Exercise For Convexity and Optimization in $\mathbb{R}^n$

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May, 2025

Contents

Dedicated to the knee scrapes, playdates, and heartaches.

1 Topics in Real Analysis

1.1 Introduction

Exercise 1.1. For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, show that $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. Interpret this relation as a statement about parallelograms in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution. To prove that $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we use the definition of the Euclidean norm and properties of the dot product. Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.
First, expand $\|\mathbf{x}+\mathbf{y}\|^2$:

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Next, expand $\|\mathbf{x}-\mathbf{y}\|^2$:

$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Add these two expressions:

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = \left(\|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2\right)+\left(\|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2+\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+\|\mathbf{y}\|^2+2(\mathbf{x}\cdot\mathbf{y})-2(\mathbf{x}\cdot\mathbf{y}).$$

The cross terms $2(\mathbf{x} \cdot \mathbf{y})$ and $-2(\mathbf{x} \cdot \mathbf{y})$ cancel, yielding:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

Thus, the equality holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this equality has a geometric interpretation related to parallelograms. Consider vectors $\mathbf{x}$ and $\mathbf{y}$ emanating from the same initial point. These vectors form two adjacent sides of a parallelogram. The vector $\mathbf{x}+\mathbf{y}$ represents one diagonal of the parallelogram, and $\mathbf{x}-\mathbf{y}$ represents the other diagonal (assuming the parallelogram is completed appropriately).
The left side of the equality, $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2$, is the sum of the squares of the lengths of the two diagonals. The right side, $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$, is twice the sum of the squares of the lengths of the two adjacent sides. Since a parallelogram has two pairs of equal sides, the sum of the squares of the lengths of all four sides is $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ (two sides of length $\|\mathbf{x}\|$ and two of length $\|\mathbf{y}\|$).
Therefore, the equality states that for any parallelogram in $\mathbb{R}^2$ or $\mathbb{R}^3$, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all four sides. This is a fundamental property of parallelograms in Euclidean geometry, often called the parallelogram law.

1.3 Algebra of Sets

Lemma 3.1. Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of subsets of a set X. Then

$$\bigcup_{\alpha \in A} S_\alpha = c\left[\bigcap_{\alpha \in A} (cS_\alpha)\right],$$

$$\bigcap_{\alpha \in A} S_\alpha = c\left[\bigcup_{\alpha \in A} (cS_\alpha)\right].$$

Proof. We establish both identities by showing mutual inclusion of the corresponding sets.

1. $\displaystyle\bigcup_{\alpha \in A} S_\alpha = c\Big[\bigcap_{\alpha \in A} (cS_\alpha)\Big]$.
(i) Subset relation $\subseteq$. Let $x \in \bigcup_{\alpha \in A} S_\alpha$. Then there exists an index $\alpha_0 \in A$ such that $x \in S_{\alpha_0}$. If x were also contained in $\bigcap_{\alpha \in A}(cS_\alpha)$, it would belong to cS_{α_0}, i.e. $x \notin S_{\alpha_0}$, a contradiction. Therefore $x \notin \bigcap_{\alpha \in A}(cS_\alpha)$, which means $x \in c\big[\bigcap_{\alpha \in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Conversely, take $x \in c\big[\bigcap_{\alpha \in A}(cS_\alpha)\big]$. Then $x \notin \bigcap_{\alpha \in A}(cS_\alpha)$, so there exists an index $\alpha_1 \in A$ with $x \notin cS_{\alpha_1}$. Equivalently, $x \in S_{\alpha_1}$, hence $x \in \bigcup_{\alpha \in A} S_\alpha$. Combining (i) and (ii) yields the desired equality.

2. $\displaystyle\bigcap_{\alpha \in A} S_\alpha = c\Big[\bigcup_{\alpha \in A} (cS_\alpha)\Big]$.
The argument is analogous.
(i) Subset relation $\subseteq$. Let $x \in \bigcap_{\alpha \in A} S_\alpha$. Then $x \in S_\alpha$ for every α. Consequently, $x \notin cS_\alpha$ for any α, which implies $x \notin \bigcup_{\alpha \in A}(cS_\alpha)$. Hence $x \in c\big[\bigcup_{\alpha \in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Let $x \in c\big[\bigcup_{\alpha \in A}(cS_\alpha)\big]$. Then $x \notin \bigcup_{\alpha \in A}(cS_\alpha)$, so for every $\alpha \in A$ we have $x \notin cS_\alpha$; equivalently $x \in S_\alpha$. Therefore $x \in \bigcap_{\alpha \in A} S_\alpha$.
Since both inclusions hold, the second identity follows. $\square$

1.4 Metric Topology of $\mathbb{R}^n$

Exercise 4.1. Use the properties of the norm to show that the function d defined by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

is a metric, or distance function, on $\mathbb{R}^n$.

Proof. To verify that $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ is a metric on $\mathbb{R}^n$, we must show that it satisfies the following three properties for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$: *(i)* non–negativity with the identity of indiscernibles, *(ii)* symmetry, and *(iii)* the triangle inequality.

1. Non–negativity and identity of indiscernibles. The Euclidean norm is always non–negative, so

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \geq 0.$$

Moreover, $d(\mathbf{x},\mathbf{y}) = 0$ if and only if $\|\mathbf{x}-\mathbf{y}\| = 0$, which occurs precisely when $\mathbf{x}-\mathbf{y} = \mathbf{0}$, i.e. when $\mathbf{x} = \mathbf{y}$.

2. Symmetry. Because $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for any vector $\mathbf{v}$,

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \|-(\mathbf{x}-\mathbf{y})\| = \|\mathbf{y}-\mathbf{x}\| = d(\mathbf{y},\mathbf{x}).$$

3. Triangle inequality. The Euclidean norm satisfies the triangle inequality $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Choosing $\mathbf{u} = \mathbf{x}-\mathbf{y}$ and $\mathbf{v} = \mathbf{y}-\mathbf{z}$ gives

$$d(\mathbf{x},\mathbf{z}) = \|\mathbf{x}-\mathbf{z}\| = \|(\mathbf{x}-\mathbf{y}) + (\mathbf{y}-\mathbf{z})\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}).$$

Since all three axioms hold, the function d is indeed a metric on $\mathbb{R}^n$. □

Exercise 4.2. (a) Sketch the graph of $y = \sin(1/x)$ for $x > 0$.
(b) Consider the graph as a set in $\mathbb{R}^2$ and find the limit points of this set.

Solution. (a) Sketch of the curve. Set $t = 1/x$ for $x > 0$. Then the graph of $y = \sin(1/x)$ for $x > 0$ corresponds to the standard sine curve $y = \sin t$ for $t > 0$, but viewed through the change of variables $x = 1/t$. As $x \to 0^+$ we have $t \to +\infty$, so the curve oscillates infinitely often between -1 and 1 while its x–coordinate approaches 0. For moderate values of x the graph resembles the usual sine curve stretched horizontally, whereas near the y–axis the oscillations become increasingly rapid, creating a "comb-like" fringe that accumulates on the interval $\{0\} \times [-1, 1]$.

(b) Limit points of the graph. Let

$$S \;:=\; \big\{\,(x, \sin(1/x)) : x > 0\,\big\} \subset \mathbb{R}^2.$$

We claim that the set of limit points of S is

$$S' \;=\; S \,\cup\, \big\{\,(0, y) : -1 \leq y \leq 1\,\big\}.$$

The verification proceeds in two steps.

Step 1: Every point of S is a limit point. Fix $(x_0, \sin(1/x_0)) \in S$ with $x_0 > 0$ and let $\varepsilon > 0$ be given. Recall the elementary inequality $|\sin u - \sin v| \leq |u - v|$ valid for all real numbers u, v. Set

$$\delta := \min\Big\{\frac{\varepsilon}{2},\, \frac{\varepsilon x_0^2}{4},\, \frac{x_0}{2}\Big\}.$$

Pick any $x_\varepsilon \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ and define $P_\varepsilon := (x_\varepsilon, \sin(1/x_\varepsilon))$. Then $P_\varepsilon \in S$ and

$$|x_\varepsilon - x_0| < \frac{\varepsilon}{2}, \qquad |\sin(1/x_\varepsilon) - \sin(1/x_0)| \leq |1/x_\varepsilon - 1/x_0| = \frac{|x_\varepsilon - x_0|}{|x_\varepsilon| x_0} \leq \frac{2\delta}{x_0^2} \leq \frac{\varepsilon}{2}.$$

Consequently

$$\big\|P_\varepsilon - (x_0, \sin(1/x_0))\big\| \leq \sqrt{\big(|x_\varepsilon - x_0|\big)^2 + \big(|\sin(1/x_\varepsilon) - \sin(1/x_0)|\big)^2} < \varepsilon,$$

proving that $(x_0, \sin(1/x_0))$ is indeed a limit point of S.

Step 2: Points of the form $(0, y)$ with $|y| \leq 1$ are limit points. Fix $y \in [-1, 1]$ and $\varepsilon > 0$. Because the sine function attains every value in $[-1, 1]$ infinitely often, we can pick $t_\varepsilon > \max\{1/\varepsilon, 0\}$ such that $|\sin t_\varepsilon - y| < \varepsilon$. Setting $x_\varepsilon = 1/t_\varepsilon$ we obtain $0 < x_\varepsilon < \varepsilon$ and

$$\big\|(x_\varepsilon, \sin(1/x_\varepsilon)) - (0, y)\big\| < \sqrt{\varepsilon^2 + \varepsilon^2} < \sqrt{2}\,\varepsilon.$$

Thus $(0, y)$ is approached by points of S distinct from itself, so it is a limit point.

Step 3: No other points are limit points. If $x < 0$, every open ball centred at (x, y) contains points whose first coordinate is negative, whereas S lies entirely in $x > 0$; thus such points cannot be limit points. If $x = 0$ but $|y| > 1$, the vertical separation $|y| - 1$ already exceeds the range of sin, so no sequence in S can approach $(0, y)$.
Finally, consider a point (x, y) with $x > 0$ that does
emphnot belong to S. Write $y_0 := \sin(1/x)$ and set $d := |y - y_0| > 0$. Choose

$$r := \min\Big\{\frac{d}{2},\, \frac{x}{2}\Big\}.$$

For any $(x', \sin(1/x')) \in S$ satisfying $|x' - x| < r$ we have $x' \geq x/2$ and hence, using $|\sin u - \sin v| \leq |u - v|$ again,

$$|\sin(1/x') - y_0| \leq \frac{|x' - x|}{x'x} \leq \frac{2r}{x^2} \leq \frac{d}{2}.$$

Therefore

$$|\sin(1/x') - y| \geq |y - y_0| - |\sin(1/x') - y_0| > \frac{d}{2},$$

so the Euclidean distance between $(x', \sin(1/x'))$ and (x, y) exceeds $d/2$. Hence the ball $B\big((x, y), d/2\big)$ contains no point of S, proving that (x, y) is not a limit point. Collecting the cases established in Steps 1–3 completes the description of S'.

Consequently, the set of all limit points of the graph is precisely $S \cup \big(\{0\} \times [-1, 1]\big)$, as asserted.

Exercise 4.3. Show that for $x \in \mathbb{R}^n$ and $r > 0$ the set $B(x, r)$ is open; that is, show that an open ball is open.

Proof. Let $y \in B(x, r)$, so by definition $\|y - x\| < r$. Define

$$\varepsilon \ := \ r - \|y - x\| \ > \ 0.$$

We claim that the entire ball $B(y, \varepsilon)$ is contained in $B(x, r)$. Indeed, if $z \in B(y, \varepsilon)$ then $\|z - y\| < \varepsilon$, and by the triangle inequality,

$$\|z - x\| \ \leq \ \|z - y\| + \|y - x\| \ < \ \varepsilon + \|y - x\| \ = \ r.$$

Hence $z \in B(x, r)$. Since every point y of $B(x, r)$ is an interior point, the set $B(x, r)$ is open. □

Remark. *Idea of the proof.* To show that the open ball $B(x, r)$ is an open set we verify that every point it contains is an interior point. Fix $y \in B(x, r)$ and measure how much "room" is left before reaching the boundary: the gap is $\varepsilon := r - \|y - x\| > 0$. Any point z that sits within this gap around y (that is, $\|z - y\| < \varepsilon$) cannot escape the larger ball, because the triangle inequality guarantees $\|z - x\| < \|z - y\| + \|y - x\| < \varepsilon + \big(r - \varepsilon\big) = r$. Thus the smaller ball $B(y, \varepsilon)$ lies completely inside $B(x, r)$, making y an interior point. Since y was arbitrary, $B(x, r)$ is open.

Exercise 4.4. Show that for $x \in \mathbb{R}^n$ and $r > 0$ the closed ball $\overline{B(x, r)}$ is closed.

Proof. Set $C := \overline{B(x, r)} = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$. We show that its complement C^c is open. Let $y \in C^c$, so $\|y - x\| > r$. Define

$$\varepsilon \ := \ \frac{\|y - x\| - r}{2} \ > \ 0.$$

If $z \in B(y, \varepsilon)$ then $\|z - y\| < \varepsilon$, and by the triangle inequality,

$$\|z - x\| \ \geq \ \|y - x\| - \|z - y\| \ > \ \big(r + 2\varepsilon\big) - \varepsilon \ = \ r + \varepsilon \ > \ r.$$

Hence $z \notin C$. Therefore $B(y, \varepsilon) \subset C^c$, proving that every point of C^c is interior. The complement of C is open, so C is closed. □

Remark. *Idea of the proof.* For a point y lying *outside* the closed ball we measure how far it is from the boundary: the surplus distance is $\delta := \|y - x\| - r > 0$. Choosing half of this surplus as a radius, $\varepsilon = \delta/2$, guarantees that the entire ball $B(y, \varepsilon)$ stays outside, because any point inside that small ball remains at least $r + \varepsilon > r$ away from x. Since such a neighbourhood exists around each exterior point, the complement of the closed ball is open, which is exactly the definition of the original set being closed.