Problem Set 6 – Estimation

Problem 1:

Given are the random variables X_1, X_2, N_1, N_2 with distribution:

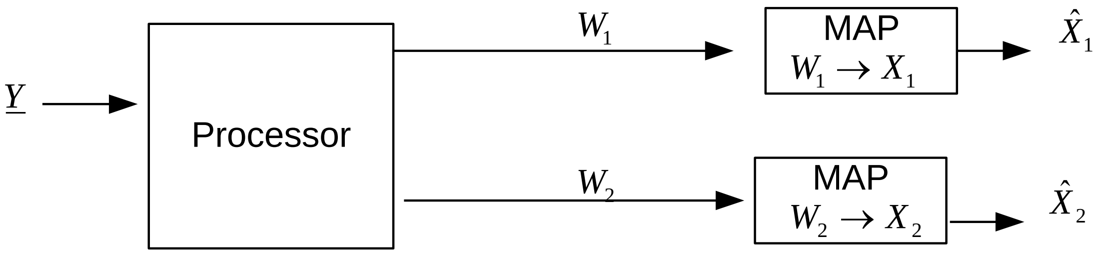
$$N_i \sim N\left(0, \sigma^2\right) \quad X_i \sim \begin{cases} +1 & w.p.\ 1-p \\ -1 & w.p.\ \ p \end{cases}$$

Further, X_1, X_2, N_1, N_2 are independent.

Let us denote $\underline{Y} = H\underline{X} + \underline{N}$, where:

$$\underline{N} = \left[N_1\ N_2\right]\ \ \underline{X} = \left[X_1\ \ X_2\right]\ \ \underline{Y} = \left[Y_1\ Y_2\right]\ H = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \qquad 0 \le \alpha < 1$$

1. For $\alpha = 0$, find the optimal estimator in the sense of minimum probability of error of X_i given $\underline{Y}$.
2. For the case $p = \frac{1}{2}$, what will the probability of error be if we were to operate the estimation function from section 1 on measurements for which $\alpha \neq 0$?
3. A simpler scheme is proposed (though sub-optimal, in the general case) for the estimator:



   The signal $\underline{Y}$ is passed through a signal processor having two outputs W_1, W_2. The optimal estimator in the sense of minimum probability of error for estimating X_i from W_i .

   The signal processing unit inverts the the channel matrix, namely:

$$\left[W_1\ W_2\right]^T = H^{-1}\underline{Y}$$

   Find $\hat{X}_1, \hat{X}_2$, find the probability of error in the estimation process, for $p = \frac{1}{2}$.

Problem 2:

Let $\underline{X}$ and $\underline{Y}$ be random vectors (of any length).

- Let $\hat{X}^{MMSE}$ be the optimal MMSE estimator of $\underline{X}$ from $\underline{Y}$, and assume the covariance matrix of the estimation error is C_{MMSE}.

- Let $\hat{X}^{BLE}$ be the optimal **linear** MMSE estimator of $\underline{X}$ from $\underline{Y}$, and assume the covariance matrix of the estimation error is C_{BLE}.

We define $W = a^T X$, where a is a known deterministic vector (or, in other words, $W = \sum_{i=1}^{n} a_i X_i$).

1. Show that the optimal MMSE estimator of W given $\underline{Y}$ is $a^T \hat{X}^{MMSE}$.
2. Provide an expression for the mean squared estimation error, as a function of a and of the covariance matrix of the error C_{MMSE}.
3. Show that the optimal **linear** MMSE estimator of W given $\underline{Y}$ is. $a^T \hat{X}^{BLE}$
4. Provide an expression for the mean squared estimation error, as a function of a and of the covariance matrix of the error C_{BLE}.
5. Prove: if $\underline{e}$ is the estimation error of some estimator of $\underline{X}$ from $\underline{Y}$ and $\underline{e}_{OPT}$ is the optimal estimator's error for the same problem, then the difference between the covariance matrices of the estimation errors is a positive semi-definite matrix: $E\{\underline{e}\underline{e}^T\} - E\{\underline{e}_{OPT}\underline{e}_{OPT}{}^T\} \geq 0$.
6. Prove: if $\underline{e}$ is the estimation error of some linear estimator of $\underline{X}$ from $\underline{Y}$ and $\underline{e}_{OPT}$ is the optimal linear estimator's error for the same problem, then the difference between the covariance matrices of the estimation errors is a positive semi-definite matrix: $E\{\underline{e}\underline{e}^T\} - E\{\underline{e}_{OPT}\underline{e}_{OPT}{}^T\} \geq 0$.

Problem 3:

Let $\{X_i\}$ be a series of i.i.d random variables with expected value 0. Let us denote:

$$Y_n = \sum_{i=0}^{n} X_i$$

1. Find the optimal MMSE estimator of Y_n given Y_m, when $m \leq n$.
2. Find the optimal MMSE estimator of Y_n given $Y_1, Y_2, ..., Y_m$, when $m \leq n$.
3. Repeat section 1 when now $n = 2$ and m is a discrete random variable that takes the values $\{1, 2\}$ with equal probability.

Problem 4:

Given is the optimal MMSE estimator of X from Y:

$$\hat{X} = \text{sgn}(Y) = \begin{cases} 1, Y > 0 \\ -1, Y \leq 0 \end{cases}$$

and the mean squared error in the estimation is:

$$E\left\{\left(\hat{X}-X\right)^2\right\}=\frac{1}{2}$$

Moreover, it is given that $Y \sim U\left(-1,1\right)$.

1. Calculate the expected value of X .
2. Calculate σ_{XY} , the covariance between X and Y .
3. Find the optimal linear MMSE estimator of X from Y .
4. Calculate the mean squared error of the estimator found in section 3.

<u>Problem 5:</u>

Let X, N_1, N_2 be three random variables. Given:

- The two random variables N_1, N_2 are **jointly normal** with expected value 0, variance σ_N^2, and correlation coefficient $-1 \le \rho \le 1$.
- The random variable X is normal with expectation 0 and variance σ_x^2, statistically independent of N_1, N_2.

Moreover, the measurements $Y_1 = X + N_1$ and $Y_2 = X + N_2$ are given.

1. Show that the five random variables X, N_1, N_2, Y_1, Y_2 are jointly Gaussian.
2. What is the conditional distribution $f_{Y_1|X}(y_1 | x)$? What is the conditional expectation? What is the conditional variance?
3. Given only Y_1, what is the best <u>linear</u> estimator $\hat{X}_{BLE}$ of X that brings to a minimum the expectation of the squared error?
4. Given only Y_1, what is the <u>optimal</u> estimator $\hat{X}_{opt}$ of X that brings to a minimum the expectation of the squared error?
5. What is the expectation of the squared error of the estimator you found in section 4?
6. Given Y_1 and Y_2, what is the best estimator $\hat{X}$ of X that brings to a minimum the expectation of the squared error?
7. What is the expectation of the squared error of the estimator you found in section 6?
8. What is ρ that brings the error calculated in section 7 to a minimum? Explain!
9. Provide an expression for the error found in section 7 for the two extreme cases $\sigma_X^2 << \sigma_N^2$ and $\sigma_X^2 >> \sigma_N^2$.