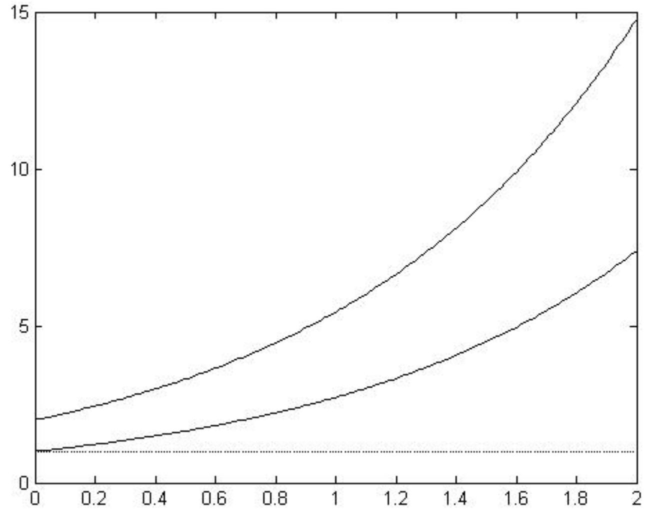
Solution to Problem Set 7

Problem 1:

1. The sample function of the process:

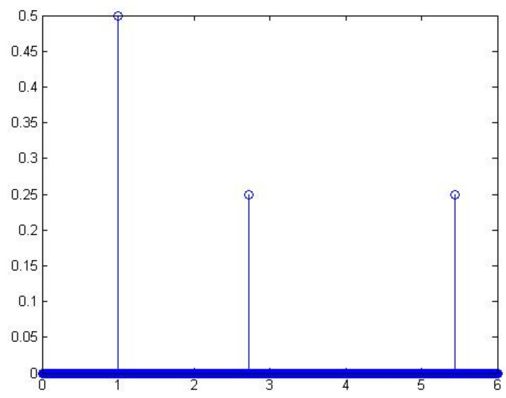


The lower graph is for $\omega = \omega_1$, the middle one is for $\omega = \omega_2$ and the upper one is for $\omega = \omega_3$.

2. We will "freeze" the time parameter at $t_0 = 1$. We get a random variable with the following distribution:

$$X\left(t_0 = 1\right) = \begin{cases} 1 & w.p.\frac{1}{2} \\ e & w.p.\frac{1}{4} \\ 2e & w.p.\frac{1}{4} \end{cases}$$

The density function of this random variable is:



3. Given that $X\left(t_0=0\right)=1$ holds, it is obvious that $\omega\in\{\omega_1,\omega_2\}$ (for $\omega=\omega_3$ we would have gotten $X\left(t_0=0\right)=2\cdot e^0=2$, contrary to the given). Therefore, for $X(1)$, there are two possible values:

$$\Pr\left(X(1)=1\,|\,\underbrace{X(0)=1}_{\omega=\omega_1}\right)=\frac{\Pr\left(X(1)=1\cap X(0)=1\right)}{\Pr\left(X(0)=1\right)}=\frac{\Pr\left(\omega=\omega_1\right)}{\Pr\left(\omega=\omega_1\right)+\Pr\left(\omega=\omega_2\right)}=\frac{2}{3}$$

$$\Pr\left(X(1)=e\,|\,\underbrace{X(0)=1}_{\omega=\omega_2}\right)=\frac{\Pr\left(X(1)=e\cap X(0)=1\right)}{\Pr\left(X(0)=1\right)}=\frac{\Pr\left(\omega=\omega_2\right)}{\Pr\left(\omega=\omega_1\right)+\Pr\left(\omega=\omega_2\right)}=\frac{1}{3}$$

Thus, overall, we get the conditional distribution:

$$X(1)|_{X(0)=1}=\begin{cases}1 & \text{w.p. } \frac{2}{3}\\ e & \text{w.p. } \frac{1}{3}\end{cases}$$

<u>Problem 2:</u>

1.

$$E\{X(t)\}=\int_{-\infty}^{\infty}e^{-at}f_A(a)da$$

2.

$$R_x(t_1,t_2)\triangleq E\{X(t_1)X(t_2)\}=\int_{-\infty}^{\infty}e^{-at_1}e^{-at_2}f_A(a)da$$

3. On us to find $f_x(t)$. $X(t)$ is a one-to-one function of A. That is to say, for every value of A, exists a constant $X(t)$, and do not exist $a\neq a'$ such that $x_a(t)$ is equal to $x_{a'}(t)$. And, therefore, based on the transition formula between two distribution functions:

$$X(t)=x\Rightarrow a=\frac{-\ln(x)}{t}$$

$$\Downarrow$$

$$f_{x(t)}(x;t)=\frac{1}{\left|\partial X(t)\big/\partial a\right|}f_A(a)\Bigg|_{a=\frac{-\ln x}{t}}=\frac{1}{|-te^{-at}|}f_A(a)\Big|_{a=\frac{-\ln(x)}{t}}=$$

$$=\frac{1}{|-tx|}f_A\left(\frac{-\ln x}{t}\right)=\frac{1}{xt}f_A\left(\frac{-\ln x}{t}\right)$$

Problem 3:

1. a. The sum of the probabilities of N_0 taking each whole number must be 1. From here:

$$\sum_{n=-\infty}^{\infty}\Pr\left(N_0=n\right)=A\sum_{n=-(M-1)}^{M-1}\left(1-\frac{|n|}{M}\right)=A+2A\sum_{n=1}^{M-1}\left(1-\frac{n}{M}\right)=$$

$$=A+2A(M-1)-\frac{2A}{M}\sum_{n=1}^{M-1}n=A+2A(M-1)-\frac{2A}{M}\frac{(M-1)M}{2}=$$

$$A\left[1+2(M-1)-(M-1)\right]=AM$$

$$\Rightarrow \boxed{A=\frac{1}{M}}$$

b. The expected value is 0, since the probability is a symmetric function:

$$\forall n.\Pr\left(N_0=n\right)=\Pr\left(N_0=-n\right)$$

c. Since the expectation is 0, we get:

$$VAR\left(N_0\right)=E\left[N_0^2\right]-E^2\left[N_0\right]=E\left[N_0^2\right]$$

$$E\left[N_0^2\right]=\sum_{n=-(M-1)}^{M-1}n^2\frac{1}{M}\left(1-\frac{|n|}{M}\right)=\frac{2}{M}\sum_{n=1}^{M-1}\left(n^2-\frac{n^3}{M}\right)=\frac{M^2-1}{6}$$

2. a.

$$E\left[N[k]\right]=E\left[N_0+\sum_{j=1}^{k}W[j]\right]=E\left[N_0\right]+\sum_{j=1}^{k}E\left[W[j]\right]=0$$

b. Let us first assume that $l\geq k$:

$$R_N\left[k,k+\Delta k\right]=E\left[N[k]N[k+\Delta k]\right]=E\left[N[k]\left(N[k]+\sum_{j=k+1}^{k+\Delta k}W[j]\right)\right]=$$

$$=E\left[N^2[k]\right]+E\left[N[k]\sum_{j=k+1}^{k+\Delta k}W[j]\right]=E\left[N^2[k]\right]=Var\left[N[k]\right].$$

$$Var\left[N[k]\right]=Var\left[N_0+\sum_{j=1}^{k}W[j]\right]\overset{(a)}{=}Var\left[N_0\right]+\sum_{j=1}^{k}Var\left[W[j]\right].$$

where (a) is from the fact that $W[k]$ is i.i.d and independent of N_0 .

$$Var\left[W[k]\right]=E\left[W^2[k]\right]-E^2\left[W[k]\right]=E\left[W^2[k]\right]=\frac{1}{4}(-1)^2+\frac{1}{2}0^2+\frac{1}{4}1^2=\frac{1}{2}.$$

Let us use the variance of N_0 from section 1:

$$R_N\left[k,k+\Delta k\right]=Var\left[N[k]\right]=Var\left[N_0\right]+\sum_{j=1}^{k}Var\left[W[j]\right]=\frac{M^2-1}{6}+\frac{1}{2}k$$

And in the general case: $R_N[k,l]=\dfrac{M^2-1}{6}+\dfrac{1}{2}\min(k,l)$.

c. The process is not stationary in either sense since its auto-correlation function is not dependent only on time difference.

3. a. The optimal estimator is, of course, the conditional expectation estimator:

$$\begin{aligned}
\hat{N}_{opt}[k] &= E\left[N[k] \mid N[k-1]\right] = E\left[N[k-1]+W[k] \mid N[k-1]\right] = \\
&= E\left[N[k-1] \mid N[k-1]\right] + E\left[W[k] \mid N[k-1]\right] = \\
&= N[k-1] + E\left[W[k]\right] = \boxed{N[k-1]}.
\end{aligned}$$

where we made use of the fact that $W[k]$ is independent of $N[k-1]$, since it is a function of N_0 and of $W[j],\ j<k$.

b. The optimal estimator is the conditional expectation estimator:

$$\begin{aligned}
&\hat{N}_{opt}[k] = E\left[N[k] \mid N[k-1],N[k-2]\right] = E\left[N[k-1]+W[k] \mid N[k-1],N[k-2]\right] \\
&= E\left[N[k-1] \mid N[k-1],N[k-2]\right] + E\left[W[k] \mid N[k-1],N[k-2]\right] \\
&= N[k-1]
\end{aligned}$$

* The last equality is due to the fact that $\{N[k-1],N[k-2]\}$ are a deterministic function of $\{N_0,W_1,\cdots,W_{k-1}\}$. $W[k]$ is independent of $\{N_0,W_1,\cdots,W_{k-1}\}$, and, thus, also of $\{N[k-1],N[k-2]\}$.

c. Notice that the optimal estimator of $N[k]$ from the pair of samples came out linear:

$$\hat{N}_{opt}[k] = [1\ \ 0]\begin{bmatrix} N[k-1] \\ N[k-2] \end{bmatrix} = N[k-1]$$

Therefore, this is also the optimal linear estimator.

<u>Calculation of the MSE:</u> for the three cases, the estimator is the same – $\hat{N}_{opt}[k]=N[k-1]$, and, thus, the MSE for all of them is:

$$MSE = E\left[\left(\hat{N}_{opt}[k]-N[k]\right)^2\right] = E\left[\left(N[k-1]-\{N[k-1]+W[k]\}\right)^2\right] = E\left[W^2[k]\right]\boxed{=\frac{1}{2}}$$

4. a. Let us first calculate the <u>optimal</u> estimator of $N[k]$ from the samples vector. For the same reason as in section 3.b., we get that $\hat{N}[k]=N[k-1]$:

$$\hat{N}_{opt}[k] = E\left[N[k] \mid N[k-1],\cdots,N[k-10]\right]$$
$$= E\left[N[k-1]+W[k] \mid N[k-1],\cdots,N[k-10]\right]$$
$$= E\left[N[k-1] \mid N[k-1],\cdots,N[k-10]\right]+E\left[W[k] \mid N[k-1],\cdots,N[k-10]\right]$$
$$= N[k-1]$$

Here, too, the estimator we got is linear, and, therefore, it is the optimal linear estimator from the samples vector.

b. Calculation of the MSE: as in the previous section:

$$MSE = E\left[\left(\hat{N}_{opt}[k]-N[k]\right)^2\right] = E\left[W^2[k]\right] = \frac{1}{2}.$$

Problem 4:

1. $X(t)$ is S.S.S, that is to say:

$$f_{X_1,\ldots,X_n}\left(x_1,\ldots,x_n;t_1,\ldots,t_n\right) = f_{X_1,\ldots,X_n}\left(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau\right) \quad \forall t_i,n,\tau \quad i\in[1,n]$$

We would like to prove that $Z(t)$ is SSS as well. To that end we will show that the joint characteristic function of $[Z(t_1),\ldots,Z(t_n)]$ is equal to the joint characteristic function of $[Z(t_1+\tau),\ldots,Z(t_n+\tau)]$ for any time set $t_1,\ldots t_n$, and any $n\in\mathbb{N},\tau\in\mathbb{R}$. We have:

$$\Phi_{Z_1,\ldots,Z_n;t_1,\ldots,t_n}(\omega_1,\ldots,\omega_n) = \mathbb{E}\Big[\exp\big(j(\omega_1 Z(t_1)+\cdots+\omega_n Z(t_n))\big)\Big]$$

$$= \int_{\mathbb{R}^n} \exp\big(j(\omega_1 z_1+\cdots+\omega_n z_n)\big) f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n;t_1,\ldots,t_n)dz_1\ldots dz_n$$

$$\overset{Z(t)=g(X(t))}{\triangleq} \int_{\mathbb{R}^n} \exp\big(j(\omega_1 g(x_1)+\cdots+\omega_n g(x_n))\big) f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;t_1,\ldots,t_n)dx_1\ldots dx_n$$

$$\overset{X(t)\ SSS}{\triangleq} \int_{\mathbb{R}^n} \exp\big(j(\omega_1 g(x_1)+\cdots+\omega_n g(x_n))\big) f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau)dx_1\ldots dx_n$$

$$\overset{Z(t)=g(X(t))}{\triangleq} \int_{\mathbb{R}^n} \exp\big(j(\omega_1 z_1+\cdots+\omega_n z_n)\big) f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n;t_1+\tau,\ldots,t_n+\tau)dz_1\ldots dz_n$$

$$= \Phi_{Z_1,\ldots,Z_n;t_1+\tau,\ldots,t_n+\tau}(\omega_1,\ldots,\omega_n)$$

Note that this is true for any $t_i,n,\tau \Rightarrow Z(t)$ is S.S.S

2. An example of a W.S.S process that operation of a memoryless system on it creates a process which is not W.S.S:

Let us define the following random process:

$$X(t) = A\cos(t)+B\sin(t)$$

where A and B are independent random variables:

$$A = \begin{cases} +1 & w.p. \quad 1/2 \\ -1 & w.p. \quad 1/2 \end{cases} \quad , \qquad B \sim N(0,1)$$

Let us check whether $X(t)$ is W.S.S:

$$E[X(t)] = E[A\cos(t) + B\sin(t)] = E[A]\cos(t) + E[B]\sin(t) = 0\cdot\cos(t) + 0\cdot\sin(t) = 0, \quad \forall t$$

$$E[X(t_1)X(t_2)] = E[A^2]\cos(t_1)\cos(t_2) + E[B^2]\sin(t_1)\sin(t_2) =$$
$$= 1\cdot\cos(t_1)\cos(t_2) + 1\cdot\sin(t_1)\sin(t_2) = \cos(t_1 - t_2) = R_X(t_1 - t_2)$$

We got that $X(t)$ is W.S.S.

Now, let us examine the random process $Z(t) = (X(t))^2$:

$$Z(t) = (A\cos(t) + B\sin(t))^2 = A^2\cos^2(t) + B^2\sin^2(t) + 2AB\cos(t)\sin(t)$$

At $t = 0$ we get:

$$Z(0) = A^2\cos^2(0) + B^2\sin^2(0) + 2AB\cos(0)\sin(0) = A^2 = 1 \qquad w.p.\ 1$$

Namely, $\text{var}(Z(0)) = 0$.

At $t = \frac{\pi}{2}$, we get:

$$Z\left(\frac{\pi}{2}\right) = A^2\cos^2\left(\frac{\pi}{2}\right) + B^2\sin^2\left(\frac{\pi}{2}\right) + 2AB\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = B^2$$

Namely, $\text{var}\left(Z(\pi/2)\right) \neq 0$.

Overall we got:

$$\text{var}(Z(0)) \neq \text{var}\left(Z\left(\frac{\pi}{2}\right)\right) \qquad (*)$$

Since variance constant in time is a necessary condition for W.S.S, it follows that $Z(t)$ is not W.S.S.

Explanation: if we negate to assume that $Z(t)$ is W.S.S, we get:

$$\text{var}(Z(0)) = E\left[Z^2(0)\right] - E\left[Z(0)\right]^2 = R_Z(0,0) - E_Z^{\ 2}(0) \overset{(1)}{=} R_Z\left(\frac{\pi}{2},\frac{\pi}{2}\right) - E_Z^{\ 2}\left(\frac{\pi}{2}\right)$$
$$= E\left[Z^2\left(\frac{\pi}{2}\right)\right] - E\left[Z\left(\frac{\pi}{2}\right)\right]^2 = \text{var}\left(Z\left(\frac{\pi}{2}\right)\right)$$

where (1) is from the negated assumption that $Z(t)$ is W.S.S (expectation constant in time and auto-correlation dependent of time difference only). Namely, we got that $\text{var}(Z(0)) = \text{var}\left(Z\left(\frac{\pi}{2}\right)\right)$, in contradiction to (*), and, thus, the assumption that $Z(t)$ is W.S.S is incorrect.

Problem 5:

1. It is clear that in this case the probability is 1, since the sine function is bounded between -1 and 1.

2. Throughout a complete cycle, *sin* necessarily takes negative values, therefore the probability is 0.

3. Notice that the sine function takes positive values throughout half a cycle and negative values throughout the following half of the cycle. We would like the given section, whose length is a quarter of a cycle, to be completely contained in the half cycle where *sin* takes positive values, and from here that the probability is 0.25.

4. The answer in this case is 0.5 (only once in a cycle the value 1 is taken. We want that a section with length of half a cycle contain this point).

5. There are two solutions in the section $\left[-\pi,\pi\right]$ to the equation $\sin\left(y\right)=\sin\left(\alpha\right)$ and they are $y_1=\alpha,\ y_2=\pi-\alpha$. Therefore, based on the given, we conclude that $\theta=\alpha,\ \theta=\pi-\alpha$, but these angles are obtained with equal probability since θ distributes uniformly in the section $\left[-\pi,\pi\right]$. Thus:

$$X\left(t\right)\Big|_{X(0)=\sin(\alpha)}=\begin{cases}\sin\left(\omega_0 t+\alpha\right) & w.p.\dfrac{1}{2}\\ \sin\left(\omega_0 t+\left(\pi-\alpha\right)\right) & w.p.\dfrac{1}{2}\end{cases}$$

6. The expected value of the process:

$$E\left[X\left(t_0\right)\right]=E\left[\sin\left(\omega_0 t_0+\theta\right)\right]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\left(\omega_0 t_0+\theta\right)d\theta=0$$

7. For simplicity of the solution, we will assume for the following section that $\omega_0=\pi$. For all *t*, when -1< *x* <1, two solutions for ϕ exist that lead to *x*:

$$\theta_i=\arcsin(x)-\pi t\Rightarrow$$

$$f_X(x;t)=\sum_{i=1}^{2}\frac{1}{\left|\dfrac{\partial x(t)}{\partial\theta_i}\right|}f_\theta(\theta_i)\,|_{\theta_i=\arcsin(x)-\pi t}=$$

$$=\sum_{i=1}^{2}\frac{1}{|\cos(\pi t+\theta)|}\frac{1}{2\pi}=\frac{1}{2\pi}\frac{2}{\cos(\arcsin(x))}=\frac{1}{\pi}\frac{1}{\sqrt{1-\sin^2(\arcsin(x))}}$$

$$=\begin{cases}\dfrac{1}{\pi}\dfrac{1}{\sqrt{1-x^2}} & -1<x<1\\ 0 & else\end{cases}$$