FUNCTIONAL DISCRETIZATION OF THE CO-ADJOINT ACTION ON THE DIFFEOMORPHISM GROUP FOR EULER-POINCARÉ EQUATIONS

1. Introduction. Geometric tools borrowed from Riemannian geometry and Lie theory have proven to be instrumental in the study of continuum motion. They have been used to establish beautiful connections between conserved quantities and symmetries [**?**], fluid stability and curvature [1, 2], and links between hydrodynamics with optimal transport [3] and quantum mechanics [4, 5]. Arnold's seminal work [6] formed a connection between solutions of the incompressible Euler equations and geodesics on the space of volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ [6] using a general framework for studying geodesics of right (or left) invariant metrics on possibly infinite-dimensional Lie groups. Kinetic systems whose Lagrangian is defined by a right-invariant metric possess a conservation of momentum as the analogous Noether quantity associated with a translational symmetry in finite-dimensions. In the context of fluid dynamics, this symmetry is referred to as a *relabelling symmetry*, being an invariance under coordinate transformation described by the action of the diffeomorphism group.

A reduction via symmetry can be performed for physical systems on $\mathrm{Diff}(M)$ whose Lagrangian exhibits relabelling symmetry. This effectively transforms the second-order Euler-Lagrange equations on the configuration space into the first-order Euler-Poincaré equations on the associated Lie algebra of smooth vector fields $\mathfrak{X}(M)$. Geometrically, the reduction in order indicates that the dynamics evolve on a submanifold of the phase space corresponding to the co-adjoint orbits

$$\mathcal{O}_{m_0} = \left\{ \mathrm{Ad}^*_\varphi \boldsymbol{m}_0 \;\middle|\; \varphi \in G \right\} , \tag{1.1}$$

where $\boldsymbol{m}_0$ is the initial momentum. This apparent constraint on the evolution, right-invariance of the metric, and Kelvin's circulation theorem are all equivalent for the case of ideal fluid flow [7]. In this setting there is a Lagrangian (moving) frame of reference where the system is at rest, defined by the translation back to the Eulerian (inertial) frame at the identity map. More concretely, the Euler-Poincaré equations, in the absence of potential energy, can be written in the form

$$\frac{d}{dt}\mathrm{Ad}^*_{\varphi(t)}(\boldsymbol{m}(t)) = 0 \,. \tag{1.2}$$

Due to properties of the group action, it follows that $\boldsymbol{m}$ is constrained to the path in (1.1) defined by the inverse of the Lagrangian map, that is

$$\boldsymbol{m}(t) = \mathrm{Ad}^*_{\varphi^{-1}} \boldsymbol{m}_0 \,, \tag{1.3}$$

defining the transformation of momentum from the moving to inertial frame. The expression (1.3) suggests that the evolution can be approximated through a discrete representation of the co-adjoint action on the group resulting in a path which remains on the orbits by construction.

1.1. Contribution and scope of this work. The numerical method we present in this work is based on a mapping-based approach which seeks an approximation in the space of diffeomorphisms. The characteristic mapping method which we analyze in this work stems from the Gradient-Augmented Level-Set [8] and Reference Mapping methods [9] and was subsequently developed for linear advection [10, 11], and for the Euler equations [12, 13, 14]. We will illustrate how the algorithmic approach taken by the CM method provides an integrator which exactly preserves the co-adjoint orbits by discretizing the evolution of the momentum through a functional approximation of the co-adjoint action. Informally, the algorithmic approach approximates the following sequence of operations

$$\varphi^{-1}(t_n) \mapsto \mathrm{Ad}^*_{\varphi^{-1}(t_n)} \boldsymbol{m}_0 = \boldsymbol{m}(t_n) \mapsto \boldsymbol{u}(t_n) \mapsto \varphi^{-1}(t_{n+1}) \,. \tag{1.4}$$

The technique possesses two key properties resulting in a powerful approach for the approximation of turbulent continuum dynamics: 1) *resolution invariance* and 2) *composite compression*. Both come as a result of the particular functional discretization that the method utilizes. Beyond its application to computing the momentum evolution, these properties arise in general from the approximation of the operator

$$K : \mathrm{Diff}(M) \times V \to V \,, \quad (\varphi, a) \mapsto \varphi^* a \,, \tag{1.5}$$

on a vector space V of functions or densities for instance. By discretizing only in the first component, the method generates a functional approximation of the pullback operator which acts pointwise on the elements of V. When $\varphi = \varphi_{[t,0]}$ this gives a means of approximating the evolution of any Lie advected quantity via a sampling operation leading to a notion of resolution invariance.

The composite compression follows from utilizing the group structure of $\mathrm{Diff}(M)$ to capture the generation of fine scale over time. In particular, the path formed by the forward and inverse Lagrangian maps can be decomposed as a composition of short-time sub interval flows. Considering an arbitrary sequence of time steps $t_0 < t_1 < \ldots < t_m = t$, and using the notation $\varphi_{[s,t]} = \varphi(t) \circ \varphi^{-1}(s)$, a temporal semi-discretization for the forward and backward maps is given by

$$\varphi(t) = \varphi_{[t_{m-1},t_m]} \circ \varphi_{[t_{m-2},t_{m-1}]} \circ \cdots \circ \varphi_{[t_1,t_2]} \circ \varphi_{[t_0,t_1]}, \tag{1.6a}$$

$$\varphi^{-1}(t) = \varphi_{[t_1,t_0]} \circ \varphi_{[t_2,t_1]} \circ \cdots \circ \varphi_{[t_{m-1},t_{m-2}]} \circ \varphi_{[t_m,t_{m-1}]}. \tag{1.6b}$$

Each of the submaps are approximated spatially in a finite-dimensional interpolation space $D_h \subset C^1\mathrm{Diff}(M)$. In turn, the entire path is an element of the composite approximation space

$$D_h^{\circ,k} = \underbrace{D_h \circ D_h \circ \cdots \circ D_h}_{\text{k-times}} . \tag{1.7}$$

This generates a discrete pullback operator of the form

$$K_h : D_h^{\circ,k} \times V \to V . \tag{1.8}$$

The approximation space is allowed to grow with the flow this composite discretization is capable of capturing an exponentially increasing resolution with a linear increase in computational resources. This has been demonstrated to be a powerful technique for Euler's equations [12, 13, 14] offering a unique means of capturing the multi-scale dynamics of turbulent fluid flows without the necessity of increasing the spatial resolution.

The geodesics of right-invariant metrics on infinite-dimensional Lie groups have been linked to a number of equations arising in mathematical physics including the Camassa-Holm equations [15, 16], the Constantin-Lax-Majda equation [17, 18], and the Hunter-Saxton equations [19, 20, 21]. They also play an important role in the solution of two-point boundary value problems on $\mathrm{Diff}(M)$ such as in the context of Large Deformation Diffeomorphic Metric Mapping (LDDMM) theory [22, 23] where one seeks a diffeomorphism transforming between two objects (images, metrics, densities) with minimal path length as measured with respect to the metric. In fluid mechanics these metrics can model continuum motion with the idealization that the density variable does not contribute to the kinetic energy. If the density is not neglected however, the metric will generally have a dependence on the position variable, leading to the setting of semi-invariant metrics, i.e. invariance only with respect to $\mathrm{Diff}_\mu(M)$. This setting has been studied in [24] where geodesic well-posedness was established for a large class of metrics. Extending the numerical formulation developed here to the more general case of semi-invariant metrics will be the subject of our future work.

Here we extend the CM method techniques to a larger class of continuum dynamics described by Euler-Poincaré equations with advected parameters. The article is structured as follows: in section 2 we provide a background on geometric hydrodynamics and the analytic details for the generalization of the CM method to a larger class of dynamics with advected parameters and source terms using a unified geometric framework. Thereafter in sections 4 we propose a generalization of the approximation techniques and study its convergence properties in the space of Sobolev H^s diffeomorphisms. We provide proofs for theorems which give theoretical understanding of the stability and accuracy of the method based on the properties of the discretization scheme as operators on the diffeomorphism group.

2. Background on geometric hydrodynamics.

2.1. Notation. Let the continuum domain be modelled as a compact Riemannian manifold (M, g). We will denote μ as the Riemannian volume form and $|\cdot| = g(\cdot,\cdot)$. The configuration space of continuum motion in the absence of shock formation is the space of diffeomorphisms $\mathrm{Diff}(M)$ of the domain. The state variables in the moving frame are the Lagrangian position map $\varphi(t) \in \mathrm{Diff}(M)$ and Lagrangian velocity $\dot\varphi(t) \in T_{\varphi(t)}\mathrm{Diff}(M)$ where the tangent spaces are defined by

$$T_\varphi \mathrm{Diff}(M) = \{\xi : M \to TM \,:\, \pi_M \circ \xi = \varphi\} \tag{2.1}$$

with $\pi_M : TM \to M$ being the tangent bundle. Elements of (2.1) are right translations of vector fields, with $T_{\mathrm{id}}\mathrm{Diff}(M) = \mathfrak{X}(M)$, allowing us to associate a $\boldsymbol{v} \in \mathfrak{X}(M)$ for any $\xi \in T_\varphi \mathrm{Diff}(M)$ such that $\xi = \boldsymbol{v} \circ \varphi$. In the static frame of reference (at the identity map) the corresponding vector field for the Lagrangian velocity is the Eulerian velocity field $\boldsymbol{u}(t) = \dot{\varphi} \circ \varphi^{-1}$. An evolution equation for the inverse map can be written in the Eulerian frame by differentiating $\mathrm{id} = \varphi_{[t,0]} \circ \varphi_{[0,t]}$ with respect to time and using the definition of $\boldsymbol{u}(t)$ we see that

$$\partial_t \varphi_{[t,0]} + D\varphi_{[t,0]}(\boldsymbol{u}(t)) = 0\,, \tag{2.2}$$

which expresses the advection of the particle labels by the flow. The mass of the continuum is modelled as an element in the space of normalized volume elements

$$\mathscr{V}(M) = \left\{ \beta \in \Omega^n(M) \,:\, \beta > 0\,, \int_M \beta = \mu(M) \right\}\,, \tag{2.3}$$

and we denote by $\varrho = \varphi_* \rho_0 \mu \coloneqq (\varphi^{-1})^* \rho_0 \mu \in \mathscr{V}(M)$ as the material density. It will be at times convenient to write $\varrho = J_\mu(\varphi^{-1}) \rho_0 \circ \varphi^{-1} \mu = \rho\mu$ where $J_\mu(\varphi^{-1})$ is the Jacobian determinant of the inverse map. Transformations which preserve the volume defined by μ are elements of the space of volume-preserving diffeomorphisms

$$\mathrm{Diff}_\mu(M) = \{\varphi \in \mathrm{Diff}(M) \,:\, \varphi^* \mu = \mu\}\,. \tag{2.4}$$

Tangent to (2.4) are the right translations of divergence-free vector fields, that is

$$T_\varphi \mathrm{Diff}_\mu(M) = \mathfrak{X}_\mu(M) \circ \varphi = \{\boldsymbol{u} \in \mathfrak{X}(M) \,:\, \mathcal{L}_{\boldsymbol{u}} \mu = 0\} \circ \varphi\,, \tag{2.5}$$

where $\mathcal{L}_{\boldsymbol{u}} : \Omega^k(M) \to \Omega^k(M)$ is the Lie derivative along the Eulerian velocity. Dual to the space of smooth vector fields is the space of currents which necessitates the introduction of the theory of distributions. Instead, it is common place [25, 4] to use a regularized dual space, identifying the dual of the Lie algebra with the cotangent space (2.6) (c.f. [26] for a particularly elucidating discussion on the regular dual). The momentum of the continuum is defined as an element of the cotangent space

$$T^*_{\mathrm{id}} \mathrm{Diff}(M) \coloneqq \Omega^1(M) \otimes \mathscr{V}(M) \tag{2.6}$$

at the identity element and all cotangent spaces $T^*_\varphi \mathrm{Diff}(M)$ are identified with (2.6). The pairing between a Lagrangian velocity element $\dot{\varphi} = \boldsymbol{u} \circ \varphi \in T_\varphi \mathrm{Diff}(M)$ and a momentum element $\boldsymbol{m} = \boldsymbol{v}^\flat \otimes \alpha \in T^*_\varphi \mathrm{Diff}(M)$ is defined by

$$\langle \boldsymbol{m}, \dot{\varphi} \rangle_\varphi = \int_M \boldsymbol{v}^\flat(\boldsymbol{u})\alpha = \int_M g(\boldsymbol{u}, \boldsymbol{v})\alpha\,. \tag{2.7}$$

The cotagent space (2.6) can be viewed as defined by the image of an inertia operator of the form

$$\tilde{A} : T_\varphi \mathrm{Diff}(M) \to T^*_\varphi \mathrm{Diff}(M)\,, \quad \boldsymbol{v} \circ \varphi \mapsto (A\boldsymbol{v})^\flat \otimes \mu\,, \tag{2.8}$$

where $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is assumed to be a positive, elliptic, and self-adjoint differential operator, defining generalized momentum elements. The configuration space is also a group with the action of composition. In the smooth setting, $\mathrm{Diff}(M)$ can be considered an infinite-dimensional Lie group with a Fréchet space topology [27]. The left and right actions of $\mathrm{Diff}(M)$ on itself will be denoted $L_\varphi(\eta) = \varphi \circ \eta$ and $R_\varphi(\eta) = \eta \circ \varphi$. Let $\eta(t) \in \mathrm{Diff}(M)$ and suppose that $\varphi \in \mathrm{Diff}(M)$ is fixed. The differentials of the left and right actions of φ acting on the point $(\eta, \dot{\eta}) \in T_\eta \mathrm{Diff}(M)$ are given by $TR_\varphi(\eta, \dot{\eta}) = (\eta \circ \varphi, \dot{\eta} \circ \varphi) \in T_{\eta\circ\varphi} \mathrm{Diff}(M)$ and $TL_\varphi(\eta, \dot{\eta}) = (\varphi \circ \eta, D\varphi \circ \eta \cdot \dot{\eta}) \in T_{\varphi\circ\eta} \mathrm{Diff}(M)$ respectively. The *right* Lie algebra of $\mathrm{Diff}(M)$ is given the space of vector fields $\mathfrak{X}(M)$ with Lie bracket defined by negative the Jacobi-Lie bracket. The group adjoint action with respect to an element $\varphi \in \mathrm{Diff}(M)$ is defined by

$$\mathrm{Ad}_\varphi : \mathfrak{X}(M) \to \mathfrak{X}(M)\,, \quad \boldsymbol{v} \mapsto dL_\varphi dR_{\varphi^{-1}}(\boldsymbol{v}) = (D\varphi \cdot \boldsymbol{v}) \circ \varphi^{-1}\,. \tag{2.9}$$

Letting $\Phi(s) \in \mathrm{Diff}(M)$ be such that $\Phi(0) = \mathrm{id}$ and $\Phi'(0) = \boldsymbol{u}$, taking the differential of (2.9) defines the algebra adjoint action as

$$\frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}_{\Phi(s)}(\boldsymbol{v}) = \mathrm{ad}_{\boldsymbol{u}}(\boldsymbol{v}) = -[\boldsymbol{u}, \boldsymbol{v}] = -\mathcal{L}_{\boldsymbol{u}}\boldsymbol{v}\,, \tag{2.10}$$

where $\mathcal{L}_{\boldsymbol{u}}$ acts on vector fields. The group coadjoint operator, dual to (2.9) with respect to the pairing (2.7), takes the explicit form

$$\mathrm{Ad}^*_{\varphi}(\boldsymbol{v}^\flat \otimes \mu) = \varphi^*\boldsymbol{v}^\flat \otimes \mu = \varphi^*(\boldsymbol{v}^\flat) \otimes J_\mu(\varphi)\mu\,. \tag{2.11}$$

Similarly defined, the algebra coadjoint operator is given by the Lie derivative on the product space (2.6)

$$\mathrm{ad}^*_{\boldsymbol{u}}(\boldsymbol{v}^\flat \otimes \mu) = \mathcal{L}_{\boldsymbol{u}}(\boldsymbol{v}^\flat \otimes \mu) = (\mathcal{L}_{\boldsymbol{u}}\boldsymbol{v}^\flat + \mathrm{div}(\boldsymbol{u})\boldsymbol{v}^\flat) \otimes \mu\,, \tag{2.12}$$

where the divergence is defined with respect to the Riemannian volume form.

2.2. Right-invariant metrics. Kinetic mechanical systems, with a Lagrangian defined by the kinetic energy, have a natural description from the viewpoint of Riemannian geometry. The canonical kinetic energy associated to many continuum dynamical systems is defined by the L^2 metric, expressed equivalently in the Lagrangian and Eulerian frames as

$$G_\varphi(\dot\varphi, \dot\varphi) = \int_M |\dot\varphi|^2\mu = \int_M \varrho|\boldsymbol{u}|^2\,. \tag{2.13}$$

In 1966, Arnold [6] established a connection between geodesic motion with respect to the metric (2.13) restricted to $\mathrm{Diff}_\mu(M)$ and solutions of the incompressible Euler equations. This arose as a special case of a general framework to study geodesics of right- (or left-) invariant metrics on possibly infinite-dimensional Lie groups as Euler equations on their Lie algebra.

DEFINITION 2.1. *A metric $G_\varphi(\cdot,\cdot) : T_\varphi \mathit{Diff}(M) \times T_\varphi \mathit{Diff}(M) \to \mathbb{R}$ is right-invariant if*

$$G_\varphi(\boldsymbol{u}\circ\varphi, \boldsymbol{v}\circ\varphi) = G_{\varphi\circ\eta}(dR_\eta(\boldsymbol{u}\circ\varphi), dR_\eta(\boldsymbol{v}\circ\varphi))\,, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathfrak{X}(M)\,, \eta \in \mathit{Diff}(M)\,. \tag{2.14}$$

Geometrically, right-invariance implies that the metric has no dependence on the position variable. Hence distances on $\mathrm{Diff}(M)$ can be measured by translating the metric back to the identity element along the path connecting any two points. This *relabelling symmetry* of the variational principle defined by the metric indicates that there is a Lagrangian frame of reference where the system is at rest, defined by the translation back to the Eulerian frame at the identity map.

A geodesic flow $t \mapsto \varphi(t) \in \mathrm{Diff}(M)$ associated to a metric $\langle\cdot,\cdot\rangle$, between two diffeomorphisms $\varphi_0, \varphi_T \in \mathrm{Diff}(M)$ is defined as a solution to the Lagrangian variational problem with fixed endpoints

$$\delta\int_0^T L(\varphi(t), \dot\varphi(t))dt = \delta\int_0^T \frac{1}{2}\langle\dot\varphi, \dot\varphi\rangle_{\varphi(t)}dt = 0\,, \quad \varphi(0) = \varphi_0\,, \quad \varphi(T) = \varphi_T\,. \tag{2.15}$$

In the case that the Lagrangian defined by the metric is right-invariant, then the second-order Euler-Lagrange equations describing the geodesic can be reduced into a first-order Euler-Poincaré equations on the (right) Lie algebra [28]. More concretely, considering right-invariant metrics of the due to the right-invariance we can define a symmetry reduced Lagrangian $\ell(\boldsymbol{u}) = L(\mathrm{id}, \dot\varphi\circ\varphi^{-1}) = L(\varphi, \dot\varphi)$ and the induced variations take the form $\delta\boldsymbol{u} = \dot{\boldsymbol{v}} + [\boldsymbol{u}, \boldsymbol{v}]$ where $\boldsymbol{v} = \delta\varphi\circ\varphi^{-1}$. Considering right-invariant metrics of the form

$$G^A_\varphi(\boldsymbol{u}\circ\varphi, \boldsymbol{v}\circ\varphi) = \int_M g(\boldsymbol{u}, A\boldsymbol{v})\mu = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle\,, \tag{2.16}$$

and letting $\boldsymbol{m} = A\boldsymbol{u}$, it can be readily observed that geodesics can be reconstructed from

$$\partial_t\boldsymbol{m} + \mathrm{ad}^*_{\boldsymbol{u}}(\boldsymbol{m}) = 0\,, \quad \dot\varphi = dR_{\varphi(t)}(\boldsymbol{u}(t))\,. \tag{2.17}$$

The equations describe geodesic motion exclusively from the dynamics governed the Eulerian frame, where the geodesic is reconstructed directly from the second equation (2.17). The reduced Euler-Poincaré equations (2.17) are the Eulerian expression of the conservation of momentum. They admit an alternative symmetry-reduced form in terms of the inverse map. Here the reduction happens by eliminating the direct dependence on the momentum variable. In particular, using the identity

$$\frac{d}{dt}\mathrm{Ad}^*_{\varphi(t)}(\boldsymbol{v}) = \mathrm{Ad}^*_{\varphi(t)}\mathrm{ad}^*_{\boldsymbol{u}}(\boldsymbol{v})\,, \tag{2.18}$$

and (1.2) gives us that

$$\boldsymbol{u}(t) = A^{-1}\mathrm{Ad}^*_{\varphi^{-1}(t)}(\boldsymbol{m}_0)\,. \tag{2.19}$$

The evolution of the path $(\varphi(t), \dot{\varphi}(t)) \in T_{\varphi(t)}\mathrm{Diff}(M)$ can thus be either by reduced to a path $\boldsymbol{m}(t) \in \mathfrak{X}^*(M)$ or as a path $\varphi^{-1}(t) \in \mathrm{Diff}(M)$ through the relation (2.19) due to the right-invariance of the Lagrangian. The inverse map satisfies the following non-linear advection equation

$$\partial_t \varphi_{[t,0]} + D\varphi_{[t,0]}\left(A^{-1}\mathrm{Ad}^*_{\varphi_{[t,0]}}(\boldsymbol{m}_0)\right) = 0\,, \quad \varphi_{[0,0]} = \mathrm{id}_M\,, \tag{2.20}$$

where $D\varphi_{[t,0]} : TM \to TM$ is its differential. By approximating the evolution of the inverse map directly through the solution of (2.20), the evolution of the can be approximated directly on the co-adjoint orbit (1.1). Before detailing the proposed numerical approach, we will first broaden the class of dynamics that we can consider by incorporating advected parameters into the Lagrangian.

2.3. Momentums maps and advected parameters. In order to extend the class of dynamical systems considered, a notion of potential energy must be incorporated into the Lagrangian. In this work we consider potential energies defined by a set of advected Eulerian quantities acting as thermodynamically isolated sub-systems without exchange of heat or mass. The class of dynamics can be viewed as a subset of Euler-Poincaré equations with advected parameters [25]. We suppose that V is a vector space on which $\mathrm{Diff}(M)$ acts via pullback. Letting $\varphi(t)$ be a curve in $\mathrm{Diff}(M)$ generated by the flow $\dot{\varphi} = \boldsymbol{u} \circ \varphi$, then $a(t) \in V$ is an *advected parameter* along $\varphi(t)$ if it satisfies the Lie advection equation

$$\partial_t a + \mathcal{L}_{\boldsymbol{u}} a = 0\,. \tag{2.21}$$

The evolution of an advected parameter can expressed via pushforward of the initial condition, that is

$$\frac{d}{dt}\varphi^* a(t) = \varphi^*(\partial_t a + \mathcal{L}_{\boldsymbol{u}} a) \iff a(t) = \varphi_* a_0\,. \tag{2.22}$$

In this sense advected parameters are "frozen into the flow" [25], transferring momentum to the continuum only through their contribution to the potential energy. Common examples of advected quantities include the mass density and specific entropy ($V = \mathscr{V}(M)$), buoyancy field or salinity ($V = C^\infty(M)$), along with magnetic fields ($V = \Omega^2(M)$). The mechanism for momentum transfer is largely dependent on how $\mathrm{Diff}(M)$ acts on the advected parameter space V which is encoded in the *momentum map* defined by the transformation.

The action of $\mathrm{Diff}(M)$ on V by pullback is a *right* action since it satisfies the property $(\varphi \circ \eta)^* a = a \cdot (\varphi \circ \eta) = (a \cdot \varphi) \cdot \eta$. The pullback induces an infinitesimal action of $\mathfrak{X}(M)$ on V given by the Lie derivative and the associated cotangent lifted action defines a momentum map of the form

$$\diamond : T^*V \cong V^* \times V \to T^*_{\mathrm{id}}\mathrm{Diff}(M)\,, \quad \langle a \diamond b, \boldsymbol{u}\rangle = -\langle b, \mathcal{L}_{\boldsymbol{u}} a\rangle_{V\times V^*}\,. \tag{2.23}$$

The momentum map assigns a "generalized momentum" in the dual Lie algebra $\mathfrak{g}^*$ to each element of the phase space T^*V. They are connected with Noether's theorem which states that these generalized momenta will be constants of motion when the Hamiltonian is invariant under the action of G. In section 3 we will indicate how the choice of geometric structure of the data precludes various simplifications to the expression for the momentum's evolution.

2.4. Equations of motion. Let $\bar{U} : \mathcal{A} \to \mathbb{R}$ be a potential energy functional defined on an arbitrary collections of advected parameters $(a^1(t), \ldots, a^k(t)) \in V_1 \times \cdots \times V_k = \mathcal{A}$. Since each $a^i(t)$ satisfy (2.21) the potential $\bar{U}$ defines an implicit potential $U : \mathrm{Diff}(M) \to \mathbb{R}$ such that

$$U(\varphi; a^1_0, \ldots, a^k_0) = \bar{U}(\varphi_*(a^1_0), \ldots, \varphi_*(a^k_0)) = \bar{U}(a^1, \ldots, a^k)\,. \tag{2.24}$$

We obtain a Lagrangian mechanical system on $\mathrm{Diff}(M)$ with Lagrangian $L : T\mathrm{Diff}(M) \to \mathbb{R}$ defined by

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}\langle \dot{\varphi}, \dot{\varphi}\rangle_\varphi - U(\varphi; a^1_0, \ldots, a^k_0)\,, \tag{2.25}$$

Transforming into the Eulerian frame induces a symmetry reduced Lagrangian

$$L(\varphi,\dot\varphi) = \ell(\boldsymbol{u},a^1,\ldots,a^k) = \frac{1}{2}\langle A\boldsymbol{u},\boldsymbol{u}\rangle - \bar U(a^1,\ldots,a^k) \tag{2.26}$$

Proceeding to apply Hamilton's variational principle to the Lagrangian, suppose that $\eta(t,s)$ is a variation of $\varphi(t)$ such that $\eta(t,0)=\varphi(t)$ and denote $\boldsymbol{v}\circ\varphi = \partial_s\eta(t,0) = \delta\varphi$ with fixed endpoints such that $\delta\varphi(0) = \delta\varphi(T) = 0$. We obtain the following variational formulation of the equations of motion

$$\begin{aligned}
\delta\int_0^T \ell(\varrho,a^1(t),\ldots,a^k(t),\boldsymbol{u})dt &= \int_0^T \langle\frac{\delta\ell}{\delta\boldsymbol{u}},\delta\boldsymbol{u}\rangle + \sum_{i=1}^k\langle\frac{\delta\ell}{\delta a^i},\,\partial_s|_{s=0}\,\eta^*(a^i)\rangle\,dt\\
&= \int_0^T \langle\frac{\delta\ell}{\delta\boldsymbol{u}},\dot{\boldsymbol{v}} - \mathrm{ad}_{\boldsymbol{u}}(\boldsymbol{v})\rangle + \sum_{i=1}^k\langle\frac{\delta\ell}{\delta a^i},\mathcal{L}_{\boldsymbol{v}}(a^i)\rangle\,dt\\
&= -\int_0^T \langle(\partial_t + \mathrm{ad}^*_{\boldsymbol{u}})\frac{\delta\ell}{\delta\boldsymbol{u}},\boldsymbol{v}\rangle + \langle\sum_{i=1}^k a^i\diamond\frac{\delta\ell}{\delta a_i},\boldsymbol{v}\rangle\,dt\,,
\end{aligned} \tag{2.27}$$

where the J_i are the momentum maps defined on T^*V_i. Setting the variation of the action functional to zero and closing the system with the Lie advection equations we obtain the equations of motion in the Eulerian frame

$$\partial_t\boldsymbol{m} + \mathrm{ad}^*_{\boldsymbol{u}}\boldsymbol{m} = -\boldsymbol{a}\diamond\frac{\delta\ell}{\delta\boldsymbol{a}}\,, \tag{2.28a}$$

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{a} = 0\,, \tag{2.28b}$$

where $\boldsymbol{a}$ and $\boldsymbol{J}$ denote the vector of advected parameters and component-wise operation of the momentum maps. The equations of motion (2.28) are a symmetry reduced form of the dynamics which has eliminated the dependence on the position variable within the configuration space. They can however be expressed in an alternative symmetry reduced form, similarly to (2.20) as follows. Applying the identity (2.18), integrating from 0 to t, and applying the co-adjoint action of the inverse map, the momentum can be expressed as

$$\begin{aligned}
\boldsymbol{m}(t) &= \mathrm{Ad}^*_{\varphi_{[t,0]}}(\boldsymbol{m}_0) - \mathrm{Ad}^*_{\varphi_{[t,0]}}\left[\int_0^t \mathrm{Ad}^*_{\varphi_{[0,s]}}\left(\boldsymbol{a}(s)\diamond\frac{\delta\ell}{\delta\boldsymbol{a}}(s)\right)ds\right].\\
&= \mathrm{Ad}^*_{\varphi_{[t,0]}}(\boldsymbol{m}_0) - \int_0^t \mathrm{Ad}^*_{\varphi_{[t,0]}}\left[\boldsymbol{a}_0\diamond\varphi_{[0,s]}\cdot\frac{\delta\ell}{\delta\boldsymbol{a}}\right]ds\\
&= \mathrm{Ad}^*_{\varphi_{[t,0]}}(\boldsymbol{m}_0) - \int_0^t \boldsymbol{a}(t)\diamond\varphi_{[t,s]}\cdot\frac{\delta\ell}{\delta\boldsymbol{a}}ds\\
&= \mathrm{Ad}^*_{\varphi_{[t,0]}}(\boldsymbol{m}_0) - \boldsymbol{a}(t)\diamond\int_0^t \varphi_{[t,s]}\cdot\frac{\delta\ell}{\delta\boldsymbol{a}}ds
\end{aligned} \tag{2.29}$$

where we have simplified the non-local in time appearing in (2.29) by applying the equivariance of the momentum map (**??**). Coupling to the transport of the inverse map (2.20) we get the following non-linear advection equation

$$\partial_t\varphi_{[t,0]} + D\varphi_{[t,0]}(A^{-1}\boldsymbol{m}(t)) = 0\,. \tag{2.30}$$

Note that due to the presence of the time integral in (2.29) and the inverse of the differential operator A, this transformation results in a loss of locality in space and time. However, at this expense, we have a direct functional relation between the inverse map and the momentum through the non-linear mapping

$$\mathrm{Ad}^* : \mathrm{Diff}(M)\times T^*\mathrm{Diff}(M)\to T^*\mathrm{Diff}(M)\,,\quad (\varphi,\boldsymbol{m})\mapsto\mathrm{Ad}^*_\varphi(\boldsymbol{m})\,. \tag{2.31}$$

This formulation informs the functional approximation technique proposed in this work.

3. Example continuum dynamics. We conclude our discussion on the theoretical background behind the formulation used in the numerical method we present in section 4 with some examples of continuum dynamical systems written in the form (2.30). We first consider the implications that arise from the incompressibility assumption in the presence of advected densities and scalar fields. Thereafter, we consider the case of ideal magneto-hydrodynamics and the Vlasov-Poisson equation to illustrate the applications in plasma physics. We conclude with an illustration in the context of boundary value problems on $\mathrm{Diff}(M)$ through the examples of LDM [29, 22] and diffeomorphic density matching [30]. The examples are not meant to be exhaustive but rather to illustrate how the functional form of the dynamics is primarily dictated by the geometry of the model due to the various simplifications on the non-local in time interaction term in (2.29) which can arise.

3.1. Incompressible hydrodynamics. Incompressible hydrodynamics arise from Hamiltonian systems defined on the space of volume-preserving diffeomorphisms. Tangent to $\mathrm{Diff}_\mu(M)$ is the space of divergence-free vector fields $\mathfrak{X}_\mu(M)$ which are tangent to the boundary of M and the momentum elements can be associated with the quotient space of one-forms modulo exact forms, that is $\mathfrak{X}^*_\mu(M) \cong \Omega^1(M)/d\Omega^0(M)$ [27]. Here the association of the dual with this quotient space is necessary to have a non-degenerate pairing since divergence-free vector fields are L^2 orthogonal to gradient vector fields. In this context, the inertia operator can be viewed as the assignment

$$A : \mathfrak{X}_\mu(M) \to \mathfrak{X}^*_\mu(M)\,, \quad \boldsymbol{u} \mapsto [\boldsymbol{u}^\flat]\,. \tag{3.1}$$

In the absence of advected parameters the momentum evolution (2.28) describe the Lie-advection of the cosets

$$(\partial_t + \mathrm{ad}^*_{\boldsymbol{u}})[\boldsymbol{u}^\flat] = (\partial_t + \mathcal{L}_{\boldsymbol{u}})[\boldsymbol{u}^\flat] = [\boldsymbol{0}] \tag{3.2}$$

where the [0] coset is the space exact one-forms. The group co-adjoint operator is given by the pullback and descends to a well-defined operator on the quotient space since it commutes with the exterior derivative. Along with (3.2), this gives us that

$$[\boldsymbol{u}^\flat(t)] = \varphi^*_{[t,0]}[\boldsymbol{u}^\flat_0] = [\varphi^*_{[t,0]}\boldsymbol{u}^\flat_0]\,. \tag{3.3}$$

The unique reconstruction of the advecting velocity field used the incompressibility condition. In particular, a unique representative in each coset is defined by the vorticity $\boldsymbol{\omega} = d\boldsymbol{u}^\flat$ and the circulation of $\boldsymbol{u}^\flat$ along all non-contractible loops on M. Recall the Hodge decomposition for a one-form

$$\Omega^1(M) = d\Omega^0(M) \oplus \delta\Omega^2(M) \oplus \mathcal{H}^1(M)\,, \tag{3.4}$$

where $\mathcal{H}^1(M)$ is the space of harmonic one form $\alpha \in \Omega^1(M)$ such that $\Delta\alpha = 0$ where $\Delta = d\delta + \delta d$ is the Hodge-Laplacian. Let the projection onto the harmonic component of (3.4) be denoted

$$P_{\mathcal{H}} : \Omega^1(M) \to \mathcal{H}^1(M)\,, \quad \boldsymbol{u}^\flat \mapsto \sum_{i=1}^{n} \langle \boldsymbol{u}^\flat, \boldsymbol{\xi}_i\rangle \boldsymbol{\xi}^\flat_i\,, \tag{3.5}$$

where $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n\}$ forms a basis for the harmonic vector fields. We can then write the inverse of (3.1) as

$$A^{-1} : \mathfrak{X}^*_\mu(M) \to \mathfrak{X}_\mu(M)\,, \quad [\boldsymbol{u}^\flat] = \boldsymbol{m} \mapsto \left(\delta\Delta^{-1} d\boldsymbol{m} + P_{\mathcal{H}}(\boldsymbol{m})\right)^\sharp \tag{3.6}$$

where $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is the codifferential and $\Delta = d\delta + \delta d$ is the Hodge Laplacian. The evolution of the inverse map can thus be written as the coupled system

$$\partial_t \varphi_{[t,0]} + D\varphi_{[t,0]}(\boldsymbol{u}(t)) = 0\,, \quad \boldsymbol{u}(t) = (\delta\Delta^{-1} d\varphi^*_{[t,0]}(\boldsymbol{u}^\flat_0) + P_{\mathcal{H}}(\varphi^*_{[t,0]}(\boldsymbol{u}^\flat_0)))^\sharp\,, \tag{3.7}$$

using the transport of the momentum (3.2).

3.1.1. Advected scalar fields. Passively advected quantities are ubiquitous in geophysical flows of the oceans and atmosphere. Common forms of active advected quantities are the buoyancy field, oceanic salinity, and specific entropy. The incompressibility assumption can be valid for the study of geophysical fluid dynamics when the energetic contribution from small scale, fast travelling gravity waves can be neglected. A prototypical example a geophysical flow with advected parameter is presented by the inviscid Boussinesq equations for stratified flows in a domain $\Omega \subset \mathbb{R}^3$ with vertical coordinate z. Under the influence of a gravitational potential, density variations influence the acceleration of the fluid the potential energy

$$U : \mathscr{V}(M) \to \mathbb{R}\,, \quad \varrho \mapsto \int_\Omega gz\varrho\,, \tag{3.8}$$

where g is the gravitational acceleration and ϱ is the mass density of the fluid. Under the incompressibility assumption, the Lie advected mass density $\varrho = \rho\mu$ can be expressed solely in terms of the scalar field ρ since $\varrho(t) = \varphi_* \rho_0 \mu = \rho_0 \circ \varphi^{-1}\mu$ and since μ remains fixed, the potential energy reduces a potential on the space of scalar fields. Dual to scalar fields is the space of densities in the $L^2(M)$ inner product and the momentum map on $C^\infty(M)$ is given by

$$\diamond : C^\infty(M) \times \mathscr{V}(M) \to \mathfrak{X}^*(M)\,, \quad (f,\mu) \mapsto \mu \otimes df\,. \tag{3.9}$$

This can be readily observed by noting that $\langle \varrho, \mathcal{L}_{\boldsymbol{u}} f\rangle = \langle \varrho \otimes df, \boldsymbol{u}\rangle$. It follows that

$$\rho \diamond \frac{\delta \ell}{\delta \rho} = g\rho \wedge dz\,, \tag{3.10}$$

where dz denotes the constant one-form point in the vertical direction and this recovers the Boussinesq term $b\hat{z}$ for buoyancy field $b = g\rho$. Applying the formula (2.29) the evolution of the velocity field is determined by

$$\boldsymbol{u}^\flat(t) = A^{-1}\left[\varphi^*_{[t,0]}(\boldsymbol{u}_0^\flat) - g\rho_0 \circ \varphi_{[t,0]} \wedge \int_0^t dz \circ \varphi_{[t,s]} ds\right]\,. \tag{3.11}$$

The time integral thus substantially simplifies to an integral along a single component of the inverse map for the gravitational potential energy with an advected scalar mass density field.

3.2. Plasma physics. The dynamics of many continuum systems related to plasma physics such as the Maxwell-Vlasov equations and the equations of ideal magnetohydrodynamics are well-known to possess a Hamiltonian structure [31, 32, 33]. Here we discuss the relation of the formulation in terms of the inverse map above to the particular case of the Vlasov-Poisson and ideal incompressible magnetohydrodynamic equations as examples of continuum dynamics with an advected (?) and two-forms respectively.

3.2.1. Vlasov-Poisson equations.

3.2.2. Ideal magnetohydrodynamics. Ideal magnetohydrodynamics (MHD) describes the motion of a perfectly conducting incompressible fluid transporting a magnetic field. We let $\boldsymbol{B} \in \mathfrak{X}(\mathbb{R}^3)$ denote the transported magnetic field associated with magnetic field two-form $\beta \in \Omega^2(\mathbb{R}^3)$ such that $\boldsymbol{B} = (\star\beta)^\sharp$. The configuration space is restricted to the space of volume-preserving diffeomorphisms and the symmetry reduced Lagrangian is given by

$$\ell(\boldsymbol{u}, \beta) = \frac{1}{2}\langle \boldsymbol{u}^\flat \otimes \mu, \boldsymbol{u}\rangle - \frac{1}{2}(\beta,\beta)_{L^2}\,. \tag{3.12}$$

In order to express the Euler-Poincaré equations associated to the MHD Lagrangian, we need only compute the momentum map associated to the transported magnetic field. Here we work with the magnetic field two-form which allows for more geometric flexibility. The divergence-free constraint on the magnetic vector field translates into closedness of the magnetic field two-form since $0 = \nabla \cdot \boldsymbol{B} = \delta \boldsymbol{B}^\flat = \star d \star \star \beta = (-1)^? \star d\beta$. Therefore the advected parameter space for the magnetic field is the space of closed two-forms $\Omega^2_{cl}(M)$ and the L^2 dual can be associated with the space $\Omega^{n-2}(M)/d\Omega^{n-3}(M)$. The momentum map is then given by the following Lemma [4]:

LEMMA 3.1. *The momentum map $J : T^*\Omega^2_{cl}(M) \to T^*_{id}\mathit{Diff}(M)$ is given by*

$$J(\beta, [\alpha]) = \iota_{\boldsymbol{v}}(\beta) \otimes \mu\,, \tag{3.13}$$

where the vector field $\boldsymbol{v} \in \mathfrak{X}(M)$ is determined by $\iota_{\boldsymbol{v}}\mu = d\alpha$.

Applying the momentum operator yields $\langle \frac{\delta l}{\delta \beta}, \xi\beta\rangle = \langle \frac{\delta l}{\delta \beta} \Diamond \beta, \xi\rangle$ which can be computed to give the Lorentz force $\frac{\delta l}{\delta \beta}\Diamond\beta = \iota_j \beta = \big(\star((\nabla \times (\star\beta)^\sharp) \times (\star\beta)^\sharp)\big)^\flat$.

3.3. Shape analysis. Diffeomorphic transformation theory has become a central theoretical and computational tool for image registration and computational anatomy [23, 22, **?**]. The registration problem seeks a transformation $\varphi \in \mathrm{Diff}(M)$ between two objects $I_0, I_1 \in V$ such that $\varphi_* I_0 = I_1$. The seminal work of Beg et al. [22] used a variational problem to compute this transformation computing the vector field minimizing the functional

$$E(\boldsymbol{u}) = \frac{1}{2}\int_0^1 \langle \boldsymbol{u}^\flat \otimes \mu, A\boldsymbol{u}\rangle dt + \frac{1}{2}\|\varphi^*_{[1,0]} I_0 - I_1\|\,, \tag{3.14}$$

such that $\boldsymbol{u} = \partial_t \varphi_{[0,t]} \circ \varphi_{[t,0]}$. Bruveris et al. [29] gave a geometric reformulation of this problem using a momentum map representation in the variational problem. In the case that the objects are images, represented by scalar fields, then $\mathrm{Diff}(M)$ acts by composition, and the dual space is given by $\mathscr{V}(M)$. The associated momentum map is given by

$$J : T^* C^\infty(M) \mapsto T^*_{\mathrm{id}}\mathrm{Diff}(M)\,, \quad J(f, \varrho) = -df \otimes \varrho\,, \tag{3.15}$$

which follows from directly from the definition (2.23). It follows that the solution the minimizer to (3.14) must satisfy [29]

$$\boldsymbol{u}(t) = A^{-1} J_\mu(\varphi_{[1,t]}) \cdot (I_0 \circ \varphi_{[t,0]} - I_1 \circ \varphi_{[1,t]})\nabla I_0 \circ \varphi_{[t,0]}\,. \tag{3.16}$$

4. Functional Discretization. In this section we give a geometric analysis of the resolution and convergence properties of the proposed numerical method. We begin by describing the numerical method used to compute the submaps, the characteristic mapping method, along with its extension to incorporate source terms. Thereafter, we elaborate on the resolution properties of the method resulting from the decompositions (**??**) and the reduction along coadjoint orbits. We then provide a generalization of the CM methods developed in [13, 12] to the dynamics described by (2.28). The section is then concluded with a convergence analysis of the numerical method, supporting the numerical examples included in section **??**.

4.1. Sobolev Diffeomorphisms. The space of diffeomorphisms can be equipped with a Banach manifold structure as the completion with respect to a norm such as the strong C^k, Hölder $C^{k,\alpha}$, or Sobolev H^s norms. Here, we opt to consider the case of Sobolev H^s diffeomorphisms in which out convergence estimates will be established. We begin by introducing some preliminary notation following [34]. Denote the Fourier transform as of an $L^2(\mathbb{R})$ function as

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad f \mapsto \hat{f} = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) dx\,. \tag{4.1}$$

Recall that for $s \in \mathbb{R}^+$ the Sobolev norm of a function $f : \mathbb{R}^n \to \mathbb{R}$ can be defined by

$$\|f\|_{H^s} \coloneqq \|(1 + |\xi|^2)^{s/2} \hat{f}\|_{L^2}\,. \tag{4.2}$$

The Sobolev space $H^s(\mathbb{R})$ is the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the H^s norm and $H^s(\mathbb{R}^n, \mathbb{R}^n)$ is the space of vector-valued functions whose components are in $H^s(\mathbb{R}^n)$. Sobolev spaces can be defined on a Riemannian manifold (M, g) using local coordinate charts, such as those formed by Riemannian normal coordinates for instance. Define a cover of M by open sets $B(x_\alpha, \epsilon) \subset M$ centred at $x_\alpha \in M$ such that the exponential map $\exp_\alpha : T_{x_\alpha} M \to U_\alpha$ is a local diffeomorphism and let $\{\rho_{x_\alpha}\}$ be a partition of unity subordinate to this cover. The Sobolev $H^s(M)$ norm of a function $f : M \to \mathbb{R}^n$ can be defined by

$$\|f\|_{H^s(M)} \coloneqq \sum_\alpha \|(\rho_{x_\alpha} \cdot f) \circ \exp_\alpha\|_{H^s(\mathbb{R}^n)}\,. \tag{4.3}$$

It can be shown changes to the partition of unity or covering induces equivalent norms [**?**] and further that the choice of Riemannian metric induces equivalent norms. In general, for fiber bundles $\pi : E \to M$, we can work in local trivializations of the bundle and we denote $H^s(M, E)$ as the Hilbert space formed by the completion of smooth sections $\sigma : M \to E$ in these norms.

Let $C^1\mathrm{Diff}(\mathbb{R}^n)$ be the space of C^1 diffeomorphisms, that is

$$C^1\mathrm{Diff}(\mathbb{R}^n) = \left\{\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \,:\, \varphi \text{ is a bijection}\,,\, \varphi^{-1} \in C^1(\mathbb{R}^n, \mathbb{R}^n)\right\} \tag{4.4}$$

In order to define the space of H^s diffeomorphisms it is required that the maps decay to the identity at infinity. The space can be defined as the translation of $H^s(\mathbb{R}^n, \mathbb{R}^n)$ maps as

$$\mathrm{Diff}^s(\mathbb{R}^n) = \left\{ \varphi \in \mathrm{id} + H^s(\mathbb{R}^n) \,:\, \varphi \in C^1\mathrm{Diff}(\mathbb{R}^n) \right\} . \tag{4.5}$$

Note here that there exists other equivalent definitions on this space [35]. Letting $s > n/2 + 1$, the space of H^s diffeomorphisms is defined as

$$\mathrm{Diff}^s(M) = C^1\mathrm{Diff}(M) \cap H^s(M, M) , \tag{4.6}$$

where $C^1\mathrm{Diff}(M) \subset C^1(M, M)$ is the space of C^1 diffeomorphisms of M. An atlas of charts can be given to the space of $H^s(M, M)$ maps via the Riemannian exponential map $\exp : TM \to M$ [36], where a chart about $\eta \in H^s(M, M)$ is constructed through the tangent spaces

$$T_\eta H^s(M, M) = \{ v \in H^s(M, TM) \,:\, \pi_M \circ v = \eta \} \tag{4.7}$$

via the Riemannian exponential map. In particular, $v \mapsto \exp \circ v$ is a chart between an open neighbourhood of zero in the tangent space (4.7) and a neighbourhood of η. Since $\mathrm{Diff}^s(M) \subset H^s(M, M)$ is open it inherits this smooth structure and the tangent space at the identity map coincides with the space of H^s vector fields [36], i.e. $T_{\mathrm{id}}\mathrm{Diff}^s(M) = H^s(TM)$.

Considering other bundles over the manifold of H^s diffeomorphisms, we will use the notation $T\mathrm{Diff}^r(M) \restriction \mathrm{Diff}^s(M)$ as the restriction of $T\mathrm{Diff}^r(M)$ to the base $\mathrm{Diff}^s(M)$. The cotangent bundle is not defined through the natural dual space $H^s(TM)^* \cong H^{-s}(T^*M)$, but instead, since we have identified the momentum elements through the inertia operator, (2.8), it is defined by

$$\mathfrak{X}^{s-k}(M)^* = \tilde{A}\left(H^s(TM)\right) = H^{s-k}(T^*M) \otimes \mathscr{V}^{s-1}(M) , \tag{4.8}$$

where $\mathscr{V}^{s-1}(M)$ is the closure of (2.3) in the H^{s-1} norm. The particular regularity is imposed in order to have a well-defined co-adjoint action on the space of momenta since the pullback action on the density component will lower the regularity by one. All cotangent spaces $T^*_\varphi\mathrm{Diff}^{s-k}(M)$ are identified with (4.8) and we can write the cotangent bundle as the product $T^*\mathrm{Diff}^s(M) = \mathrm{Diff}^s(M) \times \mathfrak{X}^{s-k}(M)^*$.

In the Sobolev setting, the space of H^s diffeomorphisms is no longer a Lie group since the inverse map and left composition operations are not smooth. An appropriate regularity for the domain and range of these operators must be defined in order to assert 4.5. In what follows we recall a collection of Lemmas, based on results given in the monograph [37], from which the proof of 4.5 follows as a corollary. We elaborate on some of the analysis needed in the convergence analysis given in section 5. In particular, we require the explicit form for the operator norm of the right-composition operator in order to justify stability of the numerical method. Recall the regularity for the multiplication of H^s functions and the composition of H^s diffeomorphisms.

LEMMA 4.1. *Suppose that $s > n/2$ and $0 \le s' \le s$. Then for any $f \in H^{s'}(M, \mathbb{R})$ and $g \in H^s(M, \mathbb{R})$ there exists a constant $C > 0$ such that*

$$\|f \cdot g\|_{s'} \le C\|f\|_{s'}\|g\|_s . \tag{4.9}$$

LEMMA 4.2. *The right composition with a diffeomorphism $\varphi \in \mathit{Diff}^s(M)$ $s > n/2 + 1$*

$$R_\varphi : H^{s'}(M, M) \to H^{s'}(M, M) , \qquad f \mapsto f \circ \varphi \tag{4.10}$$

is smooth for $s > n/2 + 1$ and $0 \le s' \le s$. For a fixed $\eta \in \mathit{Diff}^{s+k}(M)$, the left composition operator

$$L_g : \mathit{Diff}^s(M) \to \mathit{Diff}^s(M) , \quad \varphi \mapsto \varphi^* g \tag{4.11}$$

is C^k when $s > n/2 + 1$.

The Lemma 4.2 was first shown in the C^k setting by Smale and Abraham [38] and are commonly referred to as the ‘alpha-omega’ Lemmas [36]. The result for the right composition follows from the Sobolev Lemma and the fact that R_φ is a linear map and the result for the left composition in the Sobolev setting was proven by Inci et al. [37]. A more precise notion of stability for the numerical method requires an explicit bound on the operator norm of the right composition (4.10).

THEOREM 4.3. *The operator norm of the right composition map* (4.10) *is bound by*

$$\|R_\varphi\|_{H^{s'}\to H^{s'}} = \sup_{f\in H^{s'},\|f\|_{s'}=1} \|R_\varphi(f)\|_{s'} \le \|det(D\varphi^{-1})\|_\infty \cdot (1+\|D\varphi - I\|_{s-1})^{s'}\,. \tag{4.12}$$

Proof. We first consider the case that $M=\mathbb{R}^n$. The case of $s'=0$ follows by a change of variables and the Sobolev Lemma. The proof of the statement for $1\le s'\le s$ will follow by establishing

$$|f\circ\varphi|_{s'} \le \|\det(D\varphi^{-1})\|_\infty \Big(\sum_{r=1}^{s'} |f|_r^2 (1+\|D\varphi - I\|_{s-1})^{2r}\Big)^{1/2}\,. \tag{4.13}$$

Proceeding by induction, and using the seminorm

$$|f|_s = \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2} \frac{s!}{\alpha_1!\alpha_2!\cdots\alpha_d!} \tag{4.14}$$

Try (omitting the det for now)

$$\|f\circ\varphi\|_s \le \sum_{r=0}^{s} |f|_r (\|\nabla\varphi - I\|_{s-1}+1)^r \tag{4.15}$$

Induction:

$$\|f\circ\varphi\|_{s+1} = \|f\circ\varphi\|_{L^2} + \sum_i \|\partial_i(f\circ\varphi)\|_s = \|f\|_{L^2} + \sum_{i,j} \|(\partial_j f\circ\varphi)(\partial_i\varphi^j - \delta^i_j) + \delta^i_j(\partial_i f\circ\varphi)\|_s \tag{4.16}$$

$$\le \|f\|_{L^2} + \sum_j \|(\partial_j f\circ\varphi)\|_s (\|\nabla\varphi^j - 1\|_{s-1}+1) \le \|f\|_{L^2} + \sum_{r=0}^{s}\sum_j |\partial_j f|_r (\|\nabla\varphi^j - 1\|_{s-1}+1)^{r+1} \tag{4.17}$$

□

The convergence of the functional discretization of the pullback and co-adjoint operators offered by the CM method requires the specification of the correct regularity for which the co-adjoint is differentiable. In the Sobolev setting, we can extend the map (2.31) by density to obtain the following map between Banach manifolds

$$\mathrm{Ad}^* : \mathrm{Diff}^s(M)\times T^*\mathrm{Diff}^r(M) \to T^*\mathrm{Diff}^r(M)\,,\quad (\varphi,\boldsymbol{m})\mapsto \mathrm{Ad}^*_\varphi(\boldsymbol{m})\,. \tag{4.18}$$

The regularity of (4.18) follows as a corollary of the regularity of the pullback operator.

THEOREM 4.4. *The pullback operator*

$$\mathcal{K} : \mathit{Diff}^s(M)\times H^{s+k}(T^*M) \to H^s(T^*M)\,,\quad (\varphi,\alpha)\to\varphi^*\alpha \tag{4.19}$$

is C^k for $s>n/2+1$.

Proof. The result follows as consequence of the regularity of the composition operator

$$\mathcal{C} : \mathrm{Diff}^s(M)\times H^{s+k}(M) \to H^{s-1}(M)\quad (\varphi,f)\mapsto f\circ\varphi\,. \tag{4.20}$$

In [37] is was shown that (4.20) is a C^k map with $s>n/2+1$. This results readily extends to (4.19) by first noting that in local coordinates $(x_1,\ldots,x_n)$ for an open set $U\subset M$, the pullback of a differential form expressed locally as $\alpha=\alpha_i dx^i$ can be written $\partial_j\varphi^i\alpha_i\circ\varphi dx^j$. Hence the pullback operator can be written locally as the sum of operators of the form $M_{\partial_j\varphi^i}\circ\mathcal{C}(\varphi,\alpha_i)$ where M_f is the multiplication operator. The claim then follows by noting that $\partial_j\varphi^i\in H^{s-1}(M)$ and hence its multiplication operator is smooth.

COROLLARY 4.5. *The non-linear map* (4.18) *is C^k for $s>n/2+1$ and $r\ge s+k$.*

Proof. Due to the functional form (2.11), we can write the co-adjoint operator as the composition

$$\mathrm{Ad}^*(\varphi,m) = \mathrm{Ad}^*(\varphi,\boldsymbol{u}^\flat\otimes\mu) = \varphi^*(\boldsymbol{u}^\flat\otimes\mu) = J_\mu(\varphi)\cdot\varphi^*\boldsymbol{u}^\flat\otimes\mu = M_\rho\circ\mathcal{K}(\varphi,\boldsymbol{u}^\flat)\otimes\mu\,, \tag{4.21}$$

where $\rho=J_\mu(\varphi)$ and $M_f : H^r(T^*M)\to H^r(T^*M)$ is the multiplication operator with f. The result then follows by 4.4. □

4.2. The Characteristic Mapping Method.

4.2.1. Spatial Discretization.

4.2.2. Temporal Discretization.

4.2.3. Incorporating source terms.

5. Error estimates.

5.1. Adaptive Decomposition.

5.2. Stability of the numerical schemes.

5.3. Error estimates.

6. Numerical Examples.

7. Conclusion.

REFERENCES

[1] G. Misiolek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, Ph.D. thesis, State University of New York at Stony Brook (1992).

[2] G. Misiołek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, Indiana University mathematics journal (1993) 215–235.

[3] F. Otto, The geometry of dissipative evolution equations: the porous medium equation (2001).

[4] B. Khesin, G. Misiołek, K. Modin, Geometric hydrodynamics and infinite-dimensional newton's equations, Bulletin of the American Mathematical Society 58 (3) (2021) 377–442.

[5] B. Khesin, G. Misiolek, K. Modin, Geometric hydrodynamics via madelung transform, Proceedings of the National Academy of Sciences 115 (24) (2018) 6165–6170.

[6] V. I. Arnold, On the differential geometry of infinite-dimensional lie groups and its application to the hydrodynamics of perfect fluids, in: Vladimir I. Arnold-Collected Works, Springer, 1966, pp. 33–69.

[7] J. Marsden, A. Weinstein, Coadjoint orbits, vortices, and clebsch variables for incompressible fluids, Physica D: Nonlinear Phenomena 7 (1-3) (1983) 305–323.

[8] J.-C. Nave, R. R. Rosales, B. Seibold, A gradient-augmented level set method with an optimally local, coherent advection scheme, Journal of Computational Physics 229 (10) (2010) 3802–3827.

[9] K. Kamrin, C. H. Rycroft, J.-C. Nave, Reference map technique for finite-strain elasticity and fluid–solid interaction, Journal of the Mechanics and Physics of Solids 60 (11) (2012) 1952–1969.

[10] O. Mercier, X.-Y. Yin, J.-C. Nave, The Characteristic Mapping Method for the Linear Advection of Arbitrary Sets, SIAM Journal on Scientific Computing 42 (3) (2020) A1663–A1685.

[11] S. Taylor, J.-C. Nave, A projection-based Characteristic Mapping method for tracer transport on the sphere, Journal of Computational Physics 477 (2023) 111905.

[12] X.-Y. Yin, O. Mercier, B. Yadav, K. Schneider, J.-C. Nave, A characteristic mapping method for the two-dimensional incompressible euler equations, Journal of Computational Physics 424 (2021) 109781.

[13] X.-Y. Yin, K. Schneider, J.-C. Nave, A characteristic mapping method for the three-dimensional incompressible euler equations, arXiv preprint arXiv:2107.03504 (2021).

[14] S. Taylor, J.-C. Nave, A characteristic mapping method for incompressible hydrodynamics on a rotating sphere (arXiv:2302.01205) (Oct. 2023). arXiv:2302.01205.

[15] R. Camassa, D. D. Holm, An integrable shallow water equation with peaked solitons, Physical review letters 71 (11) (1993) 1661.

[16] S. Kouranbaeva, The camassa–holm equation as a geodesic flow on the diffeomorphism group, Journal of Mathematical Physics 40 (2) (1999) 857–868.

[17] P. Constantin, P. D. Lax, A. Majda, A simple one-dimensional model for the three-dimensional vorticity equation, Communications on pure and applied mathematics 38 (6) (1985) 715–724.

[18] M. Bauer, B. Kolev, S. C. Preston, Geometric investigations of a vorticity model equation, Journal of Differential Equations 260 (1) (2016) 478–516.

[19] J. K. Hunter, R. Saxton, Dynamics of director fields, SIAM Journal on Applied Mathematics 51 (6) (1991) 1498–1521.

[20] J. Lenells, The hunter–saxton equation describes the geodesic flow on a sphere, Journal of Geometry and Physics 57 (10) (2007) 2049–2064.

[21] B. Khesin, G. Misiołek, Euler equations on homogeneous spaces and virasoro orbits, Advances in Mathematics 176 (1) (2003) 116–144.

[22] M. F. Beg, M. I. Miller, A. Trouvé, L. Younes, Computing large deformation metric mappings via geodesic flows of diffeomorphisms, International journal of computer vision 61 (2005) 139–157.

[23] L. Younes, Shapes and diffeomorphisms, Vol. 171, Springer, 2010.

[24] M. Bauer, K. Modin, Semi-invariant riemannian metrics in hydrodynamics, Calculus of Variations and Partial Differential Equations 59 (2) (2020) 65.

[25] D. D. Holm, J. E. Marsden, T. S. Ratiu, The euler–poincaré equations and semidirect products with applications to continuum theories, Advances in Mathematics 137 (1) (1998) 1–81.

[26] K. Modin, M. Perlmutter, S. Marsland, R. McLachlan, On euler–arnold equations and totally geodesic subgroups, Journal of Geometry and Physics 61 (8) (2011) 1446–1461.
[27] V. I. Arnold, B. A. Khesin, Topological methods in hydrodynamics, Vol. 125, Springer Science & Business Media, 2008.
[28] J. E. Marsden, T. S. Ratiu, Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems, Vol. 17, Springer Science & Business Media, 2013.
[29] M. Bruveris, F. Gay-Balmaz, D. D. Holm, T. S. Ratiu, The momentum map representation of images, Journal of nonlinear science 21 (2011) 115–150.
[30] M. Bauer, S. Joshi, K. Modin, Diffeomorphic density matching by optimal information transport, SIAM Journal on Imaging Sciences 8 (3) (2015) 1718–1751.
[31] P. J. Morrison, The maxwell-vlasov equations as a continuous hamiltonian system, Physics Letters A 80 (5-6) (1980) 383–386.
[32] J. E. Marsden, A. Weinstein, The hamiltonian structure of the maxwell-vlasov equations, Physica D: nonlinear phenomena 4 (3) (1982) 394–406.
[33] P. J. Morrison, R. D. Hazeltine, Hamiltonian formulation of reduced magnetohydrodynamics, The Physics of fluids 27 (4) (1984) 886–897.
[34] M. Bauer, M. Bruveris, E. Cismas, J. Escher, B. Kolev, Well-posedness of the epdiff equation with a pseudo-differential inertia operator, Journal of Differential Equations 269 (1) (2020) 288–325.
[35] M. Bruveris, F.-X. Vialard, On completeness of groups of diffeomorphisms, Journal of the European Mathematical Society 19 (5) (2017) 1507–1544.
[36] D. G. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Annals of Mathematics (1970) 102–163.
[37] H. Inci, T. Kappeler, P. Topalov, On the regularity of the composition of diffeomorphisms, Vol. 226, American Mathematical Soc., 2013.
[38] R. Abraham, S. Smale, Lectures of smale on differential topology, (No Title) (1963).