

Exercise For Convexity and Optimization in \mathbb{R}^n

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Dedicated to the knee scrapes, playdates, and heartaches.

1 Topics in Real Analysis

Exercise 1.1. For any vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^n , show that $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. Interpret this relation as a statement about parallelograms in \mathbb{R}^2 and \mathbb{R}^3 .

Solution. To prove that $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we use the definition of the Euclidean norm and properties of the dot product. Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.

First, expand $\|\mathbf{x} + \mathbf{y}\|^2$:

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Next, expand $\|\mathbf{x} - \mathbf{y}\|^2$:

$$\|\mathbf{x} - \mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Add these two expressions:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = (\|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2) + (\|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2) = \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) - 2(\mathbf{x} \cdot \mathbf{y}).$$

The cross terms $2(\mathbf{x} \cdot \mathbf{y})$ and $-2(\mathbf{x} \cdot \mathbf{y})$ cancel, yielding:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

Thus, the equality holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

In \mathbb{R}^2 and \mathbb{R}^3 , this equality has a geometric interpretation related to parallelograms. Consider vectors \mathbf{x} and \mathbf{y} emanating from the same initial point. These vectors form two adjacent sides of a parallelogram. The vector $\mathbf{x} + \mathbf{y}$ represents one diagonal of the parallelogram, and $\mathbf{x} - \mathbf{y}$ represents the other diagonal (assuming the parallelogram is completed appropriately).

The left side of the equality, $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2$, is the sum of the squares of the lengths of the two diagonals. The right side, $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$, is twice the sum of the squares of the lengths of the two adjacent sides. Since a parallelogram has two pairs of equal sides, the sum of the squares of the lengths of all four sides is $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ (two sides of length $\|\mathbf{x}\|$ and two of length $\|\mathbf{y}\|$).

Therefore, the equality states that for any parallelogram in \mathbb{R}^2 or \mathbb{R}^3 , the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all four sides. This is a fundamental property of parallelograms in Euclidean geometry, often called the parallelogram law.

Exercise 4.1. Use the properties of the norm to show that the function d defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}$$

is a metric, or distance function, on \mathbb{R}^n .

Proof. To verify that $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is a metric on \mathbb{R}^n , we must show that it satisfies the following three properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$: (i) non-negativity with the identity of indiscernibles, (ii) symmetry, and (iii) the triangle inequality.

1. Non-negativity and identity of indiscernibles. The Euclidean norm is always non-negative, so

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0.$$

Moreover, $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\|\mathbf{x} - \mathbf{y}\| = 0$, which occurs precisely when $\mathbf{x} - \mathbf{y} = \mathbf{0}$, i.e. when $\mathbf{x} = \mathbf{y}$.

2. Symmetry. Because $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for any vector \mathbf{v} ,

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|-(\mathbf{y} - \mathbf{x})\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y}, \mathbf{x}).$$

3. Triangle inequality. The Euclidean norm satisfies the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Choosing $\mathbf{u} = \mathbf{x} - \mathbf{y}$ and $\mathbf{v} = \mathbf{y} - \mathbf{z}$ gives

$$d(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\| = \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

Since all three axioms hold, the function d is indeed a metric on \mathbb{R}^n . □