Problem Set 7 – Random Processes

Problem 1:

The random process $X(t) = A \cdot e^{Bt}$, $t \geq 0$ is given, where A, B are random variables.

The sample space: $\Omega = \{\omega_1, \omega_2, \omega_3\}$ $w.p. \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\}$, where the events ω_i are the following:

$$\omega_1 = \{A = 1, B = 0\} \Rightarrow X(t) \equiv 1 \quad w.p. \ \frac{1}{2}$$

$$\omega_2 = \{A = 1, B = 1\} \Rightarrow X(t) = e^t \quad w.p. \ \frac{1}{4}$$

$$\omega_3 = \{A = 2, B = 1\} \Rightarrow X(t) = 2e^t \quad w.p. \ \frac{1}{4}$$

1. Draw all the possible sample functions of the process.
2. What is the PDF of the random variable $X(t_0 = 1)$?
3. Find the conditional distribution of $X(t_0 = 1)$, given that $X(t_0 = 0) = 1$ holds.

Problem 2:

Given is the random process $X(t) = e^{-At}$, where A is a random variable with PDF $f_A(a)$.

1. Find the expectation of the process.
2. Find the auto-correlation function of the process.
3. Find the first-order PDF of the process, $f_X(x;t)$.

Problem 3:

Formulas that may be useful throughout the problem (k, m whole):

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

$$\sum_{k=1}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2$$

The random variable N_0 is given, which takes whole values (positive and negative).

It holds that:

$$P_r\left(N_0 = n\right) = \begin{cases} A\left[1 - \dfrac{|n|}{M}\right] & |n| < M \\ 0 & otherwise \end{cases}$$

where M is a known natural constant, n is a whole number and A is constant.

1. a. Calculate the value of A.
   b. Calculate the expected value of N_0.
   c. Calculate the variance of N_0.

The discrete time random process $N[k]$ is defined as follows:

$$N[k] = 0 \quad (\forall k < 0)$$
$$N[0] = N_0$$
$$N[k] = N[k-1] + W[k] \quad (\forall k > 0)$$

where $W[k]$ is an i.i.d series with the following distribution (of first order):

$$W[k] = \begin{cases} 1 & w.p\ 0.25 \\ 0 & w.p\ 0.5 \\ -1 & w.p\ 0.25 \end{cases}$$

It is given, also, that all the series $W[k]$ is independent of N_0.

2. a. Calculate the expected value of the random process $N[k]$.
   b. Calculate the auto-correlation function of $N[k]$.
   c. Is $N[k]$ a stationary process? If so, in what sense?

3. Now, $N[k]$ is to be estimated from its past samples.
   Calculate the following estimators <u>and the mean squared error</u> obtained in the estimation:
   a. The optimal estimator of $N[k]$ from $N[k-1]$.
   b. The optimal estimator of $N[k]$ <u>from the pair of samples</u> $N[k-1], N[k-2]$.
   c. The <u>optimal linear</u> estimator of $N[k]$ from the pair of samples $N[k-1], N[k-2]$.

4. a. What is the optimal linear estimator of $N[k]$ from the samples vector $\left[N[k-1], N[k-2], \ldots, N[k-10]\right]$? Prove your answer!

   b. What is the mean squared error obtained in the estimation?

Problem 4:

Let $X(t)$ be a random process and $g()$ be some real function, namely:

$$g : R \longrightarrow R$$

Let us define the random process $Z(t)$ as follows:

$$Z(t) = g\left(X(t)\right) \quad \forall t$$

Prove the following claims:

1. If $X(t)$ is S.S.S., then $Z(t)$ is also S.S.S.
2. <u>Bonus</u>: If $X(t)$ is W.S.S., then $Z(t)$ is not necessarily W.S.S.

Problem 5:

θ is a random variable that distributes uniformly in the section $[-\pi, \pi]$. $X(t)$ is a random process defined by $X(t) = \sin(\omega_0 t + \theta)$.

1. Calculate $\Pr\left(X(t_0) < 3\right)$.
2. Calculate $\Pr\left(X(t) \geq 0, \forall 0 \leq t \leq \frac{2\pi}{\omega_0}\right)$.
3. Calculate $\Pr\left(X(t) \geq 0, \forall 0 \leq t \leq \frac{\pi}{2\omega_0}\right)$.
4. Calculate $\Pr\left(\exists 0 \leq t \leq \frac{\pi}{\omega_0} : X(t) = 1\right)$.
5. What is the conditional distribution: $X(t)\big|_{X(0)=\sin\alpha}$?
6. Calculate $E\left[X(t_0)\right]$.
7. What is the first order PDF of the process?