Solution of Problem Set 2

Problem 1:

1.

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0,1,2,\ldots,n$$

$$M_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{sk} = (1-p+pe^s)^n$$

$$\frac{d}{ds} M_X(s) = npe^s (1-p+pe^s)^{n-1}$$

$$\frac{d^2}{ds^2} M_X(s) = npe^s (1-p+pe^s)^{n-1} + n(n-1)p^2 e^{2s} (1-p+pe^s)^{n-2}$$

$$E(X) = M_X'(0) = np$$

$$E(X^2) = M_X''(0) = np + n(n-1)p^2 = n^2p^2 + np(1-p)$$

$$Var(X) = E(X^2) - [E(X)]^2 = np(1-p)$$

2.

$$f_X(x) = \lambda e^{-\lambda x} u(x))^{n-k}, \qquad k = 1,2,\ldots,n$$

$$M_X(s) = \int_0^\infty \lambda e^{-\lambda x} e^{sx} = \frac{\lambda}{\lambda - s}, \qquad \mathrm{Re}\{s\} < \lambda$$

$$\frac{d}{ds} M_X(s) = \frac{\lambda}{(\lambda - s)^2}, \qquad \mathrm{Re}\{s\} < \lambda$$

$$\frac{d^2}{ds^2} M_X(s) = \frac{2\lambda}{(\lambda - s)^3}, \qquad \mathrm{Re}\{s\} < \lambda$$

$$E(X) = M_X'(0) = \tfrac{1}{\lambda}$$

$$E(X^2) = M_X''(0) = \tfrac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \tfrac{2}{\lambda^2} - \tfrac{1}{\lambda^2} = \tfrac{1}{\lambda^2}$$

Problem 2:

For a strictly increasing and differentiable (except for a finite number of points) function:

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} y \left(\frac{d}{dy} F_Y(y) \right) dy \underbrace{=}_{(1)} \int_{-\infty}^{\infty} y \frac{d}{dy} F_X\left(g^{-1}(y)\right) dy =$$

$$\int_{-\infty}^{\infty} y \frac{d}{dy}\left(\int_{-\infty}^{x=g^{-1}(y)} f_X(t)dt\right) dy \underbrace{=}_{(2)} \int_{-\infty}^{\infty} y \left(f_X\left(g^{-1}(y)\right)\frac{d}{dy}\left(g^{-1}(y)\right)\right) dy \underbrace{=}_{\substack{x=g^{-1}(y) \\ dx=\left(\frac{d}{dy}\left(g^{-1}(y)\right)\right)dy}} \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Further explanation:

(1) $F_Y(y) = P(Y \le y) = P(g(x) \le y) = P\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right)$

The 3rd equality follows from the fact that g is a strictly increasing function.

(2) The fundamental theorem of calculus claims:

$$V(x) = \int_a^x v(t)dt \quad => \quad \frac{d}{dx}V(x) = v(x)$$

And thus:

$$V(x) = \int_a^{u(x)} v(t)dt \quad => \quad \frac{\partial}{\partial x}V(x) = v(u(x)) \underbrace{\frac{\partial}{\partial x}u(x)}_{\text{נגזרת פנימית}}$$

Problem 3:

The moment-generating function of $Y = \sum_{i=1}^{n} X_i$ is

$$M_Y(s) = E\left(e^{sY}\right) = E\left(e^{s(X_1+X_2+\cdots+X_n)}\right) = E\left(\prod_{i=1}^{n} e^{sX_i}\right) = \prod_{i=1}^{n} E\left(e^{sX_i}\right) = \left(M_X(s)\right)^n$$

where we swapped the expected value of a product by a product of expected values since X_i are statistically independent, and the last equality is true due to X_i being identically distributed hence having the same moment generating function.

Problem 4

1. a. Let us use the transition formula we saw in class:

$$y = e^x$$

For $y > 0$, only one solution exists to the equation $x_1 = \ln(y)$, and for $y \le 0$, there are no solutions.

Moreover:

$$\left.\frac{de^x}{dx}\right|_{x=\ln(y)} = y$$

All in all:

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right\} u(y)$$

b. Seeing as how Y distributes like e^X, $X \sim N(\mu, \sigma^2)$, it follows that Y^r distributes like $e^{rX} = e^{X_1}$, where $X_1 = rX$. But from here we get that $X_1 \sim N(r\mu, r^2\sigma^2)$, and, therefore, based on section 1 a: $Y^r \sim LN(r\mu, r^2\sigma^2)$.

2.

$$Y_1 \sim LN(0,1)$$

$\Rightarrow$ from section 1 b:

$$Y_1^k \sim LN(0, k^2)$$

Consequently:

$$Y_1^k = e^{X_k}, \qquad X_k \sim N(0, k^2)$$

$$X_k = kZ, \qquad Z \sim N(0,1)$$

$$Y_1^k = e^{kZ}$$

$$E\left\{ Y_1^k \right\} = E\left\{ e^{kZ} \right\} = M_Z(k) = e^{\frac{k^2}{2}}$$

where $M_Z(a)$ is the moment-generating function of a standard Gaussian random variable:

$$M_Z(a) = e^{\frac{a^2}{2}}$$