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日期: /

1.1. Preliminaries.

Prop: 泰勒展开:

$$F(x_1,\cdots,x_n) = F_0(x_1,\cdots,x_{n-1}) + F_1(x_1,\cdots,x_{n-1})\cdot x_n + \cdots + F_1(x_1,\cdots,x_{n-1})\cdot x_n^d .$$

日期: /

1.2. Affine Space & Algebraic Set

def1: affine n-space: $\mathbb{A}^n = \mathbb{A}^n(k) = k^n = k\times\cdots\times k$.

def2: hypersurface $V(F) = \{(a_1,\cdots,a_n)\in\mathbb{A}^n \mid F(a_1,\cdots,a_n)=0\}$, $F\in k[x_1,\cdots,x_n]$.

当 $\deg(F)=1$, denote "hyperplane".

def3: 对 $S\subseteq k[x_1,\cdots,x_n]$

$V(S) = \{(a_1,\cdots,a_n)\in\mathbb{A}^n \mid F(a_1,\cdots,a_n)=0, \forall F\in S\}$

denote (affine) algebraic set

Props: (1) $V(S) = \bigcap\limits_{F\in S} V(F)$

(2) $S_1\subseteq S_2 \Rightarrow V(S_1)\supseteq V(S_2)$.

(3) $I=(S)\lhd k[x_1,\cdots,x_n]$, 有 $V(I)=V(S)$.

(4) $\bigcap\limits_{i\in I} V(S_i) = V(\bigcup\limits_{i\in I} S_i)$ $\quad$ $\sum I_i$ 为包含 $\bigcup I_i$ 的最小理想.

(5) $\bigcup\limits_{i=1}^{m} V(I_i) = V(I_1\cdots I_m)$. $\bigcap\limits_{i\in I} V(I_i) = V(\sum\limits_{i\in I} I_i)$. $(= V(\bigcup\limits_{i\in I} I_i))$.

(6) $\mathbb{A}^n = V(0)$. $\varnothing = V(1)$.

$\{(a_1,\cdots,a_n)\} = V(x_1-a_1,\cdots,x_n-a_n)$.

(7) $\mathbb{A}^1$ 的 $\forall$ proper alg-set is finite.

证: $\exists F\in S \Rightarrow V(F)\supset V(S)$.

又 $|V(F)|\leq \deg(F)<+\infty$.

$\therefore V(S)<+\infty$.

(反过来, finite subset in $\mathbb{A}^n$ 总是 alg. set.)

日期: /

def4: (Zariski topology):

定义其中闭集为 $\mathbb{A}^n$ 上代数集.

待补充.

代数集的交并仍为代数集.

日期: / 1.3. Ideals & Points.

def 1: $X\subseteq \mathbb{A}^n$, 定义X的理想 $I(X)=\{F\in k[x_1,\cdots,x_n] \mid F(P)=0, \forall P\in X\}$.

(即在X上消失的全体多项式.)(注意: X不必为代数集, 二者互为对偶)

Props: (1) $I(X)=\bigcap_{a_i\in X} I(a_i)$

(2) $I(\bigcup_{i\in I} X_i)=\bigcap_{i\in I} I(X_i)$

(3) $I(V_1)\cdots I(V_m)=I(\bigcup_{i=1}^{m} V_i)$, $\sum I(V_i)=I(\bigcap_{i=1}^{n} V_i)$, V为alg set.

(4) if $X\subset Y$, 则 $I(X)\supset I(Y)$.

$I(\emptyset)=k[x_1,\cdots,x_n]$

对无限域 $\mathbb{A}$, $I(\mathbb{A}^n)=0$. (可由归纳+泰勒展开得出).

对有限域 k, 不成立. eg. $k=\{a_1,\cdots,a_m\}$ $F=(x_1-a_1)\cdots(x_1-a_m)$. 有 $F\in I(k^n)$.

(5) $I(\{(a_1,\cdots,a_n)\})=(x_1-a_1,\cdots,x_n-a_n)$.

$\mapsto$ I不一定为I(X)生成 $\qquad$ $\mapsto$ X不一定为代数集.

$I\circ V(I)\supseteq I$ $\qquad$ $V\circ I(X)\supseteq X$

$\Downarrow$

$V\circ I\circ V(S)=V(S)$. $\qquad$ $I\circ V\circ I(X)=I(X)$.

S是a set of polynomials. $\qquad$ X是a set of pts.

$\Rightarrow$ $V\circ I(V)=V$. $\qquad$ $I\circ V(I)=I$.

def2: radical ideal:

$I\triangleleft R$ 且 若 $r^n\in I$, 则 $r\in I$. denote I 根式理想.

I的radical ideal为: $\sqrt{I}=\{r\in R \mid \exists n\in\mathbb{N}, s.t.\ r^n\in I\}$.

补充: (radical ideal的刻画) I为radical ideal $\Leftrightarrow$ $I=\bigcap_{P\supset I \text{且为prime}} P$

Props: (6). $\forall X$, $I(X)$ 为 radical ideal.

(7) V为 alg set. $P\in \mathbb{A}^n\setminus V$.

$I(V)\neq I(V\cup\{P\})$.

☆ 推论: $P_1,\cdots,P_n, r_1,\cdots,r_n$, $\exists F$, s.t. $F(P_i)=r_i$, $\forall i$.

$$
\begin{tikzcd}
\{\text{subset}\} \arrow[ddrr, "I"] & & \{\text{ideal}\} \arrow[ddll, "V"'] \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\
\{\text{alg set}\} \arrow[rr, shift left, "I"] & & \{\text{radical ideal}\} \arrow[ll, shift left, "V"]
\end{tikzcd}
$$

V (不满射. eg. $\mathbb{R}^n$上 $I=(x_1^2+x_2^2)$).

日期: /

1.4. Hilbert Basis Thm

Thm1: 对 $\forall$ alg set. V，存在 $F_1, \cdots, F_m$ st. $V = V(F_1, \cdots, F_m)$.

(即 $\forall$ alg set. V 有限生成).

def: if R is a Noetherian ring $\Leftrightarrow$. $\forall I \subset R$ 有限生成.

$\Leftrightarrow$. $J_1 \subset J_2 \subset \cdots$ 为一理想升链，则 $\exists N$. $J_\infty = J_N$. (即升链有限).

(引理) Thm2: (Hilbert Basis Thm):

if R = noeth. 则 $R[X]$ = noeth. (从而 $R[x_1, \cdots, x_n]$ = noeth.).

Pf: if $F = a_0 + a_1 x + \cdots + a_d x^d \in R[X]$. $a_d \neq 0$ denote a_d "leading coefficient" of F.

$J = \{I$ 中 polynomials' leading coefficients$\} \subset R$.

由 R noeth. $\therefore J$ 有限生成.

$\rightsquigarrow F_1, \cdots, F_r \in I$, 其 l-c. 生成了 J.

取 $N > \deg(F_i)$. $\forall i$.

记 $J_m = \{F$ 的 l-c. $| F \in I$ 且 $\deg(F) \leq m\}$.

$\xrightarrow{\text{同理}} \{F_{mj}\}$ 的 l-c 生成了 J_m.

构造 $I' = (\{F_i\} \cup \{F_{mj}\})$. 即由 $\{F_i\}, \{F_{mj}\}$ 有限生成.

试证 $I = I'$. 显然 $I \supset I'$.

$I \subset I'$): 假设 $I \supsetneq I'$. 记 G 为在 I 中的最小度数的 polynomial.

if $\deg(G) > N$, $\exists Q_i$ 使 $\sum Q_i F_i$ 与 G. l-c 相同

日期: /

从而 $\deg(G-\sum Q_iF_i)<\deg(G)$. $\therefore$ $G-\sum Q_iF_i\in I'$.

$\Rightarrow G\in I'$. $\Rightarrow \deg(G)<N$. 矛盾.

Cor: Thm1 显成立. 且 $V(I)=V(F_1,\cdots,F_r)=\bigcap V(F_i)$.

日期:　　/

1.5. Irreducible

Def: An alg set V 称作 reducible.

if $\exists$ alg sets V_1, V_2, s.t. $V = V_1 \cup V_2$ 且 $V_1 \subsetneq V$, $V_2 \subsetneq V$.

否则称作 irreducible.

Prop. 1: V = alg set in $\mathbb{A}^n$.

V = irr. $\Leftrightarrow$ $I(V)$ = prime

证: $\Rightarrow$: 反证. 则 $\exists\, a, b \notin I(V)$. s.t. $ab \in I(V)$

$$V_1 := V(I(V)+(a)) \subseteq V(I(V)) = V.$$

$$V_2 := V(I(V)+(b)) \subseteq V(I(V)) = V$$

$$V_1 \cup V_2 = (V \cap V(a)) \cup (V \cap V(b)).$$

$$= V \quad \Leftarrow \quad V \subseteq V(ab) = V(a) \cup V(b).$$

$a \notin I(V) \Rightarrow \exists P \in V$, s.t. $P(a) \neq 0 \Rightarrow P \notin V(I(V)+(a)) = V_1 \Rightarrow V_1 \neq V$.

$\Leftarrow$: 反证. 则 $V = V_1 \cup V_2$. $V_1 \subsetneq V$, $V_2 \subsetneq V$.

$\Rightarrow I(V_1) \supsetneq I(V)$. $I(V_2) \supsetneq I(V)$.

$\Rightarrow \exists F_1 \in I(V_1) \setminus I(V)$, $F_2 \in I(V_2) \setminus I(V)$.

$\Rightarrow F_1 F_2 \in I(V_1 \cup V_2) = I(V)$.

$\Rightarrow I(V)$ 非 prime.

日期: /

若不是 alg set, 可能可无限拆分下去.

Thm: $\forall$ alg set V, $\exists$ 唯一的有限的 irr alg sets V_i, 即 $V=\bigcup_{i=1}^{n} V_i$. where V_i irr.

并称这样的拆分为"不可约分支" 不计顺序意义下.

Prop 2: R=noeth. 对 $\forall$ $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq \{I \mid I \triangleleft R\}$.

则 $\mathcal{I}$ 有一个最大的元素. (包含序关系下).

(事实上, 这与 noeth. 环是等价的)

Cor: 对 $\forall$ $\emptyset \neq \mathcal{V}=\{v \mid v=\text{alg set in } \mathbb{A}^n\}$ 有一个极小元.

Pf: def $\mathcal{V}=\{v=\text{alg set in } \mathbb{A}^n$, 不可写为有限 irr 的并$\}$.

Thm $\Leftrightarrow$ $\mathcal{V}=\emptyset$.

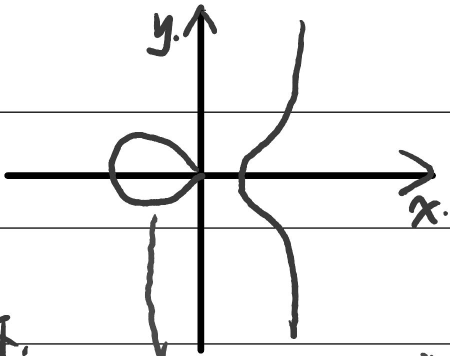
反证. 则 $\mathcal{V}\neq\emptyset$. Cor $\Rightarrow$ $\mathcal{V}$ 有极小元 V.

$V\in\mathcal{V} \Rightarrow V\neq \text{irr} \Rightarrow V=V_1\cup V_2$. 且 $V_i\subsetneq V$.

$\overset{\min}{\Longrightarrow} V_i\notin\mathcal{V} \overset{\text{def}}{\Longrightarrow} V_i$ 可写为有限 irr 的并 $\Rightarrow V\notin\mathcal{V}$. (矛盾)

eg. $y^2=x(x^2-1)$. ← V.

是 irr.



($\mathbb{R}^2$ 不是代数闭域
在 $\mathbb{C}^2$ 上即可较易看出 V 是 irr).

该图形为 $\mathbb{C}^2$ 上的一截面.

(在一般拓扑意义下可拆,
Zariski 拓扑下不可拆).

在 Zariski 拓扑下非闭集.
即非 alg set.

日期: /

代数闭域下成立.

Prop3: 非代数闭域下, $I \triangleleft k[x_1,\cdots,x_n]$ prime. $\not\Rightarrow$ $V(I)=$ irr.

(Prop1的逆)

eg. $F=x^2(x-1)^2+y^2 \in \mathbb{R}[x,y]$.

$I=(F)$ 为prime 但 $V(I)=\{(0,0),(1,0)\} \neq$ irr.

日期: /

1.6 平面 irr. 分类

Prop1: $F, G \in k[X,Y]$, if $\gcd(F,G)=1$. 则 $\#V(F,G)<+\infty$.

(注: 若 $\gcd(F,G)=H$. 则 $V(H) \subset V(F,G)$).

Pf: 引理: $R=$ UFD. $F,G \in R[X]$. $K=\mathrm{Frac}(R)$.

$$\gcd(F,G)=1 \text{ in } R[X] \Leftrightarrow \gcd(F,G)=1 \text{ in } K[X].$$

$$\therefore \gcd(F,G)=1 \text{ in } \underset{\sim}{k[X]}[Y] \Rightarrow \gcd(F,G)=1 \text{ in } \underset{\sim}{\overset{PID}{k(X)[Y]}}$$

$$\Rightarrow (F,G)=(1).$$

$$\therefore \exists\, s,t \in k(X)[Y].\ \text{s.t.}\quad sF+tG=1.$$

assume $s=\dfrac{S(X,Y)}{L(X)}$ $t=\dfrac{T(X,Y)}{L(X)}$. $\therefore$ $SF+TG=L$. in $k[X][Y]$.

$\Rightarrow V(F,G) \subset V(L(X))$. 同理, $V(F,G) \subset V(R(Y))$.

Prop2: $F \in k[X,Y]$ = irr. Assume $V(F)$ is infinite. 则 $V(F)$ = irr.

Pf:

$$\forall G \in I(V(F)). \Rightarrow \underset{\substack{\sim\\ \text{infinte}}}{V(F)} \subseteq V(F,G).$$

$$\Rightarrow (F,G) \neq 1 \Rightarrow F \mid G$$

$$\Rightarrow I(V(F)) \supset (F).$$

$$\Rightarrow I(V(F)) = (F) \to \text{prime} \Rightarrow V(F) \text{ irr.}$$

Thm: (仿射平面的 irr. 的分类)

对 $\forall$ irr. alg. set $V \in \mathbb{A}^2$. V 必为以下形式之一:

日期: /

(1) $\mathbb{A}^2$, pts.

(2) $V(F)$ where F irr 且 $\#V(F)=+\infty$.

Pf: $I(V)=0 \Rightarrow V=V(I(V))=V(0)=\mathbb{A}^2$

$I(V)\neq 0 \Rightarrow \exists$ irr $F\in I(V) \Rightarrow (F)\subseteq I(V)$.

$\forall G\in I(V) \Rightarrow \overset{\#V=+\infty}{V}\subset V(F,G) \Rightarrow \#V(F,G)=+\infty \overset{类似}{\Rightarrow} F|G$.

$\Rightarrow (F)=I(V) \Rightarrow V=V(F)$.

日期: /

1.7. Hilbert Nullstellensate.

Prop: $k=\bar{k}$. $\forall F=F_1^{a_1}\cdots F_r^{a_r}$

(1) $V(F)=V(F_1)\cup\cdots\cup V(F_r)$.

(2) $I(V(F))=(F_1,\cdots,F_r)=\sqrt{F_1^{a_1}\cdots F_r^{a_r}}$

Pf: $I(V(F))=\cap I(V(F_i))\overset{k=\bar{k}}{=}\cap F_i=(F_1\cdots F_r)$.

Think: $I\circ V(I)\supseteq\sqrt{I}$. $I=(F)\lhd k[X,Y]\overset{k=\bar{k}}{\Rightarrow} I\circ V(I)\subseteq\sqrt{I}$.

Thm1 (Hilbert Nullstellensate) $k=\bar{k}$, $\forall I\lhd k[x_1,\cdots,x_n]$.

$$I(V(I))=\sqrt{I}.$$

Thm2 (Weak Nullstellensate). $k=\bar{k}$. $\forall I\lhd k[x_1,\cdots,x_n]$ proper $\Rightarrow V(I)\neq\emptyset$.

Pf-Thm1: $I(V(I))\supset\sqrt{I}$ 显. (?)

$\forall G\in I(V(I))$, 由 $k[x_1,\cdots,x_n]$ Noeth, $I=(F_1,\cdots,F_m)$.

$\because\forall P,\ F_i(P)=0\Rightarrow G(P)=0$

构造 $J=(F_1,\cdots,F_m,\ X_{m+1}G-1)\lhd k[x_1,\cdots,x_{m+1}]$

$\forall \overset{Q=(P,a).}{Q}\in V(J).\Rightarrow F_i(P)=0.\forall i\Rightarrow P\in V(I)\Rightarrow G(P)=0.$

$\Rightarrow (X_{m+1}G-1)|_Q=-1\neq0$ 矛盾.

$\Rightarrow$. $J=\emptyset$.

由Thm2. $J=k[x_1,\cdots,x_n]$.

$\Rightarrow\exists H_1,\cdots,H_m,H$ .st. $1=H_1F_1+\cdots+H_mF_m+H\cdot(X_{m+1}G-1)$.

日期: /

取 $x_{m+1}=\frac{1}{G}$ $\Rightarrow 1=H_1(x_1,\cdots,x_m,\frac{1}{G})F_1+\cdots+H_m(x_1,\cdots,x_m,\frac{1}{G})F_m$

$\exists$ 充分大的 N, $G^N=H_1'F_1+\cdots+H_m'F_m$. (即通分).

$\Rightarrow G^N\in(F_1,\cdots,F_m)$. $\therefore G\in\sqrt{I}$.

$\therefore I(V(I))\subset\sqrt{I}$.

Cor: $k=\bar{k}$:

① $\{\text{alg set}\}\xleftrightarrow{1:1}\{\text{radical ideal}\}$

$\cup$ $\qquad$ $\cup$

$\{\text{irr. alg. set}\}\xleftrightarrow{1:1}\{\text{prime ideal}\}=\mathrm{Spec}(k)$

$\cup$ $\qquad$ $\cup$

$\mathbb{A}^n\xleftrightarrow{1:1}\{\text{pts}\}\xleftrightarrow{1:1}\{\text{maximal ideal}\}=\mathrm{Specm}(k)$.

② $\{\text{irr hyper surface}\}\xleftrightarrow{1:1}\{(F)\mid F=\text{irr}\}$.

③(有限代数集的刻画):

$V(I)=\text{finite}\iff k[x_1,\cdots,x_n]/I$ is finite 维 k 线性空间.

且 $\Rightarrow \#V(I)\le\dim_k k[x_1,\cdots,x_n]/I$.

(一元时, 即 F 零点数 $\le F$ 次数)

pf: $\Leftarrow$): $\forall p_1,\cdots,p_m\in V(I)$.

$\exists F_i\in k[x_1,\cdots,x_m]$ st. (由 $p_i\in V(I)$. 构造 $F_i\in k[x_1,\cdots,x_m]$. 并尝试证 F_i 线性无关.)

$$F_i(p_j)=\begin{cases}1, & i=j\\ 0, & i\neq j.\end{cases}$$

if $\sum a_iF_i\in I\Rightarrow a_jF_jp_j=0\Rightarrow a_j=0$. $\forall j$.

日期: /

$\Rightarrow F_1,\cdots,F_m \bmod I$ 线性无关.

(二) 并且不等式成立.

$\Rightarrow$): 设 $V(I)=\{P_1,\cdots,P_r\}$. $P_i=(a_{i1},\cdots,a_{in})$.

$$F_j=\prod_{i=1}^{n}(x_j-a_{ij})\in I(V(I))=\sqrt{I}.$$

$\Rightarrow \exists N_i$, $\underset{\in k[x_1]}{F_1^{N_1}},\cdots,\underset{\in k[x_n]}{F_n^{N_n}}\in I$.

$\Rightarrow \dim(k[x_1,\cdots,x_n]/I)\leq \dim(k[x_1,\cdots,x_n]/(F_1^{N_1},\cdots,F_n^{N_n}))$ #.

$= n\cdot\sum_{i=1}^{n}N_i$

Pf-Thm2:

Assume I = maximal, $k[x_1,\cdots,x_n]\twoheadrightarrow k[x_1,\cdots,x_n]/I\overset{\text{为域}}{=}k'$.

$a_i=x_i \pmod I$.

Fact: $\Rightarrow k'$ 是 f.g. k-alg. 且为 field

? $\Rightarrow k'$ 是 f.g. as a k-module

? $\Rightarrow \forall a\in k'$ is algebraic over $k=\bar{k}$.

$\Rightarrow a\in k$.

$\Rightarrow I=(x_1-a_1,\cdots,x_n-a_n)$.

$\Rightarrow (a_1,\cdots,a_n)\in V(I)$. $\neq\emptyset$.

日期:　　/

def: $R\to S$, s is called "integral" over R.

if s 是 $R[x]$ 中某首一多项式的根.

?

Prop: L/k 为域扩张, 若 $L=k[a_1,\cdots,a_n]$.

则 L 是 k 上有限维线性空间.

若 S 是一个 R 代数. (即 $R\hookrightarrow S$, 即 R 可视作 S 的 subring).

- S 是 f.g. as a R-module $\Leftrightarrow S=\sum_{i=1}^{m} R s_i \Leftrightarrow S\cong R^{\oplus m}/N$.

  $\Downarrow$ 　 $\not\Uparrow$ (Props).

- S 是 f.g. as a R-alg. $\Leftrightarrow S=\varphi(R)[s_1,\cdots,s_m]\cong R[x_1,\cdots,x_m]/I$

  $\Downarrow$ 　 $\not\Uparrow$

- S 是 f.g. field over R $\Leftrightarrow S=R(s_1,\cdots,s_m)$.

$I\to R[x_1,\cdots,x_m]\to R(\varphi)[s_1,\cdots,s_m]$. exact.

Props: if S,R 为 field 即 $S=L$, $R=k$, L/k.

则 L 是 f.g. as a k-alg $\Rightarrow$ 是 f.g. as a k-module.

Pf: $\Rightarrow$ $L=k(s_1,\cdots,s_m)$.

$m=1$ 时, … 成立.

由归纳, $L=k(a_1)[a_2,\cdots,a_m]$ f.g. as a $k(a_1)$-alg.

$m=m$ 时, $\Rightarrow a_2,\cdots,a_m$ 在 $k(a_1)$ 上 integral.

$\Rightarrow \exists g\neq 0\in k[a_1]$, s.t. $ga_2,\cdots,ga_m$ integral over $k[a_1]$.

$\Rightarrow$ 记 $\overline{k[a_1]}=\{\beta\in L \mid \beta \text{ is integral over } k[a_1]\}$.

日期:　　　/

则 $ga_1,\cdots,ga_m \in \overline{k[a_1]}$.

$\forall f \in L = k[a_1,\cdots,a_m]$, $f = F(a_1,\cdots,a_m)$ ← k-degree norm ? d

$\Rightarrow g^d f = g^d F(a_1,\cdots,a_m) = F(ga_1,\cdots,ga_m) \in \overline{k[a_1]}$.

① a_1 integral over k.

$\Rightarrow k(a_1) = k[a_1]$ is f.g. as a k-mod.

$\Rightarrow L = k(a_1)[a_2,\cdots,a_m]$ is f.g. as a $k(a_1)$-mod.

$\Rightarrow L$ is f.g. as a k-mod.

② a_1 不是 integral over k.

$\Rightarrow k[x] \cong k[a_1]$.

取 $f = \frac{1}{(g+1)a_1} \in L$.

$\Rightarrow g^d f = \frac{g(a_1)^d}{(g(a_1)+1)\cdot a_1} \overset{易}{=} \frac{h_1(a_1)}{h_2(a_1)}$, $(h_1,h_2) \neq 1$.

由上, $g^d f \in \overline{k[a_1]}$, 则 $h_2 | h_1$, 矛盾!

?

日期: /

Def: $V \subseteq \mathbb{A}^n$ alg. set.

V is called an affine variety (仿射代数簇) if V is irr.

$V = \text{irr.} \Leftrightarrow I(V) = \text{prime} \Leftrightarrow \Gamma(V) = k[x_1, \cdots, x_n]/I(V) = \text{domain}.$

denote $\Gamma(V)$ as coordinate ring of V (坐标环).

Props: $\left\{ W \subseteq \mathbb{A}^n \,\middle|\, \begin{array}{l} W = \text{variety} \\ W \subseteq V \end{array} \right\} \xleftrightarrow{1:1} \{ P \triangleleft \Gamma(V) \mid P = \text{prime} \}.$

Pf: $(k[x_1, \cdots, x_n]/I(V))/(I(W)/I(V)) \cong k[x_1, \cdots, x_n]/I(W).$

$I(W)/I(V)$素. $\leftarrow$ domain

P素 $\rightarrow I(W)$素.

Def: $\mathcal{F}(V,k) = \{ k\text{-valued fuction on } V \}$. 易验证 is a k-alg.

Lem: $\varphi: \underset{k\text{-alg.}}{k[x_1, \cdots, x_n]} \longrightarrow \underset{k\text{-alg}}{\mathcal{F}(V,k)}$ is a ring hom.

$F \longmapsto (P \to F(P))$

$\ker(\varphi) = \{ F \mid F(P) = 0, \forall P \in V \} = I(V).$

$\Rightarrow \Gamma(V) = k[x_1, \cdots, x_n]/I(V) \hookrightarrow \mathcal{F}(V,k).$

理解: $\Gamma(V)$中元素可视作V上的某多项式函数.

Ex: $\dashrightarrow \varphi: V \to k^n \dashrightarrow \varphi: V \to W.$ 代数簇.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi"'] & W \arrow[d, "f"] \\ & k \end{tikzcd} \quad \Leftrightarrow \quad \tilde{\varphi}: \mathcal{F}(W,k) \to \mathcal{F}(V,k), \quad f \longmapsto f \circ \varphi.$$

Lem: $\{ \varphi: V \to W \} \xleftrightarrow{1:1} \{ \tilde{\varphi}: \mathcal{F}(W,k) \to \mathcal{F}(V,k) \mid \text{ring hom} \}.$

日期: /

def: $\varphi: V \to W$, $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$. $\varphi(a_1,\dots,a_n)=(\varphi_1(a_1,\dots,a_n),\dots,\varphi_m(a_1,\dots,a_n))$.

if $\varphi_1,\dots,\varphi_m \in \Gamma(V)$. 则 φ is denoted as poly. map

Props: 若 φ为 poly map. 则 $\tilde{\varphi}(\Gamma(W)) \subset \Gamma(V)$.

Pf: $\forall f \in \Gamma(W) \Rightarrow f = F(x_1,\dots,x_m) \bmod I(W)$.

则 $f\circ\varphi(a_1,\dots,a_n) = f(T_1(a_1,\dots,a_n),\dots,T_m(a_1,\dots,a_n)) \bmod I(V)$.

$= F(\overset{\in\Gamma(V)}{\underline{T_1(x_1,\dots,x_n)}},\dots,T_m(x_1,\dots,x_n))\big|_{(a_1,\dots,a_n)} \bmod I(V)$.

$= G(x_1,\dots,x_n)\big|_{(a_1,\dots,a_n)} \bmod I(V)$.

$\Rightarrow f\circ\varphi \in \Gamma(V)$.

Lem: $\{\varphi: V \to W \mid \varphi = \text{poly map}\} \xleftrightarrow{1:1} \{\phi: \Gamma(W) \to \Gamma(V) \mid \phi = k\text{-alg. hom}\}$.

即在$\tilde{\varphi}$中保持坐标环.

Pf: $\Leftarrow$: $\phi: \Gamma(W) \to \Gamma(V)$.

def: $x_i \bmod I(W) \mapsto \varphi_i$

def: $\varphi(a_1,\dots,a_n) = (\varphi_1(a_1,\dots,a_n),\dots,\varphi_m(a_1,\dots,a_n))$

验证φ即可.

……见(照片 9.29.)

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d] \\ \mathbb{A}^n \arrow[r, "T=(T_1,\dots,T_m)"] & \mathbb{A}^m \end{tikzcd}$$

Def: a poly map $\varphi: V \to W$ is called isomorphism, if $\exists$ poly map $\psi: W \to V$, s.t. $\varphi\circ\psi = id_W$, $\psi\circ\varphi = id_V$.

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Props: $T: \mathbb{A}^n \to \mathbb{A}^m$ 有$T^{-1}(V)$也为代数簇.

$$\begin{array}{ccc} \mathbb{A}^n & \to & \mathbb{A}^m \\ \cup & & \cup \\ T^{-1}(V) & \mapsto & V \end{array}$$

Pf: $T^{-1}(V) = \{P \in \mathbb{A}^n \mid T(P) \subseteq V\}$

$= \{P \in \mathbb{A}^n \mid F(T(P)) = 0, \forall F \in I(V)\}$.

$= V(\{F \circ T \mid F \in I(V)\})$.

...

坐标变换:

Def: a poly map $T: \mathbb{A}^n \to \mathbb{A}^n$ is called "affine change of coordination" iff

- $\deg(F_i) = 1.\ \forall i$.
- T is invertible.

考虑 $\frac{F}{G} = f \in k(V) = \mathrm{Frac}(\Gamma(V))$.

def pole set of f as. $\bigcap_{\substack{f = \frac{F}{G} \\ F, G \in \Gamma(V)}} \{P \in V \mid G(P) = 0\}$.

def $\mathcal{O}_P(V) = \{f \in k(V) \mid f \text{ is defined on } P\} = \underset{\text{即分式化}}{S^{-1}\Gamma(V)}$ as. local ring of V at P.

显, $\Gamma(V) \subset \mathcal{O}_P(V) \subset k(V)$.

$\underset{\text{noeth.}}{k[x_1, \cdots, x_n]/I(V)} \simeq \underset{\text{noeth.}}{\Gamma(V)} \rightsquigarrow \underset{\text{noeth.}}{S^{-1}\Gamma(V)}$

日期: /

Props: 1) pole set of f is an alg. set

2). $\Gamma(V)=\bigcap_{p\in V}\mathcal{O}_p(V)$.

Pf: 1) def $J_F=\{G\in k[x_1,\cdots,x_n] \mid G \bmod I(V)\cdot f\in\Gamma(V)\}$.

$\Rightarrow J_F \triangleleft k[x_1,\cdots,x_n]. \Rightarrow V(J_F)=$ pole set of f.

$\subseteq$): $\forall\, P\in V(J_F)$, ?

$\supseteq$): ?

2) $\supseteq$): $\forall f\in\bigcap_{p\in V}\mathcal{O}_p(V). \Rightarrow V(J_f)=\varnothing$.

$\overset{\text{Hilbert}}{\Rightarrow} J_f=k[x_1,\cdots,x_n]$

$\Rightarrow 1\in J_f \overset{\text{def}}{\Rightarrow} f=1\cdot f\in\Gamma(V)$.

$\subseteq$): 显.

启发: 坐标环是由处处可以定义的有理函数构成.

Lem: (local ring 的等价刻画).

1) R is a local ring.

2) $R\setminus R^*$ is an ideal of R.

Lem: $\mathcal{O}_p(V)$ is a local rin, 其 maximal ideal is:

$\{f\in\mathcal{O}_p(V) \mid f(P)=0\}=\{\frac{F}{G} \mid F,G\in\Gamma(V),\ F(P)=0,\ G(P)\neq 0\}$.

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离散赋值:

Def. A discrete valuation on k is a map:

$$v: k \to \mathbb{Z}\cup\{\infty\}.$$

使得 • $v(ab)=v(a)+v(b)$.

• $v(a+b)\geq \min\{v(a), v(b)\}$.

• $v(a)=\infty \Leftrightarrow a=0$.

eg.1. $k((t)) = \mathrm{Frac}(k[[t]]) = \left\{\frac{f}{g} = \frac{t^n(a_n+a_{n+1}t+\cdots)}{t^m(b_m+b_{m+1}t+\cdots)} = t^{n-m}(c_1+c_2t+\cdots)\right\}$.

定义 $v(\frac{f}{g}) = n-m$.

eg.2 $\mathbb{Z}_p = \{a_0+a_1p+\cdots \mid a_0,a_1,\cdots\in\{0,1,\cdots,p-1\}\}$. ($p$进域)

类似地定义 $v(\frac{f}{g})=n-m$. (注意:不同于eg.1. $\mathbb{Z}_p$中"+","·"均要考虑进位)

Def: $\mathcal{O}_K=\{r\in k \mid v(r)\geq 0\}$. denote as discrete valuation ring (DVR).

Prop:(等价刻画) 若R为domain, 以下等价:

1) $R=$ DVR

2) $R=$ noeth, local with principal $\underbrace{\text{max ideal}}_{\text{记 } m}$　　order of (r), 记 $\mathrm{ord}(r)$.

3) $\exists t\in R$, s.t. $\forall r\in R$ $\exists!\mu\in R^{\times}$, $n\in\mathbb{N}$, s.t. $r=\mu t^n$.　　$t\in m\backslash m^2$.

(eg.1, eg.2 即用此法定义)　　informizing parameter (单值参数)

Rmk: 可以利用$v(\cdot)$在k上建立拓扑$\mathcal{T}_v$, 但$\mathcal{T}_v$不在代数变换下保持.

eg. $\mathbb{R} \xrightarrow{(\cdot)^2} \mathbb{R}$ 　二者$\mathcal{T}$不相容.

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齐次化与去齐次化:

齐次化: $k[x_1,\cdots,x_n] \longrightarrow k[x_1,\cdots,x_{n+1}]$

$f(x_1,\cdots,x_n) \longmapsto f^* = f(\frac{x_1}{x_{n+1}},\cdots,\frac{x_n}{x_{n+1}})\cdot x_{n+1}^d$ $\boxed{= \text{form of deg. } d,\ d=\deg(f)}$

去齐次化: $k[x_1,\cdots,x_{n+1}] \longrightarrow k[x_1,\cdots,x_n]$

$F(x_1,\cdots,x_{n+1}) \longmapsto F_* = F(x_1,\cdots,x_n,1)$.

Rmk: f^*与f_*不可逆.

Prop: 1) $(F\cdot G)_* = F_*\cdot G_*$. $(f\cdot g)^* = f^*\cdot g^*$.

2) $(F+G)_* = F_*+G_*$. $(f+g)^* \neq f^*+g^*$.

要求次数一样. eg. $1^*=1$, $x^*=x$. $(1+x)^* = x+y$.

设 $r=\deg(f)$. $s=\deg(g)$. $t=\deg(f+g)$. 则 $t\leq \max\{r,s\}$.

$(f+g)^* = (f(\frac{x_1}{x_{n+1}},\cdots,\frac{x_n}{x_{n+1}}) + g(\frac{x_1}{x_{n+1}},\cdots,\frac{x_n}{x_{n+1}}))\,x_{n+1}^t$

$= (x_{n+1}^{-r} f^* + x_{n+1}^{-s} g^*)\cdot x_{n+1}^t$. $\boxed{= \text{form of deg. } t}$

3) $\deg(f^*)=\deg(f)$

$\deg(F) \geq \deg(F_*)$ ("="成立当且仅当 $x_{n+1} \nmid F$).

4) $(f^*)_* = f$.

$x_{n+1}^{d-s}(F_*)^* = F$. $d=\deg(F)$. $s:=\deg(F_*)$.

Rmk: 对 Form. F 分解 $\rightsquigarrow$ 对 F_* 分解

日期: /

CRT:

Thm: $k=\bar{k}$, $V(I)=\{P_1,\cdots,P_N\}\subseteq \mathbb{A}^n$. $I\in k[x_1,\cdots,x_n]$. $\mathcal{O}_i=\mathcal{O}_{P_i}(\mathbb{A}^n)$. $R=k[x_1,\cdots,x_n]/I$.
(CRT)

则 $R\simeq \prod_{i=1}^{N}\mathcal{O}_i/I\cdot\mathcal{O}_i$.

Cor: 1) $\dim_k R=\sum_{i=1}^{N}\dim_k(\mathcal{O}_i/I\cdot\mathcal{O}_i)$.

2) $V(I)=\{P\}\iff k[x_1,\cdots,x_n]\simeq\mathcal{O}_P(\mathbb{A}^n)/I\cdot\mathcal{O}_P(\mathbb{A}^n)$.

Thm Pf: $\{P_i\}\subseteq V(I)\subseteq\mathbb{A}^n$. def $I_i:=I(\{P_i\})\triangleleft k[x_1,\cdots,x_n]$.

$\Rightarrow I=I(V(I))\subseteq I_i$. 且 $I_1\cdots I_N\overset{\text{def}}{\subseteq} I(V(I))\overset{\text{Hilbert}}{=}\sqrt{I}$.

$\Downarrow\ \sqrt{I}$ noeth.

$\exists d$, s.t. $(I_1\cdots I_N)^d\subseteq I$.

$\Downarrow$

$I_1^d\cdots I_N^d\equiv 0\ (\mathrm{mod}\ I)$.

$\Downarrow$

def $J_i:=I_i/I\triangleleft k[x_1,\cdots,x_n]/I=R$

$\Rightarrow J_1^d\cdots J_N^d=0\triangleleft R$.

$J_1^s+J_2^t=R\iff I_1^s+I_2^t=k[x_1,\cdots,x_n]\iff V(I_1^s+I_2^t)=\emptyset\iff \underbrace{V(I_1)}_{\{P_1\}}\cap\underbrace{V(I_2)}_{\{P_2\}}=\emptyset$.

从而 $J_i^s+J_j^t=R$ 即互素. $(\forall s,t)$.

$\therefore R\simeq R/0\simeq R/J_1^d\cdots J_N^d\overset{\text{CRT}}{\simeq}\prod_{i=1}^{N}R/J_i^d$

$\therefore$要证 $R/J_i^d\simeq\mathcal{O}_i/I\mathcal{O}_i$.

由 $R/J_i^d=k[x_1,\cdots,x_n]/I\big/(I_i/I)^d\overset{?}{\simeq}k[x_1,\cdots,x_n]/(I+I_i^d)$

lem1: $\mathcal{O}_i=R\cdot\mathcal{O}_i=I_i^s\mathcal{O}_i+I_j^t\mathcal{O}_i$?

假设 $I_i^s\mathcal{O}_i\subsetneq\mathcal{O}_i$ $\therefore I_i^s\mathcal{O}_i\subset m_i$. ($\mathcal{O}_i$的极大理想)

日期: /

从而必有 $I_j^t \mathcal{O}_i = \mathcal{O}_i$. ($I_j$ 可被 $\mathcal{O}_i$ 吸收).

Lem2: $I_1^d \cdots I_N^d \mathcal{O}_i = I \cdot \mathcal{O}_i$

$\overset{?}{\Rightarrow}$ $I_i^d \subseteq I_i^d \mathcal{O}_i \subseteq I\mathcal{O}_i$.

考虑证明: $k[x_1, \cdots x_n]/I+I_i^d \simeq \mathcal{O}_i/I\mathcal{O}_i$.

$F S \bmod I+I_i^d \longmapsto \frac{F}{G} \bmod I\mathcal{O}_i \qquad SG \equiv 1 \pmod{I\mathcal{O}_i}$.

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Plane Curve

点与切线:

Def: A plane curve is $k[X,Y]/\sim$.

其中, def $\sim$: $F\sim G \Leftrightarrow G=\lambda F,\ \lambda\in k^*$.

若 $F=F_1^{a_1}\cdots F_r^{a_r}$. 称 F_i 为 component. a_i 为其 multiplication.

$a_i=1 \Rightarrow$ simple comp~. $a_i\geq 2 \Rightarrow$ multiple comp~.

考虑: 在 $P(a,b)$ 附近将 F 等价于 $\frac{\partial F}{\partial X}\big|_P\cdot(X-a)+\frac{\partial F}{\partial Y}\big|_P\cdot(Y-b)=0$.

Assume $P(0,0)$. 作坐标平移.

展开F得. $F=F_m+F_{m+1}+\cdots+F_d$ ($F_m\neq 0$, $i=\deg(F_i)$).

$\Rightarrow$ def m as multiplication of F on P.

P上, $F=0 \sim F_m=0 \Rightarrow F_m(X,Y) \rightsquigarrow F_m=L_1^{d_1}\cdots L_r^{d_r}$

$\Updownarrow$ (P在F代表的curve上)

$F_m(X,Y)\downarrow F_m(X,1) = \prod_{i=1}^{r}(X-a_i)$

L_i: 一次齐次多项式

$\rightsquigarrow F_m=0 \Leftrightarrow L_i=0\ \forall i$. 称 L_i 为切线 tangent line. 重数为 d_i.

Prop: $F=F_1^{a_1}\cdots F_r^{a_r}$. 对切线 L.

P的重数 $m=a_1m_1+\cdots+a_rm_r$. L的重数 $d=a_1d_1+\cdots+a_rd_r$.

$m \Downarrow$ denote $m_P(F)$.

日期: /

Thm: F=irr. $P\in F$ $P\in L=aX+bY+c$(的line). 记 $\underbrace{\mathcal{O}_P(F)}_{\cup\!\!\shortmid\ \mathfrak{m}}=\mathcal{O}_P(V(F))$.

则 1) $m_P(F)=\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. $(n\gg 0)$.

$\Rightarrow$ F在P处性质可由$\mathcal{O}_P(F)$刻画.

2) P= simple pt. $\Leftrightarrow$ $\mathcal{O}_P(F)$=DVR.

(也可用于制造DVR).

3) if P= simple pt., def $l:=\underset{\sim\sim\sim}{L \bmod F}\in\mathcal{O}_P(F)$. then:

l is a 单参 (DVR). $\Leftrightarrow$ L不是F在P处的切线.

Pf: (1) $P(0,0)$. 考虑计算 $\dim(\mathcal{O}_P(F)/\mathfrak{m}^n)$.

$\mathfrak{m}=\left\{\frac{f}{g}\mid f(P)=0,\ g(P)\neq 0,\ f,g\in k[X,Y]/(F)\right\}$.

$\subset$): ① X,Y在P消失 易证

$\supset$): ② Taylor展开.

$=(X,Y)\lhd\mathcal{O}_P(F)$. (不是在$k[X,Y]$中).

$=I\,\mathcal{O}_P(F)$. $(I\triangleq(X,Y))$. (*¹)

$\rightsquigarrow$ $0\to\mathfrak{m}^n/\mathfrak{m}^{n+1}\to\mathcal{O}_P(F)/\mathfrak{m}^{n+1}\to\mathcal{O}_P(F)/\mathfrak{m}^n\to 0$ exact (同态Thm3).

$\rightsquigarrow$ $\dim(\mathcal{O}_P(F)/\mathfrak{m}^{n+1})=\dim(\mathfrak{m}^n/\mathfrak{m}^{n+1})+\dim(\mathcal{O}_P(F)/\mathfrak{m}^n)$.

$\therefore$ (1) $\Leftrightarrow$ $\exists S$, s.t. $n\gg 0$. $\dim_k(\mathcal{O}_P(F)/\mathfrak{m}^n)=m_P(F)\cdot n+S$.

$$\begin{array}{ccc} k[x,y] & \twoheadrightarrow & \Gamma(F)(=k[x,y]/(F)) \\ \downarrow & \circlearrowright & \downarrow \\ \mathcal{O}_P(\mathbb{A}^2) & \overset{\exists!}{\twoheadrightarrow} & \mathcal{O}_P(F) \end{array}$$

?

(也可用分式化的泛性质理解).

$\rightsquigarrow$ $\mathcal{O}_P(F)\simeq\mathcal{O}_P(\mathbb{A}^2)/(F)$. (*²)

$k[X,Y]/(I^n,F)\overset{CRT}{\simeq}\mathcal{O}_P(\mathbb{A}^2)/(I^n,F)\mathcal{O}(\mathbb{A}^2)\overset{\text{HomThm3}}{\simeq}\mathcal{O}_P(\mathbb{A}^2)/(F)\Big/(I^n,F)\mathcal{O}_P(\mathbb{A}^2)/(F)$ (*²) (*¹).

$V(I^n,F)=V(I)\cap V(F)=\{P\}$.

$\simeq\mathcal{O}_P(F)/\mathfrak{m}^n$.

$(I^n,F)/(F)=I^n$. $\mathcal{O}(\mathbb{A}^2)/(F)=\mathcal{O}_P(F)$.

日期: /

由 $0 \to k[X,Y]/I^{n-m} \xrightarrow[1^\circ]{\overset{G}{\cdot}\times F} k[X,Y]/I^n \longrightarrow k[X,Y]/(I^n,F) \to 0$. exact $(n \gg 0)$
                                          $\downarrow$
$\overset{}{2^\circ}$ (只要 $n \geq m$)

(Pf: 1°: $G\cdot F \in I^n$. $F = F_m + \cdots$ $G = G_r + \cdots \Rightarrow r \geq n-m \Rightarrow G \in I^{n-m}$.,
2°: 显然.).

$\rightsquigarrow \dim k[X,Y]/(I^n,F) = \dim_k(k[X,Y]/I^n) - \dim_k(k[X,Y]/I^{n-m})$.

$$= \frac{n(n+1)}{2} - \frac{(n-m)(n-m+1)}{2}.$$

$$= n\cdot m - \frac{m^2-m}{2}. \quad \#.$$

(2). $\Leftarrow)$: $\mathcal{O}_P(F) = DVR \Rightarrow m^n/m^{n+1} = k[t^n] \overset{\nearrow \text{只有 } t^n \text{ 项}}{\simeq} k$.

$\Rightarrow \dim(m^n/m^{n+1}) = 1$.
$\overset{\text{"}}{\Rightarrow} m_P(F) = 1$
$\Rightarrow P = $ simple pt.

$\Rightarrow$:) Assume 不妨 $P(0,0)$. $Y=0$ 与 F 相切 at P (坐标变换)

$\Rightarrow F = Y + ($higher terms$)$
　　　　　　　$\swarrow$ 要么含 Y, 要么含 X^2.
$= Y(1+H) + X^2 \cdot T$.

$\overset{\text{mod}F}{\rightsquigarrow} 0 = y(1+h) + x^2 t$.

$\deg(H) > 1 \Rightarrow H(P) = 0 \Rightarrow h(P) = 0 \Rightarrow (1+h)(P) \neq 0$.

$\Rightarrow 1+h \in \mathcal{O}_P(F)^\times$.

$\Rightarrow y = -x^2 \frac{t}{1+h} \in (x^2) \lhd \mathcal{O}_P(F)$.

$\Rightarrow m_P(F) = (x) \lhd \mathcal{O}_P(F)$ 为主理想 　($m = I\mathcal{O}_P(F)$.). $\leftarrow (X,Y)$ 在 $\mathcal{O}_P(F)$ 中.

(noeth 里).
$\Rightarrow \mathcal{O}_P(F) = DVR$.

日期: /

(3) $\Leftarrow$): 选系, 从而不妨 $L=X$.

同(2). $m_p(F)=(x)$.

$\hookrightarrow$ $x\in m\setminus m^2 \Leftrightarrow l=x=$ 单参

$\Rightarrow$): 假设 L 是切线, 则 $F=L+(\text{higher term})$.

$\overset{\text{mod} F}{\Rightarrow} l\in(x,y)^2$.

$\Rightarrow l\in m^2$. $\Rightarrow l$ 不是单参. 矛盾. #

Cor: ① def: $\text{ord}_p^F: O_p(F)\longrightarrow \mathbb{Z}\cup\{\infty\}$.

$r=u\cdot t^n \longmapsto n.:=\text{ord}_p^F(r)$.

② $L=$ tangent line $\Leftrightarrow \text{ord}_p F(L)\geq 2$.

$L\neq$ tangent line $\Leftrightarrow \text{ord}_p F(L)=1$.

L 不存在 (L 必须过 p) $\Leftarrow \text{ord}_p F(L)<1$.

日期: /

§ 交点

想法: · F&G intersect properly at P if F,G无common component through P.
→希望定义为有限数.

· F&G intersect transversally at P if P is simple at F&G with 不同 tangent line.
↳希望定义为1.

⇓

公理: ① $I(P, F\cap G)=\infty \Leftrightarrow$ F&G don't intersect properly.

② $I(P, F\cap G)=0 \Leftrightarrow P\notin F$ 或 $P\notin G$.

③ $I(P, F\cap G)=I(P, G\cap F)$.

④ $I(T^{-1}(P), F^T\cap G^T)=I(P, F\cap G)$.

⑤ $I(P, F\cap G)\geq m_p(F)\cdot m_p(G)$. 有冗余 → $I(P, X\cap Y)=1$.

(等号成立 ⇔ F&G没有 common tangent line.)

⑥ $I(P, \prod_i F_i \cap \prod_j G_j)=\sum_{i,j} I(P, F_i\cap G_j)$.

⑦ $I(P, F\cap G)=I(P, F\cap(G+AF))$. $\forall A\in k[X,Y]$. → 可用于作带余除法.

由此定义出的$I(P, F\cap G)$是否存在?是否唯一?

Thm: 1) $\exists!\ I(P, F\cap G)$ satisfy ①~⑦公理.

2) $I(P, F\cap G)=\dim_k \mathcal{O}_P(\mathbb{A}^2)/(F,G)$.

Pf: 1) (也是计算时的过程).

假设存在,证明唯一性) Consider: 假设$\exists I(P, F\cap G)$, 验证可由①~⑦确定值.

$I(P, F\cap G)=\infty \Leftrightarrow$ F,G有 common component. (可计算).

日期: $P \overset{④}{\rightsquigarrow} P(0,0)$.

若无，则 $\underbrace{I(P, F\cap G)}_{:=n} < \infty$. 归纳:

$n=0$ 时，可由②计算.

Assume $n<m$ 时，可被计算

则对 $n=m$, $\deg(F(x,0))=r$. $\deg(G(x,0))=s$.

不妨设 $r\le s$, 1° $r=0$. $\Rightarrow Y\mid F$

可被计算
↑⑤⑥⑦

$$\overset{⑥}{\Rightarrow} \underbrace{I(P, F\cap G)}_{=n} = \underbrace{I(P, Y\cap G)}_{\ge 1} + \underbrace{I(P, (F/Y)\cap G)}_{<n}.$$

可被计算

2° $r\ne 0$. $\overset{③⑦}{\Rightarrow}$ 辗转相除.

$\Rightarrow F', G'$, s.t. $r'=0$. $\rightsquigarrow 1^\circ$

*(Rmk: 是否可由①~⑦计算出相交值 ∈ 存在性范畴. 在存在性基础上,还可证明唯一性)

2) 验证 $\dim_k \mathcal{O}_P(\mathbb{A}^2)/(F,G)$ 满足①~⑦.

②③④⑦显.

Pf-① $H := \gcd(F,G)$.

F, G don't intersect properly $\Leftrightarrow H(P)=0$.

要证 $H(P)=0 \Leftrightarrow \dim_k \mathcal{O}/(F,G) = \infty$.

$\Rightarrow$:) $(F,G)\subset(H) \Rightarrow \dim_k \mathcal{O}/(F,G) \ge \dim_k \mathcal{O}/(H) \ge \dim_k (k[X,Y]/(H))$

日期: /

令H_1为H的不可约因子(削减域).
不妨设$H_1 \notin k[X]$

又由 $\underset{\dim_k=\infty}{k[X]} \hookrightarrow k[X,Y] \twoheadrightarrow \underset{\dim_k=\infty}{k[X,Y]/(H_1)} \twoheadleftarrow k[X,Y]/(H)$. $\rightsquigarrow$ $\dim_k=\infty$.

$\rightsquigarrow \dim_k \mathcal{O}/(F,G)=\infty$.

($\Leftarrow$:) 假设$H(P)\neq 0$. ($\Leftrightarrow H\in\mathcal{O}^\times$).

$\Rightarrow \quad \mathcal{O}/(F,G)=\mathcal{O}/(F/H,G/H) \xleftarrow[?]{CRT} k[X,Y]/(F/H,G/H)$.

由 $P\in V(F/H,G/H)=\text{finite} \Rightarrow \dim_k k[X,Y]/(F/H,G/H)=\text{finite}$.

$\Rightarrow I(P,F\cap G)<\infty$.

Pf-⑥: 只要证: $I(P,F\cap GH)=I(P,F\cap G)+I(P,F\cap H)$.

Assume $\gcd(F,GH)=1$. (若否, ∞)

$$0\to\mathcal{O}/(F,H)\xrightarrow{\times G}\mathcal{O}/(F,GH)\to\mathcal{O}/(F,G)\to 0.$$
$$\mathrm{St}(F,GH)\mapsto \mathrm{St}(F,G).$$

易验证 exact.

从而 $\dim_k\mathcal{O}/(F,GH)=\dim_k\mathcal{O}/(F,G)+\dim_k\mathcal{O}/(F,H)$.

$\Leftrightarrow I(P,F\cap GH)=I(P,F\cap G)+I(P,F\cap H)$.

Pf-⑤: $\mathcal{O}/(F,G)\twoheadrightarrow\mathcal{O}/(I^{m+n},F,G)$. $m=m_P(F)$, $n=m_P(G)$. $I=(X,Y)$.

$V(I^{m+n},F,G)\subseteq V(I^{m+n})=V(I)=\{(0,0)\}=\{P\}$.

考虑利用正合列转化为$k[X,Y]/I^r$的计算.

日期: /

$$\begin{tikzcd} \mathcal{O}/(F,G) \arrow[r, two heads, "\varphi"] & \mathcal{O}/(I^{m+n},F,G) \\ & k[X,Y]/(I^{m+n},F,G) \arrow[u, "\cong"', "CRT"] \end{tikzcd}$$

$$0 \to \ker\psi \to k[X,Y]/I^n \times k[X,Y]/I^m \xrightarrow{\psi} k[X,Y]/I^{m+n} \twoheadrightarrow k[X,Y]/(I^{m+n},F,G) \to 0.$$ 易验证exact.

$(A,B) \longmapsto AF+BG$

若 $A\in I^n$, $\rightsquigarrow AF\in I^{m+n}$.

若 $B\in I^n$, $\rightsquigarrow BG\in I^{m+n}$

$$\rightsquigarrow I(P,F\cap G) = \dim_k \mathcal{O}/(F,G) \geq \dim_k \mathcal{O}/(I^{m+n},F,G)$$
$$= \dim_k k[X,Y]/(I^{m+n},F,G)$$
$$= \dim_k [X,Y]/I^{m+n} - \dim_k [X,Y]/I^m - \dim_k [X,Y]/I^n + \dim\ker(\psi).$$
$$\geq \dim_k [X,Y]/I^{m+n} - \dim_k [X,Y]/I^m - \dim_k [X,Y]/I^n$$
$$= \frac{(m+n)(m+n+1)}{2} - \frac{m(m+1)}{2} - \frac{n(n+1)}{2}$$
$$= mn.$$

$$\text{"="} \iff \begin{cases} \varphi = \text{isomorphism} \iff (F,G) = (I^{m+n},F,G) \iff I^{m+n} \subseteq (F,G) \triangleleft \mathcal{O} \\ \psi = \text{injective} \end{cases}$$

1) F & G have no common tangent line at P. $\Rightarrow I^t \subset (F,G) \triangleleft \mathcal{O}$. $\boxed{\forall t \geq m+n-1}$

pf: lem: 若 $\left.\begin{matrix} L_1, L_2, \cdots \\ M_1, M_2, \cdots \end{matrix}\right\}$ forms of deg 1.
$L_i \neq M_j$.

则

$$\begin{matrix} & 1 & \\ & L_1 \quad M_1 & \\ & L_1L_2 \quad L_1M_1 \quad M_1M_2 & \\ & \vdots & \\ I^t & L_1L_2\cdots L_t \quad L_1\cdots L_{t-1}M_1 \quad \cdots \quad M_1M_2\cdots M_t & \end{matrix}$$

forms a basis of $k[X,Y]$.

日期:　　　/

于是考虑 $F=F_m+F_{m+1}+\cdots,\ F_m=L_1\cdots L_m.\quad L_{m+1}=L_{m+2}=\cdots:=L_m.$

$G=G_n+G_{n+1}+\cdots,\ G_n=M_1\cdots M_n.\quad M_{n+1}=M_{n+2}=\cdots:=M_n.$

F,G 无 common 切线 $\Rightarrow \gcd(F_m,G_n)=1\Rightarrow L_i\neq M_j.$

对 $\forall\ i+j=t\geq m+n-1\Rightarrow i\geq m$ 或 $j\geq n.$

assume $i\geq m$: $L_1\cdots L_i\cdot M_1\cdots M_j=F_m\cdot H.$ $\hookrightarrow$ form (deg $t-m$)

$\therefore\ F_m H=F\cdot H+(F_m-F)\cdot H.$

$\rightsquigarrow F_m H\in(F,G)\Leftrightarrow \underbrace{(F_m-F)}_{\in I^{m+1}}\underbrace{H}_{\in I^{t-m}}\in(F,G)\Leftarrow I^{t+1}\subseteq(F,G).$

$\rightsquigarrow\ I^t\subseteq(F,G)\Leftarrow I^{t+1}\subseteq(F,G)\Leftarrow\cdots\Leftarrow I^N\subseteq(F,G)$

$\therefore$ 只要证 $\exists$ 充分大 N s.t. $I^N\subseteq(F,G)\lhd\mathcal{O}.$

$V(F,G)=$ finite $\Rightarrow V(F,G)=\{P_1,P_2,\cdots,P_r\}.\ (P=(0,0)).$

$\therefore\exists\ H\in k[X,Y]$ s.t. $H(P)\neq0\quad H(P_i)=0.$

$\Rightarrow HX,HY\in I(V(F,G))=\sqrt{(F,G)}.$

$\Rightarrow\exists\ N_1\gg0$ s.t. $(HX)^{N_1},(HY)^{N_1}\in(F,G)\lhd k[X,Y].$

又 $H(P)\neq0\Rightarrow H\in\mathcal{O}_P(\mathbb{A}^2)^\times\Rightarrow\begin{cases}(HX)=(X)\lhd\mathcal{O}.\\(HY)=(Y)\lhd\mathcal{O}.\end{cases}$

$\Rightarrow X^{N_1},Y^{N_1}\in(F,G)\lhd\mathcal{O}.$

$\Rightarrow N:=2N_1,\quad (X,Y)^N\subseteq(F,G)\lhd\mathcal{O}.$ #.

2) F & G have no common tangent line at $P\Leftrightarrow\psi=$ injective.

---

pf: $\psi: k[X,Y]/I^m\times k[X,Y]/I^n\xrightarrow{\times(F,G)}k[X,Y]/I^{m+n}.$

$(\bar{A},\bar{B})\longmapsto\overline{AF+BG}$

若 $\underset{\sim}{AF+BG}\in I^{m+n}$

$=(A_r+A_{r+1}+\cdots)(F_m+F_{m+1}+\cdots)+(B_s+B_{s+1}+\cdots)(G_n+G_{n+1}+\cdots).$

最低项: 1° $A_rF_m+B_sG_n$ $(r+m=s+n)$.

2° A_rF_m或B_sG_n.

1° Assume $r+m=s+n<m+n$. $\Rightarrow r<n$ & $s<m$.

$\Rightarrow A_rF_m+B_sG_n=0$. (否则 $AF+BG\notin I^{m+n}$).

$\Rightarrow F_m|B_s$ & $G_n|A_r$ $\Rightarrow m\le s$ & $n\le r$. 矛盾.

$\therefore$ $r+m=s+n\ge m+n$. $\Rightarrow A\in I^n$, $B\in I^m$.

2° 最低项$=0$ $\Rightarrow \min(r+m, s+n)\ge m+n$. $\Rightarrow r\ge n$ & $s\ge m$.

$\Rightarrow A\in I^n, B\in I^m$.

$\Rightarrow (\bar{A},\bar{B})=(\bar{0},\bar{0})$ $\Rightarrow$ $\psi=$ injective.

反之: 若 F & G 在P处有common 切线.

$\Rightarrow \exists$ L s.t. $L\,|\gcd(F_m,G_n)$.

deg=1 form.

$\psi(\frac{G_n}{L}F, -\frac{F_m}{L}G)=(\frac{G_n}{L}F-\frac{F_m}{L}G) \bmod I^{m+n}$

$=(\frac{G_n}{L}F_m-\frac{F_m}{L}G_n)(\bmod I^{m+n})$

$=0$.

$\Rightarrow$ $0\ne(\frac{G_n}{L}F, -\frac{F_m}{L}G)\in \mathrm{Ker}(\psi)$. $\Rightarrow \mathrm{Ker}(\psi)\ne 0$. #.

Props: (相交数的性质).

(1) $p=$ simple pt on F $\Rightarrow$ $I(P, F\cap G)=\mathrm{ord}_p^F(G)$.

(2) $\gcd(F,G)=1$. $\Rightarrow \sum\limits_{P\in F\cap G} I(P,F\cap G)=\dim_k k[X,Y]/(F,G)$.

Pf: (1) $I(P,F\cap G)=\dim_k \mathcal{O}_p(\mathbb{A}^2)/(F,G)$.

$=\dim_k \mathcal{O}_p(\mathbb{A}^2)/(F) \big/ (F,G)/(F)$.

日期: /

$= \dim_k \mathcal{O}_P(F)/(g)$. $g \equiv G \pmod F$.

由 $\mathcal{O}_P(F)$ 为 DVR. $\therefore g = u \cdot t^{\mathrm{ord}_P^F(G)}$.

$= \dim_k(\mathcal{O}/t^{\mathrm{ord}_P^F(G)})$

$= \mathrm{ord}_P^F(G)$.

$\hookrightarrow 1, t, t^2, \cdots, t^{\mathrm{ord}_P^F(G)-1}$ 为基.

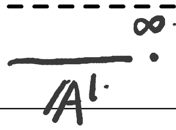
(2) $\gcd(F,G)=1 \Rightarrow \dim_k k[X,Y]/(F,G) < \infty$

$\& \ k[X,Y]/(F,G) \overset{CRT}{\simeq} \bigoplus_{P \in V(F,G)} \mathcal{O}_P(\mathbb{A}^2)/(F,G)$.

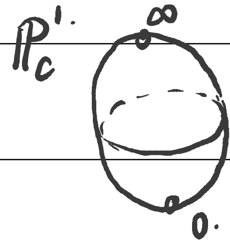
$\Rightarrow \dim_k k[X,Y]/(F,G) = \sum_{P \in V(F,G)} \mathcal{O}_P(\mathbb{A}^2)/(F,G) = \sum_{P \in V(F,G)} I(P, F \cap G)$.

日期: /

Projective Space

$\mathbb{P}^1$: 

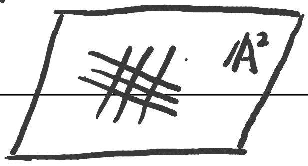
即 $\mathbb{A}^1 \cup \{\infty\}$.



拓扑意义下: 在映射 $\mathbb{A}^1\setminus\{0\} \xleftrightarrow{1:1} \mathbb{A}^1\setminus\{0\}$ 下,

$$z \longmapsto \frac{1}{z}$$

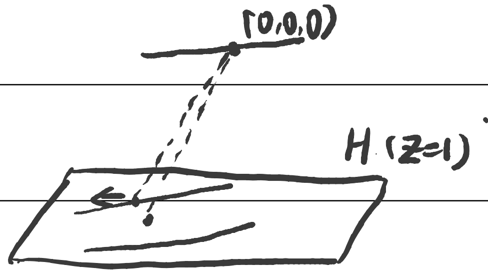
粘连 $\mathbb{A}^1 \cdot \mathbb{A}^1 = \mathbb{P}^1$.

$\mathbb{P}^2$: 

\+ 各方向无穷远点

拓扑意义: S^2 对径相粘连.

平试化的理解: $\mathbb{A}^3$.



$\mathbb{A}^2 \xrightarrow{1:1} H \hookrightarrow \{\text{lines in } \mathbb{A}^3 \text{ through } (0,0,0)\} = \mathbb{P}^2$.

Def: $\mathbb{P}^n := \{\text{lines in } \mathbb{A}^{n+1} \text{ through } (0,0,\dots,0)\}$. (Def1)

Consider: $\mathbb{A}^{n+1}\setminus\{(0,\dots,0)\} \xrightarrow{\varphi} \mathbb{P}^n$

$P \longmapsto l: OP$.

又 $\varphi(P)=\varphi(Q) \Longleftrightarrow Q=\lambda P$.

∴记 $\sim$: $P\sim Q \Longleftrightarrow Q=\lambda P$, $\lambda\in k^*$

$\Rightarrow \mathbb{P}^n \simeq (\mathbb{A}^{n+1}\setminus\{(0,\dots,0)\})/\sim$ (Def2).

$l \longmapsto [x_1:\cdots:x_{n+1}] := \overline{(\lambda x_1,\dots,\lambda x_{n+1})}$.

↓ denote as.

齐次坐标 (homog coordinate of l).

日期: /

易有 $\mathbb{A}^n \xrightarrow{1:1} H := \{(x_1,\cdots,x_{n+1}) \mid x_{n+1}=1\} \hookrightarrow \mathbb{P}^n$.

consider $\mathbb{P}^n \setminus \mathbb{A}^n := \{$ lines through $(0,0,\cdots,0)$ 平行于 $\mathbb{A}^n\}$.

$\simeq \{$lines through $(0,\cdots,0) \in \mathbb{A}^{n+1}\}. \simeq \mathbb{P}^{n-1}$

$\rightsquigarrow \underbrace{\mathbb{P}^n \simeq \mathbb{A}^n \cup \mathbb{P}^{n-1}}$ (Def3).

记 $\varphi_i: \mathbb{A}^n \xrightarrow{1:1} H_i\{(x_1,\cdots,x_{i-1},1,x_{i+1},\cdots,x_n)\}$.

$$\begin{tikzcd} \mathbb{A}^n \arrow[r, "1:1"] \arrow[dr, bend right, "\varphi_i"'] & H_i \arrow[d, hook] \\ & \mathbb{P}^n \end{tikzcd}$$

记 $U_i := \mathrm{Im}\,\varphi_i$. 则 $\underbrace{\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i}$ (Def4).

记 $H_\infty := \mathbb{P}^n \setminus U_{n+1}$.

Def: $P = [x_1,\cdots,x_{n+1}] \in \mathbb{P}^{n+1}$ is called a zero of $F \in k[x_1,\cdots,x_{n+1}]$. if

$F(\lambda x_1,\cdots,\lambda x_{n+1}) = 0.\ \forall \lambda \in k^*$.

denote as $F(P)=0$.

从而类同 $\mathbb{A}^n$ 中有,

① $V(F) := V_p(F) := \{P \in \mathbb{P}^n \mid F(P)=0\}$.

$V(S) := V_P(S) := \bigcap_{F\in S} V(F)$. (projective alg. set).

② $\forall X \subset P^n$. $I(X) := I_p(X) := \{F \in k[x_1,\cdots,x_{n+1}] \mid F(P)=0.\ \forall P \in X\}$.

Consider: $\forall F = F_r + F_{r+1} + \cdots + F_d$. ($F_i :=$ form of deg i).

$F(P)=0 \Leftrightarrow F(\lambda x_1,\cdots,\lambda x_{n+1}) = 0.\ \forall \lambda \in k^*$

日期: /

$\Leftrightarrow \lambda^r F_r(x_1,\cdots,x_{n+1})+\cdots+\lambda^d F_d(x_1,\cdots,x_{n+1})=0.\ \forall \lambda\in k^*$

$\Leftrightarrow F_r(P)=\cdots=F_d(P)=0. \rightsquigarrow V(F)=\bigcap_{i=r}^{d} V(F_i)$

($\rightsquigarrow G=$ homog. $P=[x_1,\cdots,x_{n+1}]$. $G(P)=0 \Leftrightarrow G(x_1,\cdots,x_{n+1})=0.$)

---

def: $I \triangleleft k[x_1,\cdots,x_{n+1}]$ is called homogenous if

$\forall F=F_0+\cdots+F_m\in I \Rightarrow F_i\in I,\ \forall i.$

i.e. $I=I_0\oplus\cdots\oplus I_m$, $I_i:=\{\text{al form of deg } i \in I\}$.

Fact: I homog. $\Leftrightarrow$ I is generated by a set of homog. poly.

def: $\langle S\rangle :=$ minimal homg. ideal containing S.

Prop 1: $I(X)$. homogenous & radical.

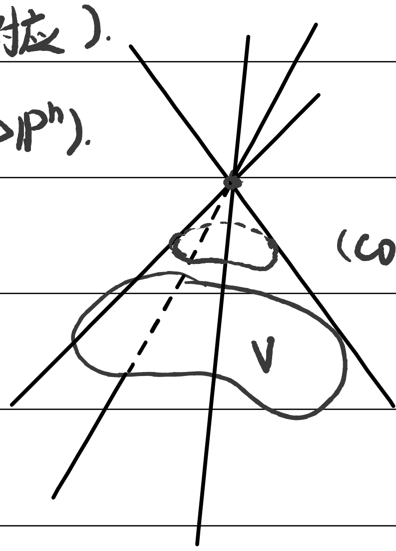
Prop 2: $V(S)=V((S))=V(\langle S\rangle)$.

$\rightsquigarrow \boxed{\{\text{homog.\&radical ideals}\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{proj. alg. set}\}}$

(Rmk: 在 affine 中. if $k=\bar{k}$, 则 V, I 为 1:1 对应).

Consider: $\forall\ V=$ proj. alg. set ($\pi: \mathbb{A}^{n+1}\setminus\{(0,\cdots,0)\} \twoheadrightarrow \mathbb{P}^n$).

def $C(V):=\pi^{-1}(V)\cup\{(0,\cdots,0)\}$.



Prop 1: $V\neq\emptyset \Rightarrow I_p(V)=I_a(C(V))$.

$I=$ homog. $\Rightarrow C(V_p(I))=V_a(I)$.
$V_p(I)\neq\emptyset$.

(Rmk: 即 $C(-)$ 与 $V_a(-)$ 或 $I_a(-)$ 可交换).

日期: /

Thm (Proj. Nullstellsatz): $k=\bar{k}$, $I \trianglelefteq k[x_1,\cdots,x_{n+1}]$ homog. then.

1) $V_p(I)=\emptyset \Leftrightarrow I$ contains all forms of $\deg \geq N$ for some N.

$\Leftrightarrow m^N \subset I$ for some N. ($m = I_a(0,\cdots,0)$).

2) $V_p(I) \neq \emptyset \Rightarrow I_p \circ V_p(I) = \sqrt{I}$.

Pf: 1) $V_p(I)=\emptyset \overset{\text{def4.}}{\Leftrightarrow} V_a(I) \subseteq \{(0,0,\cdots,0)\}$.

$\Leftrightarrow I_a \circ V_a(I) \supseteq I_a(\{(0,\cdots,0)\}) = m$.

$= \sqrt{I}$ ($k=\bar{k}$).

$\underset{\text{Noeth.}}{\Leftrightarrow} \exists N \gg 0$, s.t. $m^N \subseteq I$.

2). $I_p \circ V_p(I) = I_a(C(V_p(I))) = I_a \circ V_a(I) = \sqrt{I}$.

Cor: $\{\text{radical homog. ideals} \neq m\} \xleftrightarrow{1:1} \{\text{proj. alg. sets}\}$.

Def: proj. alg. set V is called irreducible, if:

$V = V_1 \cup V_2 \Rightarrow V = V_1$ or $V = V_2$.

- Proj. variety := Irreducible proj. alg. set

Fact: 1) $V \neq \emptyset$. V = irr. $\Leftrightarrow I(V)$ = prime

2) $V \overset{\exists !}{=} \cup V_i$. V_i = irr. $V_i \nsubseteq V_j$.

irr. component

Cor: $\{\text{homog. prime ideals} \neq m\} \xleftrightarrow{1:1} \{\text{proj. varieties}\}$

日期: /

Consider $F \in k[x_1, \cdots, x_{n+1}]$. $F = F_r + F_{r+1} + \cdots + F_d$.

$$V(F) = \bigcap_{i=r}^{d} V(F_i).$$

$V(F)$ is called projective hypersuface, if F is homog.

and: projective hyperplane, if $F =$ deg 1 form.

e.g. $x_1, \cdots, x_{n+1}$ $\quad V_p(x_i)$ coordinate hyperplane.

$$\bigcap_{i=1}^{n+1} V_p(x_i) = \varnothing. \quad 而 \bigcap_{i=1}^{n+1} V_a(x_i) = \{(0, \cdots, 0)\}.$$

坐标环:

proj. $V \subseteq \mathbb{P}^n$. irr.

define $\Gamma_p(V) := k[x_1, \cdots, x_{n+1}] / I(V)$.

$k_p(V) := \mathrm{Frac}(\Gamma_p(V))$.

Rmk: $\forall f \in \Gamma_p(V)$, f is NOT a function on V. $\rightsquigarrow$ $\forall f \in k_p(V)$, f is NOT a rational fun. on V.

即 $\begin{array}{l} V \to k \\ P \mapsto f(P) \end{array}$ 非良定义.

这与affine中大为不同.

Consider 描述 $\mathbb{P}^n$上函数:

F & G are homog. poly. with same deg.

$$\Rightarrow \frac{F}{G} : \mathbb{P}^n \setminus V(G) \longrightarrow k$$

$$[x_1 : \cdots : x_{n+1}] \longmapsto \frac{F(x_1, \cdots, x_{n+1})}{G(x_1, \cdots, x_{n+1})}.$$

日期: /

易验证 well-defined.

Define $k(\mathbb{P}^n) := \{\frac{F}{G} \mid F\&G$: homog., same deg$\}$.

$\subseteq k_p(\mathbb{P}^n)$. 为子域

Rmk: rational functions 从 $\mathbb{A}^n$上 到$\mathbb{P}^n$上 的推广.

Q: how to define $k(V)$?

↓

Q: how to define form of deg d on V?

Consider: def: $f \in \Gamma_p(V)$ is a form of deg d, if

$$f = F \bmod I(V).$$

for some F form of deg d.

Fact1: $\forall f \in \Gamma_h(V)$. $f \overset{\exists!}{=} f_0 + f_1 + \cdots + f_d$ f_i = form of deg i.

Pf: 存在.易.

唯一: 假设 $f = f_0 + f_1 + \cdots + f_d = f_0' + f_1' + \cdots + f_d'$.

$\Rightarrow (f_0 - f_0') + \cdots + (f_d - f_d') = 0.$

$\Rightarrow (F_0 - F_0') + \cdots + (F_d - F_d') \in I(V)$

$\Rightarrow F_i - F_i' \in I(V)$ $\forall i$. $\Rightarrow f_i = f_i'$ $\forall i$.

日期: /

Fact2: $\forall f, g \in \Gamma_h(V)$: forms of the same degree d. Then:

$$\frac{f}{g}: \quad V \setminus \{P \mid g(P)=0\} \longrightarrow k.$$

$$[x_1 : x_2 : \cdots : x_{n+1}] \longmapsto \frac{F(x_1, \cdots, x_{n+1})}{G(x_1, \cdots, x_{n+1})}.$$

is well-defined.

Def: $k(V) := \left\{ \frac{f}{g} \mid f \,\&\, g \text{ forms in } \Gamma_h(V) \text{ of same degree}, g \neq 0 \right\}$.

$\Rightarrow \quad k \subset k(V) \subset k_p(V)$ 为子域

denote as V上的有理函数域

(Rmk: Q: $\Gamma(V) = ?$.

extend from affine, 可定义$\Gamma(V)$为$\{ f \in k(V) \mid f \text{ is defined everywhere} \}$.

but, $\mathbb{P}^n$上如此定义的$\Gamma(V)$较小, 有$\Gamma(V) \simeq k$.)

eg. $V = \mathbb{P}^1 \Rightarrow \Gamma_p(V) = k[X, Y]$, $k_p(V) = k(X, Y)$.

由$k(V) = \left\{ \frac{F}{G} \mid F \,\&\, G \text{ has same deg in } \Gamma_h(V) \right\}$.

而 $\deg F = d = \deg G$.

$$\rightarrow \frac{F}{G} = \frac{F/Y^d}{G/Y^d} = \frac{F(\frac{X}{Y}, \frac{Y}{Y})}{G(\frac{X}{Y}, \frac{Y}{Y})} \overset{x := \frac{X}{Y}}{=\!=} \frac{F(x,1)}{G(x,1)} = \frac{F_*(x)}{G_*(x)} = \left(\frac{F_*}{G_*}\right)(x)$$

注意: 这里F_*, G_*可任取.

$\rightsquigarrow k(\mathbb{P}^1) = k(x)$.

日期: /

局部环:

$\mathcal{O}_p(V) := \{ f \in k(V) \mid f \text{ is defined at } P \}$.

Fact: $\mathcal{O}_p(V)$ is local ring. with maximal ideal

$m_p(V) = \{ \frac{f}{g} \mid f, g : \text{forms of the same degree}, g(P) \neq 0 . f(P) = 0 \}$.

坐标变换:

仿射情形下: $\mathbb{A}^{n+1} \xrightarrow{T} \mathbb{A}^{n+1}$. $T = (T_1, T_2, \cdots, T_{n+1})$. T为 change of coordinate

$\Leftrightarrow \deg(T_i) = 1$ 且 T invertible.

射影情形:

$$\begin{tikzcd} \mathbb{A}^{n+1} \setminus \{0\} \arrow[r, "T"] \arrow[d] & \mathbb{A}^{n+1} \setminus \{0\} \arrow[d] \\ \mathbb{P}^n \arrow[r, dashed, "\exists ! T"] & \mathbb{P}^n \end{tikzcd}$$

iff $F(0) = 0$.

($\Rightarrow$ F线性 $\Rightarrow F(\lambda Q) = \lambda F(Q)$).

$\Rightarrow$ T (in projective) well-defined

projective change of coordinate.

Fact: 1) $V \subseteq \mathbb{P}^n$ alg set. $\Leftrightarrow V^T$ alg set.

2) V variety $\Leftrightarrow V^T$ variety.

3) 在各类坐标环上有诱导的映射:

$\tilde{T}: \Gamma_p(V) \xrightarrow{\sim} \Gamma_p(V^T)$.

$\tilde{T}: k_p(V) \xrightarrow{\sim} k_p(V^T)$.

$\tilde{T}: k(V) \xrightarrow{\sim} k(V^T)$.

$\tilde{T}: \mathcal{O}_p(V) \xrightarrow{\sim} \mathcal{O}_{T^{-1}(P)}(V^T)$

日期: /

Affine & Project — Connections.

$$\mathbb{A}^n \xhookrightarrow{\varphi_{n+1}} \mathbb{P}^n.$$

Def: $\mathcal{J} := \{U \subseteq \mathbb{P}^n \mid \mathbb{P}^n \setminus U = \text{proj. alg set}\}$. 为 $\mathbb{P}^n$上的 Zariski top.

Aim: $\mathbb{A}^n \overset{\varphi_{n+1}}{\simeq} U_{n+1} \subseteq \mathbb{P}^n$, $U_{n+1} = \mathbb{P}^n \setminus V(X_{n+1})$.

Prop1: $\forall$ open $U \subseteq X$,

$$\{W \subseteq U \mid W \text{ close in } U\} \overset{(\cdot)}{\rightleftharpoons} \{Z \subseteq X \mid Z \text{ close in } X\}.$$ 良定义但非一一映射.

$$W \longmapsto \bar{W}.$$
$$Z \cap U \longleftarrow Z.$$ 非单

Aim: $\{\text{affine alg. set in } \mathbb{A}^n\} \underset{(\cdot)_*}{\overset{(\cdot)^*}{\rightleftharpoons}} \{\text{proj. alg. set. in } \mathbb{P}^n\}$.

$(\cdot)^*$: $\forall\, V = \text{alg. set in } \mathbb{A}^n$. def. $I^* = \langle f^* \mid f \in I(V) \rangle$.

$\Rightarrow I^* \trianglelefteq k[x_1, \cdots x_{n+1}]$ 且 homog.

则 def $V^* = V_p(I^*) \subseteq \mathbb{P}^n$.

Prop2: $V, W \subseteq \mathbb{A}^n$ alg sets. Then:

1) $V^* =$ 最小 proj. alg. set in $\mathbb{P}^n$ 包含 $\varphi_{n+1}(V)$.

对应于 Prop1.

(i.e. V^* is the closure of $\varphi_{n+1}(V)$ in $\mathbb{P}^n$ under Zariski top.)

2) $\varphi_{n+1}(V) = V^* \cap U_{n+1}$

3) $V \subseteq W \Rightarrow V^* \subseteq W^*$

4) $V = \text{irr} \Rightarrow V^* = \text{irr}$.

5) $V = \bigcup V_i$ (irr.) $\Rightarrow V^* = \bigcup V_i^*$ is irr. decomposition.

日期: /

(6) 若 $\emptyset \neq V = \bigcup V_i$ (irr.) $\subsetneq \mathbb{A}^n \Rightarrow V_i^* \not\subseteq H_\infty$ 且 $V_i^* \not\subseteq H_\infty$.

Pf: 2). $V^* \cap U_{n+1} = \{ P = [x_1 : x_2 : \cdots : x_n : 1] \in \mathbb{P}^n \mid f^*(P) = 0, \forall f \in I(V) \}$.

$= \{ P = [x_1 : x_2 : \cdots : x_n : 1] \in \mathbb{P}^n \mid f(x_1, \cdots, x_n) = 0 . \forall f \in I(V) \}$.

$= \{ P = [x_1 : x_2 : \cdots : x_n : 1] \in \mathbb{P}^n \mid (x_1, \cdots, x_n) \in V \}$.

$= \varphi_{n+1}(V)$.

3) $V \subseteq W \Leftrightarrow I(W) \subseteq I(V) \Rightarrow I^*(W) \subseteq I^*(V) \xRightarrow{V_p} V^* \subseteq W^*$

4) $V = irr. \Leftrightarrow I(V) = prime \Rightarrow I^*(V) = prime \Leftrightarrow V^* = irr.$

$\Updownarrow$

$$\boxed{\begin{array}{l}(\forall F, G \in k[x_1, \cdots, x_{n+1}]. \text{ homog. } F \cdot G \in I^*(V) \Rightarrow F \text{ or } G \in I^*(V)). \\ \Rightarrow x_{n+1}^{\alpha} (F_*)^* \cdot x_{n+1}^{\beta} (G_*)^* \in I^*(V)\end{array}}$$

1) $\varphi_{n+1}(V) \subseteq V^*$ 由(2) $\checkmark$.

∴要证 $\forall Z$ alg set st. $\varphi_{n+1}(V) \subseteq Z$, 有 $V^* \subseteq Z$.

$\Leftrightarrow I(Z) \subseteq I^*(V)$.

$\forall F \in I(Z) . \Rightarrow \forall P = (x_1, \cdots, x_n) \in V$,

$\varphi_{n+1}(P) \subset Z$.

$\Rightarrow F_*(x_1, \cdots, x_n) = F(x_1, \cdots, x_n, 1) = 0$.

$\Rightarrow F_* \in I(V)$.

$\Rightarrow F = x_{n+1}^r (F_*)^* \in I^*(V)$. #.

5) V_i irr. $\xRightarrow{4)} V_i^*$ irr.

日期: /

$V=\bigcup_i V_i \overset{1)}{\Rightarrow} V^*=\bigcup_i V_i^*$. $\Rightarrow V^*=\bigcup V_i^*$ is irr. decomposition.

$V_i\nsubseteq V_j \overset{2)}{\Rightarrow} V_i^*\nsubseteq V_j$.

6) $\emptyset\neq V=\bigcup_i V_i\subsetneq \mathbb{A}^n \Rightarrow V_i^*\nsubseteq H_\infty$ 且 $V_i^*\nsupseteq H_\infty$

suppose: $H_\infty\subseteq V_i^*$.

$\Rightarrow I^*(V_i)\subseteq I_p\circ V_p(I^*(V_i))\subseteq I_p(H_\infty)=(x_{n+1})$.

而 $V_i\neq\emptyset \Rightarrow I(V_i)\neq 0 \Rightarrow \exists F\in I(V_i)\setminus\{0\}$.

$\Rightarrow F^*=x_{n+1}^{\deg F}F(\frac{x_1}{x_{n+1}},\cdots,\frac{x_n}{x_{n+1}})\notin(x_{n+1})$.

$(\cdot)_*$: $\forall V=$ alg set in $\mathbb{P}^n$. def. $I_*=\langle F_*\mid \text{homog. } F\in I\rangle$.

则 def $V_*:=V(I_*)\subseteq\mathbb{A}^n$.

Prop: $V,W\subseteq\mathbb{P}^n$ alg set. Then:

1) $V\subseteq W\Rightarrow V_*\subseteq W_*$.

2) $V=\bigcup V_i\subseteq\mathbb{P}^n$ 且 $V_i\nsubseteq H_\infty$, $H_\infty\nsubseteq V_i$. 则:

$\emptyset\neq V_*\subsetneq\mathbb{A}^n$ 且 $(V_*)^*=V$.

3) $\boxed{Z\subseteq\mathbb{A}^n \text{ alg set} \Rightarrow (Z^*)_*=Z}$.

Pf: 1) $V\subseteq W\Rightarrow I(V)\supseteq I(W)$.

$\Rightarrow I(V)_*\supseteq I(W)_*\Rightarrow V_*\subseteq W_*$.

2) 不妨设 V irr.

日期: /

$\varphi_{n+1}(V_*) \subset V \Leftarrow \forall [x_1 : x_2 : \cdots : x_n : 1] \in \varphi_{n+1}(V_*) \subseteq \mathbb{P}^n$.

$\Leftrightarrow (x_1, \cdots, x_n) \in V_* = V(F_*)$. $F \in I(V)$ homog.

$\Leftrightarrow F_*(x_1, \cdots, x_n) = 0$. $\forall F \in I(V)$. homog.

$\Leftrightarrow F(x_1, \cdots, x_n, 1) = 0$. $\forall F \in I(V)$. homog.

$\Leftrightarrow [x_1, \cdots, x_n : 1] \in V$.

$\therefore$ 由 $(V_*)^* =$ minimal alg. set. 包含 $V_* \supseteq \varphi_{n+1}(V_*)$

$\Rightarrow (V_*)^* \subseteq V$. $(\Leftarrow \varphi_{n+1}(V_*) \subset V)$.

下证 $V \subseteq (V_*)^* \Leftrightarrow I(V_*)^* \subseteq I(V)$.

$\forall F \in I(V_*) = I(V(I(V)_*)) = \sqrt{I(V)_*}$.

$\Rightarrow \exists N \gg 0$. s.t. $F^N \in I(V)_*$

$\Rightarrow x_{n+1}^t (F^N)^* \in \overset{irr.}{I(V)} = prime$.

$\Rightarrow (F)^* \in I(V)$. #.