ME7224
Modal Analysis of Mechanical Systems

End Semester

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Contents

1 Question 1

A cantilever beam has been tested using 8 points along its length. Point #1 is at the free end, and Point #8 is near the fixed support. The accelerance FRF data, H^a_{ij} for $i = 1$ and $j = 1, 2, \ldots, 8$ is available. Please note that this data has the units $\frac{\mathrm{m}}{\mathrm{s}^2(\mathrm{lbf})}$.

Code

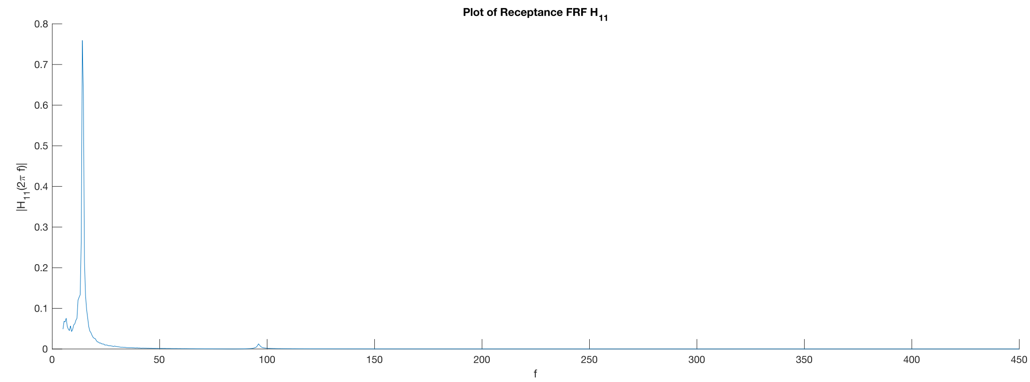
```
data = readmatrix('data.xlsx');
% data(:, 2:end) = data(:, 2:end)/4.448222;

modes = data(:, 1);
H11a = data(:, 2) + i .* data(:, 3);
H11 = H11a ./ modes.^2;
H12a = data(:, 4) + i .* data(:, 5);
H12 = H12a ./ modes.^2;
H13a = data(:, 6) + i .* data(:, 7);
H13 = H13a ./ modes.^2;
H14a = data(:, 8) + i .* data(:, 9);
H14 = H14a ./ modes.^2;
H15a = data(:, 10) + i .* data(:, 11);
H15 = H15a ./ modes.^2;
H16a = data(:, 12) + i .* data(:, 13);
H16 = H16a ./ modes.^2;
H17a = data(:, 14) + i .* data(:, 15);
H17 = H17a ./ modes.^2;
H18a = data(:, 16) + i .* data(:, 17);
H18 = H18a ./ modes.^2;
```

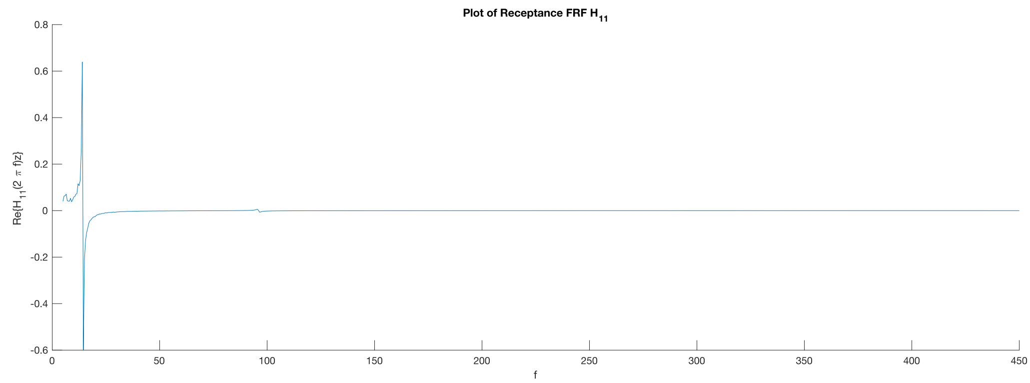
1.1 Part A

Plot the magnitude as well as the real and imaginary parts of the receptance FRFs, H_{11} and H_{14}, for the frequency range $5 \leq f \leq 450$ Hz.

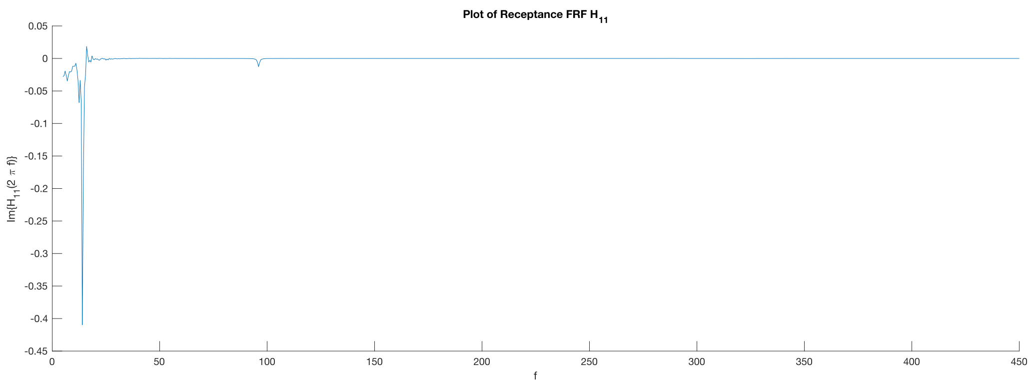
1.1.1 Plots



(a) Magnitude

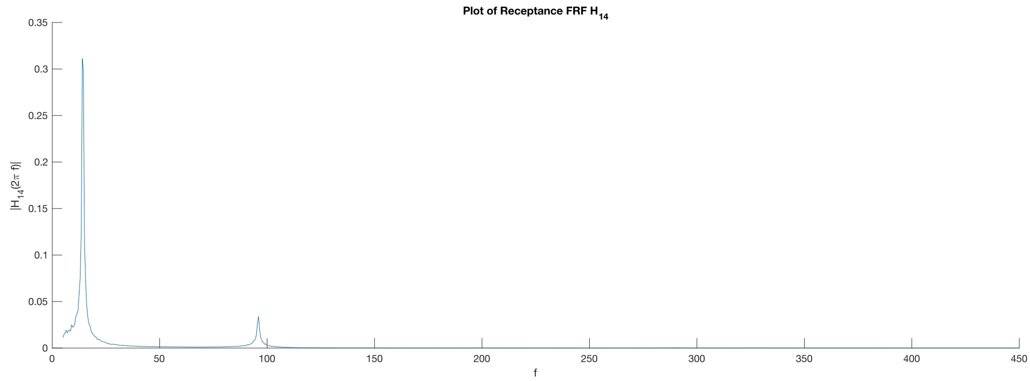


(b) Real Part

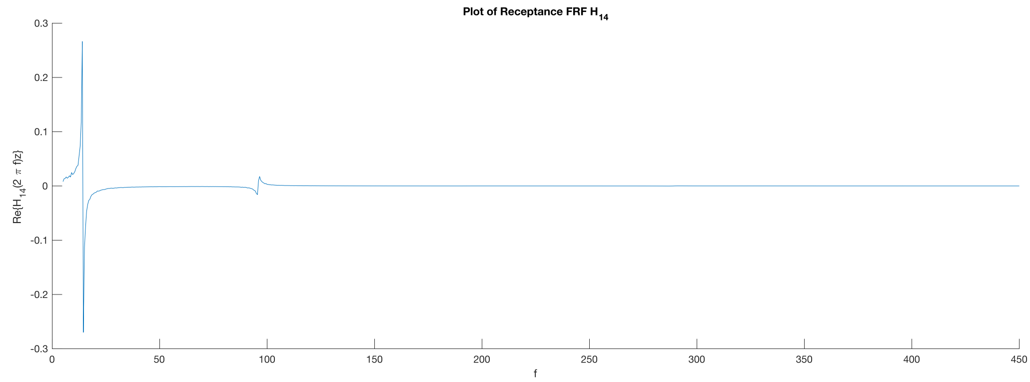


(c) Imaginary Part

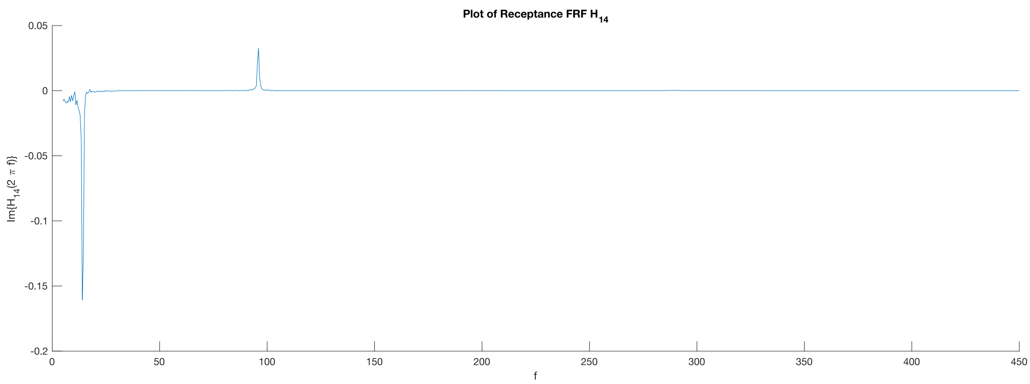
Figure 1.1: Receptance FRF $H_{11}(\omega)$ in the range $5 \leq \omega \leq 450$



(a) Magnitude



(b) Real Part



(c) Imaginary Part

Figure 1.2: Receptance FRF $H_{14}(\omega)$ in the range $5 \leq \omega \leq 450$

1.2 Part B

For the second and third modes, use peak-picking and circle fit SDoF methods to estimate the natural frequency, damping parameter (assume it is structural damping), and modal constants, plot the mode shapes.

1.2.1 Results

Mode	Method	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}	H_{16}	H_{17}	H_{18}
Mode 1	Peak Picking	14.00	14.00	14.00	14.00	14.00	14.00	14.00	15.50
	Circle Fit	13.50	15.50	13.50	13.50	13.50	13.50	13.50	14.50
Mode 2	Peak Picking	96.00	96.50	96.00	96.00	96.00	96.00	96.00	96.00
	Circle Fit	95.50	95.50	95.00	95.00	95.00	95.00	95.00	95.00
Mode 3	Peak Picking	287.50	289.00	287.00	289.50	288.50	289.00	289.50	289.50
	Circle Fit	288.00	288.50	288.00	288.50	288.50	288.50	288.50	287.00

Table 1.1: SDoF Modal Analysis for f_r

Mode	Method	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}	H_{16}	H_{17}	H_{18}
Mode 1	Peak Picking	0.0357	0.0357	0.0357	0.0357	0.0357	0.0357	0.0357	0.2903
	Circle Fit	0.1091	0.1249	0.0768	0.2058	0.2634	0.1091	0.1975	0.1052
Mode 2	Peak Picking	0.0104	0.0104	0.0104	0.0104	0.0104	0.0104	0.0104	0.0104
	Circle Fit	0.0158	0.0211	0.0158	0	0.0158	0.0158	0.0158	0.0211
Mode 3	Peak Picking	0.0157	0.0156	0.0070	0.0104	0.0173	0.0069	0.0173	0.0104
	Circle Fit	0.0173	0.0121	0.0173	0.0173	0	0	0.0017	0.0122

Table 1.2: SDoF Modal Analysis for d_r

Mode	Method	ϕ_{1r}	ϕ_{2r}	ϕ_{3r}	ϕ_{4r}	ϕ_{5r}	ϕ_{6r}	ϕ_{7r}	ϕ_{8r}
Mode 1	Peak Picking	2.3054	2.3423	1.3282	0.9445	0.8499	0.0540	0.1156	0.0458
	Circle Fit	5.7582	1.3857	1.9541	4.2664	5.1329	0.7970	0.2532	0.0471
Mode 2	Peak Picking	1.0992	0.1446	2.6906	2.9583	3.0761	2.4040	0.9760	0.2750
	Circle Fit	1.3473	0.2246	3.4645	0	3.9954	3.1275	1.2590	0.4632
Mode 3	Peak Picking	0.3305	1.5430	0.8332	0.7398	1.2738	1.7047	1.4384	0.1988
	Circle Fit	0.3923	1.3112	2.4458	0.9991	0	0	0.0884	0.1696

Table 1.3: SDoF Modal Analysis for $\phi_{jr}\phi_{kr}$ (Normalized by 2π)

1.2.2 Plots

(a) Peak Picking (b) Circle Fit

Figure 1.3: SDoF Modal Analysis for Mode Shapes

1.3 Part C

Regenerate the accelerance FRFs from the fitted data (H^a_{11} and H^a_{15}) and check how good a fit it is with the experimental accelerance FRFs. The frequency range is $50 \leq f \leq 450$ Hz. Comment on the results obtained.

Both Peak Picking and Circle Fit estimates the correct natural frequencies, and damping with slight variation. The reconstructed FRF fits reasonably well near the resonance but not on the extremes.

1.3.1 Method

Peak Picking

The FRF $H_{jk}(\omega)$ near resonance frequency ω_r is given by

$$H_{jk}(\omega) = \frac{\phi_{jr}\phi_{kr}}{(\omega_r - \omega^2) + i\eta\omega_r\omega}$$

We find the value of ω_r is identified by locating the peak of $|H_{jk}(\omega)|$ around some interval $\omega_r - \epsilon \leq \omega \leq \omega_r + \epsilon$ for some $\epsilon \in \mathbb{R}$. The peak amplitude is give by

$$\text{Peak} = \left| \frac{\phi_{jr}\phi_{kr}}{\eta\omega_r^2} \right|$$

The half power points are subsequently obtained by finding the pre-image of the FRF function for value equal to $H_{jk}(\omega_{a,b}) = \frac{\text{Peak}}{\sqrt{2}}$. The damping constant can be estimated as,

$$\eta \approx \frac{|\omega_a - \omega_b|}{\omega_r}$$

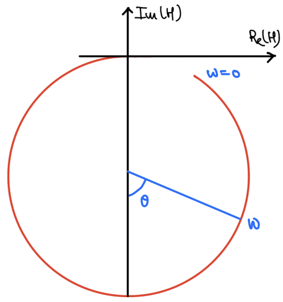
The modal constant is thus obtained as,

$$\phi_{jr}\phi_{kr} = \text{Peak} \times \eta\omega_r^2$$

Circle Fit

The FRF $H_{jk}(\omega)$ near resonance frequency ω_r is given by

$$H_{jk}(\omega) = \frac{\phi_{jr}\phi_{kr}}{(\omega_r - \omega^2) + i\eta\omega_r\omega}$$

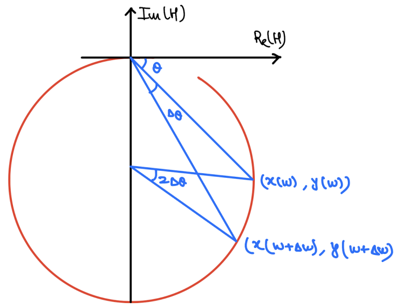


The plot of $\mathrm{Re}\{H_{jk}(\omega)\}$ vs $\mathrm{Im}\{H_{jk}(\omega)\}$ resembles a circle. The peak rate is given by

$$\frac{d\theta}{d\omega^2} = \frac{-2}{\eta\omega_r^2\left(1+\tan(\frac{\theta}{2})^2\right)}$$

For structural damping, we have

$$\mathrm{Re}\{H_{jk}(\omega)\} = \frac{1}{\omega_r^2}\frac{\left(1-\frac{\omega^2}{\omega_r^2}\right)}{\left(1-\frac{\omega^2}{\omega_r^2}\right)^2+\eta^2}$$
$$\mathrm{Im}\{H_{jk}(\omega)\} = \frac{1}{\omega_r^2}\frac{\eta}{\left(1-\frac{\omega^2}{\omega_r^2}\right)^2+\eta^2}$$



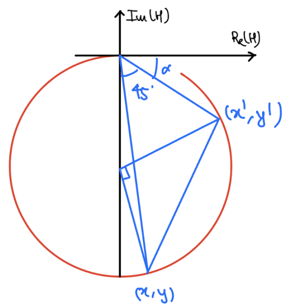
From geometry, we have

$$\tan(\theta) = \left|\frac{\mathrm{Im}\{H_{jk}(\omega)\}}{\mathrm{Re}\{H_{jk}(\omega)\}}\right|$$
$$\tan(\theta+\Delta\theta) = \left|\frac{\mathrm{Im}\{H_{jk}(\omega+\Delta\omega)\}}{\mathrm{Re}\{H_{jk}(\omega+\Delta\omega)\}}\right|$$

The rate is thus given by

$$\frac{\Delta\theta}{\Delta\omega^2} = 2\times\frac{\arctan\left(\left|\frac{\mathrm{Im}\{H_{jk}(\omega+\Delta\omega)\}}{\mathrm{Re}\{H_{jk}(\omega+\Delta\omega)\}}\right|\right) - \arctan\left(\left|\frac{\mathrm{Im}\{H_{jk}(\omega)\}}{\mathrm{Re}\{H_{jk}(\omega)\}}\right|\right)}{\Delta\omega^2}$$

We find the value of ω_r is identified by locating the maxima of the sweep rate $\frac{d\theta}{d\omega^2}$ around some interval $\omega_r - \epsilon \le \omega \le \omega_r + \epsilon$ for some $\epsilon \in \mathbb{R}$. We sweep by 270° over FRFs to obtain points and draw the best fit circle to obtain the centre (x_c, y_c) and the radius R.
The half power points can be obtained by taking the points which make 90° with the centre and the resonance point.



$$(\mathrm{Re}\{H_{jk}(\omega_a)\}, \mathrm{Im}\{H_{jk}(\omega_a)\}) = (x_c + R\sin(\alpha), y_c - R\cos(\alpha))$$
$$(\mathrm{Re}\{H_{jk}(\omega_b)\}, \mathrm{Im}\{H_{jk}(\omega_b)\}) = (x_c - R\sin(\alpha), y_c + R\cos(\alpha))$$

With the corresponding pre-images ω_a and ω_b, the damping constant can be estimated as,

$$\eta \approx \frac{|\omega_a^2 - \omega_b^2|}{2\omega_r^2}$$

The modal constant is thus obtained as,

$$\phi_{jr}\phi_{kr} = 2R\omega_r^2\eta$$

1.3.2 Code

Peak Picking

```
    function [wr, dr, phijr_phikr] = peak_picking(H, w)
        H = abs(H);
        [peak, wr] = findpeaks(H, w);
        half_power = 1/sqrt(2) * peak;
        [r,c] = size(wr);
        if (r > 1) | (c > 1)
            wr = wr(end);
            half_power = half_power(end);
        end
        ind_wr = find(w==wr);

        ind_wa = find(min(abs(H(1:ind_wr)-half_power)) == abs(H
           (1:ind_wr)-half_power));
        ind_wb = find(min(abs(H(ind_wr:end)-half_power)) == abs
           (H(ind_wr:end)-half_power)) + ind_wr-1;
        wa = w(ind_wa);
        wb = w(ind_wb);

        dr = (wb-wa)/wr;
        phijr_phikr = peak * dr * wr^2;
    end
```

Circle Fit

```
    function [wr, dr, phijr_phikr] = circle_fit(H, w)
        rh = real(H);
        ih = imag(H);

        slopes = zeros(size(w));
        for i = 1:1:size(w, 2)
            slopes(i) = abs(ih(i)/rh(i));
        end

        rate = zeros(size(w));
        for i = 1:1:size(w, 2)-1
            del_theta = (atan(slopes(i+1)) - atan(slopes(i)))
               *2;
            del_w2 = w(i+1)^2 - w(i)^2;
            rate(i) = del_theta/del_w2;
        end
        [peak, wr] = findpeaks(rate, w);

        [r,c] = size(wr);
        if (r > 1) | (c > 1)
            wr = wr(end);
        end
```

```
        ind_wr = find(w==wr);

        yrxr = slopes(ind_wr);
        y1x1 = (yrxr-tan(deg2rad(135/2)))/(1+tan(deg2rad(135/2))*yrxr);
        ind_wa = find(min(abs(slopes(1:ind_wr)-y1x1)) == abs(slopes(1:ind_wr)-y1x1));
        ind_wb = find(min(abs(slopes(ind_wr:end)-y1x1)) == abs(slopes(ind_wr:end)-y1x1)) + ind_wr-1;

        x = rh(ind_wa:ind_wb); y = ih(ind_wa:ind_wb);
        x = x(:); y = y(:);
        a = [x y ones(size(x))]\[-(x.^2+y.^2)];
        xc = -.5*a(1);
        yc = -.5*a(2);
        R  =  sqrt((a(1)^2+a(2)^2)/4-a(3));

        alpha = atan((ih(ind_wr)-yc)/(rh(ind_wr)-xc));
        xa = xc + R*sin(alpha); ya = yc - R*cos(alpha);
        xb = xc - R*sin(alpha); yb = yc + R*cos(alpha);

        ind_wa = find(min(abs(slopes(1:ind_wr)-abs(ya/xa))) == abs(slopes(1:ind_wr)-abs(ya/xa)));
        ind_wb = find(min(abs(slopes(ind_wr:end)-abs(yb/xb))) == abs(slopes(ind_wr:end)-abs(yb/xb))) + ind_wr-1;
        wa = w(ind_wa);
        wb = w(ind_wb);

        dr = (wb^2-wa^2)/(2*wr^2);
        phijr_phikr = 2 * R * wr^2 * dr;
    end
```

1.3.3 Plots

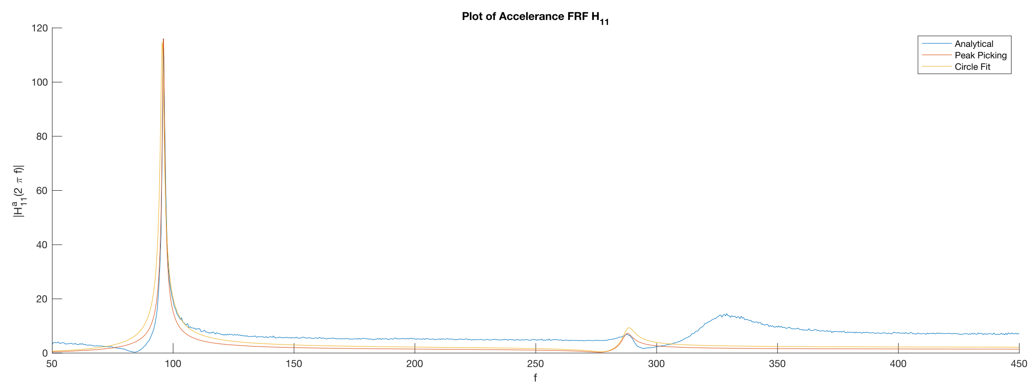


Figure 1.4: SDoF Fitting for H_{11}^{a}

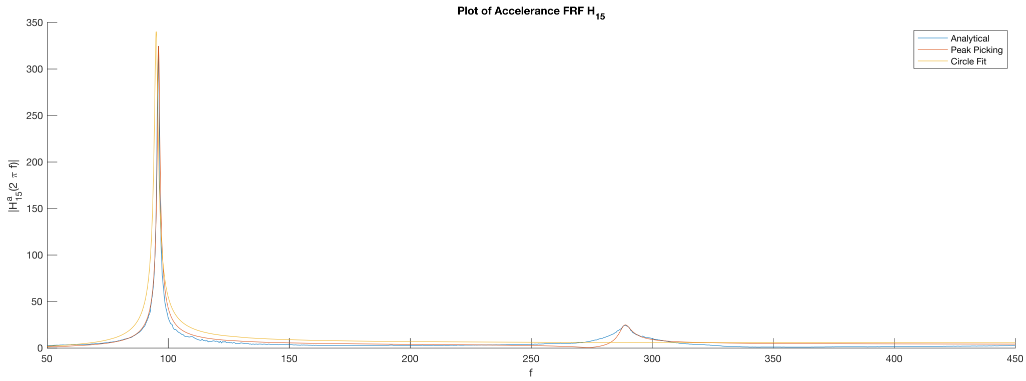


Figure 1.5: SDoF Fitting for H^a_{15}

1.4 Part D

Find out the residual stiffness and mass for improving the fit for H^a_{12} and H^a_{16}.

1.4.1 Method

The reconstructed FRF accounting for the residues

$$H_{jk}(\omega) = \frac{-1}{\omega^2 M^R_{jk}} + \sum_{m=1}^{N} \frac{\phi_{jm}\phi_{km}}{(\omega_m^2 - \omega^2) + i\eta\omega_m\omega} + \frac{1}{K^R_{jk}}$$

where the residual mass and stiffness can be found out at low and high frequency measurements of the FRF respectively.

1.4.2 Code

```
function [mr, kr] = residual(H_actual, H_generated, w)
    H_actual = abs(H_actual);
    H_generated = abs(H_generated);
    length = floor(size(w, 2)/10);
    w_low = w(1:length);
    H_actual_low = H_actual(1:length);
    H_actual_low = H_actual_low(:);
    H_generated_low = H_generated(1:length);
    H_generated_low = H_generated_low(:);

    w_high = w(end-length:end);
    H_actual_high = H_actual(end-length:end);
    H_actual_high = H_actual_high(:);
    H_generated_high = H_generated(end-length:end);
    H_generated_high = H_generated_high(:);

    kr = 1/ mean((H_actual_high-H_generated_high));
    mr = mean(-1./w_low.^2) / (mean(H_actual_low-
       H_generated_low)-1/kr);
end
```

1.4.3 Results

Method	Residue	H^a_{12}	H^a_{16}
Peak Picking	Mass	-4.5316×10^{-5}	-9.1850×10^{-5}
	Stiffness	0.3330	-2.9780
Circle Fit	Mass	-5.9514×10^{-5}	9.3397×10^{-5}
	Stiffness	0.3513	-1.0487

Table 1.4: SDoF Modal Analysis for d_r

1.4.4 Plots

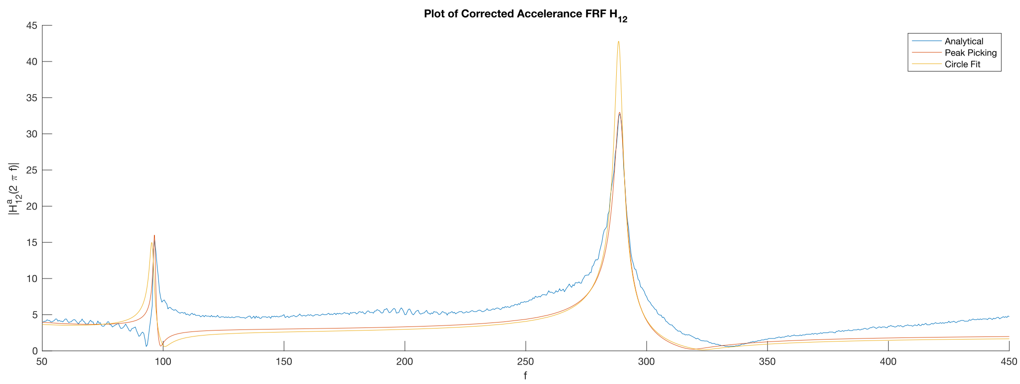


Figure 1.6: Corrected SDoF Fitting for H^a_{12}

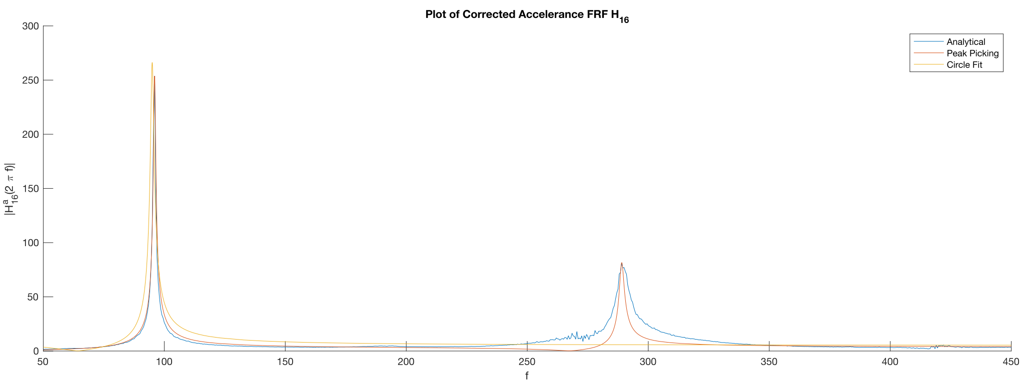


Figure 1.7: Corrected SDoF Fitting for H^a_{16}

2 Question 2

Use the same experimental data from the previous question and do the following.

2.1 Part A

Use the rational fraction polynomial (RFP) method (based on orthogonal polynomials) to extract the natural frequencies, damping ratios (assume viscous damping), and mode shape parameters, in the frequency range $5 \leq f \leq 125$ Hz using the receptance FRFs. Assume $m = 2$ for this exercise.

2.1.1 Method

Writing FRF as rational fraction polynomial form,

$$H_{ij}(\omega_i) = \frac{\sum\limits_{k=0}^{2\times m-1} c_k \phi_k^+(\omega_i)}{\sum\limits_{k=0}^{2\times m} d_k \theta_k^+(\omega_i)}$$

where the polymomials ϕ_k^+ and θ_k^+ are orthogonal in the sense

$$\sum_{i=1}^{L} \phi_j^+(\omega_i)^* \phi_k^+(\omega_i) = \delta_{jk} \qquad \sum_{i=1}^{L} |H_{j1}(\omega_i)|^2 \theta_j^+(\omega_i)^* \theta_k^+(\omega_i) = \delta_{jk}$$

The coefficients c_k and d_k are determined by minimizing the error

$$\{\varepsilon_{ij}\} = [P]\{c\} - [T_{ij}]\{d\} - \{w_{ij}\}$$

where

$$[P] = \begin{bmatrix} \phi_0^+(\omega_1) & \phi_1^+(\omega_1) & \dots & \phi_{2m-1}^+(\omega_1) \\ \phi_0^+(\omega_2) & \phi_1^+(\omega_2) & \dots & \phi_{2m-1}^+(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0^+(\omega_L) & \phi_1^+(\omega_L) & \dots & \phi_{2m-1}^+(\omega_L) \end{bmatrix}$$

$$[T] = \begin{bmatrix} H_{ij}(\omega_1)\phi_0^+(\omega_1) & H_{ij}(\omega_1)\phi_1^+(\omega_1) & \dots & H_{ij}(\omega_1)\phi_{2m-1}^+(\omega_1) \\ H_{ij}(\omega_2)\phi_0^+(\omega_2) & H_{ij}(\omega_2)\phi_1^+(\omega_2) & \dots & H_{ij}(\omega_2)\phi_{2m-1}^+(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ H_{ij}(\omega_L)\phi_0^+(\omega_L) & H_{ij}(\omega_L)\phi_1^+(\omega_L) & \dots & H_{ij}(\omega_L)\phi_{2m-1}^+(\omega_L) \end{bmatrix}$$

$$\{w_{ij}\} = \begin{Bmatrix} H_{ij}(\omega_1)\phi_{2m}^+(\omega_1) \\ H_{ij}(\omega_2)\phi_{2m}^+(\omega_2) \\ \vdots \\ H_{ij}(\omega_L)\phi_{2m}^+(\omega_L) \end{Bmatrix}$$

and

$$\{c\} = \begin{Bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2m-1} \end{Bmatrix} \qquad \{d\} = \begin{Bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{2m-1} \end{Bmatrix} \qquad d_{2m} = 1$$

The minimization of error ε leads to

$$\begin{bmatrix} [I] & [X] \\ [X^T] & [I] \end{bmatrix} \begin{Bmatrix} \{c\} \\ \{d\} \end{Bmatrix} = \begin{Bmatrix} \{g\} \\ \{0\} \end{Bmatrix}$$

where $[X] = \mathrm{Re}\{[P]^*[T]\}$ and $\{h\} = \mathrm{Re}\{[P]^*\{w\}\}$ and $[I] = [I]_{2m-1\times 2m-1}$.

Simplifying, we have

$$\begin{gathered} \left[[I] - [X]^T[X]\right]\{d\} = -[X]^T h \\ \{c\} = \{h\} - [X]\{d\} \end{gathered}$$

The natural frequencies ω_n are determined by identifying the poles p_n and the modal constants from the residues r_n of the rational fraction polynomial.

$$H_{ij}(\omega) = \frac{\sum_{k=0}^{2\times m-1} c_k \phi_k^+(\omega)}{\sum_{k=0}^{2\times m} d_k \theta_k^+(\omega)} = \sum_{n=0}^{m} \frac{r_n}{\omega^2 - p_n^2}$$

2.1.2 Code

The RFP was computed by the method described in [1] using the code shared on Moodle.

2.1.3 Results

$H_{ij}(\omega)$	Method	Natural Frequencies
$H_{11}(\omega)$	RFP ($m = 2$)	14.1093, 95.4778
$H_{12}(\omega)$	RFP ($m = 2$)	14.1234, 82.0227
$H_{13}(\omega)$	RFP ($m = 2$)	14.1722, 95.8854
$H_{14}(\omega)$	RFP ($m = 2$)	14.2129, 95.8901
$H_{15}(\omega)$	RFP ($m = 2$)	14.1533, 95.8726
$H_{16}(\omega)$	RFP ($m = 2$)	14.1116, 95.8623
$H_{17}(\omega)$	RFP ($m = 2$)	13.0223, 95.8796
$H_{18}(\omega)$	RFP ($m = 2$)	12.0126, 95.9112

Table 2.1: MDoF Modal Analysis for f_r

$H_{ij}(\omega)$	Method	Damping Coefficients
$H_{11}(\omega)$	RFP ($m = 2$)	0.0096, 0.0050
$H_{12}(\omega)$	RFP ($m = 2$)	0.0108, 0.0090
$H_{13}(\omega)$	RFP ($m = 2$)	0.0108, 0.0090
$H_{14}(\omega)$	RFP ($m = 2$)	0.0110, 0.0039
$H_{15}(\omega)$	RFP ($m = 2$)	0.0091, 0.0044
$H_{16}(\omega)$	RFP ($m = 2$)	0.0114, 0.0041
$H_{17}(\omega)$	RFP ($m = 2$)	0.0253, 0.0043
$H_{18}(\omega)$	RFP ($m = 2$)	0.0206, 0.0044

Table 2.2: MDoF Modal Analysis for d_r

$H_{ij}(\omega)$	Method	Mode Shapes
$H_{11}(\omega)$	RFP ($m = 2$)	4.5011, 1.1439
$H_{12}(\omega)$	RFP ($m = 2$)	4.5683, 0.2283
$H_{13}(\omega)$	RFP ($m = 2$)	2.7933, 2.2860
$H_{14}(\omega)$	RFP ($m = 2$)	2.3158, 2.6622
$H_{15}(\omega)$	RFP ($m = 2$)	1.6808, 2.9461
$H_{16}(\omega)$	RFP ($m = 2$)	0.8037, 2.2607
$H_{17}(\omega)$	RFP ($m = 2$)	0.2436, 0.9457
$H_{18}(\omega)$	RFP ($m = 2$)	0.0528, 0.2660

Table 2.3: MDoF Modal Analysis for $\phi_{ir}\phi_{jr}$ (Normalized by 2π)

2.2 Part B

Now vary m from 3 to 6 and generate the parameters as before. Comment on the results you obtain.

The plot of the measured FRF shows two significant peaks in the frequency range. The RFP extraction for $m > 2$ leads to erroneous modes, which are amplified (unlike the extraction on a forward problem) due to the presence of noise during measurement.

2.2.1 Results

$H_{ij}(\omega)$	Method	Natural Frequencies
$H_{11}(\omega)$	RFP ($m = 3$)	10.4415, 14.3382, 96.0472
	RFP ($m = 4$)	7.8463, 14.2598, 96.0438, 111.4589
	RFP ($m = 5$)	6.7488, 14.2377, 61.2456, 96.0431, 118.0244
	RFP ($m = 6$)	5.5579, 14.2337, 45.6187, 96.0414, 105.536, 120.9981
$H_{12}(\omega)$	RFP ($m = 3$)	11.6811, 14.3327, 96.2046
	RFP ($m = 4$)	9.9952, 14.2578, 89.2128, 96.4994
	RFP ($m = 5$)	8.5518, 14.2217, 60.1191, 96.1887, 115.8381
	RFP ($m = 6$)	7.3057, 14.2207, 46.9128, 84.9083, 96.241, 119.8048
$H_{13}(\omega)$	RFP ($m = 3$)	14.1738, 95.7515, 103.9853
	RFP ($m = 4$)	14.1713, 26.6875, 95.863, 123.419
	RFP ($m = 5$)	12.8401, 14.2751, 95.7326, 97.2494, 124.7085
	RFP ($m = 6$)	9.3758, 14.2147, 63.8863, 95.8436, 101.1223, 124.802
$H_{14}(\omega)$	RFP ($m = 3$)	14.214, 61.0498, 95.9033
	RFP ($m = 4$)	14.1929, 21.2834, 95.8749, 109.6208
	RFP ($m = 5$)	13.4766, 14.4151, 90.3101, 95.9212, 116.4271
	RFP ($m = 6$)	10.9149, 14.2456, 69.7619, 95.8632, 103.3689, 119.95
$H_{15}(\omega)$	RFP ($m = 3$)	14.1354, 51.0337, 95.8316
	RFP ($m = 4$)	11.4857, 14.3851, 95.8497, 116.4295
	RFP ($m = 5$)	9.2496, 14.2628, 95.8137, 101.0025, 119.33
	RFP ($m = 6$)	7.5253, 14.2454, 65.4945, 95.8427, 109.855, 121.4306
$H_{16}(\omega)$	RFP ($m = 3$)	13.7488, 20.2463, 95.8402
	RFP ($m = 4$)	7.7017, 14.4188, 95.8597, 104.9106
	RFP ($m = 5$)	5.4098, 14.281, 77.4557, 95.8423, 115.6755
	RFP ($m = 6$)	5.1684, 14.2576, 65.9829, 95.8567, 101.8442, 120.3223
$H_{17}(\omega)$	RFP ($m = 3$)	7.4936, 15.9643, 95.851
	RFP ($m = 4$)	4.4817, 15.0437, 95.863, 96.2197
	RFP ($m = 5$)	8.6306, 14.325, 64.4705, 95.8562, 120.23
	RFP ($m = 6$)	3.9129, 14.221, 62.6247, 95.8698, 104.0425, 122.0378
$H_{18}(\omega)$	RFP ($m = 3$)	11.074, 29.9839, 95.8904
	RFP ($m = 4$)	9.7669, 18.367, 81.1508, 95.8858
	RFP ($m = 5$)	4.7747, 14.3965, 64.0019, 95.8801, 115.923
	RFP ($m = 6$)	5.5471, 14.2363, 63.6071, 95.8833, 100.6198, 118.4792

Table 2.4: MDoF Modal Analysis for f_r

$H_{ij}(\omega)$	Method	Damping Coefficients
$H_{11}(\omega)$	RFP ($m = 3$)	-0.0148, 0.0075, 0.005
	RFP ($m = 4$)	-0.0689, 0.0084, 0.0051, 0.0003
	RFP ($m = 5$)	-0.2316, 0.0082, -0.0036, 0.005, 0.0003
	RFP ($m = 6$)	-0.3599, 0.0084, -0.0161, 0.005, -0.0005, -0.001
$H_{12}(\omega)$	RFP ($m = 3$)	0.0327, 0.0024, 0.007
	RFP ($m = 4$)	0.0321, 0.006, -0.0059, 0.0057
	RFP ($m = 5$)	-0.059, 0.0075, -0.0115, 0.0074, -0.0049
	RFP ($m = 6$)	-0.1685, 0.0086, -0.0679, -0.0093, 0.0072, -0.0002
$H_{13}(\omega)$	RFP ($m = 3$)	0.01, 0.0029, 0.0033
	RFP ($m = 4$)	0.011, 0.0145, 0.0037, 0.0003
	RFP ($m = 5$)	0.0114, 0.0041, 0.0042, -0.0001, 0.0006
	RFP ($m = 6$)	-0.0512, 0.0085, -0.0089, 0.0039, 0.0007, 0.0004
$H_{14}(\omega)$	RFP ($m = 3$)	0.0115, 0.006, 0.004
	RFP ($m = 4$)	0.0122, 0.0121, 0.004, 0.001
	RFP ($m = 5$)	0.0329, 0.0017, 0.0032, 0.0041, -0.0001
	RFP ($m = 6$)	-0.0275, 0.0081, -0.0017, 0.0039, 0.0021, -0.0001
$H_{15}(\omega)$	RFP ($m = 3$)	0.0091, 0.0742, 0.0045
	RFP ($m = 4$)	0.1228, 0.0197, 0.0042, 0.0054
	RFP ($m = 5$)	-0.0196, 0.0076, 0.0036, 0.0204, 0.0039
	RFP ($m = 6$)	-0.1645, 0.008, 0.0151, 0.0041, 0.0029, 0.0022
$H_{16}(\omega)$	RFP ($m = 3$)	0.0004, 0.1077, 0.0041
	RFP ($m = 4$)	-0.1156, 0.0183, 0.004, -0.0055
	RFP ($m = 5$)	-0.5751, 0.0114, 0.015, 0.0041, -0.0034
	RFP ($m = 6$)	-0.5396, 0.0109, 0.0356, 0.004, 0.0017, -0.0011
$H_{17}(\omega)$	RFP ($m = 3$)	0.2257, 0.23, 0.0041
	RFP ($m = 4$)	-0.7355, 0.0408, 0.0062, -0.0118
	RFP ($m = 5$)	-1, 0.0255, 0.0743, 0.0041, 0.0072
	RFP ($m = 6$)	-0.67, 0.0235, 0.0742, 0.0042, 0.003, 0.0086
$H_{18}(\omega)$	RFP ($m = 3$)	-0.0138, 0.0812, 0.0041
	RFP ($m = 4$)	-0.1419, -0.0093, -0.0057, 0.0042
	RFP ($m = 5$)	-0.4379, 0.0256, 0.0458, 0.0042, 0.0011
	RFP ($m = 6$)	-0.116, 0.0222, 0.0475, 0.0042, -0.0359, 0.0031

Table 2.5: MDoF Modal Analysis for d_r

$H_{ij}(\omega)$	Method	Mode Shapes
$H_{11}(\omega)$	RFP ($m = 3$)	0.7674, 4.5016, 1.1609
	RFP ($m = 4$)	0.9607, 4.6933, 1.1704, 0.0101
	RFP ($m = 5$)	1.2175, 4.9379, 0.0344, 1.1687, 0.0015
	RFP ($m = 6$)	5.5579, 14.2337, 45.6187, 96.0414, 105.536, 120.9981
$H_{12}(\omega)$	RFP ($m = 3$)	3.1264, 1.0481, 0.239
	RFP ($m = 4$)	1.0967, 2.9088, 0.073, 0.2104
	RFP ($m = 5$)	0.4231, 4.0871, 0.052, 0.2436, 0.008
	RFP ($m = 6$)	7.3057, 14.2207, 46.9128, 84.9083, 96.241, 119.8048
$H_{13}(\omega)$	RFP ($m = 3$)	2.8621, 1.8745, 0.2779
	RFP ($m = 4$)	3.1042, 0.2989, 2.2696, 0.0232
	RFP ($m = 5$)	1.1107, 1.6654, 2.3871, 0.4678, 0.0276
	RFP ($m = 6$)	9.3758, 14.2147, 63.8863, 95.8436, 101.1223, 124.802
$H_{14}(\omega)$	RFP ($m = 3$)	2.3746, 0.1095, 2.6961
	RFP ($m = 4$)	2.4519, 0.2017, 2.7104, 0.0482
	RFP ($m = 5$)	1.408, 0.8717, 0.0726, 2.6836, 0.0086
	RFP ($m = 6$)	10.9149, 14.2456, 69.7619, 95.8632, 103.3689, 119.95
$H_{15}(\omega)$	RFP ($m = 3$)	1.7601, 0.3562, 2.9358
	RFP ($m = 4$)	0.9125, 2.384, 2.8724, 0.0754
	RFP ($m = 5$)	0.0662, 1.6138, 2.7218, 0.2781, 0.0456
	RFP ($m = 6$)	7.5253, 14.2454, 65.4945, 95.8427, 109.855, 121.4306
$H_{16}(\omega)$	RFP ($m = 3$)	0.5872, 0.9113, 2.2235
	RFP ($m = 4$)	0.9297, 1.4581, 2.2273, 0.0434
	RFP ($m = 5$)	0.4708, 1.0614, 0.0574, 2.2422, 0.0135
	RFP ($m = 6$)	5.1684, 14.2576, 65.9829, 95.8567, 101.8442, 120.3223
$H_{17}(\omega)$	RFP ($m = 3$)	1.51, 1.4554, 0.9115
	RFP ($m = 4$)	0.7837, 0.6977, 0.9521, 0.2494
	RFP ($m = 5$)	1.0887, 0.4404, 0.0567, 0.9073, 0.0053
	RFP ($m = 6$)	3.9129, 14.221, 62.6247, 95.8698, 104.0425, 122.0378
$H_{18}(\omega)$	RFP ($m = 3$)	0.0661, 0.0608, 0.2542
	RFP ($m = 4$)	0.1774, 0.1501, 0.0304, 0.2505
	RFP ($m = 5$)	0.1372, 0.1249, 0.0346, 0.2544, 0.0025
	RFP ($m = 6$)	5.5471, 14.2363, 63.6071, 95.8833, 100.6198, 118.4792

Table 2.6: MDoF Modal Analysis for $\phi_{ir}\phi_{jr}$ (Normalized by 2π)

2.3 Part C

Regenerate the accelerance FRFs and compare it with experimental data for H_{11} and H_{17} and comment on your results.

2.3.1 Results

The RFP method performs better for H_{11} but fails to capture the system's behavior of H_{17} for small frequencies. With the increase in m, the RFP fit for H_{17} improves. The erroneous mode shapes can be eliminated by estimating the variance of the mode shapes.

2.3.2 Plots

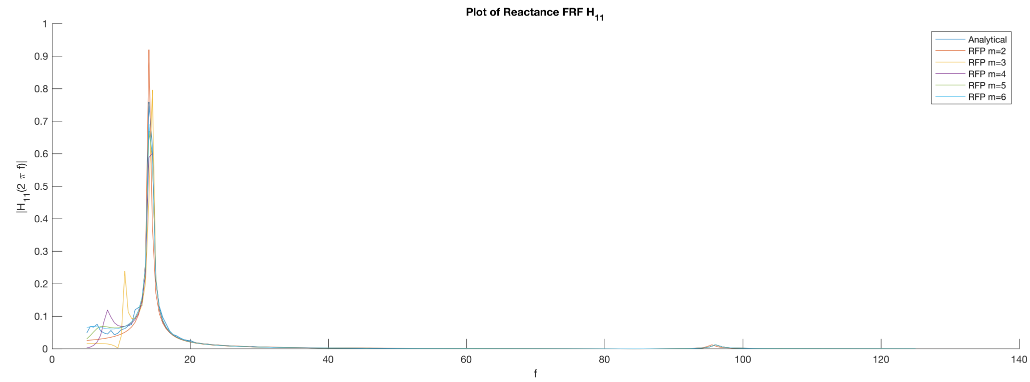


Figure 2.1: MDoF Fitting for H_{11}

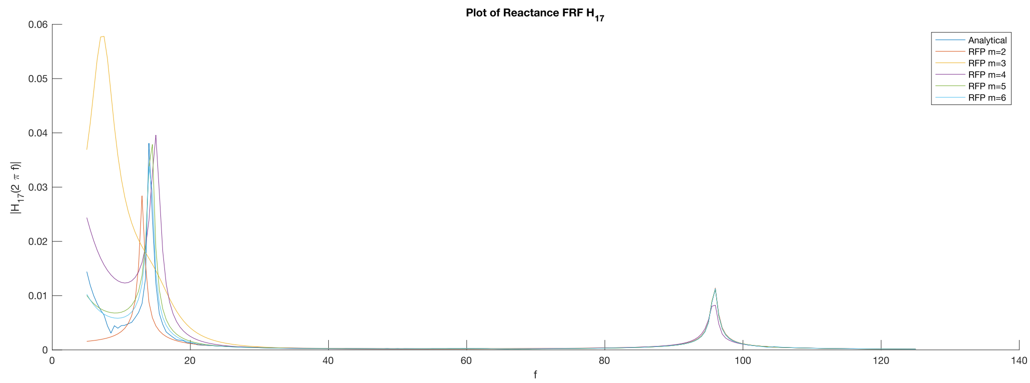


Figure 2.2: MDoF Fitting for H_{17}

References

[1] M. H. Richardson and D. L. Formenti, "Parameter estimation from frequency response measurements using rational fraction polynomials," in *1ºIMAC Conference*, (Orlando, FL), 1982.