计算方法 第 6 章习题答案

6.1 设函数 $f(x)$ 在区间 $[a,b]$ 上二阶导数连续, 试导出以下求积公式:

(1) $\int_a^b f(x)dx = (b-a)f(a) + \frac{(b-a)^2}{2}f'(\eta),\ \ a<\eta<b$;

(2) $\int_a^b f(x)dx = (b-a)f(a) - \frac{(b-a)^2}{2}f'(\eta),\ \ a<\eta<b$;

(3) $\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\eta),\ \ a<\eta<b$.

解: (1) 左矩形公式: $f(x)\approx f(a)$, $\int_a^b f(x)dx \approx \int_a^b f(a)dx = f(a)(b-a)$, 因此

$$\begin{aligned}&\int_a^b f(x)dx - f(a)(b-a) = \int_a^b f(x)dx - \int_a^b f(a)dx = \int_a^b (f(x)-f(a))dx\\ =&\int_a^b f'(\xi)(x-a)dx = f'(\eta)\int_a^b (x-a)dx = \frac{1}{2}(b-a)^2 f'(\eta),\quad \xi,\eta\in(a,b)\end{aligned}$$

(2) 右矩形公式: $f(x)\approx f(b)$, $\int_a^b f(x)dx \approx \int_a^b f(b)dx = f(b)(b-a)$, 因此

$$\begin{aligned}&\int_a^b f(x)dx - f(b)(b-a) = \int_a^b f(x)dx - \int_a^b f(b)dx = \int_a^b (f(x)-f(b))dx\\ =&\int_a^b f'(\xi)(x-b)dx = f'(\eta)\int_a^b (x-b)dx = -\frac{1}{2}(b-a)^2 f'(\eta),\quad \xi,\eta\in(a,b)\end{aligned}$$

(3) 中矩形公式: $f(x)\approx f\left(\frac{a+b}{2}\right)$, 则

$$\int_a^b f(x)dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)(b-a)$$

因此

$$\begin{aligned}&\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) = \int_a^b f(x)dx - \int_a^b f\left(\frac{a+b}{2}\right)dx = \int_a^b \left(f(x)-f\left(\frac{a+b}{2}\right)\right)dx\\ =&\int_a^b \left(f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right) + \frac{1}{2}f''(\xi)\left(x-\frac{a+b}{2}\right)^2\right)dx\\ =&f'\left(\frac{a+b}{2}\right)\int_a^b \left(x-\frac{a+b}{2}\right)dx + \frac{1}{2}f''(\eta)\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx\\ =&\frac{1}{24}f''(\eta)(b-a)^3,\quad \xi,\eta\in(a,b)\end{aligned}$$

6.7 导出下列求积公式及其截断误差估计式

(1) $\int_0^h f(x)dx \approx A_0 f(0) + B_0 f'(0) + A_1 f(h) + B_1 f'(h)$;

(2) $\int_0^{2h} f(x)dx \approx A_0 f(0) + A_1 f(h) + A_2 f(2h)$;

(3) $\int_{-1}^1 x^2 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$;

(4) $\int_0^1 \sqrt{x} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$;

(5) $\int_0^{+\infty} \mathrm{e}^{-x} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$.

解: (1) 已知该式有四个待定系数, 故令 $f(x) = 1, x, x^2, x^3$, 代入求积公式使其精确成立, 得到

$$A_0 + A_1 = h,\ \ B_0 + A_1 h + B_1 = \frac{1}{2}h^2,\ \ A_1 h^2 + 2B_1 h = \frac{1}{3}h^3,\ \ A_1 h^3 + 3B_1 h^2 = \frac{1}{4}h^4,$$

解得

$$A_0 = \frac{h}{2},\quad A_1 = \frac{h}{2},\quad B_0 = \frac{h^2}{12},\quad B_1 = -\frac{h^2}{12}.$$

因此

$$\int_0^h f(x)dx \approx Q[f] = \frac{h}{2}f(0) + \frac{h^2}{12}f'(0) + \frac{h}{2}f(h) - \frac{h^2}{12}f'(h)$$

当 $f(x) = x^4$ 时,

$$I[f] = \int_0^h x^4 dx = \frac{1}{5}h^5,\quad Q[f] = \frac{1}{6}h^5 \neq I[f],$$

因此该公式具有3次代数精度. 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{4!}f^{(4)}(\xi)x^2(x-h)^2,$$

此时有 $e(0) = e'(0) = e(h) = e'(h) = 0$,

$$\begin{aligned}R[f] =&R[e(x)] = \int_0^h e(x)dx - \frac{h}{12}[6e(0) + he'(0) + 6e(h) - he'(h)]\\ =&\int_0^h e(x)dx = \int_0^h \frac{1}{4!}f^{(4)}(\xi)x^2(x-h)^2 dx\\ =&\frac{1}{4!}f^{(4)}(\eta)\int_0^h x^2(x-h)^2 dx = \frac{h^5}{720}f^{(4)}(\eta)\end{aligned}$$

(2) 令 $f(x) = 1, x, x^2$, 代入求积公式使其精确成立, 得到

$$A_0 + A_1 + A_2 = 2h,\quad A_1 h + 2A_2 h = 2h^2,\quad A_1 h^2 + 4A_2 h^2 = \frac{8}{3}h^3,$$

解得

$$A_0 = \frac{h}{3},\quad A_1 = \frac{4h}{3},\quad A_2 = \frac{h}{3}.$$

因此

$$\int_0^{2h} f(x)dx \approx Q[f] = \frac{h}{3}f(0) + \frac{4h}{3}f(h) + \frac{h}{3}f(2h)$$

当 $f(x) = x^3$ 时,

$$I[f] = \int_0^{2h} x^3 dx = 4h^4,\quad Q[f] = \frac{h}{3}[4h^3 + 8h^3] = 4h^4 = I[f].$$

当 $f(x)=x^4$ 时,

$$I[f]=\int_0^{2h} x^4 dx=\frac{32}{5}h^5,\ \ Q[f]=\frac{h}{3}[4h^4+16h^4]=\frac{20}{3}h^5\neq I[f].$$

因此该求积公式的代数精度为 $m=3$. 根据广义佩亚诺定理, 取

$$e(x)=\frac{1}{4!}f^{(4)}(\xi)x(x-h)^2(x-2h),$$

此时 $e(0)=e(h)=e(2h)=0$, 且

$$\begin{aligned}R[f]=&R[e(x)]=\int_0^{2h} e(x)dx-\frac{h}{3}[e(0)+4e(h)+e(2h)]\\=&\int_0^{2h} e(x)dx=\int_0^{2h}\frac{1}{4!}f^{(4)}(\xi)x(x-h)^2(x-2h)dx\\=&\frac{1}{4!}f^{(4)}(\eta)\int_0^{2h} x(x-h)^2(x-2h)dx=\frac{h^5}{90}f^{(4)}(\eta).\end{aligned}$$

(3) 令 $f(x)=1,x,x^2,x^3$, 代入求积公式使其精确成立, 有

$$A_0-A_1=\frac{2}{3},\ \ -A_0x_0+A_1x_1=0,\ \ A_0x_0^2+A_1x_1^2=\frac{2}{5},\ \ A_0x_0^3+A_1x_1^3=0,$$

解得

$$x_{0,1}=\pm\frac{\sqrt{15}}{5},\ \ A_0=A_1=\frac{1}{3}.$$

因此求积公式为

$$\int_{-1}^{1} x^2 f(x)dx=\frac{1}{3}f\left(\frac{\sqrt{15}}{5}\right)+\frac{1}{3}f\left(-\frac{\sqrt{15}}{5}\right).$$

又 $R[x^4]=\frac{4}{25}\neq 0$, 因此代数精度为3. 根据广义佩亚诺定理取

$$e(x)=\frac{1}{4!}f^{(4)}(\xi)(x+\frac{\sqrt{15}}{5})^2(x-\frac{\sqrt{15}}{5})^2,$$

此时 $e(-\frac{\sqrt{15}}{5})=e(\frac{\sqrt{15}}{5})=0$, 且

$$R[f]=R[e]=\int_{-1}^{1} x^2e(x)dx=\int_{-1}^{1}\frac{1}{4!}f^{(4)}(\xi)x^2(x+\frac{\sqrt{15}}{5})^2(x-\frac{\sqrt{15}}{5})^2dx=\frac{1}{525}f^{(4)}(\eta)$$

(4) 截断误差 $R[f]=\int_0^1\sqrt{x}f(x)dx-A_0f(x_0)-A_1f(x_1)$. 令

$$\begin{cases}R[1]=\dfrac{2}{3}-A_0-A_1=0\\ R[x]=\dfrac{2}{5}-A_0x_0-A_1x_1=0\\ R[x^2]=\dfrac{2}{7}-A_0x_0^2-A_1x_1^2=0\\ R[x^3]=\dfrac{2}{9}-A_0x_0^3-A_1x_1^3=0\end{cases}$$

解得

$$x_0=\frac{5}{9}-\frac{\sqrt{280}}{63},\ \ x_1=\frac{5}{9}+\frac{\sqrt{280}}{63},\ \ A_0=\frac{1}{3}-\frac{\sqrt{280}}{300},\ \ A_1=\frac{1}{3}+\frac{\sqrt{280}}{300}.$$

因此求积公式为

$$\int_0^1 \sqrt{x} f(x) dx \approx \left(\frac{1}{3}-\frac{\sqrt{280}}{300}\right) f\left(\frac{5}{9}-\frac{\sqrt{280}}{63}\right)+\left(\frac{1}{3}+\frac{\sqrt{280}}{300}\right) f\left(\frac{5}{9}+\frac{\sqrt{280}}{63}\right)$$

且

$$R[x^4]=\frac{2}{11}-\left(\frac{1}{3}-\frac{\sqrt{280}}{300}\right)\left(\frac{5}{9}-\frac{\sqrt{280}}{63}\right)^4-\left(\frac{1}{3}+\frac{\sqrt{280}}{300}\right)\left(\frac{5}{9}+\frac{\sqrt{280}}{63}\right)^4=\frac{128}{43659}\neq 0$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x)=\frac{1}{4!} f^{(4)}(\xi)\big(x-\frac{5}{9}+\frac{\sqrt{280}}{63}\big)^2\big(x-\frac{5}{9}-\frac{\sqrt{280}}{63}\big)^2,$$

此时有 $e(\frac{5}{9}-\frac{\sqrt{280}}{63})=e(\frac{5}{9}+\frac{\sqrt{280}}{63})=0$, 则

$$\begin{aligned}
R[f]=&R[e(x)]=\int_0^1 \sqrt{x} e(x) dx-\left(\frac{1}{3}-\frac{\sqrt{280}}{300}\right) e\left(\frac{5}{9}-\frac{\sqrt{280}}{63}\right)-\left(\frac{1}{3}+\frac{\sqrt{280}}{300}\right) e\left(\frac{5}{9}+\frac{\sqrt{280}}{63}\right)\\
=&\int_0^1 \sqrt{x} e(x) dx=\int_0^1 \frac{1}{4!} f^{(4)}(\xi) \sqrt{x}\big(x-\frac{5}{9}+\frac{\sqrt{280}}{63}\big)^2\big(x-\frac{5}{9}-\frac{\sqrt{280}}{63}\big)^2 dx\\
=&\frac{1}{24} f^{(4)}(\eta) \int_0^1 \sqrt{x}\big(x-\frac{5}{9}+\frac{\sqrt{280}}{63}\big)^2\big(x-\frac{5}{9}-\frac{\sqrt{280}}{63}\big)^2 dx=\frac{16}{130977} f^{(4)}(\eta).
\end{aligned}$$

(5) 截断误差 $R[f]=\int_0^{+\infty} \mathrm{e}^{-x} f(x) dx-A_0 f(x_0)-A_1 f(x_1)$. 令

$$\begin{cases}
R[1]=1-A_0-A_1=0\\
R[x]=1-A_0 x_0-A_1 x_1=0\\
R[x^2]=2-A_0 x_0^2-A_1 x_1^2=0\\
R[x^3]=6-A_0 x_0^3-A_1 x_1^3=0
\end{cases}$$

解得

$$x_0=2-\sqrt{2},\ \ x_1=2+\sqrt{2},\ \ A_0=\frac{2+\sqrt{2}}{4},\ \ A_1=\frac{2-\sqrt{2}}{4}.$$

因此求积公式为

$$\int_0^{+\infty} \mathrm{e}^{-x} f(x) dx \approx \frac{2+\sqrt{2}}{4} f(2-\sqrt{2})+\frac{2-\sqrt{2}}{4} f(2+\sqrt{2}).$$

由于

$$R[x^4]=24-\frac{2+\sqrt{2}}{4}(2-\sqrt{2})^4-\frac{2-\sqrt{2}}{4}(2+\sqrt{2})^4=4\neq 0,$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x)=\frac{1}{4!} f^{(4)}(\xi)\big(x-2+\sqrt{2}\big)^2\big(x-2-\sqrt{2}\big)^2,$$

此时有 $e(2-\sqrt{2})=e(2+\sqrt{2})=0$, 则

$$\begin{aligned}R[f]=&R[e(x)]=\int_0^{+\infty}\mathrm{e}^{-x}e(x)dx\\=&\int_0^{+\infty}\frac{1}{4!}f^{(4)}(\xi)\mathrm{e}^{-x}\left(x-2+\sqrt{2}\right)^2\left(x-2-\sqrt{2}\right)^2dx\\=&\frac{1}{24}f^{(4)}(\eta)\int_0^{+\infty}\mathrm{e}^{-x}\left(x-2+\sqrt{2}\right)^2\left(x-2-\sqrt{2}\right)^2dx=\frac{1}{6}f^{(4)}(\eta).\end{aligned}$$

6.10 确定下列数值微分公式的系数, 导出截断误差简单表达式

(1) $f'(0)\approx af(-h)+bf(0)+cf(h)$

(2) $f'(h)\approx af'(0)+b[f(2h)-f(h)]$

解: (1) 截断误差为

$$R[f]=f'(0)-af(-h)-bf(0)-cf(h).$$

分别取 $f=1,x,x^2$, 令 $R[f]=0$ 得到方程组

$$\begin{cases}a+b+c=0\\(c-a)h=1\\(a+c)h^2=0\end{cases}$$

解得

$$a=-\frac{1}{2h},\quad b=0,\quad c=\frac{1}{2h}.$$

故数值微分公式为

$$f'(0)\approx-\frac{1}{2h}f(-h)+\frac{1}{2h}f(h).$$

由于 $R[x^3]=h^2\neq0$, 所以代数精度 $m=2$. 根据广义佩亚诺定理, 取

$$e(x)=\frac{1}{6}f'''(\xi)x(x+h)(x-h),$$

$$R[f]=R[e]=e'(0)=-\frac{h^2}{6}f'''(\xi).$$

(2) 截断误差为

$$R[f]=f'(h)-af'(0)-b[f(2h)-f(h)].$$

分别取 $f=1,x,x^2$, 令 $R[f]=0$ 得到方程组

$$\begin{cases}0=0\\a+bh=1\\3bh^2=2h\end{cases}$$

解得 $a=\frac{1}{3}$, $b=\frac{2}{3h}$. 因此

$$f'(h)\approx\frac{1}{3}f'(0)+\frac{2}{3h}\big[f(2h)-f(h)\big].$$

由于 $R[x^3]=-\frac{5}{3}h^2\neq 0$, 故代数精度 $m=2$. 根据广义佩亚诺定理, 取

$$e(x)=\frac{1}{6}f'''(\xi)x^2(x-2h),$$

注意到 $e(0)=e'(0)=e(2h)=0$, 于是

$$R[f]=R[e]=e'(h)-\frac{1}{3}e'(0)-\frac{2}{3h}\big[e(2h)-e(h)\big]=e'(h)+\frac{2}{3h}e(h)=-\frac{5}{18}h^2f'''(\xi)-\frac{1}{24}h^3f^{(4)}(\bar{\xi}).$$