Maxwell's Equations:

$$\oint E \cdot ds = -\frac{d}{dt} \int B \cdot dA$$

$$\oint H \cdot ds = \int \frac{d\vec{D}}{dt} \cdot dA$$

Material Relations:

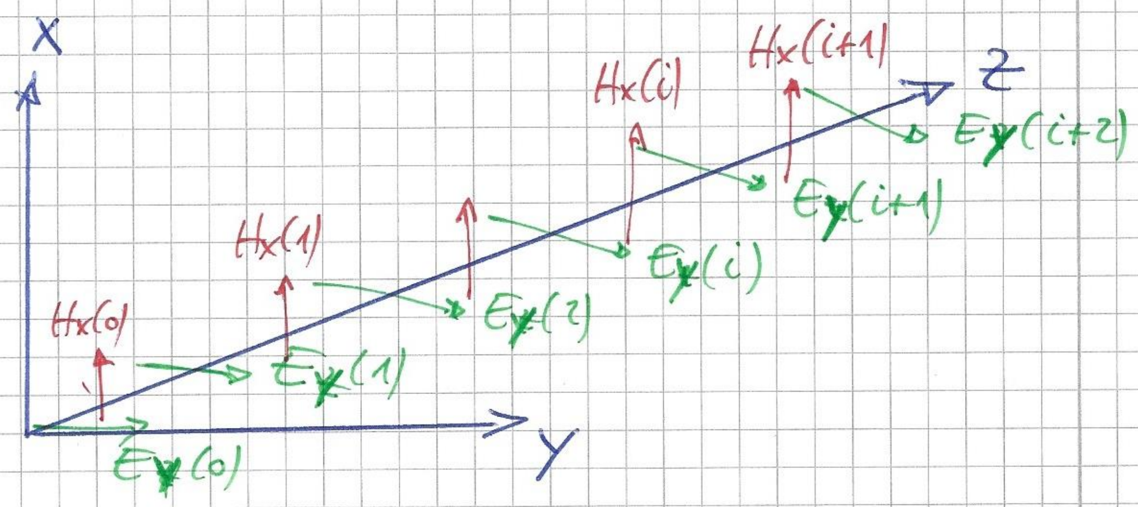
$$B = \mu \cdot H$$

$$D = \varepsilon \cdot E$$

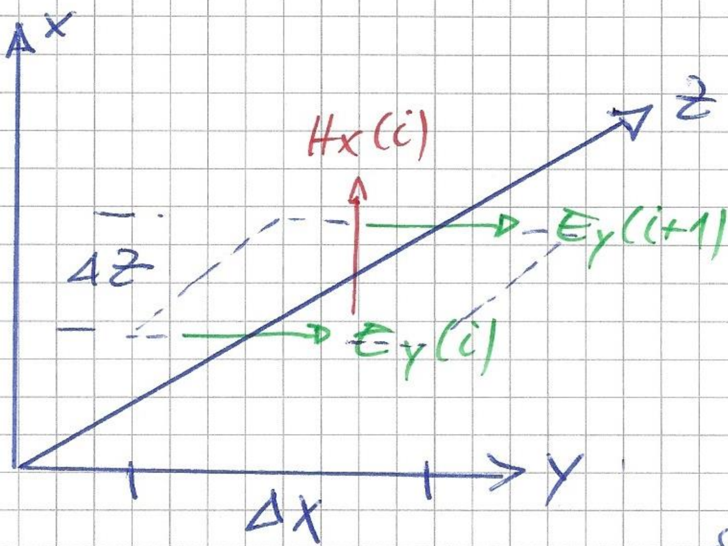
$$\Rightarrow \quad \oint E\,ds = -\frac{d}{dt} \int \mu \cdot H \cdot dA \qquad (1)$$

$$\oint H \cdot ds = \frac{d}{dt} \iint \varepsilon \cdot E \cdot dA \qquad (2)$$

1D Discretization: (TE Mode: E_y, H_x)



Let's look into one loop: $\oint E\,ds$
for Eq. (1):



since $E_z = 0$

$$\oint E\,ds = E_y(i)\cdot\Delta x + 0 - E_y(i+1)\,\Delta x - 0$$

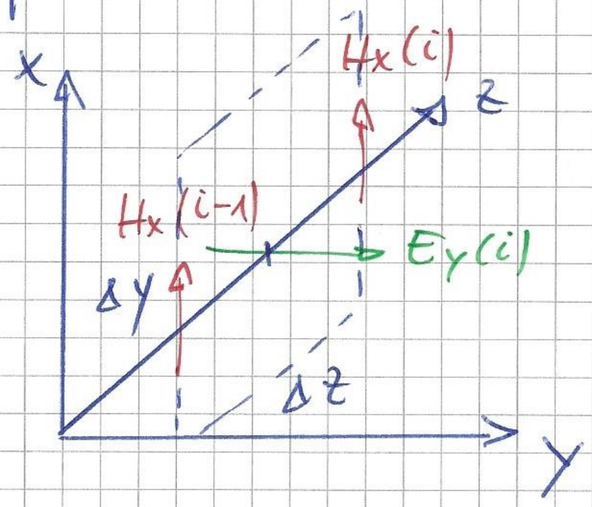
$$\int \mu\cdot H\,dA = \mu\cdot\int H\cdot dA = \mu\cdot H_x(i)\cdot\Delta x\cdot\Delta z$$

together:

$$\Delta x\left(E_y(i) - E_y(i+1)\right) = -\frac{d}{dt}\left(\mu\,H_x(i)\cdot\Delta x\cdot\Delta z\right)$$

$$E_y(i) - E_y(i+1) = -\mu\cdot\Delta z\,\frac{d\,H_x(i)}{dt} \qquad (3)$$

Now look into a loop around E for Eq. (2):



$$H_z = 0$$

$$\oint H ds = H_x(i) \cdot \Delta y - 0 - H_x(i-1) \cdot \Delta y + 0$$

$$= \Delta y \left(H_x(i) - H_x(i-1) \right)$$

$$\int \varepsilon \cdot E dA = \varepsilon \cdot E_y(i) \cdot \Delta z \cdot \Delta y$$

$$\Rightarrow \Delta y \left(H_x(i) - H_x(i-1) \right) = \frac{d}{dt} \left(\varepsilon \, \Delta z \cdot \Delta y \cdot E_y(i) \right)$$

$$H_x(i) - H_x(i-1) = \varepsilon \cdot \Delta z \, \frac{d}{dt} E_y(i) \qquad (4)$$

Now we use a "leapfrog" time integration where the components are allocated in a staggered way along the time axis.

We use an indexing scheme:

$$E_{i,j} = E(i \cdot \Delta z, j \cdot \Delta t)$$

$$H_{i,j} = H((i+\tfrac{1}{2})\Delta z, j \cdot \Delta t + \tfrac{1}{2}\Delta t)$$

We get the following time derivative:

$$\left.\frac{d E_y}{dt}\right|_{\substack{t=(j+\frac{1}{2})\Delta t \\ z = i \cdot \Delta z}} = \frac{E_{y\,i,j+1} - E_{y\,i,j}}{\Delta t}$$

$$\left.\frac{d H_x}{dt}\right|_{\substack{t=j \cdot \Delta t \\ z = (i+1/2)\Delta z}} = \frac{H_{x\,i,j} - H_{x\,i,j-1}}{\Delta t}$$

From Eq. (3) we get:

$$E_y(i \cdot \Delta z) - E_y((i+1)\Delta z) = -\mu \cdot \Delta z \cdot \frac{d\, H_x((i+1/2)\Delta z)}{dt}$$

Evaluation at $t = j \cdot \Delta t$

$$E_{y_{i,j}} - E_{y_{i+1,j}} = -\mu \cdot \Delta z \cdot \frac{H_{x_{i,j}} - H_{x_{i,j-1}}}{\Delta t}$$

We now solve for $H_{x_{i,j}}$:

$$\boxed{H_{x_{i,j}} = -\frac{\Delta t}{\mu \cdot \Delta z}\left(E_{y_{i,j}} - E_{y_{i+1,j}}\right) + H_{x_{i,j-1}}}$$

Same for Eq. (4):

$$H_x((i+1/2)\Delta z) - H_x((i-1/2)\Delta z) = \varepsilon\, \Delta z \cdot \frac{dE_y(i\, \Delta z)}{dt}$$

Evaluation at $t = (j+1/2)\Delta t$

$$H_{x_{i,j}} - H_{x_{i-1,j}} = \varepsilon \cdot \Delta z \cdot \frac{E_{y_{i,j+1}} - E_{y_{i,j}}}{\Delta t}$$

Solve for $E_{y_{i,j+1}}$

$$\boxed{E_{y_{i,j+1}} = \frac{\Delta t}{\varepsilon\, \Delta z}\left(H_{x_{i,j}} - H_{x_{i-1,j}}\right) + E_{y_{i,j}}}$$