计算方法 第 4 章习题答案

4.1 已知函数 $y=f(x)$ 在若干点处的函数值如下表所示, 求其拉格朗日插值多项式和牛顿插值多项式, 并写截断误差表达式.

(1)

x_i	0	1	2	5
y_i	2	3	12	127

(2)

x_i	-2	-1	0	1
y_i	15	4	5	24

解 (1) 由题设可知 $n=3$, 拉格朗日插值基函数有 4 个, 分别为:

$$\begin{aligned}
l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{1}{10}(x-1)(x-2)(x-5),\\
l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{1}{4}x(x-2)(x-5),\\
l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{6}x(x-1)(x-5),\\
l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{1}{60}x(x-1)(x-2),
\end{aligned}$$

因此拉格朗日插值多项式为:

$$L_3(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 + l_3(x)y_3 = x^3 + x^2 - x + 2,$$

截断误差为:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) = \frac{f^{(4)}(\xi)}{24}x(x-1)(x-2)(x-5), \quad \xi \in (0,5).$$

对于牛顿插值多项式, 先列差商表如下:

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
0	2			
1	3	1		
2	12	9	4	
5	147	45	9	1

由此得牛顿插值多项式为:

$$\begin{aligned}
N_3(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)\\
&\quad + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)\\
&= 2 + x + 4x(x-1) + x(x-1)(x-2) = x^3 + x^2 - x + 2,
\end{aligned}$$

截断误差为:

$$\begin{aligned}
R_3(x) &= f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\\
&= f[0,1,2,3,x]x(x-1)(x-2)(x-5).
\end{aligned}$$

(2) 由题设可知 $n=3$, 拉格朗日插值基函数有 4 个, 分别为:

$$l_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}=-\frac{1}{6}x(x-1)(x+1),$$
$$l_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}=\frac{1}{2}x(x+2)(x-1),$$
$$l_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}=-\frac{1}{2}(x+2)(x+1)(x-1),$$
$$l_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{1}{6}x(x+2)(x+1),$$

因此拉格朗日插值多项式为:

$$L_3(x)=l_0(x)y_0+l_1(x)y_1+l_2(x)y_2+l_3(x)y_3=x^3+9x^2+9x+5,$$

截断误差为:

$$R_3(x)=\frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)=\frac{f^{(4)}(\xi)}{24}x(x+2)(x+1)(x-1),\ \ \xi\in(-2,1).$$

对于牛顿插值多项式, 先列差商表如下:

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
-2	15			
-1	4	-11		
0	5	1	6	
1	24	19	9	1

由此得牛顿插值多项式为:

$$\begin{aligned}N_3(x)&=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)\\&\quad+f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)\\&=15-11(x+2)+6(x+2)(x+1)+x(x+2)(x+1)=x^3+9x^2+9x+5,\end{aligned}$$

截断误差为:

$$\begin{aligned}R_3(x)&=f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\\&=f[-2,-1,0,1,x]x(x+2)(x+1)(x-1).\end{aligned}$$

4.3 在 $x=-1,1,2$ 处, $f(x)=-3,0,4$, 求 $f(x)$ 的二次插值多项式.

(1) $p_2(x)=a_0+a_1x+a_2x^2$;

(2) 拉格朗日插值多项式;

(3) 牛顿插值多项式.

证明上述三种方法求得的插值多项式是相同的.

解 (1) 根据插值条件有

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_0 - a_1 + a_2 = -3 \\ a_0 + 2a_1 + 4a_2 = 4 \end{cases}$$

解得

$$a_0 = -\frac{7}{3}, \quad a_1 = \frac{3}{2}, \quad a_2 = \frac{5}{6}.$$

因此

$$p_2(x) = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

(2) 拉格朗日插值多项式: 由于

$$\begin{aligned} l_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = -\frac{1}{2}(x+1)(x+2), \\ l_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{1}{6}(x-1)(x-2), \\ l_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{3}(x-1)(x+1), \end{aligned}$$

故

$$L_2(x) = \sum_{k=0}^{2} l_k(x) y_k = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

(3) 牛顿插值多项式: 由于

$$\begin{aligned} & f[x_0] = -3, \quad f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3}{2}, \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 4, \\ & f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{5}{6}, \end{aligned}$$

因此

$$N_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

根据插值多项式的存在唯一性, 可知上述三种方法得到的插值多项式是相同的.

4.6 证明 k 阶差商的如下性质:

(1) 设 $p(x) = cf(x)$, 则

$$p[x_0, x_1, \cdots, x_k] = cf[x_0, x_1, \cdots, x_k];$$

(2) 设 $p(x) = f(x) + g(x)$, 则

$$p[x_0, x_1, \cdots, x_k] = f[x_0, x_1, \cdots, x_k] + g[x_0, x_1, \cdots, x_k].$$

证明 (1) 用数学归纳法证明. 当 $k=0$ 时,

$$p[x_0]=p(x_0)=cf(x_0)=cf[x_0].$$

当 $k=1$ 时,

$$p[x_0,x_1]=\frac{p[x_1]-p[x_0]}{x_1-x_0}=\frac{cf[x_1]-cf[x_0]}{x_1-x_0}=c\frac{f[x_1]-f[x_0]}{x_1-x_0}=cf[x_0,x_1].$$

假设当 $k=n$ 时结论成立, 即有

$$p[x_0,x_1,\cdots,x_n]=cf[x_0,x_1,\cdots,x_n].$$

则当 $k=n+1$ 时,

$$\begin{aligned}p[x_0,x_1,\cdots,x_n,x_{n+1}]&=\frac{p[x_1,\cdots,x_n,x_{n+1}]-p[x_0,x_1,\cdots,x_n]}{x_{n+1}-x_0}\\&=\frac{cf[x_1,\cdots,x_n,x_{n+1}]-cf[x_0,x_1,\cdots,x_n]}{x_{n+1}-x_0}=cf[x_0,x_1,\cdots,x_n,x_{n+1}].\end{aligned}$$

因此结论对任意的 $k\geqslant 0$ 都成立.

(2) 亦可用数学归纳法证明. 当 $k=0$ 时,

$$p[x_0]=p(x_0)=f(x_0)+g(x_0)=f[x_0]+g[x_0].$$

当 $k=1$ 时,

$$\begin{aligned}p[x_0,x_1]=\frac{p[x_1]-p[x_0]}{x_1-x_0}&=\frac{f[x_1]+g[x_1]-f[x_0]-g[x_0]}{x_1-x_0}\\&=\frac{f[x_1]-f[x_0]}{x_1-x_0}+\frac{g[x_1]-g[x_0]}{x_1-x_0}=f[x_0,x_1]+g[x_0,x_1].\end{aligned}$$

假设当 $k=n$ 时结论成立, 即有

$$p[x_0,x_1,\cdots,x_n]=f[x_0,x_1,\cdots,x_n]+g[x_0,x_1,\cdots,x_n].$$

则当 $k=n+1$ 时,

$$\begin{aligned}p[x_0,x_1,\cdots,x_n,x_{n+1}]&=\frac{p[x_1,\cdots,x_n,x_{n+1}]-p[x_0,x_1,\cdots,x_n]}{x_{n+1}-x_0}\\&=\frac{f[x_1,\cdots,x_n,x_{n+1}]+g[x_1,\cdots,x_n,x_{n+1}]-f[x_0,x_1,\cdots,x_n]-g[x_0,x_1,\cdots,x_n]}{x_{n+1}-x_0}\\&=f[x_0,x_1,\cdots,x_n,x_{n+1}]+g[x_0,x_1,\cdots,x_n,x_{n+1}].\end{aligned}$$

因此结论对任意的 $k\geqslant 0$ 都成立.

上述两条性质说明差商是线性算子.

4.7 $f(x)=x^7+x^4+3x+1$, 求 $f[2^0,2^1,\cdots,2^7]$ 和$f[2^0,2^1,\cdots,2^8]$.

解 根据差商与导数的关系,

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f^{(k)}(\xi)}{k!},$$

可知

$$f[2^0, 2^1, \ldots, 2^7] = \frac{f^{(7)}(\xi)}{7!} = 1, \quad f[2^0, 2^1, \ldots, 2^8] = \frac{f^{(8)}(\xi)}{8!} = 0.$$

4.9 已知函数 $y = f(x)$ 在若干点处的函数值、导数值如下表所示, 求埃尔米特插值多项式和截断误差表达式:

(1)

x_i	-1	0	1
y_i	-1	0	1
y_i'	0	0	0

(2)

x_i	0	1	2
y_i	1	-1	0
y_i'	0	0	
y_i''	2		

解 (1) 设拉格朗日型埃尔米特插值多项式的形式为

$$H_5(x) = \sum_{i=0}^{2} h_i(x) f(x_i) + \sum_{i=0}^{2} \overline{h}_i(x) f'(x_i) = \sum_{i=0}^{2} h_i(x) f(x_i),$$

其中

$$h_i(x) = \left[1 - 2(x - x_i) \sum_{\substack{j=0 \\ j \neq i}}^{2} \frac{1}{x_i - x_j}\right] l_i^2(x).$$

由于

$$l_0(x) = \frac{x(x-1)}{2}, \quad l_1(x) = (x+1)(x-1), \quad l_2(x) = \frac{x(x+1)}{2},$$

因此

$$h_0(x) = \frac{1}{4} x^2 (x-1)^2 (3x+4),$$
$$h_1(x) = (x+1)^2 (x-1)^2,$$
$$h_2(x) = \frac{1}{4} x^2 (x+1)^2 (4-3x),$$

代入得埃米尔特插值多项式为

$$H_5(x) = -\frac{1}{4} x^2 (x-1)^2 (3x+4) + \frac{1}{4} x^2 (x+1)^2 (4-3x) = -\frac{1}{2} x^3 (3x^2 - 5).$$

进一步, 其截断误差估计式为

$$R_5(x) = f(x) - H_5(x) = \frac{f^{(6)}(\xi)}{6!} (x+1)^2 x^2 (x-1)^2, \quad \xi \in (-1, 1).$$

(2) 由差商与导数的关系可知

$$f[0,0] = f'(0) = 0, \;\; f[0,0,0] = \frac{f''(0)}{2!} = 1, \;\; f[1,1] = f'(1) = 0,$$

作差商表

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$	$\cdots$	$\cdots$
0	1					
0	1	0				
0	1	0	1			
1	-1	-2	-2	-3		
1	-1	0	2	4	7	
2	0	1	1	$-\frac{1}{2}$	$-\frac{9}{4}$	$-\frac{37}{8}$

所以

$$
\begin{aligned}
H_5(x) &= 1 + x^2 - 3x^3 + 7x^3(x-1) - \frac{37}{8}x^3(x-1)^2 \\
&= \frac{1}{8}\big(-37x^5 + 130x^4 - 117x^3 + 8x^2 + 8\big), \\
R_5(x) &= f[0,0,0,1,1,2,x]x^3(x-1)^2(x-2).
\end{aligned}
$$

4.10 设 $x_i\ (i=0,1,\cdots,n)$ 是互不相同的插值节点, $l_i(x)\ (i=0,1,\cdots,n)$ 是拉格朗日插值基函数. 证明:

(1) $\sum\limits_{i=1}^{n} l_i(x) = 1$;

(2) $\sum\limits_{i=1}^{n} l_i(x)x_i^k = x^k \quad (k=1,2,\ldots,n)$;

(3) $\sum\limits_{i=1}^{n} l_i(x)(x_i-x)^k = 0 \quad (k=1,2,\ldots,n)$;

(4) $\sum\limits_{i=1}^{n} l_i(0)x_i^k = \begin{cases} 1, & k=0, \\ 0, & k=1,2,\cdots,n, \\ (-1)^n x_0x_1\ldots x_n, & k=n+1. \end{cases}$

证明 已知对任意给定的 $n+1$ 个插值节点, n 次拉格朗日插值多项式及其截断误差可以表示为:

$$
L_n(x) = \sum_{i=0}^{n} l_i(x)f(x_i), \quad R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)(\xi)}}{(n+1)!}\pi_{n+1}(x).
$$

(1) 根据题目可知, 被插函数 $f(x)=1$, 因此有

$$
f^{(n+1)}(x) = 0 \quad \Longrightarrow \quad f(x) - L_n(x) = R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x) = 0,
$$

从而有

$$
L_n(x) = \sum_{i=0}^{n} l_i(x) = f(x) = 1.
$$

(2) 根据题目可知, 被插函数 $f(x)=x^k\ (k\leqslant n)$, 因此有

$$f^{(n+1)}(x)=0 \quad\Longrightarrow\quad f(x)-L_n(x)=R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)=0,$$

从而有

$$L_n(x)=\sum_{i=0}^{n} l_i(x)x_i^k=f(x)=x^k.$$

(3) 解法 1: 先变量替换, 将 x 换成 t, 则等式结果不发生改变. 此时可知被插函数为 $f(x)=(x-t)^k\ (k\leqslant n)$, 因此有

$$f^{(n+1)}(x)=0 \quad\Longrightarrow\quad R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)=0,$$

即

$$L_n(x)=\sum_{i=0}^{n} l_i(x)f(x_i)=\sum_{i=0}^{n} l_i(x)(x_i-t)^k=f(x),$$

特别地,

$$L_n(t)=\sum_{i=0}^{n} l_i(t)(x_i-t)^k=f(t)=0.$$

解法 2: 将 $(x_i-x)^k$ 按二次项展开得

$$(x_i-x)^k=\sum_{j=0}^{k} C_k^j x_i^j(-x)^{k-j},$$

因此

$$\sum_{i=0}^{n} l_i(x)(x_i-x)^k=\sum_{i=0}^{n}\left(\sum_{j=0}^{k} C_k^j x_i^j(-x)^{k-j}\right)l_i(x)=\sum_{j=0}^{k} C_k^j(-x)^{k-j}\sum_{i=0}^{n} x_i^j l_i(x),$$

由 (2) 得 $\sum\limits_{i=0}^{n} x_i^j l_i(x)=x^j$, 因此

$$\sum_{i=0}^{n} l_i(x)(x_i-x)^k=\sum_{j=0}^{k} C_k^j(-x)^{k-j}x^j=x^k\sum_{j=0}^{k}(-1)^{k-j}C_k^j=0.$$

(4) 根据题目可知, 被插函数 $f(x)=x^k$, 因此有

$$f^{(n+1)}(x)=\begin{cases} 0, & k=0,1,\cdots,n, \\ (n+1)!, & k=n+1, \end{cases}$$

从而有

$$R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)=\begin{cases} 0, & k=0,1,\cdots,n, \\ \pi_{n+1}(x), & k=n+1, \end{cases}$$

因此

$$L_n(x) = \sum_{i=0}^{n} l_i(x) x_i^k = f(x) - R_n(x) = \begin{cases} x^k, & k = 0, 1, \cdots, n, \\ x^k - \pi_{n+1}(x), & k = n+1. \end{cases}$$

特别地,

$$L_n(0) = \sum_{i=0}^{n} l_i(0) x_i^k = f(0) - R_n(0) = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \cdots, n, \\ (-1)^n x_0 x_1 \cdots x_n, & k = n+1. \end{cases}$$

4.13 已知函数 $y = f(x)$ 在若干点处的函数值如下表所示, 求满足以下端点条件的三次样条插值函数:

(1) $S'(0) = 0,\ \ S'(4) = 48;$

(2) $S''(0) = 0,\ \ S''(4) = 24.$

x_i	0	1	2	3	4
y_i	-8	-7	0	19	56

解 由 $h_i = x_i - x_{i-1}$ 得

$$h_1 = h_2 = h_3 = h_4 = 1.$$

设三次样条插值函数 $S(x)$ 在节点 x_i 处的二阶导数值为 $S''(x_i) = M_i\ (i = 0, 1, \cdots, 4)$, 则 $S(x)$ 在子区间 $[x_{i-1}, x_i]\ (i = 1, \cdots, 4)$ 上的表达式为:

$$S(x) = \frac{(x_i - 3)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) \frac{x_i - x}{h_i} + \left(y_i - \frac{h_i^2}{6} M_i \right) \frac{x - x_{i-1}}{h_i}.$$

根据 $\mu_i = \frac{h_i}{h_i + h_{i+1}},\ \lambda_i = 1 - \mu_i\ (i = 1, 2, 3)$ 得

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{2},\ \ \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}.$$

再根据 $d_i = 6f[x_{i-1}, x_i, x_{i+1}]\ (i = 1, 2, 3)$ 可得

$$d_1 = 18, \quad d_2 = 36, \quad d_3 = 54.$$

(1) 由第二类边界条件得

$$d_0 = 6f[x_0, x_0, x_1] = 6, \quad d_4 = 6f[x_3, x_4, x_4] = 66$$

进一步可以得到关于第二类边界条件的三弯矩方程组

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \\ 36 \\ 54 \\ 66 \end{pmatrix}.$$

解得

$$M_0 = 0, \quad M_1 = 6, \quad M_2 = 12, \quad M_3 = 18, \quad M_4 = 24,$$

因此三次样条插值函数 $S(x)$ 为

$$S(x) = \begin{cases} x^3 - 8, & 0 \leqslant x \leqslant 1, \\ x^3 - 8, & 1 \leqslant x \leqslant 2, \\ x^3 - 8, & 2 \leqslant x \leqslant 3, \\ x^3 - 8, & 3 \leqslant x \leqslant 4, \end{cases}$$

即

$$S(x) = x^3 - 8, \quad x \in [0, 4].$$

(2) 由第一类边界条件得 $M_0 = 0,\ M_4 = 24,$ 进一步可得其三弯矩方程组为：

$$\begin{pmatrix} 2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \\ 42 \end{pmatrix},$$

解得

$$M_1 = 6, \quad M_2 = 12, \quad M_3 = 18,$$

因此三次样条插值函数 $S(x)$ 为

$$S(x) = \begin{cases} x^3 - 8, & 0 \leqslant x \leqslant 1, \\ x^3 - 8, & 1 \leqslant x \leqslant 2, \\ x^3 - 8, & 2 \leqslant x \leqslant 3, \\ x^3 - 8, & 3 \leqslant x \leqslant 4, \end{cases}$$

即有

$$S(x) = x^3 - 8, \quad x \in [0, 4].$$