Problem Set 8 – Stationarity, Power Spectral Density

Problem 1:

1. $X(t)$ is a WSS random process with auto-correlation function $R_X(\tau) = e^{-\alpha|\tau|}$. Calculate its PSD.
2. The process $Y(t)$ has the following PSD: $S_Y(\omega) = \begin{cases} 1 & |\omega| \le B \\ 0 & |\omega| > B \end{cases}$.

   Calculate its auto-correlation function.

Problem 2:

A random process $X(t)$ with expectation zero, auto-correlation function $R_X(\tau)$ and PSD $S_X(\omega)$ is given.

1. We define $Y(t) = X(t+a) - X(t-a)$. Express $Y(t)$ PSD and auto-correlation function using $R_X(\tau)$ and $S_X(\omega)$.
2. We now define $Y(t) = X(t) \cdot G(t)$ where $G(t)$ is a random process, independent of $X(t)$, with expectation 1 and auto-correlation function $R_G(\tau)$. Express $R_{XY}(t, t+\tau)$, $R_Y(t, t+\tau)$ and $E\{Y(t)\}$ with $R_X(\tau)$ and $R_G(\tau)$.
   Are $X(t)$ and $Y(t)$ **JWSS**?

Problem 3:

An AR process of the form $X_n = 0.5X_{n-1} + W_n$ is given, where W_n is an i.i.d process with expectation 0 and variance 1. It is assumed that the process exists from time $-\infty$ such that the asymptotical conditions take place and it may be assumed that X_n is a WSS process.

1. Find the optimal MMSE estimator of X_{n+5} given the sample X_n.
2. Find the optimal MMSE estimator of X_{n+5} given all the samples $\{X_k\}_{k=-\infty}^{n}$.

Now, the process $Y_n = X_n + Z_n$ is defined, where Z_n is an i.i.d process independent of W_n , with expectation 0 and variance 1.

3. Find the optimal linear MMSE estimator of Y_{n+5} given the sample Y_n.
4. Find the optimal linear MMSE estimator of Y_{n+5} given the samples Y_n, Y_{n-1}.

Problem 4:

Consider the following A.R. linear random process:

$$X_n = \frac{1}{2} X_{n-1} + W_n \qquad (*)$$

where:

$$X_0 = 0, \qquad W_n = \begin{cases} 1 & w.p. \quad 1/2 \\ 0 & w.p. \quad 1/2 \end{cases} \quad , \quad \{W_n\} \ i.i.d.$$

1. Find the distribution of X_n.
2. Find an expression for X_n as a function of the random process $\{W_i\}$.
3. Find the stationary distribution of X_n that suits the equation (*).