Bayesian Capital Asset Pricing Model (CAPM)

1. Executive Summary

When making rational investment decisions, we need to formulate our expectations for future growth of stock prices, measured by expected returns (ignoring dividends)

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \qquad E[R_t] = \frac{E[P_t] - P_{t-1}}{P_{t-1}}$$

where P_t stands for stock price at time t; R_t stands for stock return during period $t-1$ to t. Buying high-return stocks and getting rid of low-return stocks, we will end up with sound investment results.

The first well-kown attemp to tackle such return expectation problem dates back to William F. Sharpe (1964), where he postulates that individual stock return (R_i) can be explained by overall market return (R_m):

$$E[R_j] = R_f + \beta_j(E[R_M] - R_f))$$

The above formula is widely known as the Capital Asset Pricing Model (CAPM). Interested reader can further refer to the Wikipedia page. Even though Sharpe's model is widely cited and accepted as a solid banchmark for return expectation, it has many drawbacks (widely documentated in the literature, link to an example). In this project, let's tackle one of these problems: the randomness imbedded CAPM parameter β.

Traditional CAPM assumes that there is a (constant) true model parameter β (frequentist point of view). However, intuition tells us that such β is far from being a constant number. The relationship between individual stock return and overall market return during financial crisis is drastically different from their relationship during a bull market. Bayesian modeling can help us capture such uncertainty in β by allowing it to be a random variable.

Goal: As a summary, in the following sections, we will try to improve Sharpe's CAPM model and tackle the expected stock return formulation problem from a Bayesian perspective.

Conclusion: ExxonMobil (XOM) has the highest expected return 5.7%, while Citigroup (C) has the lowest 3.2%.

2. Data Collection

```
## 'data.frame':    252 obs. of  11 variables:
##  $ GM   : num  36.8 36.4 35.7 36 35.6 ...
##  $ F    : num  13.7 13.6 13.4 13.4 13.6 ...
##  $ UTX  : num  50 49.4 49.1 49.1 48.6 ...
##  $ CAT  : num  46.1 45.6 44.8 45.4 45.3 ...
##  $ MRK  : num  28.8 28.7 28.9 29.1 28.7 ...
##  $ PFE  : num  24.9 24.9 24.8 24.8 24.8 ...
##  $ IBM  : num  95.9 94.8 94.6 94.3 93.9 ...
##  $ MSFT : num  26.1 26.2 26.2 26.1 26.1 ...
##  $ C    : num  45 44.7 45.2 45.7 45.4 ...
##  $ XOM  : num  48.4 48 47.8 48.4 48.1 ...
##  $ SP500: num  1202 1188 1184 1188 1186 ...
```

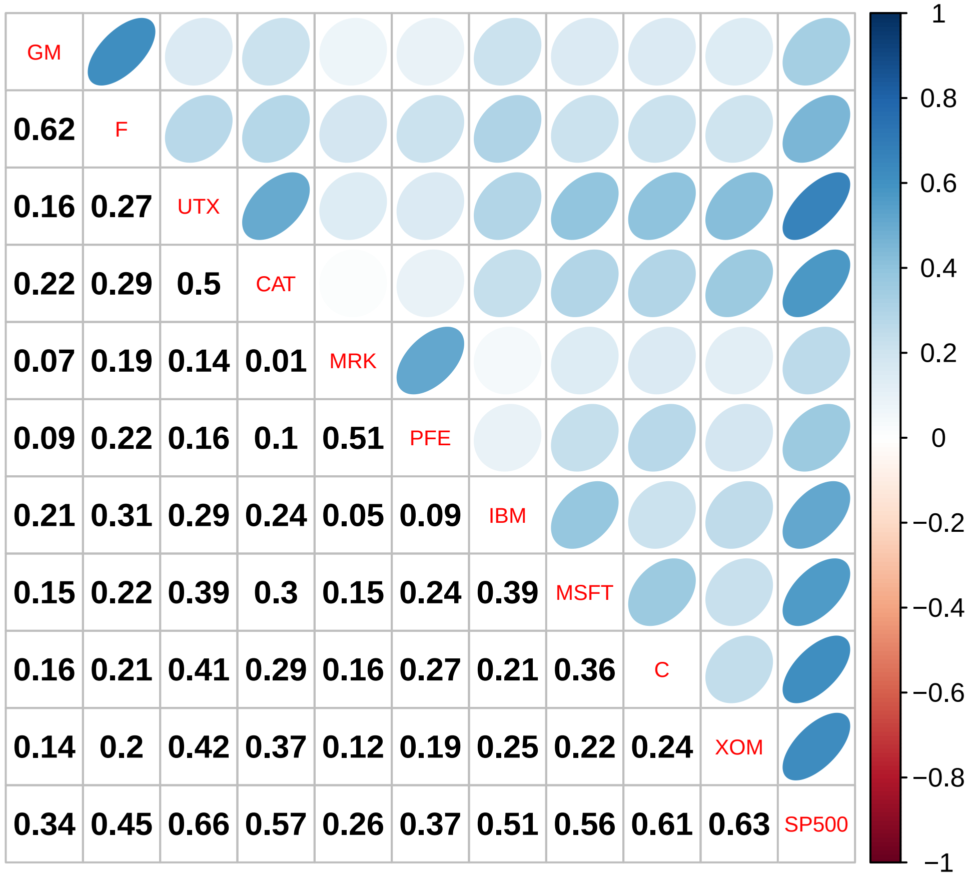
Thanks to the electronification of financial markets, data is usually clean, tidy, and widely available. Example sources include: Yahoo Finance and Google Finance. For our purposes, we need a collection of time series return data for different stocks. Here, I am using the data provided Professor David Ruppert and David S. Matteson from Cornell University. Such data set is publicly available through the following link.

The above block shows summary statistics of adjusted daily closing prices of 10 stocks (GM, F, UTX etc.) and the market (S&P 500) in year 2005. After checking that: there is no outlier and missing value, we are ready to move on.

3. Data Exploration

Given daily adjusted closing prices, we can calculate daily returns of each security:$R_{j,t} = \frac{P_{j,t} - P_{j,t-1}}{P_{j,t-1}}$

The following pair-wise correlation plot shows relationships between stock and market returns. We can observe (from the last column) that, strong linear relationships do exist between stock daily returns and market (S&P 500) daily returns, which justifies the linear model postulated in Sharpe (1964).



4. Model Postulation

4.1 Benchmark Model (OLS Regression)

The simplest estimation approach is to run 10 independent OLS regressions of each stock returns on the market returns. Estiamted coefficients are listed below.

```
##     GM      F    UTX    CAT    MRK    PFE    IBM   MSFT      C    XOM
## 1.4031 1.2696 1.0750 1.4066 0.6986 0.8172 0.8882 0.7891 0.7174 1.4165
```

This approach, however, suffers from two main drawbacks: (1) as we mentioned before, β_j is treated as a constant number, which is far from being true in reality. (2) each OLS regression is run independently from each other. When estimating coefficients for stock i, data for all the other stocks are thrown away, which is a great loss of information.

These two problems can be handled well in the following hierarchical Bayesian linear regression model.

4.2 Bayesian Hierarchical Linear Regression

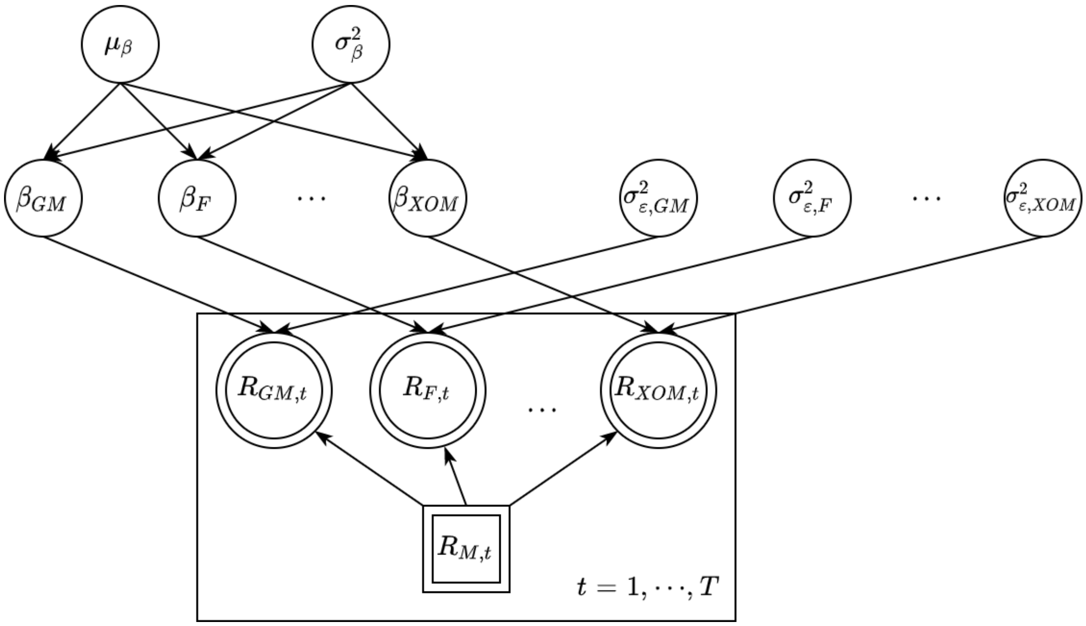


Figure 1: Bayesian CAPM

To make things easier, we assume the risk-free rate (R_f) is zero. Then, the stock return generating process becomes:

$$R_{j,t} = \beta_j \cdot R_{M,t} + \epsilon_{j,t} \qquad j \in \{\text{GM}, \text{F}, \cdots, \text{XOM}\}$$

$$\epsilon_{j,t} \sim N(0, \sigma^2_{\epsilon,j}) \qquad \beta_j \sim N(\mu_\beta, \sigma^2_\beta)$$

$$\sigma^2_{\epsilon,j} \sim \text{InvGamma}(0.1, 0.001) \qquad \mu_\beta \sim N(1, 10^6) \qquad \sigma^2_\beta \sim \text{Uniform}(1, 100)$$

Here, we put a hierarchical structure on $\{\beta_j\}$: rather than fitting a linear model independently for each stock, we assume that $\{\beta_j\}$ come from a joint distribution. This choice (1) aligns with our intuition that each stock (as a small group) sits in the universe of all tradable assets, and (2) accounts for the relationships between return data from different stocks.

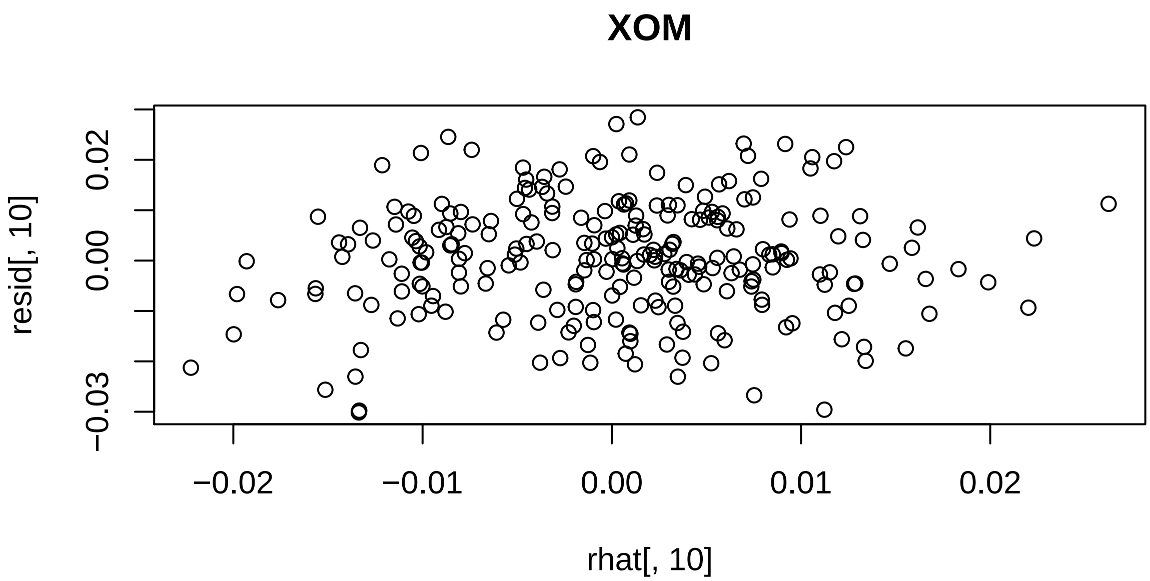
5. Model Fitting

```
bayesian_capm = "model
{
  for(t in 1:N)
  {
    for(j in 1:m)
    {
      R[t,j] ~ dnorm(beta[j]*mkt[t], tau_e[j])
    }
  }
  for(j in 1:m)
  {
    beta[j] ~ dnorm(mean_b, tau_b)
    tau_e[j] ~ dgamma(0.1, 0.001)
  }
  mean_b ~ dnorm(1, 1e-6)
  tau_b ~ dunif(1, 100)
}"
```

With the above model string, we can use `JAGS` to fit our Bayesian Capital Asset Pricing Model.

6. Model Checking

Trace plots and Gelman diagnostics show that the burn-in period is sufficiently long and the underlying Markov chains reach stationarity. No significant auto-correlation is found and the effective same sizes are all at the level of thousands, which shows that our posterior estimations are fairly accurate.



Residual plots also look reasonably good (above is an example for XOM), which shows that our models assumptions approximate the reality well.

7. Model Usage

```
##          GM      F    UTX    CAT    MRK    PFE    IBM   MSFT      C    XOM
## Mean 1.1889 1.1733 1.0682 1.3046 0.8138 0.8703 0.9095 0.8166 0.7437 1.3303
## SD   0.1847 0.1387 0.0773 0.1228 0.1399 0.1175 0.0908 0.0759 0.0643 0.1124
```

Given the above estimations for β_j, we can formulate rational estimations for expected stock returns.

$$E[R_j] = \beta_j \cdot E[R_M]$$

```
##      GM       F     UTX     CAT     MRK     PFE     IBM    MSFT       C     XOM
## 0.05127 0.05060 0.04607 0.05626 0.03510 0.03753 0.03922 0.03522 0.03207 0.05737
```

Caveats for our Bayesian Capital Asset Pricing Model: (1) here, we did not consider uncertainties in risk-free rates (R_f). In practice, R_f is affected by the monetary policy of central banks and should not be ignored. (2) CAPM is a single-factor model, which only considers market risk R_M. The real world is much more complicated than this simple model and investors should take more factors into considerations.

References

- Ruppert, David. Statistics and data analysis for financial engineering. Vol. 13. New York: Springer, 2011.
- Sharpe, William F. "Capital asset prices: A theory of market equilibrium under conditions of risk." The journal of finance 19, no. 3 (1964): 425-442.