计算方法 第 7 章习题答案

7.2 方程 $x^3 - x^2 - 1 = 0$ 在 $x_0 = 1.5$ 附近有根, 将方程作三种同解变形, 可得三种迭代格式

(1) $x = 1 + \frac{1}{x^2}$, $x_{k+1} = 1 + \frac{1}{x_k^2}$

(2) $x^3 = 1 + x^2$, $x_{k+1} = \sqrt[3]{1 + x_k^2}$

(3) $x = \frac{1}{x-1}$, $x_{k+1} = \frac{1}{\sqrt{x_k - 1}}$

判断各迭代格式在 $x_0 = 1.5$ 附近的收敛性, 选一种收敛最快的迭代格式, 计算在 $x_0 = 1.5$ 附近的根, 准确到4位小数.

解 取 $x_0 = 1.5$ 的邻域 $[1.4, 1.6]$ 作分析.

    (1) $\varphi(x) = 1 + \frac{1}{x^2}$, $|\varphi'(x)| = |-\frac{2}{x^3}| \leq \frac{2}{1.4^3} = L_1 = 0.7289 < 1$, 所以迭代收敛.

    (2) $\varphi(x) = \sqrt[3]{1 + x^2}$, $|\varphi'(x)| = \frac{2x}{3\sqrt[3]{(1+x^2)^2}} < \frac{2 \times 1.6}{3\sqrt[3]{(1+1.6^2)^2}} = L_2 = 0.5174 < 1$, 所以迭代收敛.

    (3) $\varphi(x) = \frac{1}{\sqrt{(x-1)}}$, $|\varphi'(x)| = \frac{1}{2\sqrt{(x-1)^3}} > \frac{1}{2 \times 0.6^{\frac{3}{2}}} = L_3 = 1.0758 > 1$, 所以迭代发散.

由于 $L_2 < L_1 < L_3$, 故迭代格式 (2) 收敛速度最快, 选取迭代格式 (2) 进行迭代. 要使结果具有4位小数, 只需要

$$|x^* - x_k| \leq \frac{L_2}{1 - L_2}|x_{k+1} - x_k| < \frac{1}{2} \times 10^{-4}$$

即 $|x_k - x_{k-1}| < \frac{1 - L_2}{L_2} \times \frac{1}{2} \times 10^{-4} < \frac{1}{2} \times 10^{-4}$. 取 $x_0 = 1.5$, 计算结果如下：

$$x_1 = 1.48125, \ x_2 = 1.47272, \ x_3 = 1.46882, \ x_4 = 1.46705, \ x_5 = 1.46624,$$

$$x_6 = 1.46588, \ x_7 = 1.46571, \ x_8 = 1.46563, \ x_9 = 1.46560.$$

由于 $|x_9 - x_8| < \frac{1}{2} \times 10^{-4}$, 取 $x^* \approx x_9$.

7.3 设在区间 $[a, b]$ 上 $0 < m \leq f'(x) \leq M$, 证明对任意 $\lambda \in (0, \frac{2}{M})$, 取 $x_0 \in [a, b]$ 时, 由迭代格式

$$x_{k+1} = x_k - \lambda f(x_k), k = 0, 1, 2, \dots$$

产生的迭代序列 x_k 收敛于方程 $f(x) = 0$ 的根 x^*.

证明 设 $\varphi(x) = x - \lambda f(x)$, 可知 $\varphi(x)$ 在 $(-\infty, +\infty)$ 上可导. 对于任意给定的 λ 值, 满足条件 $0 < m \leq f'(x) \leq M$ 时,

$$\varphi'(x) = 1 - \lambda f'(x) \implies 1 - \lambda M \leq \varphi'(x) \leq 1 - \lambda m < 1$$

又 $0 < \lambda < \frac{2}{M}$, 则 $0 < \lambda M < 2$ 时, $-1 < 1 - \lambda M < 1$, 所以 $-1 < 1 - \lambda M \leq \varphi'(x) \leq 1 - \lambda m < 1$. 若令 $L = \max\{|1 - \lambda M|, |1 - \lambda m|\}$, 则可知 $|\varphi'(x)| \leq L < 1$.

$$|x_k - x^*| \leq L|x_{k-1} - x^*| \leq \dots \leq L^k |x_0 - x^*| \to 0$$

即 $\lim_{k \to \infty} x_k = x^*$.

7.4 利用迭代法的思想证明

$$\lim_{k\to\infty} \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}_{\text{开方 } k \text{ 次}} = 2.$$

证明　令

$$x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}_{\text{开方 } k \text{ 次}}$$

则 x_{k+1} 可看作是初值为 $x_0 = \sqrt{2}$ 的迭代格式 $x_{k+1} = \sqrt{2+x_k}$ 生成的迭代序列, 其中迭代函数为 $\phi(x) = \sqrt{2+x}$.

在区间 $[1,3]$ 上考虑该迭代格式的收敛性. 由于

$$\phi'(x) = \frac{1}{2\sqrt{2+x}} \implies |\phi'(x)| < 1, \ \forall x \in [1,2]$$

因此以 $\phi(x)$ 为迭代函数的简单迭代法 $x_{k+1} = \phi(x_k)$ 在区间 $[1,3]$ 上收敛, 且收敛于的 $\phi(x)$ 的不动点 x^*. 易知 $\phi(2) = 2$, 即 $x^* = 2$,

$$\lim_{k\to\infty} \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}_{\text{开方 } k \text{ 次}} = x^* = 2.$$

7.8 怎样选取函数 $h(x)$, 使得当

$$\varphi(x) = x - \frac{f(x)}{f'(x)} + h(x)\left[\frac{f(x)}{f'(x)}\right]^2$$

时, 简单迭代格式至少三阶收敛.

解　令 $p(x) = -\frac{1}{f'(x)}$, $q(x) = \frac{h(x)}{f'^2(x)}$, 则

$$\varphi(x) = x + p(x)f(x) + q(x)f^2(x).$$

设 x^* 满足 $f(x^*) = 0$, 则有 $\varphi(x^*) = x^*$, 即不动点存在. 若要简单迭代格式 $x_{k+1} = \varphi(x_k)$ 至少三阶收敛到 $f(x) = 0$ 的根 x^*. 根据收敛阶定理, 应有

$$\varphi'(x^*) = 0, \ \varphi''(x^*) = 0.$$

于是

$$\varphi'(x^*) = 1 + p(x^*)f'(x^*) + p'(x^*)f(x^*) + q'(x^*)f^2(x^*) + 2q(x^*)f(x^*)f'(x^*)$$

$$= 1 + p(x^*)f'(x^*) = 1 - \frac{f'(x^*)}{f'(x^*)} = 1 - 1 = 0,$$

$$\varphi''(x^*) = p''(x^*)f(x^*) + 2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + q''(x^*)f^2(x^*)$$

$$+ 4f(x^*)f'(x^*)q'(x^*) + 2q(x^*)\left(f(x^*)f''(x^*) + f'^2(x^*)\right)$$

$$= 2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + 2q(x^*)f'^2(x^*) = 0,$$

即 $p(x), q(x)$ 满足关系

$$2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + 2q(x^*)f'^2(x^*) = 0.$$

由于

$$p(x^*) = -\frac{1}{f'(x^*)}, \ \ p'(x^*) = \frac{f''(x^*)}{f'^2(x^*)}, \ \ q(x^*) = \frac{h(x^*)}{f'^2(x^*)},$$

代入得到

$$2\frac{f''(x^*)}{f'(x^*)} - \frac{f''(x^*)}{f'(x^*)} + 2h(x^*) = 0 \implies h(x^*) = -\frac{f''(x^*)}{2f'(x^*)}.$$

由 $f(x)$ 及 x^* 的任意性, 可知当 $h(x) = -\frac{1}{2}\frac{f''(x)}{f'(x)}$ 时, 上述简单迭代法至少三阶收敛.

7.9 用松弛法和斯特森加速技术求解习题 7.2.

解 以习题 7.2 中的迭代格式 (1) 为例考虑加速技术. 松弛法:

$$\varphi(x) = 1 + \frac{1}{x^2}, \ \ \varphi'(x) = -\frac{2}{x^3}, \ \ \varphi'(x_0) = \varphi'(1.5) = -\frac{2}{1.5^3} = -\frac{16}{27}$$

迭代格式为

$$x_{k+1} = \frac{1 + \frac{1}{x_k^2} + \frac{16}{27}x_k}{1 + \frac{16}{27}}$$

计算如下:

$$x_1 = 1.46512, \ \ x_2 = 1.46558, \ \ x_3 = 1.46557,$$

故 $x^* = 1.4656$.

斯特森加速技术:

$$x_1 = \varphi(x_0) = 1.44444, \ \ x_2 = \varphi(x_1) = 1.47929, \ \ \overline{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46586$$

$$x_0 = \overline{x} = 1.46586, \ \ x_1 = \varphi(x_0) = 1.46539, \ \ x_2 = \varphi(x_1) = 1.46569, \ \ \overline{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46557$$

$$x_0 = \overline{x} = 1.46557, \ \ x_1 = \varphi(x_0) = 1.46557, \ \ x_2 = \varphi(x_1) = 1.46557, \ \ \overline{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46557$$

由于 $|\overline{x} - x_2| < \frac{1}{2} \times 10^{-5}$, 故取 $x^* \approx \overline{x} = 1.4656$.

7.13 证明 $f(x) = (x - x^*)^m g(x)$, 且 $g(x^*) \neq 0$, 当 $m > 1$ 时, 求解方程 $f(x) = 0$ 的牛顿法一阶收敛, 但令 $u = f(x)/f'(x)$ 时, 迭代法

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)} \ \ \text{或者} \ \ x_{k+1} = x_k - \frac{u(x_k)}{u'(x_k)}$$

二阶收敛.

证明 令 $\varphi(x) = x - \frac{f(x)}{f'(x)}$, 由于

$$0 < \varphi'(x^*) = 1 - \frac{1}{m} < 1,$$

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所以一阶收敛.

令
$$\varphi(x) = x - m\frac{f(x)}{f'(x)} = x - m\frac{(x-x^*)g(x)}{mg(x) + (x-x^*)g'(x)},$$

则
$$\varphi(x^*) = x^*, \ \varphi'(x^*) = 0,$$

所以至少二阶收敛.

7.17 给定 $x^{(0)} = (0.6, 0.8)^{\mathrm{T}}$, 用布洛依登法求解如下方程组的近似解, 准确到两位小数
$$x_1^2 + x_2^2 - 1 = 0, \quad 2x_1 + x_2 - 1 = 0.$$

解 由题知
$$f(x) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ 2x_1 + x_2 - 1 \end{pmatrix}, \quad f(x^{(0)}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$J_f(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2 & 1 \end{pmatrix}, \quad A_0 = J_f(x^{(0)}) = \begin{pmatrix} 1.2 & 1.6 \\ 2 & 1 \end{pmatrix}, \quad A_0^{-1} = \begin{pmatrix} -0.5 & 0.8 \\ 1 & -0.6 \end{pmatrix}$$

$$x^{(1)} = x^{(0)} - A_0^{-1}f(x^{(0)}) = (-0.2, 1.4)^{\mathrm{T}}, \ f(x^{(1)}) = (0,1)^{\mathrm{T}},$$

$$s^{(1)} = x^{(1)} - x^{(0)} = (-0.8, 0.6)^{\mathrm{T}}, \ y^{(1)} = f(x^{(1)}) - f(x^{(0)}) = (1,-1)^{\mathrm{T}},$$

$$A_1^{-1} = A_0^{-1} + \frac{(s^{(1)} - A_0^{-1}y^{(1)})s^{(1)\mathrm{T}}A_0^{-1}}{s^{(1)\mathrm{T}}A_0^{-1}y^{(1)}} = \begin{pmatrix} -0.25 & 0.55 \\ 0.5 & -0.1 \end{pmatrix}$$

$$x^{(2)} = x^{(1)} - A_1^{-1}f(x^{(1)}) = (0.05, 0.9)^{\mathrm{T}}, \ f(x^{(2)}) = (1.0, 0.0)^{\mathrm{T}},$$

$$s^{(2)} = x^{(2)} - x^{(1)} = (0.25, -0.5)^{\mathrm{T}}, \ y^{(2)} = f(x^{(2)}) - f(x^{(1)}) = (-1.1875, 0)^{\mathrm{T}},$$

$$A_2^{-1} = A_1^{-1} + \frac{(s^{(2)} - A_1^{-1}y^{(2)})s^{(2)\mathrm{T}}A_1^{-1}}{s^{(2)\mathrm{T}}A_1^{-1}y^{(2)}} = \begin{pmatrix} -0.2105 & 0.5263 \\ 0.4211 & -0.0526 \end{pmatrix}$$

$$x^{(3)} = x^{(2)} - A_2^{-1}f(x^{(2)}) = (0.0105, 0.9789)^{\mathrm{T}}, \ f(x^{(3)}) = (-0.1875, 0)^{\mathrm{T}},$$

$$s^{(3)} = x^{(3)} - x^{(2)} = (-0.0395, 0.0789)^{\mathrm{T}}, \ y^{(3)} = f(x^{(3)}) - f(x^{(2)}) = (0.1459, 0)^{\mathrm{T}},$$

$$A_3^{-1} = A_2^{-1} + \frac{(s^{(3)} - A_2^{-1}y^{(3)})s^{(3)\mathrm{T}}A_2^{-1}}{s^{(3)\mathrm{T}}A_2^{-1}y^{(3)}} = \begin{pmatrix} -0.2705 & 0.5623 \\ 0.5409 & -0.1246 \end{pmatrix}$$

$$x^{(4)} = x^{(3)} - A_3^{-1}f(x^{(3)}) = (-0.0007, 1.0014)^{\mathrm{T}}, \ f(x^{(4)}) = (-0.0416, 0)^{\mathrm{T}},$$

$$s^{(4)} = x^{(4)} - x^{(3)} = (-0.0124, 0.0248)^{\mathrm{T}}, \ y^{(4)} = f(x^{(4)}) - f(x^{(3)}) = (0.0444, 0)^{\mathrm{T}},$$

$$A_4^{-1} = A_3^{-1} + \frac{(s^{(4)} - A_3^{-1}y^{(4)})s^{(4)\mathrm{T}}A_3^{-1}}{s^{(4)\mathrm{T}}A_3^{-1}y^{(4)}} = \begin{pmatrix} -0.2531 & 0.5519 \\ 0.5062 & -0.1037 \end{pmatrix}$$

于是有
$$x^{(5)} \approx (9.48 \times 10^{-6}, 0.99998)^{\mathrm{T}}, \ \|x^{(5)} - x^{(4)}\|_\infty \leqslant \frac{1}{2} \times 10^{-2},$$

达到给定的精度, 因此取 $x^* \approx x^{(5)}$.