

计算方法 第5章习题答案

5.1 求下列给定区间上关于权函数 $\omega(x)$ 的正交多项式 $g_0(x), g_1(x), g_2(x)$.

- (1) $[0, 1], \omega(x) = \sqrt{x};$ (2) $[-1, 1], \omega(x) = 1 + x^2;$
 (3) $[0, 1], \omega(x) = \sqrt{x(1-x)};$ (4) $[-1, 1], \omega(x) = \sqrt{1-x^2}.$

解 根据**三项递推关系式**构造正交多项式. 注意计算内积时需带入权函数.

(1) 取 $g_0(x) = 1$, 计算

$$\beta_0 = (xg_0, g_0) = \int_0^1 x\sqrt{x}dx = \frac{2}{5}, \quad \gamma_0 = (g_0, g_0) = \int_0^1 \sqrt{x}dx = \frac{2}{3},$$

因此 $g_1(x) = x - \beta_0/\gamma_0 = x - \frac{3}{5}$. 进一步计算

$$\begin{aligned} \beta_1 &= (xg_1, g_1) = \int_0^1 x\left(x - \frac{3}{5}\right)^2 \sqrt{x}dx = \frac{184}{7875}, \\ \gamma_1 &= (g_1, g_1) = \int_0^1 \left(x - \frac{3}{5}\right)^2 \sqrt{x}dx = \frac{8}{175}, \end{aligned}$$

因此

$$\begin{cases} b_1 = \frac{\beta_1}{\gamma_1} = \frac{23}{45}, \\ c_1 = \frac{\gamma_1}{\gamma_0} = \frac{12}{175}, \end{cases} \implies g_2(x) = (x - b_1)g_1(x) - c_1g_0(x) = x^2 - \frac{10}{9}x + \frac{5}{21}.$$

(2) 取 $g_0(x) = 1$, 计算

$$\beta_0 = (xg_0, g_0) = \int_{-1}^1 x(1+x^2)dx = 0, \quad \gamma_0 = (g_0, g_0) = \int_{-1}^1 (1+x^2)dx = \frac{8}{3},$$

因此 $g_1(x) = x - \beta_0/\gamma_0 = x$. 进一步计算

$$\beta_1 = (xg_1, g_1) = \int_{-1}^1 x^3(1+x^2)dx = 0, \quad \gamma_1 = (g_1, g_1) = \int_{-1}^1 x^2(1+x^2)dx = \frac{16}{15},$$

因此

$$\begin{cases} b_1 = \frac{\beta_1}{\gamma_1} = 0, \\ c_1 = \frac{\gamma_1}{\gamma_0} = \frac{2}{5}, \end{cases} \implies g_2(x) = (x - b_1)g_1(x) - c_1g_0(x) = x^2 - \frac{2}{5}.$$

(3) 取 $g_0(x) = 1$, 计算

$$\beta_0 = (xg_0, g_0) = \int_0^1 x\sqrt{x(1-x)}dx = \frac{\pi}{16}, \quad \gamma_0 = (g_0, g_0) = \int_0^1 \sqrt{x(1-x)}dx = \frac{\pi}{8},$$

因此 $g_1(x) = x - \beta_0/\gamma_0 = x - \frac{1}{2}$. 进一步计算

$$\begin{aligned} \beta_1 &= (xg_1, g_1) = \int_0^1 x\left(x - \frac{1}{2}\right)^2 \sqrt{x(1-x)}dx = \frac{\pi}{256}, \\ \gamma_1 &= (g_1, g_1) = \int_0^1 \left(x - \frac{1}{2}\right)^2 \sqrt{x(1-x)}dx = \frac{\pi}{128}, \end{aligned}$$

因此

$$\begin{cases} b_1 = \frac{\beta_1}{\gamma_1} = \frac{1}{2}, \\ c_1 = \frac{\gamma_1}{\gamma_0} = \frac{1}{16}, \end{cases} \implies g_2(x) = (x - b_1)g_1(x) - c_1g_0(x) = x^2 - x + \frac{3}{16}.$$

(4) 取 $g_0(x) = 1$, 计算

$$\beta_0 = (xg_0, g_0) = \int_{-1}^1 x\sqrt{1-x^2}dx = 0, \quad \gamma_0 = (g_0, g_0) = \int_{-1}^1 \sqrt{1-x^2}dx = \frac{\pi}{2},$$

因此 $g_1(x) = x - \beta_0/\gamma_0 = x$. 进一步计算

$$\beta_1 = (xg_1, g_1) = \int_{-1}^1 x^3\sqrt{1-x^2}dx = 0, \quad \gamma_1 = (g_1, g_1) = \int_{-1}^1 x^2\sqrt{1-x^2}dx = \frac{\pi}{8},$$

因此

$$\begin{cases} b_1 = \frac{\beta_1}{\gamma_1} = 0, \\ c_1 = \frac{\gamma_1}{\gamma_0} = \frac{1}{4}, \end{cases} \implies g_2(x) = (x - b_1)g_1(x) - c_1g_0(x) = x^2 - \frac{1}{4}.$$

5.3 求下列函数在指定区间上的最优平方逼近一次多项式:

$$(1) y = \sqrt{x}, [0, 1]; \quad (2) y = e^x, [-1, 1].$$

解 (1) 设最优平方逼近一次多项式为 $p_1(x) = c_0 + c_1x$, 则有 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned} (\varphi_i, \varphi_j) &= \int_0^1 x^{i+j}dx = \frac{1}{i+j+1}, \quad i, j = 0, 1 \\ (\varphi_0, f) &= \int_0^1 \sqrt{x}dx = \frac{2}{3}, \quad (\varphi_1, f) = \int_0^1 x^{\frac{3}{2}}dx = \frac{2}{5}. \end{aligned}$$

于是对应的正规方程组为

$$\begin{pmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} (\varphi_0, f) \\ (\varphi_1, f) \end{pmatrix},$$

即

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{5} \end{pmatrix},$$

解得 $c_0 = \frac{4}{15}, c_1 = \frac{4}{5}$, 因此有

$$p_1(x) = \frac{4}{15} + \frac{4}{5}x.$$

(2) 设最优平方逼近一次多项式为 $p_1(x) = c_0 + c_1x$, 则有 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned} (\varphi_i, \varphi_j) &= \int_{-1}^1 x^{i+j}dx = \frac{1 + (-1)^{i+j}}{i+j+1}, \quad i, j = 0, 1 \\ (\varphi_0, f) &= \int_{-1}^1 e^x dx = e - e^{-1}, \quad (\varphi_1, f) = \int_0^1 xe^x dx = 2e^{-1}. \end{aligned}$$

于是对应的正规方程组为

$$\begin{pmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} (\varphi_0, f) \\ (\varphi_1, f) \end{pmatrix},$$

即

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} e - e^{-1} \\ 2e^{-1} \end{pmatrix},$$

解得 $c_0 = \frac{e-e^{-1}}{2}$, $c_1 = 3e^{-1}$, 因此有

$$p_1(x) = \frac{e - e^{-1}}{2} + 3e^{-1}x.$$

5.4 用正交多项式求下列函数的最优平方逼近二次多项式:

$$(1) \begin{array}{c|cccccccccc} x_i & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline y_i & 2 & 7 & 8 & 10 & 11 & 11 & 10 & 9 & 8 \end{array} \quad (2) y = \arcsin x, [0, 1].$$

解 (1) 设最优平方逼近二次多项式为 $p_2(x) = a_0 + a_1g_1(x) + a_2g_2(x)$. 利用三项递推关系构造正交多项式. 取

$$g_0(x) = 1, \quad g_1(x) = x - \frac{\beta_0}{\gamma_0}, \quad g_2(x) = (x - b_1)g_1(x) - c_1g_0(x),$$

而

$$\gamma_0 = (g_0, g_0) = (1, 1) = \sum_{i=1}^9 1 = 9, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \sum_{i=1}^9 x_i = 53,$$

故 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{53}{9}$. 又因

$$\begin{aligned} \gamma_1 &= (g_1, g_1) = \sum_{i=1}^9 \left(x_i - \frac{53}{9}\right)^2 = \frac{620}{9}, \\ \beta_1 &= (xg_1, g_1) = \sum_{i=1}^9 x_i \left(x_i - \frac{53}{9}\right)^2 = \frac{29780}{81}, \end{aligned}$$

因此

$$\begin{cases} b_1 = \frac{\beta_1}{\gamma_1} = \frac{1489}{279}, \\ c_1 = \frac{\gamma_1}{\gamma_0} = \frac{620}{81}, \end{cases} \implies g_2(x) = (x - b_1)g_1(x) - c_1g_0(x) = x^2 - \frac{348}{31}x + \frac{737}{31}.$$

另一方面, 由于

$$\begin{aligned} (g_2, g_2) &= \sum_{i=1}^9 \left(x_i^2 - \frac{348}{31}x_i + \frac{737}{31}\right)^2 = \frac{15708}{31}, \quad (g_0, f) = \sum_{i=1}^9 y_i = 76, \\ (g_1, f) &= \sum_{i=1}^9 \left(x_i - \frac{53}{9}\right)y_i = \frac{373}{9}, \quad (g_2, f) = \sum_{i=1}^9 \left(x_i^2 - \frac{348}{31}x_i + \frac{737}{31}\right)y_i = -\frac{4203}{31}, \\ a_0 &= \frac{(g_0, f)}{(g_0, g_0)} = \frac{76}{9}, \quad a_1 = \frac{(g_1, f)}{(g_1, g_1)} = \frac{373}{620}, \quad a_2 = \frac{(g_2, f)}{(g_2, g_2)} = -\frac{1401}{5236}, \end{aligned}$$

故

$$p(x) = a_0 g_0(x) + a_1 g_1(x) + a_2 g_2(x) = -\frac{1737}{1190} + \frac{94387}{26180}x - \frac{1401}{5236}x^2.$$

(2) 解法一: 设 $p_2(x) = c_0 + c_1x + c_2x^2$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x, \varphi_2(x) = x^2$.

$$\begin{aligned} (\varphi_i, \varphi_j) &= \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, & (\varphi_0, f) &= \int_0^1 \arcsin x dx = \frac{\pi}{2} - 1, \\ (\varphi_1, f) &= \int_0^1 x \arcsin x dx = \frac{\pi}{8}, & (\varphi_2, f) &= \int_0^1 x^2 \arcsin x dx = \frac{\pi}{6} - \frac{2}{9}. \end{aligned}$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} - 1 \\ \frac{\pi}{8} \\ \frac{\pi}{6} - \frac{2}{9} \end{pmatrix}$$

解得

$$c_0 = 5\pi - \frac{47}{3}, \quad c_1 = 76 - 24\pi, \quad c_2 = \frac{45\pi}{2} - 70,$$

因此

$$p_2(x) = 5\pi - \frac{47}{3} + (76 - 24\pi)x + \left(\frac{45\pi}{2} - 70\right)x^2.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_0^1 1 dx = 1, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \int_0^1 x dx = \frac{1}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{1}{2}$. 又因为

$$\gamma_1 = (g_1, g_1) = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}, \quad \beta_1 = (xg_1, g_1) = \int_0^1 x \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{24},$$

因此

$$g_2(x) = \left(x - \frac{\beta_1}{\gamma_1}\right)g_1(x) - \frac{\gamma_1}{\gamma_0}g_0(x) = \left(x - \frac{1}{2}\right)^2 - \frac{1}{12} = x^2 - x + \frac{1}{6}.$$

另一方面, 由于

$$\begin{aligned} (g_2, g_2) &= \int_0^1 \left(x^2 - x + \frac{1}{6}\right) dx = \frac{1}{180}, & (g_0, f) &= \int_0^1 \arcsin x dx = \frac{\pi}{2} - 1, \\ (g_1, f) &= \int_0^1 \left(x - \frac{1}{2}\right) \arcsin x dx = \frac{1}{2} - \frac{\pi}{8}, \\ (g_2, f) &= \int_0^1 \left(x^2 - x + \frac{1}{6}\right) \arcsin x dx = \frac{\pi}{8} - \frac{7}{18}, \end{aligned}$$

从而得

$$\begin{aligned} p_2(x) &= c_0 g_0(x) + c_1 g_1(x) + c_2 g_2(x) \\ &= \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) + \frac{(g_2, f)}{(g_2, g_2)} g_2(x) \\ &= 5\pi - \frac{47}{3} + (76 - 24\pi)x + \left(\frac{45\pi}{2} - 70\right)x^2. \end{aligned}$$

解法三：取 Legendre 正交多项式作为基函数，先作变换 $x = \frac{t+1}{2} (-1 \leq t \leq 1)$ ，则可令 $g(t) = \arcsin\left(\frac{t+1}{2}\right)$ 。前三个 Legendre 正交多项式为

$$p_0(t) = 1, \quad p_1(t) = t, \quad p_2(t) = \frac{1}{2}(3t^2 - 1).$$

设 $g(t)$ 的最优平方逼近二次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t) + c_2 p_2(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}, (k = 0, 1, 2)$ 。同时可得

$$\begin{aligned} (p_0, g) &= \int_{-1}^1 \arcsin\left(\frac{t+1}{2}\right) dt = \pi - 2, & (p_1, g) &= \int_{-1}^1 t \arcsin\left(\frac{t+1}{2}\right) dt = 2 - \frac{\pi}{2}, \\ (p_2, g) &= \frac{1}{2} \int_{-1}^1 (3t^2 - 1) \arcsin\left(\frac{t+1}{2}\right) dt = \frac{3\pi}{2} - \frac{14}{3}, \end{aligned}$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{\pi}{2} - 1, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = 3 - \frac{3\pi}{4}, \quad c_2 = \frac{(p_2, g)}{(p_2, p_2)} = \frac{15\pi}{4} - \frac{35}{3},$$

故

$$q(t) = \frac{29}{6} - \frac{11\pi}{8} + \left(3 - \frac{3\pi}{4}\right)t + \left(\frac{45\pi}{8} - \frac{35}{2}\right)t^2.$$

再将 $t = 2x - 1$ 带入上式，则得到 $f(x)$ 的最优平方逼近二次多项式

$$p(x) = 5\pi - \frac{47}{3} + (76 - 24\pi)x + \left(\frac{45\pi}{2} - 70\right)x^2.$$

5.6 求下列函数在指定区间上的最优平方逼近一次多项式：

- (1) $y = \sqrt{x}, \quad [\frac{1}{4}, 1];$ (2) $y = \ln x, \quad [1, 2];$
 (3) $y = e^x, \quad [0, 1];$ (4) $y = \sqrt{1+x^2}, \quad [0, 1].$

解 (1) 解法一：设 $p_1(x) = c_0 + c_1 x$ ，则 $\varphi_0(x) = 1, \varphi_1(x) = x$ 。

$$\begin{aligned} (\varphi_i, \varphi_j) &= \int_{\frac{1}{4}}^1 x^{i+j} dx = \frac{1 - (\frac{1}{4})^{i+j+1}}{i+j+1}, & (\varphi_0, f) &= \int_{\frac{1}{4}}^1 \sqrt{x} dx = \frac{7}{12}, \\ (\varphi_1, f) &= \int_{\frac{1}{4}}^1 x \sqrt{x} dx = \frac{31}{80}, \end{aligned}$$

由正规方程组有

$$\begin{pmatrix} \frac{3}{4} & \frac{15}{32} \\ \frac{15}{32} & \frac{21}{64} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{7}{12} \\ \frac{31}{80} \end{pmatrix}$$

解得

$$c_0 = \frac{10}{27}, \quad c_1 = \frac{88}{135},$$

因此

$$p_1(x) = \frac{10}{27} + \frac{88}{135}x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_{\frac{1}{4}}^1 1dx = \frac{3}{4}, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \int_{\frac{1}{4}}^1 xdx = \frac{15}{32},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{5}{8}$. 另一方面, 由于

$$\begin{aligned}(g_0, f) &= \int_{\frac{1}{4}}^1 \sqrt{x}dx = \frac{7}{12}, \quad (g_1, f) = \int_{\frac{1}{4}}^1 (x - \frac{5}{8})\sqrt{x}dx = \frac{11}{480}, \\(g_1, g_1) &= \int_{\frac{1}{4}}^1 (x - \frac{5}{8})^2 dx = \frac{9}{256},\end{aligned}$$

因此得

$$p_1(x) = c_0 g_0(x) + c_1 g_1(x) = \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) = \frac{10}{27} + \frac{88}{135}x.$$

解法三: 取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{3t+5}{8} (-1 \leq t \leq 1)$, 则可令 $g(t) = \sqrt{\frac{3t+5}{8}}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \quad p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}$, $(k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^1 \sqrt{\frac{3t+5}{8}} dt = \frac{14}{9}, \quad (p_1, g) = \int_{-1}^1 t \sqrt{\frac{3t+5}{8}} dt = \frac{22}{135},$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{7}{9}, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{11}{35},$$

故

$$q(t) = \frac{7}{9} + \frac{11}{35}t.$$

再将 $t = \frac{8x-5}{3}$ 带入上式, 则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = \frac{10}{27} + \frac{88}{135}x.$$

(2) 解法一: 设 $p_1(x) = c_0 + c_1 x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned}(\varphi_i, \varphi_j) &= \int_1^2 x^{i+j} dx = \frac{2^{i+j+1} - 1}{i+j+1}, \quad (\varphi_0, f) = \int_1^2 \ln x dx = 2 \ln 2 - 1, \\(\varphi_1, f) &= \int_1^2 x \ln x dx = 2 \ln 2 - \frac{3}{4},\end{aligned}$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \ln 2 - 1 \\ 2 \ln 2 - \frac{3}{4} \end{pmatrix}$$

解得

$$c_0 = 20 \ln 2 - \frac{29}{2}, \quad c_1 = 9 - 12 \ln 2,$$

因此

$$p_1(x) = 20 \ln 2 - \frac{29}{2} + (9 - 12 \ln 2)x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_1^2 1 dx = 1, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \int_1^2 x dx = \frac{3}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{3}{2}$. 另一方面, 由于

$$\begin{aligned} (g_0, f) &= \int_1^2 \ln x dx = 2 \ln 2 - 1, \quad (g_1, f) = \int_1^2 \left(x - \frac{3}{2}\right) \ln x dx = \frac{3}{4} - \ln 2, \\ (g_1, g_1) &= \int_1^2 \left(x - \frac{3}{2}\right)^2 dx = \frac{1}{12}, \end{aligned}$$

因此得

$$p_1(x) = c_0 g_0(x) + c_1 g_1(x) = \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) = 20 \ln 2 - \frac{29}{2} + (9 - 12 \ln 2)x.$$

解法三: 取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+3}{2} (-1 \leq t \leq 1)$, 则可令 $g(t) = \ln \frac{t+3}{2}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \quad p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}$, $(k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^1 \ln \frac{t+3}{2} dt = 4 \ln 2 - 2, \quad (p_1, g) = \int_{-1}^1 t \ln \frac{t+3}{2} dt = 3 - 4 \ln 2,$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = 2 \ln 2 - 1, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{9}{2} - 6 \ln 2,$$

故

$$q(t) = 2 \ln 2 - 1 + \left(\frac{9}{2} - 6 \ln 2\right)t.$$

再将 $t = 2x - 3$ 带入上式, 则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 20 \ln 2 - \frac{29}{2} + (9 - 12 \ln 2)x.$$

(3) 解法一: 设 $p_1(x) = c_0 + c_1x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned}(\varphi_i, \varphi_j) &= \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, & (\varphi_0, f) &= \int_0^1 e^x dx = e - 1, \\(\varphi_1, f) &= \int_0^1 x e^x dx = 1,\end{aligned}$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} e - 1 \\ 1 \end{pmatrix}$$

解得

$$c_0 = 4e - 10, \quad c_1 = 18 - 6e,$$

因此

$$p_1(x) = 4e - 10 + (18 - 6e)x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_0^1 1 dx = 1, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \int_0^1 x dx = \frac{1}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{1}{2}$. 另一方面, 由于

$$\begin{aligned}(g_0, f) &= \int_0^1 e^x dx = e - 1, & (g_1, f) &= \int_0^1 (x - \frac{1}{2})e^x dx = \frac{3 - e}{2}, \\(g_1, g_1) &= \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12},\end{aligned}$$

因此得

$$p_1(x) = c_0 g_0(x) + c_1 g_1(x) = \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) = 4e - 10 + (18 - 6e)x.$$

解法三: 取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+1}{2} (-1 \leq t \leq 1)$, 则可令 $g(t) = e^{\frac{t+1}{2}}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \quad p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}, (k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^1 e^{\frac{t+1}{2}} dt = 2e - 2, \quad (p_1, g) = \int_{-1}^1 t e^{\frac{t+1}{2}} dt = 6 - 2e,$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = e - 1, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = 9 - 3e,$$

故

$$q(t) = e - 1 + (9 - 3e)t.$$

再将 $t = 2x - 1$ 带入上式, 则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 4e - 10 + (18 - 6e)x.$$

(4) 解法一: 设 $p_1(x) = c_0 + c_1x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned}(\varphi_i, \varphi_j) &= \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, & (\varphi_0, f) &= \int_0^1 \sqrt{1+x^2} dx = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})), \\(\varphi_1, f) &= \int_0^1 x\sqrt{1+x^2} dx = \frac{1}{3}(2\sqrt{2} - 1),\end{aligned}$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})) \\ \frac{1}{3}(2\sqrt{2} - 1) \end{pmatrix}$$

解得

$$c_0 = 2(1 - \sqrt{2} + \ln(1+\sqrt{2})), \quad c_1 = 5\sqrt{2} - 4 - 3\ln(1+\sqrt{2}),$$

因此

$$p_1(x) = 2(1 - \sqrt{2} + \ln(1+\sqrt{2})) + (5\sqrt{2} - 4 - 3\ln(1+\sqrt{2}))x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_0^1 1 dx = 1, \quad \beta_0 = (xg_0, g_0) = (x, 1) = \int_0^1 x dx = \frac{1}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{1}{2}$. 另一方面, 由于

$$\begin{aligned}(g_0, f) &= \int_0^1 \sqrt{1+x^2} dx = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})), & (g_1, g_1) &= \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}, \\(g_1, f) &= \int_0^1 (x - \frac{1}{2})\sqrt{1+x^2} dx = \frac{1}{12}(5\sqrt{2} - 4 - 3\ln(1+\sqrt{2})),\end{aligned}$$

因此得

$$\begin{aligned}p_1(x) &= c_0 g_0(x) + c_1 g_1(x) = \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) \\&= 2(1 - \sqrt{2} + \ln(1+\sqrt{2})) + (5\sqrt{2} - 4 - 3\ln(1+\sqrt{2}))x.\end{aligned}$$

解法三: 取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+1}{2} (-1 \leq t \leq 1)$, 则可令 $g(t) = \sqrt{1 + \frac{1}{4}(t+1)^2}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \quad p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}$, $(k = 0, 1)$. 同时可得

$$\begin{aligned}(p_0, g) &= \int_{-1}^1 \sqrt{1 + \frac{1}{4}(t+1)^2} dt = \sqrt{2} + \ln(1 + \sqrt{2}), \\(p_1, g) &= \int_{-1}^1 t \sqrt{1 + \frac{1}{4}(t+1)^2} dt = \frac{1}{3}(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})),\end{aligned}$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2})), \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{1}{2}(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})),$$

故

$$q(t) = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2})) + \frac{1}{2}(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2}))t.$$

再将 $t = 2x - 1$ 带入上式, 则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 2(1 - \sqrt{2} + \ln(1 + \sqrt{2})) + (5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2}))x.$$