Exercise For Convexity and Optimization in $\mathbb{R}^n$

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Contents

Dedicated to the knee scrapes, playdates, and heartaches.

1 Topics in Real Analysis

Exercise 1.1. For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, show that $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$. Interpret this relation as a statement about parallelograms in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution. To prove that $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$ for any vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, we use the definition of the Euclidean norm and properties of the dot product. Recall that $\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x}$.
First, expand $\|\mathbf{x}+\mathbf{y}\|^2$:

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}+2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2+2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2.$$

Next, expand $\|\mathbf{x}-\mathbf{y}\|^2$:

$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}-2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2-2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2.$$

Add these two expressions:

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = \left(\|\mathbf{x}\|^2+2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2\right)+\left(\|\mathbf{x}\|^2-2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2+\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+\|\mathbf{y}\|^2+2(\mathbf{x}\cdot\mathbf{y})-2(\mathbf{x}\cdot\mathbf{y}).$$

The cross terms $2(\mathbf{x}\cdot\mathbf{y})$ and $-2(\mathbf{x}\cdot\mathbf{y})$ cancel, yielding:

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2.$$

Thus, the equality holds for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this equality has a geometric interpretation related to parallelograms. Consider vectors $\mathbf{x}$ and $\mathbf{y}$ emanating from the same initial point. These vectors form two adjacent sides of a parallelogram. The vector $\mathbf{x}+\mathbf{y}$ represents one diagonal of the parallelogram, and $\mathbf{x}-\mathbf{y}$ represents the other diagonal (assuming the parallelogram is completed appropriately).
The left side of the equality, $\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2$, is the sum of the squares of the lengths of the two diagonals. The right side, $2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$, is twice the sum of the squares of the lengths of the two adjacent sides. Since a parallelogram has two pairs of equal sides, the sum of the squares of the lengths of all four sides is $2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2$ (two sides of length $\|\mathbf{x}\|$ and two of length $\|\mathbf{y}\|$).
Therefore, the equality states that for any parallelogram in $\mathbb{R}^2$ or $\mathbb{R}^3$, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all four sides. This is a fundamental property of parallelograms in Euclidean geometry, often called the parallelogram law.

Lemma 3.1. Let $\{S_\alpha\}_{\alpha\in A}$ be a collection of subsets of a set X. Then

$$\bigcup_{\alpha\in A} S_\alpha = c\left[\bigcap_{\alpha\in A}(cS_\alpha)\right],$$

$$\bigcap_{\alpha\in A} S_\alpha = c\left[\bigcup_{\alpha\in A}(cS_\alpha)\right].$$

Proof. We establish both identities by showing mutual inclusion of the corresponding sets.

1. $\displaystyle\bigcup_{\alpha\in A} S_\alpha = c\Big[\bigcap_{\alpha\in A}(cS_\alpha)\Big]$**.**
(i) Subset relation $\subseteq$. Let $x \in \bigcup_{\alpha\in A} S_\alpha$. Then there exists an index $\alpha_0 \in A$ such that $x \in S_{\alpha_0}$. If x were also contained in $\bigcap_{\alpha\in A}(cS_\alpha)$, it would belong to cS_{α_0}, i.e. $x \notin S_{\alpha_0}$, a contradiction. Therefore $x \notin \bigcap_{\alpha\in A}(cS_\alpha)$, which means $x \in c\big[\bigcap_{\alpha\in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Conversely, take $x \in c\big[\bigcap_{\alpha\in A}(cS_\alpha)\big]$. Then $x \notin \bigcap_{\alpha\in A}(cS_\alpha)$, so there exists an index $\alpha_1 \in A$ with $x \notin cS_{\alpha_1}$. Equivalently, $x \in S_{\alpha_1}$, hence $x \in \bigcup_{\alpha\in A} S_\alpha$. Combining (i) and (ii) yields the desired equality.

2. $\displaystyle\bigcap_{\alpha\in A} S_\alpha = c\Big[\bigcup_{\alpha\in A}(cS_\alpha)\Big]$**.**
The argument is analogous.
(i) Subset relation $\subseteq$. Let $x \in \bigcap_{\alpha\in A} S_\alpha$. Then $x \in S_\alpha$ for every α. Consequently, $x \notin cS_\alpha$ for any α, which implies $x \notin \bigcup_{\alpha\in A}(cS_\alpha)$. Hence $x \in c\big[\bigcup_{\alpha\in A}(cS_\alpha)\big]$.
(ii) Subset relation $\supseteq$. Let $x \in c\big[\bigcup_{\alpha\in A}(cS_\alpha)\big]$. Then $x \notin \bigcup_{\alpha\in A}(cS_\alpha)$, so for every $\alpha \in A$ we have $x \notin cS_\alpha$; equivalently $x \in S_\alpha$. Therefore $x \in \bigcap_{\alpha\in A} S_\alpha$.
Since both inclusions hold, the second identity follows. □

Exercise 4.1. Use the properties of the norm to show that the function d defined by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

is a metric, or distance function, on $\mathbb{R}^n$.

Proof. To verify that $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is a metric on $\mathbb{R}^n$, we must show that it satisfies the following three properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$: *(i)* non–negativity with the identity of indiscernibles, *(ii)* symmetry, and *(iii)* the triangle inequality.

1. Non–negativity and identity of indiscernibles. The Euclidean norm is always non–negative, so

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0.$$

Moreover, $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\|\mathbf{x} - \mathbf{y}\| = 0$, which occurs precisely when $\mathbf{x} - \mathbf{y} = \mathbf{0}$, i.e. when $\mathbf{x} = \mathbf{y}$.

2. Symmetry. Because $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for any vector $\mathbf{v}$,

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|-(\mathbf{x} - \mathbf{y})\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y}, \mathbf{x}).$$

3. Triangle inequality. The Euclidean norm satisfies the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Choosing $\mathbf{u} = \mathbf{x} - \mathbf{y}$ and $\mathbf{v} = \mathbf{y} - \mathbf{z}$ gives

$$d(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\| = \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

Since all three axioms hold, the function d is indeed a metric on $\mathbb{R}^n$. $\square$