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# NUMERICAL INVESTIGATION ON NATURAL CONVECTION IN ANNULAR FINS

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## ABSTRACT

*Steady-state temperature distribution of annular fins is investigated for various fin parameters. The fin is subjected to natural convection and modeled using a cylindrical meshing scheme. The Finite Volume Method (FVM) is used to discretize the computational domain for the solution of the General Scalar Transport Equation equation. The proposed solver is based on the cyclic tri-diagonal matrix algorithm (cTDMA) routine, which is validated using the derived analytical solution for an axis-symmetric case along the center plane.*

## NOMENCLATURE

| | |
|---|---|
| $k$ | Coefficient of Thermal Conductivity |
| $h$ | Coefficient of Natural Convection |
| $r_{\text{in}}$ | Inner Radius of the fin |
| $r_{\text{out}}$ | Outer Radius of the fin |
| $t$ | Thickness of the fin |
| $T$ | Temperature |
| $T_{r_{\text{in}}}$ | Temperature on the inner radial surface of the fin |
| $T_{r_{\text{out}}}$ | Temperature on the outer radial surface of the fin |
| $T_{z_{\text{up}}}$ | Temperature on the top surface of the fin |
| $T_{z_{\text{down}}}$ | Temperature on the bottom surface of the fin |
| $\alpha$ | Under-relaxation coefficient |

## INTRODUCTION

In recent years, the demand for efficient heat transfer devices has increased significantly due to rapid advancements in various industrial and technological fields [1–3]. Annular fins are widely used in heat exchangers and other heat management systems due to their ability to improve heat transfer rates by enlarging the surface area for convection [4, 5]. A thorough understanding of the steady-state temperature distribution in annular fins is crucial for optimizing their design and performance.

This paper aims to investigate the steady-state temperature distribution of annular fins subjected to natural convection for various fin parameters. A detailed numerical study is conducted using a cylindrical meshing scheme to model the annular fin geometry. The Finite Volume Method (FVM) is employed to discretize the computational domain [6, 7]. The proposed solver is based on the cyclic tri-diagonal matrix algorithm [8] routine, which is validated by comparing the numerical results with the derived analytical solution for an axis-symmetric case along the center plane.

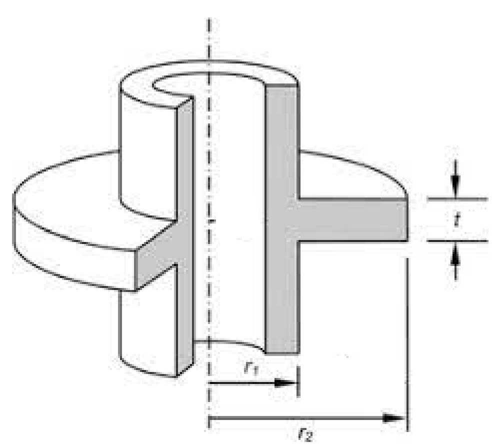


**FIGURE 1.** Annular Fin

## GENERALIZED SCALAR TRANSPORT EQUATION

The Generalized Scalar Transport Equation [9] is given by

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\phi\mathbf{u}) = \nabla\cdot(\Gamma\nabla\phi) + S_\phi \tag{1}$$

For steady diffusion,

$$\nabla\cdot(\Gamma\nabla\phi) + S_\phi = 0 \tag{2}$$

For the analysis of temperature distribution on annular fins (with no heat generation),

$$\nabla\cdot(k\nabla T) = 0 \tag{3}$$

## NUMERICAL SOLUTION

### Discretization

Finite Volume Method has been used to discretize the governing equation in the cylindrical mesh. The governing equation 3 is discretized as follows

$$\int_{\mathrm{CV}} \nabla\cdot(k\nabla T)\,d\upsilon = 0$$
$$\implies \sum_f (k\nabla T)_f \cdot A_f = 0$$

Reducing to the standard form (in 3 Dimensions) with a second-order accurate scheme $\nabla T = \frac{T_{\mathrm{nb}} - T_P}{\Delta}$, applied to the diffusion terms,

$$a_P T_P = \sum_f a_{\mathrm{nb}} T_{\mathrm{nb}} \tag{4}$$

For an isotropic fin (on the interior region),

$$a_{r_+} = \frac{kr_+\Delta\theta\Delta z}{\Delta r} \qquad a_{r_-} = \frac{kr_-\Delta\theta\Delta z}{\Delta r}$$
$$a_{\theta_+} = \frac{k\Delta r\Delta z}{r\Delta\theta} \qquad a_{\theta_-} = \frac{k\Delta r\Delta z}{r\Delta\theta}$$
$$a_{z_+} = \frac{kr\Delta r\Delta\theta}{\Delta z} \qquad a_{z_-} = \frac{kr\Delta r\Delta\theta}{\Delta z}$$
$$a_P = \sum_f a_{\mathrm{nb}}$$

The differential quantities $\Delta r$, $\Delta\theta$ and $\Delta z$ are defined according to the mesh parameters $n$, $m$ and $l$ as,

$$\Delta r = \frac{r_{\mathrm{out}} - r_{\mathrm{in}}}{n};\ \Delta\theta = \frac{2\pi}{m};\ \Delta z = \frac{t}{l}$$

### Boundary Conditions

The numerical simulations have been carried out for annular fins subjected to Dirichlet Boundary Condition on the inner radial surface and Neumann Boundary Condition on the top, bottom and the outer radial surfaces [10]. The domain is solved for Natural Convection.

### Algorithm

The discretized equations (4) are solved until convergence using cyclic tri-diagonal matrix algorithm, a varient of the general Thomas algorithm [8].

The general TDMA solver is modified to adapt to the periodic boundary conditions along the $\theta$ direction using Sherman–Morrison formula [11].

**Algorithm 1** cyclic Tri-Diagonal Matrix Algorithm

```
repeat
    for z = 0 to z = t do
        repeat
            for r = r_in to r = r_out do
                solve for T(θ; r, z) using TDMA
            end for
            solve for T(r, θ; z)
        until convergence
    end for
    solve for T(r, θ, z)
until convergence
```

## ANALYTICAL SOLUTION

The governing equation (3) is solved (on 1 Dimension) along the radial direction. The fin is subjected to an axis-symmetric boundary condition. For the thin fin, the convection on the top and bottom surfaces are neglected in comparison to the diffusion along the radial and the azimuthal direction. The modified governing equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0$$

Solving for the boundary conditions $T_{r_{\mathrm{in}}}$ and $-k\frac{dT}{dr} = h(T - T_{r_{\mathrm{out}},\infty})$ leads to

$$T = c_1\ln(r) + c_2$$

where

$$c_1\ln(r_{\mathrm{in}}) + c_2 = T_{r_{\mathrm{in}}}$$
$$c_1\left(\ln(r_{\mathrm{out}}) + \frac{k}{hr_{\mathrm{out}}}\right) + c_2 = T_{r_{\mathrm{out}},\infty}$$

## RESULTS

The numerical studies, were conducted for mesh parameters $n = 32$, $m = 64$ and $l = 8$. The thickness $t$ was varied keeping $r_{\text{in}} = 1$, $r_{\text{out}} = 4$ fixed. The boundary conditions imposed were $T_{r_{\text{in}}} = 1000$, $T_{r_{\text{out}},\infty} = 40$, $T_{z_{\text{up}},\infty} = 40$ and $T_{z_{\text{down}},\infty} = 40$. The fin was assumed to be isotropic with coefficient of conductivity $k = 25$ W/mK and coefficient of convection $h = 10$ W/m$^2$K. The equations were solved iteratively till the overall error is less than $10^{-6}$.

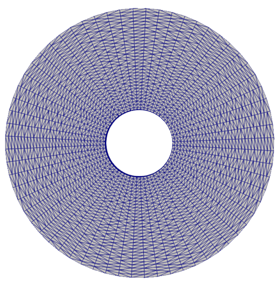

**FIGURE 2.** Meshed Domain

The temperature distribution on the surface and along the mid-plane of the fin are plotted for multiple thicknesses.

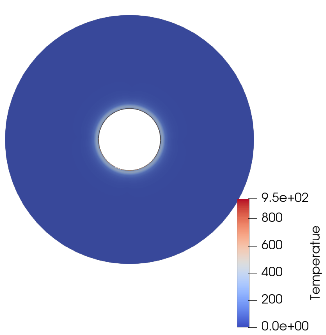


**FIGURE 3.** Temperature Distribution on Surface for Dirichlet BC on all faces

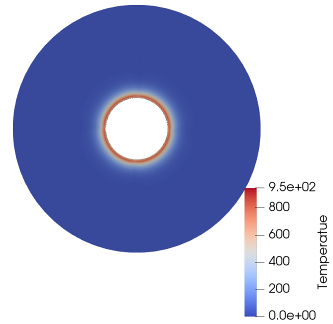


**FIGURE 4.** Temperature Distribution on Mid-Plane for Dirichlet BC on all faces

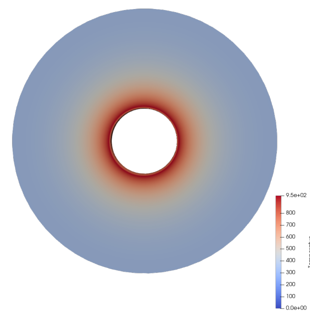

**FIGURE 5.** Temperature Distribution for $t = 0.5$ on Surface

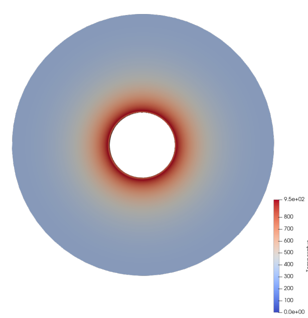

**FIGURE 6.** Temperature Distribution for $t = 0.5$ on Mid-Plane

The plot of the numerical solution and the analytical solution is asymptotic on both radial boundaries, validating the efficacy of the proposed numerical solution.

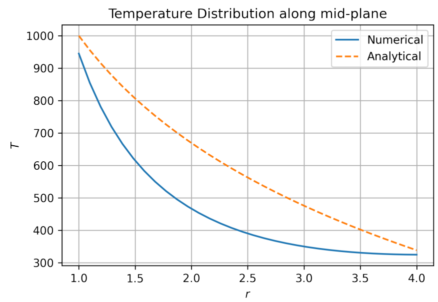


**FIGURE 7**. The variance of the Numerical Solution from the Analytical Solution

The variance of the numerical solution from the analytical solution (with approximations) along the mid-plane suggests that neglecting convection on the top and bottom surfaces indeed affects the distribution.

## 2D Approximation

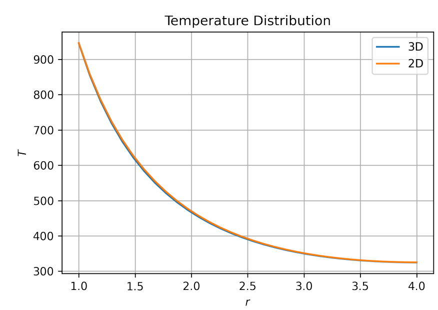


**FIGURE 8**. Deviation due to 2D approximation

The plot suggests that a 2-dimensional approximation essential relationships as a 3D-dimensional meshing scheme.

## Ablation Studies

The fin thickness $t$ was systematically varied to investigate to study the performance of fins keeping the boundary conditions invariant.

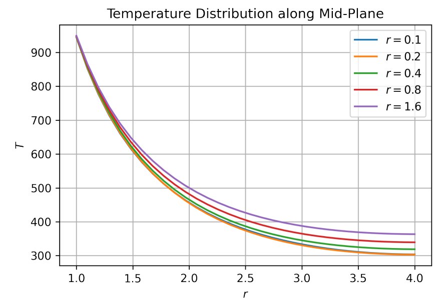


**FIGURE 9**. Temperature Distribution on Mid-Plane of Fins for various $t$

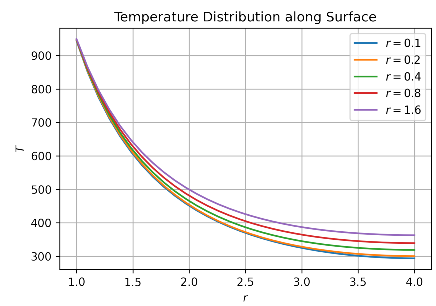


**FIGURE 10**. Temperature Distribution on the Surface of Fins for various $t$

The fin efficiencies are calculated based on the assumption that the maximum possible heat transfer is when fin temperature is same as the base temperature $T_{r_{\text{in}}}$.

| Thickness $t$ | Efficiency $\eta$ |
|---|---|
| 0.1 | 37.10 % |
| 0.2 | 37.19 % |
| 0.4 | 38.41 % |
| 0.8 | 40.26 % |
| 1.6 | 42.41 % |

**TABLE 1**. Efficiency of Fins for various $t$

From the plots, it is evident that thinner fins exhibit superior performance compared to thicker fins. This observation can be explained by the influence of fin geometry on thermal performance.

## DISCUSSION & CONCLUSION

This study investigates the steady-state temperature distribution of annular fins with varying fin parameters, considering natural convection as the mode of heat transfer. The use of a cylindrical meshing scheme provides an appropriate modeling framework for this problem. The Finite Volume Method (FVM) is employed to discretize the computational domain, and the General Scalar Transport Equation is solved to obtain the temperature distribution within the fin.

The proposed solver utilizes the cyclic tri-diagonal matrix algorithm (cTDMA) routine, offering an efficient and reliable solution method. The validity of this approach is demonstrated by comparing the results with the derived analytical solution for an axis-symmetric case along the center plane. This comparison confirms the accuracy and robustness of the cTDMA-based solver in predicting the temperature distribution in annular fins.

The findings of this study provide valuable insights into the thermal performance of annular fins, which have important implications for the design and optimization of heat exchangers and other heat transfer devices.

## ACKNOWLEDGMENT

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