Recitation 9 – WSS Processes through LTI Systems, Wiener Filter

WSS Random Process through an LTI System

Let $X(t)$ be a WSS random process. $X(t)$ is set as the input of an LTI system with impulse response $h(t)$, whose Fourier transform is $H(\omega)$. In the output of the system is the random process $Y(t)$, that is JWSS with $X(t)$. Furthermore:

$$\eta_Y = E\{Y(t)\} = E\left\{\int_{-\infty}^{+\infty} h(\alpha) X(t-\alpha)\, d\alpha\right\} = \eta_X \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha = \eta_X H(\omega=0)$$

$$R_{YX}(\tau) = E\{Y(t) X(t-\tau)\} = E\{(h(t) * X(t)) X(t-\tau)\} = E\left\{\left(\int_{-\infty}^{+\infty} h(\alpha) X(t-\alpha)\, d\alpha\right) X(t-\tau)\right\}$$

$$= \int_{-\infty}^{+\infty} h(\alpha) E\{X(t-\alpha) X(t-\tau)\}\, d\alpha = \int_{-\infty}^{+\infty} h(\alpha) R_{XX}(\tau-\alpha)\, d\alpha = h(\tau) * R_{XX}(\tau)$$

$$R_{XY}(\tau) = R_{YX}(-\tau) = \int_{-\infty}^{+\infty} h(\alpha) R_{XX}(-\tau-\alpha)\, d\alpha = R_{XX}(-\tau) * h(-\tau) = R_{XX}(\tau) * h(-\tau)$$

$$R_{YY}(\tau) = E\{Y(t) Y(t-\tau)\} = E\left\{\left(\int_{-\infty}^{+\infty} h(\alpha) X(t-\alpha)\, d\alpha\right) Y(t-\tau)\right\} = \int_{-\infty}^{+\infty} h(\alpha) E\{X(t-\alpha) Y(t-\tau)\}\, d\alpha$$

$$= \int_{-\infty}^{+\infty} h(\alpha) R_{XY}(\tau-\alpha)\, d\alpha = h(\tau) * R_{XY}(\tau) = h(\tau) * R_{XX}(\tau) * h(-\tau)$$

$$\Rightarrow \quad S_{YX}(\omega) = H(\omega) S_{XX}(\omega), \quad S_{XY}(\omega) = S_{XX}(\omega) H^*(\omega), \quad S_{YY}(\omega) = S_{XX}(\omega) \left| H(\omega) \right|^2$$

Similarly, these can be developed for discrete time random processes and systems:

$$R_{YX}[k] = h[k] * R_{XX}[k]$$
$$R_{XY}[k] = R_{YX}[-k] = R_{XX}[k] * h[-k]$$
$$R_{YY}[k] = h[k] * R_{XX}[k] * h[-k]$$

$$\Rightarrow \quad S_{YX}(e^{j\omega}) = H(e^{j\omega}) S_{XX}(e^{j\omega}), \quad S_{XY}(e^{j\omega}) = S_{XX}(e^{j\omega}) H^*(e^{j\omega}), \quad S_{YY}(e^{j\omega}) = S_{XX}(e^{j\omega}) \left| H(e^{j\omega}) \right|^2$$

Definition: White Noise

Random process $X(t)$ is called white noise in continuous time iff it is WSS and satisfies $R_{XX}(\tau) = \sigma^2 \delta(\tau)$.

Random process Y_n is called white noise in discrete time iff it is WSS and satisfies $R_{YY}[k] = \sigma^2 \delta[k]$.

Notice that in both cases, every pair of different samples of the random process is uncorrelated. For a Gaussian white noise random process, we get that every pair of different samples of the random process are statistically independent.

Optimal Linear Estimation (for two JWSS Processes) – Wiener Filter

Given two JWSS random processes, $X(t)$ and $Y(t)$ with expectations zero, we would like to calculate the optimal linear MMSE estimator of $X(t)$ from the samples $\{Y(t), t \in \Re\}$, i.e., we want to find a filter $h(\bullet)$ such that $\hat{X}_{LMMSE}(t) = \int_{-\infty}^{+\infty} h(t,s)Y(s)ds$.

The resulting solution is:

$$\hat{X}_{LMMSE}(t) = h_{Wiener}(t) * Y(t)$$

where $h_{Wiener}(t)$ is given by:

$$H_{Wiener}(\omega) = \frac{S_{XY}(\omega)}{S_{YY}(\omega)}$$

The estimation error process of the optimal linear estimator $e(t) = X(t) - \hat{X}_{LMMSE}(t)$ is orthogonal to all linear functions of the samples. That is to say:

$$E\{e(t')g(t)*Y(t)\} = 0 \qquad \forall t, t'$$

for any $g(t)$. Furthermore, $\hat{X}_{LMMSE}(t)$ is a WSS process, since it is the result of passing a WSS process $Y(t)$ through an LTI system.

The auto-correlation function of the estimation error is:

$$R_{ee}(\tau) = E\left[e(t+\tau)e(t)\right] = E\left[e(t+\tau)\left(X(t) - \hat{X}_{LMMSE}(t)\right)\right]$$

$$\overset{(1)}{=} E\left[e(t+\tau)X(t)\right] = E\left[\left(X(t+\tau) - \hat{X}_{LMMSE}(t+\tau)\right)X(t)\right]$$

$$= R_{XX}(\tau) - E\left[\hat{X}_{LMMSE}(t+\tau)X(t)\right] = R_{XX}(\tau) - E\left[\hat{X}_{LMMSE}(t+\tau)\left(e(t) + \hat{X}_{LMMSE}(t)\right)\right]$$

$$\overset{(2)}{=} R_{XX}(\tau) - R_{\hat{X}\hat{X}}(\tau)$$

where (1)-(2) follow since the estimation error process is orthogonal to all linear functions of the samples, and hence also to $\hat{X}_{LMMSE}(t)$.

The spectrum of the estimation error:

$$S_{ee}(\omega) = S_{XX}(\omega) - S_{\hat{X}\hat{X}}(\omega) = S_{XX}(\omega) - H_{Wiener}(\omega)S^*_{XY}(\omega) \overset{(1)}{=} S_{XX}(\omega) - \frac{|S_{XY}(\omega)|^2}{S_{YY}(\omega)}$$

(1) if $S_{YY}(\omega) \neq 0$.

And the expectation of the squared error is, of course:

$$E\left\{e^2(t)\right\} = R_{ee}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ee}(\omega)d\omega$$

Problem 1:

Given are two JWSS random processes $X(t), Y(t)$ with expected value zero. Let us denote by $H_{Y\to X}(\omega)$ the optimal linear estimation filter of $X(t)$ from the random process $Y(t)$ (Wiener filter).

1. Consider the process $Z(t)$ that is created by the transition of the random process $X(t)$ through an LTI system with frequency response $G(\omega)$. What is the optimal linear estimation filter of $Z(t)$ from $Y(t)$ and what is the spectrum of the estimation error?
2. Given the random process $\tilde{Y}(t)$ created by passing the random process $Y(t)$ through an LTI system with frequency response $F(\omega) \neq 0 \ \forall\omega$. What is the optimal linear estimation filter of $X(t)$ from $\tilde{Y}(t)$ and what is the spectrum of the estimation error?

Solution:

1. First, we will show that $Y(t), Z(t)$ are JWSS. The process $Z(t)$ is WSS since it is created from the passing the process $X(t)$ through an LTI system. $Y(t)$ is WSS from the given. The only condition left to check is whether $R_{ZY}(t+\tau,t) = R_{ZY}(\tau)$.

$$R_{ZY}(t+\tau,t) = E\{Z(t+\tau)Y(t)\} = E\left\{\int_{-\infty}^{\infty} g(\alpha)X(t+\tau-\alpha)Y(t)\,d\alpha\right\} = \int_{-\infty}^{\infty} g(\alpha)R_{XY}(\tau-\alpha)\,d\alpha$$
$$= g(\tau) * R_{XY}(\tau)$$

Since the processes are JWSS, the optimal estimation filter is Wiener filter:

$$H_{Y\to Z}(\omega) = \frac{S_{ZY}(\omega)}{S_Y(\omega)} = \frac{G(\omega)S_{XY}(\omega)}{S_Y(\omega)} = G(\omega)H_{Y\to X}(\omega)$$

The spectrum of the estimation error is:

$$S_{e,Y\to Z}(\omega) = S_Z(\omega) - \frac{|S_{ZY}(\omega)|^2}{S_Y(\omega)} = |G(\omega)|^2 S_X(\omega) - |G(\omega)|^2 \frac{|S_{XY}(\omega)|^2}{S_Y(\omega)} = |G(\omega)|^2 S_{e,Y\to X}(\omega)$$

where $S_{e,Y\to Z}(\omega)$ is the spectrum of the error of the optimal linear estimator of $Z(t)$ from $Y(t)$ and $S_{e,Y\to X}(\omega)$ is the spectrum of the error of the optimal linear estimator of $X(t)$ from $Y(t)$.

Conclusion: The Wiener filter of a random process that is the output of an LTI system, is the Wiener filter of the input process of the system, concatenated to the system.

2. We will show that $X(t),\tilde{Y}(t)$ are JWSS. The process $\tilde{Y}(t)$ is WSS since it is created by the transition of the process $Y(t)$ through an LTI system. $X(t)$ is WSS based on the given. We are left to check whether $R_{X\tilde{Y}}(t+\tau,t)=R_{X\tilde{Y}}(\tau)$.

$$R_{X\tilde{Y}}(t+\tau,t)=E\left\{ X(t+\tau)\tilde{Y}(t)\right\}=E\left\{ X(t+\tau)\int_{-\infty}^{+\infty} f(\alpha)Y(t-\alpha)d\alpha \right\}=\int_{-\infty}^{+\infty} f(\alpha)R_{XY}(\tau+\alpha)d\alpha=$$
$$=R_{XY}(\tau)*f(-\tau)$$

Therefore, here, too, the solution is Wiener filter:

$$H_{\tilde{Y}\to X}(\omega)=\frac{S_{X\tilde{Y}}(\omega)}{S_{\tilde{Y}}(\omega)}=\frac{S_{XY}(\omega)F^*(\omega)}{|F(\omega)|^2 S_Y(\omega)}=\frac{1}{F(\omega)}\cdot H_{Y\to X}(\omega)$$

The estimation error's spectrum:

$$S_{e,\tilde{Y}\to X}(\omega)=S_X(\omega)-H_{\tilde{Y}\to X}(\omega)S^*_{X\tilde{Y}}(\omega)=S_X(\omega)-\frac{1}{F(\omega)}H_{Y\to X}(\omega)\cdot\left(S_{XY}(\omega)F^*(\omega)\right)^*$$
$$=S_X(\omega)-H_{Y\to X}(\omega)S^*_{XY}(\omega)$$

The conclusion is that, in order to estimate $X(t)$ from the random process $\tilde{Y}(t)$, we must invert the system that created $\tilde{Y}(t)$, and to estimate $X(t)$ from the output by the optimal filter $H_{Y\to X}(\omega)$.

Moreover, notice that:

$$S_{e,\tilde{Y}\to X}(\omega)=S_{e,Y\to X}(\omega)$$

and, in particular, due to the invertibility of $F(\omega)$:

$$MSE_{Linear,\tilde{Y}\to X}=MSE_{Linear,Y\to X}$$

Problem 2 (Exam B, Winter 2010)

Consider the process $Y[n]=X[n]+W[n]$, where:

$$X[n]=\begin{cases} A & \text{w.p. } 1/2 \\ -A & \text{w.p. } 1/2 \end{cases} \quad \text{,iid}$$
$$Z[n]\sim N(0,1) \quad \text{,iid}$$
$$W[n]=\alpha Z[n]+\beta Z[n-1]$$

It is given that $X[n],Z[n]$ are independent, $A>0$ and $\alpha^2+\beta^2=1$.

1. Find the expectation of the processes $X[n],Y[n],W[n]$.
2. Find the autocorrelation function of the processes $X[n],Y[n],W[n]$.
3. What is the optimal linear MMSE estimator of the process $W[n]$ from the entire process $Y[n]$?

4. What is the <u>optimal estimator in the sense of probability of error</u> of $X[n_0]$ from $Y[n_0]$ (i.e. from only the n_0-th sample of the process $Y[n]$)?
5. Find the optimal <u>MMSE</u> estimator of $W[n_0]$ from $Y[n_0]$?

<u>Solution:</u>

1.

$$\eta_x = E[X_n] = \sum_i \Pr(X = x_i)\cdot x_i = 0.5A - 0.5A = 0$$

$$\eta_W = E[W_n] = E[\alpha\cdot Z_n + \beta\cdot Z_{n-1}] = \alpha\cdot E[Z_n] + \beta\cdot E[Z_{n-1}] = 0$$

$$\eta_Y = E[Y_n] = E[X_n + W_n] = E[X_n] + E[W_n] = 0 + 0 = 0$$

2. The autocorrelation function of the process $X[n]$:

   It is given that the process $X[n]$ is i.i.d and, thus, there is no correlation between samples of different times of the process. Thus, the process $X[n]$ is white noise with autocorrelation function $R_X[k] = \sigma_X^2 \cdot \delta[k]$, where:

$$\sigma_X^2 = Var(X_n) = E[X_n^2] = 0.5(A)^2 + 0.5(-A)^2 = A^2$$

   Let us find the autocorrelation function of the process $W[n]$ by calculating the power spectrum of $W[n]$ and then use the inverse Fourier transform. We will treat $W[n]$ as the output of an LTI system, whose input is the process $Z[n]$:

$$Z[n] \longrightarrow \boxed{h[n] = \alpha\cdot\delta[n] + \beta\cdot\delta[n-1]} \longrightarrow W[n]$$

   $Z[n]$ is white noise with variance 1. Its spectrum is given by the Fourier transform of delta function: $R_Z[k] = \delta[k] \quad \Rightarrow \quad S_Z(e^{j\omega}) = 1$

   The spectrum of the process $W[n]$ can be found by using the formula of a random process passing through an LTI system:

$$S_W(e^{jw}) = S_Z(e^{j\omega})\left|H(e^{j\omega})\right|^2 = (\alpha + \beta\cdot e^{j\omega})(\alpha + \beta\cdot e^{-j\omega}) = 1 + \alpha\beta(e^{j\omega} + e^{-j\omega})$$

   where we made use of the given $\alpha^2 + \beta^2 = 1$.

   By using inverse Fourier transform, we can find the autocorrelation function:

$$R_W[k] = \delta[k] + \alpha\beta\cdot\delta[k-1] + \alpha\beta\cdot\delta[k+1]$$

   The autocorrelation function of the process $Y[n]$:

$$R_Y[k] = E[Y[n+k]Y[n]] = E[(X[n+k] + W[n+k])\cdot(X[n] + W[n])] =$$

$$= E(X[n]X[n-k]) + E(W[n+k]W[n]) + E(X[n+k]W[n]) + E(W[n+k]X[n]) \overset{(1)}{=} R_X[k] + R_W[k]$$

$$R_Y[k] = (A^2 + 1)\delta[k] + \alpha\beta\cdot\delta[k-1] + \alpha\beta\cdot\delta[k+1]$$

(1) $X[n]$ and $Z[n]$ are independent and, therefore, also $X[n]$ and $W[n]$ are independent. As a result, the cross-correlation between them is zero.

3. First we show that $W\left[n\right]$ and $Y\left[n\right]$ are JWSS. From the previous sections, the expectation and autocorrelation functions of the above processes are not dependent of n, thus each one of them is WSS. Moreover:

$$R_{WY}[k] = E(W[n+k]Y[n]) = E(W[n+k](X[n]+W[n])) = R_W[k]$$

In order to find the optimal linear MMSE estimator of the process $W[n]$ from the process $Y[n]$, we will use Wiener filter:

$$H(e^{j\omega}) = \frac{S_{WY}(e^{j\omega})}{S_Y(e^{j\omega})}$$

Taking the Fourier Transform of $R_Y[k]$ and $R_{WY}[k]$ we have:

$$H(e^{j\omega}) = \frac{S_{WY}(e^{j\omega})}{S_Y(e^{j\omega})} = \frac{1+\alpha\beta\cdot e^{-j\omega}+\alpha\beta\cdot e^{j\omega}}{1+A^2+\alpha\beta\cdot e^{-j\omega}+\alpha\beta\cdot e^{j\omega}} = \frac{1+2\alpha\beta\cdot\cos(\omega)}{1+A^2+2\alpha\beta\cdot\cos(\omega)}$$

<u>Notes:</u>

- For $A\to 0$ we get $H(e^{j\omega})=1$, that is to say $h[n]=\delta[n]$.
  This is clearly correct seeing as how, when $A\to 0$, we get that $Y[n]\approx W[n]$ and certainly the best estimator is $\hat{W}[n]=Y[n]$.

- For $A\to\infty$ we get $H(e^{j\omega})=0$, namely $h[n]=0$. This is true since, in this case, $X[n]$ constitutes "noise" with infinite variance to $W[n]$ - therefore, the best estimator of $W[n]$ is its expectation (zero).

4. The optimal estimator in the sense of probability of error is the MAP estimator. Denote $Y_{n_0}=Y[n_0],\ X_{n_0}=X[n_0]$.

$$\hat{X}_{n_0}^{MAP}(Y_{n_0}) = \arg\max_{x\in\{\pm A\}} \Pr(X_{n_0}=x\,|\,Y_{n_0}) = \arg\max_{x\in\{\pm A\}} \frac{\Pr(X_{n_0}=x)\cdot f_{Y_{n_0}|X_{n_0}}(Y_{n_0}\,|\,X_{n_0}=x)}{f_{Y_{n_0}}(y)}$$

The denominator is independent of x and does not affect the maximization. Moreover, $\Pr(X_{n_0}=A)=\Pr(X_{n_0}=-A)$ so we can eliminate the left element of the numerator. Hence, the expression we would like to maximize is:

$$\hat{X}_{n_0}^{MAP}(Y_{n_0}) = \arg\max_{x\in\{\pm A\}} f_{Y_{n_0}|X_{n_0}}(Y_{n_0}\,|\,X_{n_0}=x)$$

It is given that $W[n]=\alpha Z[n]+\beta Z[n-1]$ and $Z[n]\sim N(0,1)$ is i.i.d. Therefore, $\left[Z[n]\ Z[n-1]\right]$ constitute a Gaussian random vector. $W[n]$ is a Gaussian random variable as a linear combination of the above vector with expectation zero and variance $\alpha^2+\beta^2=1$, therefore, $W[n]\sim N(0,1)$.

Consequently:

$$Y_{n_0} \mid \left(X_{n_0} = x \right) = x + W_n \sim N(x,1)$$

$$\Rightarrow f_{Y_{n_0}|X_{n_0}}(Y_{n_0} = y \mid X_{n_0} = x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}$$

This estimation problem was encountered in previous lessons! The MAP estimator is, once again, $\hat{X}_{n_0}^{MAP}(Y_{n_0}) = A \cdot sign(Y_{n_0})$.

5. Let us denote $\hat{X}_{n_0,MMSE} = E[X_{n_0} \mid Y_{n_0}]$ and $\hat{W}_{n_0,MMSE} = E[W_{n_0} \mid Y_{n_0}]$.

$$\hat{W}_{n_0,MMSE} = E[W_{n_0} \mid Y_{n_0}] \overset{W_{n_0} = Y_{n_0} - X_{n_0}}{=} E[Y_{n_0} - X_{n_0} \mid Y_{n_0}] = E[Y_{n_0} \mid Y_{n_0}] - E[X_{n_0} \mid Y_{n_0}] = Y_{n_0} - \hat{X}_{n_0,MMSE}$$

In order to complete the solution, we must find $\hat{X}_{n_0,MMSE}$.

Let us find the conditional probability $\Pr\left(X_{n_0} = x \mid Y_{n_0} = y \right)$ and calculate the expectation based on this probability to get the estimator:

$$f_{Y_{n_0}}(y) = \frac{1}{2} f_{Y_{n_0}|X_{n_0}=A}(y \mid x = A) + \frac{1}{2} f_{Y_{n_0}|X_{n_0}=-A}(y \mid x = -A) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}(e^{-\frac{(y-A)^2}{2}} + e^{-\frac{(y+A)^2}{2}})$$

Based on Bayes' theorem:

$$\Pr\left(X_{n_0} = x \mid Y_{n_0} = y \right) = \frac{f_{Y_{n_0}|X_{n_0}}(y \mid x)\Pr\left(X_{n_0} = x \right)}{f_{Y_{n_0}}(y)} = \frac{e^{-\frac{(y-x)^2}{2}} \Pr\left(X_{n_0} = x \right)}{\frac{1}{2}(e^{-\frac{(y-A)^2}{2}} + e^{-\frac{(y+A)^2}{2}})} = \frac{e^{-\frac{(y-x)^2}{2}}}{(e^{-\frac{(y-A)^2}{2}} + e^{-\frac{(y+A)^2}{2}})}$$

Finally, we will find the expectation of the random variable $X_{n_0} \mid y$:

$$\hat{X}_{n_0,MMSE} = \sum_{x=\pm A} x \Pr\left(X_{n_0} = x \middle| Y_{n_0} = y \right) = \frac{Ae^{-\frac{(y-A)^2}{2}} - Ae^{-\frac{(y+A)^2}{2}}}{e^{-\frac{(y-A)^2}{2}} + e^{-\frac{(y+A)^2}{2}}} =$$

$$= A \cdot \frac{e^{-\frac{y^2+A^2}{2}}}{e^{-\frac{y^2+A^2}{2}}} \cdot \frac{e^{Ay} - e^{-Ay}}{e^{Ay} + e^{-Ay}} = A \cdot \frac{e^{Ay} - e^{-Ay}}{e^{Ay} + e^{-Ay}} \qquad \Rightarrow \hat{X}_{n_0,MMSE} = A \cdot \tanh\left(A \cdot Y_{n_0} \right)$$

By substituting back into the estimator $\hat{W}_{n_0,MMSE}$ we get:

$$\hat{W}_{n_0,MMSE} = Y_{n_0} - A \cdot \tanh(A \cdot Y_{n_0})$$

Extra Questions

Problem 3:

Given is a WSS random process $X(t)$ with expectation zero.

Let us define a new random process:

$$Y(t)=\frac{1}{2T}\int_{t-T}^{t+T}X(\alpha)\cdot d\alpha$$

1. Express $Y(t)$ as a transition of $X(t)$ through an LTI system. Write down the impulse response of the system, $h(t)$.
2. Express $S_{YY}(\omega)$ and $S_{YX}(\omega)$ using $S_{XX}(\omega)$.
3. What is the expected value and variance of the random variable $Y(0)$?

Solution:

1. The integration limits can be expressed by a multiplication by a window – this is the filter:

$$Y(t)\overset{\tau=t-\alpha}{=}\frac{1}{2T}\cdot\int_{-T}^{T}X(t-\tau)\cdot d\tau=\int_{-\infty}^{\infty}X(t-\tau)\cdot\left(\frac{1}{2T}\cdot rect\left(\frac{\tau}{T}\right)\right)\cdot d\tau=X(t)*h(t)$$

where:

$$h(t)=\frac{1}{2T}\cdot rect\left(\frac{t}{T}\right)\overset{FT}{\leftrightarrow}\frac{\sin(\omega\cdot T)}{\omega\cdot T}\qquad;\qquad rect(x)=\begin{cases}1 & |x|<1\\ 0 & o.w.\end{cases}$$

2. We can immediately write:

$$S_{YX}(\omega)=S_{XX}(\omega)\frac{\sin(\omega\cdot T)}{\omega\cdot T},\qquad S_{YY}(\omega)=S_{XX}(\omega)|H(\omega)|^2=S_{XX}(\omega)\frac{\sin^2(\omega\cdot T)}{(\omega\cdot T)^2}$$

3. Note that $Y(0)$ is essentially the estimator of the expectation of the process $X(t)$, as defined in previous recitation:

$$Y(0)=\frac{1}{2T}\int_{-T}^{T}X(\alpha)\cdot d\alpha\quad\left(=\hat{\eta}_X(T)\right)$$

$$E\{Y(0)\}=E\left\{\frac{1}{2T}\int_{-T}^{T}X(\alpha)\cdot d\alpha\right\}=\frac{1}{2T}\int_{-T}^{T}E\{X(\alpha)\}d\alpha\underset{\underset{W.S.S.}{\uparrow}}{=}\frac{1}{2T}\int_{-T}^{T}\eta_X d\alpha=\eta_X=0$$

$$Var\{Y(0)\}=Var\{Y(t)\}\underset{\eta_X=0}{=}R_{YY}(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{YY}(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{XX}(\omega)\frac{\sin^2(\omega\cdot T)}{(\omega\cdot T)^2}d\omega$$