

# Exercise For Convexity and Optimization in $\mathbb{R}^n$

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Dedicated to the knee scrapes, playdates, and heartaches.

# 1 Topics in Real Analysis

## 1.1 Introduction

**Exercise 1.1.** For any vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$ , show that  $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ . Interpret this relation as a statement about parallelograms in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

**Solution.** To prove that  $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$  for any vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , we use the definition of the Euclidean norm and properties of the dot product. Recall that  $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$ .

First, expand  $\|\mathbf{x} + \mathbf{y}\|^2$ :

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Next, expand  $\|\mathbf{x} - \mathbf{y}\|^2$ :

$$\|\mathbf{x} - \mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2.$$

Add these two expressions:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = (\|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2) + (\|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2) = \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) - 2(\mathbf{x} \cdot \mathbf{y}).$$

The cross terms  $2(\mathbf{x} \cdot \mathbf{y})$  and  $-2(\mathbf{x} \cdot \mathbf{y})$  cancel, yielding:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

Thus, the equality holds for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .

In  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , this equality has a geometric interpretation related to parallelograms. Consider vectors  $\mathbf{x}$  and  $\mathbf{y}$  emanating from the same initial point. These vectors form two adjacent sides of a parallelogram. The vector  $\mathbf{x} + \mathbf{y}$  represents one diagonal of the parallelogram, and  $\mathbf{x} - \mathbf{y}$  represents the other diagonal (assuming the parallelogram is completed appropriately).

The left side of the equality,  $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2$ , is the sum of the squares of the lengths of the two diagonals. The right side,  $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ , is twice the sum of the squares of the lengths of the two adjacent sides. Since a parallelogram has two pairs of equal sides, the sum of the squares of the lengths of all four sides is  $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$  (two sides of length  $\|\mathbf{x}\|$  and two of length  $\|\mathbf{y}\|$ ).

Therefore, the equality states that for any parallelogram in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all four sides. This is a fundamental property of parallelograms in Euclidean geometry, often called the parallelogram law.

## 1.3 Algebra of Sets

**Lemma 3.1.** Let  $\{S_\alpha\}_{\alpha \in A}$  be a collection of subsets of a set  $X$ . Then

$$\bigcup_{\alpha \in A} S_\alpha = c \left[ \bigcap_{\alpha \in A} (cS_\alpha) \right],$$

$$\bigcap_{\alpha \in A} S_\alpha = c \left[ \bigcup_{\alpha \in A} (cS_\alpha) \right].$$

**Proof.** We establish both identities by showing mutual inclusion of the corresponding sets.

$$1. \bigcup_{\alpha \in A} S_\alpha = c \left[ \bigcap_{\alpha \in A} (cS_\alpha) \right].$$

(i) *Subset relation  $\subseteq$ .* Let  $x \in \bigcup_{\alpha \in A} S_\alpha$ . Then there exists an index  $\alpha_0 \in A$  such that  $x \in S_{\alpha_0}$ . If  $x$  were also contained in  $\bigcap_{\alpha \in A} (cS_\alpha)$ , it would belong to  $cS_{\alpha_0}$ , i.e.  $x \notin S_{\alpha_0}$ , a contradiction. Therefore  $x \notin \bigcap_{\alpha \in A} (cS_\alpha)$ , which means  $x \in c[\bigcap_{\alpha \in A} (cS_\alpha)]$ .

(ii) *Subset relation  $\supseteq$ .* Conversely, take  $x \in c[\bigcap_{\alpha \in A} (cS_\alpha)]$ . Then  $x \notin \bigcap_{\alpha \in A} (cS_\alpha)$ , so there exists an index  $\alpha_1 \in A$  with  $x \notin cS_{\alpha_1}$ . Equivalently,  $x \in S_{\alpha_1}$ , hence  $x \in \bigcup_{\alpha \in A} S_\alpha$ . Combining (i) and (ii) yields the desired equality.

$$2. \bigcap_{\alpha \in A} S_\alpha = c \left[ \bigcup_{\alpha \in A} (cS_\alpha) \right].$$

The argument is analogous.

(i) *Subset relation  $\subseteq$ .* Let  $x \in \bigcap_{\alpha \in A} S_\alpha$ . Then  $x \in S_\alpha$  for every  $\alpha$ . Consequently,  $x \notin cS_\alpha$  for any  $\alpha$ , which implies  $x \notin \bigcup_{\alpha \in A} (cS_\alpha)$ . Hence  $x \in c[\bigcup_{\alpha \in A} (cS_\alpha)]$ .

(ii) *Subset relation  $\supseteq$ .* Let  $x \in c[\bigcup_{\alpha \in A} (cS_\alpha)]$ . Then  $x \notin \bigcup_{\alpha \in A} (cS_\alpha)$ , so for every  $\alpha \in A$  we have  $x \notin cS_\alpha$ ; equivalently  $x \in S_\alpha$ . Therefore  $x \in \bigcap_{\alpha \in A} S_\alpha$ .

Since both inclusions hold, the second identity follows.  $\square$

## 1.4 Metric Topology of $\mathbb{R}^n$

**Exercise 4.1.** Use the properties of the norm to show that the function  $d$  defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}$$

is a metric, or distance function, on  $\mathbb{R}^n$ .

**Proof.** To verify that  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$  is a metric on  $\mathbb{R}^n$ , we must show that it satisfies the following three properties for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ : (i) non-negativity with the identity of indiscernibles, (ii) symmetry, and (iii) the triangle inequality.

**1. Non-negativity and identity of indiscernibles.** The Euclidean norm is always non-negative, so

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0.$$

Moreover,  $d(\mathbf{x}, \mathbf{y}) = 0$  if and only if  $\|\mathbf{x} - \mathbf{y}\| = 0$ , which occurs precisely when  $\mathbf{x} - \mathbf{y} = \mathbf{0}$ , i.e. when  $\mathbf{x} = \mathbf{y}$ .

**2. Symmetry.** Because  $\|\mathbf{v}\| = \|-\mathbf{v}\|$  for any vector  $\mathbf{v}$ ,

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|-(\mathbf{y} - \mathbf{x})\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y}, \mathbf{x}).$$

**3. Triangle inequality.** The Euclidean norm satisfies the triangle inequality  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ . Choosing  $\mathbf{u} = \mathbf{x} - \mathbf{y}$  and  $\mathbf{v} = \mathbf{y} - \mathbf{z}$  gives

$$d(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\| = \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

Since all three axioms hold, the function  $d$  is indeed a metric on  $\mathbb{R}^n$ .  $\square$

**Exercise 4.2.** (a) Sketch the graph of  $y = \sin(1/x)$  for  $x > 0$ .

(b) Consider the graph as a set in  $\mathbb{R}^2$  and find the limit points of this set.

**Solution.** (a) **Sketch of the curve.** Set  $t = 1/x$  for  $x > 0$ . Then the graph of  $y = \sin(1/x)$  for  $x > 0$  corresponds to the standard sine curve  $y = \sin t$  for  $t > 0$ , but viewed through the change of variables  $x = 1/t$ . As  $x \rightarrow 0^+$  we have  $t \rightarrow +\infty$ , so the curve oscillates infinitely often between  $-1$  and  $1$  while its  $x$ -coordinate approaches  $0$ . For moderate values of  $x$  the graph resembles the usual sine curve stretched horizontally, whereas near the  $y$ -axis the oscillations become increasingly rapid, creating a "comb-like" fringe that accumulates on the interval  $\{0\} \times [-1, 1]$ .

(b) **Limit points of the graph.** Let

$$S := \{ (x, \sin(1/x)) : x > 0 \} \subset \mathbb{R}^2.$$

We claim that the set of limit points of  $S$  is

$$S' = S \cup \{ (0, y) : -1 \leq y \leq 1 \}.$$

The verification proceeds in two steps.

*Step 1: Every point of  $S$  is a limit point.* Fix  $(x_0, \sin(1/x_0)) \in S$  with  $x_0 > 0$  and let  $\varepsilon > 0$  be given. Recall the elementary inequality  $|\sin u - \sin v| \leq |u - v|$  valid for all real numbers  $u, v$ . Set

$$\delta := \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon x_0^2}{4}, \frac{x_0}{2} \right\}.$$

Pick any  $x_\varepsilon \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$  and define  $P_\varepsilon := (x_\varepsilon, \sin(1/x_\varepsilon))$ . Then  $P_\varepsilon \in S$  and

$$|x_\varepsilon - x_0| < \frac{\varepsilon}{2}, \quad |\sin(1/x_\varepsilon) - \sin(1/x_0)| \leq |1/x_\varepsilon - 1/x_0| = \frac{|x_\varepsilon - x_0|}{|x_\varepsilon x_0|} \leq \frac{2\delta}{x_0^2} \leq \frac{\varepsilon}{2}.$$

Consequently

$$\|P_\varepsilon - (x_0, \sin(1/x_0))\| \leq \sqrt{(|x_\varepsilon - x_0|)^2 + (|\sin(1/x_\varepsilon) - \sin(1/x_0)|)^2} < \varepsilon,$$

proving that  $(x_0, \sin(1/x_0))$  is indeed a limit point of  $S$ .

*Step 2: Points of the form  $(0, y)$  with  $|y| \leq 1$  are limit points.* Fix  $y \in [-1, 1]$  and  $\varepsilon > 0$ . Because the sine function attains every value in  $[-1, 1]$  infinitely often, we can pick  $t_\varepsilon > \max\{1/\varepsilon, 0\}$  such that  $|\sin t_\varepsilon - y| < \varepsilon$ . Setting  $x_\varepsilon = 1/t_\varepsilon$  we obtain  $0 < x_\varepsilon < \varepsilon$  and

$$\|(x_\varepsilon, \sin(1/x_\varepsilon)) - (0, y)\| < \sqrt{\varepsilon^2 + \varepsilon^2} < \sqrt{2}\varepsilon.$$

Thus  $(0, y)$  is approached by points of  $S$  distinct from itself, so it is a limit point.

*Step 3: No other points are limit points.* If  $x < 0$ , every open ball centred at  $(x, y)$  contains points whose first coordinate is negative, whereas  $S$  lies entirely in  $x > 0$ ; thus such points cannot be limit points. If  $x = 0$  but  $|y| > 1$ , the vertical separation  $|y| - 1$  already exceeds the range of  $\sin$ , so no sequence in  $S$  can approach  $(0, y)$ .

Finally, consider a point  $(x, y)$  with  $x > 0$  that does

emphnot belong to  $S$ . Write  $y_0 := \sin(1/x)$  and set  $d := |y - y_0| > 0$ . Choose

$$r := \min \left\{ \frac{d}{2}, \frac{x}{2} \right\}.$$

For any  $(x', \sin(1/x')) \in S$  satisfying  $|x' - x| < r$  we have  $x' \geq x/2$  and hence, using  $|\sin u - \sin v| \leq |u - v|$  again,

$$|\sin(1/x') - y_0| \leq \frac{|x' - x|}{x'x} \leq \frac{2r}{x^2} \leq \frac{d}{2}.$$

Therefore

$$|\sin(1/x') - y| \geq |y - y_0| - |\sin(1/x') - y_0| > \frac{d}{2},$$

so the Euclidean distance between  $(x', \sin(1/x'))$  and  $(x, y)$  exceeds  $d/2$ . Hence the ball  $B((x, y), d/2)$  contains no point of  $S$ , proving that  $(x, y)$  is not a limit point. Collecting the cases established in Steps 1–3 completes the description of  $S'$ .

Consequently, the set of all limit points of the graph is precisely  $S \cup (\{0\} \times [-1, 1])$ , as asserted.

**Exercise 4.3.** Show that for  $x \in \mathbb{R}^n$  and  $r > 0$  the set  $B(x, r)$  is open; that is, show that an open ball is open.

**Proof.** Let  $y \in B(x, r)$ , so by definition  $\|y - x\| < r$ . Define

$$\varepsilon := r - \|y - x\| > 0.$$

We claim that the entire ball  $B(y, \varepsilon)$  is contained in  $B(x, r)$ . Indeed, if  $z \in B(y, \varepsilon)$  then  $\|z - y\| < \varepsilon$ , and by the triangle inequality,

$$\|z - x\| \leq \|z - y\| + \|y - x\| < \varepsilon + \|y - x\| = r.$$

Hence  $z \in B(x, r)$ . Since every point  $y$  of  $B(x, r)$  is an interior point, the set  $B(x, r)$  is open.  $\square$

**Remark. Idea of the proof.** To show that the open ball  $B(x, r)$  is an open set we verify that every point it contains is an interior point. Fix  $y \in B(x, r)$  and measure how much "room" is left before reaching the boundary: the gap is  $\varepsilon := r - \|y - x\| > 0$ . Any point  $z$  that sits within this gap around  $y$  (that is,  $\|z - y\| < \varepsilon$ ) cannot escape the larger ball, because the triangle inequality guarantees  $\|z - x\| < \|z - y\| + \|y - x\| < \varepsilon + (r - \varepsilon) = r$ . Thus the smaller ball  $B(y, \varepsilon)$  lies completely inside  $B(x, r)$ , making  $y$  an interior point. Since  $y$  was arbitrary,  $B(x, r)$  is open.

**Exercise 4.4.** Show that for  $x \in \mathbb{R}^n$  and  $r > 0$  the closed ball  $\overline{B(x, r)}$  is closed.

**Proof.** Set  $C := \overline{B(x, r)} = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ . We show that its complement  $C^c$  is open. Let  $y \in C^c$ , so  $\|y - x\| > r$ . Define

$$\varepsilon := \frac{\|y - x\| - r}{2} > 0.$$

If  $z \in B(y, \varepsilon)$  then  $\|z - y\| < \varepsilon$ , and by the triangle inequality,

$$\|z - x\| \geq \|y - x\| - \|z - y\| > (r + 2\varepsilon) - \varepsilon = r + \varepsilon > r.$$

Hence  $z \notin C$ . Therefore  $B(y, \varepsilon) \subset C^c$ , proving that every point of  $C^c$  is interior. The complement of  $C$  is open, so  $C$  is closed.  $\square$

**Remark. Idea of the proof.** For a point  $y$  lying *outside* the closed ball we measure how far it is from the boundary: the surplus distance is  $\delta := \|y - x\| - r > 0$ . Choosing half of this surplus as a radius,  $\varepsilon = \delta/2$ , guarantees that the entire ball  $B(y, \varepsilon)$  stays outside, because any point inside that small ball remains at least  $r + \varepsilon > r$  away from  $x$ . Since such a neighbourhood exists around each exterior point, the complement of the closed ball is open, which is exactly the definition of the original set being closed.

**Exercise 4.5.** Show that any finite set of points  $x_1, \dots, x_k$  in  $\mathbb{R}^n$  is closed.

**Proof.** Denote the finite set by  $F := \{x_1, \dots, x_k\}$ . We show that its complement  $F^c$  is open. Let  $y \in F^c$ ; then  $y \neq x_i$  for every  $i$ . Define the positive distances

$$d_i := \|y - x_i\| > 0, \quad i = 1, \dots, k, \quad \text{and set} \quad \varepsilon := \frac{1}{2} \min_{1 \leq i \leq k} d_i > 0.$$

For any  $z \in B(y, \varepsilon)$  we have  $\|z - y\| < \varepsilon \leq d_i/2$ , hence by the triangle inequality

$$\|z - x_i\| \geq \|y - x_i\| - \|z - y\| > d_i - \varepsilon \geq \frac{1}{2}d_i > 0 \quad (1 \leq i \leq k).$$

Consequently  $z \neq x_i$  for every  $i$ , i.e.  $z \in F^c$ . Thus  $B(y, \varepsilon) \subset F^c$ , so every exterior point is interior to the complement;  $F^c$  is open and  $F$  is closed.

An alternative argument is to note that each singleton  $\{x_i\}$  is closed (apply the previous step with  $k = 1$ ) and that a finite union of closed sets remains closed:

$$F = \bigcup_{i=1}^k \{x_i\}. \quad \square$$

**Remark. Idea of the proof.** For a point  $y$  not in the finite set we compute its distances to each listed point. The smallest of these distances is still positive; taking half of it as the radius gives a neighbourhood around  $y$  that misses the entire finite set, showing the complement is open. Equivalently, observe that singletons are closed and a finite union of closed sets is closed.

**Exercise 4.6.** Show that in  $\mathbb{R}^n$  no point  $x$  with  $\|x\| = 1$  is an interior point of the closed unit ball  $\overline{B(0, 1)}$ .

**Proof.** Let  $x \in \mathbb{R}^n$  satisfy  $\|x\| = 1$ . To show that  $x$  is *not* an interior point of  $\overline{B(0, 1)}$  we must prove that every open ball centred at  $x$  contains points that lie *outside*  $\overline{B(0, 1)}$ .

Fix an arbitrary radius  $r > 0$  and define

$$y := \left(1 + \frac{r}{2}\right) x.$$

Then

$$\|y - x\| = \left|1 + \frac{r}{2} - 1\right| \|x\| = \frac{r}{2} < r, \quad \text{but} \quad \|y\| = 1 + \frac{r}{2} > 1.$$

Hence  $y$  lies within the ball  $B(x, r)$  yet satisfies  $\|y\| > 1$ , so  $y \notin \overline{B(0, 1)}$ . Consequently  $B(x, r)$  is *not* contained in  $\overline{B(0, 1)}$ . Because  $r > 0$  was arbitrary, no open neighbourhood of  $x$  sits inside the closed unit ball, and thus  $x$  is not an interior point.  $\square$

**Remark.** *Idea of the proof.* Any boundary point  $x$  with  $\|x\| = 1$  can be pushed slightly outward along its own direction: scaling by  $1 + \varepsilon$  moves the point a distance roughly  $\varepsilon$  while immediately leaving the unit ball. No matter how small a radius we choose, such an outward perturbation fits inside the radius yet escapes the ball, so the original point cannot be interior.

**Exercise 4.7.** Show that for any set  $S$  in  $\mathbb{R}^n$  the set  $\bar{S}$  (the closure of  $S$ ) is closed.

**Proof.** Recall that the closure of  $S$  is defined by

$$\bar{S} := \{x \in \mathbb{R}^n : \forall r > 0, B(x, r) \cap S \neq \emptyset\}.$$

We shall prove that its complement  $\bar{S}^c$  is open.

Let  $y \in \bar{S}^c$ . By definition of the complement, there exists a radius  $r_0 > 0$  such that the open ball  $B(y, r_0)$  does *not* meet  $S$ ; otherwise  $y$  would belong to the closure. In particular, for this same  $r_0$  we have

$$B(y, r_0) \cap S = \emptyset \implies B(y, r_0) \cap \bar{S} = \emptyset,$$

because  $\bar{S}$  contains  $S$ . Hence  $B(y, r_0)$  lies entirely inside  $\bar{S}^c$ , showing that  $y$  is an interior point of the complement. Since every point of  $\bar{S}^c$  is interior, the complement is open; equivalently,  $\bar{S}$  is closed.  $\square$

**Remark.** *Idea of the proof.* A point fails to belong to the closure precisely when some open ball around it avoids  $S$ . But that same ball automatically avoids  $\bar{S}$  as well, so it sits inside the complement. Thus every exterior point has a protective open neighbourhood—the hallmark of an open set—so the complement is open and the closure is closed.

**Exercise 4.8.** Show that for any set  $S$  the closure  $\bar{S}$  is equal to the intersection of all closed sets containing  $S$ .

**Proof.** Denote by

$$\mathcal{F} := \{F \subset \mathbb{R}^n : F \text{ is closed and } S \subset F\}, \quad K := \bigcap_{F \in \mathcal{F}} F.$$

The intersection of any family of closed sets is closed, hence  $K$  is closed. Because  $S \subset F$  for every  $F \in \mathcal{F}$ , we also have  $S \subset K$ .

1.  $\bar{S} \subset K$ . Let  $F \in \mathcal{F}$ . Since  $F$  is closed and contains  $S$ , the closure property of  $\bar{S}$  implies  $\bar{S} \subset F$ . This holds for every  $F \in \mathcal{F}$ , therefore  $\bar{S} \subset K$ .

2.  $K \subset \bar{S}$ . The closure  $\bar{S}$  itself is closed and contains  $S$ , so  $\bar{S} \in \mathcal{F}$ . Intersecting over  $\mathcal{F}$  cannot yield a set larger than one of its members, hence  $K \subset \bar{S}$ .

Combining the two inclusions gives  $K = \bar{S}$ . That is, the closure of  $S$  coincides with the intersection of all closed sets that contain  $S$ .  $\square$

**Remark.** *Idea of the proof.* The family of all closed supersets of  $S$  has a natural "smallest" member given by their intersection  $K$ . Because  $K$  is itself closed and contains  $S$ , the closure  $\bar{S}$ —which is the minimal closed set containing  $S$ —must sit inside  $K$ . On the other hand,  $\bar{S}$  belongs to the family, so the intersection  $K$  cannot fall outside  $\bar{S}$ . The two sets therefore match exactly.

## 1.5 Limits and Continuity

**Exercise 5.1.** (i) The limit of a convergent sequence is unique.

(ii) Let  $\lim_{k \rightarrow \infty} x_k = x_0$ , let  $\lim_{k \rightarrow \infty} y_k = y_0$ , and let  $\lim_{k \rightarrow \infty} \alpha_k = \alpha$ , where  $\{\alpha_k\}$  is a sequence of scalars. Then  $\lim_{k \rightarrow \infty} (x_k + y_k)$  exists and equals  $x_0 + y_0$ , and  $\lim_{k \rightarrow \infty} \alpha_k x_k$  exists and equals  $\alpha x_0$ .

(iii) A convergent sequence is bounded.

**Proof.** We work in  $\mathbb{R}^n$  and use the standard  $\|\cdot\|$  norm; limits are taken with respect to this norm.

(i) **Uniqueness of the limit.** Suppose a sequence  $\{x_k\}$  converges to both  $\ell_1$  and  $\ell_2$ . Fix  $\varepsilon > 0$  and choose

$N_1, N_2 \in \mathbb{N}$  such that

$$k \geq N_1 \implies \|x_k - \ell_1\| < \varepsilon/2, \quad k \geq N_2 \implies \|x_k - \ell_2\| < \varepsilon/2.$$

For  $k \geq \max\{N_1, N_2\}$  we then have

$$\|\ell_1 - \ell_2\| \leq \|\ell_1 - x_k\| + \|x_k - \ell_2\| < \varepsilon.$$

Because  $\varepsilon$  was arbitrary,  $\|\ell_1 - \ell_2\| = 0$ , whence  $\ell_1 = \ell_2$ . The limit is therefore unique.

**(ii) Limits of sums and scalar multiples.**

1. Let  $\varepsilon > 0$ . Choose  $N_1, N_2$  such that  $k \geq N_1$  implies  $\|x_k - x_0\| < \varepsilon/2$  and  $k \geq N_2$  implies  $\|y_k - y_0\| < \varepsilon/2$ . For  $k \geq N := \max\{N_1, N_2\}$  we have

$$\|(x_k + y_k) - (x_0 + y_0)\| \leq \|x_k - x_0\| + \|y_k - y_0\| < \varepsilon.$$

Hence  $x_k + y_k \rightarrow x_0 + y_0$ .

2. Given  $\varepsilon > 0$ , convergence of  $\alpha_k$  means there exists  $N_1$  with  $|\alpha_k - \alpha| < \varepsilon/(2(1 + \|x_0\|))$  for  $k \geq N_1$ . Convergence of  $x_k$  gives  $N_2$  with  $\|x_k - x_0\| < \varepsilon/(2(1 + |\alpha|))$  for  $k \geq N_2$ . For  $k \geq N := \max\{N_1, N_2\}$  we estimate

$$\|\alpha_k x_k - \alpha x_0\| \leq |\alpha_k - \alpha| \|x_k\| + |\alpha| \|x_k - x_0\|.$$

The triangle inequality together with  $\|x_k\| \leq \|x_k - x_0\| + \|x_0\|$  yields

$$\|\alpha_k x_k - \alpha x_0\| < \frac{\varepsilon}{2} \frac{\|x_k - x_0\| + \|x_0\|}{1 + \|x_0\|} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus  $\alpha_k x_k \rightarrow \alpha x_0$ .

**(iii) A convergent sequence is bounded.** Let  $x_k \rightarrow x_0$ . Take  $\varepsilon := 1$  and choose  $N$  so that  $\|x_k - x_0\| < 1$  for all  $k \geq N$ . Then

$$\|x_k\| \leq \|x_k - x_0\| + \|x_0\| < 1 + \|x_0\| \quad (k \geq N).$$

Define

$$M := \max\{\|x_1\|, \dots, \|x_{N-1}\|, 1 + \|x_0\|\}.$$

Every term of the sequence satisfies  $\|x_k\| \leq M$ , so the sequence is bounded.  $\square$

**Remark.** *Idea of the proof.* Uniqueness comes from squeezing two putative limits together via the triangle inequality. The limit laws follow by splitting the error of a sum or product into individually controllable parts. Boundedness is immediate once the tail of the sequence lies in a fixed ball around its limit, while the finitely many initial terms can be absorbed into a single bound.

**Exercise 5.2.** Show that if  $\lim_{x \rightarrow s} f(x) = L$ , then  $L$  is unique.

**Proof.** Assume that  $f$  approaches two limits  $L_1$  and  $L_2$  at the point  $s$ . We show that they must coincide. Suppose, for contradiction, that  $L_1 \neq L_2$ . Set

$$\varepsilon := \frac{|L_1 - L_2|}{3} > 0.$$

Because  $\lim_{x \rightarrow s} f(x) = L_1$ , there exists  $\delta_1 > 0$  such that

$$0 < |x - s| < \delta_1 \implies |f(x) - L_1| < \varepsilon.$$

Likewise, convergence to  $L_2$  furnishes  $\delta_2 > 0$  with

$$0 < |x - s| < \delta_2 \implies |f(x) - L_2| < \varepsilon.$$

Now choose any  $x$  satisfying

$$0 < |x - s| < \delta := \min\{\delta_1, \delta_2\}.$$

Then both inequalities hold simultaneously, and the triangle inequality gives

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon = \frac{2}{3}|L_1 - L_2|,$$

which is impossible. Hence our assumption  $L_1 \neq L_2$  must be false; therefore  $L_1 = L_2$ . The limit value is unique.  $\square$

**Remark.** *Idea of the proof.* If two distinct numbers both pretend to be the limit, pick an  $\varepsilon$  slightly smaller than their separation. Close enough to  $s$ , the function has to lie within that  $\varepsilon$ -tube around each candidate limit, but those two tubes do not overlap—contradiction. The triangle inequality formalises this intuition and forces the two limits to coincide.

**Exercise 5.3.** Prove the following theorem.

Let  $\lim_{x \rightarrow s} f(x) = L$  and let  $\lim_{x \rightarrow s} g(x) = M$ . Then

$$\lim_{x \rightarrow s} [f(x) + g(x)]$$

exists and equals  $L + M$ . Also  $\lim_{x \rightarrow s} [\alpha f(x)]$  exists and equals  $\alpha L$ . A useful sequential criterion for the existence of a limit is now given.

**Proof.** We verify the two limit laws using the  $\varepsilon$ - $\delta$  definition.

**1. Sum rule.** Fix  $\varepsilon > 0$  and set  $\varepsilon_0 := \varepsilon/2$ . Because  $\lim_{x \rightarrow s} f(x) = L$ , there is  $\delta_1 > 0$  such that

$$0 < |x - s| < \delta_1 \implies |f(x) - L| < \varepsilon_0.$$

Likewise,  $\lim_{x \rightarrow s} g(x) = M$  yields  $\delta_2 > 0$  with

$$0 < |x - s| < \delta_2 \implies |g(x) - M| < \varepsilon_0.$$

Let  $\delta := \min\{\delta_1, \delta_2\}$ . Then for  $0 < |x - s| < \delta$  we have

$$|[f(x) + g(x)] - (L + M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon_0 + \varepsilon_0 = \varepsilon.$$

Hence  $\lim_{x \rightarrow s} [f(x) + g(x)] = L + M$ .

**2. Scalar-multiple rule.** Let  $\alpha \in \mathbb{R}$  and fix  $\varepsilon > 0$ . Because  $f(x) \rightarrow L$ , there exists  $\delta > 0$  such that  $0 < |x - s| < \delta$  implies  $|f(x) - L| < \varepsilon/|\alpha|$  if  $\alpha \neq 0$ ; when  $\alpha = 0$  the conclusion is immediate. Then

$$|\alpha f(x) - \alpha L| = |\alpha| |f(x) - L| < \varepsilon,$$

establishing  $\lim_{x \rightarrow s} [\alpha f(x)] = \alpha L$ .

**3. Sequential criterion.** We claim that a limit  $\lim_{x \rightarrow s} h(x) = H$  exists exactly when for every sequence  $\{x_n\}$  with  $x_n \neq s$  and  $x_n \rightarrow s$ , the sequence  $h(x_n)$  converges to  $H$ .

( $\Rightarrow$ ) If the limit exists, then given  $\varepsilon > 0$  the  $\varepsilon$ - $\delta$  condition furnishes  $\delta > 0$ . Eventually  $|x_n - s| < \delta$ , forcing  $|h(x_n) - H| < \varepsilon$ , so  $h(x_n) \rightarrow H$ .

( $\Leftarrow$ ) Conversely, suppose  $h$  does not converge to  $H$ . Then there exists  $\varepsilon_0 > 0$  such that for every  $\delta > 0$  we can pick  $x$  with  $0 < |x - s| < \delta$  and  $|h(x) - H| \geq \varepsilon_0$ . Construct  $x_n$  inductively with  $|x_n - s| < 1/n$  and  $|h(x_n) - H| \geq \varepsilon_0$ . Then  $x_n \rightarrow s$  yet  $h(x_n)$  stays  $\varepsilon_0$  away from  $H$ , contradicting the sequential hypothesis. Hence the  $\varepsilon$ - $\delta$  definition must hold.  $\square$

**Remark.** *Idea of the proof.* The sum rule splits the error of  $f + g$  into the errors of  $f$  and  $g$ , each made arbitrarily small; the scalar rule simply factors out  $\alpha$ . The sequential criterion states that functions inherit their limits along every path approaching  $s$ , and conversely any failure of the  $\varepsilon$ - $\delta$  condition can be translated into a "bad" sequence that prevents convergence.

**Exercise 5.4.** Let  $f$  be a real-valued function defined on a set  $S$  in  $\mathbb{R}^n$ . Show that if  $f$  is continuous at a point  $x_0$  in  $S$  and if  $f(x_0) < 0$ , then there exists a  $\delta > 0$  such that  $f(x) < 0$  for all  $x$  in  $B(x_0, \delta) \cap S$ .

**Proof.** Because  $f$  is continuous at  $x_0 \in S$  and  $f(x_0) < 0$ , let us set

$$\varepsilon := \frac{-f(x_0)}{2} > 0.$$

By continuity, there exists  $\delta > 0$  such that for every  $x \in S$  with  $\|x - x_0\| < \delta$  we have

$$|f(x) - f(x_0)| < \varepsilon.$$

For such  $x$  we estimate

$$f(x) = f(x_0) + (f(x) - f(x_0)) < f(x_0) + \varepsilon = f(x_0) - \frac{1}{2}f(x_0) = \frac{1}{2}f(x_0) < 0.$$

Hence  $f(x) < 0$  for all  $x \in B(x_0, \delta) \cap S$ , as required.  $\square$

**Remark.** *Idea of the proof.* Continuity lets us control  $f(x)$  by forcing it to stay within an  $\varepsilon$ -band around  $f(x_0)$ . Choosing  $\varepsilon$  to be less than half the magnitude of the negative value  $f(x_0)$  guarantees that  $f(x)$  remains negative throughout a small ball centred at  $x_0$ .