Recitation 8 – Joint stationarity, Power Spectral Density

Joint stationarity:

Two random processes $X(t), Y(t)$ are called **jointly strict sense stationary** (JSSS) iff for all values of n, any deterministic time series $t_1, \cdots, t_n$ and any constant τ, the two following random vectors have the same distribution:

$$\left[X(t_1), Y(t_1) \cdots, X(t_n), Y(t_n)\right]$$

$$\left[X(t_1+\tau), Y(t_1+\tau) \cdots, X(t_n+\tau), Y(t_n+\tau)\right]$$

Two random processes $X(t), Y(t)$ are called **jointly wide sense stationary** (JWSS) iff the following conditions are met:

- $X(t)$ and $Y(t)$ are both WSS
- The cross-correlation function of $X(t), Y(t)$ depends only on time difference

$$R_{XY}(t_1, t_2) = E(X(t_1)Y(t_2)) = R_{XY}(t_1 - t_2)$$

Properties:

- if $X(t), Y(t)$ are two independent SSS processes they are also JSSS.
- if $X(t), Y(t)$ are two independent WSS processes they are also JWSS.
- if $X(t), Y(t)$ are two JSSS processes they are also JWSS.
- if $X(t), Y(t)$ are two **jointly Gaussian** JWSS processes they are also JSSS.

Problem 1:

Let X and Y be i.i.d random variables, such that $X, Y \sim N\left(0, \sigma^2\right)$.

Let us define the following random process:

$$Z(t) = X \cdot \cos(4t) + Y \cdot \sin(4t)$$
$$A(t) = X \cdot \cos(4t)$$

1. Find the expectation function and the auto-covariance function of the processes $A(t)$, $Z(t)$.

2. Are $A(t), Z(t)$ Gaussian random processes?

3. Are $A(t), Z(t)$ **jointly** Gaussian random processes?

4. Calculate the optimal MMSE estimator of the random variable X given only the sample $Z(t_0)$, and the respective MSE.

5. Calculate the optimal MMSE estimator of the random variable X given $\{Z(t)\}_{t=-\infty}^{t=\infty}$ and the respective MSE.
6. For each of the processes $A(t), Z(t)$ determined whether they are stationary. If so, in what sense?

Solution:

1.

$$\eta_Z(t) = E\left\{X \cdot \cos(4t) + Y \cdot \sin(4t)\right\} = 0$$

$$R_Z(t_1,t_2) = E\left\{\left(X \cdot \cos(4t_1) + Y \cdot \sin(4t_1)\right) \cdot \left(X \cdot \cos(4t_2) + Y \cdot \sin(4t_2)\right)\right\} =$$

$$\sigma^2 \cos(4t_1)\cos(4t_2) + E\{X \cdot Y\} \cdot \left(\cos(4t_1) \cdot \sin(4t_2) + \cos(4t_2) \cdot \sin(4t_1)\right) + \sigma^2 \sin(4t_1)\sin(4t_2) =$$

$$\sigma^2 \cos(4t_1)\cos(4t_2) + \sigma^2 \sin(4t_1)\sin(4t_2) = \sigma^2 \cos\left(4(t_1 - t_2)\right)$$

$$\eta_A(t) = E\left\{X \cdot \cos(4t_1)\right\} = 0$$

$$R_A(t_1,t_2) = E\left\{X \cdot \cos(4t_1) \cdot X \cdot \cos(4t_2)\right\} = E\left\{X^2\right\} \cdot \cos(4t_1) \cdot \cos(4t_2) = \sigma^2 \cos(4t_1)\cos(4t_2)$$

2. The pair of variables (X,Y) are jointly Gaussian. For any time series $t_1,\cdots,t_n$, the vector $\left[Z(t_1),\cdots,Z(t_n)\right]$ constitutes a linear transformation of (X,Y):

$$\begin{bmatrix} Z(t_1) \\ \vdots \\ Z(t_n) \end{bmatrix} = \begin{bmatrix} \cos(4t_1) & \sin(4t_1) \\ \vdots & \vdots \\ \cos(4t_n) & \sin(4t_n) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

   Therefore, the variables $\left[Z(t_1),\cdots,Z(t_n)\right]$ are jointly Gaussian – namely, $Z(t)$ is a Gaussian random process. The same goes for $A(t)$.

3. Both of the processes are jointly Gaussian – since the set of variables $\left[Z(t_1),\cdots,Z(t_n), A(t_1),\cdots,A(t_n)\right]$ constitute a linear transformation of (X,Y), and, thus, they are jointly Gaussian.

4. The pair of variables $\left(X, Z(t_0)\right)$ are jointly Gaussian, therefore the optimal MMSE estimator is the optimal linear estimator:

$$\hat{X}_{MMSE}\left(Z(t_0)\right) = E(X) + \frac{\text{Cov}(X,Z(t_0))}{\text{Var}(Z(t_0))}\left[Z(t_0) - E\left(Z(t_0)\right)\right] = \frac{\text{Cov}(X,Z(t_0))}{\text{Var}(Z(t_0))} Z(t_0)$$

   It holds that:

$$\text{cov}(X,Z(t_0)) = \text{cov}(X, X \cdot \cos(4t_0) + Y \cdot \sin(4t)) = \sigma^2 \cos(4t_0)$$

$$\text{var}(Z(t_0)) = \text{var}(X \cdot \cos(4t_0) + Y \cdot \sin(4t_0)) = \sigma^2 \cos^2(4t_0) + \sigma^2 \sin^2(4t_0) = \sigma^2$$

   Thus:

$$\hat{X}_{MMSE}\left(Z\left(t_0\right)\right)=\cos(4t_0)Z(t_0)$$

$$MSE=\mathrm{Var}(X)-\mathrm{Var}\left(\hat{X}_{MMSE}\right)=\sigma^2-\cos^2(4t_0)\mathrm{Var}\left(Z(t_0)\right)=\sigma^2-\cos^2(4t_0)\sigma^2=\sigma^2\sin^2(4t_0)$$

5. Now all the measurements $\left\{Z(t)\right\}_{t\in R}$ are given, and, in particular, $Z(0)$ is also given. Therefore, based on section 4, we can estimate X with no error whatsoever:

$$\hat{X}_{MMSE}=Z\left(0\right)=X$$
$$\mathrm{MSE}=0$$

6. $A(t)$ is not WSS since its auto-correlation function does not depend only on the time difference:

$$R_A\left(t_1,t_2\right)=\sigma^2\cos\left(4t_1\right)\cos\left(4t_2\right)$$

So, for example:

$$R_A\left(0,\pi\right)\neq R_A\left(0+\tfrac{\pi}{2},\pi+\tfrac{\pi}{2}\right)$$

Obviously, it is not SSS as well.

$Z(t)$ is WSS since its expectation function is constant **and** its auto-correlation function depends only on the time difference:

$$\eta_Z(t)=0$$
$$R_Z\left(t_1,t_2\right)=\sigma^2\cos\left(4\left(t_1-t_2\right)\right)$$

Since it is a Gaussian random process, it is also SSS.

Power Spectral Density of a WSS Random Process

The Power Spectral Density (PSD) of a WSS random process is defined as:

$$S_{XX}\left(\omega\right)=\int_{-\infty}^{+\infty}R_{XX}\left(\tau\right)e^{-j\omega\tau}d\tau$$

Properties of the Power Spectral Density:

- Deterministic function
- Real and even function
- $\forall\omega\ S_X\left(\omega\right)\geq 0$
- Inverse Fourier transform:

$$R_{XX}\left(\tau\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{XX}\left(\omega\right)e^{j\omega\tau}d\omega$$

  Thus:

$$E\left\{X^2\left(t\right)\right\}=R_{XX}\left(0\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{XX}\left(\omega\right)d\omega$$

- The PSD is a measure of the second order statistics of the random process. Thus, it is possible for two different random processes to have the same PSD.

Definition of Cross-Spectral Density:

For $X(t), Y(t)$ two real JWSS random processes, the cross-spectral density $S_{XY}(\omega)$ is defined as:

$$R_{XY}(\tau) \xleftrightarrow{FT} S_{XY}(\omega)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

Properties of the Cross-Spectrum:

- Deterministic function
- Not necessarily real
- The symmetric conjugate property: $S_{XY}(\omega) = S_{XY}^{*}(-\omega)$

  Proof:

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau = \overline{\int_{-\infty}^{+\infty} \overline{R_{XY}(\tau)\, e^{-j\omega\tau}} d\tau} = \overline{\int_{-\infty}^{+\infty} R_{XY}(\tau)\, e^{-j(-\omega)\tau} d\tau} = S_{XY}^{*}(-\omega)$$

- Inverse Fourier transform:

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

- The following property holds (it is easy to see by definition):

$$S_{YX}(\omega) = S_{XY}^{*}(\omega) = S_{XY}(-\omega)$$

$$S_{YX}(\omega) = \int_{-\infty}^{+\infty} R_{YX}(\tau)\, e^{-j\omega\tau} d\tau = \int_{-\infty}^{+\infty} R_{XY}(-\tau)\, e^{-j\omega\tau} d\tau = \int_{-\infty}^{+\infty} R_{XY}(\alpha)\, e^{j\omega\alpha} d\alpha = \overline{\int_{-\infty}^{+\infty} R_{XY}(\alpha)\, e^{-j\omega\alpha} d\tau} = S_{XY}^{*}(\omega)$$

Discrete Time Definitions:

The definitions for a random process in discrete time are very similar, where the continuous Fourier transforms are exchanged by DTFT. If X_n is a real and WSS random process in discrete time, then:

$$R_{XX}[k] \xleftrightarrow{DTFT} S_{XX}\left(e^{j\omega}\right)$$

$$S_{XX}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} R_{XX}[k] e^{-j\omega k} \qquad\qquad R_{XX}[k] = \frac{1}{2\pi} \int_{0}^{2\pi} S_{XX}\left(e^{j\omega}\right) e^{j\omega k} d\omega$$

If X_n, Y_n are real JWSS random processes in discrete time, then:

$$R_{XY}[k] \xleftrightarrow{DTFT} S_{XY}\left(e^{j\omega}\right)$$

$$S_{XY}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} R_{XY}[k] e^{-j\omega k} \qquad\qquad R_{XY}[k] = \frac{1}{2\pi} \int_{0}^{2\pi} S_{XY}\left(e^{j\omega}\right) e^{j\omega k} d\omega$$

Problem 2:

Given is the following random process:

$$X(t) = \sin(Wt + \theta)$$

where W is a random variable with PDF $f_W(w)$ and θ is a random phase that distributes uniformly on the section $[0,2\pi)$. W and θ are independent.

1. Prove that the process $X(t)$ is a WSS process.

Now it is given that $W \sim Unif[W_{\min}, W_{\max})$, $0 < W_{\min} < W_{\max}$.

2. Find $R_X(\tau)$.
3. Find the PSD of the process $X(t)$.

Solution:

1. We must show that the expectation and autocorrelation function are not dependent of time.

   First, we will show that the expectation is zero for all t:

$$E\{X(t)\} \underbrace{=}_{\text{החלקה}} E\{E\{X(t)\,|\,W\}\} = E\left\{\int_{-\infty}^{\infty} X(t) f_{\theta|W}(\theta\,|\,W)d\theta\right\} \underbrace{=}_{\substack{W,\theta\text{ are}\\ \text{statistically}\\ \text{independent}}} E\left\{\int_{-\infty}^{\infty} X(t) f_{\theta}(\theta)d\theta\right\}$$

$$= E\left\{\int_0^{2\pi} \sin(Wt+\theta)\frac{1}{2\pi}d\theta\right\} = E\left\{\frac{1}{2\pi}\int_0^{2\pi}\sin(Wt+\theta)d\theta\right\} = E\{0\} = 0$$

   Now, we will show that the autocorrelation function is dependent of time difference only:

$$R_X(t+\tau,t) = E\{X(t+\tau)X(t)\} = E\{\sin(Wt+W\tau+\theta)\sin(Wt+\theta)\} =$$

$$= \frac{1}{2}E\{\cos(W\tau) - \cos(2Wt+W\tau+2\theta)\}$$

   where, in the last equality, the following trigonometric identity was used:

$$\sin\alpha\sin\beta = 0.5\cdot(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

   We will make use of the smoothing theorem once more in order to calculate $E\{\cos(2Wt + W\tau + \theta)\}$:

$$E\{\cos(2Wt+W\tau+2\theta)\} = E\{E\{\cos(2Wt+W\tau+2\theta)\,|\,W\}\} =$$

$$= E\left\{\frac{1}{2\pi}\int_0^{2\pi}\cos(2Wt+W\tau+2\theta)\,d\theta\right\} = E\{0\} = 0$$

   All in all, we got that:

$$R_X(t+\tau,t) = \frac{1}{2}E\{\cos(W\tau)\} = R_X(\tau)$$

Notice that, in order to calculate $R_X(\tau)$ explicitly, we need the distribution of W. However, we were able to show that for **any** distribution of W, the autocorrelation function is independent of time.

2. It is given that $W \sim Unif[W_{\min}, W_{\max})$, that is to say:

$$f_W(w) = \begin{cases} \dfrac{1}{W_{\max} - W_{\min}} & if \;\; W \in [W_{\min}, W_{\max}) \\ 0 & o.w. \end{cases}$$

$$R_X(\tau) = \frac{1}{2} E\{\cos(W\tau)\} = \frac{1}{2}\int_{W_{\min}}^{W_{\max}} \frac{1}{W_{\max} - W_{\min}} \cos(W\tau) dW = \frac{1}{2(W_{\max} - W_{\min})} \frac{\sin(W\tau)}{\tau}\Bigg|_{W_{\min}}^{W_{\max}}$$

$$= \frac{1}{2(W_{\max} - W_{\min})} \frac{\sin(W_{\max}\tau)}{\tau} - \frac{1}{2(W_{\max} - W_{\min})} \frac{\sin(W_{\min}\tau)}{\tau}$$

3. We need to find the Fourier transform of the autocorrelation function we found in the previous section.

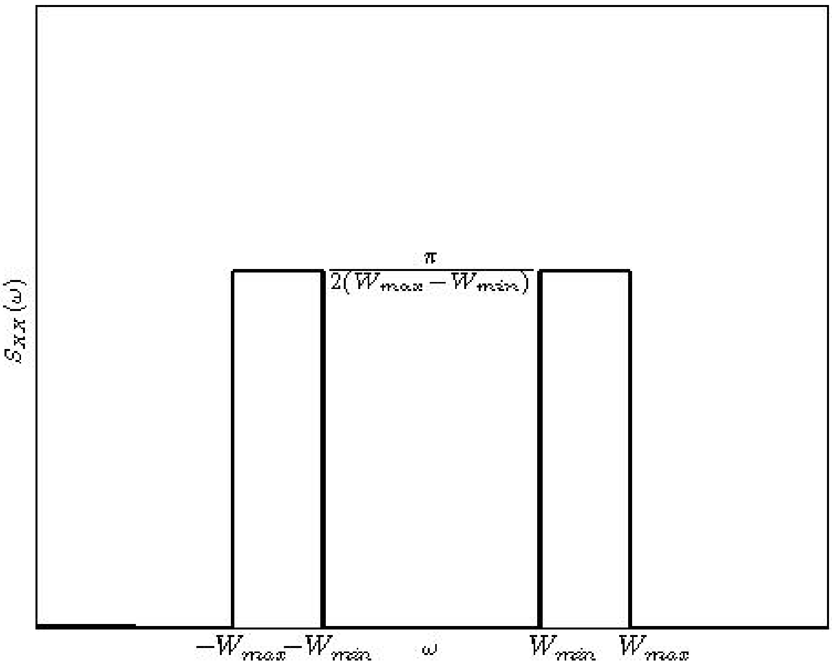
$$\frac{\sin(W_{\min}\tau)}{\tau} \xrightarrow{FT} \begin{cases} \pi & |\omega| \le W_{\min} \\ 0 & |\omega| > W_{\min} \end{cases}$$

$$\frac{\sin(W_{\max}\tau)}{\tau} \xrightarrow{FT} \begin{cases} \pi & |\omega| \le W_{\max} \\ 0 & |\omega| > W_{\max} \end{cases}$$

Therefore:

$$S_X(\omega) = F\{R_X(\tau)\} = \frac{1}{2(W_{\max} - W_{\min})}\left(F\left\{\frac{\sin(W_{\max}\tau)}{\tau}\right\} - F\left\{\frac{\sin(W_{\min}\tau)}{\tau}\right\}\right) =$$

$$\frac{\pi}{2(W_{\max} - W_{\min})} \begin{cases} 1 & |\omega| \in [W_{\min}, W_{\max}) \\ 0 & |\omega| \notin [W_{\min}, W_{\max}) \end{cases}$$



Extra Questions

Problem 3:

Let $X[n]$ be a Gaussian WSS process with expectation 0 and autocorrelation function $R_X[k]$, and let $Z[n], M[n]$ be two i.i.d standard Gaussian processes, that is:

$$Z[n], M[n] \sim N(0,1)$$

The processes $X[n], Z[n], M[n]$ are independent.
We define the processes:

$$Y[n] = X[n] + Z[n]$$

$$V[n] = X[n-5] + M[n]$$

1. Is $Y[n]$ stationary? If so, in what sense?
2. Are $Y[n]$ and $V[n]$ jointly stationary? If so, in what sense?
3. Is the process $U[n] = (-1)^n X[n]$ stationary? If so, in what sense?
4. Are the processes $X[n]$ and $U[n]$ jointly stationary? If so, in what sense?

Solution:

1. $X[n]$ is a WSS Gaussian random process and hence SSS. $Z[n]$ is also an SSS Gaussian random process. $Y[n]$ is the sum of two independent Gaussian random processes hence it is Gaussian. In addition $Y[n]$ is WSS since it is the sum of two **independent** WSS processes. From this is follows that $Y[n]$ is a Gaussian WSS process and hence SSS.

2. $Y[n]$ is a Gaussian SSS process as we saw in last section. $V[n]$ is a Gaussian SSS process from the same reasons. We check that $Y[n]$ and $V[n]$ are JWSS:

$$R_{YV}[k] = E\left[Y[n+k]V[n]\right] = E\left[\left(X[n+k]+Z[n+k]\right)\left(X[n-5]+M[n]\right)\right]$$
$$= E\left[X[n+k]X[n-5]\right] = R_{XX}[k+5]$$

where we used the fact that $X[n], M[n], Z[n]$ are independent and $X[n]$ is WSS. Since $Y[n]$ and $V[n]$ are JWSS Gaussian processes they are also JSSS.

3. First let's check that $U[n]$ is WSS:

$$E\left(U[n]\right) = E\left((-1)^n X[n]\right) = (-1)^n E\left(X[n]\right) = 0$$

$$R_U[n+k,n] = E\left(U[n+k]U[n]\right) = E\left((-1)^{n+k} X[n+k](-1)^n X[n]\right)$$

$$= (-1)^{2n+k} R_X[k] = (-1)^k R_X[k]$$

We got that the expectation and auto-correlation functions do not depend on time so $U[n]$ is WSS. Now, since $X[n]$ is Gaussian, and every sample set of $U[n]$ is generated via linear transformation on a sample set of $X[n]$, every sample set of $U[n]$ consists a Gaussian random vector. From here it follows that $U[n]$ is a Gaussian random process, so it is also SSS.

4. The processes are not jointly stationary. We evaluate their cross-correlation function and show that it is dependent on n:

   $$R_{XU}[n+k,n]=E\left(X[n+k]U[n]\right)=E\left(X[n+k](-1)^n X[n]\right)=(-1)^n R_X[k]$$

   Since they are not JWSS they are certainly not JSSS.