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Chapter 6

Finite Element Methods for Boundary Value Problems

6.1 Elliptic Equations for Boundary Value Problems(BVPs)

In this section, we are concerned with numerical methods for BVPs. As in the previous sections, we adopt a simple form for the purpose of introducing the numerical methods

$$u" = f(x, u, u'), a < x < b. \tag{6.1}$$

The Dirichlet boundary condition is $u(a) = \alpha, u(b) = \beta$ is considered for convenience. It is worth to mention that the choice of β will play an important role on the uniqueness and stability. A theoretical proof will be more difficult than IVPs, let us explain it simply with an example as following.

例6.1.1. Let $f(x, u, u') = -u$, $u(0) = 0$ in above equation. In this sense, the exact solution of this problem is written as $u(x) = c \sin x$, where c is any constant. It is easy to find out that there are infinite solutions of this boundary problem when $\beta = 0$, and it is unsolvable when $\beta \neq 0$, if $b = n\pi$.

6.1.1 Finite Difference Approximation

Firstly, let us consider the one spatial dimensional BVPs for illustration.

例6.1.2. Consider a uniform grid

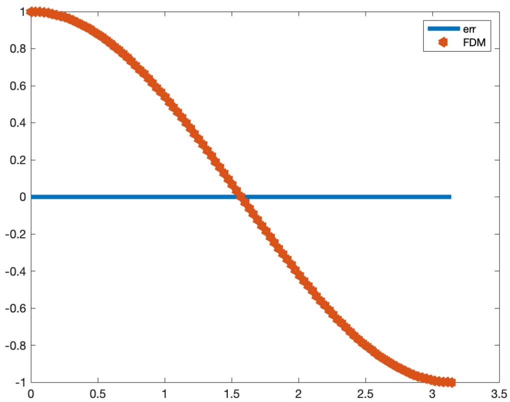
$$x_i = a + ih, i = 0, 1, \ldots, n+1$$

for interval $[a, b]$, where $h = (b-a)/(n+1)$ is the grid size. The purpose of numerical solution to two-point boundary value problem

$$u'' = -3u + 2\cos x, u(0) = 1, u(\pi) = -1,$$

is to find approximations $u_i \approx u(x_i), i = 1, \ldots, n$ in the condition of given boundary value $u_0 = u(a) = \alpha$ and $u_{n+1} = u(b) = \beta$. A center divided scheme yields

$$u_{i-1} - (2+3h^2)u_i + u_{i+1} = -2h^2 \cos x_i, i = 1, 2, \ldots, n.$$



The right figure shows the solution for $n = 20$.

Two Dimensional BVPs are constantly used for validation and practical using. 记$u_{i,j}$表示网格点(i,j)上待求函数的近似值，then

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h_x^2} + O(h_x^2) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} + O(h_y^2).
\end{aligned}$$

因此在二维情形（设$h_x = h_y = h$，舍去高阶无穷小项后）可得

$$-\triangle u \approx \frac{1}{h^2} \begin{pmatrix} & -u_{i,j+1} & \\ -u_{i-1,j} & 4u_{i,j} & -u_{i+1,j} \\ & -u_{i,j-1} & \end{pmatrix} \tag{6.2}$$

稀疏Laplace算子-Matlab表示:上述稀疏Laplace算子在Matlab中的实现通常有两中不同的做法

- kron — 基于内部未知点($(n-1) \times (n-1)$个)；
- spLaplacian5.m — 基于全部点($(n+1) \times (n+1)$个)。

推荐采用后一种方案，代价是将边界条件也作为方程，优点是算子离散和边界条件的处理相对独立，因此数学上更直观。

例6.1.3. 考虑如下Poisson方程的Dirichlet边值问题

$$-\triangle u = f, \quad (x, y) \in [0, 1]^2. \tag{6.3}$$

取解析函数

$$u(x, y) = \sin(\pi x)\sin(\pi y), \tag{6.4}$$

作为测试解，相应地取$f = 2\pi^2 u(x, y)$。

实现细节参考testPoisson.m以及相关.m文件。

6.1.2 Parameterization Methods

The first important method is the collocation method.

对于边界条件 $u(a)=\alpha, u(b)=\beta$ 下的一维两点边值问题

$$u''=f(t,u,u'), \qquad a<t<b,$$

我们要寻找形如

$$u(t) \approx v(t,\boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$$

的近似解，其中 $\phi_i(t)$ 是定义在 $[a,b]$ 上的基函数，$\boldsymbol{x}$ 是待定的 n 维参数向量.

为了确定参数向量 $\boldsymbol{x}$，我们定义 n 个点 $a<t_1<\cdots<t_n=b$，称为配置点

基函数后，在每个内部配置点上将近似解及其导数代入常微分方程，则得到

程组

$$v''(t_i,\boldsymbol{x})=f(t_i,v(t_i,\boldsymbol{x}),v'(t_i,\boldsymbol{x})), \qquad i=2,\cdots,n-1,$$

由边界条件，又得到两个附加的方程

$$v(t_1,\boldsymbol{x})=\alpha, \quad v(t_n,\boldsymbol{x})=\beta.$$

这样产生了含有 n 个未知数的 n 个方程. 这个方程组线性与否依赖于 f 关于 u

线性，求解这个方程组，确定出向量参数 $\boldsymbol{x}$，就得到了近似解函数 v.

例6.1.4 (collocation 课本练习题10.8). Using the method of collocation so solve the following boundary problem:

$$u''=-3u+2\cos t, u(0)=1, u(\pi)=-1,$$

The second strategy is the Galerkin Method, The advantage of the Galerkin method is a good property in approximation theory.

6.1.3 Shooting Method

The fundamental idea of the shooting method is to fully utilizing the efficiency of explicit scheme.

打靶法是将给定的边值问题用一系列的初值问题代替.

一阶两点边值问题

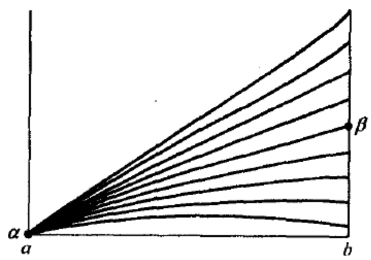
$$\boldsymbol{y}'=\boldsymbol{f}(t,\boldsymbol{y}), \qquad a<t<b,$$

满足约束条件

$$\boldsymbol{g}(\boldsymbol{y}(a),\boldsymbol{y}(b))=\boldsymbol{0},$$

这等价于非线性代数方程组

$$\boldsymbol{h}(\boldsymbol{x}) \equiv \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}(b;\boldsymbol{x}))=\boldsymbol{0},$$



例6.1.5 (shooting)**.** Using the method of shooting so solve the following boundary problem:

$$u'' = -3u + 2\cos t, u(0) = 1, u(\pi) = -1,$$

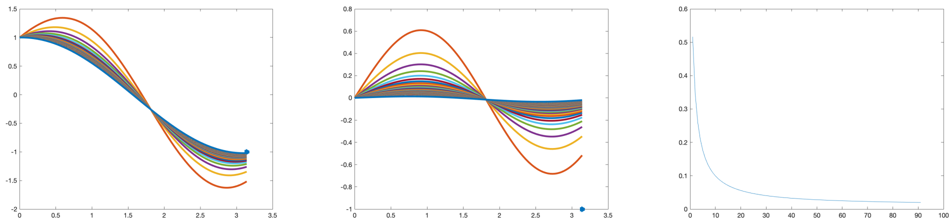


Figure 6.1: (Left)The solution of shooting method, (Middle) Errors, (Right) the objective function

6.1.4 Newton's method for nonlinear problems

考虑如下形式的"半"线性问题

$$-\triangle u + f(\mathbf{x}, u) = g, \quad \mathbf{x} \in \Omega \subset \mathbf{R}^2 \tag{6.5}$$

仍取前例中的解析函数作为测试解，此外令

$$f(\mathbf{x}, u) = u^3. \tag{6.6}$$

右端项g根据u和f的表达式可计算得到，参考fung.m

可在所有内点$i, j = 1, \cdots, n-1$上建立方程

$$\frac{1}{h^2}\begin{pmatrix} & -u_{i,j+1} & \\ -u_{i-1,j} & 4u_{i,j} & -u_{i+1,j} \\ & -u_{i,j-1} & \end{pmatrix} + f(x_i, y_j, u_{i,j}) = g_{i,j}.$$

如果f中不包含一阶微分算子(反应-扩散问题)，那么f(线性化后)只对$u_{i,j}$有贡献，若考虑带一阶微分算子项的f,称为对流-扩散问题。不妨将上述代数方程简记为

$$A\mathbf{u} + f(\mathbf{u}) = \mathbf{g} \tag{6.7}$$

令$F(\mathbf{u}) = A\mathbf{u} + f(\mathbf{u}) - \mathbf{g}$ 则有Newton迭代

$$\mathbf{u}^{new} = \mathbf{u}^{old} - F'(\mathbf{u}^{old})^{-1}F(\mathbf{u}^{old}) \tag{6.8}$$

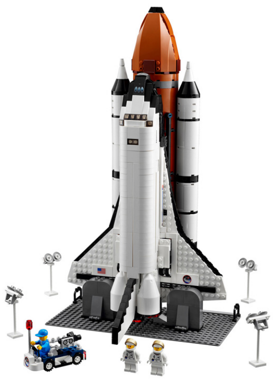
其中$F'(\mathbf{u}) = A + f'(\mathbf{u})$. Matlab实现的时候会引进一些额外的变量以及向量化，请参考test_semi_newton.m

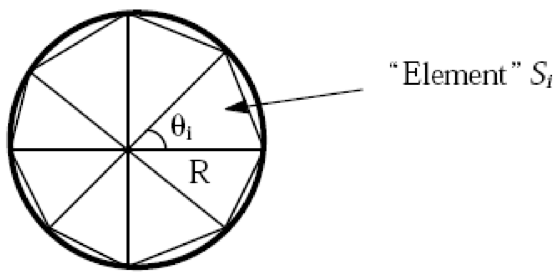
[一些练习]

- 修改算例一(用不同u)，记录$h = 10, 20, 40, 80, 160, 320$时数值解的$L_2$或$L_\infty$误差，计算收敛阶
- 修改半线性问题算例，计算Newton迭代法的收敛阶
- 尝试将上述方法扩展至x-和y-方向的区间和步长不一致的情形：$h_x \neq h_y$以及$(x, y) \in [a, b] \times [c, d]$
- 尝试变更方程编号的顺序（列优先或其他感兴趣的顺序）
- 进一步研究如何利用Newton迭代法求逆矩阵，以及其他可能加速求解线性方程足的方法

6.2 Finite Element Approximations for BVPs

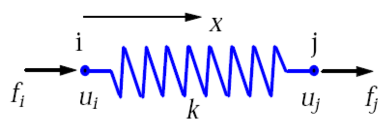
A more mathematical illustration of piecewise. Approximation of the area of a circle S with $N-$polygon where the area(S_i)= $\frac{1}{2}R^2\sin(\theta_i)$. A further formulation is $S = \lim_{N\to\infty}\sum_{i=1}^{N} S_i =$





,

$\dfrac{R^2}{2}\lim_{N\to\infty} N\sin\left(\dfrac{2\pi}{N}\right)$, which sum up to πR^2.



例6.2.1 (Formulation of a Spring System)**.** To keep the equilibrium, additional fores is required at node i and j

$$
\begin{aligned}
f_i = -F = -k(u_j - u_i) &= ku_i - ku_j \\
f_j = F = k(u_j - u_i) &= -ku_i + ku_j
\end{aligned}
$$

In a matrix form(known as the stiff matrix of the spring)

$$
\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} f_i \\ f_j \end{bmatrix} \tag{6.9}
$$

The finite element methods are widely applied in numerically solving partial differential equaitons. As an example, let us considering the simplest form of Poisson's equation

$$
\begin{cases} -(u_{xx} + u_{yy}) = f(x,y), & (x,y) \in \Omega \\ \qquad u(x,y) = u_b(x,y), & (x,y) \in \partial\Omega \end{cases}
$$

The purpose of this notes is Understand the fundamental ideas of the FEM, and Know the behavior and usage of classical finite elements.Solve elliptic boundary value problem with FEM routines, and interpret and evaluate the quality of the results. On the other hand, it is also essential to understand the limitations of the FEM, so that classical results on theoretical analysis are also presented.

Variation formulations

Find $u(x,y) \in H^1(\Omega)$, such that $u(x,y) = u_b(x,y)$ at $\partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Omega f v d\Omega, \quad \forall v(x,y) \in H_0^1(\Omega), \tag{6.10}$$

and also referred as $a(u,v) = F(v)$ briefly. Thed different variational frameworks for BVP is actually the Minimum of the energy variational. Other Variational methods are also popular for various applications, such as the Galerkin methods and the Ritz methods, which are all equivallent in the case of Poisson's e quaiton.

The finite element formulations

Find $u_h(x,y) \in H^1(\Omega) \cap V_h(\triangle)$, such that $u_h(x,y) = u_b(x,y)$ at the boundary of the domain and

$$\int_\triangle \nabla u_h \cdot \nabla v_h dxdy = \int_\triangle f v_h dxdy, \quad \forall v_h(x,y) \in H_0^1(\Omega) \cap V_h(\triangle). \tag{6.11}$$

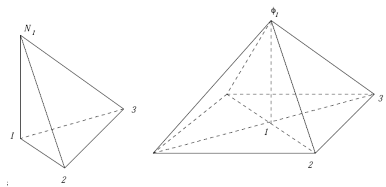
Remark: This is a discretized version of the weak formulations.

The Finite Element(FE) Space $V_h(\triangle)$

The key feature of finite element approximation is to construct the approximating space $V_h(\triangle)$ on the given mesh $\triangle$ with N nodes. A carefully chosen basis function set $\{\phi_i(x,y)\}_{i=1}^N$ is possible to approximate any desired function $u(x,y)$ by

$$u(x,y) \approx u_h(x,y) = \sum_{i=1}^N u_i \phi_i(x,y), \quad \forall (x,y) \in \triangle.$$

In practical calculation, the nodal basis function is preferred for simplicity.



例6.2.2 (P^1 - linear shape functions)**.** Denote $N_j(x,y), \forall j = 1,2,3..$ On element $\triangle^e \in \triangle$ with vertices $\langle v_i(x_i,y_i), v_j(x_j,y_j), v_k(x_k,y_k) \rangle$, The linear shape function $N_i(x,y) := \alpha_i + \beta_i x + \gamma_i y, (x,y) \in \triangle^e$ associated with vertex i satisfies

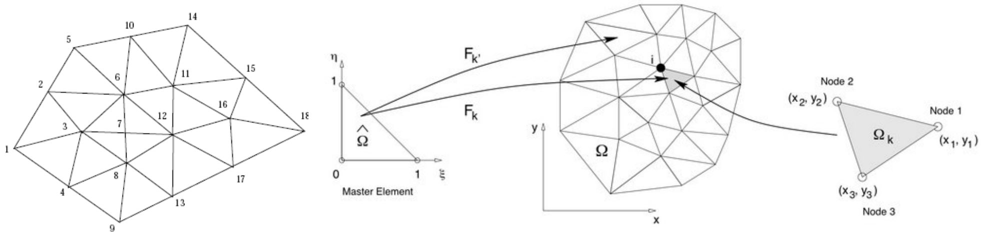
$$N_i(x_i,y_i) = 1, \quad N_i(x_j,y_j) = 0, \quad N_i(x_k,y_k) = 0.$$

The above 3×3 linear system gives

$$\alpha_i = \frac{x_j y_k - x_k y_j}{2|\triangle^e|}, \quad \beta_i = \frac{y_j - y_k}{2|\triangle^e|}, \quad \gamma_i = \frac{x_k - x_j}{2|\triangle^e|}$$

For example

$$N_2(x,y) = \frac{1}{2|\triangle^e|}\big(x_3 y_1 - x_1 y_3\big) + \big(y_3 - y_1\big)x + \big(x_1 - x_3\big)y.$$



例6.2.3 (P^2 - quadratic shape functions)**.** Consider a second order approximation on given element $\triangle^e$. Let

$$\lambda_1 = \frac{\begin{vmatrix} 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_2 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & \mathbf{x} & \mathbf{y} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}},$$

then it is straightforward to prove that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and

$$P^2(\triangle^e) = span\{\lambda_1^2, \lambda_2^2, \lambda_3^2, 4\lambda_1\lambda_2, 4\lambda_1\lambda_3, 4\lambda_2\lambda_3\},$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is referred as the *barycentric coordinate*. Let us leave the P^3 - cubic shape functions cases as the exercises.

Abstract definition for $V_h(\triangle)$

A Finite Element space can be described by

1. Finite Element Mesh $\triangle$
2. Degree Of Freedoms Σ
3. Master Element $\hat{\Omega}$
4. Affine Mapping F_k

Ciarlet's definition of a *finite element*$(K, \mathcal{P}, \Sigma)$

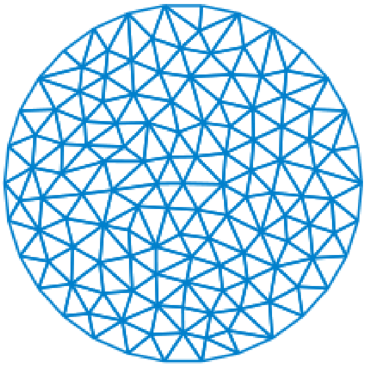
An alternative mathematical definition for $V_h(\triangle)$.

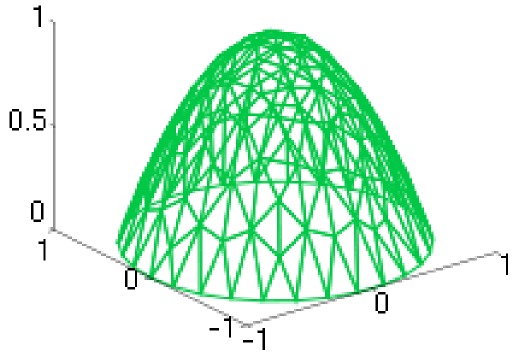
1. *element domain*: $K \subset \mathcal{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary
2. space of *shape functions*: $\mathcal{P}$ be a finite-dimensional space of functions on K
3. *degree of freedoms*: $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\}$ be a basis for $\mathcal{P}'$

P. G. Ciarlet: The Finite Element Method for Elliptic Equation, 1978.

例6.2.4. Interpolation

$$u(x, y) = 1 - x^2 - y^2, \quad (x, y) \in x^2 + y^2 < 1.$$





Apply Boundary Conditions

To close linear system, boundary should be applied. In general

- Dirichlet BC on the entire boundary, *i.e.*, $u(x,y)|_{\partial\Omega} = u_0(x,y)$ is given.
- Neumann BC on the entire boundary, *i.e.*, $\partial u/\partial n|_{\partial\Omega} = g(x,y)$ is given.
  In this case, the solution to a Poisson equation may not be unique or even exist, depending upon whether a compatibility condition is satisfied. Integrating the Poisson equation over the domain, we have

  $$\iint_\Omega f dxdy = \iint_\Omega \Delta u\, dxdy = \iint_\Omega \nabla\cdot\nabla u\, dxdy = \int_{\partial\Omega} u_n\, ds = \int_{\partial\Omega} g(x,y)\, ds = 0\,,$$

  which is the compatibility condition to be satisfied for the solution to exist. If a solution does exist, it is not unique as it is determined within an arbitrary constant.
- Mixed BC on the entire boundary, *i.e.*,

  $$\alpha(x,y)u(x,y) + \beta(x,y)\frac{\partial u}{\partial n} = \gamma(x,y)$$

  is given, where $\alpha(x,y)$, $\beta(x,y)$, and $\gamma(x,y)$ are known functions.
- Dirichlet, Neumann, and Mixed BC on some parts of the boundary.

稀疏性

The discrete variational problem results in a sparse linear system

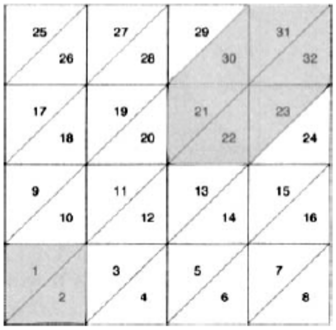
$$K\mathbf{u} = \mathbf{f}$$

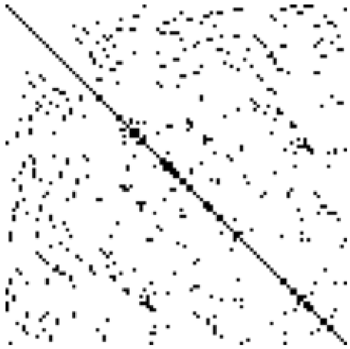
using P^1-FE space with N vertex(degree of freedoms), where the stiffness matrix K is $N\times N$ and its entries

$$K_{i,j} = a(\phi_i,\phi_j) := \int_\Omega \nabla\phi_i\cdot\nabla\phi_j, \quad \forall i,j = 1,2,\ldots,N.$$

Assemble

a element-by-element technique





$$\left[\begin{array}{cccc} 1500 & 0 & -1500 & 0 \\ 0 & 0 & 0 & 0 \\ -1500 & 0 & 1500 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} 3 \\ 4 \\ 5 \\ 6 \end{array} \qquad \left[\begin{array}{cccccccc} 1500 & 0 & -1500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} \\ \\ 3 \\ 4 \\ 5 \\ 6 \\ \\ \\ \end{array} \quad (25.$$

Element (3) joins nodes 1 and 4. Its EFT is { 1,2,7,8 }. Matrices $\mathbf{K}^{(3)}$ and $\mathbf{K}$ upon merge are

$$\left[\begin{array}{cccc} 768 & -576 & -768 & 576 \\ -576 & 432 & 576 & -432 \\ -768 & 576 & 768 & -576 \\ 576 & -432 & -576 & 432 \end{array}\right]\begin{array}{l} 1 \\ 2 \\ 7 \\ 8 \end{array} \qquad \left[\begin{array}{cccccccc} 2268 & -576 & -1500 & 0 & 0 & 0 & -768 & 576 \\ -576 & 432 & 0 & 0 & 0 & 0 & 576 & -432 \\ -1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & 576 & 0 & 0 & 0 & 0 & 768 & -576 \\ 576 & -432 & 0 & 0 & 0 & 0 & -576 & 432 \end{array}\right]\begin{array}{l} 1 \\ 2 \\ \\ \\ \\ \\ 7 \\ 8 \end{array}$$

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Parallel Realization of the Element-by-Element FEM Technique by CUDA

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The utilization of Graphical Processing Units (GPUs) for the element-by-element (EbE) finite element method (FEM) is demonstrated. EbE FEM is a long known technique, by which a conjugate gradient (CG) type iterative solution scheme can be entirely decomposed into computations on the element level, i.e., without assembling the global system matrix. In our implementation NVIDIA's parallel computing solution, the Compute Unified Device Architecture (CUDA) is used to perform the required element-wise computations in parallel. Since element matrices need not be stored, the memory requirement can be kept extremely low. It is shown that this low-storage but computation-intensive technique is better suited for GPUs than those requiring the massive manipulation of large data sets.

例6.2.5. In this example, $u(x,y) = x/(x^2+y^2)$ is known at $\partial\Omega$ for the Dirichlet BVP. A sequence of meshes with different mesh size used.

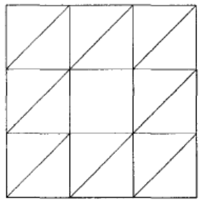
It is easy to obtain the error of the FE approximation u_h

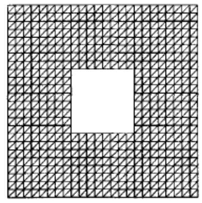
例6.2.6. In the second example, we consider more general form BVP

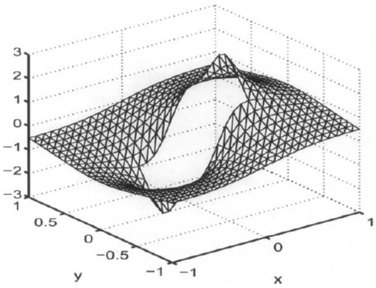
$$\nabla \cdot (\kappa(x,y)\nabla u) = f(x,y) \qquad in \quad \Omega,$$

where the coefficient $\kappa(x,y) = 1 + xy^2$ is nonconstant. The analytic solution $u(x,y) = xy(1-x)(1-y)$ is considered as a test problem. In this case, u is obviously zero at the boundary and

$$f(x,y) = -y^3 + y^4 + 4y^3x - 4y^4x + 2y - 2y^2 - 2x^2y + 6x^2y^2 + 2x^3y - 6x^3y^2 + 2x - 2x^2.$$







h	$\|u-u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
0.9428	3.592	—
0.4714	2.513	1.429
0.2357	1.453	1.730
0.1179	0.7630	1.904

6.2.1 Error estimation for Elliptic Equation

Existence of a weak solution

定理6.2.1 (Lax-Milgram)**.** Given a Hilbert space V, a continuous, coercive bilinear form $a(\cdot,\cdot)$ and a continuous linear functional $F \in V'$, there exists a unique $u \in V$ such that

$$a(u,v) = F(v) \qquad \forall v \in V.$$

Convergence

定理6.2.2 (Céa)**.** Suppose a Hilbert space V and V_h is a (closed) subspace of V, additionally, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive(not necessarily symmetric) on V. Then for the finite element variation problem (6.11) we have

$$\|u-u_h\|_V \leq \frac{C}{\alpha} \min_{v\in V_h} \|u-v\|_V,$$

where C is the continuity constant and α -dis the coercivity constant of $a(\cdot,\cdot)$.

It shows that u_h is quasi-optimal, and also a priori error estimation is derived.

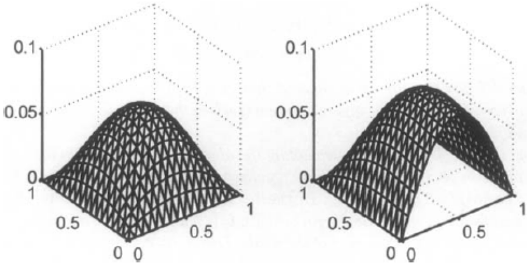
6.2.2 Higher Order Finite Elements

High Order Finite Elements (on Triangles) Continuous piecewise polynomial functions of degree d is straightforward on Lagrange triangles On each edge there must be $d+1$ nodes, and dimension of polynomial space in two variables with degree d:

$$1+2+\cdots+(d+1) = \frac{(d+1)(d+2)}{2}$$

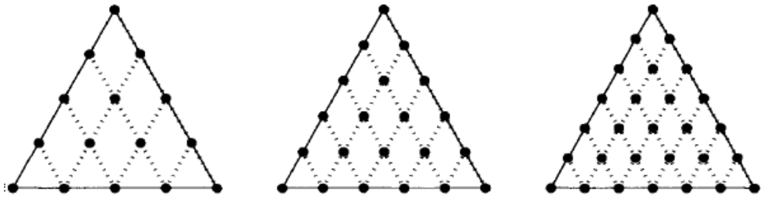
Domain with Curved Edges and Non-uniform Meshes

The errors of P^1 FE approximations for $u(x,y) = xy(1-x^2-y^2)$



h	$\|u-u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
$\frac{\sqrt{2}}{2}$	0.1128	—
$\frac{\sqrt{2}}{4}$	0.06275	1.797
$\frac{\sqrt{2}}{8}$	0.03232	1.941
$\frac{\sqrt{2}}{16}$	0.01629	1.985
$\frac{\sqrt{2}}{32}$	0.008159	1.996

Figure 6.2: The solution with Dirichlet conditions(Left) and its errors when using different size meshes(Right). The solution when Neumann conditions $\frac{\partial u}{\partial n} = 0$ are applied at only one side(Middle)

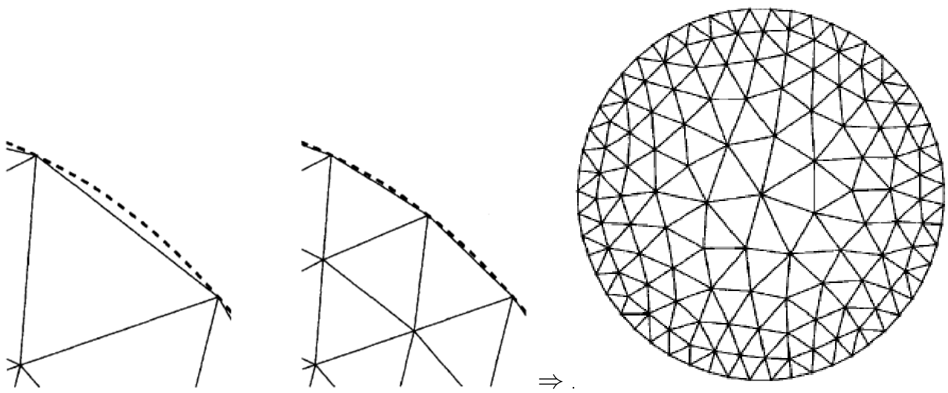


6.2.3 Adaptivity

FreeFem++是一个基于有限元方法，数值求解偏微分方程的免费软件，它是一个拥有自己高级编程语言的集成化产品。值得一提的是，它针对二维问题实现了网格重分功能，且算法的效率经过优化，在应用数学问题研究中具有一定的实用价值。http://www.freefem.org. Hecht, F. New development in freefem++. J. Numer. Math. 20 (2012), no. 3-4, 251–265.

6.3 Mixed and non-conforming FEMs

Exercises



k	*Error on* Ω_k	*Error on* $\Omega \setminus \Omega_k$	*Error on* Ω
1	$7.737 \cdot 10^{-2}$	$1.927 \cdot 10^{-1}$	$2.076 \cdot 10^{-1}$
2	$2.573 \cdot 10^{-2}$	$9.941 \cdot 10^{-2}$	$1.027 \cdot 10^{-1}$
3	$8.597 \cdot 10^{-3}$	$5.010 \cdot 10^{-2}$	$5.049 \cdot 10^{-2}$
4	$2.179 \cdot 10^{-3}$	$2.510 \cdot 10^{-2}$	$2.519 \cdot 10^{-2}$

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