$C_{SW}^{(1)}$

(Written by Maximilian Ammer, 2023)
This notebook calculates the one-loop value of the improvement coefficient C_{SW}.
By choosing c1=0 or c1=-1/12 one can choose between the plaquette-(Wilson) gluon action or the Lüscher-Weisz gluon action.
By loading the appropriate files for the definition of the vertices V1, V2, V3 one can choose between the Wilson and the Brillouin fermion actions.

```
SetDirectory["~path~"];
```

In[ ]:=
```
Get["myLPT.m"]
```

Definitions

Parameters

In[ ]:=
```
(* Wilson parameter*)
```

In[ ]:=
```
r = 1;
```

In[ ]:=
```
(* Parameters of the Lüscher-Weisz gluon action*)
```

In[ ]:=
```
c1 = 0;(*-1/12;*)
c0 = 1 - 8 c1;
```

In[ ]:=
```
(* Parameters of the Brillouin action *)
λ0 = -240 / 64; λ1 = 8 / 64; λ2 = 4 / 64; λ3 = 2 / 64; λ4 = 1 / 64;
ϱ1 = 64 / 432; ϱ2 = 16 / 432; ϱ3 = 4 / 432; ϱ4 = 1 / 432;
```

Feynman Rules

Propagators

In[ ]:=
```
(* Gluon Propagator*)
```

In[ ]:=
```
G[mu_, nu_, k_] := GLW[mu, nu, k]
```

In[ ]:=
```
(* Fermion Propagator (Wilson or Brillouin)*)
S[k_] := SWil[k](*SBri[k]*)
```

Vertices

In[ ]:=
```
(* ggg-vertex *)
Vg3[mu_, nu_, rho_, k1_, k2_, k3_] := Vg3LW[mu, nu, rho, k1, k2, k3]
```

In[ ]:=
```
(* qqg-vertex *)
```

```
In[ ]:= DumpGet["V1_Wilson.mx"]
        DumpGet["V1_Brillouin.mx"]
        DumpGet["V1_Clover.mx"];

In[ ]:= V1[mu_, p_, q_] :=
         (V1Wilson(*V1Brillouin*)[mu, p, q] + V1Clover[mu, p, q]) // myVecExpand

In[ ]:= (* qqgg-vertex *)

In[ ]:= DumpGet["V2_Wilson.mx"]
        DumpGet["V2_Brillouin.mx"]
        DumpGet["V2_Clover.mx"]

In[ ]:= V2[mu_, nu_, p_, q_, k1_, k2_] :=
         a (V2Wilson[mu, nu, p, q] (*V2Brillouin[mu,nu,p,q,k1,k2]*) +
             V2Clover[mu, nu, p, q, k1, k2]) // myVecExpand

In[ ]:= (* qqggg-vertex *)

In[ ]:= DumpGet["V3_Wilson.mx"]
        DumpGet["V3_Brillouin.mx"]
        DumpGet["V3_Clover.mx"]

In[ ]:= V3[mu_, nu_, rho_, p_, q_, k1_, k2_, k3_] :=
         a (V3Wilson[mu, nu, rho, p, q] (*V3Brillouin[mu,nu,rho,p,q,k1,k2,k3]*) +
             V3Clover[mu, nu, rho, p, q, k1, k2, k3]) // myVecExpand
```

Calculation

Diagram (a)

```
In[ ]:= (* Color Factor *)
        G1aColor = T[b, a, b] // myColor
```

Out[ ]=

$$-\frac{T[a]}{2\,Nc}$$

```
In[ ]:= (* Diagram *)
        Λa[mu_] := G1aColor Sum[V1[nu, q + k, q].G[nu, rho, k].S[q + k].
            V1[mu, p + k, q + k].S[p + k].V1[rho, p, p + k] , {nu, 4}, {rho, 4}]

In[ ]:= (* Lattice integrand (divergent) *)
        G1al = G1[Λa, 1, 2] /. g0 → 1 /. T[a] → 1; // Timing

Out[ ]= {7.51368, Null}
```

In[ ]:=
```
(* Coefficient of divergence *)
(Ja = SeriesCoefficient[
    G1al /. k_ /; MemberQ[{k1, k2, k3, k4}, k] :> a k, {a, 0, 0}]) // Timing
```
Out[ ]=

$$\left\{184.867, \frac{-k1^2 + 3\,csw\,k1^2 + k2^2 - 3\,csw\,k2^2 + 2\,k3^2 - 4\,csw\,k3^2 + 2\,k4^2 - 4\,csw\,k4^2}{2\left(k1^2 + k2^2 + k3^2 + k4^2\right)^3 Nc}\right\}$$

In[ ]:=
```
G1ac = Coefficient[Ja /. k_^2 /; MemberQ[{k1, k2, k3, k4}, k] :> ksq / 4, 1 / (ksq^2)]
```
Out[ ]=

$$\frac{1 - 2\,csw}{2\,Nc}$$

Diagram (b)

In[ ]:=
```
(* Color Factor *)
G1bColor = T[b, c] × f[b, c, a] // myColor
```
Out[ ]=

$$\frac{1}{2}\,i\,Nc\,T[a]$$

In[ ]:=
```
(* Diagram *)
Λb[mu_] :=
 6 G1bColor Sum[V1[nu, q - k, q].G[nu, rho, k].G[sigma, tau, p - q + k].S[q - k].
     Vg3[rho, sigma, mu, -k, p - q + k, q - p].V1[tau, p, q - k],
   {nu, 4}, {rho, 4}, {sigma, 4}, {tau, 4}]
```

In[ ]:=
```
(* Lattice integrand (divergent) *)
G1bl = G1[Λb, 1, 2] /. g0 → 1 /. T[a] → 1; // Timing
```
Out[ ]=

```
{62.3412, Null}
```

```
(* Coefficient of divergence *)
(Jb = SeriesCoefficient[
    G1bl /. k_ /; MemberQ[{k1, k2, k3, k4}, k] :> a k, {a, 0, 0}]) // Timing
```
Out[ ]=

$$\Big\{32.7246, -\Big(\big(2\,k1^2\,Nc + csw\,k1^2\,Nc - 2\,k2^2\,Nc + 5\,csw\,k2^2\,Nc - 6\,k3^2\,Nc + 9\,csw\,k3^2\,Nc - 6\,k4^2\,Nc + 9\,csw\,k4^2\,Nc\big) \Big/ \Big(4\left(k1^2 + k2^2 + k3^2 + k4^2\right)^3\Big)\Big)\Big\}$$

In[ ]:=
```
G1bc = Coefficient[Jb /. k_^2 /; MemberQ[{k1, k2, k3, k4}, k] :> ksq / 4, 1 / (ksq^2)]
```
Out[ ]=

$$\frac{1}{4}\,(3\,Nc - 6\,csw\,Nc)$$

Diagram (c)

In[•]:=
```
(* Color Factor *)
G1cColor = T[b, c] × f[b, c, a] // myColor
```
Out[•]=

$$\frac{1}{2}\, i\, Nc\, T[a]$$

In[•]:=
```
(* Diagram *)
Λc[mu_] :=
 6 G1cColor Sum[V2[nu, sigma, p, q, q - p - k, k].G[sigma, tau, k].G[nu, rho, p - q + k].
    Vg3[rho, tau, mu, p - q + k, -k, q - p], {nu, 4}, {rho, 4}, {sigma, 4}, {tau, 4}]
```

In[•]:=
```
(* Lattice integrand (divergent) *)
G1cl = G1[Λc, 1, 2] /. g0 → 1 /. T[a] → 1; // Timing
```
Out[•]=

{55.6218, Null}

```
(* Coefficient of divergence *)
(Jc = SeriesCoefficient[
    G1cl /. k_ /; MemberQ[{k1, k2, k3, k4}, k] :> a k, {a, 0, 0}]) // Timing
```
Out[•]=

$$\left\{0.043016, \frac{3\, csw\, Nc}{4\left(k1^2 + k2^2 + k3^2 + k4^2\right)^2}\right\}$$

In[•]:=
```
G1cc = Coefficient[Jc /. k_^2 /; MemberQ[{k1, k2, k3, k4}, k] :> ksq / 4, 1 / (ksq^2)]
```
Out[•]=

$$\frac{3\, csw\, Nc}{4}$$

Diagram (d)

In[•]:=
```
(* Color Factor *)
G1cColor1 = T[a, b, b] // myColor
G1cColor2 = T[b, a, b] // myColor
G1cColor3 = T[b, b, a] // myColor
```
Out[•]=

CF T[a]

Out[•]=

$$-\frac{T[a]}{2\, Nc}$$

Out[•]=

CF T[a]

In[•]:=
```
(* Diagram *)
Λd[mu_] := Sum[
   G1cColor1 V3[mu, nu, rho, p, q, q - p, -k, k].G[nu, rho, k]
    + G1cColor2 V3[nu, mu, rho, p, q, -k, q - p, k].G[nu, rho, k]
    + G1cColor3 V3[nu, rho, mu, p, q, -k, k, q - p].G[nu, rho, k], {nu, 4}, {rho, 4}]
```

In[ ]:=
```
(* Lattice integrand *)
G1dl = G1[Λd, 1, 2] /. g0 → 1 /. T[a] → 1; // Timing
```
Out[ ]=
```
{6.61484, Null}
```

Diagram (e/f)

In[ ]:=
```
(* Color Factor *)
G1eColor1 = T[b, a, b] // myColor
G1eColor2 = T[b, b, a] // myColor
```
Out[ ]=

$$-\frac{T[a]}{2\ Nc}$$

Out[ ]=

$$CF\ T[a]$$

```
(* Diagram *)
Λe[mu_] :=
 G1eColor1 Sum[V1[nu, q + k, q].S[q + k].G[nu, rho, k].V2[mu, rho, p, q + k, q - p, k],
    {nu, 4}, {rho, 4}] + G1eColor2 Sum[V1[nu, q + k, q].S[q + k].
     G[nu, rho, k].V2[rho, mu, p, q + k, k, q - p], {nu, 4}, {rho, 4}]
```

In[ ]:=
```
(* Lattice integrand (divergent) *)
G1el = G1[Λe, 1, 2] /. g0 → 1 /. T[a] → 1; // Timing
```
Out[ ]=
```
{3.8933, Null}
```

In[ ]:=
```
(* Coefficient of divergence *)
(Je = SeriesCoefficient[G1el /. k_ /; MemberQ[{k1, k2, k3, k4}, k] :→ a k,
     {a, 0, 0}] // Simplify) // Timing
```
Out[ ]=

$$\left\{5.00988,\ -\frac{\left(k3^2+k4^2\right)\ (-1+2\ CF\ Nc)+csw\ \left(2\ k1^2-2\ k2^2-k3^2-k4^2\right)\ (1+2\ CF\ Nc)}{4\ \left(k1^2+k2^2+k3^2+k4^2\right)^3\ Nc}\right\}$$

In[ ]:=
```
G1ec = Coefficient[
   Je /. k_^2 /; MemberQ[{k1, k2, k3, k4}, k] :→ ksq / 4, 1 / (ksq^2)] // Simplify
```
Out[ ]=

$$\frac{1+csw-2\ CF\ Nc+2\ CF\ csw\ Nc}{8\ Nc}$$

In[ ]:=
```
G1fl = G1el;
G1fc = G1ec;
```

Directly integrating the sum of all diagrams

```
(* ! Attention ! The following results
  do not match the precision given in the paper ! *)
(* Integrations with higher accuracy and further
 error analysis is needed to reproduce those results *)
```

In[ ]:=
```
G1totalIntegrand = G1al + G1bl + G1cl + G1dl + G1el + G1fl /. a → 1;
```

In[ ]:=
```
(G1total = myNumInt[
    G1totalIntegrand /. csw → 1 /. CF → (Nc^2 - 1) / (2 Nc) /. Nc → 3, 6]) // Timing
```

Out[ ]=

$\{37.0228, \{0.134293, 1.58226 \times 10^{-8}\}\}$

In[ ]:=
```
csw1 = 2 G1total[[1]]
```

Out[ ]=

0.268586

In[ ]:=
```
G1totalIntegrandCF = Coefficient[G1totalIntegrand /. csw → 1, CF];
G1totalIntegrandNc = Coefficient[G1totalIntegrand /. csw → 1, Nc];
G1totalIntegrandNcInv = Coefficient[G1totalIntegrand /. csw → 1, 1 / Nc];
```

In[ ]:=
```
(G1totalCF = myNumInt[G1totalIntegrandCF, 6]) // Timing
(G1totalNc = myNumInt[G1totalIntegrandNc, 6]) // Timing
(G1totalNcInv = myNumInt[G1totalIntegrandNcInv, 6]) // Timing
```

Out[ ]=

$\{2.29786, \{0.0982489, 2.39687 \times 10^{-8}\}\}$

Out[ ]=

$\{14.3646, \{0.000295176, 6.30234 \times 10^{-9}\}\}$

Out[ ]=

$\{16.4593, \{0.00721709, 8.31377 \times 10^{-9}\}\}$

In[ ]:=
```
2 (CF G1totalCF[[1]] + Nc G1totalNc[[1]] + 1 / Nc G1totalNcInv[[1]]) // Expand
% /. CF → (Nc^2 - 1) / (2 Nc) // Expand
csw1Nc = Coefficient[%, Nc];
csw1NcInv = Coefficient[%%, 1 / Nc];
%%% /. Nc → 3
```

Out[ ]=

$$0.196498\,\mathrm{CF} + \frac{0.0144342}{\mathrm{Nc}} + 0.000590352\,\mathrm{Nc}$$

Out[ ]=

$$-\frac{0.0838147}{\mathrm{Nc}} + 0.0988393\,\mathrm{Nc}$$

Out[ ]=

0.26858

Integrating each diagram

```
B2 = 1 / (DB[k]^2);
```

Diagram (a)

In[ ]:=
```
G1aN = myNumInt[G1al - G1ac B2 /. a → 1 /. csw → 1 /. Nc → 3, 6]
```
Out[ ]=

$\{0.00492614, 3.04384 \times 10^{-9}\}$

In[ ]:=
```
csw1a = 2 (G1ac B2 + G1aN[[1]] /. csw → 1 /. Nc → 3) // Expand
```
Out[ ]=

$0.00985229 - \frac{B_2}{3}$

Diagram (b)

In[ ]:=
```
G1bN = myNumInt[G1bl - G1bc B2 /. a → 1 /. csw → 1 /. Nc → 3, 6]
```
Out[ ]=

$\{0.062948, 7.5626 \times 10^{-9}\}$

In[ ]:=
```
csw1b = 2 (G1bc B2 + G1bN[[1]] /. csw → 1 /. Nc → 3) // Expand
```
Out[ ]=

$0.125896 - \frac{9 B_2}{2}$

Diagram (c)

In[ ]:=
```
G1cN = myNumInt[G1cl - G1cc B2 /. a → 1 /. csw → 1 /. Nc → 3, 6]
```
Out[ ]=

$\{-0.0620628, 2.64696 \times 10^{-8}\}$

In[ ]:=
```
csw1c = 2 (G1cc B2 + G1cN[[1]] /. csw → 1 /. Nc → 3) // Expand
```
Out[ ]=

$-0.124126 + \frac{9 B_2}{2}$

Diagram (d)

In[ ]:=
```
G1dN = myNumInt[G1dl /. a → 1 /. csw → 1 /. CF → (Nc^2 - 1) / (2 Nc) /. Nc → 3, 6]
```
Out[ ]=

$\{0.148697, 2.77571 \times 10^{-8}\}$

In[ ]:=
```
csw1d = 2 (G1dN[[1]] /. csw → 1 /. CF → (Nc^2 - 1) / (2 Nc) /. Nc → 3) // Expand
```
Out[ ]=

0.297395

Diagram (e/f)

In[ ]:=
```
G1eN = myNumInt[G1el - G1ec B2 /. a → 1 /. csw → 1 /. CF → (Nc^2 - 1) / (2 Nc) /. Nc → 3, 6]
```
Out[ ]=

$\{-0.0101075, 1.96431 \times 10^{-9}\}$

In[ ]:=
```
csw1e = 2 (G1ec B2 + G1eN[[1]] /. csw → 1 /. CF → (Nc^2 - 1) / (2 Nc) /. Nc → 3) // Expand
```
Out[ ]=

$$-0.0202149 + \frac{B_2}{6}$$

Sum

In[ ]:=
```
csw1a + csw1b + csw1c + csw1d + 2 csw1e // Expand
```
Out[ ]=

0.268587