ 0 & \text{else}\end{cases}$$

3. Let us substitute the spectrum we found in section 2 in the general expression of a Wiener filter:

$$H(\omega)=\frac{S_{N_1N_2}(\omega)}{S_{N_2}(\omega)}=\begin{cases} 1 & \omega_2\le|\omega|\le\min(\omega_1,2) \\ 0 & \text{else}\end{cases}$$

The mean squared error:

$$E\left\{\varepsilon^2\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(S_{N_1}(\omega)-S_{N_1N_2}(\omega)H(\omega)\right)d\omega=\frac{a}{\pi}\min(\omega_1,\omega_2,2)$$

<u>Meaning:</u> the contribution to the error comes from the frequency intervals in which $N_1$ exists but $N_2$ does not.

4. The minimum error is zeroed when $\omega_1=0$ (there is no signal to estimate) or when $\omega_2=0$ (i.e. $N_2(t)\equiv V(t)$).
5. The maximal error is achieved when $\omega_1,\omega_2\ge 2$. Namely, the source $N_1$ is maximal but there is no correlation between $N_1$ and $N_2$. Alternatively, it can be said that when $\omega_2\ge\min(\omega_1,2)$, the error is equal to the whole signal $N_1$.

**<u>Problem 4:</u>**

1. It holds that:

$$Z(t)=X_2(t)+X_1(t)\cos(\omega_0t+\theta)$$
$$R_Z(t+\tau,t)=R_{X_2}(\tau)+R_{X_1}(\tau)E\left\{\cos(\omega_0t+\omega_0\tau+\theta)\cos(\omega_0t+\theta)\right\}$$
$$=R_{X_2}(\tau)+R_{X_1}(\tau)\left(\frac{1}{2}E\left\{\cos(\omega_0\tau)\right\}+\frac{1}{2}E\left\{\cos(2\omega_0t+\omega_0\tau+2\theta)\right\}\right)$$

In problem 1 we showed that:

$$E\left\{\cos(2\omega_0t+\omega_0\tau+2\theta)\right\}=0$$

and, therefore:

$$R_Z(\tau)=R_{X_2}(\tau)+\frac{1}{2}R_{X_1}(\tau)\cos(\omega_0\tau)$$
$$S_Z(\omega)=S_{X_2}(\omega)+\frac{1}{4}\left\{S_{X_1}(\omega-\omega_0)+S_{X_1}(\omega+\omega_0)\right\}$$

2. 

$$S_{\hat{X}_2}(\omega)=|H(\omega)|^2\,S_Z(\omega)=\begin{cases} S_Z(\omega) & |\omega|<B \\ 0 & O.W.\end{cases}$$

3.

$$R_{X_2\hat{X}_2}(t+\tau,t)=E\left\{X_2(t+\tau)\hat{X}_2(t)\right\}$$
$$=E\left\{X_2(t+\tau)\left(\int_{-\infty}^{\infty}h(\alpha)Z(t-\alpha)d\alpha\right)\right\}=\int_{-\infty}^{\infty}h(\alpha)E\left\{X_2(t+\tau)Z(t-\alpha)\right\}d\alpha$$
$$=\int_{-\infty}^{\infty}h(\alpha)E\left\{X_2(t+\tau)\left[Y_1(t-\alpha)+X_2(t-\alpha)\right]\right\}d\alpha\overset{(1)}{=}\int_{-\infty}^{\infty}h(\alpha)E\left\{X_2(t+\tau)X_2(t-\alpha)\right\}d\alpha$$
$$=\int_{-\infty}^{\infty}h(\alpha)R_{X_2}(\tau+\alpha)d\alpha=R_{X_2}(\tau)*h(-\tau)=R_{X_2\hat{X}_2}(\tau)$$

(1) $X_2(t)$ is independent of $Y_1(t)$.

$$\Rightarrow S_{X_2\hat{X}_2}(\omega)=S_{X_2}(\omega)\cdot H^*(\omega)=\begin{cases}e^{-|\omega|} & |\omega|<B\\ 0 & O.W.\end{cases}$$

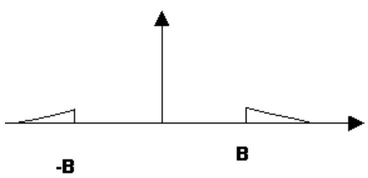
4. Let us find the error spectrum through the autocorrelation function of the error (notice that the filter we used $H(\omega)$ is not the optimal filter and, therefore, the formulas for the spectrum of the estimation error for the Wiener filter are not valid here).

$$R_{ee}(\tau)=E\left\{\left(X_2(t+\tau)-\hat{X}_2(t+\tau)\right)\cdot\left(X_2(t)-\hat{X}_2(t)\right)\right\}$$
$$=R_{X_2}(\tau)-R_{X_2\hat{X}_2}(\tau)-R_{\hat{X}_2X_2}(\tau)+R_{\hat{X}_2}(\tau)$$
$$S_{ee}(\omega)=S_{X_2}(\omega)-S_{X_2\hat{X}_2}(\omega)-S_{\hat{X}_2X_2}(w)+S_{\hat{X}_2}(\omega)$$
$$=S_{X_2}(\omega)-S_{X_2\hat{X}_2}(\omega)-S^*_{X_2\hat{X}_2}(w)+S_{\hat{X}_2}(\omega)$$
$$=S_{X_2}(\omega)-2\,\mathrm{Re}\left\{S_{X_2\hat{X}_2}(\omega)\right\}+S_{\hat{X}_2}(\omega)$$
$$=S_{X_2}(\omega)-2S_{X_2}(\omega)H(\omega)+H(\omega)\cdot\left(S_{X_2}(\omega)+\frac{1}{4}\left(S_{X_1}(\omega-\omega_0)+S_{X_1}(\omega+\omega_0)\right)\right)$$
$$=S_{X_2}(\omega)\left(1-H(\omega)\right)+\frac{1}{4}H(\omega)\cdot\left(S_{X_1}(\omega-\omega_0)+S_{X_1}(\omega+\omega_0)\right)$$

Let us distinguish between three cases:

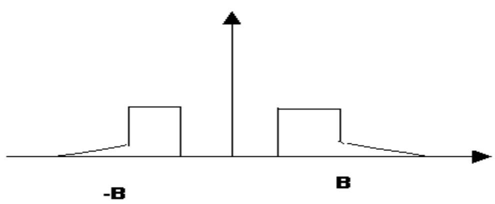
a. $B<\frac{1}{2}$:

$$S_e(\omega)=\begin{cases}e^{-|\omega|} & |\omega|>B\\ 0 & O.W.\end{cases}$$



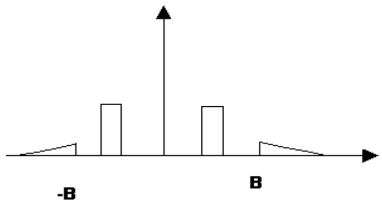

b. $\frac{1}{2}<B<2\frac{1}{2}$:

$$S_e(\omega)=\begin{cases}0 & |\omega|<\frac{1}{2}\\ p/4 & \frac{1}{2}<|\omega|<B\\ e^{-|\omega|} & |\omega|>B\end{cases}$$



c. $B > 2\frac{1}{2}$:

$$S_e(\omega) = \begin{cases} 0 & |\omega| < \frac{1}{2}, \frac{5}{2} < |\omega| < B \\ {}^{p}\!/_{4} & \frac{1}{2} < |\omega| < \frac{5}{2} \\ e^{-|\omega|} & |\omega| > B \end{cases}$$




5.

$$R_{X_1}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{X_1}(\omega)d\omega = \frac{1}{2\pi}\cdot 2P \underbrace{=}_{\text{נתון}} \frac{4}{\pi e} \Rightarrow P = \frac{4}{e}$$

The error is:

$$MSE \triangleq E\{e^2(t)\} = R_e(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_e(\omega)d\omega$$

Let us check the value of the error in each of the different ranges:

$$\underline{\underline{B < \frac{1}{2}}}$$

$$MSE = \frac{2}{2\pi}\int_{B}^{\infty} e^{-\omega}d\omega = \frac{1}{\pi}e^{-B} \Rightarrow \min\{MSE\} = \frac{1}{\pi}e^{-\frac{1}{2}}; B = \frac{1}{2}$$

$$\underline{\underline{\frac{1}{2} < B < 2\frac{1}{2}}}$$

$$MSE = \frac{1}{\pi}e^{-B} + 2\left(B - \frac{1}{2}\right)\frac{P}{4}\cdot\frac{1}{2\pi} = \frac{1}{\pi}\left(e^{-B} + \left(B - \frac{1}{2}\right)\frac{P}{4}\right) \underset{P=4/e}{=} \frac{1}{\pi}\left(e^{-B} + \left(B - \frac{1}{2}\right)\cdot\frac{1}{e}\right)$$

$$\frac{\partial MSE}{\partial B} = -e^{-B} + \frac{1}{e} = 0 \Rightarrow B = 1; \quad MSE = \frac{1.5}{\pi e}$$

$$\underline{\underline{B > 2\frac{1}{2}}}$$

$$MSE = \frac{1}{2\pi}\cdot 2\cdot 2\cdot\frac{P}{4} + \frac{1}{\pi}e^{-B} = \frac{P}{2\pi} + \frac{1}{\pi}e^{-B} = \frac{2}{\pi e} + \frac{1}{\pi}e^{-B}$$

$$\Rightarrow MSE_{\min} = \frac{2}{\pi e}, B \to \infty$$

Namely, the value of $B$ that gives the minimum squared error in the reproduction is:

$$B=1,\quad MSE_{\min}=\frac{1.5}{\pi e}$$

6. We saw already in the previous sections that $X_2(t)$ and $Z(t)$ are JWSS. We will use Wiener filter (with expectation zero):

$$H^{opt}(\omega)=\frac{S_{X_2Z}(\omega)}{S_Z(\omega)}=\frac{S_{X_2}(\omega)}{S_{X_2}(\omega)+\frac{1}{4}\left(S_{X_1}(\omega-\omega_0)+S_{X_1}(\omega+\omega_0)\right)}=$$

$$=\begin{cases}1 & if\ |\omega|\notin[0.5,2.5) \\ \dfrac{e^{-|\omega|}}{e^{-|\omega|}+P/4} & f\ |\omega|\in[0.5,2.5)\end{cases}$$

The spectrum of the error is given by:

$$S_e(\omega)=S_{X_2}(\omega)-\frac{|S_{X_2Z}(\omega)|^2}{S_Z(\omega)}=S_{X_2}(\omega)-\frac{|S_{X_2}(\omega)|^2}{S_Z(\omega)}=S_{X_2}(\omega)\left(1-\frac{S_{X_2}(\omega)}{S_Z(\omega)}\right)$$

$$=\begin{cases}0 & if\ |\omega|\notin[0.5,2.5) \\ \dfrac{e^{-|\omega|}P/4}{e^{-|\omega|}+P/4} & f\ |\omega|\in[0.5,2.5)\end{cases}$$

Notice that the $S_e(\omega)$ that this filter achieves is smaller than the one that $H(\omega)$ achieves for every $\omega$ and any election of $B$.

**Problem 5:**

1. $X[n]$ can be written in the following way:

$$X[n]=h_0[n]*W[n]$$

$$H_0(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Moreover:

$$H_1(e^{j\omega})=-0.5+e^{-j\omega}$$

$$\left|H_1(e^{j\omega})\right|^2=(-0.5+e^{-j\omega})(-0.5+e^{-j\omega})=0.25+1-0.5(e^{j\omega}+e^{-j\omega})=1.25-\cos\omega$$

$$H_2(e^{j\omega})=\frac{1}{-0.5+e^{-j\omega}}=e^{j\omega}\cdot\frac{1}{1-0.5e^{j\omega}}=e^{j\omega}H_0^*(e^{j\omega})$$

And it holds that:

$$Y_2[n]=h_2[n]*(X[n]+Z[n])=\underbrace{h_2[n]*h_0[n]*W[n]}_{\tilde{W}[n]}+\underbrace{h_2[n]*Z[n]}_{\tilde{Z}[n]}$$

The processes $Z[n]$ and $W[n]$ are independent with expectations zero, thus:

$$R_{Y_2}[l] = E\left\{\left(\tilde{W}[n+l] + \tilde{Z}[n+l]\right)\left(\tilde{W}[n] + \tilde{Z}[n]\right)\right\} = R_{\tilde{W}}[l] + R_{\tilde{Z}}[l]$$

$$R_{Y_2W}[l] = R_{\tilde{W}W}[l]$$

$$S_{Y_2}\left(e^{j\omega}\right) = \left|H_0\left(e^{j\omega}\right)H_2\left(e^{j\omega}\right)\right|^2 \underbrace{S_W\left(e^{j\omega}\right)}_{=1} + \left|H_2\left(e^{j\omega}\right)\right|^2 \underbrace{S_Z\left(e^{j\omega}\right)}_{=1} = \left|H_0\left(e^{j\omega}\right)H_2\left(e^{j\omega}\right)\right|^2 + \left|H_2\left(e^{j\omega}\right)\right|^2 =$$

$$= \left|H_0\left(e^{j\omega}\right)\right|^2\left(\left|H_0\left(e^{j\omega}\right)\right|^2 + 1\right)$$

$$S_{Y_2W}\left(e^{j\omega}\right) = H_2\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)S_W\left(e^{j\omega}\right) = e^{j\omega}\left|H_0\left(e^{j\omega}\right)\right|^2$$

2. $W[n]$ is an i.i.d process and, more specifically, WSS. We saw in the previous section that $Y_2[n]$ is WSS and that $R_{Y_2W}[n+l,n] = R_{Y_2W}[l]$. Therefore, the processes are JWSS and the optimal linear estimator is achieved by passing $Y_2[n]$ through a Wiener filter whose frequency response is:

$$F\left(e^{j\omega}\right) = \frac{S_{WY_2}\left(e^{j\omega}\right)}{S_{Y_2}\left(e^{j\omega}\right)} = \frac{e^{-j\omega}}{1 + \left|H_0\left(e^{j\omega}\right)\right|^2} = \frac{5 - 4\cos\omega}{9 - 4\cos\omega}e^{-j\omega}$$

3. Notice that $Y_1[n]$ is obtained from $Y_2[n]$ by passing through a system whose frequency response is:

$$Y_1[n] = h_3[n] * Y_2[n]$$

$$H_3\left(e^{j\omega}\right) = \frac{1}{\left(H_2\left(e^{j\omega}\right)\right)} \cdot H_1\left(e^{j\omega}\right) = \left(H_1\left(e^{j\omega}\right)\right)^2$$

$H_3\left(e^{j\omega}\right)$ is an invertible LTI system and, thus, it holds that (based on section 2 in problem 2 in the class exercise) the spectrum of the error of the optimal linear estimator of $W[n]$ from the random process $Y_1[n]$ is equal to the spectrum of the error of the estimator of $W[n]$ from the random process $Y_2[n]$, and, specifically, also the MSE of both of the estimators is equal.