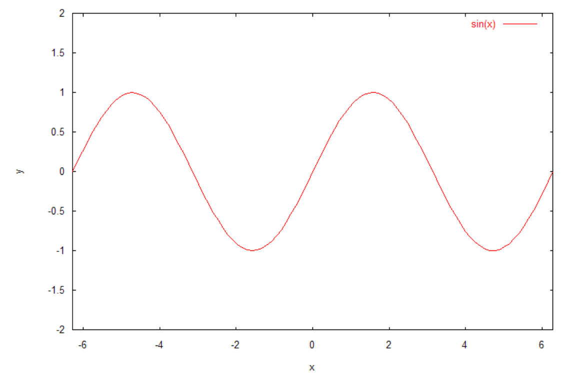
ELEC 474 – Machine Vision

FOURIER ANALYSIS

Fourier Analysis

- Can add two periodic functions, to get another periodic function:



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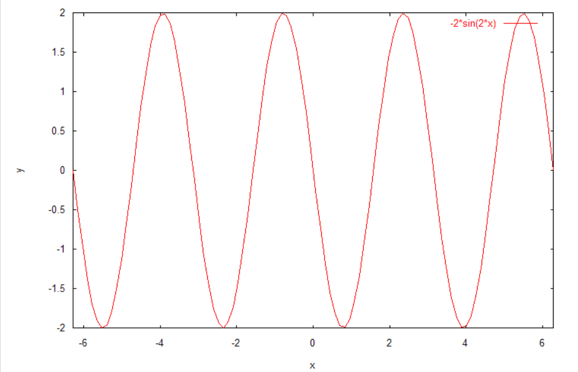

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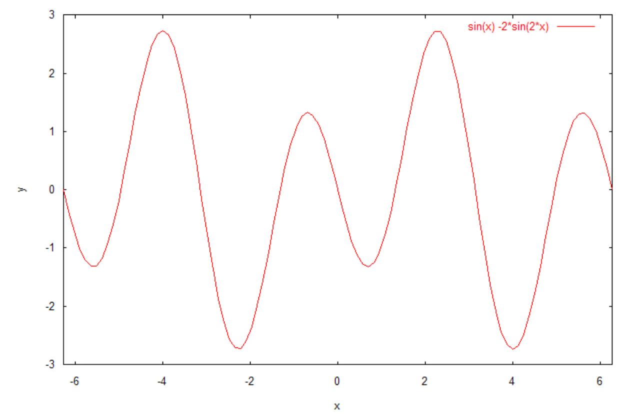


Fourier Analysis

Q1/ Can we describe any periodic function as the sum of other, simpler functions?

Ans/ Yes! Fourier Series!

Q2/ Can we describe non-periodic functions this way too?

Ans/ Yes! Fourier Transform!

Fourier Analysis

The general form of the Fourier Series Expansion is:

$$f(t) = \sum_{k=0}^{\infty}(A_k \cos 2\pi w_k t + B_k \sin 2\pi w_k t)$$

for $w_k = k/T,\ k = 0, 1, 2, \ldots$, where T is the period of $f(t)$.

The objective of Fourier Analysis is to decompose a signal into its Fourier Series by determining its Fourier coefficients A_k and B_k. This decomposition will give a different insight into the nature of $f(t)$, and will provide guidance on the design of filters to selectively remove or enhance certain frequencies.

Fourier Analysis

The *Fourier Coefficients* are determined as:

$$A_k = 2/T \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi k t}{T} \, dt, \;\; k = 1, 2, 3, \ldots$$

$$A_0 = 1/T \int_{-T/2}^{T/2} f(t) \, dt$$

$$B_k = 2/T \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi k t}{T} \, dt, \;\; k = 1, 2, 3, \ldots$$

$$B_0 = 0$$

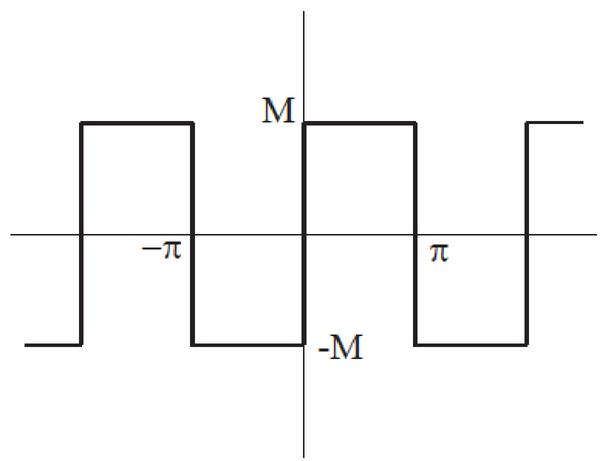
Fourier Analysis

Example:

Calculate the Fourier Series for the following square wave signal:

$$f(t) = \begin{cases} -M \text{ if } -\pi < t \leq 0 \\ M \text{ if } 0 < t \leq \pi \end{cases}$$

with $T = 2\pi$.



Fourier Analysis

$$A_k = 2/T \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi kt}{T} \, dt$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \cos \frac{2\pi kt}{2\pi} \, dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -M \cos kt \, dt + \int_{0}^{\pi} M \cos kt \, dt \right]$$

$$= \frac{1}{\pi} \left[-M \left(\frac{\sin kt}{k} \bigg|_{-\pi}^{0} \right) + M \left(\frac{\sin kt}{k} \bigg|_{0}^{\pi} \right) \right]$$

$$= \frac{1}{\pi} \left[-M \left(0 - \left(\frac{-1}{k} \right) \right) + M \left(\frac{1}{k} - 0 \right) \right]$$

$$= \frac{1}{\pi} [-M/k + M/k]$$

$$= 0$$

There is therefore no cosine term to the Fourier Series expansion, which means that the function is "odd", i.e. $f(-t) = -f(t)$. Similarly, for even functions (i.e. $f(-t) = f(t)$), $B_k = B_0 = 0$.

Fourier Analysis

$$A_0 = 1/T \int_{-T/2}^{T/2} f(t)\, dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -M\ dt + \int_{0}^{\pi} M\ dt\right]$$

$$= \frac{1}{2\pi}\left[-Mt\Big|_{-\pi}^{0} + Mt\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-M\pi + M\pi\right]$$

$$= 0$$

This showthat there is no steady-state “DC” term to this signal, i.e. the average is zero.

Fourier Analysis

$$B_k = 2/T \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi kt}{T}\, dt$$

$$= \frac{2}{2\pi} \left[\int_{-\pi}^{0} -M \sin kt\, dt + \int_{0}^{\pi} M \sin kt\, dt \right]$$

$$= \frac{1}{\pi} \left[\frac{M}{k} \cos kt \Big|_{-\pi}^{0} - \frac{M}{k} \cos kt \Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{M}{k} (\cos 0 - \cos(-k\pi)) - \frac{M}{k} (\cos(k\pi) - \cos 0) \right]$$

$$= \frac{M}{k\pi} [1 - \cos k\pi - \cos k\pi + 1]$$

$$= \frac{2M}{k\pi} (1 - \cos k\pi)$$

Fourier Analysis

Finally, $B_0 = 0$, as always. Putting this all together, we have the Fourier Series Expansion for $f(t)$ as:

$$f(t) = \sum_{k=0}^{\infty} (A_k \cos kt + B_k \sin kt)$$

$$= \sum_{k=0}^{\infty} \frac{2M}{\pi k} (1 - \cos k\pi) \sin kt$$

Fourier Analysis

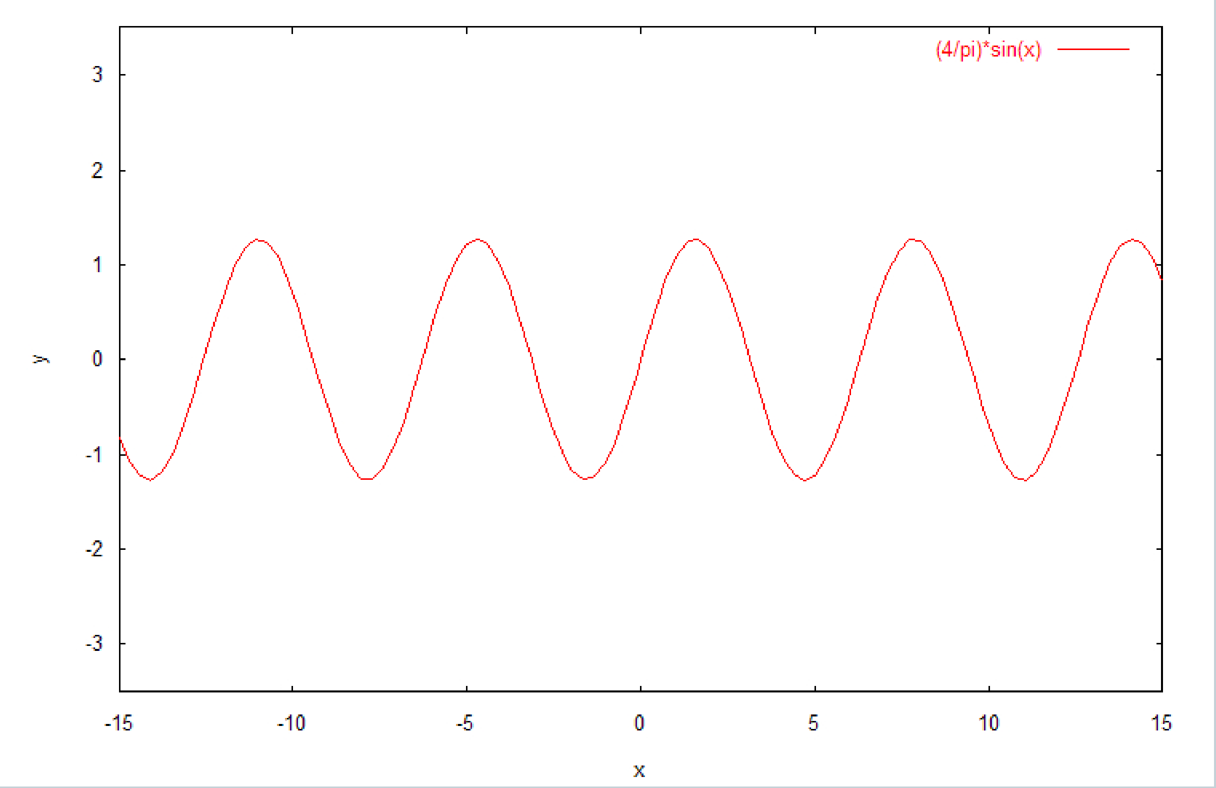
k	$\cos k\pi$	B_k
1	-1	$4M/\pi$
2	1	0
3	-1	$4M/3\pi$
4	1	0
5	-1	$4M/5\pi$
$\vdots$	$\vdots$	$\vdots$

Expanding the infinite series gives:

$$f(t) = \frac{4M}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$$

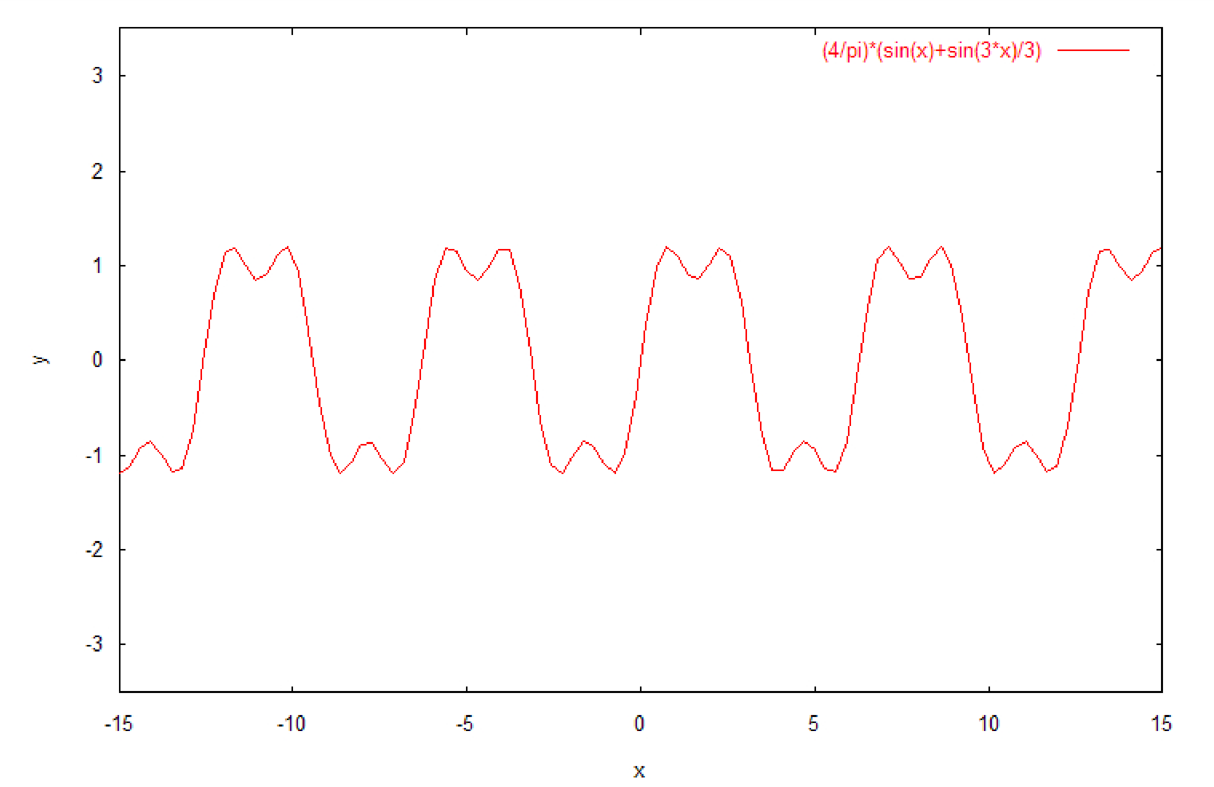
Fourier Analysis

k = 1



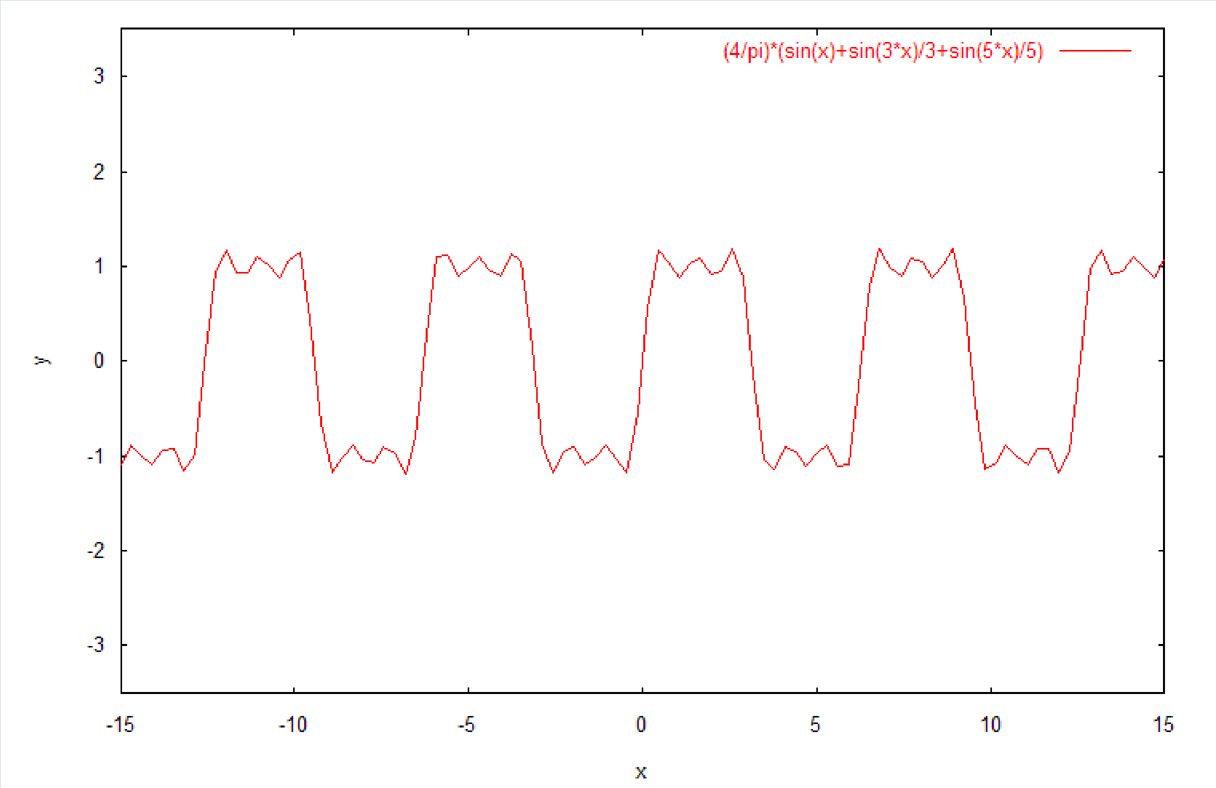
Fourier Analysis

k = 3



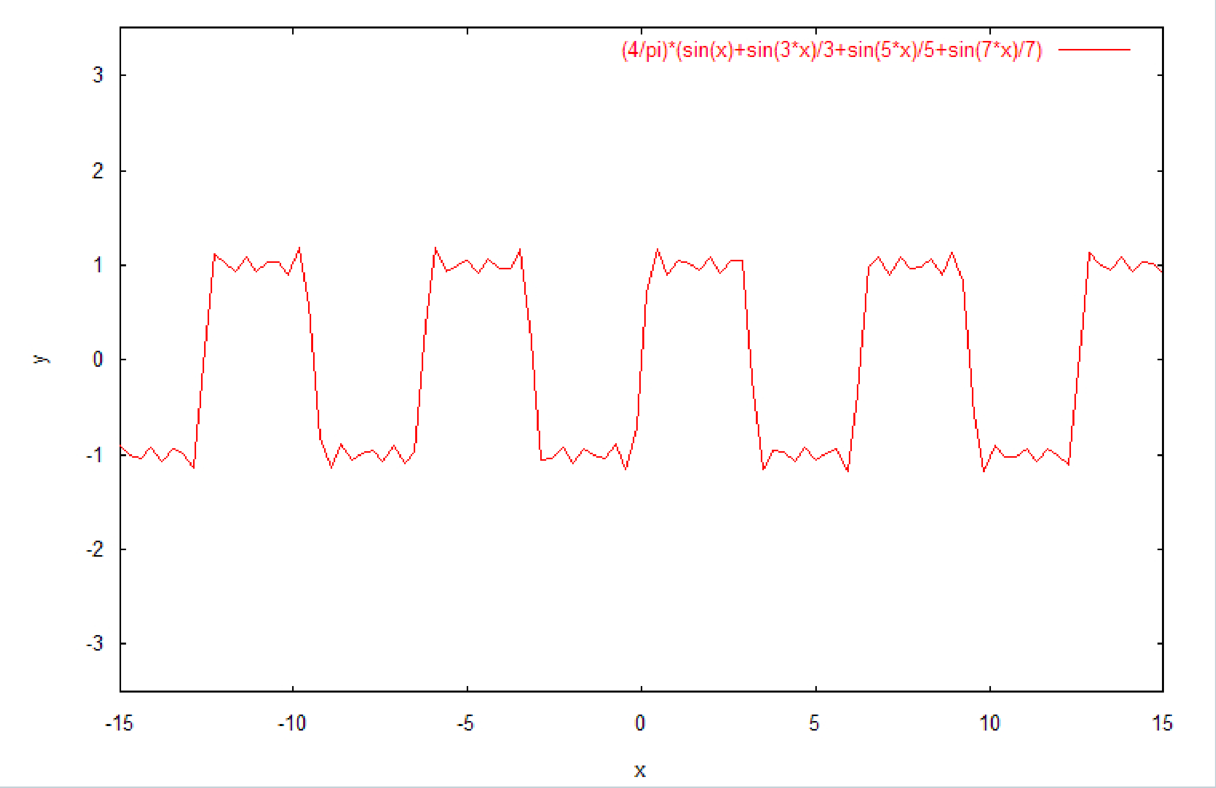
Fourier Analysis

k = 5



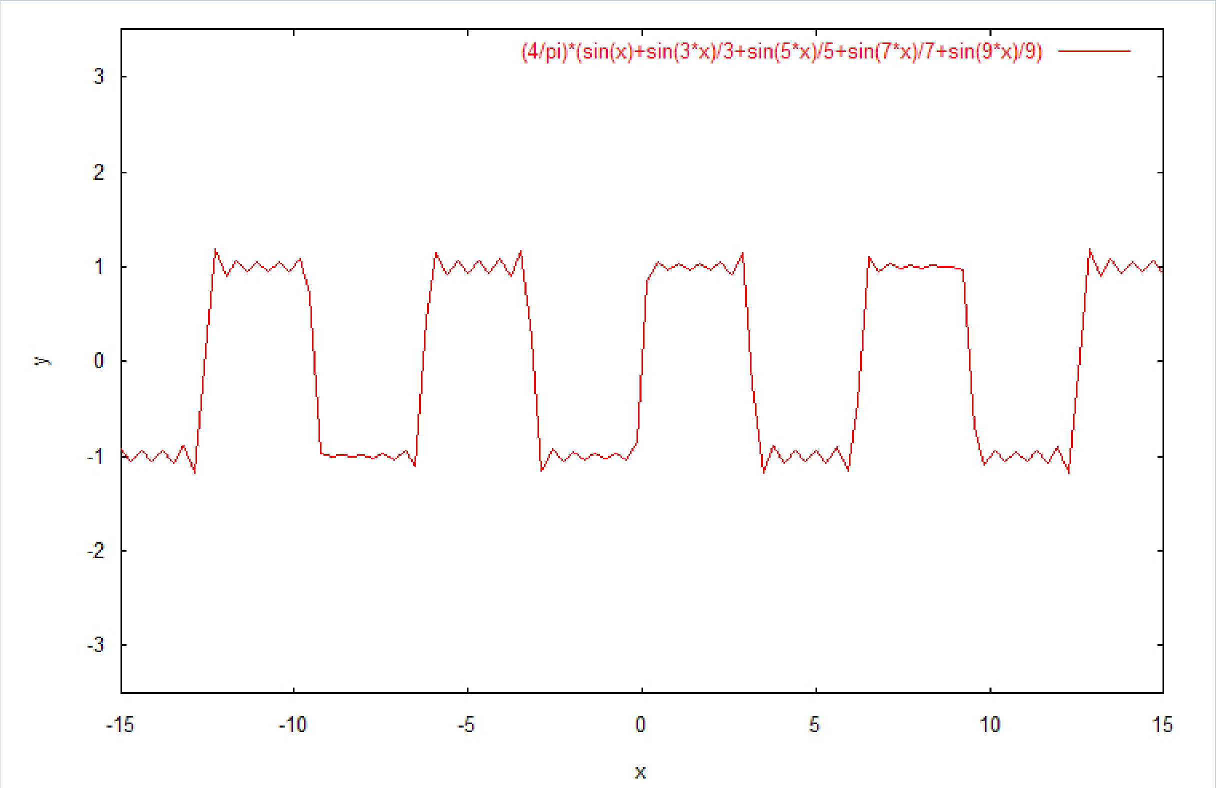
Fourier Analysis

k = 7



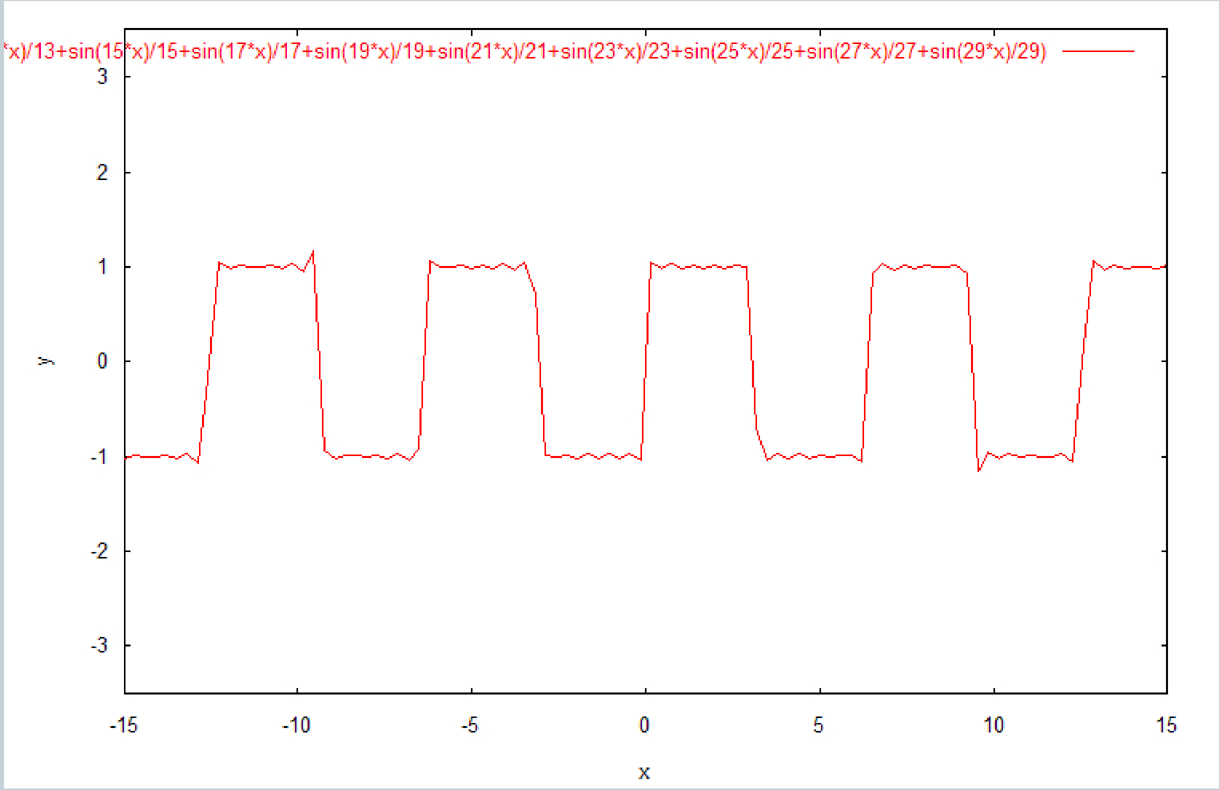
Fourier Analysis

k = 9



Fourier Example

k = 29



Fourier Analysis

For non-periodic functions, the Fourier Transform is derived from the Fourier Series, by expressing it in complex form, and taking the limit as the period goes to infinity:

$$\begin{cases} F(w) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i w t} dt \\ \\ f(t) = \int_{-\infty}^{\infty} F(w) e^{2\pi i w t} dw \end{cases}$$

Here, $F(w)$ and $f(t)$ are called the *Fourier Transform* and *Inverse Fourier Transform*, respectively.

They are alternately called *Fourier Transform Pairs*, and are often denoted as:

$$\begin{cases} \mathcal{F}[f(t)] = F(w) \\ \\ \mathcal{F}^{-1}[F(w)] = f(t) \end{cases}$$

Fourier Analysis

If the domain is discrete, then we have the *Discrete Fourier Transform (DFT)*. The motivation of the DFT is that, in practise, most signals and images that we process are sampled, and are therefore discrete. For a sequence of N items $\{f(\bar{x})\}$, $\bar{x} = 0, 1, ..., N-1$, the DFT pair is defined as:

$$\begin{cases} F(\bar{w}) = \frac{1}{N}\sum_{\bar{x}=0}^{N-1} f(\bar{x})e^{-2\pi i\bar{w}\bar{x}}, \ \bar{w} \in [0, N-1] \\ \\ f(\bar{x}) = \sum_{\bar{w}=0}^{N-1} F(\bar{w})e^{2\pi i\bar{w}\bar{x}/N}, \ \bar{x} \in [0, N-1] \end{cases}$$

Fourier Analysis

Extension to 2D is straightforward:

$$\begin{cases} F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(ux+vy)}dxdy \\ \\ f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)e^{2\pi i(ux+vy)}dudv \end{cases}$$

In 2-D, the DFT is given as:

$$\begin{cases} F(\bar{u},\bar{v}) = \sum_{\bar{x}=0}^{M-1} \sum \bar{y} = 0^{N-1} f(\bar{x},\bar{y})e^{-2\pi i(\bar{u}\bar{x}/M+\bar{v}\bar{y}/N)} \\ \\ f(\bar{x},\bar{y}) = \sum_{\bar{u}=0}^{M-1} \sum \bar{v} = 0^{N-1} F(\bar{u},\bar{v})e^{2\pi i(\bar{u}\bar{x}/M+\bar{v}\bar{y}/N)} \end{cases}$$

Here, $f(\bar{x},\bar{y})$ is an $M \times N$ image, and $F(\bar{u},\bar{v})$ is also an $M \times N$ image.

Recall that in the continuous domain, the Fourier Transform is not an approximation, but is rather an exact (albeit alternative) representation of $f(x,y)$. Similarly, $f(\bar{x},\bar{y})$ can be exactly recovered from $F(\bar{u},\bar{v})$ (subject to numerical effects, e.g. rounding). In this way, $f(\bar{x},\bar{y})$ and $F(\bar{u},\bar{v})$ contain the exact same information.

Fourier Analysis

from: http://cse19-iiith.vlabs.ac.in/theory.php?exp=fourier

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