Exercise For Convexity and Optimization in $\mathbb{R}^n$

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Contents

Dedicated to the knee scrapes, playdates, and heartaches.

1 Topics in Real Analysis

1.1 Introduction

Exercise 1.1. For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, show that $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$. Interpret this relation as a statement about parallelograms in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution. To prove that $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we use the definition of the Euclidean norm and properties of the dot product. Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.
First, expand $\|\mathbf{x}+\mathbf{y}\|^2$:

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2.$$

Next, expand $\|\mathbf{x}-\mathbf{y}\|^2$:

$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2 - 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2.$$

Add these two expressions:

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = \left(\|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2\right)+\left(\|\mathbf{x}\|^2 - 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2+\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+\|\mathbf{y}\|^2+2(\mathbf{x}\cdot\mathbf{y})-2(\mathbf{x}\cdot\mathbf{y}).$$

The cross terms $2(\mathbf{x}\cdot\mathbf{y})$ and $-2(\mathbf{x}\cdot\mathbf{y})$ cancel, yielding:

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2.$$

Thus, the equality holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this equality has a geometric interpretation related to parallelograms. Consider vectors $\mathbf{x}$ and $\mathbf{y}$ emanating from the same initial point. These vectors form two adjacent sides of a parallelogram. The vector $\mathbf{x}+\mathbf{y}$ represents one diagonal of the parallelogram, and $\mathbf{x}-\mathbf{y}$ represents the other diagonal (assuming the parallelogram is completed appropriately).
The left side of the equality, $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2$, is the sum of the squares of the lengths of the two diagonals. The right side, $2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$, is twice the sum of the squares of the lengths of the two adjacent sides. Since a parallelogram has two pairs of equal sides, the sum of the squares of the lengths of all four sides is $2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$ (two sides of length $\|\mathbf{x}\|$ and two of length $\|\mathbf{y}\|$).
Therefore, the equality states that for any parallelogram in $\mathbb{R}^2$ or $\mathbb{R}^3$, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all four sides. This is a fundamental property of parallelograms in Euclidean geometry, often called the parallelogram law.

1.3 Algebra of Sets

Lemma 3.1. Let $\{S_\alpha\}_{\alpha\in A}$ be a collection of subsets of a set X. Then

$$\bigcup_{\alpha\in A} S_\alpha = c\left[\bigcap_{\alpha\in A}(cS_\alpha)\right],$$

$$\bigcap_{\alpha\in A} S_\alpha = c\left[\bigcup_{\alpha\in A}(cS_\alpha)\right].$$

Proof. We establish both identities by showing mutual inclusion of the corresponding sets.

1. $\displaystyle\bigcup_{\alpha\in A} S_\alpha = c\Big[\bigcap_{\alpha\in A}(cS_\alpha)\Big]$**.**
(i) Subset relation $\subseteq$. Let $x \in \bigcup_{\alpha\in A} S_\alpha$. Then there exists an index $\alpha_0 \in A$ such that $x \in S_{\alpha_0}$. If x were also contained in $\bigcap_{\alpha\in A}(cS_\alpha)$, it would belong to cS_{α_0}, i.e. $x \notin S_{\alpha_0}$, a contradiction. Therefore $x \notin \bigcap_{\alpha\in A}(cS_\alpha)$, which means $x \in c\big[\bigcap_{\alpha\in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Conversely, take $x \in c\big[\bigcap_{\alpha\in A}(cS_\alpha)\big]$. Then $x \notin \bigcap_{\alpha\in A}(cS_\alpha)$, so there exists an index $\alpha_1 \in A$ with $x \notin cS_{\alpha_1}$. Equivalently, $x \in S_{\alpha_1}$, hence $x \in \bigcup_{\alpha\in A} S_\alpha$. Combining (i) and (ii) yields the desired equality.

2. $\displaystyle\bigcap_{\alpha\in A} S_\alpha = c\Big[\bigcup_{\alpha\in A}(cS_\alpha)\Big]$**.**
The argument is analogous.
(i) Subset relation $\subseteq$. Let $x \in \bigcap_{\alpha\in A} S_\alpha$. Then $x \in S_\alpha$ for every α. Consequently, $x \notin cS_\alpha$ for any α, which implies $x \notin \bigcup_{\alpha\in A}(cS_\alpha)$. Hence $x \in c\big[\bigcup_{\alpha\in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Let $x \in c\big[\bigcup_{\alpha\in A}(cS_\alpha)\big]$. Then $x \notin \bigcup_{\alpha\in A}(cS_\alpha)$, so for every $\alpha \in A$ we have $x \notin cS_\alpha$; equivalently $x \in S_\alpha$. Therefore $x \in \bigcap_{\alpha\in A} S_\alpha$.
Since both inclusions hold, the second identity follows. $\square$

1.4 Metric Topology of $\mathbb{R}^n$

Exercise 4.1. Use the properties of the norm to show that the function d defined by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

is a metric, or distance function, on $\mathbb{R}^n$.

Proof. To verify that $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ is a metric on $\mathbb{R}^n$, we must show that it satisfies the following three properties for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$: *(i)* non–negativity with the identity of indiscernibles, *(ii)* symmetry, and *(iii)* the triangle inequality.

1. Non–negativity and identity of indiscernibles. The Euclidean norm is always non–negative, so

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \geq 0.$$

Moreover, $d(\mathbf{x},\mathbf{y}) = 0$ if and only if $\|\mathbf{x}-\mathbf{y}\| = 0$, which occurs precisely when $\mathbf{x}-\mathbf{y} = \mathbf{0}$, i.e. when $\mathbf{x} = \mathbf{y}$.

2. Symmetry. Because $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for any vector $\mathbf{v}$,

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \|-(\mathbf{x}-\mathbf{y})\| = \|\mathbf{y}-\mathbf{x}\| = d(\mathbf{y},\mathbf{x}).$$

3. Triangle inequality. The Euclidean norm satisfies the triangle inequality $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Choosing $\mathbf{u} = \mathbf{x}-\mathbf{y}$ and $\mathbf{v} = \mathbf{y}-\mathbf{z}$ gives

$$d(\mathbf{x},\mathbf{z}) = \|\mathbf{x}-\mathbf{z}\| = \|(\mathbf{x}-\mathbf{y}) + (\mathbf{y}-\mathbf{z})\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}).$$

Since all three axioms hold, the function d is indeed a metric on $\mathbb{R}^n$. $\square$

Exercise 4.2. (a) Sketch the graph of $y = \sin(1/x)$ for $x > 0$.
(b) Consider the graph as a set in $\mathbb{R}^2$ and find the limit points of this set.

Solution. (a) Sketch of the curve. Set $t = 1/x$ for $x > 0$. Then the graph of $y = \sin(1/x)$ for $x > 0$ corresponds to the standard sine curve $y = \sin t$ for $t > 0$, but viewed through the change of variables $x = 1/t$. As $x \to 0^+$ we have $t \to +\infty$, so the curve oscillates infinitely often between -1 and 1 while its x–coordinate approaches 0. For moderate values of x the graph resembles the usual sine curve stretched horizontally, whereas near the y–axis the oscillations become increasingly rapid, creating a "comb-like" fringe that accumulates on the interval $\{0\} \times [-1,1]$.

(b) Limit points of the graph. Let

$$S := \big\{ (x, \sin(1/x)) : x > 0 \big\} \subset \mathbb{R}^2.$$

We claim that the set of limit points of S is

$$S' = S \cup \big\{ (0,y) : -1 \leq y \leq 1 \big\}.$$

The verification proceeds in two steps.

Step 1: Every point of S is a limit point. Fix $(x_0, \sin(1/x_0)) \in S$ with $x_0 > 0$ and let $\varepsilon > 0$ be given. Recall the elementary inequality $|\sin u - \sin v| \leq |u - v|$ valid for all real numbers u, v. Set

$$\delta := \min\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon x_0^2}{4}, \frac{x_0}{2}\Big\}.$$

Pick any $x_\varepsilon \in (x_0-\delta, x_0+\delta) \setminus \{x_0\}$ and define $P_\varepsilon := (x_\varepsilon, \sin(1/x_\varepsilon))$. Then $P_\varepsilon \in S$ and

$$|x_\varepsilon - x_0| < \frac{\varepsilon}{2}, \qquad |\sin(1/x_\varepsilon) - \sin(1/x_0)| \leq |1/x_\varepsilon - 1/x_0| = \frac{|x_\varepsilon - x_0|}{|x_\varepsilon| x_0} \leq \frac{2\delta}{x_0^2} \leq \frac{\varepsilon}{2}.$$

Consequently

$$\big\|P_\varepsilon - (x_0, \sin(1/x_0))\big\| \leq \sqrt{\big(|x_\varepsilon - x_0|\big)^2 + \big(|\sin(1/x_\varepsilon) - \sin(1/x_0)|\big)^2} < \varepsilon,$$

proving that $(x_0, \sin(1/x_0))$ is indeed a limit point of S.

Step 2: Points of the form $(0,y)$ with $|y| \leq 1$ are limit points. Fix $y \in [-1,1]$ and $\varepsilon > 0$. Because the sine function attains every value in $[-1,1]$ infinitely often, we can pick $t_\varepsilon > \max\{1/\varepsilon, 0\}$ such that $|\sin t_\varepsilon - y| < \varepsilon$. Setting $x_\varepsilon = 1/t_\varepsilon$ we obtain $0 < x_\varepsilon < \varepsilon$ and

$$\big\|(x_\varepsilon, \sin(1/x_\varepsilon)) - (0,y)\big\| < \sqrt{\varepsilon^2 + \varepsilon^2} < \sqrt{2}\,\varepsilon.$$

Thus $(0,y)$ is approached by points of S distinct from itself, so it is a limit point.

Step 3: No other points are limit points. If $x < 0$, every open ball centred at (x,y) contains points whose first coordinate is negative, whereas S lies entirely in $x > 0$; thus such points cannot be limit points. If $x = 0$ but $|y| > 1$, the vertical separation $|y| - 1$ already exceeds the range of sin, so no sequence in S can approach $(0,y)$.
Finally, consider a point (x,y) with $x > 0$ that does
emphnot belong to S. Write $y_0 := \sin(1/x)$ and set $d := |y - y_0| > 0$. Choose

$$r := \min\Big\{\frac{d}{2}, \frac{x}{2}\Big\}.$$

For any $(x', \sin(1/x')) \in S$ satisfying $|x' - x| < r$ we have $x' \geq x/2$ and hence, using $|\sin u - \sin v| \leq |u - v|$ again,

$$|\sin(1/x') - y_0| \leq \frac{|x' - x|}{x'x} \leq \frac{2r}{x^2} \leq \frac{d}{2}.$$

Therefore

$$|\sin(1/x') - y| \geq |y - y_0| - |\sin(1/x') - y_0| > \frac{d}{2},$$

so the Euclidean distance between $(x', \sin(1/x'))$ and (x, y) exceeds $d/2$. Hence the ball $B\big((x, y), d/2\big)$ contains no point of S, proving that (x, y) is not a limit point. Collecting the cases established in Steps 1–3 completes the description of S'.

Consequently, the set of all limit points of the graph is precisely $S \cup \big(\{0\} \times [-1, 1]\big)$, as asserted.

Exercise 4.3. Show that for $x \in \mathbb{R}^n$ and $r > 0$ the set $B(x, r)$ is open; that is, show that an open ball is open.

Proof. Let $y \in B(x, r)$, so by definition $\|y - x\| < r$. Define

$$\varepsilon \;:=\; r - \|y - x\| \;>\; 0.$$

We claim that the entire ball $B(y, \varepsilon)$ is contained in $B(x, r)$. Indeed, if $z \in B(y, \varepsilon)$ then $\|z - y\| < \varepsilon$, and by the triangle inequality,

$$\|z - x\| \;\leq\; \|z - y\| + \|y - x\| \;<\; \varepsilon + \|y - x\| \;=\; r.$$

Hence $z \in B(x, r)$. Since every point y of $B(x, r)$ is an interior point, the set $B(x, r)$ is open. □

Remark. *Idea of the proof.* To show that the open ball $B(x, r)$ is an open set we verify that every point it contains is an interior point. Fix $y \in B(x, r)$ and measure how much "room" is left before reaching the boundary: the gap is $\varepsilon := r - \|y - x\| > 0$. Any point z that sits within this gap around y (that is, $\|z - y\| < \varepsilon$) cannot escape the larger ball, because the triangle inequality guarantees $\|z - x\| < \|z - y\| + \|y - x\| < \varepsilon + \big(r - \varepsilon\big) = r$. Thus the smaller ball $B(y, \varepsilon)$ lies completely inside $B(x, r)$, making y an interior point. Since y was arbitrary, $B(x, r)$ is open.

Exercise 4.4. Show that for $x \in \mathbb{R}^n$ and $r > 0$ the closed ball $\overline{B(x, r)}$ is closed.

Proof. Set $C := \overline{B(x, r)} = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$. We show that its complement C^c is open. Let $y \in C^c$, so $\|y - x\| > r$. Define

$$\varepsilon \;:=\; \frac{\|y - x\| - r}{2} \;>\; 0.$$

If $z \in B(y, \varepsilon)$ then $\|z - y\| < \varepsilon$, and by the triangle inequality,

$$\|z - x\| \;\geq\; \|y - x\| - \|z - y\| \;>\; \big(r + 2\varepsilon\big) - \varepsilon \;=\; r + \varepsilon \;>\; r.$$

Hence $z \notin C$. Therefore $B(y, \varepsilon) \subset C^c$, proving that every point of C^c is interior. The complement of C is open, so C is closed. □

Remark. *Idea of the proof.* For a point y lying *outside* the closed ball we measure how far it is from the boundary: the surplus distance is $\delta := \|y - x\| - r > 0$. Choosing half of this surplus as a radius, $\varepsilon = \delta/2$, guarantees that the entire ball $B(y, \varepsilon)$ stays outside, because any point inside that small ball remains at least $r + \varepsilon > r$ away from x. Since such a neighbourhood exists around each exterior point, the complement of the closed ball is open, which is exactly the definition of the original set being closed.

Exercise 4.5. Show that any finite set of points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ is closed.

Proof. Denote the finite set by $F := \{x_1, \ldots, x_k\}$. We show that its complement F^c is open. Let $y \in F^c$; then $y \neq x_i$ for every i. Define the positive distances

$$d_i := \|y - x_i\| > 0, \qquad i = 1, \ldots, k, \quad \text{and set} \quad \varepsilon := \tfrac{1}{2} \min_{1 \leq i \leq k} d_i > 0.$$

For any $z \in B(y, \varepsilon)$ we have $\|z - y\| < \varepsilon \leq d_i/2$, hence by the triangle inequality

$$\|z - x_i\| \;\geq\; \|y - x_i\| - \|z - y\| \;>\; d_i - \varepsilon \;\geq\; \tfrac{1}{2} d_i > 0 \quad (1 \leq i \leq k).$$

Consequently $z \neq x_i$ for every i, i.e. $z \in F^c$. Thus $B(y, \varepsilon) \subset F^c$, so every exterior point is interior to the complement; F^c is open and F is closed.

An alternative argument is to note that each singleton $\{x_i\}$ is closed (apply the previous step with $k = 1$) and that a finite union of closed sets remains closed:

$$F \;=\; \bigcup_{i=1}^{k} \{x_i\}.$$ □

Remark. *Idea of the proof.* For a point y not in the finite set we compute its distances to each listed point. The smallest of these distances is still positive; taking half of it as the radius gives a neighbourhood around y that misses the entire finite set, showing the complement is open. Equivalently, observe that singletons are closed and a finite union of closed sets is closed.

Exercise 4.6. Show that in $\mathbb{R}^n$ no point x with $\|x\| = 1$ is an interior point of the closed unit ball $\overline{B(0,1)}$.

Proof. Let $x \in \mathbb{R}^n$ satisfy $\|x\| = 1$. To show that x is *not* an interior point of $\overline{B(0,1)}$ we must prove that every open ball centred at x contains points that lie *outside* $\overline{B(0,1)}$.
Fix an arbitrary radius $r > 0$ and define

$$y := \left(1 + \tfrac{r}{2}\right) x.$$

Then

$$\|y - x\| = \left|1 + \tfrac{r}{2} - 1\right| \|x\| = \frac{r}{2} < r, \qquad \text{but} \qquad \|y\| = 1 + \tfrac{r}{2} > 1.$$

Hence y lies within the ball $B(x,r)$ yet satisfies $\|y\| > 1$, so $y \notin \overline{B(0,1)}$. Consequently $B(x,r)$ is *not* contained in $\overline{B(0,1)}$. Because $r > 0$ was arbitrary, no open neighbourhood of x sits inside the closed unit ball, and thus x is not an interior point. ☐

Remark. *Idea of the proof.* Any boundary point x with $\|x\| = 1$ can be pushed slightly outward along its own direction: scaling by $1 + \varepsilon$ moves the point a distance roughly ε while immediately leaving the unit ball. No matter how small a radius we choose, such an outward perturbation fits inside the radius yet escapes the ball, so the original point cannot be interior.

Exercise 4.7. Show that for any set S in $\mathbb{R}^n$ the set $\bar{S}$ (the closure of S) is closed.

Proof. Recall that the closure of S is defined by

$$\bar{S} := \{x \in \mathbb{R}^n : \forall r > 0,\ B(x,r) \cap S \neq \varnothing\}.$$

We shall prove that its complement $\bar{S}^{c}$ is open.
Let $y \in \bar{S}^{c}$. By definition of the complement, there exists a radius $r_0 > 0$ such that the open ball $B(y, r_0)$ does *not* meet S; otherwise y would belong to the closure. In particular, for this same r_0 we have

$$B(y, r_0) \cap S = \varnothing \quad \Longrightarrow \quad B(y, r_0) \cap \bar{S} = \varnothing,$$

because $\bar{S}$ contains S. Hence $B(y, r_0)$ lies entirely inside $\bar{S}^{c}$, showing that y is an interior point of the complement. Since every point of $\bar{S}^{c}$ is interior, the complement is open; equivalently, $\bar{S}$ is closed. ☐

Remark. *Idea of the proof.* A point fails to belong to the closure precisely when some open ball around it avoids S. But that same ball automatically avoids $\bar{S}$ as well, so it sits inside the complement. Thus every exterior point has a protective open neighbourhood—the hallmark of an open set—so the complement is open and the closure is closed.

Exercise 4.8. Show that for any set S the closure $\bar{S}$ is equal to the intersection of all closed sets containing S.

Proof. Denote by

$$\mathcal{F} := \{\, F \subset \mathbb{R}^n : F \text{ is closed and } S \subset F \,\}, \qquad K := \bigcap_{F \in \mathcal{F}} F.$$

The intersection of any family of closed sets is closed, hence K is closed. Because $S \subset F$ for every $F \in \mathcal{F}$, we also have $S \subset K$.

1. $\bar{S} \subset K$**.** Let $F \in \mathcal{F}$. Since F is closed and contains S, the closure property of $\bar{S}$ implies $\bar{S} \subset F$. This holds for every $F \in \mathcal{F}$, therefore $\bar{S} \subset K$.

2. $K \subset \bar{S}$**.** The closure $\bar{S}$ itself is closed and contains S, so $\bar{S} \in \mathcal{F}$. Intersecting over $\mathcal{F}$ cannot yield a set larger than one of its members, hence $K \subset \bar{S}$.

Combining the two inclusions gives $K = \bar{S}$. That is, the closure of S coincides with the intersection of all closed sets that contain S. ☐

Remark. *Idea of the proof.* The family of all closed supersets of S has a natural "smallest" member given by their intersection K. Because K is itself closed and contains S, the closure $\bar{S}$—which is the minimal closed set containing S—must sit inside K. On the other hand, $\bar{S}$ belongs to the family, so the intersection K cannot fall outside $\bar{S}$. The two sets therefore match exactly.

1.5 Limits and Continuity

Exercise 5.1. (i) The limit of a convergent sequence is unique.
(ii) Let $\lim_{k\to\infty} x_k = x_0$, let $\lim_{k\to\infty} y_k = y_0$, and let $\lim_{k\to\infty} \alpha_k = \alpha$, where $\{\alpha_k\}$ is a sequence of scalars. Then $\lim_{k\to\infty}(x_k + y_k)$ exists and equals $x_0 + y_0$, and $\lim_{k\to\infty} \alpha_k x_k$ exists and equals αx_0.
(iii) A convergent sequence is bounded.

Proof. We work in $\mathbb{R}^n$ and use the standard $\|\cdot\|$ norm; limits are taken with respect to this norm.

(i) Uniqueness of the limit. Suppose a sequence $\{x_k\}$ converges to both ℓ_1 and ℓ_2. Fix $\varepsilon > 0$ and choose

$N_1, N_2 \in \mathbb{N}$ such that

$$k \geq N_1 \implies \|x_k - \ell_1\| < \varepsilon/2, \qquad k \geq N_2 \implies \|x_k - \ell_2\| < \varepsilon/2.$$

For $k \geq \max\{N_1, N_2\}$ we then have

$$\|\ell_1 - \ell_2\| \;\leq\; \|\ell_1 - x_k\| + \|x_k - \ell_2\| < \varepsilon.$$

Because ε was arbitrary, $\|\ell_1 - \ell_2\| = 0$, whence $\ell_1 = \ell_2$. The limit is therefore unique.

(ii) Limits of sums and scalar multiples.

1. Let $\varepsilon > 0$. Choose N_1, N_2 such that $k \geq N_1$ implies $\|x_k - x_0\| < \varepsilon/2$ and $k \geq N_2$ implies $\|y_k - y_0\| < \varepsilon/2$. For $k \geq N := \max\{N_1, N_2\}$ we have

$$\|(x_k + y_k) - (x_0 + y_0)\| \;\leq\; \|x_k - x_0\| + \|y_k - y_0\| < \varepsilon.$$

   Hence $x_k + y_k \to x_0 + y_0$.

2. Given $\varepsilon > 0$, convergence of α_k means there exists N_1 with $|\alpha_k - \alpha| < \varepsilon/(2(1+\|x_0\|))$ for $k \geq N_1$. Convergence of x_k gives N_2 with $\|x_k - x_0\| < \varepsilon/(2(1+|\alpha|))$ for $k \geq N_2$. For $k \geq N := \max\{N_1, N_2\}$ we estimate

$$\|\alpha_k x_k - \alpha x_0\| \;\leq\; |\alpha_k - \alpha|\,\|x_k\| + |\alpha|\,\|x_k - x_0\|.$$

   The triangle inequality together with $\|x_k\| \leq \|x_k - x_0\| + \|x_0\|$ yields

$$\|\alpha_k x_k - \alpha x_0\| < \frac{\varepsilon}{2}\,\frac{\|x_k - x_0\| + \|x_0\|}{1 + \|x_0\|} + \frac{\varepsilon}{2} = \varepsilon.$$

   Thus $\alpha_k x_k \to \alpha x_0$.

(iii) A convergent sequence is bounded. Let $x_k \to x_0$. Take $\varepsilon := 1$ and choose N so that $\|x_k - x_0\| < 1$ for all $k \geq N$. Then

$$\|x_k\| \leq \|x_k - x_0\| + \|x_0\| < 1 + \|x_0\| \quad (k \geq N).$$

Define

$$M \;:=\; \max\bigl\{\,\|x_1\|, \ldots, \|x_{N-1}\|,\, 1 + \|x_0\|\bigr\}.$$

Every term of the sequence satisfies $\|x_k\| \leq M$, so the sequence is bounded. □

Remark. *Idea of the proof.* Uniqueness comes from squeezing two putative limits together via the triangle inequality. The limit laws follow by splitting the error of a sum or product into individually controllable parts. Boundedness is immediate once the tail of the sequence lies in a fixed ball around its limit, while the finitely many initial terms can be absorbed into a single bound.

Exercise 5.2. Show that if $\lim_{x\to s} f(x) = L$, then L is unique.

Proof. Assume that f approaches two limits L_1 and L_2 at the point s. We show that they must coincide. Suppose, for contradiction, that $L_1 \neq L_2$. Set

$$\varepsilon \;:=\; \frac{|L_1 - L_2|}{3} \;>\; 0.$$

Because $\lim_{x\to s} f(x) = L_1$, there exists $\delta_1 > 0$ such that

$$0 < |x - s| < \delta_1 \implies |f(x) - L_1| < \varepsilon.$$

Likewise, convergence to L_2 furnishes $\delta_2 > 0$ with

$$0 < |x - s| < \delta_2 \implies |f(x) - L_2| < \varepsilon.$$

Now choose any x satisfying

$$0 < |x - s| < \delta \;:=\; \min\{\delta_1, \delta_2\}.$$

Then both inequalities hold simultaneously, and the triangle inequality gives

$$|L_1 - L_2| \;\leq\; |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon = \tfrac{2}{3}|L_1 - L_2|,$$

which is impossible. Hence our assumption $L_1 \neq L_2$ must be false; therefore $L_1 = L_2$. The limit value is unique. □

Remark. *Idea of the proof.* If two distinct numbers both pretend to be the limit, pick an ε slightly smaller than their separation. Close enough to s, the function has to lie within that ε-tube around each candidate limit, but those two tubes do not overlap—contradiction. The triangle inequality formalises this intuition and forces the two limits to coincide.

Exercise 5.3. Prove the following theorem.
Let $\lim_{x\to s} f(x) = L$ and let $\lim_{x\to s} g(x) = M$. Then

$$\lim_{x\to s}[f(x) + g(x)]$$

exists and equals $L + M$. Also $\lim_{x\to s}[\alpha f(x)]$ exists and equals αL. A useful sequential criterion for the existence of a limit is now given.

Proof. We verify the two limit laws using the ε–δ definition.

1. Sum rule. Fix $\varepsilon > 0$ and set $\varepsilon_0 := \varepsilon/2$. Because $\lim_{x\to s} f(x) = L$, there is $\delta_1 > 0$ such that

$$0 < |x - s| < \delta_1 \implies |f(x) - L| < \varepsilon_0.$$

Likewise, $\lim_{x\to s} g(x) = M$ yields $\delta_2 > 0$ with

$$0 < |x - s| < \delta_2 \implies |g(x) - M| < \varepsilon_0.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then for $0 < |x - s| < \delta$ we have

$$|[f(x) + g(x)] - (L + M)| \ \le\ |f(x) - L| + |g(x) - M| < \varepsilon_0 + \varepsilon_0 = \varepsilon.$$

Hence $\lim_{x\to s}[f(x) + g(x)] = L + M$.

2. Scalar–multiple rule. Let $\alpha \in \mathbb{R}$ and fix $\varepsilon > 0$. Because $f(x) \to L$, there exists $\delta > 0$ such that $0 < |x - s| < \delta$ implies $|f(x) - L| < \varepsilon/|\alpha|$ if $\alpha \neq 0$; when $\alpha = 0$ the conclusion is immediate. Then

$$|\alpha f(x) - \alpha L| = |\alpha|\,|f(x) - L| < \varepsilon,$$

establishing $\lim_{x\to s}[\alpha f(x)] = \alpha L$.

3. Sequential criterion. We claim that a limit $\lim_{x\to s} h(x) = H$ exists exactly when for every sequence $\{x_n\}$ with $x_n \neq s$ and $x_n \to s$, the sequence $h(x_n)$ converges to H.
($\Rightarrow$) If the limit exists, then given $\varepsilon > 0$ the ε–δ condition furnishes $\delta > 0$. Eventually $|x_n - s| < \delta$, forcing $|h(x_n) - H| < \varepsilon$, so $h(x_n) \to H$.
($\Leftarrow$) Conversely, suppose h does not converge to H. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ we can pick x with $0 < |x - s| < \delta$ and $|h(x) - H| \ge \varepsilon_0$. Construct x_n inductively with $|x_n - s| < 1/n$ and $|h(x_n) - H| \ge \varepsilon_0$. Then $x_n \to s$ yet $h(x_n)$ stays ε_0 away from H, contradicting the sequential hypothesis. Hence the ε–δ definition must hold. $\square$

Remark. *Idea of the proof.* The sum rule splits the error of $f + g$ into the errors of f and g, each made arbitrarily small; the scalar rule simply factors out α. The sequential criterion states that functions inherit their limits along every path approaching s, and conversely any failure of the ε–δ condition can be translated into a "bad" sequence that prevents convergence.

Exercise 5.4. Let f be a real-valued function defined on a set S in $\mathbb{R}^n$. Show that if f is continuous at a point x_0 in S and if $f(x_0) < 0$, then there exists a $\delta > 0$ such that $f(x) < 0$ for all x in $B(x_0, \delta) \cap S$.

Proof. Because f is continuous at $x_0 \in S$ and $f(x_0) < 0$, let us set

$$\varepsilon \ :=\ \frac{-f(x_0)}{2}\ >\ 0.$$

By continuity, there exists $\delta > 0$ such that for every $x \in S$ with $\|x - x_0\| < \delta$ we have

$$|f(x) - f(x_0)| < \varepsilon.$$

For such x we estimate

$$f(x)\ =\ f(x_0) + \big(f(x) - f(x_0)\big)\ <\ f(x_0) + \varepsilon\ =\ f(x_0) - \tfrac{1}{2}f(x_0)\ =\ \tfrac{1}{2}f(x_0)\ <\ 0.$$

Hence $f(x) < 0$ for all $x \in B(x_0, \delta) \cap S$, as required. $\square$

Remark. *Idea of the proof.* Continuity lets us control $f(x)$ by forcing it to stay within an ε-band around $f(x_0)$. Choosing ε to be less than half the magnitude of the negative value $f(x_0)$ guarantees that $f(x)$ remains negative throughout a small ball centred at x_0.

Exercise 5.5. Show that the real-valued function $f : \mathbb{R}^n \to \mathbb{R}^1$ defined by $f(x) = \|x\|$ is continuous on $\mathbb{R}^n$ (i.e., show that the norm is a continuous function). Hint: Use the triangle inequality to show that, for any pair of vectors x and y, $|\|x\| - \|y\|| \leqslant \|x - y\|$.

Proof. We first observe the following key inequality, valid for all $x, y \in \mathbb{R}^n$:

$$\big|\,\|x\| - \|y\|\,\big|\ \le\ \|x - y\|. \tag{$*$}$$

Indeed, the triangle inequality gives $\|x\| = \|(x - y) + y\| \le \|x - y\| + \|y\|$, hence $\|x\| - \|y\| \le \|x - y\|$. Interchanging x and y yields $\|y\| - \|x\| \le \|x - y\|$. Combining the two estimates proves $(*)$.

Fix an arbitrary point $x_0 \in \mathbb{R}^n$ and let $\varepsilon > 0$ be given. Choose

$$\delta\ :=\ \varepsilon.$$

Then, whenever $\|x - x_0\| < \delta$, inequality $(*)$ implies

$$\big|\,\|x\| - \|x_0\|\,\big|\ \le\ \|x - x_0\|\ <\ \delta\ =\ \varepsilon.$$

Therefore, for every $x_0 \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $\delta = \varepsilon$ such that the ε–δ criterion for continuity is satisfied. Hence the norm function $f(x) = \|x\|$ is continuous at every point of $\mathbb{R}^n$; in other words, it is continuous on $\mathbb{R}^n$. $\square$

Remark. *Idea of the proof.* The norm is in fact *Lipschitz* with constant 1: the distance between its values never exceeds the distance between the inputs. This strong inequality not only establishes pointwise continuity via the standard ε–δ argument (take $\delta = \varepsilon$), but also shows that the norm is uniformly continuous on the whole space.

1.6 Basic Property of Real Numbers

Exercise 6.1. Let $L =$ g.l.b. of a set S. Show that there exists a sequence of points $\{x_k\}$ in S such that $x_k \to L$. Show that the sequence $\{x_k\}$ can be taken to be nonincreasing, that is, $x_{k+1} \leqslant x_k$ for every k. Does L have to be a limit point of S?

Proof. Because $L = \inf S$, by definition we have $L \leq x$ for every $x \in S$ and, moreover, for each $\varepsilon > 0$ the interval $(L, L+\varepsilon)$ contains at least one point of S.

1. Construction of a convergent sequence. For each integer $k \geq 1$ choose $x_k \in S$ such that

$$L \ \leq \ x_k \ < \ L + \frac{1}{k}.$$

Then $0 \leq x_k - L < 1/k \to 0$, so $x_k \to L$.

2. Making the sequence nonincreasing. Define recursively

$$y_1 := x_1, \qquad y_k := \min\{y_{k-1}, x_k\} \quad (k \geq 2).$$

Each y_k belongs to S (because it equals either y_{k-1} or x_k) and by construction $y_{k+1} \leq y_k$, so the sequence $\{y_k\}$ is nonincreasing. Furthermore,

$$L \leq y_k \leq x_k < L + \frac{1}{k} \longrightarrow L,$$

hence $y_k \to L$. Thus we have produced a nonincreasing sequence in S converging to L.

3. Is L necessarily a limit point of S? Not always. Take for example the set

$$S \ := \ \{0\} \cup \big\{1 + 1/n : n \in \mathbb{N}\big\} \subset \mathbb{R}.$$

Its greatest lower bound is $L = 0$, which belongs to S. However, there exists $\varepsilon = 1/2$ such that $(0, \varepsilon)$ contains no points of S other than 0 itself; hence 0 is *not* a limit point of S. (It is an isolated point.) Therefore the g.l.b. of a set need not be a limit point. $\square$

Remark. *Idea of the proof.* The infimum can be approached arbitrarily closely from above by points of S; picking one such point in each shrinking band $(L, L+1/k)$ yields convergence. Enforcing monotonicity is achieved by successively taking the smaller of the previous choice and the new candidate. The final example shows that possessing sequences converging to L does not force L to be an accumulation point—this fails whenever L is an isolated minimum of S.

Exercise 6.2. Let S be a set in $\mathbb{R}^1$. We define $-S$ to be $\{x : -x \in S\}$. Thus $-S$ is the set that we obtain by replacing each element x in S by the element $-x$. Show that S is bounded below if and only if $-S$ is bounded above. Show that α is the g.l.b. of S if and only if $-\alpha$ is the l.u.b. of $-S$.

Proof. Recall that for a subset $A \subset \mathbb{R}$ the set $-A$ is defined by $-A := \{-x : x \in A\}$.

(i) Boundedness. Suppose first that S is bounded below; that is, there exists $m \in \mathbb{R}$ such that $m \leq x$ for every $x \in S$. Then for any $y \in -S$ we can write $y = -x$ with $x \in S$, whence

$$y = -x \leq -m.$$

Hence $-m$ is an upper bound of $-S$, so $-S$ is bounded above. The converse follows by applying the same argument to $-S$ in place of S (note that $-(-S) = S$): if $-S$ is bounded above, say $y \leq M$ for all $y \in -S$, then $x = -y$ satisfies $-M \leq x$ for all $x \in S$, proving that S is bounded below.

(ii) Greatest lower bound vs. least upper bound. Assume α is the greatest lower bound (infimum) of S.

1. *$-\alpha$ is an upper bound of $-S$.* For every $y \in -S$ there exists $x \in S$ with $y = -x \leq -\alpha$ because $\alpha \leq x$.

2. *Minimality.* Let u be any upper bound of $-S$. Then $-u$ is a lower bound of S, so $\alpha \leq -u$ by optimality of α. Multiplying by -1 reverses the inequality, giving $-\alpha \geq u$. Therefore $-\alpha$ is the *least* upper bound of $-S$.

Consequently $-\alpha = \sup(-S)$.
The converse direction is identical: if $-\alpha$ is the least upper bound of $-S$, then multiplying all relevant inequalities by -1 shows that α is the greatest lower bound of S. $\square$

Remark. *Idea of the proof.* Multiplication by -1 reflects the real line about the origin, swapping the roles of "above" and "below" while preserving order up to sign. Thus lower bounds of S correspond exactly to upper bounds of $-S$, and extremal (least/ greatest) bounds correspond under the same reflection.

Exercise 6.3. Let A and B be two bounded sets of real numbers with $A \subseteq B$. Show that
$$\sup\{a : a \in A\} \leqslant \sup\{b : b \in B\},$$
$$\inf\{a : a \in A\} \geqslant \inf\{b : b \in B\}.$$

Proof. Because $A \subseteq B$, every element $a \in A$ is also an element of B.

1. Supremum. Let $M := \sup\{b : b \in B\}$. Since M is an upper bound of B, it is in particular an upper bound of A. The least upper bound of A cannot exceed any of its upper bounds; hence
$$\sup A \;\le\; M \;=\; \sup B.$$

2. Infimum. Let $m := \inf\{b : b \in B\}$. Then m is a lower bound of B, so $m \le a$ for every $a \in A$. Thus m is also a lower bound of A. The greatest lower bound of A dominates all its lower bounds, yielding
$$\inf A \;\ge\; m \;=\; \inf B.$$

Combining the two parts proves the claimed inequalities. □

Remark. *Idea of the proof.* Enlarging a set cannot lower its least upper bound nor raise its greatest lower bound: new elements can only push the supremum upward and the infimum downward, never in the opposite direction.

Exercise 6.4. Let A and B be two sets of real numbers. (i) Show that if A and B are bounded above, then
$$\sup\{(a+b) : a \in A, b \in B\} = \sup\{a : a \in A\} + \sup\{b : b \in B\}$$
(ii) Show that if A and B are bounded below, then
$$\inf\{(a+b) : a \in A, b \in B\} = \inf\{a : a \in A\} + \inf\{b : b \in B\}$$

Proof. Denote
$$S := \{a+b : a \in A,\; b \in B\}, \quad \alpha := \sup A,\; \beta := \sup B, \quad \gamma := \inf A,\; \delta := \inf B.$$

(i) Supremum of sums. Assume A and B are bounded above, so $\alpha, \beta \in \mathbb{R}$.

1. *Upper bound.* For any $a \in A$ and $b \in B$, we have $a \le \alpha$ and $b \le \beta$, hence $a + b \le \alpha + \beta$. Thus $\alpha + \beta$ is an upper bound of S, implying $\sup S \le \alpha + \beta$.
2. *Attaining the bound arbitrarily closely.* Since $\alpha = \sup A$, for each $k \ge 1$ there exists $a_k \in A$ with $\alpha - 1/k < a_k \le \alpha$. Similarly, choose $b_k \in B$ with $\beta - 1/k < b_k \le \beta$. Then
$$a_k + b_k > \alpha + \beta - 2/k \longrightarrow \alpha + \beta.$$
Consequently, $\alpha + \beta$ is *the least* upper bound, so $\sup S = \alpha + \beta$.

(ii) Infimum of sums. Assume A and B are bounded below, so $\gamma, \delta \in \mathbb{R}$.

1. *Lower bound.* For any $a \in A$ and $b \in B$, we have $a \ge \gamma$ and $b \ge \delta$, hence $a + b \ge \gamma + \delta$. Thus $\gamma + \delta$ is a lower bound of S, giving $\inf S \ge \gamma + \delta$.
2. *Approaching the bound.* For each $k \ge 1$, choose $a_k \in A$ with $\gamma \le a_k < \gamma + 1/k$ and $b_k \in B$ with $\delta \le b_k < \delta + 1/k$. Then
$$a_k + b_k < \gamma + \delta + 2/k \longrightarrow \gamma + \delta.$$
Therefore, $\gamma + \delta$ is the *greatest* lower bound, and $\inf S = \gamma + \delta$.

□