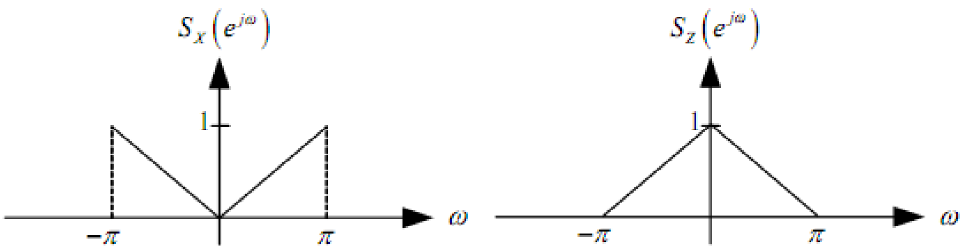
<u>Problem Set 9 – Wiener Filter</u>

<u>Problem 1:</u>

Two independent WSS Gaussian random processes $X[n], Z[n]$, are given. The two processes have zero expectation and the following Power Spectral Densities:

$$S_X\left(e^{j\omega}\right) = \frac{|\omega|}{\pi}, -\pi < \omega \le \pi \qquad S_Z\left(e^{j\omega}\right) = 1 - \frac{|\omega|}{\pi}, -\pi < \omega \le \pi$$



We are interested in calculating $X[n]$ out of $Y[n] = X[n] + Z[n]$.

1. Are $X[n]$ and $Y[n]$ JWSS?
2. Calculate the MMSE estimator of the process $X[n]$ from the process $Y[n]$.
3. What is the mean squared error (MSE) of the estimator from section 2?

We define the processes $Y^{'}[n] = (-1)^n X[n] + Z[n]$ and $Y^{"}[n] = (-1)^n Y^{'}[n]$

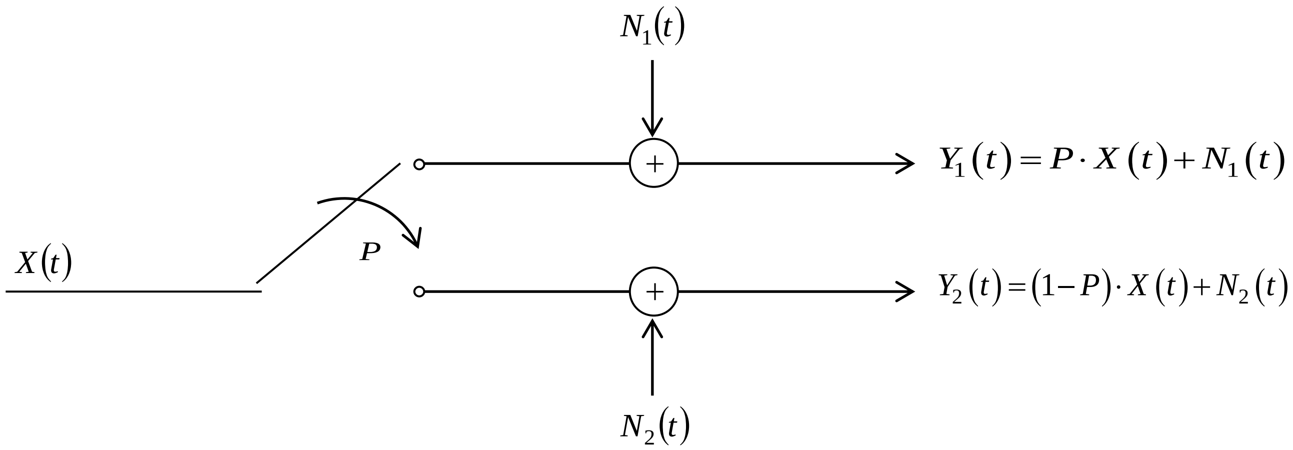
4. Are $X[n]$ and $Y^{"}[n]$ JWSS?
5. What is the MMSE (not necessarily linear) estimator of the process $X[n]$ from the process $Y^{"}[n]$?
6. Are $X[n]$ and $Y^{'}[n]$ JWSS?
7. What is the MMSE (not necessarily linear) estimator of the process $X[n]$ from the process $Y^{'}[n]$?

<u>Problem 2:</u>

Given are three independent random processes $X(t)$, $N_1(t)$ and $N_2(t)$.

The three processes are stationary, with expectation zero and spectrums $S_X(\omega)$, $S_1(\omega)$ and $S_2(\omega)$, respectively. Also a random variable P is given, independent of the three processes $X(t)$, $N_1(t)$ and $N_2(t)$, with the following distribution:

$$P = \begin{cases} 1 & w.p \quad p \\ 0 & w.p \quad 1-p \end{cases} \qquad 0 \le p \le 1$$



1. Show that the three processes $X(t)$, $Y_1(t)$ and $Y_2(t)$ are JWSS in pairs (meaning that each pair of them is JWSS).
2. Calculate $\hat{X}_1(t)$, which is the optimal linear MMSE estimator of the process $X(t)$ from the process $Y_1(t)$.
3. Calculate $\hat{X}_2(t)$, which is the optimal linear MMSE estimator of the process $X(t)$ from the process $Y_2(t)$.

We define the estimator $\hat{X}(t) = \hat{X}_1(t) + \hat{X}_2(t)$ and estimation error $\varepsilon(t) = X(t) - \hat{X}(t)$

4. Show that the cross-correlation between $\varepsilon(t)$ and any one of the two processes $Y_1(t)$, $Y_2(t)$ is zero.
5. Deduce that $\hat{X}(t)$ is the optimal linear estimator of the process $X(t)$ from both processes $\{Y_1(t), Y_2(t)\}_{t=-\infty}^{t=\infty}$.

Problem 3:

Two JWSS random processes $N_1(t), N_2(t)$ are given, each with expectation zero, autocorrelation and cross-correlation functions $R_{N_1}(\tau), R_{N_2}(\tau), R_{N_1,N_2}(\tau)$ and spectrums $S_{N_1}(\omega), S_{N_2}(\omega), S_{N_1,N_2}(\omega)$, respectively.

1. Find the optimal linear estimator $H(\omega)$ that uses all the sample signal $N_2(t)$:

$$\hat{N}_1(t_0) = \int_{-\infty}^{\infty} h(t_0 - t) \cdot N_2(t) \cdot dt$$

Now, it is given that $N_1(t) = V(t) * g_1(t)$ $N_2(t) = V(t) * g_2(t)$, where $V(t)$ is a WSS process with expectation zero and spectrum:

$$S_V(\omega) = \begin{cases} \text{a} & |\omega| \le 2 \\ 0 & |\omega| > 2 \end{cases}$$

The frequency responses of the filters are given by:

$$G_1(\omega)=\begin{cases}1 & |\omega|\le\omega_1\\ 0 & |\omega|>\omega_1\end{cases},G_2(\omega)=\begin{cases}0 & |\omega|\le\omega_2\\ 1 & |\omega|>\omega_2\end{cases}$$

2. Explain why indeed $N_1(t),N_2(t)$ are JWSS with expectation zero and calculate $S_{N_1}(\omega),S_{N_2}(\omega),S_{N_1,N_2}(\omega)$.

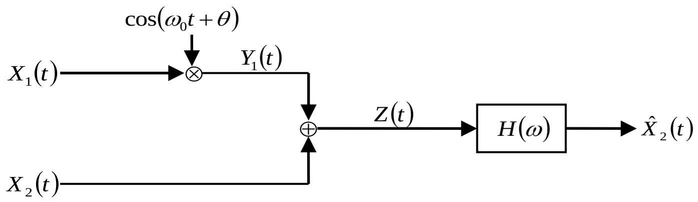
3. Calculate the estimator you found in section 1 and its mean squared error.

4. For what values of the parameters ω_1,ω_2 is the estimation error zero?

5. For what values of the parameters ω_1,ω_2 is the estimation error maximal? What is the estimator in this case and what is the estimation error?

Problem 4:

Consider the following system:



The system in the figure represents a communication channel designated to transmit two signals, $X_1(t)$ and $X_2(t)$, at the same time. It is assumed that $X_1(t)$ and $X_2(t)$ are independent random processes, wide-sense stationary with power spectrum:

$$S_{X_2}(\omega)=e^{-|\omega|} \qquad\qquad S_{X_1}(\omega)=\begin{cases}P & |\omega|\le 1\\ 0 & o.w.\end{cases}$$

Assume that $\omega_0=1.5$ and that θ is a random variable that distributes uniformly in the section $(-\pi,\pi)$ and is independent of $X_1(t)$ and $X_2(t)$.

In order to reproduce the signal $X_2(t)$, an LPF is used in the receiver, whose transfer function is:

$$H(\omega)=\begin{cases}1 & |\omega|<B\\ 0 & O.W.\end{cases}$$

This creates the signal $\hat{X}_2(t)$.

1. Calculate and draw the power spectrum of $Z(t)$.

2. Find the power spectrum of $\hat{X}_2(t)$, using the parameter B (the bandwidth of the filter in the receiver).

3. Show that $S_{X_2\hat{X}_2}(\omega)=S_{X_2}(\omega)H(\omega)$.
4. Draw the power spectrum of the reproduction error process $e(t)=X_2(t)-\hat{X}_2(t)$ for each of the cases: $B=0.25, B=2, B=3$.
5. Assuming $R_{X_1}(0)=\frac{4}{\pi e}$, find B such that the mean squared error in the reproduction, $E\{e^2(t)\}$, is minimal.
6. Find the Wiener filter for the estimation of $X_2(t)$ from $Z(t)$.

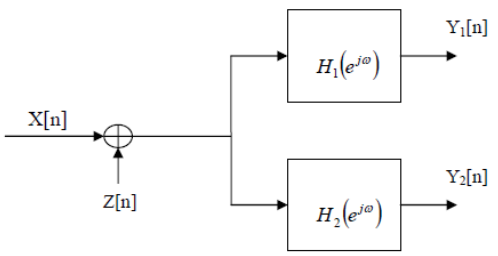
Problem 5:

Given is the linear AR random process $X[n]$, which is WSS:

$$X[n]=0.5X[n-1]+W[n]$$

where the innovations process $W[n]$ is i.i.d with expectation 0 and variance 1. An i.i.d noise $Z[n]$, independent of $W[n]$, with expectation 0 and variance 1 is added to $X[n]$. After the noise addition, the signal is passed in parallel through two filters that satisfy:

$$h_1[n]=-0.5\delta[n]+\delta[n-1]$$
$$H_2\left(e^{jw}\right)=\frac{1}{H_1\left(e^{jw}\right)}$$



1. Find $S_{Y_2}\left(e^{j\omega}\right)$, the spectrum of the process $Y_2[n]$, and the cross-spectrum between $Y_2[n]$ and the innovations process $W[n]$,. $S_{Y_2W}\left(e^{j\omega}\right)$
(**Hint:** represent the process $X[n]$ as a transition of the process $W[n]$ through an LTI system which will be marked as $H_0\left(e^{j\omega}\right)$.)
2. Calculate the optimal linear estimator of the process $W[n]$ from the $Y_2[n]$.
3. Prove that the expectation of the squared error of the optimal linear estimator of the process $W[n]$ from the process $Y_1[n]$ is identical to the expectation of the squared error of the estimator found in section 2.
(**Hint:** it can be proved without finding the estimator explicitly.)