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Subspace Identification of Individual Systems Operating in a Network (SI^{2}ON)

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***Abstract*—This note studies the identification of individual systems operating in a large-scale distributed network by considering the interconnection signals between neighboring systems to be unmeasurable. The unmeasurable interconnections act as unknown system inputs to the individual systems in a network, which poses a challenge for the identification problem. A subspace identification framework is proposed in this note for the consistent identification of individual systems using only local input and output information. The key step of this identification framework is the accurate estimation of the unknown system inputs of individual systems using local observations. Sufficient identifiability conditions are provided for the proposed identification framework and a simulation example is given to demonstrate its performance.**

***Index Terms*—Large-scale distributed network, subspace identification, blind system identification.**

I. Introduction

In this note, we consider the identification of a single system operating in a heterogeneous network using local observations. Examples of the concerned networks are discretized partial-differential-equation (PDE) systems such as fluid mechanics [1], flexible structures [2] and large telescope mirrors [3]. For these networks, the local system state as well as the interconnection signals between neighbouring systems are unmeasurable, resulting in a challenge for the single system identification problem using local measurements only. It is noted that the concerned identification problem is fundamentally different from those parametric network identification methods [4]–[7] for which the interconnections between neighboring systems are assumed to be measurable.

To date, there have been several local identification methods for state-space represented heterogeneous networks. A subspace identification algorithm was developed by approximating the unknown interconnections using a linear combination of local input and output observations [8]. This approach however requires an exhaustive search of all input and output data combinations. In other words, it is still an open problem on the selection of the local observations for the approximation of unknown interconnection signals. In [9], a nuclear norm algorithm was developed to identify a system by exploiting the different rank and order properties between the considered local and global dynamics; however, this approach is not able to identify the interconnections between neighboring systems.

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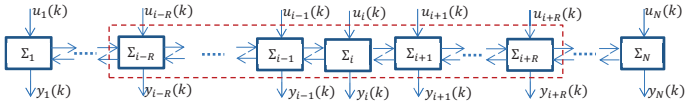

Fig. 1. Demonstration of a 1D network consisting of distinct LTI systems. The local cluster of systems $\{\Sigma_j\}_{j=i-R}^{i+R}$ centered by Σ_i is encircled by a dashed rectangle.

In this note, the *consistent* identification of individual systems operating in a heterogenous network is to be investigated. Key to the proposed identification framework is the accurate estimation of the unknown inputs of the system to be identified. To this end, a subspace-intersection based estimation algorithm is developed and a sufficient condition for the reconstruction of the unknown inputs (up to a similarity transformation) is given. This approach addresses the problem in [8] on the selection of local observations for the approximation of the unknown interconnection signals.

The following notations will be used throughout the note. For a sequence $x(k)$, we denote $\text{Row}\,[x(k)]$ as the row subspace of $[x(k)\ x(k+1)\ \cdots\ x(k+h-1)]$ with h being a positive integer. $\text{Row}\,[x_1(k)\cap x_2(k)]$ and $\text{Row}\,[x_1(k)\cup x_2(k)]$ stand for the intersection and union of $\text{Row}\,[x_1(k)]$ and $\text{Row}\,[x_2(k)]$, respectively.

The structured block matrices that will be used in the note are abbreviated as follows:

$$x_{j_1:j_2}(k)=\left[x_{j_1}^T(k),\cdots,x_{j_2}^T(k)\right]^T \text{ for } j_2\geq j_1.$$

$$x_j(k_1:k_2)=\left[x_j^T(k_1),\cdots,x_j^T(k_2)\right]^T \text{ for } k_2\geq k_1.$$

$$\text{Diag}(D_1,\cdots,D_d)=\begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_d \end{bmatrix}.$$

$$\mathcal{O}_s(C,A)=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}.$$

II. Problem formulation

In order to focus on the essence of the new identification method and for the sake of notational simplicity, the identification of a spatially varying 1D networked system, as shown

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in Fig. 1, will be investigated in detail. For this 1D network model, the local systems Σ_i for $i = 1, \cdots, N$ are represented as

$$\begin{array}{lll} \Sigma_1: & x_1(k+1) & = A_1x_1(k) + A_{1,r}x_2(k) + B_1u_1(k) \\ & y_1(k) & = C_1x_1(k) + e_1(k), \\ \Sigma_i: & x_i(k+1) & = A_ix_i(k) + A_{i,l}x_{i-1}(k) + A_{i,r}x_{i+1}(k) \\ & & \quad + B_iu_i(k) \\ & y_i(k) & = C_ix_i(k) + e_i(k), \\ i = 2, \cdots, N-1 & & \\ \Sigma_N: & x_N(k+1) & = A_Nx_N(k) + A_{N,l}x_{N-1}(k) \\ & & \quad + B_Nu_N(k) \\ & y_N(k) & = C_Nx_N(k) + e_N(k), \end{array} \tag{1}$$

where $x_i(k) \in \mathbb{R}^{n\times 1}$, $u_i(k) \in \mathbb{R}^{m\times 1}$, $y_i(k) \in \mathbb{R}^{p\times 1}$ and $e_i(k) \in \mathbb{R}^{p\times 1}$ are respectively the state, input, output and measurement noise of the system Σ_i. The system matrices $A_i, A_{i,l}, A_{i,r}, B_i$ and C_i have appropriate sizes. For the above network model, we assume that $N \geq n$ and $p, m < n$.

By denoting $u(k) = u_{1:N}(k)$, $x(k) = x_{1:N}(k)$, $y(k) = y_{1:N}(k)$ and $e(k) = e_{1:N}(k)$, the matrix form of (1) can be written as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + e(k), \end{aligned} \tag{2}$$

where $A = \begin{bmatrix} A_1 & A_{1,r} & & \\ A_{2,l} & A_2 & \ddots & \\ & \ddots & \ddots & A_{N-1,r} \\ & & A_{N,l} & A_N \end{bmatrix}$,

$$B = \text{Diag}\left(B_1, B_2, \cdots, B_N\right),$$
$$C = \text{Diag}\left(C_1, C_2, \cdots, C_N\right).$$

The standard assumptions of the network model in (1) are made as follows.

- A1. The global system model in (2) and individual systems in (1) are minimal.
- A2. The input vector $u(k)$ is a quasi-stationary signal and is persistently exciting of any finite order.
- A3. The measurement noise $e(k)$ is a white noise sequence satisfying $e(k) \sim \mathcal{N}(0, \sigma^2 I)$ and is uncorrelated with $u(k)$.

In this note, the *problem of interest* is to identify the system matrices $\{C_i, A_i, A_{i,l}, A_{i,r}, B_i\}$ of the system Σ_i up to a similarity transformation, as defined in Definition 1, using the input and output measurements from a *local cluster* consisting of the systems, $\{\Sigma_j\}_{j=i-R}^{i+R}$, centered by Σ_i with R being the radius of the concerned local cluster. Due to the unmeasurable interconnections between neighboring systems, the local system model has two unknown inputs, causing the identification problem to be quite challenging.

Definition 1. *The system matrix tuple* $\{C_i, A_i, A_{i,l}, A_{i,r}, B_i\}$ *is* similarly equivalent *to* $\{\hat{C}_i, \hat{A}_i, \hat{A}_{i,l}, \hat{A}_{i,r}, \hat{B}_i\}$ *if there exist non-singular matrices* $Q_i, Q_{i-1}, Q_{i+1} \in \mathbb{R}^{n\times n}$ *such that*

$$\hat{A}_i = Q_i^{-1}A_iQ_i, \hat{A}_{i,l} = Q_i^{-1}A_{i,l}Q_{i-1}, \hat{A}_{i,r} = Q_i^{-1}A_{i,r}Q_{i+1},$$
$$\hat{C}_i = C_iQ_i, \hat{B}_i = Q_i^{-1}B_i.$$

The rest of the note is organized as follows. In Section III, given a noisy estimate of the unknown interconnection signal, the identification of a single subsystem is formulated as an errors-in-variables system identification problem, which is then consistently identified using the instrumental-variable method [10]. In order to estimate the unknown interconnection signals, in Section IV, a subspace-based estimation method is developed by taking into account the local network structure.

III. Consistent identification of a single subsystem

We consider the identification of the system matrices, $\{C_i, A_i, A_{i,l}, A_{i,r}, B_i\}$, of the system Σ_i:

$$\begin{aligned} x_i(k+1) &= A_ix_i(k) + [A_{i,l}\ A_{i,r}]\begin{bmatrix} x_{i-1}(k) \\ x_{i+1}(k) \end{bmatrix} + B_iu_i(k) \\ y_i(k) &= C_ix_i(k) + e_i(k). \end{aligned} \tag{3}$$

To identify the above system, it is necessary to estimate the subspace spanned by the sequence of unknown system inputs $x_{i-1}(k)$ and $x_{i+1}(k)$ from the local input and output observations.

Before proceeding to discover knowledge about the unknown system inputs, we provide a subspace identification framework for the local system Σ_i by assuming that the estimates, $\hat{x}_{i\pm1}(k)$, of the unknown inputs, $x_{i\pm1}(k)$, are available and satisfy

$$\begin{aligned} \hat{x}_{i-1}(k) &= \Gamma_lx_{i-1}(k) + v_{i-1}(k) \\ \hat{x}_{i+1}(k) &= \Gamma_rx_{i+1}(k) + v_{i+1}(k), \end{aligned} \tag{4}$$

where $\Gamma_l, \Gamma_r \in \mathbb{R}^{n\times n}$ are unknown but nonsingular square coefficient matrices, and $v_{i\pm1}(k) \in \mathbb{R}^n$ are zero-mean ergodic stochastic processes that are uncorrelated with the input $u_j(k)$ and the state $x_j(k)$ for all $j = 1, \cdots, N$.

Substituting equation (4) into equation (3) yields

$$\begin{aligned} x_i(k+1) &= A_ix_i(k) + B_iu_i(k) + A_{i,lr}\left(\bar{x}_i(k) - \bar{v}_i(k)\right) \\ y_i(k) &= C_ix_i(k) + e_i(k), \end{aligned} \tag{5}$$

where $A_{i,lr} = \left[A_{i,l}\Gamma_l^{-1}\ A_{i,r}\Gamma_r^{-1}\right]$, $\bar{x}_i(k) = \begin{bmatrix} \hat{x}_{i-1}(k) \\ \hat{x}_{i+1}(k) \end{bmatrix}$ and $\bar{v}_i(k) = \begin{bmatrix} v_{i-1}(k) \\ v_{i+1}(k) \end{bmatrix}$. It is noted that the ambiguity matrices Γ_l, Γ_r do not affect the similarity transformation that is defined in Definition 1. Therefore, we will focus on the identification of the system model (5) instead of (3).

The identification of the system Σ_i then boils down to identifying the system matrices $(C_i, A_i, A_{i,lr}, B_i)$ using the values of $u_i(k), \bar{x}_i(k)$ and $y_i(k)$. Due to the noisy input and output measurements, the system model in (5) is an errors-in-variables (EIV) model. In order to achieve a consistent identification of (5), as defined in Definition 2, the instrumental-variable method [10] is adopted in this note.

Definition 2. *Suppose that the true system matrices of* (5) *are* $\{A_i^*, [B_i^*\ A_{i,lr}^*], C_i^*\}$ *and the transfer function is* $H^*(z) = C_i^*(zI - A_i^*)^{-1}[B_i^*\ A_{i,lr}^*]$ *with* $z \in \mathbb{C}$. *The estimates of the system matrices, denoted by* $(\hat{A}_i,\ [\hat{B}_i\ \hat{A}_{i,lr}]\ , \hat{C}_i)$, *are consistent if they satisfy*

$$\hat{C}_i\left(zI - \hat{A}_i\right)^{-1}[\hat{B}_i\ \hat{A}_{i,lr}] = H^*(z)$$

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as the length of input-output data tends to infinity.

As discussed above, in order to achieve a consistent identification of (3), it is crucial to obtain the estimates of the unknown inputs $x_{i\pm1}(k)$ such that equation (4) is satisfied. This will be investigated in the next section.

IV. Subspace information of unknown inputs

In this section, the state $x_{i+1}(k)$ that is one of the unknown inputs to the model in (3) will be estimated using the input-output data of the local systems $\{\Sigma_j\}_{j=i-R+2}^{i+R}$. More explicitly, the sequence $x_{i+1}(k)$ will be estimated as the intersection of $x_{i-R+2:i+1}(k)$ and $x_{i+1:i+R}(k)$. In the sequel, we will subsequently form a data equation, estimate the augmented state sequences $x_{i-R+2:i+1}(k)$ and $x_{i+1:i+R}(k)$, and compute the subspace intersection.

A. Data-equation construction

For the sake of brevity, only the estimation of $x_{i+1:i+R}(k)$ will be investigated in detail. The lifted state-space model of the cluster consisting of the systems $\{\Sigma_j\}_{j=i+1}^{i+R}$ can be written as

$$\begin{aligned}\underline{x}_i(k+1) &= \underline{A}_i\underline{x}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{D}_i\underline{v}_i(k)\\ \underline{y}_i(k) &= \underline{C}_i\underline{x}_i(k) + \underline{e}_i(k),\end{aligned} \tag{6}$$

where $\underline{y}_i(k) = y_{i+1:i+R}(k)$, $\underline{x}_i(k) = x_{i+1:i+R}(k)$, $\underline{u}_i(k) = u_{i+1:i+R}(k)$ and $\underline{e}_i(k) = e_{i+1:i+R}(k)$,

$$\underline{A}_i = \begin{bmatrix} A_{i+1} & A_{i+1,r} & & & \\ A_{i+2,l} & A_{i+2} & A_{i+2,r} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{i+R-1,l} & A_{i+R-1} & A_{i+R-1,r} \\ & & & A_{i+R,l} & A_{i+R} \end{bmatrix}$$

$$\underline{B}_i = \text{Diag}(B_{i+1}, B_{i+2}, \cdots, B_{i+R});$$

$$\underline{C}_i = \text{Diag}(C_{i+1}, C_{i+2}, \cdots, C_{i+R});$$

$$\underline{D}_i = \begin{bmatrix} A_{i+1,l} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & A_{i+R,r} \end{bmatrix} \qquad \underline{v}_i(k) = \begin{bmatrix} x_i(k) \\ x_{i+R+1}(k) \end{bmatrix}.$$

The difficulty of estimating the state $\underline{x}_i(k)$ (or $x_{i+1:i+R}(k)$) of the system (6) lies at the unknown system input $\underline{v}_i(k)$. The transfer function of (6) can be written as

$$\begin{aligned}\underline{y}_i(k) = &\underline{C}_i\left(qI - \underline{A}_i\right)^{-1} q\underline{x}_i(0)\\ &+ \underline{C}_i\left(qI - \underline{A}_i\right)^{-1}\left(\underline{B}_i\underline{u}_i(k) + \underline{D}_i\underline{v}_i(k)\right),\end{aligned} \tag{7}$$

where q represents a forward shift operator. As shown in Lemma 2 of Appendix A, the unknown input $\underline{v}_i(k)$ is unidentifiable in the sense that the row space spanned by the sequence $\underline{v}_i(k)$ cannot be recovered from the measurements of $\underline{u}_i(k)$ and $\underline{y}_i(k)$. Since the state $\underline{x}_i(k)$ for $k > 0$ depends on the input data $\underline{u}_i(\tau)$ and $\underline{v}_i(\tau)$ for all $\tau < k$, the unidentifiability of $\underline{v}_i(k)$ may affect the accurate estimation of the state $\underline{x}_i(k)$.

In order to estimate the state $\underline{x}_i(k)$ in (6) without the influence of the unknown input, the structural property of $\underline{D}_i$ will be used. Remark that the unidentifiable quantity $\underline{v}_i(k)$ only influences the first and last block row of the state equation (6); therefore, we could remove the corresponding first and last block row of (6), turning this state-space model into a time-varying model with its state vector being of varying dimension. This time-varying model is defined as

$$\begin{aligned}\mathbf{x}_i(k_0+t+1) &= \mathscr{A}_{i,t}\mathbf{x}_i(k_0+t) + \mathscr{B}_{i,t}\mathbf{u}_i(k_0+t)\\ \mathbf{y}_i(k_0+t) &= \mathscr{C}_{i,t}\mathbf{x}_i(k_0+t) + \mathbf{e}_i(k_0+t)\end{aligned} \tag{8}$$

where k_0 is a positive time-index which can be chosen arbitrarily by the user; $t = 0, 1, \cdots, \lfloor\frac{R}{2}\rfloor$; the vectors $\mathbf{x}_i(k_0+t)$, $\mathbf{u}_i(k_0+t)$, $\mathbf{y}_i(k_0+t)$ and $\mathbf{e}_i(k_0+t)$ are defined as

$$\begin{aligned}\mathbf{u}_i(k_0+t) &= u_{i+t+2:i+R-1-t}(k_0+t)\\ \mathbf{x}_i(k_0+t) &= x_{i+t+1:i+R-t}(k_0+t)\\ \mathbf{y}_i(k_0+t) &= y_{i+t+1:i+R-t}(k_0+t)\\ \mathbf{e}_i(k_0+t) &= e_{i+t+1:i+R-t}(k_0+t);\end{aligned}$$

the time-varying matrix $\mathscr{A}_{i,t}$ for $0 \le t \le \lfloor\frac{R}{2}\rfloor$ is defined as

$$\mathscr{A}_{i,t} = \underbrace{\begin{bmatrix} A_{i+t+2,l} & A_{i+t+2} & A_{i+t+2,r} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{i+R-1-t,l} & A_{i+R-1-t} & A_{i+R-1-t,r} \end{bmatrix}}_{\text{it contains } R-2t \text{ block columns}}; \tag{9}$$

the matrices $\mathscr{B}_{i,t}$ and $\mathscr{C}_{i,t}$ are defined as

$$\begin{aligned}\mathscr{B}_{i,t} &= \text{Diag}\left(B_{i+t+2}, \cdots, B_{i+R-1-t}\right)\\ \mathscr{C}_{i,t} &= \text{Diag}\left(C_{i+t+1}, \cdots, C_{i+R-t}\right).\end{aligned} \tag{10}$$

It can be seen from the time-varying system (8) that the dimensions of $\mathbf{x}_i(k_0+t)$ and $\mathbf{y}_i(k_0+t)$ decrease along with the increase of t and the state $x_i(k_0+t)$ will be void when $t > \frac{R}{2}$.

Next, we shall derive a data equation for the time-varying model in (8). Define the state transition matrix $\Psi_i(t_2, t_1)$, for $t_2 \ge t_1$, as

$$\Psi_i(t_2, t_1) = \mathscr{A}_{i,t_2}\mathscr{A}_{i,t_2-1}\cdots\mathscr{A}_{i,t_1},$$

which is the consecutive product of matrices $\mathscr{A}_{i,t}$ for $t = t_1, t_1+1, \cdots, t_2$. The stacked output of (8) can be compactly represented as

$$\underline{\mathbf{y}}_i(k_0) = \mathbf{O}^i\mathbf{x}_i(k_0) + \mathbf{T}^i\underline{\mathbf{u}}_i(k_0) + \underline{\mathbf{e}}_i(k_0), \tag{11}$$

where the dimension parameter s satisfies $s \le \lfloor\frac{R}{2}\rfloor$,

$$\begin{aligned}\underline{\mathbf{y}}_i(k_0) &= \mathbf{y}_i(k_0 : k_0+s-1),\\ \underline{\mathbf{u}}_i(k_0) &= \mathbf{u}_i(k_0 : k_0+s-2),\\ \underline{\mathbf{e}}_i(k_0) &= \mathbf{e}_i(k_0 : k_0+s-1).\end{aligned}$$

The matrices $\mathbf{O}^i \in \mathbb{R}^{ps(R-s+1)\times Rn}$ and $\mathbf{T}^i \in \mathbb{R}^{ps(R-s+1)\times ms(R-s-1)}$ are defined as

$$\mathbf{O}^i = \begin{bmatrix} \mathscr{C}_{i,0} \\ \mathscr{C}_{i,1}\Psi_i(0,0) \\ \vdots \\ \mathscr{C}_{i,s-1}\Psi_i(s-2,0) \end{bmatrix}, \tag{12}$$

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$$\mathbf{T}^i = \begin{bmatrix} 0 & & \\ \mathscr{C}_{i,1}\mathscr{B}_{i,0} & \ddots & \\ \vdots & \ddots & 0 \\ \mathscr{C}_{i,s-1}\Psi_i(s-2,1)\mathscr{B}_{i,0} & \cdots & \mathscr{C}_{i,s-1}\mathscr{B}_{i,s-2} \end{bmatrix}.$$

By concatenating the equations in (11) for $k_0 = k, k+1, \cdots, k+h-1$ with $h \gg s$, we can form the following data equation:

$$\underbrace{\left[\underline{\mathbf{y}}_i(k) \cdots \underline{\mathbf{y}}_i(k+h-1)\right]}_{\mathbf{Y}_k^i} = \mathbf{O}^i \underbrace{[\mathbf{x}_i(k) \cdots \mathbf{x}_i(k+h-1)]}_{\mathbf{X}_k^i} + \mathbf{T}^i \underbrace{[\underline{\mathbf{u}}_i(k) \cdots \underline{\mathbf{u}}_i(k+h-1)]}_{\mathbf{U}_k^i} + \underbrace{[\underline{\mathbf{e}}_i(k) \cdots \underline{\mathbf{e}}_i(k+h-1)]}_{\mathbf{E}_k^i}. \tag{13}$$

B. Estimation of augmented state $\mathbf{X}_k^i$ (or $x_{i+1:i+R}(k)$)

To obtain the row space of $\mathbf{X}_k^i$ from the data equation (13), it requires the extended observability matrix $\mathbf{O}^i$ to have full column rank. For that purpose, the following assumption is made.

A4. There exist positive integers R and s such that the extended observability matrix $\mathbf{O}^i$ defined in (12) has full column rank for all $R+1 \leq i \leq N-R$.

Based on Assumption A4, the local state information can be obtained from the future local observations that are not influenced by the unknown inputs.

Remark 1. *The observability matrix $\mathbf{O}^i$ is a subpart of the observability matrix $\mathcal{O}_s(\underline{C}_i, \underline{A}_i)$. It is remarked that Assumption A4 means that the rows of the observability matrix $\mathcal{O}_s(\underline{C}_i, \underline{A}_i)$ has a basis of $\mathbb{R}^{Rn}$ that is selected from the rows of $\mathbf{O}^i$. More information on how to select such a basis from an observability matrix can be found in [11]. In this regard, we can see that observability of $(\underline{C}_i, \underline{A}_i)$ is a necessary condition for Assumption A4.*

According to the structure of $\mathbf{O}^i$, a valid dimension parameter s should satisfy $s \leq \lfloor \frac{R+1}{2} \rfloor$. In order to ensure $\mathbf{O}^i$ to be a tall matrix, the spatial-dimension parameter R has to satisfy that

$$R \geq \frac{4n}{p} - 1. \tag{14}$$

Random numerical simulations, such as reported in Section V, suggest that Assumption A4 always holds when the parameter values are set to $m = 1, p = 1, n = 2, s = 6$ and $R = 13$.

By Assumption A4, the matrix $\mathbf{X}_k^i$ in equation (13) can be expressed as

$$\mathbf{X}_k^i = \left(\mathbf{O}^i\right)^\dagger \left(\mathbf{Y}_k^i - \mathbf{E}_k^i - \boldsymbol{T}^i \mathbf{U}_k^i\right). \tag{15}$$

From the above equation, we can obtain the row space property of $\mathbf{X}_k^i$ (or $x_{i+1:i+R}(k)$) as follows

$$\text{Row}\left[\mathbf{X}_k^i\right] \subseteq \text{Row}\begin{bmatrix} \mathbf{Y}_k^i - \mathbf{E}_k^i \\ \mathbf{U}_k^i \end{bmatrix}. \tag{16}$$

Using the same strategy as described above, the row space property of $\mathbf{X}_k^{i-R+1}$ (or $x_{i-R+2:i+1}(k)$) can be derived as follows

$$\text{Row}\left[\mathbf{X}_k^{i-R+1}\right] \subseteq \text{Row}\begin{bmatrix} \mathbf{Y}_k^{i-R+1} - \mathbf{E}_k^{i-R+1} \\ \mathbf{U}_k^{i-R+1} \end{bmatrix}.$$

where $\mathbf{X}_k^{i-R+1}, \mathbf{Y}_k^{i-R+1}, \mathbf{E}_k^{i-R+1}, \mathbf{U}_k^{i-R+1}$ are defined in (13) but with different spatial indices.

C. Estimation of $x_{i+1}(k)$ by subspace intersection

Denote

$$X_k^{i+1} = [x_{i+1}(k) \ \cdots \ x_{i+1}(k+h-1)].$$

The row space of X_k^{i+1} (or $x_{i+1}(k)$) will be estimated as the intersection of the row subspaces of $\mathbf{X}_k^i$ (or $x_{i+1:i+R}(k)$) and $\mathbf{X}_k^{i-R+1}$ (or $x_{i-R+2:i+1}(k)$). More explicitly, we have that

$$\text{Row}\left[X_k^{i+1}\right] \subseteq \text{Row}\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i - \mathbf{E}_k^i \end{bmatrix} \cap \text{Row}\begin{bmatrix} \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} - \mathbf{E}_k^{i-R+1} \end{bmatrix}. \tag{17}$$

In next lemma, it will be shown that the row subspaces on both hand sides of equation (17) are equal.

Lemma 1. *Under the Assumptions A1-A2 and A4, we have that*

$$\textit{Row}\left[X_k^{i+1}\right] = \textit{Row}\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i - \mathbf{E}_k^i \end{bmatrix} \cap \textit{Row}\begin{bmatrix} \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} - \mathbf{E}_k^{i-R+1} \end{bmatrix}. \tag{18}$$

Proof: By Assumption A4 and equation (13), it is easy to see that

$$\text{Row}\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i - \mathbf{E}_k^i \end{bmatrix} = \text{Row}\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{X}_k^i \end{bmatrix}, \tag{19}$$

and

$$\text{Row}\begin{bmatrix} \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} - \mathbf{E}_k^{i-R+1} \end{bmatrix} = \text{Row}\begin{bmatrix} \mathbf{U}_k^{i-R+1} \\ \mathbf{X}_k^{i-R+1} \end{bmatrix}. \tag{20}$$

Under Assumptions A1-A2 and by Lemma 10.4 in [12], it can be established that the following matrix has full row rank

$$\begin{bmatrix} \mathbf{U}_k^{i-R+1} \\ \mathbf{U}_k^i \\ X_k^{i-R+2} \\ \vdots \\ X_k^{i+R} \end{bmatrix}. \tag{21}$$

Then, by combining the equations (19)-(20) with the facts that

$$\mathbf{X}_k^i = \begin{bmatrix} X_k^{i+1} \\ \vdots \\ X_k^{i+R} \end{bmatrix} \quad \text{and} \quad \mathbf{X}_k^{i-R+1} = \begin{bmatrix} X_k^{i-R+2} \\ \vdots \\ X_k^{i+1} \end{bmatrix},$$

the result of the lemma is straightforward. ■

Remark 2. *Equation* (18) *provides a criterion on how to select the local observations for an accurate approximation of the unknown system input $x_{i+1}(k)$ in* (3)*, which was an unsolved problem in [8].*

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Next, we shall develop a numerical method to compute the subspace intersection in (18). More importantly, we need to find an estimate of $x_{i+1}(k)$ satisfying equation (4).

As illustrated by Corollary 8 in [13], in order to compute the row-subspace intersection in (18), we need to estimate the orthogonal complement of the column subspace of

$$\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i - \mathbf{E}_k^i \\ \hline \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} - \mathbf{E}_k^{i-R+1} \end{bmatrix}. \tag{22}$$

Denote

$$\Delta = \lim_{h\to\infty} \frac{1}{h} \begin{bmatrix} 0 \\ \mathbf{E}_k^i \\ 0 \\ \mathbf{E}_k^{i-R+1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{E}_k^i \\ 0 \\ \mathbf{E}_k^{i-R+1} \end{bmatrix}^T.$$

In order to obtain a consistent estimate of the orthogonal complement of the column subspace of (22) where the noise terms $\mathbf{E}_k^i$ and $\mathbf{E}_k^{i-R+1}$ are unknown, the noise variance σ^2 in Assumption A3 needs to be estimated. This can be done by exploiting the rank deficiency of $\begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i - \mathbf{E}_k^i \end{bmatrix}$ as shown in equation (19).

Given the estimate of the noise variance σ^2, the matrix Δ is known. Then an estimate of $x_{i+1}(k)$ satisfying equation (4) can be obtained, which will be shown in the following theorem.

Theorem 1. *Denote*

$$\mathbf{R} = \lim_{h\to\infty} \frac{1}{h} \begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i \\ \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_k^i \\ \mathbf{Y}_k^i \\ \mathbf{U}_k^{i-R+1} \\ \mathbf{Y}_k^{i-R+1} \end{bmatrix}^T - \Delta. \tag{23}$$

Let the SVD of $\mathbf{R}$ *be given as*

$$\mathbf{R} = [U_1 \;\; U_2] \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \tag{24}$$

where U_2 *contains the left singular vectors corresponding to the smallest* n *singular values. Partition* U_2 *as*

$$U_2 = \begin{bmatrix} U_{21}^T & U_{22}^T & U_{23}^T & U_{24}^T \end{bmatrix}^T,$$

where the dimensions of U_{2j} *for* $j = 1, \cdots, 4$ *accord with the block rows of the matrix in* (22)*.*

Under Assumption A3, the following estimate of $x_{i+1}(k)$ *satisfies equation* (4)*:*

$$\hat{x}_{i+1}(k) = U_{21}^T \underline{\mathbf{u}}_i(k) + U_{22}^T \underline{\mathbf{y}}_i(k), \tag{25}$$

where $\underline{\mathbf{u}}_i(k)$ *and* $\underline{\mathbf{y}}_i(k)$ *are defined in* (11)*.*

Proof: Since U_2 spans the orthogonal complement of the column subspace of the matrix in (22), by the result of Corollary 8 in [13], we can obtain that

$$\text{Row}[x_{i+1}(k)] = \text{Row}\left[U_{21}^T \underline{\mathbf{u}}_i(k) + U_{22}^T \left(\underline{\mathbf{y}}_i(k) - \underline{\mathbf{e}}_i(k)\right)\right].$$

From the above equation, we can derive that the estimate $\hat{x}_{i+1}(k)$ in (25) satisfies equation (4). ■

To ease the reference, the estimation scheme for the unmeasurable inputs is summarized in Algorithm 1.

Algorithm 1: Subspace estimation of unmeasurable inputs	
Step 1	Construct the data equation (13);
Step 2	Form the matrix $\mathbf{R}$ in (23) and compute its SVD in (24);
Step 3	Compute the estimate of $x_{i+1}(k)$ as shown in (25).

V. Numerical simulations

In this section, one simulation example is provided to validate the proposed identification method. The network model in the simulation example includes 40 systems in a line and the 20-th subsystem is to be identified. The system matrices of individual systems are generated by the superposition of fixed matrices and random matrices with small entries, where the fixed matrices are given by

$$A_i = \begin{bmatrix} 0.2728 & -0.2068 \\ 0.1068 & 0.2728 \end{bmatrix}, A_{i,l} = \begin{bmatrix} -0.1195 & -0.3565 \\ 0.0874 & -0.1048 \end{bmatrix}$$
$$A_{i,r} = \begin{bmatrix} 0.0699 & -0.4278 \\ 0.3842 & 0.1135 \end{bmatrix}, B_i = \begin{bmatrix} 0.3870 \\ -1.2705 \end{bmatrix}$$
$$C_i = \begin{bmatrix} -0.9075 & -1.3651 \end{bmatrix} \text{ for } i = 1, \cdots, 40.$$

The system input and measurement noise are randomly generated as white Gaussian noise. The dimension parameters s and R involved in the proposed identification method are respectively set to $s = 10$ and $R = 8$.

To measure the identification performance, the impulse-response-fitting (IRF) criterion for the i-th system is defined as

$$\text{IRF} = \frac{1}{N} \sum_{k=1}^{N} \frac{\sum_{j=1}^{10} \|\hat{C}_i^k (\hat{A}_i^k)^j \hat{B}_i^k - C_i^* (A_i^*)^j B_j^*\|_F^2}{\sum_{j=1}^{10} \|C_i^* (A_i^*)^j B_i^*\|_F^2}, \tag{26}$$

where N is the number of Monte-Carlo trials which is set to 200; A_i^*, B_i^*, C_i^* are true system matrices; $A_i^k, \hat{B}_i^k, \hat{C}_i^k$ are the estimated system matrices at the k-th Monte-Carlo trial. In order to show the influence of the measurement noise to the identification performance, the signal-to-noise ration (SNR) is defined as

$$\text{SNR (dB)} = 10 \log \frac{\text{var}(y_i(k) - e_i(k))}{\text{var}(e_i(k))}.$$

First, in order to demonstrate the effectiveness of the proposed method, the IRF curve with respect to SNR is shown in Fig. 2. The data length in this simulation is set to 2000. It can be seen that the IRF values decay to zero along with the increase of SNR, indicating that the proposed algorithm can perfectly address the local identification problem in the absence of measurement noise.

Second, we will provide the experimental evidence for the consistent identification result. In this simulation, we set SNR=70 dB. The IRF curve against the length of input-output data is shown in Fig. 3, where we can see that IRF values decrease along with the increase of data length. In addition, the system poles of the 20-th system are estimated by performing 200 independent simulation trials. The distributions of the estimated poles at data lengths 2000, 4000 and 8000 are given in Fig. 4. It can be observed that, as the data length increases, the estimated poles are more concentrated at their true values.

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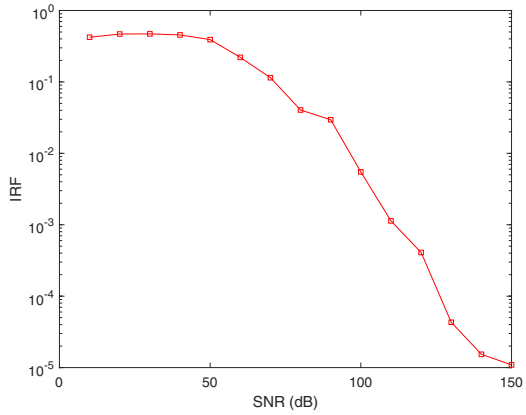

Fig. 2. IRF of the identified system with respect to the SNR.

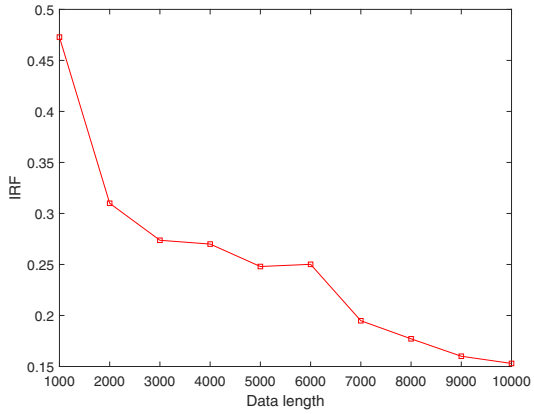

Fig. 3. IRF of the identified system with respect to the data length.

VI. Conclusion

In this note, we have presented subspace identification methods for individual systems operating in a large-scale network. Compared with the existing local system identification methods, the proposed algorithm can yield consistent identification results. The novelty of our work lies at the accurate estimation of the unknown system inputs of individual systems operating in a network. A simulation example has been given to show the effectiveness of the proposed identification algorithm.

Appendix A
Lemma 2

Lemma 2. *Consider the dynamical system in* (7)*, where only* $\underline{y}_i(k)$ *and* $\underline{u}_i(k)$ *are available. The unknown input* $\underline{v}_i(k)$ *is unidentifiable in the sense that the row space spanned by the sequence* $\underline{v}_i(k)$ *cannot be uniquely determined.*

Proof: Denote

$$\underline{\Delta}_i = \text{Diag}\left(A_{i+1,l}\Delta_l, 0, \cdots, 0, A_{i+R,r}\Delta_r\right),$$
$$\bar{\Delta}_i = \begin{bmatrix} \Delta_l & 0 \\ 0 & \Delta_r \end{bmatrix},$$

where $\Delta_l, \Delta_r \in \mathbb{R}^{n\times m}$ are arbitrary real matrices. We have

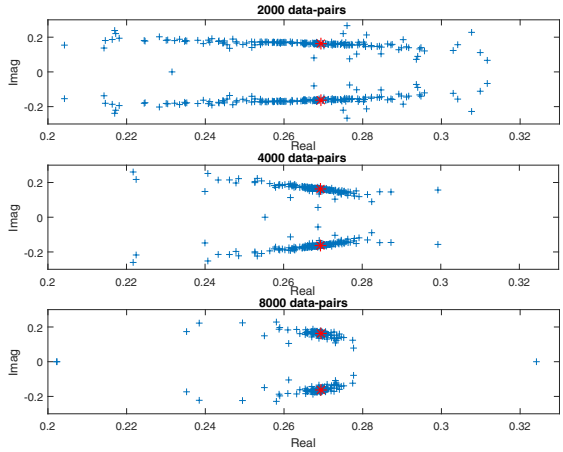

Fig. 4. Estimated poles of 20-th system by 200 Monte-Carlo trials. The red stars represent true poles, while the blue crosses denote estimated poles.

the following equation:

$$\underline{B}_i\underline{u}_i(k) + \underline{D}_i\underline{v}_i(k) = (\underline{B}_i + \underline{\Delta}_i)\underline{u}_i(k) + \underline{D}_i\left(\underline{v}_i(k) - \bar{\Delta}_i\begin{bmatrix} u_{i+1}(k) \\ u_{i+R}(k) \end{bmatrix}\right).$$

From the equation above, we can see that $(\underline{B}_i + \underline{\Delta}_i)$ can be regarded as an estimate of $\underline{B}_i$, and $\left(\underline{v}_i(k) - \bar{\Delta}_i\begin{bmatrix} u_{i+1}(k) \\ u_{i+R}(k) \end{bmatrix}\right)$ can be an estimate of $\underline{v}_i(k)$. Since Δ_l, Δ_r can be any real matrices, it can be concluded that $\underline{v}_i(k)$ is unidentifiable. ■

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