Problem Set 5 – Estimation of Random Variables

Problem 1:

We would like to estimate the random variable X from Y with distortion measure $d(e)=|e|=|X-\hat{X}|$.

1. Suppose X and Y have the joint PDF $f_{XY}(x,y)$. Show that the expectation of the distortion measure can be expressed as follows:

$$E[\,|X-\hat{X}|\,]=\int_{-\infty}^{\infty}dy\, f_Y(y)\left[\int_{-\infty}^{g(y)}(g(y)-x)\,f_{X|Y}(x\,|\,y)\,dx+\int_{g(y)}^{\infty}(x-g(y))\,f_{X|Y}(x\,|\,y)dx\right]$$

2. Find a condition for the optimal estimator in this case (Hint: by deriving with respect to $g(y)$).

Problem 2:

Let X be a uniform random variable in the section $[0,a]$, $a>0$. Let us define the random variable Y, which, given $X=x$, distributes uniformly in $[0,x]$.

1. Find the MMSE estimator of Y from X.
2. Find the MMSE estimator of X from Y.
3. Find the linear MMSE estimator of Y from X.
4. Find the linear MMSE estimator of X from Y.

Problem 3:

Given is the following system: $X\rightarrow Q(\cdot)\rightarrow Y$

X is an exponentially distributed random variable with parameter $\lambda>0$, namely:

$$f_X(x)=\begin{cases}\lambda\cdot e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$$

The function $Q(\cdot)$ takes discrete values such that:

$$y=Q(x)=\lfloor x\rfloor$$

1. Calculate $\mathrm{P}(Y=n),\quad n\in Z$.
2. Calculate $f_{X|Y}(x\,|\,y)$.

3. Find the optimal estimator in the sense of MSE and in the sense of minimum probability of error of Y given X.
4. Find the MMSE estimator of X given Y. You may use the identity:

$$\int xe^{-\lambda x}dx = -\frac{e^{-\lambda x}(\lambda x+1)}{\lambda^2}$$

5. Find the linear MMSE estimator of X given Y.

Define $Z = X - Y$.

6. Find $f_{Z|Y}(z \mid y)$.
7. Are Z and Y uncorrelated? Independent?
8. Find the MMSE and the linear MMSE estimators of Z given Y.

Problem 4 (Exam 2019):

Given is a random matrix $A = \begin{pmatrix} X & \rho Y \\ \rho Y & X \end{pmatrix}$ where $X, Y \sim N(0,1)$ are independent of each other, and ρ is a deterministic constant, $|\rho| \leq 1$.

1. Let $v = (0 \ \ 1)^T$. Is $v^T A v$ a Gaussian random variable for any $|\rho| \leq 1$?
2. Is $v^T A$ a Gaussian random vector, for **any** vector $v \in \mathbb{R}^2$ and any $|\rho| \leq 1$?

From now on assume that $\rho = 1$.

In the following sections we define several linear transformations, such that in each section we get a vector in $\mathbb{R}^2$, which we denote as $Z = (Z_1 \ \ Z_2)^T$. In each section, find the MMSE estimator of Z_2 given Z_1 and the corresponding MSE.

3. $Z = (1 \ -1) \cdot A$
4. $Z = (0 \ \ 1) \cdot A \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
5. $Z = (1 \ \ 2) \cdot A$

Problem 5:

Prove that if the random variables X, Y, Z are jointly Gaussian with expected value zero, then $\mathrm{E}[XYZ] = 0$.

Problem 6 – (channel with interferences):

In this question we will deal with information transference through a channel that has interferences: the transmitter wants to transfer a message represented by the random variable X, and does this through a transmission medium (the channel), which connects the transmitter to the receiver. Since the medium might be non-ideal, there exists a possibility of the information sent to be distorted by the channel, and, thus, in the end of the medium we get the random variable Y (which is statistically dependent of X). In this question, we will deal with two typical channels.

$$\textit{Transmitter} \xrightarrow{X} \boxed{\quad \textit{Channel} \quad} \xrightarrow{Y} \textit{Receiver}$$

<u>Binary Symmetric Channel (BSC):</u>

Given:

$$X \sim Ber(p)$$ [1]

$$N \sim Ber(\varepsilon) \qquad Y = X \oplus N \qquad \oplus = XOR$$

X, N are independent and $0 < p, \varepsilon < 1$.

1. Find the following probabilities (for $x, y \in \{0,1\}$):

$$\Pr\{Y = y \mid X = x\}, \quad \Pr\{Y = y\}, \quad \Pr\{X = x \mid Y = y\}$$

2. Find the probability of error in the output of the channel, i.e.:

$$\Pr\{X \neq Y\}$$

<u>Binary Channel with Additive Gaussian Noise (BIAGN):</u>

Given:

$$I \sim Ber(p)$$
$$X = (-1)^{(I+1)}$$
$$N \sim N(0, \sigma^2) \qquad Y = X + N$$

I, N are independent and $0 < p < 1$.

3. Find the distribution of X and the distribution of Y. Draw the PDF of each one of the distributions.

4. At the output of the channel, an attempt at guessing the transmitted message is made as follows:

$$Z = sign(Y) = \begin{cases} -1 & Y \leq 0 \\ 1 & Y > 0 \end{cases}$$

Calculate the probability of error for the decision of X, that is, $\Pr\{Z \neq X\}$

Make use of the Q-function in your final answer.