Lecture 2
Fundamental in Signal & System

Complex Numbers – the basics

The notes cover the basic definitions and properties of complex numbers (Boas 2.1-2.5). The story starts from finding solutions for the simple algebraic equation $x^2 + 1 = 0$. There is no real solution to the equation. But, if we introduce the notion of imaginary numbers,

$$i \equiv \sqrt{-1}, \quad \text{with} \quad i^2 = -1$$

one can write down the solutions $x = \pm i$. Going beyond the pure imaginary numbers, one can introduce the complex number with real and imaginary parts,

$$z = x + iy$$

where x and y are real numbers, representing the real and imaginary parts respectively.

Complex Numbers – the basics

A complex number is a number written in the form

$z = a + bi$

where a and b are real numbers and i is a formal symbol satisfying the relation $i^2 = -1$. The number a is the **real part** of z, denoted by $\operatorname{Re} z$, and b is the **imaginary part** of z, denoted by $\operatorname{Im} z$.

Two complex numbers are considered equal if and only if their real and imaginary parts are equal.

For example, if $z = 5 + (-2)i$, then $\operatorname{Re} z = 5$ and $\operatorname{Im} z = -2$. For simplicity, we write $z = 5 - 2i$.

Complex Numbers – the basics

The complex number system, denoted by $\mathbb{C}$, is the set of all complex numbers, together with the following operations of addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad (1)$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \qquad (2)$$

These rules reduce to ordinary addition and multiplication of real numbers when b and d are zero in (1) and (2). It is readily checked that the usual laws of arithmetic for $\mathbb{R}$ also hold for $\mathbb{C}$. For this reason, multiplication is usually computed by algebraic expansion, as in the following example.

Example

$$
\begin{aligned}
(5-2i)+(3+4i) &= 15+20i-6i-8i^2 \\
&= 15+14-8(-1) \\
&= 23+14i
\end{aligned}
$$

That is, multiply each term of $5-2i$ by each term of $3+4i$, use $i^2 = -1$, and write the result in the form $a+bi$.

Subtraction of complex numbers z_1 and z_2 is defined by

$$(z_1 - z_2) = z_1 + (-1)z_2$$

In particular, we write $-z$ in place of $(-1)z$.

The **conjugate** of $z = a + bi$ is the complex number $\bar{z}$ (read as "z bar"), defined by

$$\bar{z} = a - bi$$

Obtain $\bar{z}$ from z by reversing the sign of the imaginary part.

Modulus

The conjugate of $-3+4i$ is $-3-4i$; write $\overline{-3+4i}=-3-4i$.

Observed that if $z=a+bi$,then

$$z\bar{z}=(a+bi)(a-bi)=a^2-abi+bai-b^2i^2=a^2+b^2 \quad (3)$$

Since $z\bar{z}$ is real and nonnegative, it has a square root. The **absolute value** (or **modulus**) of z is the real number $|z|$ defined by

$$|z|=\sqrt{z\bar{z}}=\sqrt{a^2+b^2}$$

If z is a real number, then $z=a+0i$, and $|z|=\sqrt{a^2}$, which equals the ordinary absolute value of a.

Some Properties

Some useful properties of conjugates and absolute value are listed below; ω and z denote complex numbers.

1. $\bar{z} = z$, if and only if z is real number.
2. $\overline{\omega + z} = \bar{\omega} + \bar{z}$.
3. $\overline{\omega z} = \bar{\omega}\ \bar{z}$; in particular, $\overline{rz} = r\bar{z}$ if r is real number.
4. $z\bar{z} = |z|^2 \geq 0$.
5. $|\omega z| = |\omega||z|$.
6. $|\omega + z| \leq |\omega| + |z|$.

If $z \neq 0$, then $|z| > 0$ and z has a multiplicative inverse, denoted by $1/z$ or z^{-1} and given by

$$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}$$

Of course, a quotient ω/z simply means $\omega \cdot (1/z)$.

Example

Let $\omega = 3 + 4i$ and $z = 5 - 2i$. Compute $z\bar{z}$, $|z|$, and $\omega/\ z$.

From equation(3),

$$z\bar{z} = 5^2 + (-2)^2 = 25 + 4 = 29$$

For the absolute value, $|z| = \sqrt{z\bar{z}} = \sqrt{29}$. To compute $\omega/\ z$, first multiply both the numerator and the denominator by $\bar{z}$, the conjugate of the denominator.

Example

Because of (3), this eliminates the i in the denominator:

$$
\begin{aligned}
\frac{\omega}{z} &= \frac{3+4i}{5-2i} \\
&= \frac{3+4i}{5-2i} \cdot \frac{5+2i}{5+2i} \\
&= \frac{15+6i+20i-8}{5^2+(-2)^2} \\
&= \frac{7+26i}{28} \\
&= \frac{7}{29} + \frac{26}{29} i
\end{aligned}
$$

Complex and looks different

For complex numbers, an amazing identity arises

$$e^{i\pi} = -1$$

It is quite remarkable that two irrational numbers e and π can be related by the magic number i! One may question such beauty is only constrained within the territory of mathematics. This is not so. Consider light reflected by the mirror in one dimension. The incoming and outgoing waves can be written as

$$\Psi_{in}(x,t) = A_{in}\cos(kx - \omega t)$$

$$\Psi_{out}(x,t) = A_{out}\cos(kx + \omega t)$$

Complex Numbers – the basics

where k is the wave number (real!) and ω is the angular frequency (also real!). But, what happens inside the wall? In fact, the wave becomes evanescent and decays exponentially,

$$\Psi_{out}(x,t) \sim e^{-\alpha x}$$

Skipping the detail derivations, the main physics is – the wave number becomes imaginary $k = i\alpha$!

Geometric Interpretation

Each complex number $z = a + bi$ corresponds to a point (a, b) in the plane $\mathbb{R}^2$, as in Figure 1. The horizontal axis is called the **real axis** because the points $(a, 0)$ on it correspond to the real numbers. The vertical axis is the **imaginary axis** because the points $(0, b)$ on it correspond to the **pure imaginary numbers** of the form $0+bi$, or simply bi. The conjugate of z is the mirror image of z in the real axis. The absolute value of z is the distance from (a, b) to the origin.

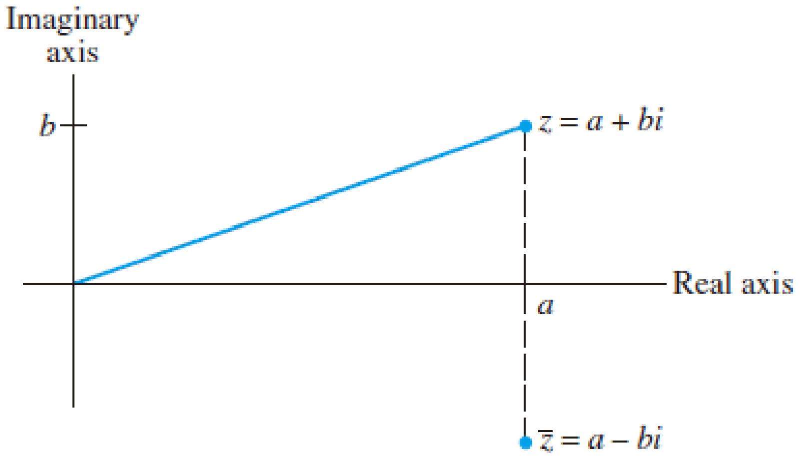


FIGURE 1 The complex conjugate is a mirror image.

Addition

Addition of complex numbers $z = a + bi$ and $\omega = c + di$ corresponds to vector addition of (a, b) and (c, d) in $\mathbb{R}^2$, as in Figure 2.

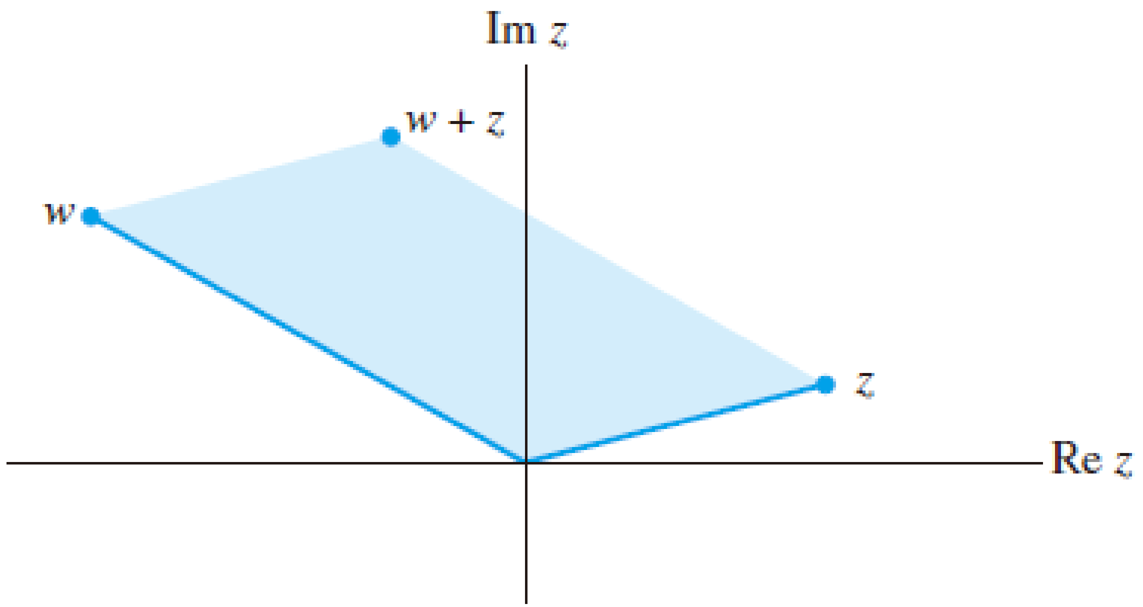


FIGURE 2 Addition of complex numbers.

Complex plane

To give a graphical representation of complex multiplication, we use **polar coordinates** in $\mathbb{R}^2$. Given a nonzero complex number $z = a + bi$, let φ be the angle between the positive real axis and the point (a, b) , as in Figure 3 where -π<$\varphi \leq$ π. The angle φ is called the **argument** of z; we write $\varphi = arg z$. From trigonometry,

$$a = |z||z|\sin\varphi, \ \ \mathrm{b} = |z|\sin\varphi$$

and so

$$z = a + bi = |z|\,(|z|\cos\varphi + i\sin\varphi)$$

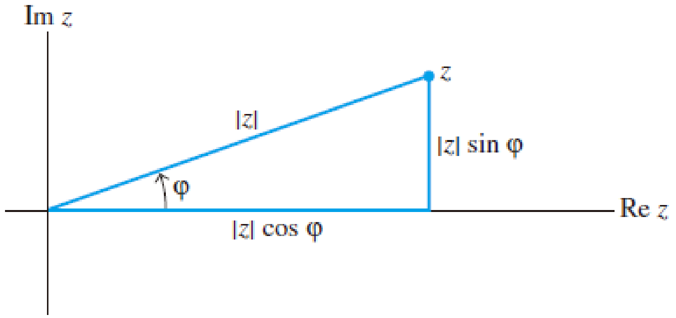


FIGURE 3 Polar coordinates of z.

Complex plane

It is convenient to plot $z = x + iy$ in the two-dimensional complex plane, using the real part x and the imaginary part y as Cartesian coordinates. Clearly, one can also write the same complex number in polar form,

$$z = x + iy = r\cos\theta + ir\sin\theta$$

Here $r = \sqrt{x^2 + y^2} = |z|$ is the absolute value of z (distance to the origin) and $\theta = \tan^{-1}(y/x)$ is the corresponding angle. Complex conjugate is defined as mirror mapping to the $x-axis$, i.e.

$$\overline{z} = x - iy = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)]$$

Since we can plot a complex number in the two dimensional plane, a natural question pops out: Is a complex number a two-dimensional vector?

Complex Numbers – the basics

Compute the following product

$$\bar{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = (x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)$$

You can convince yourself that the real part is just the inner product and the imaginary part is the outer product for two-dimensional vectors.

If ω is another nonzero complex number, say,

$$\omega = |\omega|(\cos\vartheta + i\sin\vartheta)$$

then, using standard trigonometric identities for the sine and cosine of the sum of two angles, one can verify that

$$\omega z = |\omega||z|\left[\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)\right] \tag{4}$$

See Figure 4. A similar formula may be written for quotients in polar form. The formulas for products and quotients can be stated in words as follows.

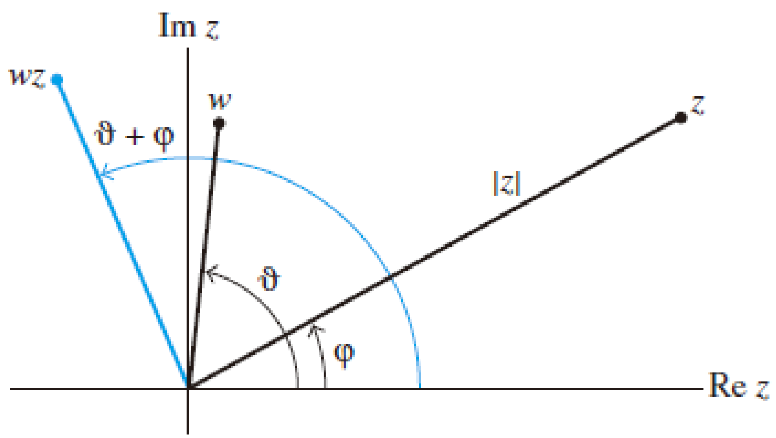
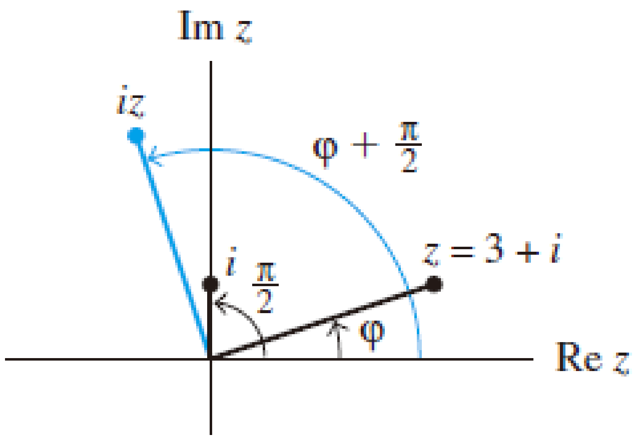


FIGURE 4 Multiplication with polar coordinates.

Example

a. If ω has absolute value 1, then $\omega = \cos\vartheta + i\sin\vartheta$, where ϑ is the argument of ω. Multiplication of any nonzero number z by ω simply rotates z through the angle ϑ.

b. The argument of i itself is $\pi/2$ radians, so multiplication of z by i rotates z through an angle of $\pi/2$ radians. For example, $3+i$ is rotated into $(3+i)i = -1+3i$.



Multiplication by i.

Taylor expansions

Now I would like to establish an important identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The easiest way to prove the above identity is through Taylor expansions,

$$\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Choose the variable $x = i\theta$ in the Taylor expansion for the exponential function and one can check that it equals the sum of the Taylor series for the sinusoidal functions.

Complex equations

Since a complex number contain real and imaginary parts, a complex equation amount to two real equations. For instance, $z^2 = 2i$ can be decomposed into two equations,

$$x^2 - y^2 = 0$$
$$2xy = 2$$

Note that both x and y are real. Thus, two solutions $x = y = 1$ and $x = y = -1$ are found for the complex equation $z^2 = 2i$.

Complex Series

Infinite series

Consider the following infinite series

$$S(z) = 1 - z + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} + \dots$$

How can we know whether the infinite series S is convergent? The simplest method is to compare with the well-known geometric series, i.e. the ratio test. For the series S , the ratio test gives

$$\rho = \lim_{n\to\infty}\left|\frac{zn}{n+1}\right| = z$$

Thus, the series is convergent for $|z| < 1$. This method can be generalized to more complicate series and helps us find out the radius of convergent.

Complex functions in series

Elementary functions of complex variable can often be expanded by complex series. For instance, the Taylor expansion for exponential function is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = e^x(\cos y + i \sin y)$$

Make us of ratio test, one can show that the series is convergent for all z. As a side remark, the Taylor expansion does not always converges to the original function. For instance, the tunneling probability through a finite barrier is

$$P \sim e^{-2\alpha \vartriangle x/h}$$

where $\alpha = \sqrt{2m(V_0 - E)}$. In the classical limit $(h \to 0)$,the probability goes to zero as expected.

Powers of a Complex Number

Formula

$$\omega z = |\omega||z|\,[\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)]$$

applies when $\mathrm{z} = \omega = \mathrm{r}(\cos\varphi + i\sin\varphi)$.

In this case

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

And

$$z^3 = z \cdot z^2$$
$$= \mathrm{r}(\cos\varphi + i\sin\varphi)\cdot r^2(\cos 2\varphi + i\sin 2\varphi)$$
$$= r^3\ (\cos 3\varphi + i\sin 3\varphi)$$

In general, for any positive integer k,

$$z^k = r^k(\cos k\varphi + i\sin k\varphi)$$

This fact is known *as De Moivre's Theorem.*

DeMoivre's theorem

In the previous lecture, we introduce Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$.
Using the rule of multiplication, the DeMoivre's theorem can be derived,

$$e^{in\theta} = \cos(n\theta) + \sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

This theorem is quite useful for deriving trigonometric identities. For instance, choose $n = 3$ in the DeMoivre's theorem and compare the real parts on both sides,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

DeMoivre’s theorem

The theorem also provides a natural way to compute roots of complex numbers,

$$z^{1/n} = \sqrt[n]{r}(\cos\frac{\theta}{n} + \sin\frac{\theta}{n})$$

However, subtleties arise for complex roots – there are more than one possible values. Therefore, when computing the n-th root of a complex numbers, there are n possible values.

Complex Functions

Complex Functions

The notes introduce the elementary complex functions (Chap 2, Sections 11-16 in Boas). In particular, we would encounter the multi-valued logarithmic function and learn to live with its indefinite values. Then, we can sensibly answer what a complex root like $(1+i)^{1-i}$ means. At the end, we apply complex algebra to find the interference pattern for the multiple-slip experiment.

Elementary functions

Starting from our good old friend, the exponential function can be decomposed into two parts,

$$e^z = e^x(\cos y + i\sin y)$$

It is important to emphasize that the complex exponential function is not monotonic anymore. If you walk along the imaginary axis, the exponential function is basically the sinusoidal function. Similarly, one can generalize the trigonometric functions in the entire complex plane,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Elementary functions

It should be straightforward to convince yourself that both $\sin z$ and $\cos z$ are not bounded anymore. However, the identity $\sin^2 z + \cos^2 z = 1$ remains valid, going beyond its original geometric meaning.

Muti-valued logarithmic function

As we proved in the previous lecture,

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

it implies that its inverse function satisfies the relation

$$\ln z_1 + \ln z_2 = \ln(z_1 z_2)$$

In the polar representation, the logarithmic function on the complex plane is

$$\ln z = \ln r + i(\theta + 2n\pi)$$

where $n = 0, \pm 1, \pm 2, \ldots$ is an arbitrary integer. This means that the logarithmic function is not single-valued. For instance, taking logarithm of the simple integer $z = -1$, you end up with infinite imaginary numbers,

$$\ln(-1) = \ln 1 + i(2n+1)\pi = \pm \pi i, \pm 3\pi i, \pm 5\pi i, \ldots$$

Fundamental in Signal & System

Periodic - Aperiodic

- A signal f(t) is *periodic* if there exists a positive constant T_0 such that

$$f(t+T_0)=f(t) \qquad \forall t$$

  - The *smallest* value of T_0 which satisfies such relation is said the *period* of the function f(t)
  - A periodic signal remains unchanged when *time-shifted* of integer multiples of the period
  - Therefore, by definition, it starts at minus infinity and lasts forever

$$-\infty \le t \le +\infty \qquad t \in {}^{\circ}$$

$$-\infty \le n \le +\infty \qquad n \in \mathbf{Z}$$

  - Periodic signals can be generated by *periodical extension*

Periodic Signal

$$x(t) = x(t+T) \quad , \quad T : period$$

$$x(t) = x(t+mT) \quad , \quad m : \text{integer}$$

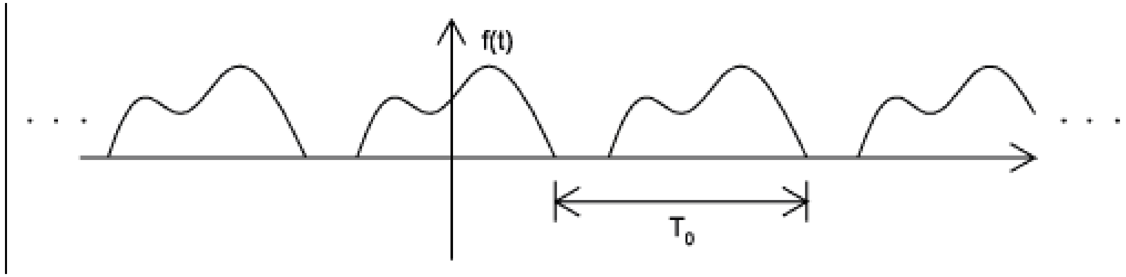
T_0 : Fundamental period : the smallest positive value of T

aperiodic : NOT periodic

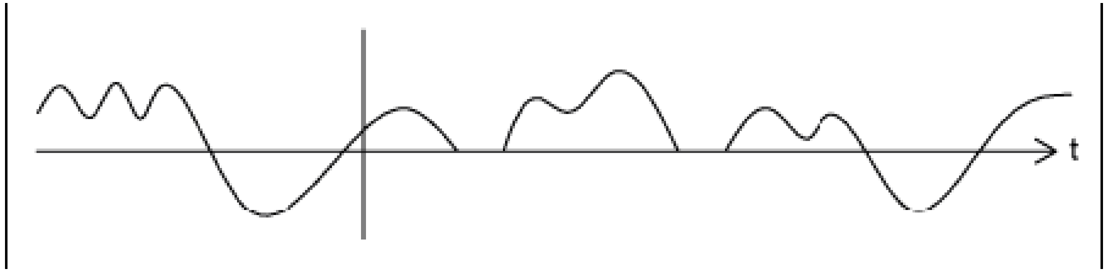
$$x[n] = x[n+N] = x[n+mN] \quad , \quad N_0$$

Examples

- Periodic signal with period T_0



- Aperiodic signal

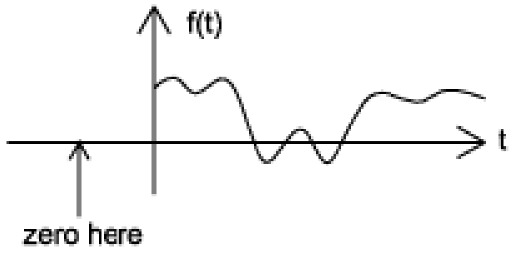


Causal and non-causal signals

- Causal signals

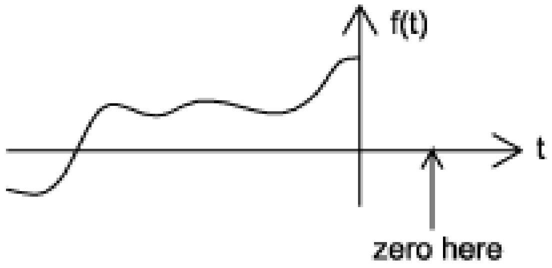
$f(t)=0 \qquad t<0$

- *Causal* signals are signals that are *zero for all negative time (or spatial positions)*, while



- Anticausals signals
- $f(t)=0 \qquad t\geq 0$

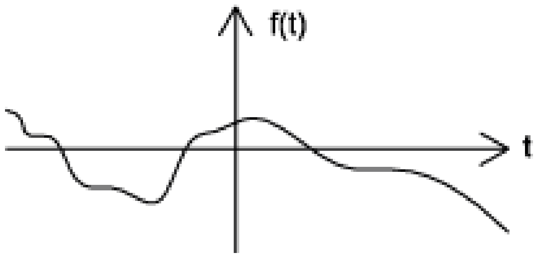
- *Anticausal* are signals that are zero for all *positive time (or spatial positions)*.



- Non-causal signals

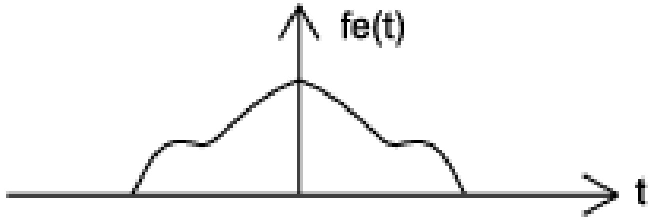
$\exists t_1<0: \qquad f(t_1)=0$

- *Noncausal* signals are signals that have nonzero values in both positive and negative time



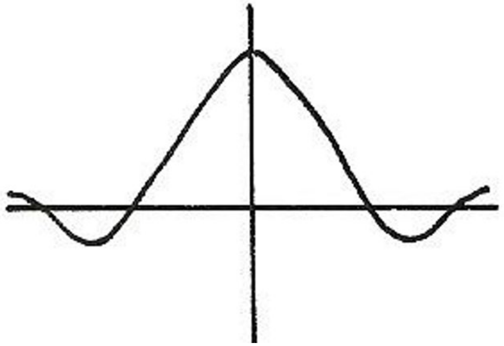
Even and Odd signals

- An even signal is any signal f such that f (t) = f (-t). Even signals can be easily spotted as they are symmetric around the vertical axis.

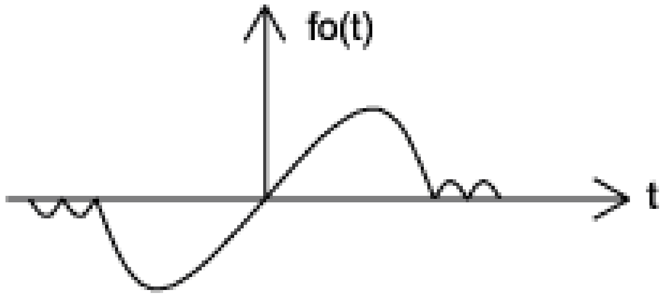


$$x(-t)=x(t)$$

Even

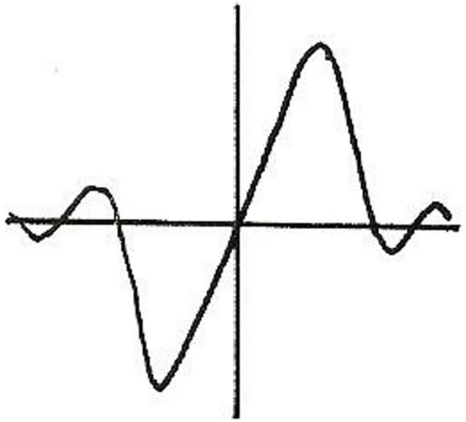


- An odd signal, on the other hand, is a signal f such that f (t)= - (f (-t))



$$x(-t)=-x(t)$$

Odd



Even/Odd Signals

- Even $x(-t) = x(t)$, $x[-n] = x[n]$
- Odd $x(-t) = -x(t)$, $x[-n] = -x[n]$
- Any signal can be discomposed into a sum of an even and an odd

$$x_1(t) = \frac{1}{2}[x(t) + x(-t)], \quad x_2(t) = \frac{1}{2}[x(t) - x(-t)]$$

Decomposition in even and odd components

- Any signal can be written as a combination of an even and an odd signals
  - Even and odd components

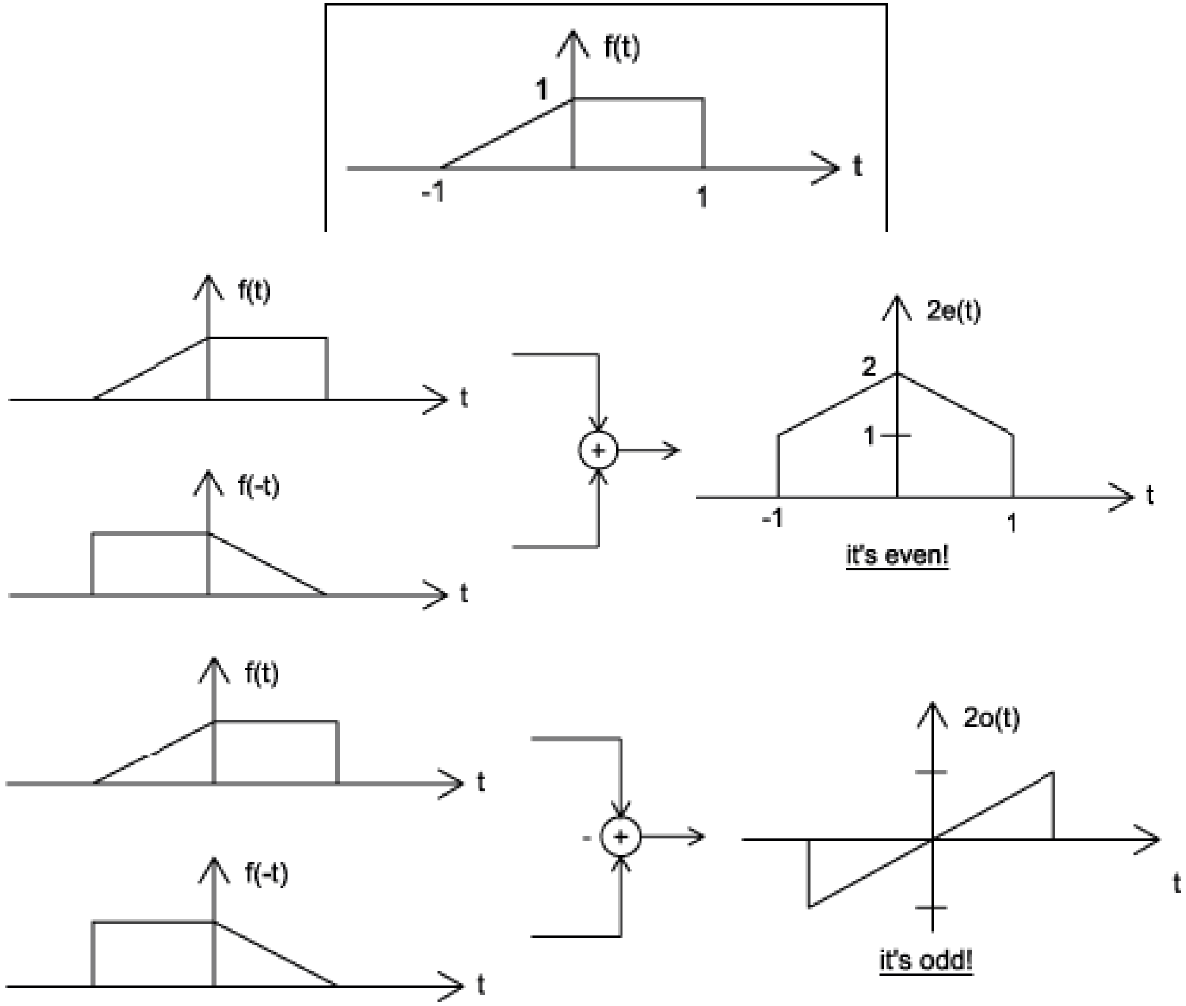
$$f(t)=\frac{1}{2}\left(f(t)+f(-t)\right)+\frac{1}{2}\left(f(t)-f(-t)\right)$$

$$f_e(t)=\frac{1}{2}\left(f(t)+f(-t)\right) \quad \text{even component}$$

$$f_o(t)=\frac{1}{2}\left(f(t)-f(-t)\right) \quad \text{odd component}$$

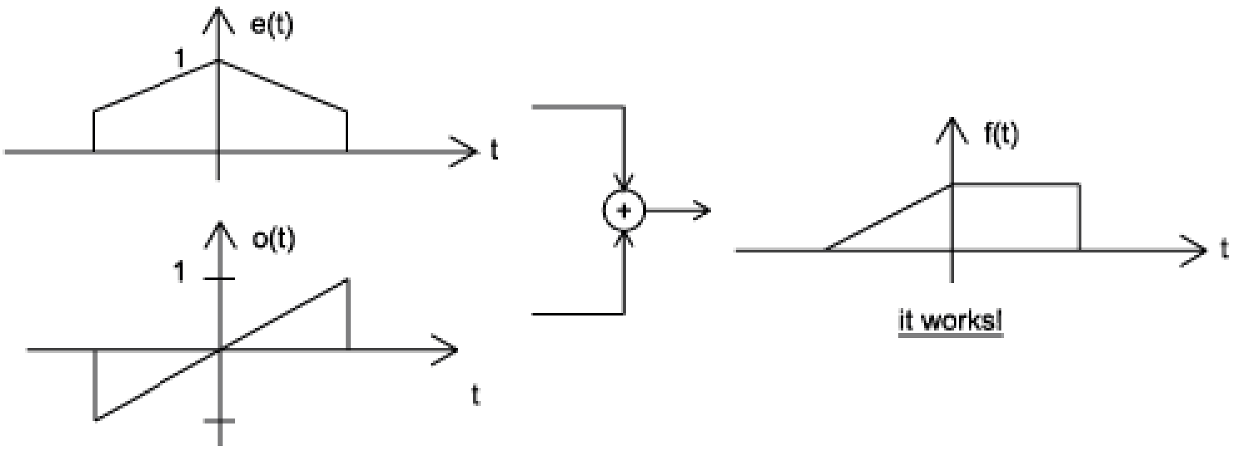
$$f(t)=f_e(t)+f_o(t)$$

Example



Example

- Proof



Some properties of even and odd functions

- even function x odd function = odd function
- odd function x odd function = even function
- even function x even function = even function
- Area

$$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t)\,dt$$

$$\int_{-a}^{a} f_e(t)\,dt = 0$$

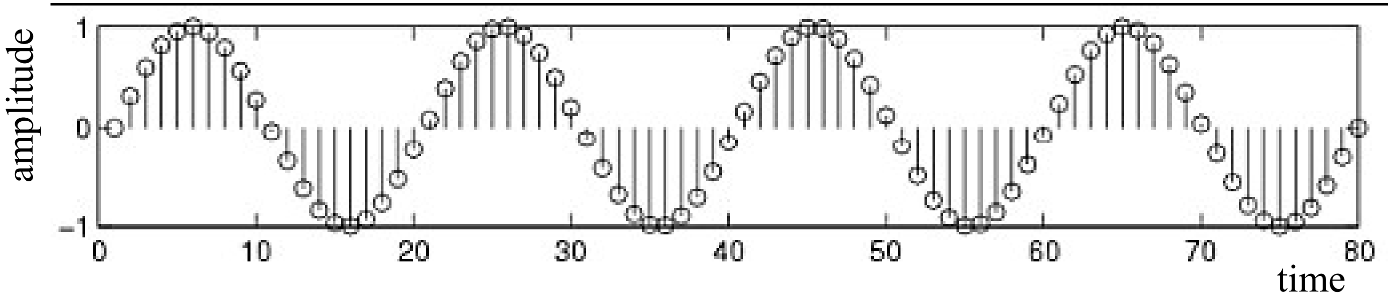
Deterministic - Probabilistic

- Deterministic signal: a signal whose *physical description* in known completely
- A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table.
- Because of this the future values of the signal can be calculated from past values with complete confidence.
  - There is *no uncertainty* about its amplitude values
  - Examples: signals defined through a mathematical function or graph

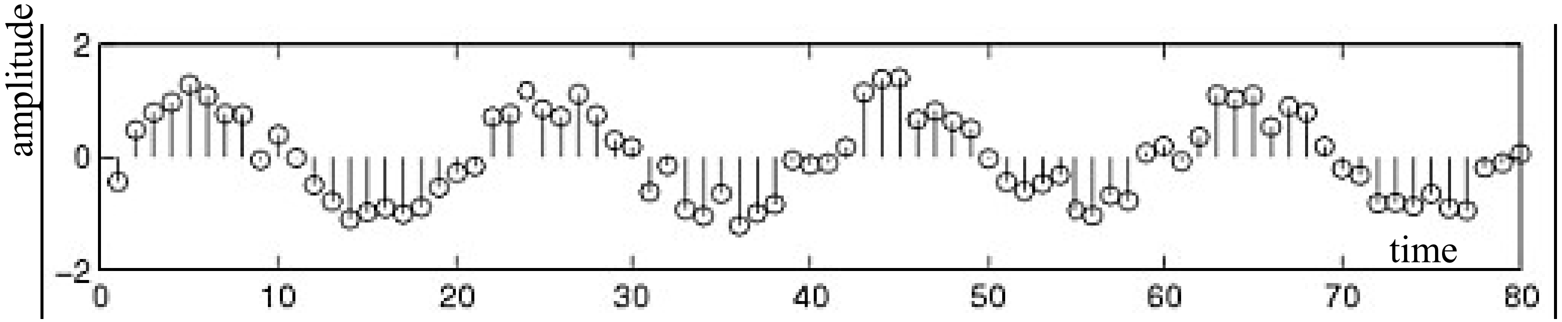
- Probabilistic (or random) signals: the amplitude values *cannot be predicted precisely* but are known only in terms of probabilistic descriptors
- The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals
  - They are realization of a stochastic process for which a model could be available
  - Examples: EEG, evocated potentials, noise in CCD capture devices for digital cameras

Example

- Deterministic signal



- Random signal



Finite and Infinite length signals

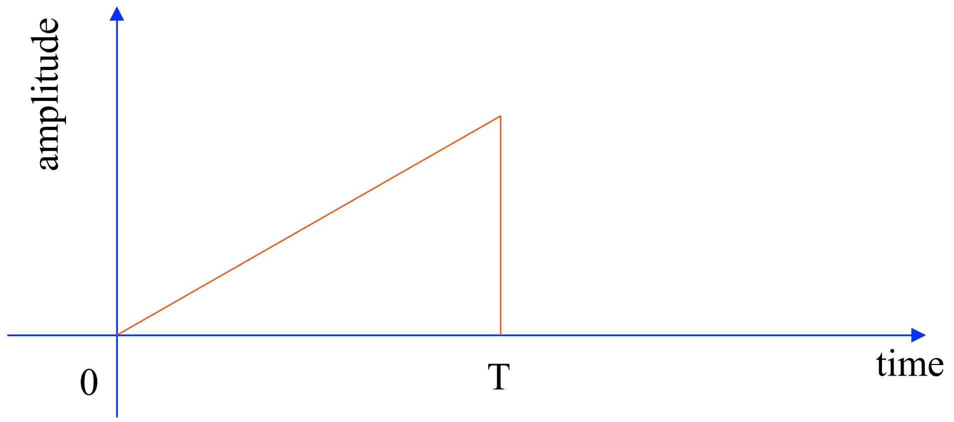
- A finite length signal is non-zero over a finite set of values of the independent variable

$$f = f(t), \forall t : t_1 \le t \le t_2$$
$$t_1 > -\infty, t_2 < +\infty$$

- An infinite length signal is non zero over an infinite set of values of the independent variable
  - For instance, a sinusoid f(t)=sin(ωt) is an infinite length signal

Size of a signal: Norms

- "Size" indicates largeness or strength.
- We will use the mathematical concept of the norm to quantify this notion for both continuous-time and discrete-time signals.
- The energy is represented by the area under the curve (of the squared signal)



Energy & Power

- Signal energy

$$E_f = \int_{-\infty}^{+\infty} f^2(t)dt$$

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt$$

- Generalized energy : L_p norm
  - For p=2 we get the energy (L_2 norm)

$$\|f(t)\| = \left(\int \left(|f(t)|\right)^p dt\right)^{1/p}$$

$$1 \le p < +\infty$$

- Power
  - The power is the time average (mean) of the squared signal amplitude, that is the *mean-squared* value of f(t)

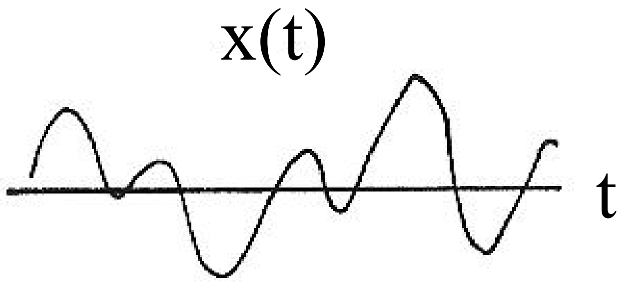
$$P_f = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^2(t)dt$$

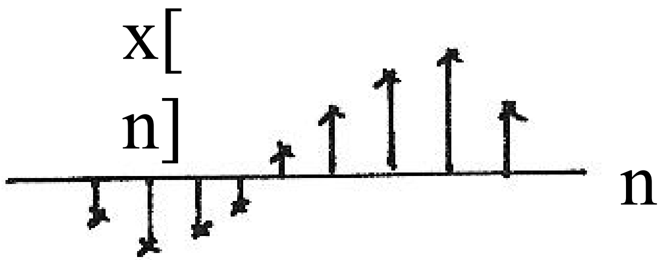
$$P_f = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 \, dt$$

Fundamental Concepts （Signals）

Continuous/Discrete-time Signals

$x(t), x[n]$





Signal Energy/Power

$$E = \int_{t_1}^{t_2} | x(t) |^2 \, dt \,, \quad E = \sum_{n=n_1}^{n_2} | x[n] |^2$$

$$P = E/(t_2 - t_1) \quad , \quad P = E/(n_2 - n_1 + 1)$$

“Electrical” Signal Energy & Power

- It is often useful to characterise signals by measures such as **energy** and **power**

For example, the **instantaneous power** of a resistor is:

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

- and the **total energy** expanded over the interval $[t_1, t_2]$ is:

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- and the **average energy** is:

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- How are these concepts defined for any continuous or discrete time signal?

Generic Signal Energy and Power

- **Total energy** of a continuous signal $x(t)$ over $[t_1, t_2]$ is:

$$E = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

- where |.| denote the magnitude of the (complex) number.

Similarly for a discrete time signal $x[n]$ over $[n_1, n_2]$:

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2$$

- By dividing the quantities by $(t_2$-$t_1)$ and $(n_2$-n_1+1), respectively, gives the **average power**, P

- Note that these are similar to the electrical analogies (voltage), but they are different, both value and dimension.

Energy and Power over Infinite Time

- For many signals, we're interested in examining the power and energy over an infinite time interval (-∞, ∞). These quantities are therefore defined by:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- If the sums or integrals do not converge, the energy of such a signal is infinite

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

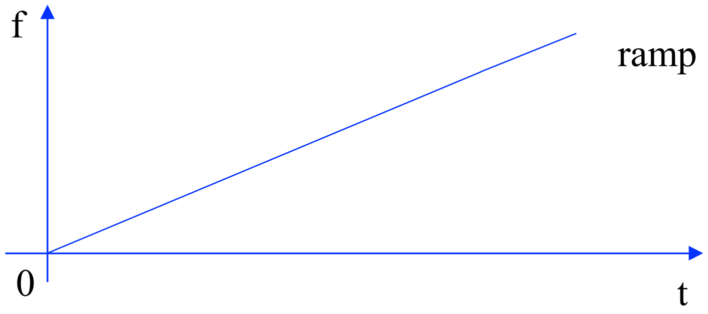
- Two important (sub)classes of signals
    1. Finite total energy (and therefore zero average power)
    2. Finite average power (and therefore infinite total energy)
- Signal analysis over infinite time, all depends on the "tails" (limiting behaviour)

Power - Energy

- The square root of the power is the root mean square (*rms*) value
  - This is a very important quantity as it is the most widespread measure of similarity/dissimilarity among signals
  - It is the basis for the definition of the Signal to Noise Ratio (SNR)

$$SNR = 20\log_{10}\left(\sqrt{\frac{P_{signal}}{P_{noise}}}\right)$$

  - It is such that a constant signal whose amplitude is =rms holds the same power content of the signal itself
- There exists signals for which neither the energy nor the power are finite



Energy and Power signals

- A signal with finite energy is an energy signal
  - Necessary condition for a signal to be of energy type is that the amplitude goes to zero as the independent variable tends to infinity
- A signal with finite and different from zero power is a power signal
  - The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
  - A power signal has infinite energy and an energy signal has zero power
  - There exist signals that are neither power nor energy, such as the ramp
- All practical signals have finite energy and thus are energy signals
  - It is impossible to generate a real power signal because this would have infinite duration and infinite energy, which is not doable.

Transformation of A Signal

- Time Shift

$$x(t) \to x(t - t_0) \quad , \quad x[n] \to x[n - n_0]$$

- Time Reversal

$$x(t) \to x(-t) \quad , \quad x[n] \to x[-n]$$

- Time Scaling

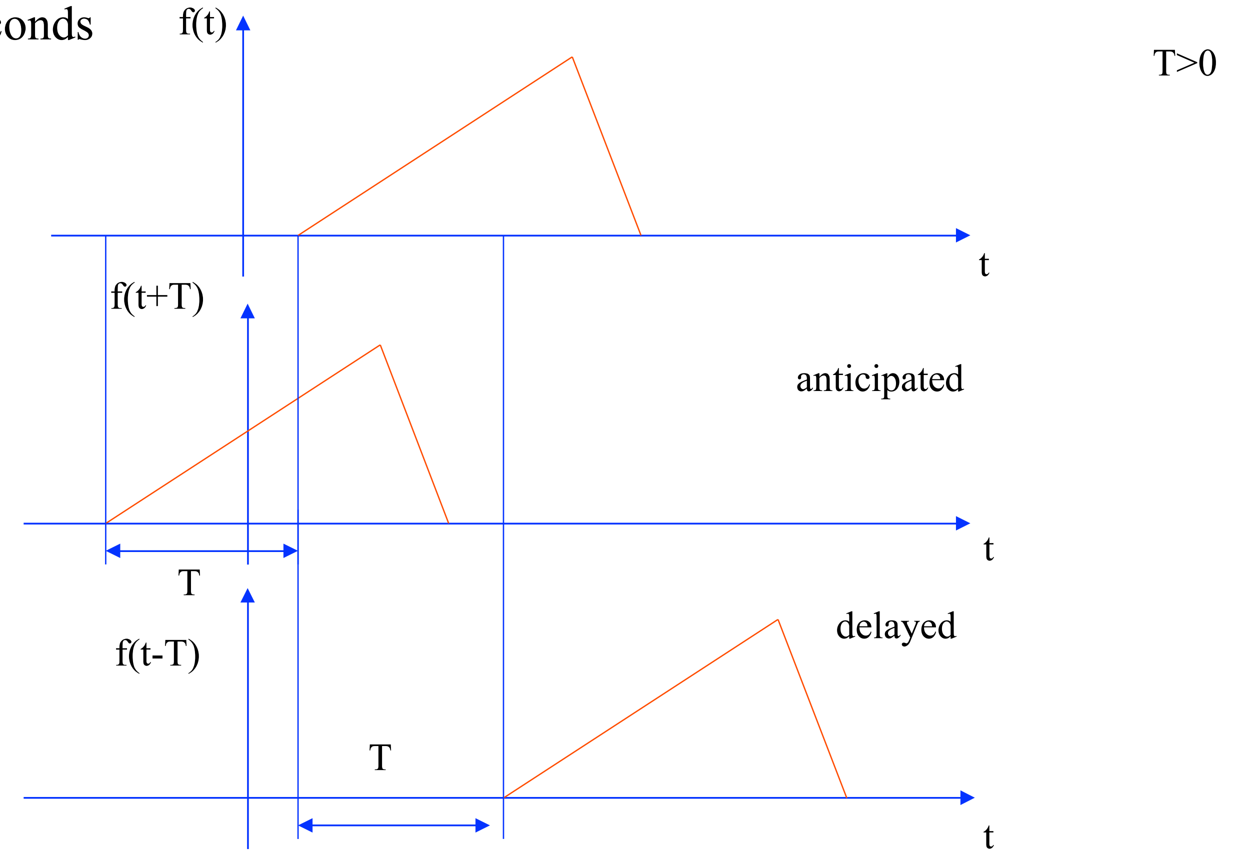
$$x(t) \to x(at) \quad , \quad x[n] \to ?$$

- Combination

$$x(t) \to x(at + b) \quad , \quad x[n] \to ?$$

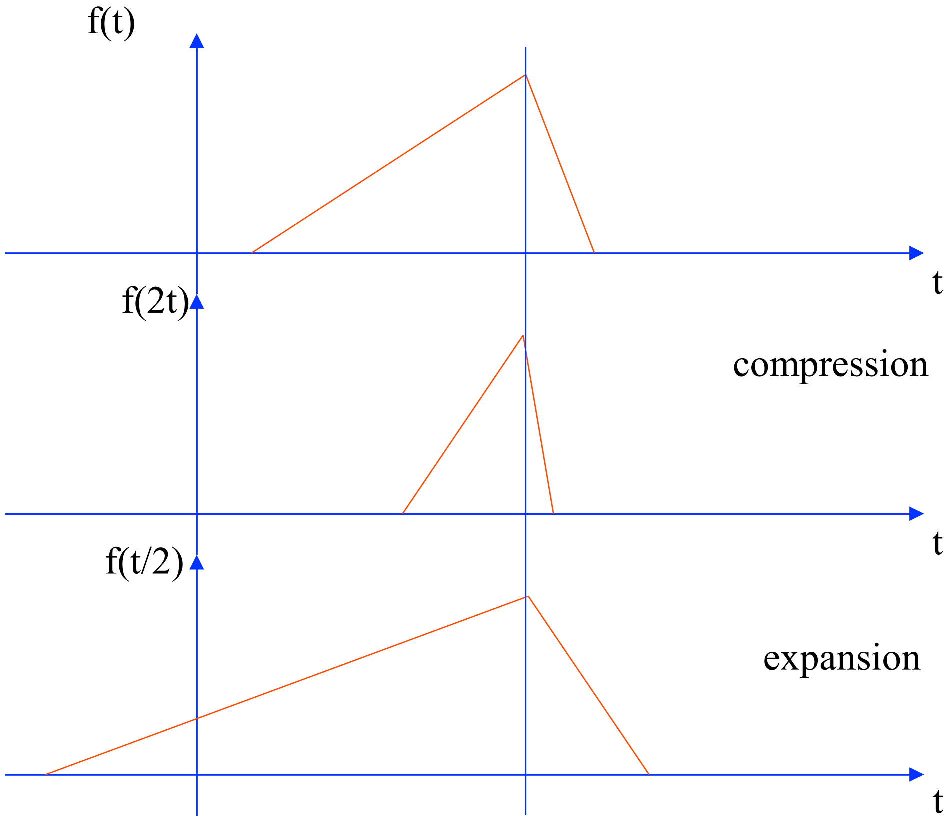
Useful signal operations: shifting, scaling, inversion

- **Shifting**: consider a signal f(t) and the same signal delayed/anticipated by T seconds



Useful signal operations: shifting, scaling, inversion

- (Time) Scaling: compression or expansion of a signal in time



compression $\varphi(t) = f(2t)$

expansion $\varphi(t) = f(t/2)$

Useful signal operations: shifting, scaling, inversion

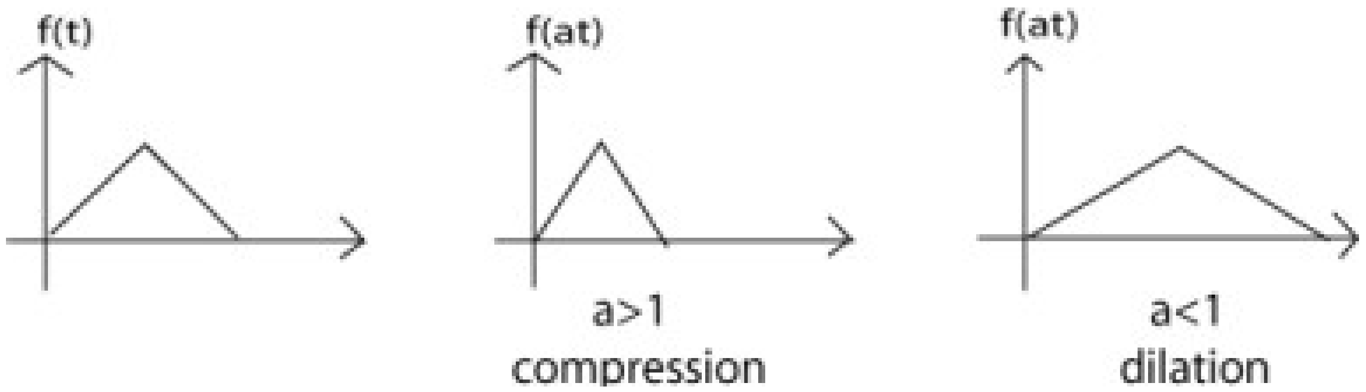
- Scaling: generalization

$a > 1$

$\varphi(t) = f(at) \rightarrow$ compressed version

$\varphi(t) = f\left(\frac{t}{a}\right) \rightarrow$ dilated (or expanded) version

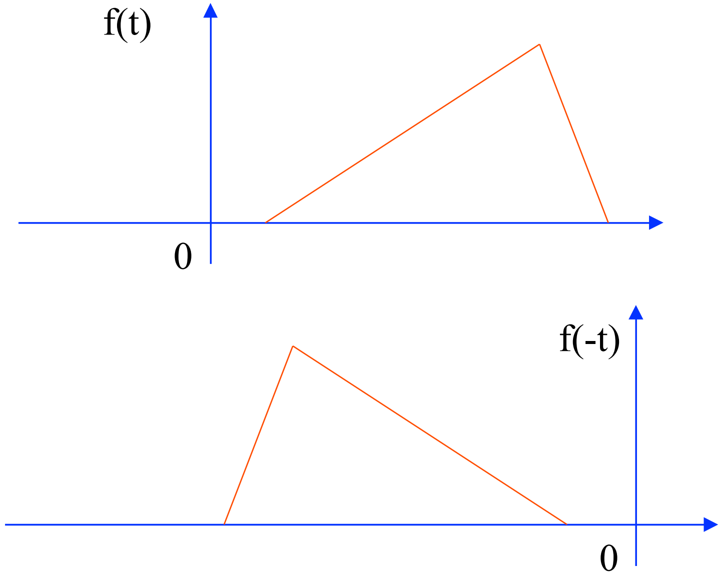
Viceversa for $a < 1$

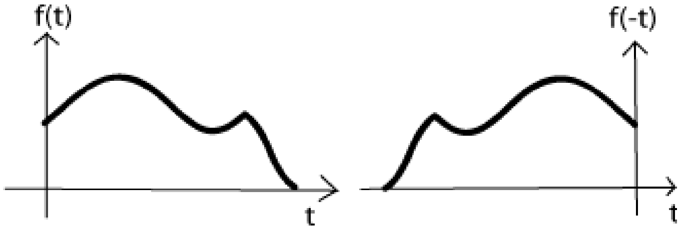


Useful signal operations: shifting, scaling, inversion

- (Time) inversion: mirror image of f(t) about the vertical axis

$$\varphi(t) = f(-t)$$





Useful signal operations: shifting, scaling, inversion

- Combined operations: $f(t) \rightarrow f(at\text{-}b)$
- Two possible sequences of operations

1. Time shift f(t) by to obtain f(t-b). Now time scale the shifted signal f(t-b) by a to obtain f(at-b).
2. Time scale f(t) by a to obtain f(at). Now time shift f(at) by b/a to obtain f(at-b).
    - Note that you have to replace t by (t-b/a) to obtain f(at-b) from f(at) when replacing t by the translated argument (namely t-b/a))

Types in Signals

Useful functions

- Unit step function
  - Useful for representing causal signals

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

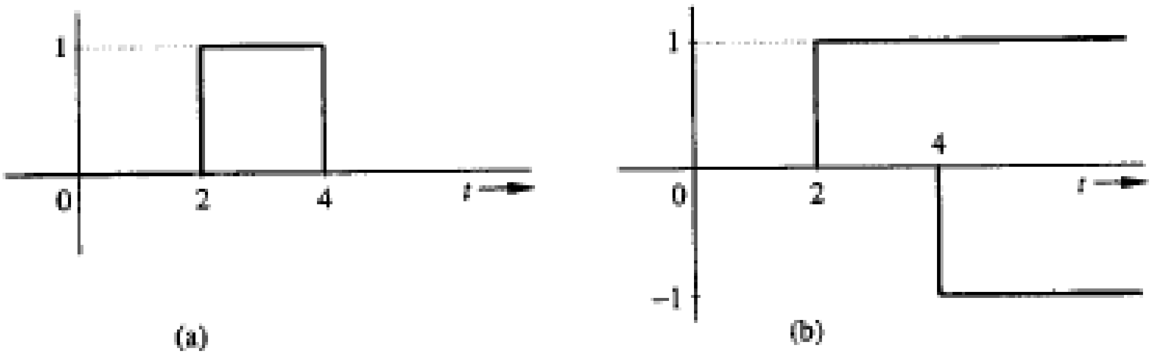
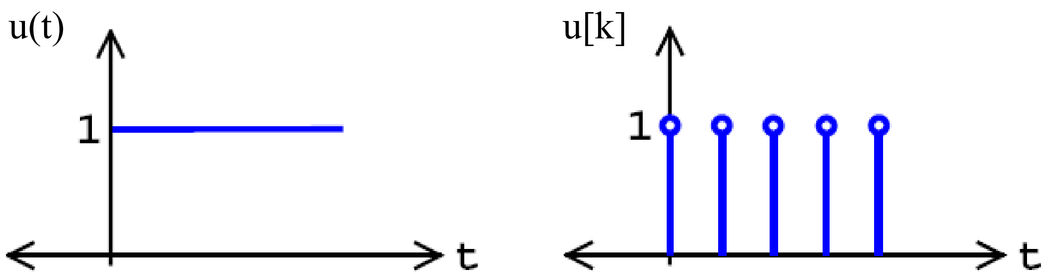


Fig. 1.15 Representation of a rectangular pulse by step functions.

$$f(t)=u(t-2)-u(t-4)$$

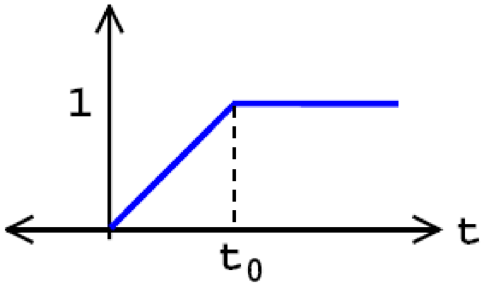
- Continuous and discrete time unit step functions



Useful functions

- Ramp function (continuous time)

$$r(t) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0} \text{ if } 0 \leq t \leq t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

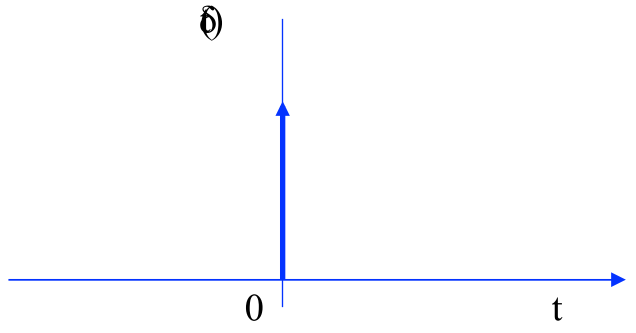


Useful functions

- Unit impulse function

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



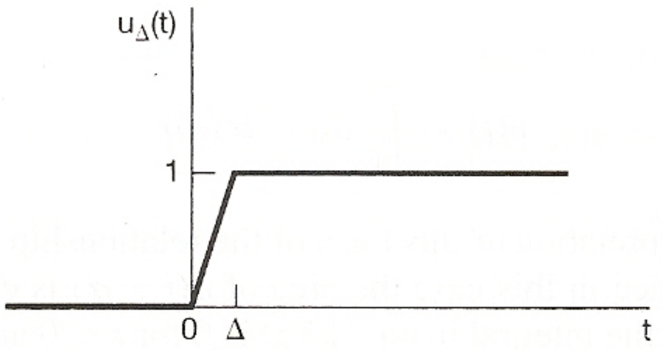


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

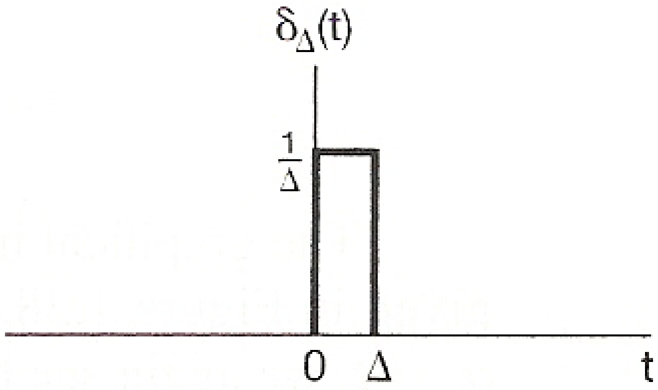


Figure 1.34 Derivative of $u_\Delta(t)$.

Properties of the unit impulse function

- Multiplication of a function by impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

- Sampling property of the unit function

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \int_{-\infty}^{+\infty} \phi(0)\delta(t)dt = \phi(0)\int_{-\infty}^{+\infty} \delta(t)dt = \phi(0)$$

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t-T)dt = \phi(T)$$

  – The area under the curve obtained by the product of the unit impulse function shifted by T and φ(t) is the value of the function φ(t) for t=T

Properties of the unit impulse function

- The unit step function is the integral of the unit impulse function

$$\frac{du}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t)$$

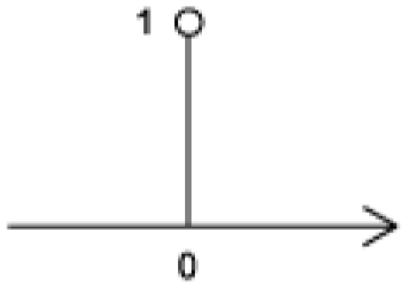
  - Thus

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Properties of the unit impulse function

- Discrete time impulse function

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$



Useful functions

- Continuous time complex exponential

$$f(t) = Ae^{j\omega t}$$

- Euler's relations

$$Ae^{j\omega t} = A\cos(\omega t) + j\left(A\sin(\omega t)\right)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-(j\omega t)}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-(j\omega t)}}{2j}$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Discrete time complex exponential
  - k=nT

$$\begin{aligned} f[n] &= Be^{snT} \\ &= Be^{j\omega nT} \end{aligned}$$

Useful functions

- Exponential function e^{st}
  - Generalization of the function $e^{j\omega t}$

$$s = \sigma + j\omega$$

Therefore

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}(\cos \omega t + j \sin \omega t) \tag{1.30a}$$

If $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^* t} = e^{\sigma - j\omega} = e^{\sigma t} e^{-j\omega t} = e^{\sigma t}(\cos \omega t - j \sin \omega t) \tag{1.30b}$$

and

$$e^{\sigma t} \cos \omega t = \frac{1}{2}(e^{st} + e^{s^* t}) \tag{1.30c}$$

Exponential/Sinusoidal Signals

- Basic Building Blocks from which one can construct many different signals and define frameworks for analyzing many different signals efficiently

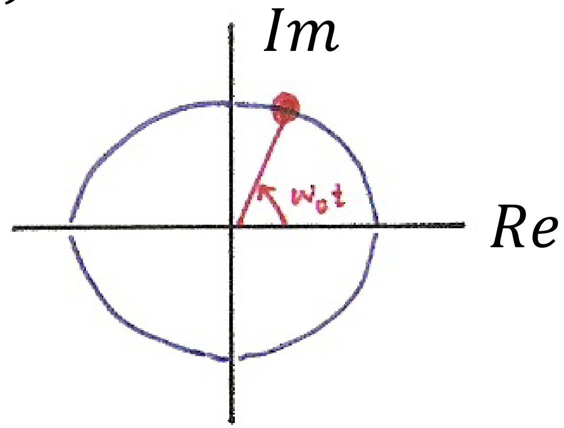
$$x(t) = e^{j\omega_0 t}, \quad \text{fundamental period} \quad T_0 = \frac{2\pi}{|\omega_0|}$$

$$\text{fundamental frequency} \quad \omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 : rad / \sec$$

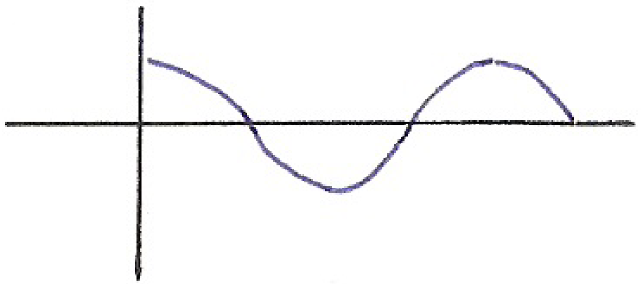
Exponential/Sinusoidal Signals

$$x(t) = e^{j\omega_0 t}$$

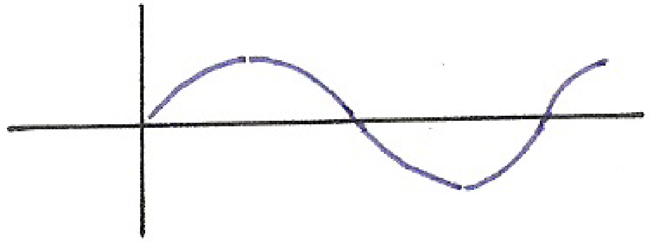


$$e^{jx} = \cos x + j \sin x$$

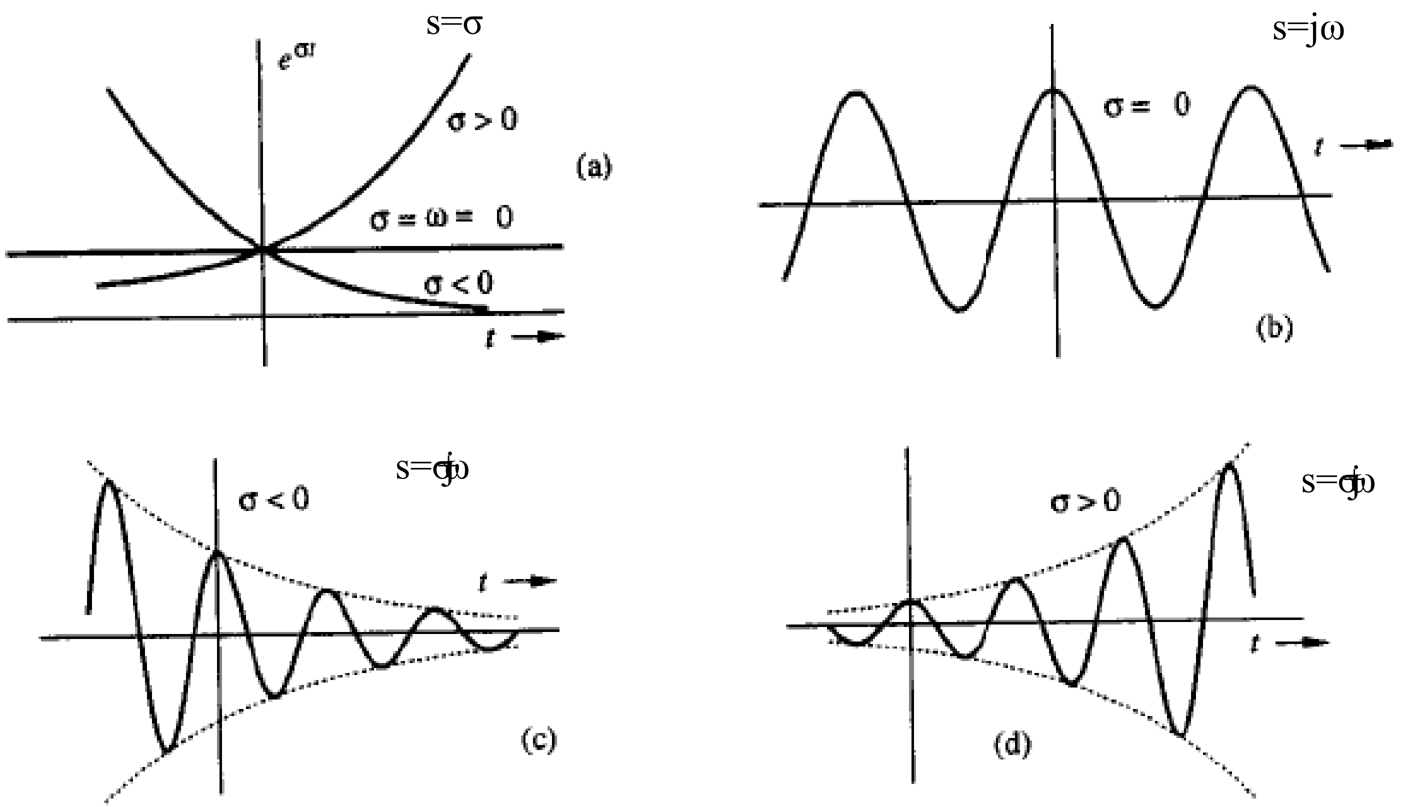
$$Re\{e^{j\omega_0 t}\} = \cos \omega_0 t$$



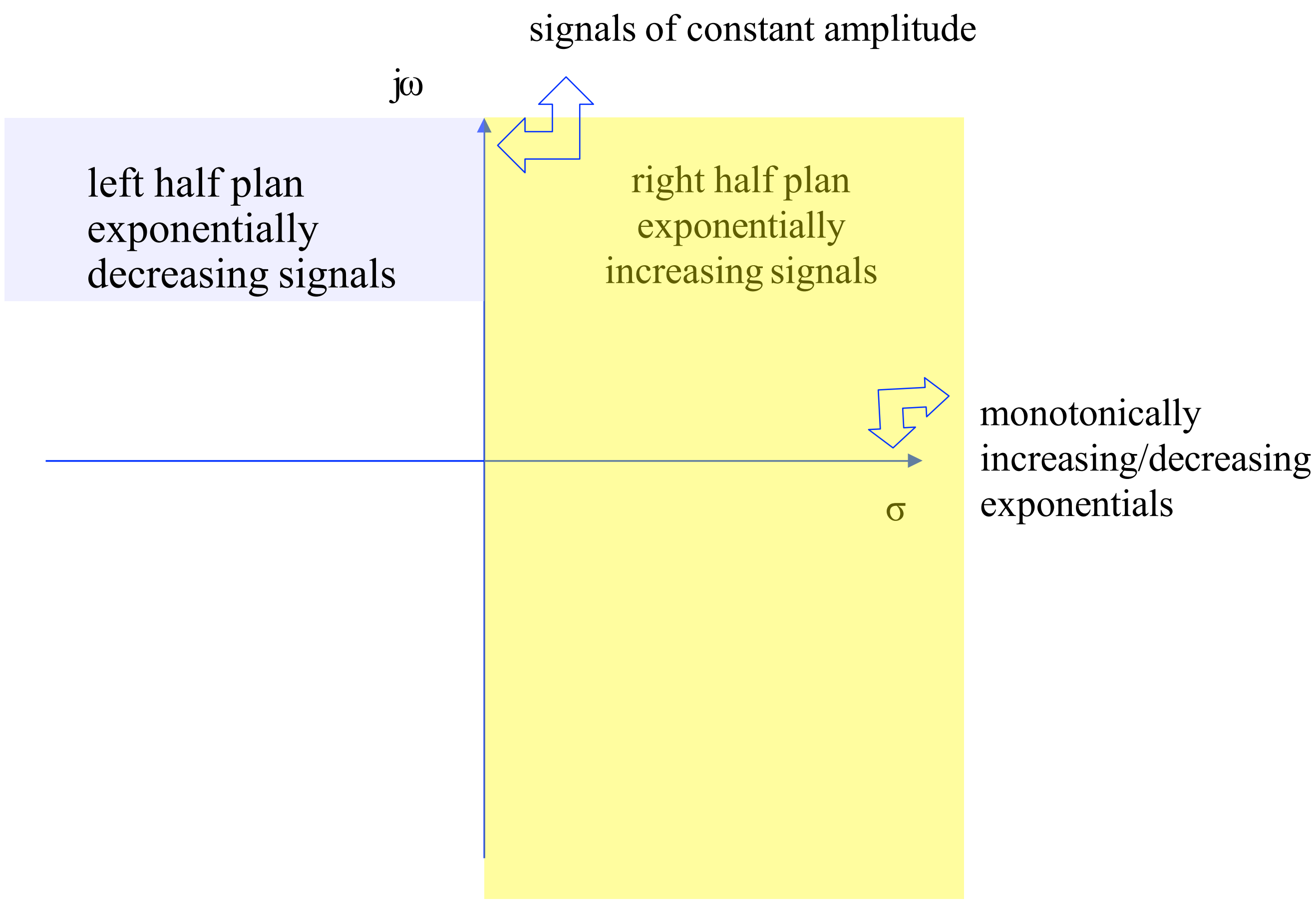
$$Im\{e^{j\omega_0 t}\} = \sin \omega_0 t$$



The exponential function



Complex frequency plan

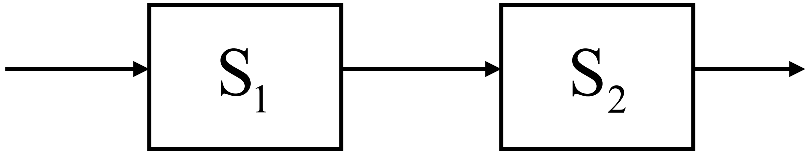


Systems

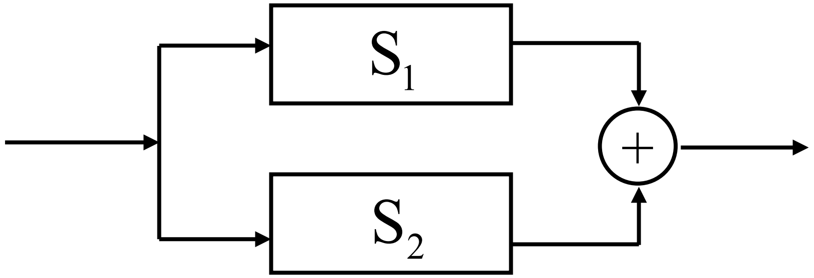
Fundamental Concepts (Systems)

- Continuous/Discrete-time Systems

- Interconnections of Systems
  - Series



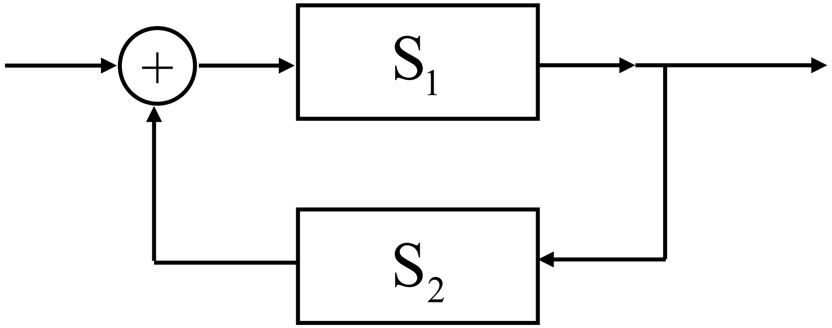
  - Parallel



Fundamental Concepts (Systems)

● Interconnections of Systems

– Feedback



– Combinations

Fundamental Concepts (Systems)

- Stability
  - stable : bounded inputs lead to bounded outputs
- Time Invariance
  - time invariant : behavior and characteristic of the system are fixed over time

Thank you for your listening!