SIMULATION METHODS

FACULTY 2, M.SC. PROGRAM HIGH INTEGRITY SYSTEMS

Winter-Semester 2020/2021, Prof. Dr.-Ing. Peter Thoma

FRANKFURT UNIVERSITY OF APPLIED SCIENCES

CONTENTS OF THIS MODULE

CLASSES OF DIFFERENTIAL EQUATIONS

ODE - ORDINARY DIFFERENTIAL EQUATIONS

Phenomena which can be described by a **single variable** (= function of time), e.g. rigid body motion (apple drop), electronic circuits, control systems (= signal flow), …

Solutions are **functions of time, position or frequency**

Examples:

- Rigid body motion - like the apple (functions of time or position)
- Variation of glucose level in human body due to sugar infusion (functions of time)
- Control system output as a function of time triggered by inputs as functions of time
- Behavior (voltages / currents) of electronic circuits as function of time or frequency
- Temperature at a room sensor due to sun illumination at windows as function of time
- …

CLASSES OF DIFFERENTIAL EQUATIONS

ODE - ORDINARY DIFFERENTIAL EQUATIONS

Phenomena which can be described by a **single variable** (= function of time), e.g. rigid body motion (apple drop), electronic circuits, control systems (= signal flow), …

Solutions are **functions of time, position or frequency**

Unable to describe:

Problems depending on more than one variable, e.g. (x, y, z, t), e.g.:

- Distribution of temperature in room as function of (x, y, z, t)
- Behavior of water waves, e.g. depending on (x, y, t)
- Airflow in a room as function of (x, y, z, t)
- Distortion of an airplane wing
- Vibration of a guitar string

⇨ **Partial differential equations**

CLASSES OF DIFFERENTIAL EQUATIONS

ODE - ORDINARY DIFFERENTIAL EQUATIONS

Phenomena which can be described by a **single variable** (= function of time), e.g. rigid body motion (apple drop), electronic circuits, control systems (= signal flow), …

Solutions are **functions of time, position or frequency**

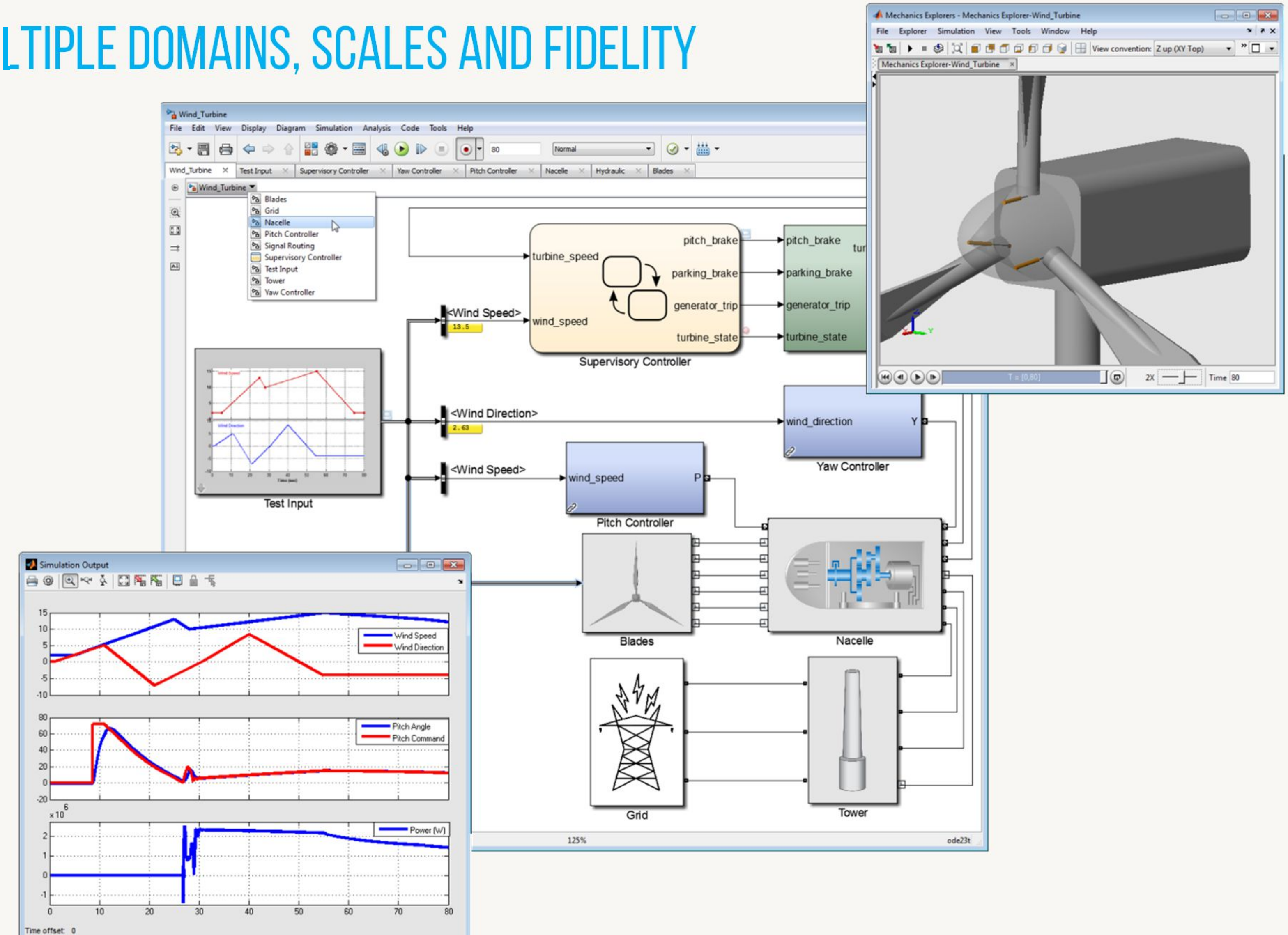
PDE - PARTIAL DIFFERENTIAL EQUATIONS

Phenomena which are described by **multiple variables** (= functions of time), e.g. structural stress, temperature, flows, electromagnetics, …

Solutions are **spatially distributed DOFs**, each of which being a **function of time or frequency**

SYSTEM SIMULATION EXAMPLE

MULTIPLE DOMAINS, SCALES AND FIDELITY



SYSTEM SIMULATION EXAMPLE

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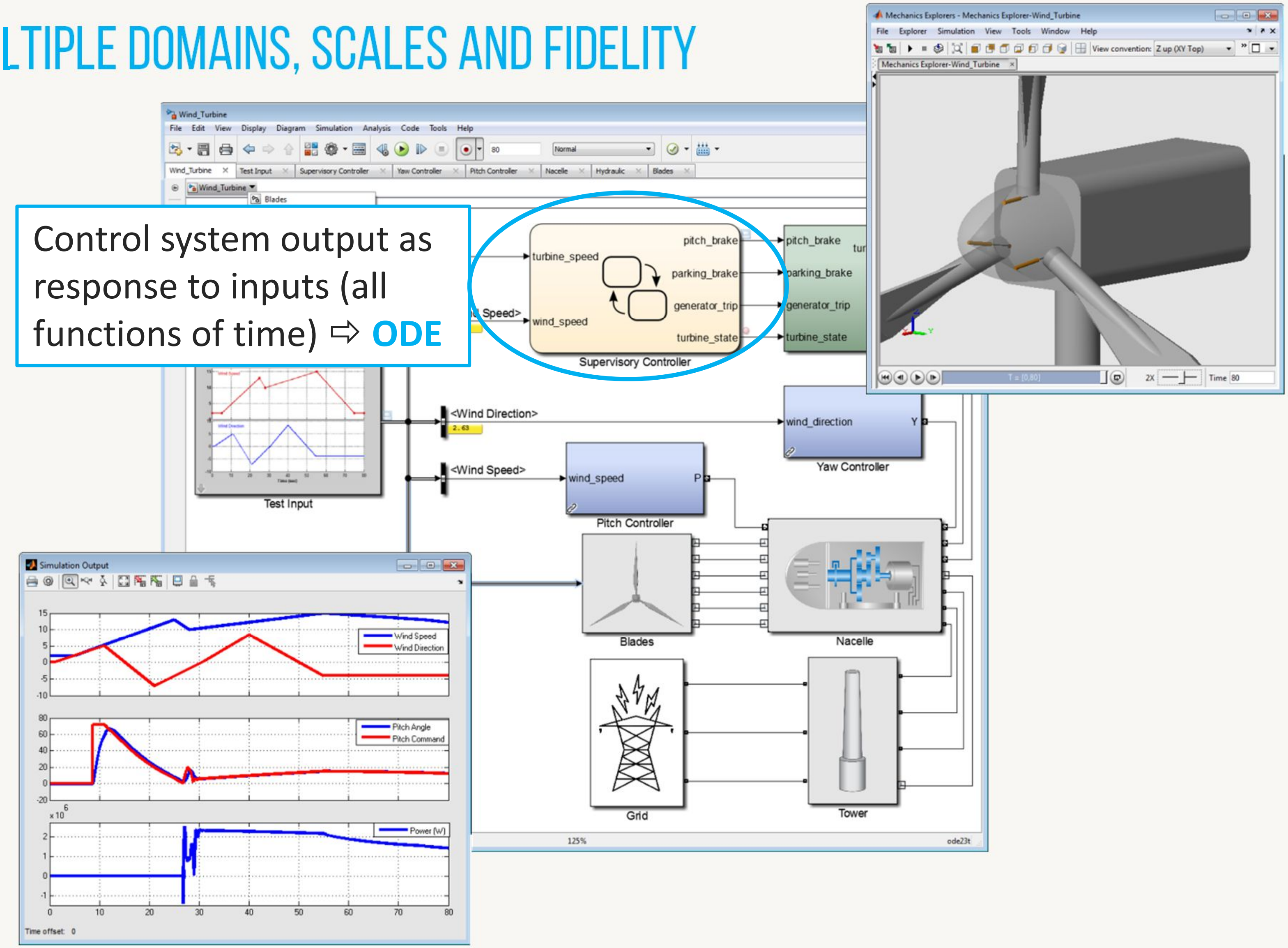

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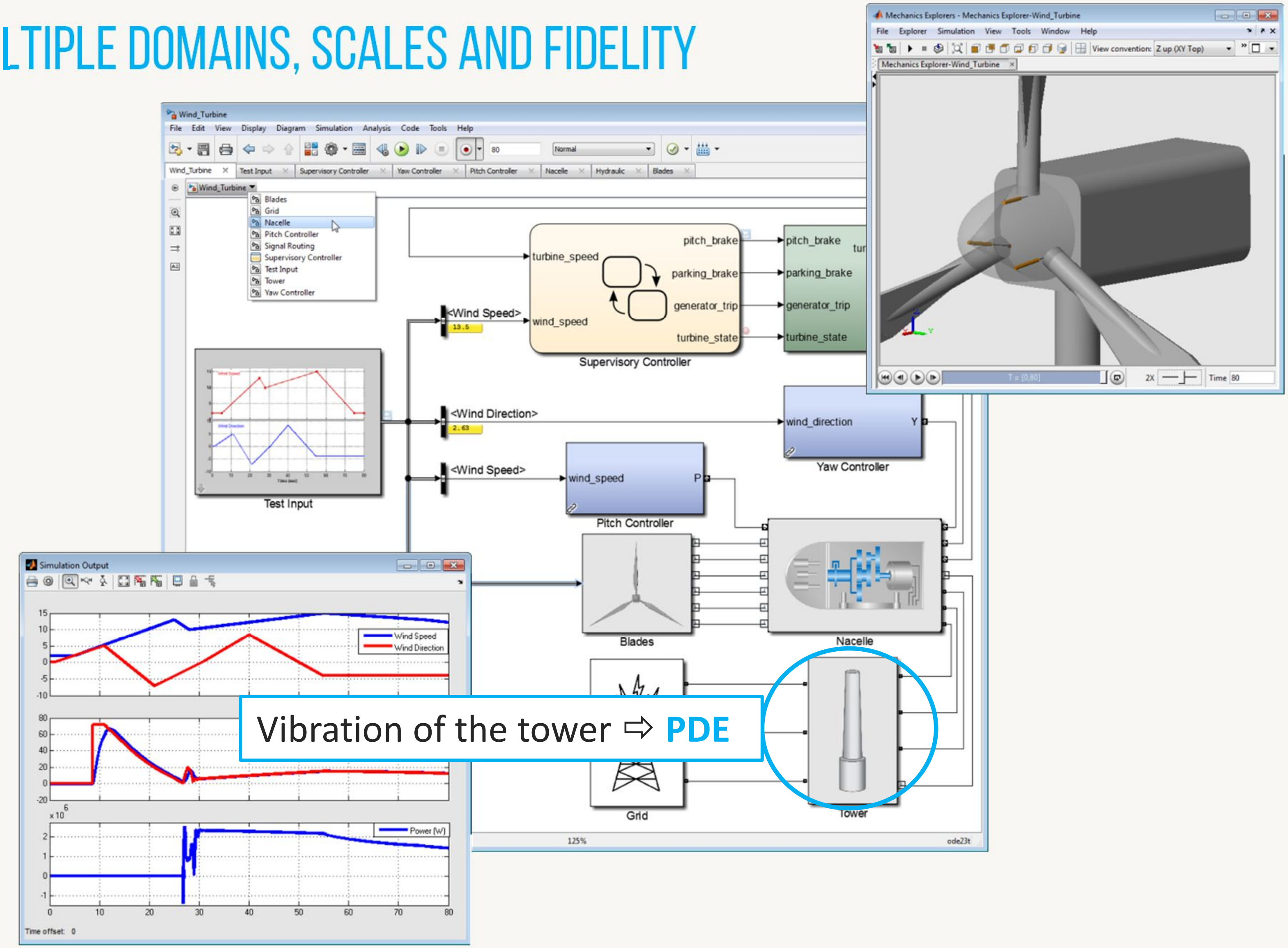

SYSTEM SIMULATION EXAMPLE

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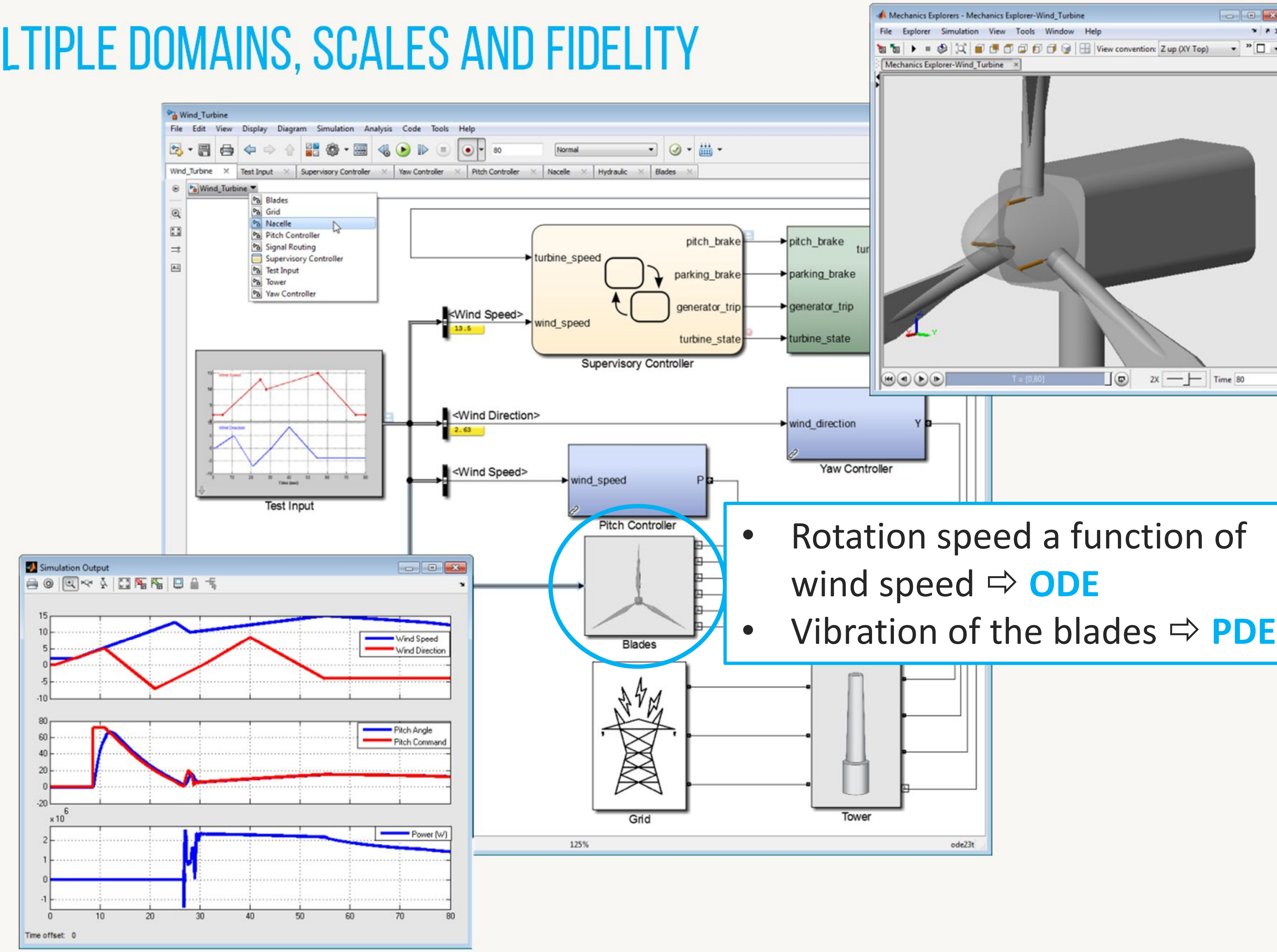


- Rotation speed a function of wind speed ⇨ **ODE**
- Vibration of the blades ⇨ **PDE**

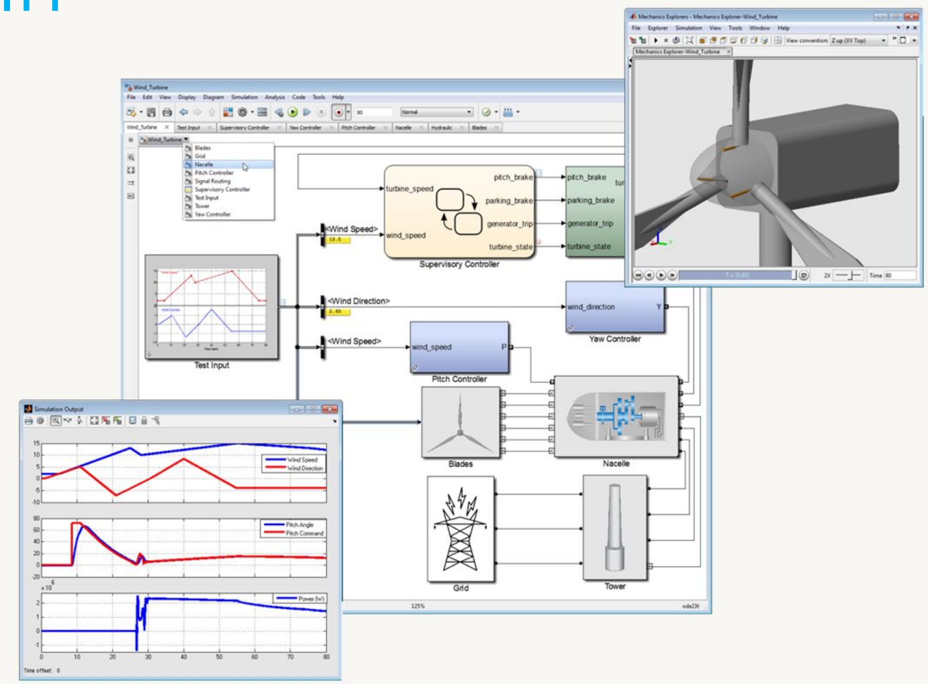
SYSTEM SIMULATION EXAMPLE

MULTIPLE DOMAINS, SCALES AND FIDELITY

- In many cases, **complex systems** are described by a **combination** of Ordinary Differential Equations (**ODE's**) and Partial Differential Equations **PDE's**

- So far, we have investigated the numerical solution of **ODE's**

- In the **following**, we will study how **PDE's** can be solved **numerically**



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6 OUTLOOK

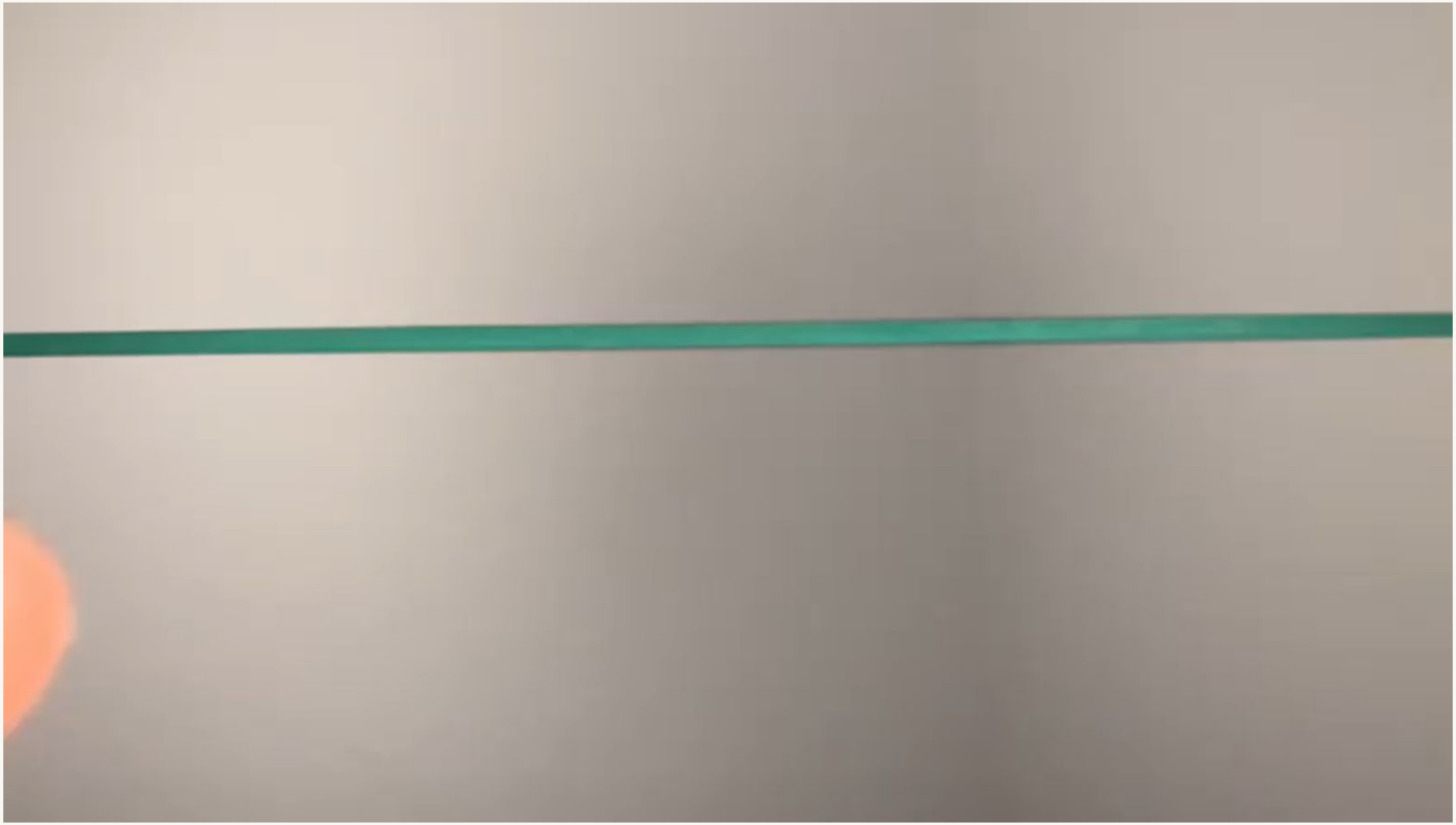
THE FINITE DIFFERENCE METHOD

THE ONE-DIMENSIONAL WAVE EQUATION

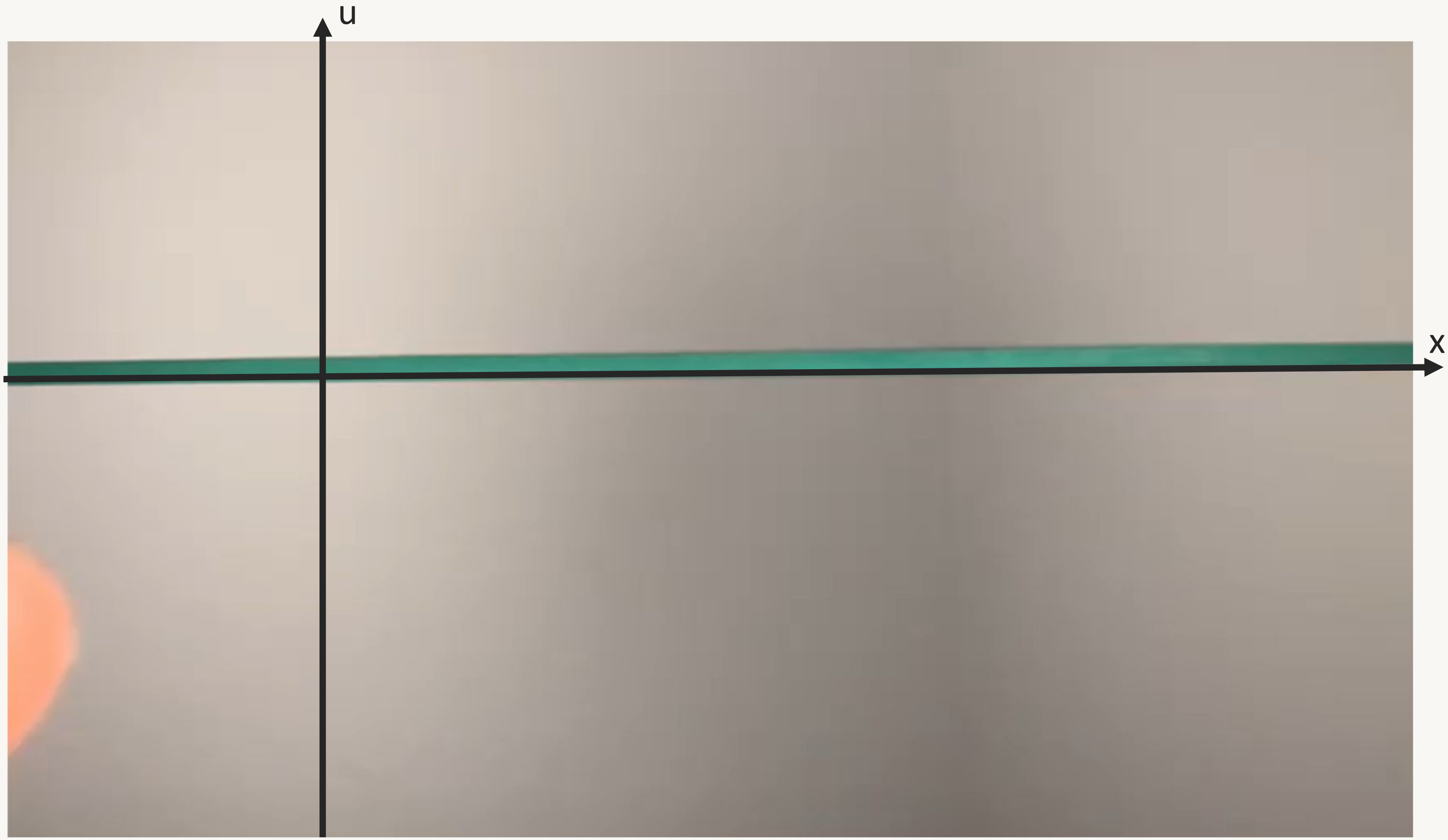
FINITE DIFFERENCE METHOD AGENDA

1. Example for oscillations of a string
2. Wave Equation for a guitar string
   2.1 Discretization of the wave equation
   2.2 Stability of iterative schemes
   2.3 Iterative schemes in matrix form
3. Summary initial value problems
4. Eigenmodes or natural resonances
5. Eigenmodes of a guitar string
   5.1 Frequency Domain formulations
   5.2 Wave equation in Frequency Domain
6. Summary

1. EXAMPLE FOR OSCILLATION OF A STRING

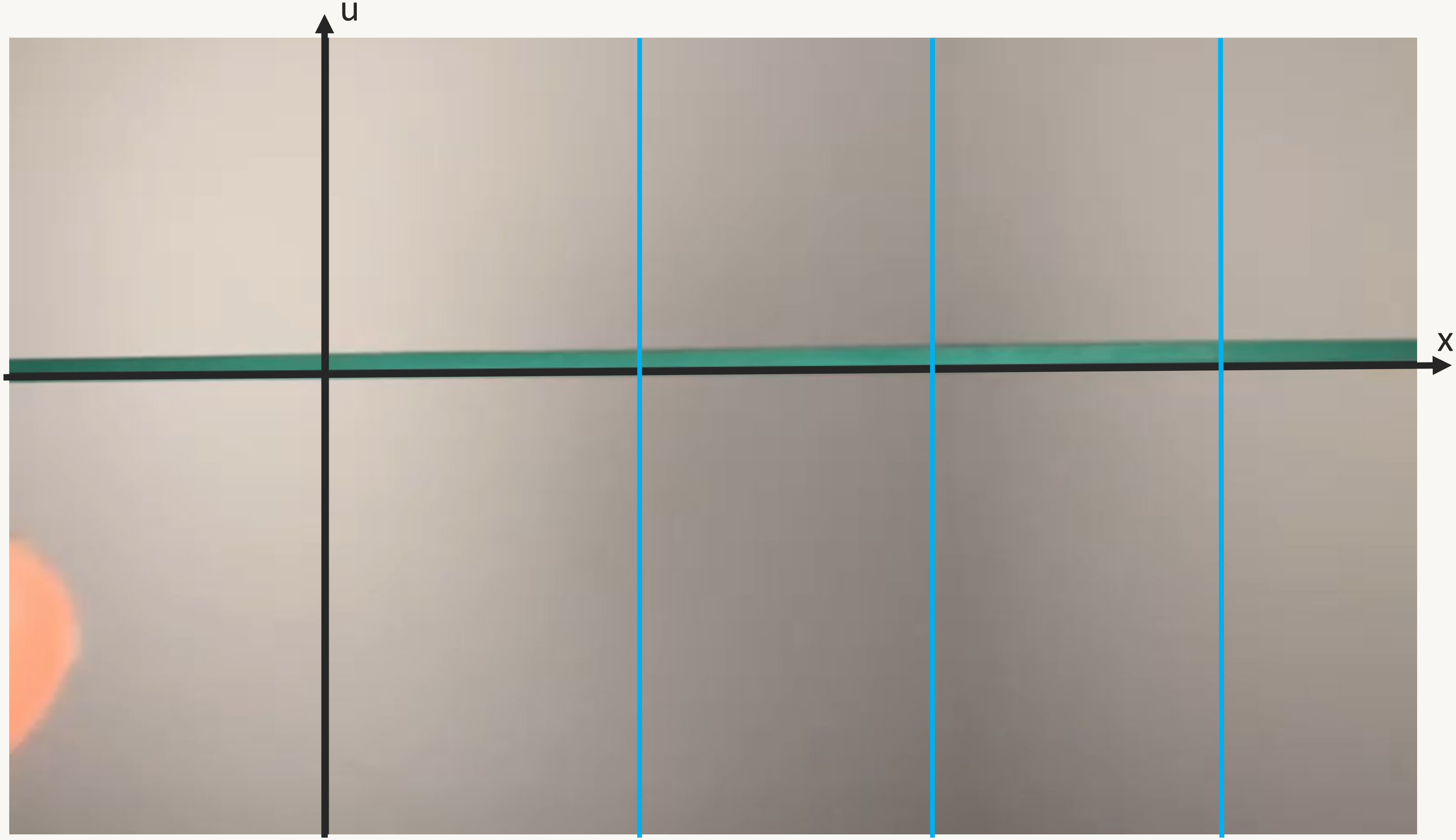


1. EXAMPLE FOR OSCILLATION OF A STRING



At each coordinate x, a point on the string performs a **harmonic oscillation** (up and down)

1. EXAMPLE FOR OSCILLATION OF A STRING



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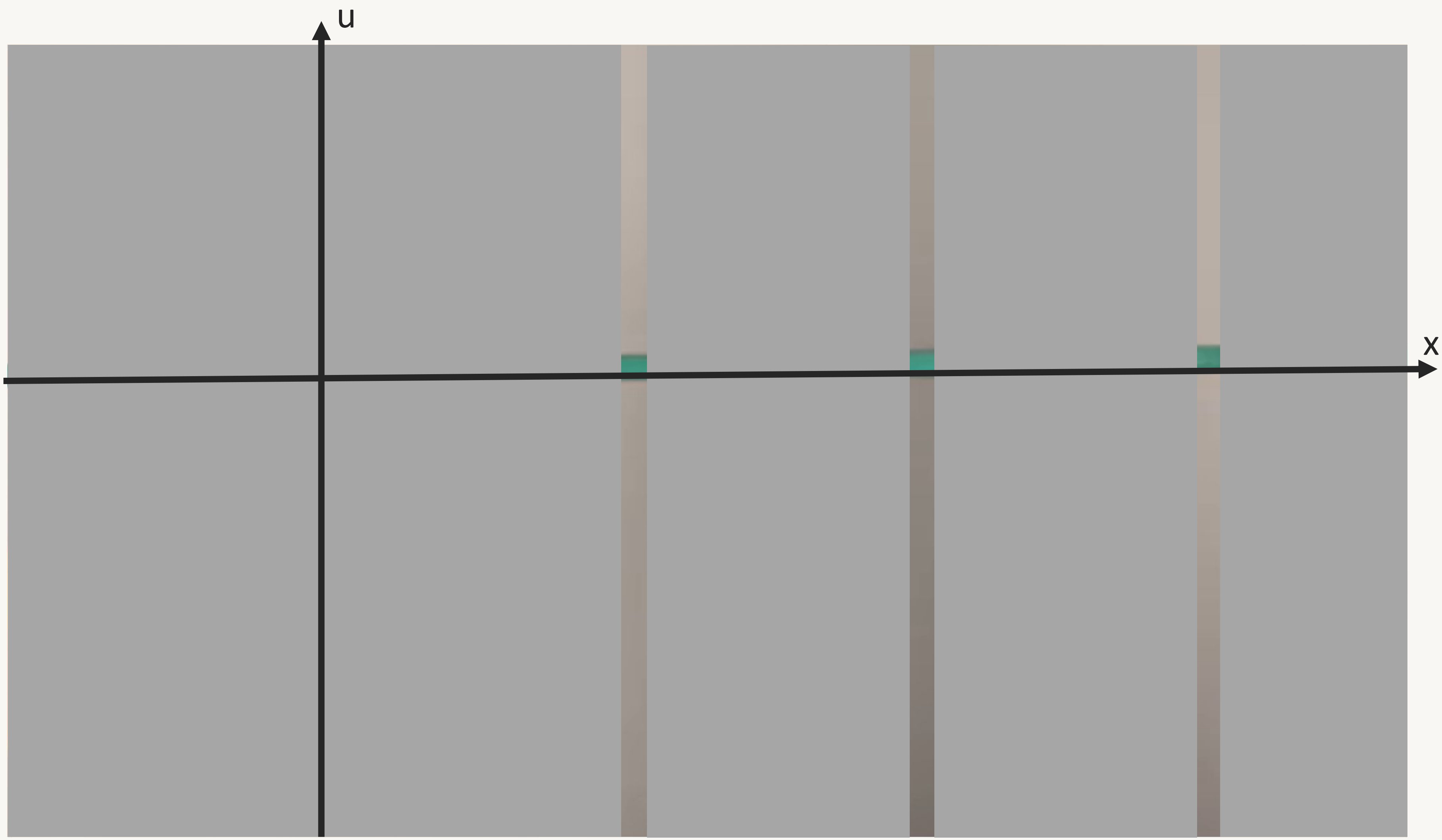
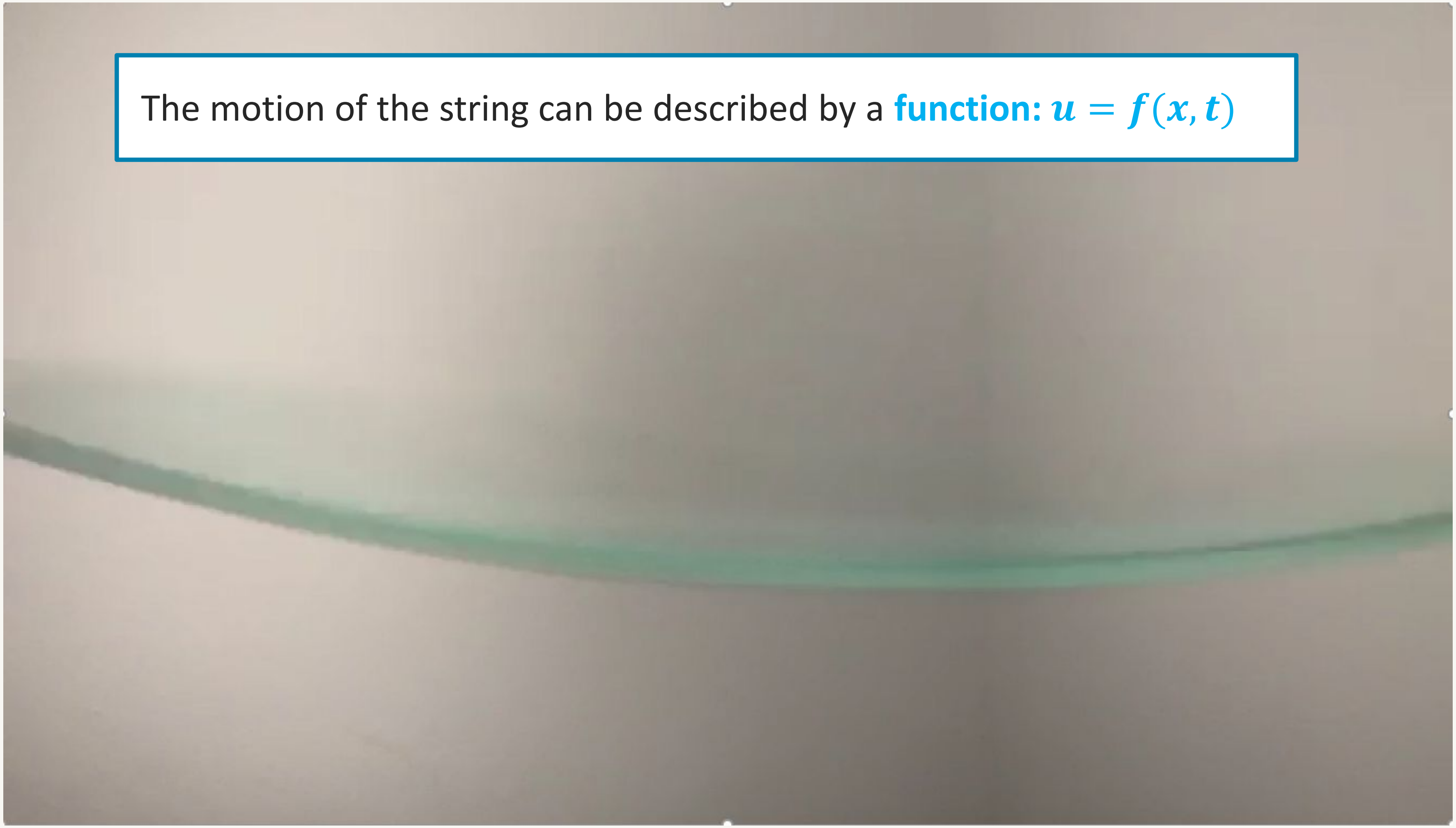


At each coordinate x, a point on the string performs a **harmonic oscillation** (up and down)

1. EXAMPLE FOR OSCILLATION OF A STRING

The motion of the string can be described by a **function:** $\boldsymbol{u = f(x,t)}$



1. EXAMPLE FOR OSCILLATION OF A STRING

FUNCTIONS WITH ONLY ONE VARIABLE

$$\frac{d^2 s(t)}{dt^2} = g$$

Function only depends on variable t

This equation is called **ordinary differential equation**

FUNCTIONS WITH MULTIPLE VARIABLES

$$u = f(x, t)$$

Function depends on variables x and t

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

This equation is called
partial differential equation

$\frac{d}{dt}$ is called **Derivative**

$\frac{\partial}{\partial t}$ is called **Partial Derivative**

FINITE DIFFERENCE METHOD AGENDA

✓ 1. Example for oscillations of a string

2. Wave Equation for a guitar string

   2.1 Discretization of the wave equation

   2.2 Stability of iterative schemes

   2.3 Iterative schemes in matrix form

3. Summary initial value problems

4. Eigenmodes or natural resonances

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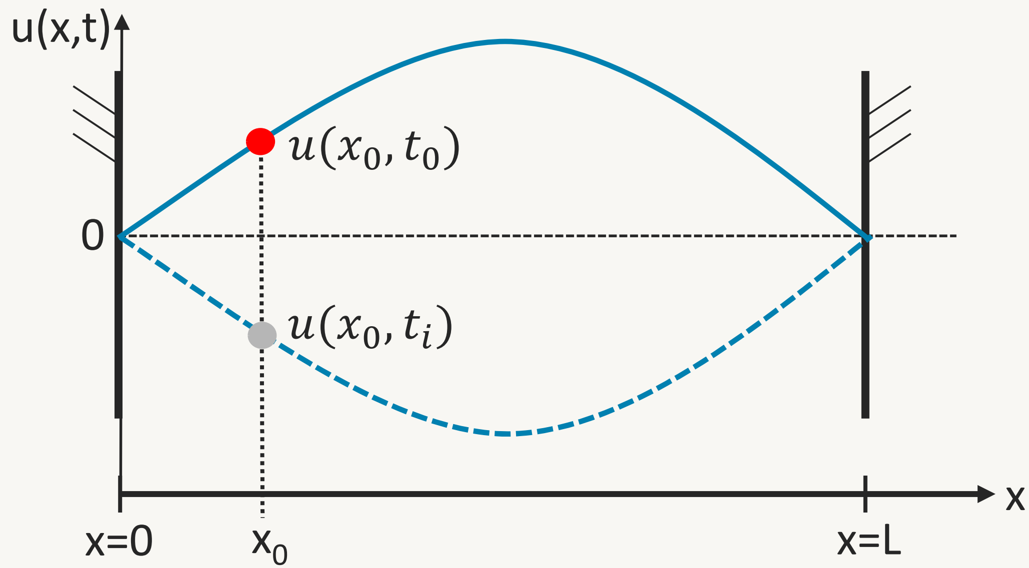
6. Summary

2. WAVE EQUATION FOR A GUITAR STRING

OSCILLATIONS OF A GUITAR STRING

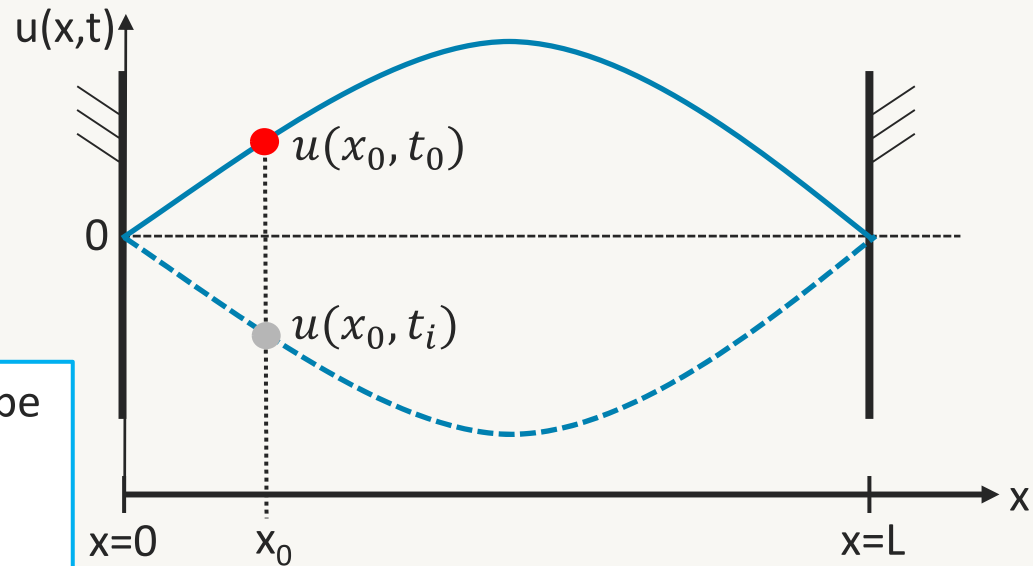


Wave Equation (partial differential equation):

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\,\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

2. WAVE EQUATION FOR A GUITAR STRING

OSCILLATIONS OF A GUITAR STRING



Please note: This wave equation can be derived from the **rules of mechanics**

In this course, we assume that the **differential equations are known**

The **derivation** of the differential equation itself is application domain **specific**, but the **simulation methods** described here are **general**

Wave Equation (partial differential equation):

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

2. WAVE EQUATION FOR A GUITAR STRING

OSCILLATIONS OF A GUITAR STRING

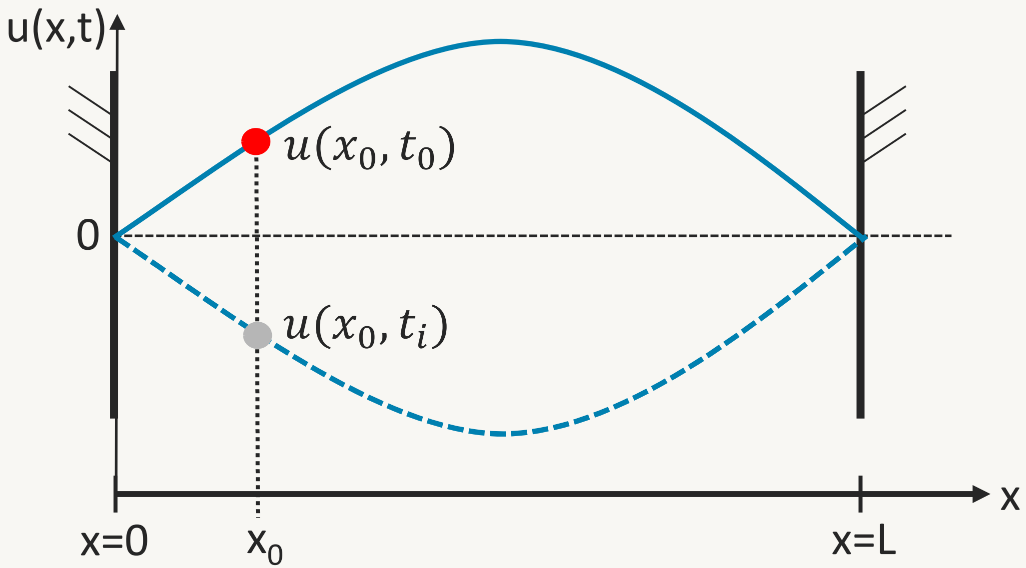
Boundary conditions:

$$u(0,t) = u(L,t) = 0$$

Initial conditions:

$$u(x,t=0) = f(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$



Wave Equation (partial differential equation):

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\,\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

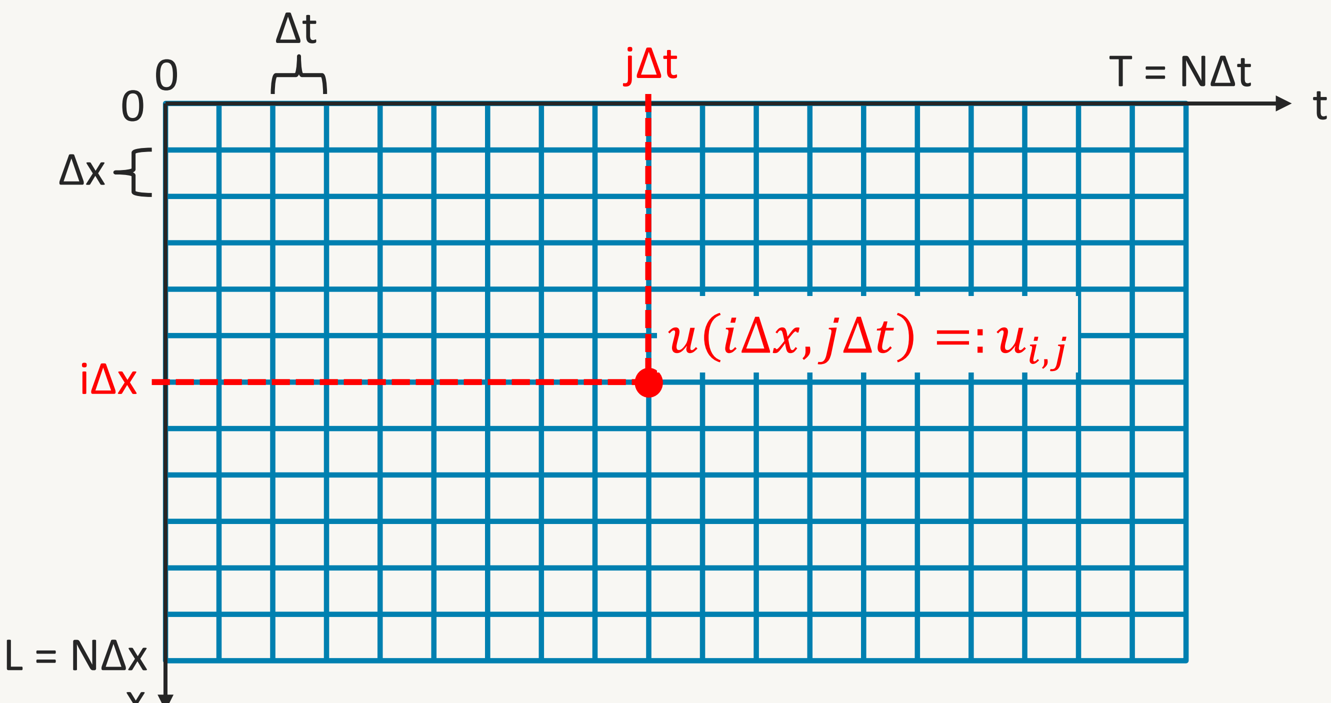
FINITE DIFFERENCE METHOD AGENDA

2.1 DISCRETIZATION OF THE WAVE EQUATION

WAVE EQUATION

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

The function depends on two variables *x* and *t* ⇨ **two-dimensional discretization**

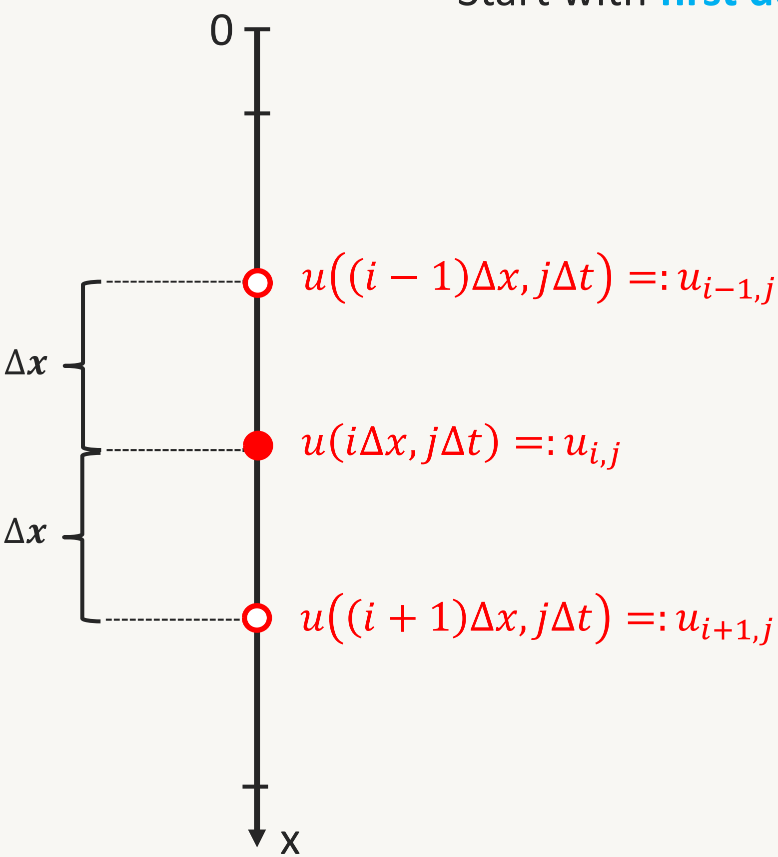


2.1 DISCRETIZATION OF THE WAVE EQUATION

SPATIAL DISCRETIZATION

$$\frac{\partial^2 u(x,t)}{\partial x^2}$$ **Partial differentiation in x**

Start with **first derivative** $\frac{\partial u(x,t)}{\partial x}$ and apply **central differencing**



$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$

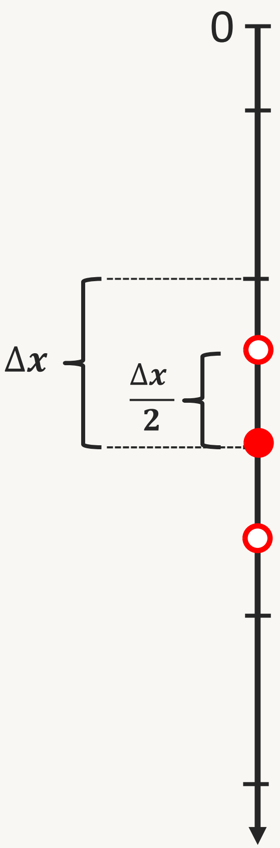
Problem: Large stencil of the operator

2.1 DISCRETIZATION OF THE WAVE EQUATION

SPATIAL DISCRETIZATION

$$\frac{\partial^2 u(x,t)}{\partial x^2}$$ **Partial differentiation in x**

Start with **first derivative** $\frac{\partial u(x,t)}{\partial x}$ and apply **central differencing**



$$u\left(\left(i-\frac{1}{2}\right)\Delta x, j\Delta t\right) =: u_{i-\frac{1}{2},j}$$

$$u(i\Delta x, j\Delta t) =: u_{i,j}$$

$$u\left(\left(i+\frac{1}{2}\right)\Delta x, j\Delta t\right) =: u_{i+\frac{1}{2},j}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x \\ t=j\Delta t}} \approx \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x}$$

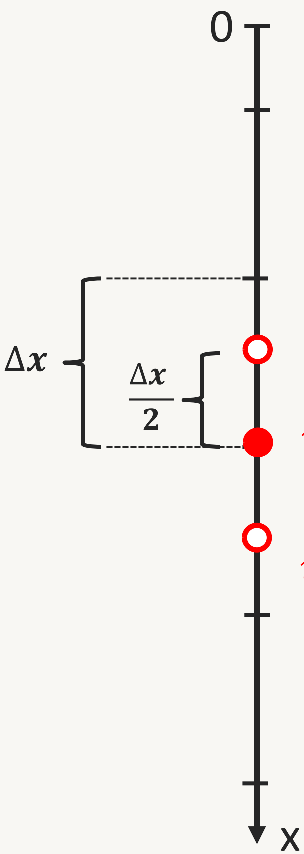
Trick: Use half grid step for discretization of derivative

2.1 DISCRETIZATION OF THE WAVE EQUATION

SPATIAL DISCRETIZATION

$$\frac{\partial^2 u(x,t)}{\partial x^2}$$ **Partial differentiation in x**

Calculate the derivative at **three different locations**



$$u\left(\left(i-\frac{1}{2}\right)\Delta x, j\Delta t\right) =: u_{i-\frac{1}{2},j}$$

$$u(i\Delta x, j\Delta t) =: u_{i,j}$$

$$u\left(\left(i+\frac{1}{2}\right)\Delta x, j\Delta t\right) =: u_{i+\frac{1}{2},j}$$

2 $$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i-\frac{1}{2}\right)\Delta x \\ t=j\Delta t}} \approx \frac{u_{i,j}-u_{i-1,j}}{\Delta x}$$

1 $$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x \\ t=j\Delta t}} \approx \frac{u_{i+\frac{1}{2},j}-u_{i-\frac{1}{2},j}}{\Delta x}$$

3 $$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i+\frac{1}{2}\right)\Delta x \\ t=j\Delta t}} \approx \frac{u_{i+1,j}-u_{i,j}}{\Delta x}$$

2.1 DISCRETIZATION OF THE WAVE EQUATION

SPATIAL DISCRETIZATION

Now calculate the **second derivative**:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i-\frac{1}{2}\right)\Delta x\\ t=j\Delta t}} \approx \frac{u_{i,j}-u_{i-1,j}}{\Delta x}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i+\frac{1}{2},j}-u_{i-\frac{1}{2},j}}{\Delta x}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i+\frac{1}{2}\right)\Delta x\\ t=j\Delta t}} \approx \frac{u_{i+1,j}-u_{i,j}}{\Delta x}$$

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} = \left.\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\right)\right|_{\substack{x=i\Delta x\\ t=j\Delta t}}$$

$$\approx \frac{\partial}{\partial x}\left(\frac{u_{i+\frac{1}{2},j}-u_{i-\frac{1}{2},j}}{\Delta x}\right) = \frac{1}{\Delta x}\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i+\frac{1}{2}\right)\Delta x\\ t=j\Delta t}} - \frac{1}{\Delta x}\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=\left(i-\frac{1}{2}\right)\Delta x\\ t=j\Delta t}}$$

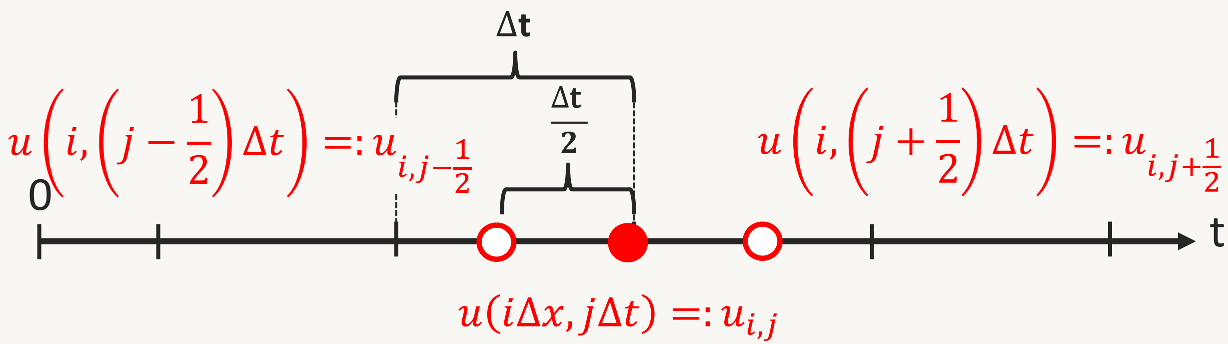
$$\approx \frac{u_{i+1,j}-u_{i,j}}{\Delta x^2} - \frac{u_{i,j}-u_{i-1,j}}{\Delta x^2} = \frac{u_{i+1,j}-u_{i,j}-u_{i,j}+u_{i-1,j}}{\Delta x^2}$$

$$\boxed{\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}}$$

2.1 DISCRETIZATION OF THE WAVE EQUATION

TEMPORAL DISCRETIZATION $\frac{\partial^2 u(x,t)}{\partial t^2}$ **Partial differentiation in t**

$\frac{\partial u(x,t)}{\partial t}$ **Trick:** Use **half grid step** for discretization of derivative, **central differencing**



$$\boxed{\left.\frac{\partial u(x,t)}{\partial t}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{\Delta t}}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=\left(j-\frac{1}{2}\right)\Delta t}} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta t}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=\left(j+\frac{1}{2}\right)\Delta t}} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

2.1 DISCRETIZATION OF THE WAVE EQUATION

TEMPORAL DISCRETIZATION

$$\left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} = \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial t}\right)\Bigg|_{\substack{x=i\Delta x\\ t=j\Delta t}}$$

$$\approx \frac{\partial}{\partial t}\left(\frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{\Delta t}\right)$$

$$\approx \frac{u_{i,j+1} - u_{i,j} - u_{i,j} + u_{i,j-1}}{\Delta t^2}$$

$$\boxed{\left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=\left(j-\frac{1}{2}\right)\Delta t}} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta t}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i,j+\frac{1}{2}} - u_{j,i-\frac{1}{2}}}{\Delta t}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\substack{x=i\Delta x\\ t=\left(j+\frac{1}{2}\right)\Delta t}} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

2.1 DISCRETIZATION OF THE WAVE EQUATION

WAVE EQUATION

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{\substack{x=i\Delta x\\ t=j\Delta t}} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}$$

$$\boxed{\frac{\partial^2 u(x,t)}{\partial x^2}} - k\boxed{\frac{\partial^2 u(x,t)}{\partial t^2}} = 0$$

$$\Longrightarrow \boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}} - k\boxed{\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}} = 0$$

Substitute the results

From the **initial condition** we know the shape of the string for $j = 0$

$$u(x, t = 0) = f(x) \quad \Rightarrow \quad u_{i,0} = f_i$$

We will now derive an equation which allows us to **iteratively** determine **future values** $u_{i,j+1}$ when all **previous values** $u_{i,j}$ are know

⇨ **Solve for $u_{i,j+1}$ in the equation above**

2.1 DISCRETIZATION OF THE WAVE EQUATION

WAVE EQUATION

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} - k\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2} = 0$$

$$\Rightarrow u_{i,j+1} = cu_{i+1,j} + cu_{i-1,j} + 2(1-c)u_{i,j} - u_{i,j-1}$$

$$\boxed{\Rightarrow u_{i,j} = cu_{i+1,j-1} + cu_{i-1,j-1} + 2(1-c)u_{i,j-1} - u_{i,j-2} \quad \text{with} \quad c = \frac{\Delta t^2}{k\Delta x^2}}$$

We have N steps in x direction, so i is in the range of 0,…,N: $u_{0,j}, u_{1,j}, \dots, u_{N,j}$

We don't need to compute $u_{0,j}$ and $u_{N,j}$ due to the **boundary conditions**:

$$u(0,t) = u(L,t) = 0 \quad \Longrightarrow \quad \boxed{u_{0,j} = u_{N,j} = 0}$$

Therefore, the **range** of i for the computation is $\boxed{i = 1, 2, \dots, N-1}$

2.1 DISCRETIZATION OF THE WAVE EQUATION

ITERATION SCHEME

$$u_{i,j} = cu_{i+1,j-1} + cu_{i-1,j-1} + 2(1-c)u_{i,j-1} - u_{i,j-2} \quad \text{with} \quad c = \frac{\Delta t^2}{k\Delta x^2}$$

Iteration scheme:

$i = 1, \dots, N-1$: $u_{i,0} = f_i$ **Initial condition**

$i = 1, \dots, N-1$: $u_{i,1} = cu_{i+1,0} + cu_{i-1,0} + 2(1-c)u_{i,0} - \boxed{u_{i,-1}}$

$i = 1, \dots, N-1, j = 2, \dots$: $u_{i,j} = cu_{i+1,j-1} + cu_{i-1,j-1} + 2(1-c)u_{i,j-1} - u_{i,j-2}$

This **iteration scheme** equation for $j = 1$ refers to the **unknown $u_{i,-1}$**

This value can be determined from the **second initial condition** $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$

Applying **backward differencing**, we obtain: $\frac{u_{i,0} - u_{i,-1}}{\Delta t} = 0 \;\Rightarrow\; \boxed{u_{i,-1} = u_{i,0}}$

2.1 DISCRETIZATION OF THE WAVE EQUATION

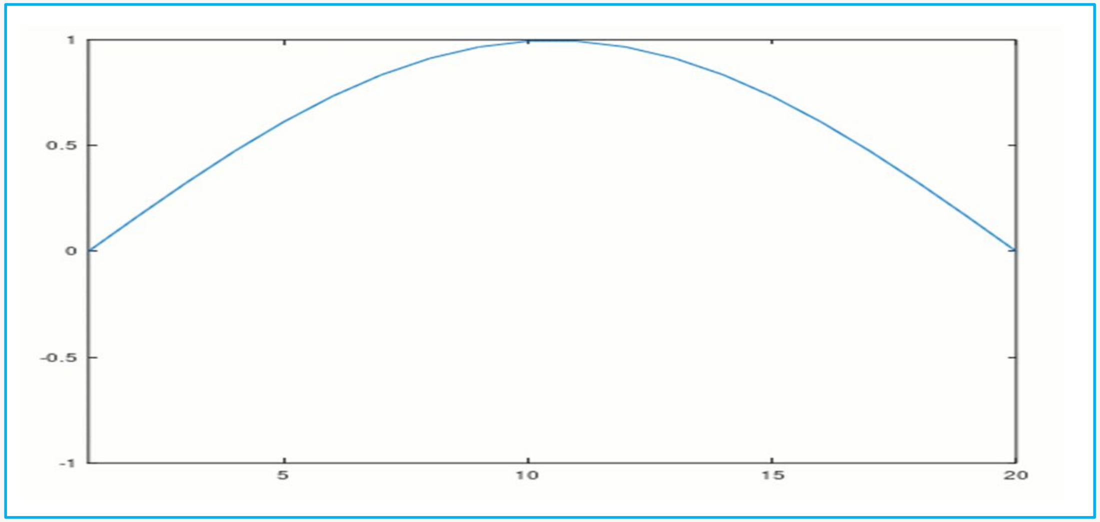
FINAL ITERATION SCHEME

$$c = \frac{\Delta t^2}{k \Delta x^2}$$

$$i = 1, \dots, N-1 \quad : \quad u_{i,0} = f_i$$

$$i = 1, \dots, N-1 \quad : \quad u_{i,1} = c u_{i+1,0} + c u_{i-1,0} + 2(1-c) u_{i,0} - u_{i,0}$$

$$i = 1, \dots, N-1, j = 2, \dots : \quad u_{i,j} = c u_{i+1,j-1} + c u_{i-1,j-1} + 2(1-c) u_{i,j-1} - u_{i,j-2}$$



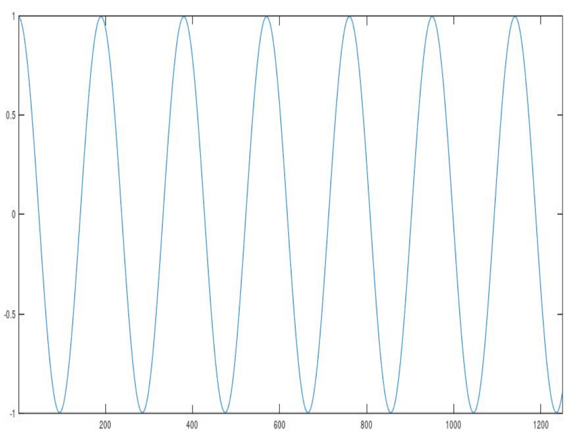
FINITE DIFFERENCE METHOD AGENDA

2.2 STABILITY OF ITERATIVE SCHEMES

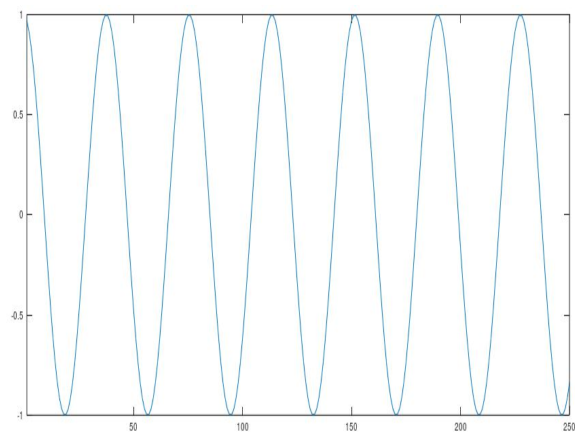
STABILITY

Consider the **time evolutions** of the single degree of freedom $u_{N/2,j}$, $k = 1.0, \Delta x = 1.0$

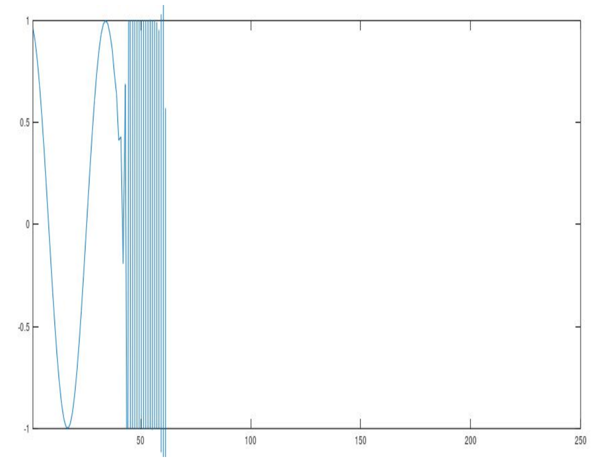
Δt = 0.2



Δt = 1.0



Δt = 1.1



The iteration is **stable** for:

$$\Delta t \leq \Delta t_{max}$$

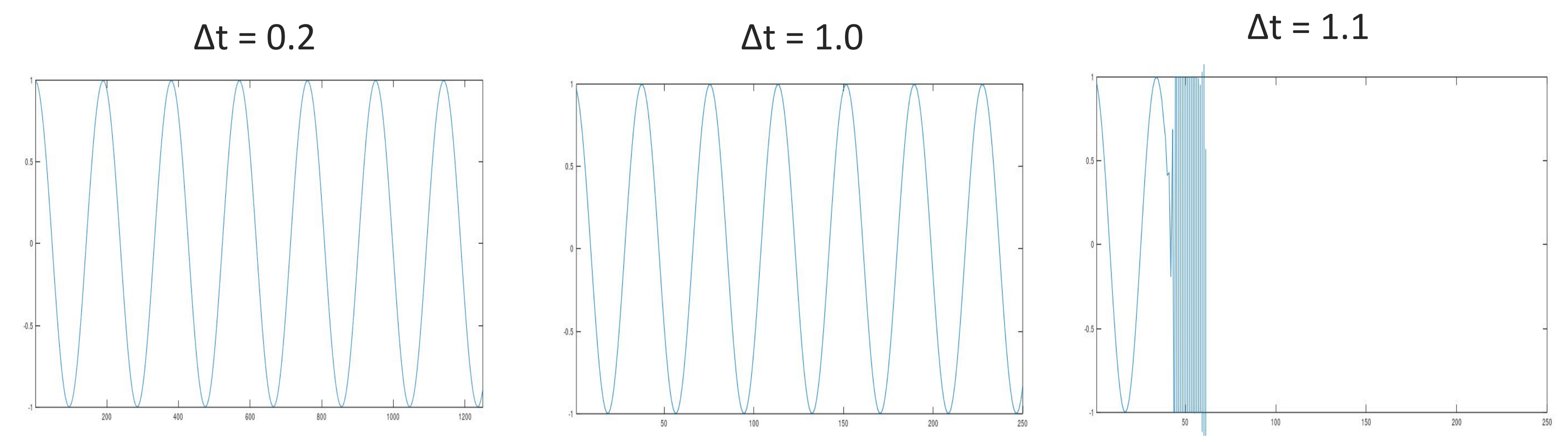
The simulation gets **unstable** for **large time steps**

The method is **conditionally stable**

2.2 STABILITY OF ITERATIVE SCHEMES

STABILITY

Consider the **time evolutions** of the single degree of freedom $u_{N/2,j}$, $k = 1.0, \Delta x = 1.0$



The iteration is **stable** for:

$$\Delta t \leq \Delta t_{max}$$

$$\Delta t_{max} = \sqrt{k}\,\Delta x$$

Courant-Friedrich-Levy Criterion

(Derivation see appendix)

FINITE DIFFERENCE METHOD AGENDA

2.3 ITERATIVE SCHEMES IN MATRIX FORM

ITERATION SCHEME

$$
\boxed{
\begin{aligned}
i &= 1, \dots, N-1 &&: \quad u_{i,0} = f_i \\
i &= 1, \dots, N-1 &&: \quad u_{i,1} = c u_{i+1,0} + c u_{i-1,0} + 2(1-c) u_{i,0} - u_{i,0} \\
i &= 1, \dots, N-1, j = 2, \dots &&: \quad u_{i,j} = c u_{i+1,j-1} + c u_{i-1,j-1} + 2(1-c) u_{i,j-1} - u_{i,j-2}
\end{aligned}
}
$$

We introduce a **vector of degrees of freedom**:

$$
\bar{u}_j = (\underbrace{u_{1,j}, u_{2,j}, \dots, u_{N-1,j}}_{\boldsymbol{u_{i,j}}}, \underbrace{u_{1,j-1}, u_{2,j-1}, \dots, u_{N-1,j-1}}_{\boldsymbol{u_{i,j-1}}})^T
$$

With the **initial conditions** we obtain:

$$
\bar{u}_0 = (\underbrace{f_1, f_2, \dots, f_{N-1}}_{\boldsymbol{u_{i,0} = f_i}}, \underbrace{f_1, f_2, \dots, f_{N-1}}_{\boldsymbol{u_{i,-1} = u_{i,0} = f_i}})^T
$$

2.3 ITERATIVE SCHEMES IN MATRIX FORM

MATRIX FORMULATION

$$\boxed{u_{i,j} = c u_{i+1,j-1} + c u_{i-1,j-1} + 2(1-c) u_{i,j-1} - u_{i,j-2}}$$

$$\bar{u}_j = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-1,j} \\ u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{N-1,j-1} \end{pmatrix} = \underbrace{\left(\begin{array}{ccccc|ccc} & & \overbrace{\vdots}^{N-1} & & & \overbrace{\vdots}^{N-1} & \\ \dots & c & 2(1-c) & c & \dots & -1 & \\ \hline & & \vdots & & & \vdots & \\ & \dots & 1 & \dots & \dots & 0 & \dots \end{array} \right)}_{\begin{pmatrix} C & -I \\ I & 0 \end{pmatrix} = M} \underbrace{\begin{pmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{N-1,j-1} \\ u_{1,j-2} \\ u_{2,j-2} \\ \vdots \\ u_{N-1,j-2} \end{pmatrix}}_{\bar{u}_{j-1}}$$

(the block matrices C, I are (N-1)x(N-1) matrices)

Or as **iteration scheme** in **matrix representation**: $\bar{u}_j = M\,\bar{u}_{j-1}$

2.3 ITERATIVE SCHEMES IN MATRIX FORM

ITERATION SCHEME IN MATRIX FORM

$$C = \begin{pmatrix} 2(1-c) & c & & & & & \\ c & 2(1-c) & c & & & & \\ & c & 2(1-c) & c & & & \\ & & & \ddots & & & \\ & & & & c & 2(1-c) & c \\ & & & & & c & 2(1-c) \end{pmatrix}$$

$$\bar{u}_0 = (f_1, f_2, \dots, f_{N-1}, f_1, f_2, \dots, f_{N-1})^T$$

$$\bar{u}_j = \begin{pmatrix} C & -I \\ I & 0 \end{pmatrix} \bar{u}_{j-1}$$

This method is also **called fixed-point iteration**. According to the **Banach fixed point theorem**, it is stable, if the **spectral radius** (=absolute value of the largest eigenvalue) of the iteration matrix is ≤ 1

FINITE DIFFERENCE METHOD AGENDA

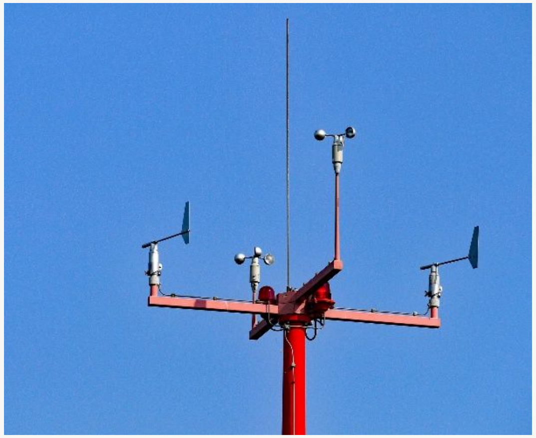
3. SUMMARY INITIAL VALUE PROBLEMS

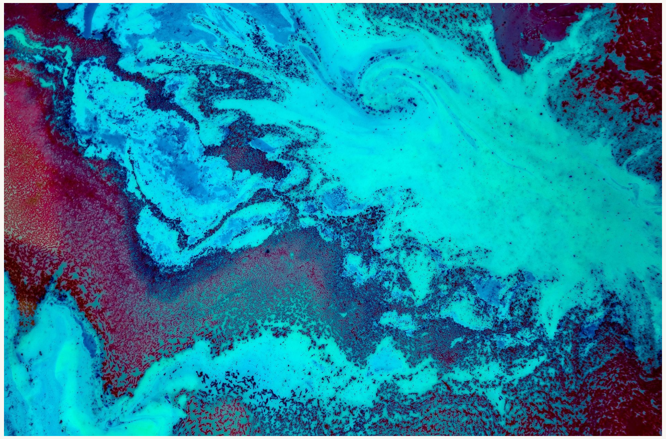
The guitar string example is an **initial value problem**:

1. We know a **model** for the system (the wave equation)
2. We know the **initial state** of the system (at $t = 0$)
3. We **simulate** the system starting from this initial condition **without further interaction** with the system

Other example for initial value problem: **Weather forecast**

- Determine (measure) all parameters of the system at a **certain point in time**
- Then simulate the **future behavior** starting from this initial condition





FINITE DIFFERENCE METHOD AGENDA

✓ 1. Example for oscillations of a string

✓ 2. Wave Equation for a guitar string

  ✓ 2.1 Discretization of the wave equation

  ✓ 2.2 Stability of iterative schemes

  ✓ 2.3 Iterative schemes in matrix form

✓ 3. Summary initial value problems

4. Eigenmodes or natural resonances

5. Eigenmodes of a guitar string

  5.1 Frequency Domain formulations

  5.2 Wave equation in Frequency Domain

6. Summary

4. EIGENMODES OR NATURAL RESONANCES

Everything has a **natural resonance**:

After an “excitation”, the gong vibrates at a characteristic frequency, its **natural resonance**

4. Eigenmodes or Natural Resonances

Everything has a **natural resonance**:

- **Musical instruments** (structural resonance, sound)
- **Mechanical clocks** (clock pendulum, accuracy)
- **Digital clocks** (electrical quartz resonance, accuracy)
- **Washing machines** (structural resonance, noise)
- **Cars** (structural resonance – chassis suspension, noise, safety)
- **Antennas** (electrical resonance, performance)
- **Buildings** (structural resonance, stability)

In mathematical terms, a **natural resonance** is called an **eigenmode** of an object

4. EIGENMODES OR NATURAL RESONANCES

TACOMA NARROWS BRIDGE (1940)

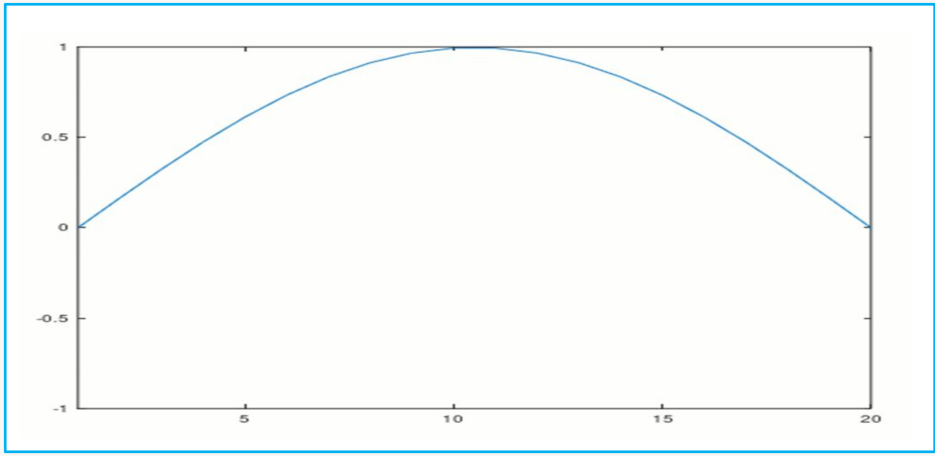
FINITE DIFFERENCE METHOD AGENDA

5. EIGENMODES OF THE GUITAR STRING

TIME DOMAIN FORMULATION

So far, we have solved the differential equation in Time Domain as initial value problem

$$\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$



EIGENMODE FORMULATION

- Now we want to calculate the **natural resonances** (=eigenmodes) of the guitar string
- Each resonance corresponds to a **characteristic resonance frequency** (= eigenvalue)
- As we are looking for resonance frequencies and not the motion of the string as function of time, it makes sense to switch to a **Frequency Domain formulation**

FINITE DIFFERENCE METHOD AGENDA

5.1 FREQUENCY DOMAIN FORMULATIONS

FREQUENCY DOMAIN FORMULATION

Let's consider a general **time harmonic** function:
(according to Fourier, any continuous function can be composed by an infinite number of time harmonic solutions)

$\boxed{u(x,t) = u(x)\cos(\omega t + \varphi(x))}$ with $\omega = 2\,\pi\,f$, f = frequency

With $e^{iz} = \cos(z) + i\sin(z)$ we can write $\cos(z) = Re\{e^{iz}\}$ or:

$$u(x,t) = Re\{u(x)\cos\big(\omega t + \phi(x)\big) + i\,u(x)\sin\big(\omega t + \phi(x)\big)\}$$

$$= u(x)\,Re\{\cos\big(\omega t + \phi(x)\big) + i\,\sin\big(\omega t + \phi(x)\big)\}$$ ($u(x)$ is real-valued)

$$= u(x)\,Re\{e^{i(\omega t+\phi(x))}\} = Re\{u(x)\,e^{i(\omega t+\phi(x))}\} = Re\{\underbrace{u(x)e^{i\,\phi(x)}}_{\underline{u}(x)}\,e^{i\omega t}\}$$

$\underline{u}(x)$ **Complex Phasor**

In summary we can write:

$\boxed{u(x,t) = Re\{\,\underline{u}(x)\;e^{i\omega t}\}}$ **General time harmonic function**

5.1 FREQUENCY DOMAIN FORMULATIONS

FREQUENCY DOMAIN FORMULATION

So far, we have shown that a general **time harmonic** function can be written as:

$$u(x,t) = u(x)\cos(\omega t + \varphi(x)) \qquad \Leftrightarrow \qquad u(x,t) = Re\left\{\underline{u}(x)\; e^{i\omega t}\right\}$$

We can now define a **complex function** such that:

$$\boxed{\underline{u}(x,t) = \underline{u}(x)\, e^{i\omega t} \qquad \Rightarrow \qquad u(x,t) = Re\left\{\underline{u}(x,t)\right\}}$$

The **time derivative** of $\underline{u}(x,t)$ can then simply be calculated as:

$$\frac{\partial \underline{u}(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(\underline{u}(x)\, e^{i\omega t}\right) = \underline{u}(x)\frac{\partial}{\partial t}e^{i\omega t} = \underline{u}(x)\, i\omega\, e^{i\omega t} = i\omega\, \underline{u}(x)e^{i\omega t} = i\omega\, \underline{u}(x,t)$$

Therefore, calculating a **time derivative** in the complex formulation simply becomes a **multiplication by $i\omega$**

FINITE DIFFERENCE METHOD AGENDA

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

We can now substitute a general **time harmonic** function $\underline{u}(x,t) = \underline{u}(x)\, e^{i\omega t}$ into the wave equation and obtain:

$$\frac{\partial^2 \underline{u}(x,t)}{\partial x^2} - k\,\frac{\partial^2 \underline{u}(x,t)}{\partial t^2} = 0$$

$$\Leftrightarrow \frac{\partial^2 \left(\underline{u}(x)\, e^{i\omega t}\right)}{\partial x^2} - k\,\frac{\partial^2 \left(\underline{u}(x)\, e^{i\omega t}\right)}{\partial t^2} = 0$$

$$\Leftrightarrow \frac{\partial^2 \underline{u}(x)}{\partial x^2}\, e^{i\omega t} - k\, \underline{u}(x)\,\frac{\partial^2 \left(e^{i\omega t}\right)}{\partial t^2} = 0$$

$$\Leftrightarrow \frac{\partial^2 \underline{u}(x)}{\partial x^2}\, e^{i\omega t} - k\,(i\omega)^2\, \underline{u}(x)\; e^{i\omega t} = 0$$

$$\Leftrightarrow \boxed{\frac{\partial^2 \underline{u}(x)}{\partial x^2} + k\,\omega^2\, \underline{u}(x) = 0}$$

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

In summary:

$$\frac{\partial^2 \underline{u}(x,t)}{\partial x^2} - k\,\frac{\partial^2 \underline{u}(x,t)}{\partial t^2} = 0 \quad \Leftrightarrow \quad \frac{\partial^2 \underline{u}(x)}{\partial x^2} + k\,\omega^2\,\underline{u}(x) = 0$$

This means that by **solving the equation** $\frac{\partial^2 \underline{u}(x)}{\partial x^2} + k\,\omega^2\,\underline{u}(x) = 0$

we obtain a **general time harmonic** (complex) solution of the wave equation:

$$\underline{u}(x,t) = \underline{u}(x)\,e^{i\omega t}$$

We can then get the corresponding **real-valued** solution by:

$$u(x,t) = Re\{\,\underline{u}(x,t)\} = Re\{\,\underline{u}(x)\,e^{i\omega t}\,\}$$

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

Starting from

$$\frac{\partial^2 \underline{u}(x)}{\partial x^2} + k\,\omega^2\,\underline{u}(x) = 0$$

we can **discretize** the x-axis as before: $\underline{u}_k = \underline{u}(k\,\Delta x)$ and then apply a **central difference** scheme to **approximate the derivatives**:

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\underline{u}(x)\right)\right]_{x=k\,\Delta x} + k\,\omega^2\,\underline{u}_k = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial x}\left(\frac{\underline{u}_{k+1/2} - \underline{u}_{k-1/2}}{\Delta x}\right)\right]_{x=k\,\Delta x} + k\,\omega^2\,\underline{u}_k = 0$$

$$\Rightarrow \boxed{\frac{\underline{u}_{k+1} - 2\underline{u}_k + \underline{u}_{k-1}}{\Delta x^2} + k\,\omega^2\underline{u}_k = 0}$$

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

Starting from

$$\frac{\underline{u}_{k+1} - 2\underline{u}_k + \underline{u}_{k-1}}{\Delta x^2} + k\,\omega^2 \underline{u}_k = 0$$

we can again group the degrees of freedom $\underline{u}_k$ in a **vector**

$$\underline{u} = (\underline{u}_1, \underline{u}_2, \dots, \underline{u}_{N-1})^T$$

and write the relation above as **matrix equation**:

$$\boxed{\begin{pmatrix} \ddots & & & \\ & 1 & -2 & 1 \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_{N-1} \end{pmatrix} + k\,\omega^2 \Delta x^2 \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_{N-1} \end{pmatrix} = 0}$$

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

$$\underbrace{\begin{pmatrix} \ddots & & & \\ & 1 & -2 & 1 \\ & & & \ddots \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_{N-1} \end{pmatrix}}_{\underline{u}} + \underbrace{k\,\omega^2 \Delta x^2}_{\lambda} \underbrace{\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_{N-1} \end{pmatrix}}_{\underline{u}} = 0$$

This **equation system** can then be written as:

$$\boxed{M\,\underline{u} - \lambda\,\underline{u} = 0 \quad \Leftrightarrow M\,\underline{u} = \lambda\,\underline{u}}$$

Mathematical eigenvalue problem

Solutions $\underline{u}$ which satisfy this relation are called **eigenvectors** (or **eigenmodes**). The corresponding scalar values λ are called **eigenvalues**

The corresponding **resonance frequency** can be calculated from an eigenvalue by

$$\lambda = -k\,\omega^2 \Delta x^2 \quad \Rightarrow \boxed{\omega = \sqrt{\frac{-\lambda}{k\,\Delta x^2}}}$$

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

We are searching for the solution of the **eigenvalue** problem $M\,\underline{u} = \lambda\,\underline{u}$

with $$M = \begin{pmatrix} \ddots & & & & \\ & 1 & -2 & 1 & \\ & & & & \ddots \end{pmatrix}$$

In general, the solution can be calculated by solving the **eigenvalue problem numerically** (e.g. with MATLAB)

In this case, the matrix M is a **tridiagonal Toeplitz matrix** (constant elements on three diagonals).

The **eigenvalues** and **eigenvectors** (or **eigenmodes**) of such matrices are known from linear algebra.

5.2 WAVE EQUATION IN FREQUENCY DOMAIN

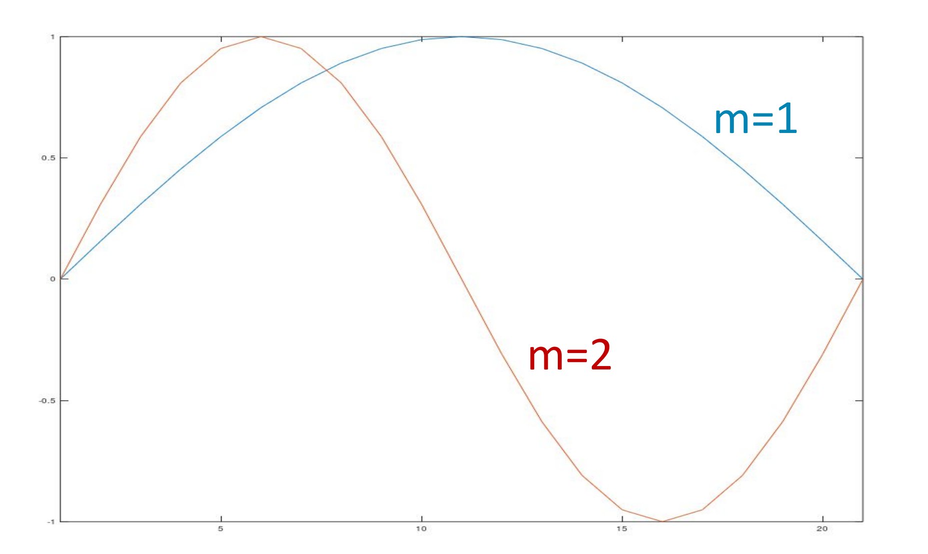
FREQUENCY DOMAIN FORMULATION OF THE WAVE EQUATION

$$M\,\underline{u} = \lambda\,\underline{u} \qquad \text{with} \qquad M = \begin{pmatrix} \ddots & & & & \\ & 1 & -2 & 1 & \\ & & & & \ddots \end{pmatrix}$$

Eigenvalues and **eigenvectors** of the tridiagonal **Toeplitz matrix:**

$$\lambda_m = -2 \; - 2\sqrt{1\cdot 1}\, cos\left(\frac{\pi\, m}{N}\right) = -2\left(1 - cos\left(\frac{\pi\, m}{N}\right)\right)$$

$$\underline{u}_m = \begin{pmatrix} sin\left(\frac{1\pi m}{N}\right) \\ sin\left(\frac{2\pi m}{N}\right) \\ \vdots \\ sin\left(\frac{(N-1)\pi m}{N}\right) \end{pmatrix}$$



MATLAB USEFUL TIPS AND TRICKS

COMPUTING EIGENVALUES AND EIGENVECTORS OF A MATRIX + USEFUL OPERATIONS

```
e = eig(A) % compute a vector containing the eigenvalues
           % of the square matrix A

[V,D] = eig(A)    % compute a matrix V whose columns are
                  % the eigenvectors of the square matrix A
                  % D is a diagonal matrix with the
                  % eigenvalues of A on the diagonal

v = A(:, i)  % extract the i-th column of the matrix A and
             % store the result in vector v

v = A(i, :)  % extract the i-th row of the matrix A and
             % store the result in vector v

v = [0; a; 0] % add zero values to the beginning and end
              % of vector a
```

FINITE DIFFERENCE METHOD AGENDA

6. SUMMARY

$$\boxed{\frac{\partial^2 u(x,t)}{\partial x^2} - k\,\frac{\partial^2 u(x,t)}{\partial t^2} = 0}$$

Wave Equation

Initial value problem	**Eigenvalue problem**
Time Domain simulation	Frequency Domain simulation

Initial value problem

- Time Domain simulation
- Start from an initial state and simulate how the system behaves as function of time (iterative)
- No further excitation during the simulation

Eigenvalue problem

- Frequency Domain simulation
- Determine natural resonances of a system (solve eigenvalue problem)
- No excitation at all
- Eigenvectors (eigenmodes) describe the shapes of the resonances
- Eigenvalues describe the resonance frequencies

THANK YOU!

See you in the exercise!

STABILITY OF ITERATIVE SCHEMES

DETERMINE THE MAXIMUM STABLE TIME STEP Δt_{max}

Our **iteration scheme** is as follows:

$$u_{k,l+1} = cu_{k+1,l} + cu_{k-1,l} + 2(1-c)u_{k,l} - u_{k,l-1} \quad \text{with} \quad c = \frac{\Delta t^2}{k\Delta x^2}$$

We consider a time and space **harmonic solution**:

$$u_{k,l} = U_0 \cos(k_x k\Delta x - \varphi_0 + \omega l\Delta t)$$

Every solution of the wave equation can be written as **superposition** of these harmonic solutions. Therefore it is sufficient to demonstrate stability for the **harmonic solutions**

In **complex form**, we write (with i being the imaginary unit):

$$\underline{u}_{k,l} = \underbrace{U_0 e^{-i\varphi_0}}_{\underline{U}_0} e^{ik_x k\Delta x} e^{i\omega l\Delta t} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}$$

STABILITY OF ITERATIVE SCHEMES

DETERMINE THE MAXIMUM STABLE TIME STEP Δt_{max}

$$\underline{u}_{k,l} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}$$

$$\underline{u}_{k,l+1} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega(l+1)\Delta t} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}\, e^{i\omega\Delta t} = \underline{u}_{k,l}\, e^{i\omega\Delta t}$$

$$\underline{u}_{k,l-1} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega(l-1)\Delta t} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}\, e^{-i\omega\Delta t} = \underline{u}_{k,l}\, e^{-i\omega\Delta t}$$

$$\underline{u}_{k+1,l} = \underline{U}_0\, e^{ik_x(k+1)\Delta x} e^{i\omega l\Delta t} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}\, e^{ik_x\Delta x} = \underline{u}_{k,l}\, e^{ik_x\Delta x}$$

$$\underline{u}_{k-1,l} = \underline{U}_0\, e^{ik_x(k-1)\Delta x} e^{i\omega l\Delta t} = \underline{U}_0\, e^{ik_x k\Delta x} e^{i\omega l\Delta t}\, e^{-ik_x\Delta x} = \underline{u}_{k,l}\, e^{-ik_x\Delta x}$$

Now introduce these descriptions into our **iteration scheme**:

$$u_{k,l+1} = cu_{k+1,l} + cu_{k-1,l} + 2(1-c)u_{k,l} - u_{k,l-1}$$

$$\Rightarrow \underline{u}_{k,l}\, e^{i\omega\Delta t} = c\, \underline{u}_{k,l}\, e^{ik_x\Delta x} + c\, \underline{u}_{k,l}\, e^{-ik_x\Delta x} + 2(1-c)\underline{u}_{k,l} - \underline{u}_{k,l}\, e^{-i\omega\Delta t}$$

$$\Rightarrow e^{i\omega\Delta t} = c\, e^{ik_x\Delta x} + c\, e^{-ik_x\Delta x} + 2(1-c) - e^{-i\omega\Delta t}$$

STABILITY OF ITERATIVE SCHEMES

DETERMINE THE MAXIMUM STABLE TIME STEP Δt_{max}

$$e^{i\omega\Delta t} = c\, e^{ik_x\Delta x} + c\, e^{-ik_x\Delta x} + 2(1-c) - e^{-i\omega\Delta t}$$

$$\Rightarrow \underbrace{e^{i\omega\Delta t} + e^{-i\omega\Delta t} - 2}_{2\cos(\omega\Delta t) - 2} = c\underbrace{\left(e^{ik_x\Delta x} + e^{-ik_x\Delta x} - 2\right)}_{= c\,(2\cos(k_x\Delta x) - 2)}$$

$$2\cos(\omega\Delta t) - 2 \qquad = c\,(2\cos(k_x\Delta x) - 2)$$

With the relation: $\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$$\Rightarrow -4\sin^2\left(\frac{\omega\Delta t}{2}\right) = -4\,c\,\sin^2\left(\frac{k_x\Delta x}{2}\right)$$

$$\boxed{c = \frac{\Delta t^2}{k\Delta x^2}}$$

$$\Rightarrow \boxed{\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\left(\frac{\Delta t}{2}\right)^2} = \frac{1}{k}\,\frac{\sin^2\left(\frac{k_x\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2}}$$

Dispersion Relation

Describes the relation of spatial and temporal propagation

STABILITY OF ITERATIVE SCHEMES

DETERMINE THE MAXIMUM STABLE TIME STEP Δt_{max}

$$\frac{sin^2\left(\frac{\omega\Delta t}{2}\right)}{\left(\frac{\Delta t}{2}\right)^2} = \frac{1}{k}\frac{sin^2\left(\frac{k_x\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2}$$

For **stable oscillations**, we require ω to be real-valued. This leads to:

$$sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{\Delta t^2}{4k}\frac{sin^2\left(\frac{k_x\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2} \leq 1 \qquad \Rightarrow \frac{\Delta t}{\sqrt{k}\,\Delta x}\left|sin\left(\frac{k_x\Delta x}{2}\right)\right| \leq 1$$

This condition needs to be satisfied for **all solutions** k_x

$$\Rightarrow \frac{\Delta t}{\sqrt{k}\,\Delta x} \leq 1$$

and finally:

$$\boxed{\Delta t_{max} = \sqrt{k}\,\Delta x}$$

Courant-Friedrich-Levy Criterion