# Solution of Problem Set 1

## Problem 1:

1.

$$P(A) = 1 - P(TT...T) = 1 - \left(\tfrac{1}{2}\right)^n$$
$$P(B) = P(HTT..T) + P(THT..T) + ... + P(TTT..H) = n\left(\tfrac{1}{2}\right)^n$$
$$P(C) = P(HTT..T) + P(THT..T) + ... + P(TTT..H) + P(TTT...T) = (n+1)\left(\tfrac{1}{2}\right)^n$$
$$P(D) = 1 - P(TTT...T) - P(HHH..H) = 1 - \left(\tfrac{1}{2}\right)^{n-1}$$

2. Let us check for which values of $n$ the equation $P(C \cap D) = P(C)P(D)$ holds.

For $n=1$: we get that $P(C \cap D) = 0$ and $P(C)P(D) = 0$, namely the events C and D are independent.

For $n \geq 2$:

$$P(C)P(D) = (n+1)\left(\tfrac{1}{2}\right)^n \left(1 - \left(\tfrac{1}{2}\right)^{n-1}\right)$$
$$P(C \cap D) = P(B) = n\left(\tfrac{1}{2}\right)^n$$

This means that events C and D are independent if and only if $(n+1)\left(1 - \left(\tfrac{1}{2}\right)^{n-1}\right) = n$ holds, and its only solution is $n = 3$.

All in all we got that events C and D are independent iff $n = 1$ or $n = 3$.

## Problem 2:

1. $P(X \leq 0) = F_X(0) = \dfrac{0.25}{2} = 0.125$
2. $F_X(x)$ is continuous at 0 and thus: $P(X < 0) = P(X \leq 0) = 0.125$
3. $P(X \leq 1) = F_X(1) \overset{(1)}{=} F_X(1^+) = 0.5$
   where (1) is due to the right-continuity property of the CDF.
4. $P(X < 1) = F_X(1^-) = 0.25$
5. $P(0 \leq X < 1) = F_X(1^-) - F_X(0^-) = 0.25 - 0.125 = 0.125$
6. $P(0 < X \leq 1.75) = F_X(1.75) - F_X(0) = 0.75 - 0.125 = 0.625$
7. $P(X > 1) = 1 - P(X \leq 1) = 1 - 0.5 = 0.5$
8. $P(X \geq 2.5) = 1 - P(X < 2.5) = 1 - P(X \leq 2.5) = 1 - 1 = 0$

**Problem 3:**

If $Y = sign(X)$ then $Y$ can take only two values, with the following probabilities:

$$p_Y(1) = P(Y=1) = P(X>0) = 0.7$$
$$p_Y(-1) = P(Y=-1) = P(X<0) = 0.3$$

Therefore, the PDF consists of two delta functions:

$$f_Y(y) = 0.3\delta(y+1) + 0.7\delta(y-1)$$

whereas the CDF is a step function:

$$F_Y(y) = P(Y \le y) = \begin{cases} 0 & y < -1 \\ 0.3 & -1 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

**Problem 4:**

1. $Y = X + 17 \Rightarrow F_Y(y) = F_X(y-17) \Rightarrow f_Y(y) = f_X(y-17)$
2. $Y = -19X \Rightarrow F_Y(y) = 1 - F_X(-\tfrac{y}{19}) \Rightarrow f_Y(y) = \tfrac{1}{19} f_X(-\tfrac{y}{19})$
3. $Y = \max(X,0) = \begin{cases} 0 & X < 0 \\ X & X \ge 0 \end{cases}$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(y) & y \ge 0 \end{cases}$$

$$\Rightarrow f_Y(y) = f_X(y)u(y) + F_X(0)\delta(y)$$

4. $Y = |X| \Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(y) - F_X(-y) & y \ge 0 \end{cases}$

$$\Rightarrow f_Y(y) = \left[ f_X(y) + f_X(-y) \right] u(y)$$

**Problem 5:**

1. $P(X > 2) = Q(2)$

   $P(X < 7) = 1 - Q(7)$

   $P(X = 3) = 0$

2. $\lim_{x\to\infty} Q(x) = 0$

   $Q(0) = \tfrac{1}{2}$

   $\lim_{x\to-\infty} Q(x) = 1$

3. $f_Y(y) = \dfrac{1}{|a|} f_X\left(\dfrac{y-b}{a}\right) = \dfrac{1}{\sqrt{2\pi a^2}} e^{-\frac{(y-b)^2}{2a^2}}$