Optimally Robust Kalman Filtering at Work: AO-, IO-, and Simultaneously IO- and AO- Robust Filters

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Abstract

We take up optimality results for robust Kalman filtering from Ruckdeschel (2001, 2010) where robustness is understood in a distributional sense, i.e.; we enlarge the distribution assumptions made in the ideal model by suitable neighborhoods, allowing for outliers which in our context may be system-endogenous/propagating or -exogenous/non-propagating, inducing the somewhat conflicting goals of tracking and attenuation. Correspondingly, the cited references provide optimally-robust procedures to deal with each type of outliers separately, but in case of IO-robustness does not say much about the implementation. We discuss this in more detail in this paper. Most importantly, we define a hybrid filter combining AO- and IO-optimal ones, which is able to treat both types of outliers simultaneously, albeit with a certain delay. We check our filters at a reference state space model, and compare the results with those obtained by the ACM filter Masreliez and Martin (1977), Martin (1979) and non-parametric, repeated-median based filters Fried et al. (2006), Fried et al. (2007).

Keywords: robustness, Kalman Filter, innovation outlier, additive outlier, minimax robustness;
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1. Introduction

Robustness is an "old" problem in Kalman filtering, with first (non-verified) hits on a quick search for "robust Kalman filter" on `scholar.google.com` as early as 1962, i.e.; even before the seminal Huber (1964) paper, often referred to as birthday of Robust Statistics.

The amount of literature on this topic is huge, and we do not attempt to give a comprehensive account here. Instead, we refer the reader to the excellent surveys given in Ershov and Lipster (1978), Kassam and Poor (1985), Stockinger and Dutter (1987), Schick and Mitter (1994), Künsch (2001), and to some extent Ruckdeschel (2001, Sect. 1.5).

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The mere notion of robustness in the context of filtering is non-standard, most frequently it is used to describe stability of the procedure w.r.t. certain variations of the "input parameters". The choice of the "input parameters" to look at varies from notion to notion. In this paper we are concerned with (distributional) **minimax robustness**; i.e.; we allow for deviations from the ideal distributional model assumptions, defining suitable neighborhoods about this ideal model. For these neighborhoods, we consider procedures minimizing the maximal predictive inaccuracy on these neighborhoods, measured in terms of mean squared error (MSE).

These minimax filters derived in Ruckdeschel (2010) come as closed-form saddle-points consisting of an optimally-robust procedure and a corresponding least favorable outlier situation. Their optimality holds in a surprisingly general setup of state space models, which is not limited to a Euclidean or time-discrete framework.

The important condition is that MSE makes sense for the range of the states, so these results cover general Hidden Markov Models for arbitrary observation space, dynamic (generalized) linear models as discussed in West et al. (1985) and West and Harrison (1989), as well as continuous time settings, as those used in applications of Mathematical Finance, compare, e.g. Nielsen et al. (2000) and Singer (2002) for respective model formulations. For the application, we need to linearize the corresponding functions transition and observation functions suitably, e.g. to give the *(continuous-discrete) Extended Kalman Filter (EKF)*. We even cover applications such as optimal portfolio selection, with corresponding controls U_t (i.e.; buying and selling operations) and, in principle, indirectly observed random fields.

In this paper though, to clarify ideas we limit ourselves to the linear, Euclidean, time discrete state space model (SSM), and in the ideal model assume normality. More details on these models can be found in many textbooks, cf. e.g. Anderson and Moore (1979), Harvey (1991), Hamilton (1993), and Durbin and Koopman (2001).

2. Setup

2.1. Ideal model

Let us start with some definitions and assumptions. Our SSM consists in an unobservable p-dimensional state X_t evolving according to a possibly time-inhomogeneous vector autoregressive model of order 1 (VAR(1)) with innovations v_t and transition matrices F_t, i.e.,

$$X_t = F_t X_{t-1} + v_t \tag{1}$$

We only observe a q-dimensional linear transformation Y_t of X_t involving an additional observation error ε_t,

$$Y_t = Z_t X_t + \varepsilon_t \tag{2}$$

In the ideal model we work in a Gaussian context, that is we assume

$$v_t \overset{\text{indep.}}{\sim} \mathcal{N}_p(0, Q_t), \qquad \varepsilon_t \overset{\text{indep.}}{\sim} \mathcal{N}_q(0, V_t), \qquad X_0 \sim \mathcal{N}_p(a_0, Q_0), \tag{3}$$

$$\{X_0, v_s, \varepsilon_t, \ s, t \in \mathbb{N}\} \text{ stochastically independent} \tag{4}$$

For this paper, we assume the hyper–parameters F_t, Z_t, Q_t, V_t, a_0 to be known.

2.2. Deviations from the ideal model

As announced, these ideal model assumptions for robustness considerations are extended by allowing (small) deviations, most prominently generated by outliers. In our notation, suffix "id" indicates the *id*eal setting, "di" the *di*storting (contaminating) situation, "re" the *re*alistic, contaminated situation.

AO's and IO's. We follow the terminology of Fox (1972), who distinguishes *innovation outliers* (or IO's) and *additive outliers* (or AO's). Historically, AO's and IO's denote gross errors affecting the observation errors and the innovations, respectively. For consistency with literature, we stick to this distinction, but rather use these terms in a wider sense: *IO*'s stand for general endogenous outliers entering the state equation, hence with propagated distortion, also covering level shifts or linear trends which would not be included in the original definition. Similarly wide-sense *AO*'s denote general exogenous outliers which do not propagate, like substitutive outliers or *SO*'s as defined in equations (18)–(20).

Different and competing goals induced by AO's and IO's. Due to their different nature, as a rule, a different reaction in the presence of IO's and AO's is required. As AO's are exogenous, we would like to ignore them as far as possible, damping their effect, while when there are IO's, something has happened in the system, so the usual goal will be to detect these structural changes as fast as possible.

A situation where both AO's and IO's may occur is more difficult: We are faced with an identification problem, as we cannot distinguish IO from AO type immediately after a suspicious observation; hence a simultaneous treatment is only possible with a certain delay—see section 4.

Another task, not pursued further in this paper, consists in recovering the situation without structural changes in an off-line situation. An example is spectral analysis of inter-individual heart frequency spectra, which requires a cleaning from both (wide-sense) IO's and AO's; after this cleaning the powerful instruments of spectral analysis will be available; cf. Spangl (2008).

3. Kalman filter and robust alternatives

3.1. Classical Method: Kalman–Filter

Filter Problem. The most important problem in SSM formulation is the reconstruction of the unobservable states X_t by means of the observations Y_t. For abbreviation let us denote

$$Y_{1:t} = (Y_1, \ldots, Y_t), \quad Y_{1:0} := \emptyset \tag{5}$$

Using MSE risk, the optimal reconstruction is the solution to

$$\mathrm{E} \left| X_t - f_t \right|^2 = \min_{f_t}, \qquad f_t \text{ measurable w.r.t. } \sigma(Y_{1:s}) \tag{6}$$

We focus on filtering ($s = t$) in this paper, while $s < t$ makes for a prediction, and $s > t$ for a smoothing problem.

Kalman–Filter. The general solution to (6), the corresponding conditional expectation $\mathrm{E}[X_t|Y_{1:s}]$ in general is rather expensive to compute. Hence as in the Gauss-Markov setting, restriction to linear filters is a common way out. In this context, Kalman (1960) introduced a recursive scheme to compute this optimal linear filter reproduced here for later reference:

$$\text{Initialization:} \quad X_{0|0} = a_0, \quad\quad\quad\quad \Sigma_{0|0} = Q_0 \quad\quad\quad (7)$$

$$\text{Prediction:} \quad X_{t|t-1} = F_t X_{t-1|t-1}, \quad\quad \Sigma_{t|t-1} = F_t \Sigma_{t-1|t-1} F_t^\tau + Q_t \quad (8)$$

$$\text{Correction:} \quad X_{t|t} = X_{t|t-1} + M_t^0 \Delta Y_t, \quad \Delta Y_t = Y_t - Z_t x_{t|t-1},$$

$$M_t^0 = \Sigma_{t|t-1} Z_t^\tau \Delta_t^{-1}, \quad\quad \Sigma_{t|t} = (\mathbb{I}_p - M_t^0 Z_t)\Sigma_{t|t-1},$$

$$\Delta_t = Z_t \Sigma_{t|t-1} Z_t^\tau + V_t \quad\quad\quad\quad\quad\quad\quad (9)$$

where $\Sigma_{t|t} = \mathrm{Cov}(X_t - X_{t|t})$, $\Sigma_{t|t-1} = \mathrm{Cov}(X_t - X_{t|t-1})$, and M_t^0 is the so-called *Kalman gain*.

The Kalman filter has a clear-cut structure with an initialization, a prediction, and a correction step. Evaluation and interpretation is easy, as all steps are linear. The strict recursivity / Markovian structure of the state equation allows one to concentrate all information from the past useful for the future in $X_{t|t-1}$.

This linearity is also the reason for its non-robustness, as observations y enter unbounded into the correction step. A good robustification has to be bounded in the observations, otherwise preserving the advantages of the Kalman filer as far as possible.

3.2. The rLS as optimally robust filter

The idea of the procedures we discuss in this paper are based on *r*obustifying *r*ecursive *L*east *S*quares: rLS. Let us begin with (wide-sense) AO's.

With only AO's, there is no need for robustification in the initialization and prediction step, as no (new) observations enter. For the correction step, we use the following robustification, compare Ruckdeschel (2000): Instead of $M^0\Delta Y$, we use a Huberization of this correction

$$H_b(M^0\Delta Y) = M^0\Delta Y \min\{1, b/\big|M^0\Delta Y\big|\} \quad\quad\quad (10)$$

for some suitably chosen clipping height b. While this is a bounded substitute for the correction step in the classical Kalman filter, it still remains reasonably simple, is non iterative and hence especially useful for online-purposes.

However it should be noted that, departing from the Kalman filter and at the same time insisting on strict recursivity, we possibly exclude "better" non-recursive procedures. These procedures on the other hand would be much more expensive to compute.

Remark 3.1. $\big| \cdot \big|$ in expression $\big|M^0\Delta Y\big|$ denotes the Euclidean norm of $\mathbb{R}^q$; instead, however you could also use other norms like Mahalanobis-type ones. The choice of a quadratic-form-type norm for H_b does not affect the optimality statements of Theorem 3.2 below, provided that the same norm be used in the MSE.

Choice of the clipping height b. For the choice of b, we have two proposals. Both are based on the simplifying assumption that $\mathrm{E}_{\mathrm{id}}[\Delta X|\Delta Y]$ is linear, which in fact turns out to

only be approximately correct. The first one chooses $b = b(\delta)$ according to an Anscombe criterion,

$$\mathrm{E}_{\mathrm{id}}\left|\Delta X - H_b(M^0 \Delta Y)\right|^2 \overset{!}{=} (1 + \delta)\,\mathrm{E}_{\mathrm{id}}\left|\Delta X - M^0 \Delta Y\right|^2 \tag{11}$$

where δ may be interpreted as "insurance premium" to be paid in terms of efficiency.

The second criterion uses the radius $r \in [0, 1]$ of the neighborhood $\mathcal{U}^{\mathrm{so}}(r)$ (defined in (20)) and determines $b = b(r)$ such that

$$(1 - r)\,\mathrm{E}_{\mathrm{id}}(|M^0 \Delta Y| - b)_+ \overset{!}{=} rb \tag{12}$$

This will produce the minimax-MSE procedure for $\mathcal{U}^{\mathrm{so}}(r)$ according to Theorem 3.2 below.

If r is unknown, which is almost always the case, we translate an idea worked out in Rieder et al. (2008) into our setting: Assume we know that $r \in [r_l, r_u]$, $0 \leq r_l < r_u \leq 1$. Then we define a *least favorable radius* r_0 such that the procedure $\mathrm{rLS}(b(r_0))$ with clipping height $b = b(r_0)$ minimizes the maximal inefficiency—w.r.t. the procedure knowing the respective radius—among all procedures $\mathrm{rLS}(b(r))$, i.e.; each rLS for some clipping height $b(r) \neq b(r_0)$ has a larger inefficiency for some $r' \in [r_l, r_u]$. Radius r_0 can be computed quite effectively by a bisection method: Let

$$A_r = \mathrm{E}_{\mathrm{id}}\left[\mathrm{tr}\,\mathrm{Cov}_{\mathrm{id}}[\Delta X | \Delta Y^{\mathrm{id}}] + (|M^0 \Delta Y^{\mathrm{id}}| - b(r))_+^2\right] \tag{13}$$

$$B_r = \mathrm{E}_{\mathrm{id}}\left[|M^0 \Delta Y^{\mathrm{id}}|^2 - (|M^0 \Delta Y^{\mathrm{id}}| - b(r))_+^2\right] + b(r)^2 \tag{14}$$

Then r_0 solves

$$A_{r_0}/A_{r_l} = B_{r_0}/B_{r_u} \tag{15}$$

compare Ruckdeschel (2010, Lemma 3.1).

From another point of view, for given b, you may interpret criteria (11) and (12) the other way round, giving the efficiency loss in the ideal model, or the size of the SO neighborhood, for which the corresponding procedure is MSE-minimax.

(One-Step)-Optimality of the rLS. The rLS filter is optimally-robust in some sense: To see this, in a first step we boil down our SSM to the following simplified model[1]. We have an unobservable but interesting state $X \sim P^X(dx)$, where for technical reasons we assume that in the ideal model $\mathrm{E}\,|X|^2 < \infty$. Instead of X we rather observe the sum

$$Y = X + \varepsilon \tag{16}$$

of X and a stochastically independent error ε. We assume that in the ideal model, the conditional distribution of Y given X must allow for densities w.r.t. some measure μ, i.e.;

$$P^{Y|X=x}(dy) = p^{\varepsilon}(y - x)\,\mu(dy) \quad P^X(dx)\text{-a.e.} \tag{17}$$

As (wide-sense) AO model, we consider an SO outlier model already used by Birmiwal and Shen (1993) and Birmiwal and Papantoni-Kazakos (1994)

$$Y^{\mathrm{re}} = (1 - U)Y^{\mathrm{id}} + UY^{\mathrm{di}}, \qquad U \sim \mathrm{Bin}(1, r) \tag{18}$$

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[1]Instead of this simplification, we could even use a more general "Bayesian" model as simplification, compare Ruckdeschel (2010, section 3.2).

for SO-contamination radius $0 \leq r \leq 1$ specifying the size of the corresponding neighborhood, i.e.; the probability for an SO. U is assumed independent of (X, Y^{id}) and (X, Y^{di}) as well as

$$Y^{\mathrm{di}}, \; X \quad \text{independent} \tag{19}$$

where $\mathcal{L}(Y^{\mathrm{di}})$ is arbitrary, unknown and uncontrollable (a.u.u.). The corresponding neighborhood is defined as

$$\mathcal{U}^{\mathrm{SO}}(r) = \bigcup_{0 \leq s \leq r} \left\{ \mathcal{L}(X, Y^{\mathrm{re}}) \,|\, Y^{\mathrm{re}} \text{ acc. to (18) and (19) with radius } s \right\} \tag{20}$$

With this setting we may formulate two typical robust optimization problems:

Minimax-SO problem. Minimize the maximal MSE on an SO-neighborhood, i.e.; find a measurable reconstruction f_0 for X s.t.

$$\max_{\mathcal{U}} \, \mathrm{E}_{\mathrm{re}} \, |X - f(Y^{\mathrm{re}})|^2 = \min_f ! \tag{21}$$

Lemma5-SO problem. In the spirit of Hampel (1968, Lemma 5), minimize the MSE in the ideal model but subject to a bound on the bias to be fulfilled on the whole neighborhood, i.e.; find a measurable reconstruction f_0 for X s.t.

$$\mathrm{E}_{\mathrm{id}} \, |X - f(Y^{\mathrm{id}})|^2 = \min_f ! \quad \text{s.t. } \sup_{\mathcal{U}} \left| \mathrm{E}_{\mathrm{re}} \, f(Y^{\mathrm{re}}) - \mathrm{E} \, X \right| \leq b \tag{22}$$

The solution to both problems can be summarized as

Theorem 3.2 (Minimax-SO, Lemma5-SO). *(1) In this situation, there is a* **saddle-point** *$(f_0, P_0^{Y^{\mathrm{di}}})$ for Problem (21)*

$$
\begin{aligned}
f_0(y) &:= \mathrm{E} \, X + D(y) w_r(D(y)), \qquad w_r(z) = \min\{1, \rho/|z|\} \tag{23} \\
P_0^{Y^{\mathrm{di}}}(dy) &:= \tfrac{1-r}{r} \big(|D(y)|/\rho - 1 \big)_+ \, P^{Y^{\mathrm{id}}}(dy) \tag{24}
\end{aligned}
$$

where $\rho > 0$ ensures that $\int P_0^{Y^{\mathrm{di}}}(dy) = 1$ and $D(y) = \mathrm{E}_{\mathrm{id}}[X|Y=y] - \mathrm{E} \, X$.
(2) f_0 from (23) also is the solution to Problem (22) for $b = \rho/r$.
(3) If $\mathrm{E}_{\mathrm{id}}[X|Y]$ is linear in Y, i.e.; $\mathrm{E}_{\mathrm{id}}[X|Y] = MY$ for some matrix M, then necessarily

$$M = M^0 = \mathrm{Cov}(X,Y) \, \mathrm{Var} \, Y^- \tag{25}$$

or in SSM formulation: M^0 is just the classical Kalman gain and f_0 the (one-step) rLS.

The proof to this theorem is given in Ruckdeschel (2010, Thm. 3.2).

Model (16) already covers our normal SSM model: we only have to identify X in model (17) with ΔX_t and replace $p^\varepsilon(y-x) \, \mu(dy)$ with $\mathcal{N}(Z_t \Delta X_t, V_t)(dy)$. If even ΔX_t is normal, (3) applies and rLS is SO-optimal.

Remark 3.3. (a) The ACM filter by Masreliez and Martin (1977), which is a competitor in this study, by analogy applies Huber (1964)'s minimax variance result to the "random location parameter X" setting of (16). They come up with redescenders as filter f. Hence the ACM filter is not so much vulnerable in the extreme tails but rather where the corresponding ψ function takes its maximum in absolute value. Care has to be taken, as such "inliers" producing the least favorable situation for the ACM are much harder to detect on naïve data inspection, in particular in higher dimensions.

(b) For exact SO-optimality of the rLS-filter, linearity of the ideal conditional expectation is crucial. However, one can show Ruckdeschel (2010, Prop. 3.6) that $\mathrm{E}_{\mathrm{id}}[\Delta X|\Delta Y]$ is linear iff ΔX is normal, but, having used the rLS-filter in the ΔX-past, normality cannot hold, compare (Ruckdeschel, 2010, Prop 3.4).

Back in the ΔX Model for $t > 1$: eSO-Neighborhoods. As noted in the last remark, rLS fails to be SO-optimal for $t > 1$. Nevertheless rLS performs quite well at both simulations and real data. One explanation for this is to consider yet an extension of the original SO-neighborhoods—the *extended SO* or *eSO–model* (compare Ruckdeschel (2010, Sect. 3.4)), where we allow X to be suitably corrupted as well. In fact, the optimality of pair $(f_0, P_0^{Y^{\mathrm{di}}})$ from Theorem 3.2 extended to $(f_0, P_0^{Y^{\mathrm{di}}} \otimes P_0^{X^{\mathrm{di}}})$ for any $P_0^{X^{\mathrm{di}}}$ with $\mathrm{E}_{\mathrm{di}} |X^{\mathrm{di}}|^2 = G$, remains a saddle-point in the corresponding Minimax-Problem on the eSO-neighborhood of same radius, cf. (Ruckdeschel, 2010, Thm. 3.10). As a consequence, (compare (Ruckdeschel, 2010, Prop. 3.11)), in the Gaussian setup, instead of focussing on the (SO–) saddle-point solution to an $\mathcal{U}(r)$-neighborhood around $\mathcal{L}(\Delta X)$ stemming from an rLS-past, we use the following coupling-type idea:

We assume that for each time t, there is a *(fictive)* random variable $\Delta X^{\mathcal{N}} \sim \mathcal{N}_p(0, \Sigma)$ such that $\Delta X_t^{\mathrm{rLS}}$ stemming from an rLS-past can be considered a contaminating X^{di} in the corresponding eSO-neighborhood around $\Delta X^{\mathcal{N}}$ with radius r. From this perspective, the rLS is exactly minimax for each time t.

3.3. IO-optimality

As noted, in the presence of IO's, we want to follow an IO outlier as fast as possible. The Kalman filter in this situation does not behave as bad as in the AO situation, but still tends to be too inert. To improve upon this, let us go back to (16) which reveals a useful symmetry of X and ε: Apparently

$$\mathrm{E}[X|Y] = Y - \mathrm{E}[\varepsilon|Y] \tag{26}$$

Hence we follow Y more closely if we damp estimation of ε, for which we use the rLS-filer. We should note that doing so, we rely on "clean", i.e., ideally distributed errors ε. With the obvious replacements, Theorem 3.2 translates word by word to a corresponding minimax Theorem for IO's, compare Ruckdeschel (2010, Thm. 4.1).

rLS.IO. In analogy to the definition of the rLS in equation (10), we set up an IO-robust version of the rLS as follows: We retain the initialization and prediction step of the classical Kalman filter and, assuming Z_t invertible for the moment, replace the correction step by

$$X_{t|t} = X_{t|t-1} + Z_t^{-1}[\Delta Y_t - H_b\Big((\mathbb{I}_q - Z_t M_t^0)\Delta Y_t\Big)] \tag{27}$$

where the same arguments for the choice of the norm and the clipping height apply as for the AO-robust version of the rLS.

To better distinguish IO- and AO-robust filters, let us call the IO-robust version *rLS.IO* and (for distinction) the AO-robust filter *rLS.AO* in the sequel.

Invertibility problem. Back in the (linear, discrete-time, Euclidean) SSM the approach just described faces the problem that in general matrix Z_t will not be invertible, so we cannot reconstruct X injectively from Y and ε. Under a certain full-rank condition, this problem can be solved by passing to corresponding rLS-type smoothers. The assumption we need is a version of *complete constructibility*, cf. Anderson and Moore (1979, Appendix) adopted to the time-inhomogeneous case: Denoting the product $F_{t+p}F_{t+p-1} \cdot \ldots \cdot F_t$ by $F_{t+p:t}$ we assume that for each t, $F_{t+p-1:t}(\mathbb{R}^p)$ is contained in $[Z_t^\tau, F_t^\tau Z_{t+1}^\tau, F_{t+1:t}^\tau Z_{t+2}^\tau, \ldots, F_{t+p-1:t}^\tau Z_{t+p-1}^\tau](\mathbb{R}^q)$. Details will be given in a subsequent paper.

Remark 3.4. (a)   It is worth noting that also our IO-robust version is a filter, hence does not use information of observations made after the state to reconstruct; rLS.IO is strictly recursive and non iterative, hence well-suited for online applications.
(b)   An alias to rLS.IO could be *"hysteric filter"* as it hysterically follows any changes in the Y's.

4. Simultaneous Treatment of AO's and IO's

This section is less backed by theoretical results than the ones on pure AO or IO situation; rather it proposes a heuristic to achieve both types of outlier robustnesses. As already mentioned, simultaneous treatment of AO's and IO's is only possible with a certain delay. With this delay, we can base our decision of whether there was an AO or an IO on the size of subsequent $|\Delta Y_t|$'s—if there was an AO this should result in only one "large" $|\Delta Y_t|$ in a row, whereas in case of an IO due to propagation, there should be a whole sequence of large $|\Delta Y_t|$'s. So a hybrid filter (called rLS.IOAO for simplicity) could be designed as follows:

To a given delay window width w, we run in parallel rLS.AO and rLS.IO (but only store the last w values of rLS.IO). By default we return the rLS.AO values. Whenever there is a run of w "large" $|\Delta Y_t^{\mathrm{rLS.AO}}|$'s we replace the last w filter values by the corresponding rLS.IO values and use these ones to continue with the rLS.AO.

Apparently rLS.IOAO gets into trouble in windows where we have both IO's and AO's; here some modeling of the type of structural change (local constant, or better: local linear) as in Fried et al. (2007) should be helpful; we do not treat this in this paper, though.

Operationalization. In the ideal (Gaussian) model, the ΔY_t's should be independent, so a reasonable decision on whether a sequence of $|\Delta Y_t^{\mathrm{rLS.AO}}|$'s is "large" could be based on corresponding quantiles of $|\Delta Y_t^{\mathrm{rLS.AO}}|$, in the ideal model, or, somewhat easier, of $\Delta Y_t^\tau \Delta_t^{-1} \Delta Y_t$ which, assuming $\mathcal{L}(\Delta Y_t)$ to be approximately normal, is approximately $\chi_q^2(0)$-distributed. Relaxing this condition a little, we already switch to rLS.IO when a high percentage h of the w $|\Delta Y_t|$'s are larger than this given quantile.

This leaves us to determine several tuning parameters: window-width w (this should really be chosen according to the application; we have obtained good results in our examples with $w = 5$), the clipping heights for rLS.IO and rLS.AO (proposal: according to (versions of) (11) or (12)), the percentage h (default: 80% of the last w instances of $|\Delta Y_t^{\mathrm{rLS.AO}}|$), and the corresponding quantile (default 99%) assuming that $\Delta Y_t \sim \mathcal{N}_q(0, \Delta_t)$.

Remark 4.1. (a)   Note that although the decision whether we issue the rLS.IO or the rLS.AO values is made w observations after the state which is to be reconstructed, we still only use filters, hence the information of Y_{t+j}, $j = 1, \ldots, w - 1$ is not used to improve the reconstruction so far, as this would involve corresponding (yet-to-be-robustified) smoothers. Once the corresponding work on robust smoothing will be done, we could surely use this additional information.

(b)   In a time-invariant linear SSM (i.e., with hyperparameters F, Z, Q, and V constant in time), in general Δ_t will converge in t exponentially fast—also if one uses bounded rLS-steps—so these tuning parameters will only have to be determined for a small number of time instances t, compare Ruckdeschel (2001, chap. 7). In fact, setting them time-invariant right from the beginning will often do a reasonable job already.

5. Simulation Example: Steady State Model

Our running example will be a one-dimensional steady state model with hyperparameters

$$p = q = 1, \quad F_t = Z_t = 1, \qquad \text{in the ideal model: } v_t, \varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1) \qquad (28)$$

We consider performance of classical Kalman filter, rLS.AO, rLS.IO, and rLS.IOAO in this model and under AO's and IO's. More specifically, we have generated deterministic AO's in observations 10,15,23, and IO's in observations 20–25 (a local linear trend) and 37–42 (level shift).

As competitors, we include the ACM filter by Martin (1979) implemented by B. Spangl in R package `robKalman`, and a variant hybr$_{\text{PRMH}}$ of `robfilter`, cf. Fried and Schettlinger (2008) concerning its implementation and Fried et al. (2006) regarding its definition, which is a non-parametric filter fitting local levels and linear trends based on repeated medians. The results are plotted in Figures 1–4, where in the plots, we confine ourselves to the rLS-variants, which already makes for five curves to be plotted in one panel.

In the ideal situation, all filters perform well, with slight advantages for the classical Kalman filter (which has smallest theoretical MSE), but closely followed (and in the prediction case slightly beaten) by the rLS.IO.

In the IO situation, the "hysteric" rLS.IO filter performs best, beating the classical Kalman filter; both rLS.IOAO and hybr$_{\text{PRMH}}$ perform reasonably well, while the AO-robust filters ACM and rLS.AO are not able to track the IO at all (as they can only perform bounded correction steps) and hence, like a hanging slope, only closely recover the changed situation.

In the AO situation, we have the complementary image; here ACM performs best (see also Remark 5.1(a)), but rLS.AO only performs slightly weaker. rLS.IOAO is a little worse, and with a certain gap, but still reasonably well follows hybr$_{\text{PRMH}}$, while both classical Kalman filter and rLS.IO (the latter even worse) perform drastically bad.

Finally, in the mixed IO and AO situation, hybr$_{\text{PRMH}}$ is by far the best solution, then followed with a certain gap by the rLS.IOAO, while all other filters perform unacceptably bad. By construction, rLS.IOAO assumes that at every time instance there only can be either an AO or an IO. Otherwise the corresponding MSE gets unbounded on every neighborhood $\mathcal{U}(r)$ for $r > 0$. Hence the AO in observation 23 really confuses rLS.IOAO completely, compare Figures 1–4: it has just switched to "hysteric" IO behavior and
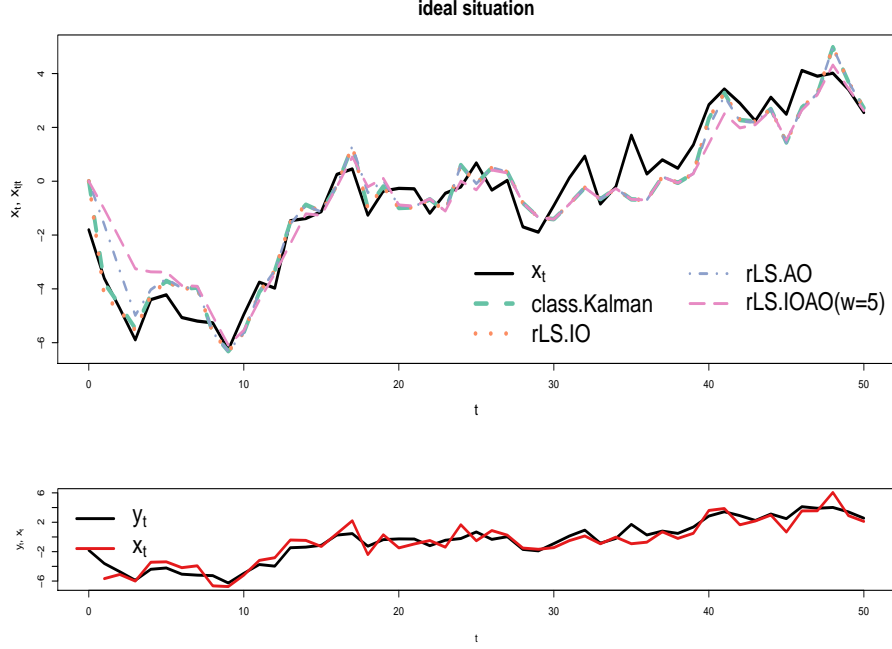
Figure 1: rLS-filter variants in model (28) in the ideal model; in the panel below (note the different y-scale) both actual states (black) and observations (red) are plotted.

hence faithfully follows the AO. Repeated-median-based $\text{hybr}_{\text{PRMH}}$ does not have this problem, as the median even stays stable under (almost) arbitrary substitutive outliers, hence it is able to keep the local linear trend. Omitting observation 23 results in a much better performance of rLS.IOAO, which then even beats $\text{hybr}_{\text{PRMH}}$, cf. Table 2.

Averaging over time in one realization of the SSM, we get the "ergodic" empirical MSEs as displayed in Tables 1, 2.

Remark 5.1. (a) We can explain why the ACM filter beats the rLS in the AO-situation by the fact that the contamination in this study clearly covers the worst-case behavior of the rLS but not the one of the ACM filter, compare Remark 3.3(e), and also fails to do so for $\text{hybr}_{\text{PRMH}}$.
(b) rLS.IOAO really has its advantages in higher dimensions where median-based filters are much harder to define and get computationally expensive. One might even think of combining rLS.IOAO and $\text{hybr}_{\text{PRMH}}$ in these settings: first let rLS.IOAO do a preliminary, fast, and dimension-independent cleaning, and then let $\text{hybr}_{\text{PRMH}}$ polish this result coordinate-wise.
(c) It is still an open question whether we can improve on the rLS.IOAO behavior, using the SSM

$$Z_t = (1, t), X_t = (a_t, b_t)^\tau, F_t = \mathbb{I}_2, Q_t = 0.1\mathbb{I}_2, V_t = 1$$

which (up to the specification of error/innovation variance) is essentially the model in the
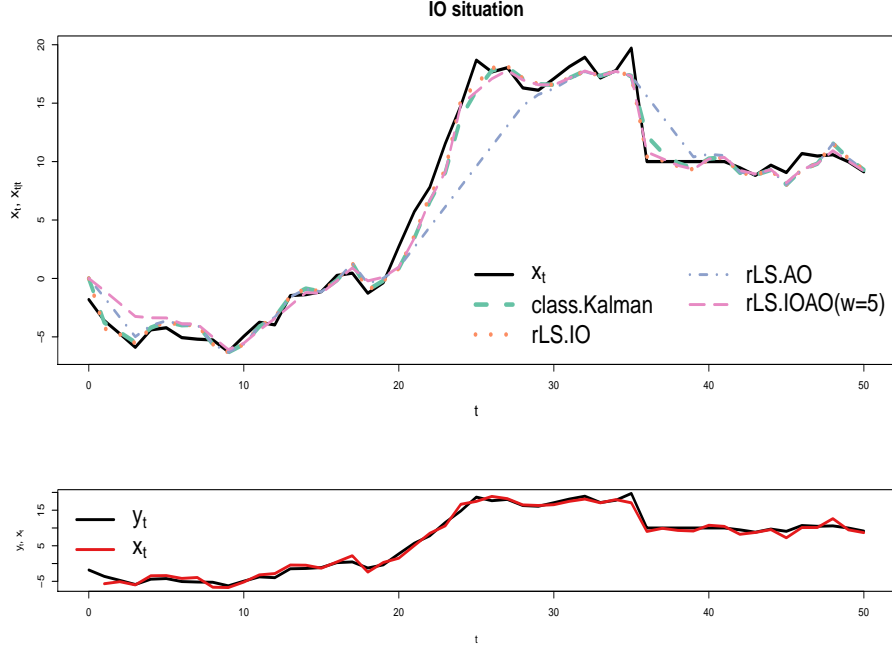
10

IO situation

Figure 2: rLS-filter variants in model (28) with IO's: a local linear trend at X_{20}–X_{25} and a level shift for states X_{37}–X_{42}; the panel below (note the different y-scale) is as in Figure 1.

empirical MSE

Situation	Type	Kalman	rLS$_{IO}$	rLS$_{AO}$	rLS$_{IOAO}$	ACM	hybr$_{PRMH}$
ideal	filter	**0.59**	0.60	0.75	1.08	0.77	1.41
	pred	1.69	**1.67**	1.96	2.26	2.01	
IO	filter	1.04	**0.83**	6.54	1.36	25.19	1.36
	pred	5.28	**4.71**	12.17	5.42	32.16	
AO	filter	15.25	30.38	0.91	1.16	**0.82**	1.79
	pred	15.15	29.68	2.00	2.25	**2.05**	
IO&AO	filter	17.00	30.52	12.89	**7.78**	28.76	**1.53**
	pred	21.94	34.56	19.23	**13.87**	36.08	

Table 1: "ergodic" estimates for the MSE of the variants of the rLS and the ACM and hybr$_{PRMH}$ in the situation described in the text; best results are printed in bold face.
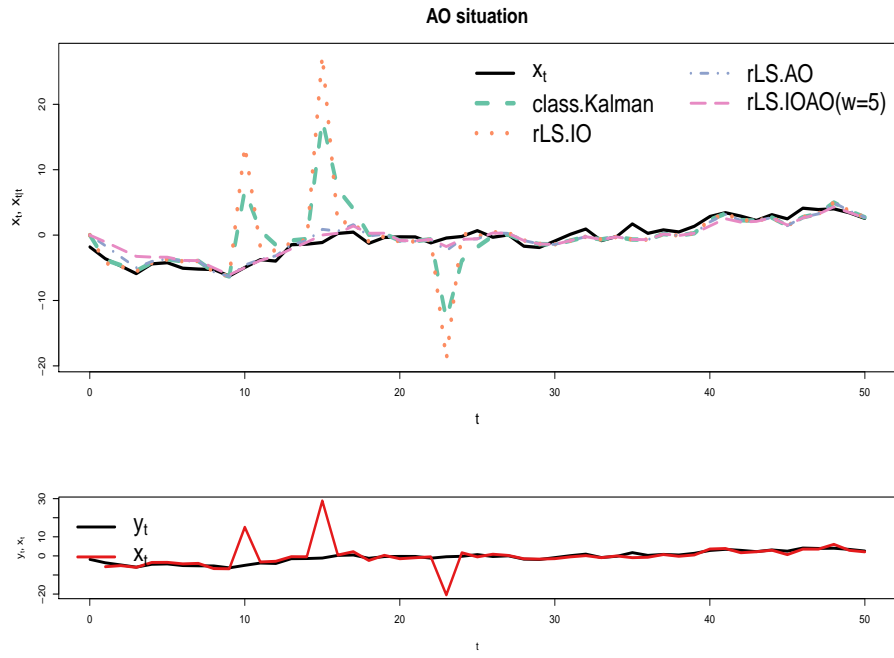
Figure 3: rLS-filter variants in model (28) with AO's in observations 10,15,23; the panel below (note the different y-scale) is as in Figure 1.
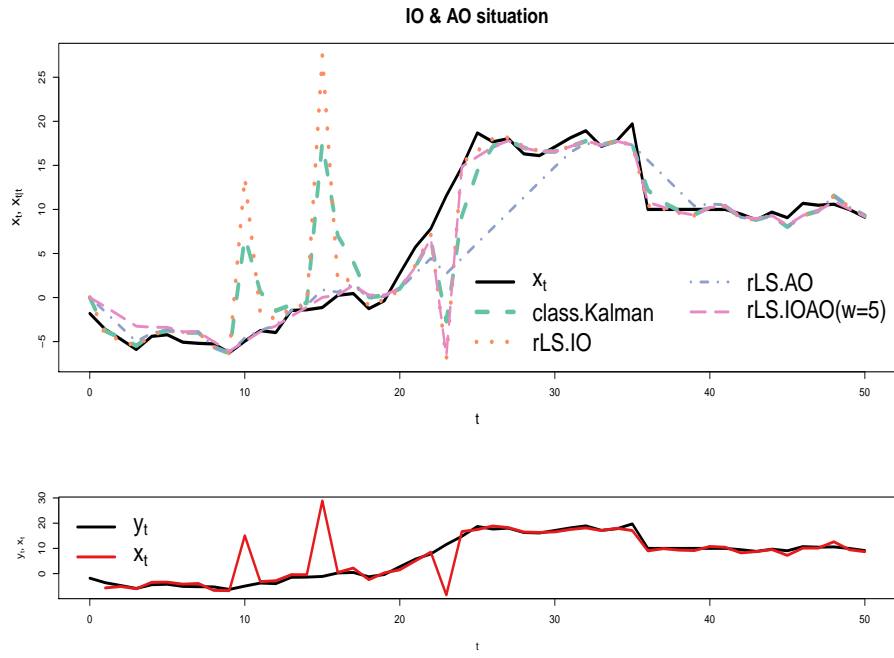
Figure 4: rLS-filter variants in model (28) with both IO's as in Figure 2 and AO's as in Figure 3; the panel below (note the different y-scale) is as in Figure 1.

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empirical MSE—without obs. 23

Situation	Type	Kalm	rLS$_{IO}$	rLS$_{AO}$	rLS$_{IOAO}$	ACM	hybr$_{PRMH}$
ideal	filter	**0.59**	0.60	0.75	1.10	0.78	1.43
	pred	1.71	**1.69**	1.99	2.29	2.03	
IO	filter	0.94	**0.74**	6.08	1.26	24.48	1.38
	pred	5.59	**4.98**	12.05	5.73	31.66	
AO	filter	12.46	24.07	0.86	1.15	**0.84**	1.83
	pred	12.18	23.05	1.94	2.25	**2.10**	
IO&AO	filter	13.28	24.21	11.58	**1.31**	27.93	1.56
	pred	17.01	26.34	17.80	**5.63**	35.38	

Table 2: results as in Table 1, but excluding the values for observation 23, where we had coincidence of (wide-sense) IO and AO, a situation not covered in the design of rLS.IOAO.

background of hybr$_{PRMH}$. In this setting Z_t is not invertible, but the model is completely constructible, so passing to smoothers might help.

Remark 5.2. A careful reader might object that we have not included any real world data application. We have done so, because most data sets we have analyzed come with unkown hyper-parameters. The setting of this paper assumes knowledge of these hyper-parameters, so we would have to estimate them somehow—possibly by the EM algorithm of Shumway and Stoffer (1982) or again by more recent refinements. But, since the robustness properties of combining a non-robust M-step (i.e., maximum likelihood) and a robust E-step (achieved using the procedures of this paper) are not evident, and a corresponding robustification of the M-step would have been out of scope, we have rather confined us to simulations.

6. Implementation: R-package `robKalman`

rLS.AO was originally implemented to `XploRe`, compare Ruckdeschel (2000). In an ongoing project with Bernhard Spangl, BOKU, Vienna, and Irina Ursachi (ITWM), we are about to implement all the rLS filter to R, see R Development Core Team (2010), more specifically to an R-package `robKalman`, the development of which is done under `r-forge` project `https://r-forge.r-project.org/projects/robkalman/`, see also R-Forge Administration and Development Team (2008). Under this address you will also find a preliminary version available for download.

7. Conclusion

In the extremely flexible class of dynamic models consisting in SSMs, we apply (distributional) robust optimality results for filtering. We could show that contrary to common prejudice a simultaneous treatment of (wide-sense) IO's and AO's is possible in SSM's—albeit with minor delay. The filters that we propose are model based (in contrast to the

non-parametric hybr$_{\text{PRMH}}$) which means that we need a higher degree of model specification in that we possibly have to estimate the hyper-parameters, but which also could help to get more precise in the ideal model (in particular for higher dimensions). Our filters are non-iterative, recursive, hence fast, and valid for higher dimensions. They are available in R in some devel versions and hopefully on CRAN soon.

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