Chapter 5

有限元方法

5.1 边值问题

5.1.1 两点边值问题

的两点边值问题形如

$$\boldsymbol{y}'=\boldsymbol{f}(t,\boldsymbol{y}),\qquad a<t<b,$$

满足边界条件

$$\boldsymbol{g}(\boldsymbol{y}(a),\boldsymbol{y}(b))=\boldsymbol{0},$$

其中 $\boldsymbol{f}:\mathbb{R}^{n+1}\to\mathbb{R}^n$,$\boldsymbol{g}:\mathbb{R}^{2n}\to\mathbb{R}^n$. 如果 $\boldsymbol{g}$ 的任意分量只涉及解在 a 或 b 处的值,并不同时涉及两者,称边界条件为独立的. 如果边界条件形如

$$\boldsymbol{B}_a\boldsymbol{y}(a)+\boldsymbol{B}_b\boldsymbol{y}(b)=\boldsymbol{c},$$

其中 $\boldsymbol{B}_a,\boldsymbol{B}_b\in\mathbb{R}^{n\times n}$,$\boldsymbol{c}\in\mathbb{R}^n$,称边界条件为线性的. 如果边界条件既独立又线性,则对每个 i,$1\leqslant i\leqslant n$,或者 $\boldsymbol{B}_a$ 的第 i 行或者 $\boldsymbol{B}_b$ 的第 i 行将仅有零元素. 如果常微分方程及边界条件都为线性,则称边值问题为线性的.

例5.1.1 (boundary problem)**.** The problem is given as:

$$u''=f(t,u,u'),a<t<b,$$

where the boundary condition is $u(a)=\alpha,u(b)=\beta$.

The choice of β will play an important role on the uniqueness and stability. For example, let $f(t,u,u')=-u$, $u(0)=0$ in above equation. In this sense, the exact solution of this problem is written as $u(t)=csint$, where c is any constant. It is easy to find out that there are infinite solutions of this boundary problem when $\beta=0$, and it is unsolvable when $\beta\neq 0$, if $b=n\pi$.

上题中β不能随意给，很容易造成存在性不成立！存在性和唯一性比IVP困难得多！举一例说明：

稳定性

例 10.3 存在性、惟一性 考虑两点边值问题

$$u''=-u, \qquad 0<t<b,$$

满足边界条件

$$u(0)=0, \qquad u(b)=\beta.$$

对这个常微分方程，满足 $u(0)=0$ 的通解为 $u(t)=c\sin t$，c 为任意常数. 如果 b 是 π 的整数倍，则对任意 c，$c\sin b=0$，所以当 $\beta=0$ 时，这个边值问题有无穷多个解，而若 $\beta\neq 0$，则无解. ■

接下来考虑扰动问题

$$\hat{\boldsymbol{y}}'=\boldsymbol{A}(t)\hat{\boldsymbol{y}}+\hat{\boldsymbol{b}}(t), \qquad a<t<b,$$

满足边界条件

$$\boldsymbol{B}_a\hat{\boldsymbol{y}}(a)+\boldsymbol{B}_b\hat{\boldsymbol{y}}(b)=\hat{\boldsymbol{c}}.$$

则解的扰动 $\boldsymbol{z}(t)=\hat{\boldsymbol{y}}(t)-\boldsymbol{y}(t)$，满足边值问题

$$\boldsymbol{z}'=\boldsymbol{A}(t)\boldsymbol{z}+\Delta\boldsymbol{b}(t), \qquad a<t<b,$$

边界条件为

$$\boldsymbol{B}_a\boldsymbol{y}(a)+\boldsymbol{B}_b\boldsymbol{y}(b)=\Delta\boldsymbol{c},$$

其中 $\Delta\boldsymbol{b}(t)=\hat{\boldsymbol{b}}(t)-\boldsymbol{b}(t)$，$\Delta\boldsymbol{c}=\hat{\boldsymbol{c}}-\boldsymbol{c}$. 由此可以得到解的扰动估计式

$$\|\boldsymbol{z}\|_\infty\leqslant\kappa\left(|\Delta\boldsymbol{c}|+\int_a^b|\Delta\boldsymbol{b}(s)|\,\mathrm{d}s\right).$$

这样，κ 是边值问题的关于常微分方程中非齐次项及边界条件扰动的绝对条件数.

5.1.2 边值问题常用数值方法

打靶法、有限差分法、配点法、Galerkin方法

1.打靶法

例5.1.2 (shooting). Using the method of shooting so solve the following boundary problem:

$$u''=-3u+2\cos t, u(0)=1, u(\pi)=-1,$$

2.有限差分法

例5.1.3 (finite differential method(FDM) 课本练习题10.7). Using the method of FDM so solve the following boundary problem:

$$u''=-3u+2\cos t, u(0)=1, u(\pi)=-1,$$

打靶法是将给定的边值问题用一系列的初值问题代替.

一阶两点边值问题

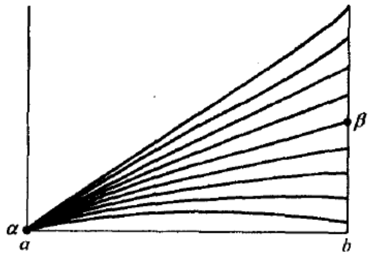
$$y'=f(t,y),\qquad a<t<b,$$

满足约束条件

$$g(y(a),y(b))=0,$$

这等价于非线性代数方程组

$$h(x)\equiv g(x,y(b;x))=0,$$



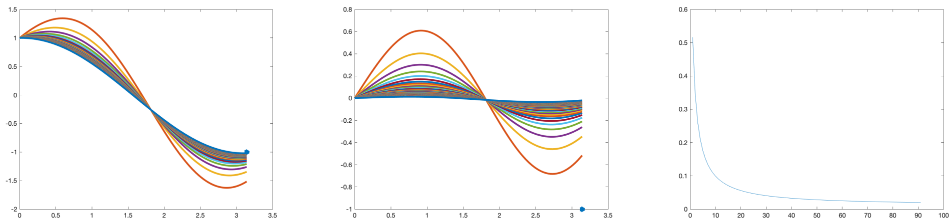


Figure 5.1: (Left)The solution of shooting method, (Middle) Errors, (Right) the objective function

的一维两点边值问题

$$u''=f(t,u,u'),\qquad a<t<b,$$

引入分割点 $t_i=a+ih, i=0,1,\cdots,n+1$,其中 $h=(b-a)/(n+1)$,希望求得的是解的近似值 $y_i\approx u(t_i), i=0,1,\cdots,n$. 我们已经知道的是 $y_0=u(a)=\alpha$ 及 $y_{n+1}=u(b)=\beta$. 接下来将常微分方程中的导数用有限差分近似代替(见 8.6.1 节),如

$$u'(t_i)\approx\frac{y_{i+1}-y_{i-1}}{2h}\quad 及\quad u''(t_i)\approx\frac{y_{i+1}-2y_i+y_{i-1}}{h^2},$$

由两点边值问题的有限差分法得到的代数方程组线性与否依赖于 f 关于 u 和 u' 是否线性. 如果方程组是非线性的,则求解时需要使用牛顿法之类的迭代法,这时需要在 $(a,\alpha),(b,\beta)$之间的直线上选择一个合理的值作为迭代的初始值. 在以上给出的例子中,每个方程都只含有三个相邻的未知数,说明线性方程组的系数矩阵或非线性方程组的雅可比矩阵是三对角的,因而,与一般的方程组相比既可以节省存储量,又可以节省工作量. 这种节约是有限差分法的特点:由于每个方程只与分割点附近的几个变量有关,所以这种方法产生的是一个稀疏矩阵.

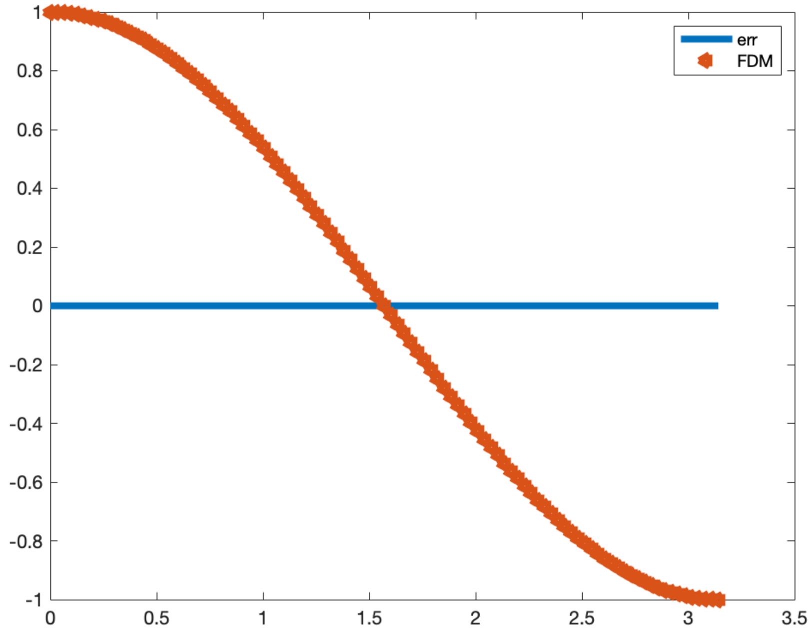


Figure 5.2: The FDM solution with its errors

3. 配点(Collocation)法

对于边界条件 $u(a)=\alpha, u(b)=\beta$ 下的一维两点边值问题

$$u''=f(t,u,u'),\qquad a<t<b,$$

我们要寻找形如

$$u(t)\approx v(t,\boldsymbol{x})=\sum_{i=1}^{n}x_i\phi_i(t)$$

的近似解，其中 $\phi_i(t)$ 是定义在 $[a,b]$ 上的基函数，$\boldsymbol{x}$ 是待定的 n 维参数向量.

为了确定参数向量 $\boldsymbol{x}$，我们定义 n 个点 $a<t_1<\cdots<t_n=b$，称为配置点

基函数后，在每个内部配置点上将近似解及其导数代入常微分方程，则得到

程组

$$v''(t_i,\boldsymbol{x})=f(t_i,v(t_i,\boldsymbol{x}),v'(t_i,\boldsymbol{x})),\qquad i=2,\cdots,n-1,$$

由边界条件，又得到两个附加的方程

$$v(t_1,\boldsymbol{x})=\alpha,\quad v(t_n,\boldsymbol{x})=\beta.$$

这样产生了含有 n 个未知数的 n 个方程. 这个方程组线性与否依赖于 f 关于 u 线性，求解这个方程组，确定出向量参数 $\boldsymbol{x}$，就得到了近似解函数 v.

例5.1.4 (collocation 课本练习题10.8). Using the method of collocation so solve the following boundary problem:

$$u''=-3u+2\cos t, u(0)=1, u(\pi)=-1,$$

4.Galerkin方法

例5.1.5 (Galerkin 课本练习题10.9). Using the method of FEM so solve the following boundary problem:

$$u''=-u+2\cos t, u(\frac{\pi}{2})=\frac{\pi}{2}, u(\frac{3\pi}{2})=-\frac{3\pi}{2},$$

5.1.3 高维问题差分方法

记$u_{i,j}$表示网格点(i,j)上待求函数的近似值，那么

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{-u_{i-1,j}+2u_{i,j}-u_{i+1,j}}{h_x^2}+O(h_x^2)\\ \frac{\partial^2 u}{\partial y^2} &= \frac{-u_{i,j-1}+2u_{i,j}-u_{i,j+1}}{h_y^2}+O(h_y^2).\end{aligned}$$

因此在二维情形（设$h_x=h_y=h$，舍去高阶无穷小项后）可得

$$-\triangle u\approx\frac{1}{h^2}\begin{pmatrix} & -u_{i,j+1} & \\ -u_{i-1,j} & 4u_{i,j} & -u_{i+1,j}\\ & -u_{i,j-1} & \end{pmatrix}\tag{5.1}$$

我们要寻找的仍是基函数的线性组合形式的近似解

$$u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t).$$

将近似解代入微分方程，并定义残量

$$r(t, \boldsymbol{x}) = v''(t, \boldsymbol{x}) - f(t) = \sum_{i=1}^{n} x_i \phi_i''(t) - f(t).$$

用最小二乘法，令函数

$$F(\boldsymbol{x}) = \frac{1}{2}\int_a^b r(t, \boldsymbol{x})^2 \mathrm{d}t$$

的梯度向量的每个分量都为零，从而使 $F(\boldsymbol{x})$ 极小化，即对 $i=1,\cdots,n$，取

$$\begin{aligned} 0 = \frac{\partial F}{\partial x_i} &= \int_a^b r(t, \boldsymbol{x}) \frac{\partial r}{\partial x_i} \mathrm{d}t = \int_a^b r(t, \boldsymbol{x}) \varphi_i''(t) \mathrm{d}t \\ &= \sum_{j=1}^{n} \left(\int_a^b \phi_j''(t) \phi_i''(t) \mathrm{d}t \right) x_j - \int_a^b f(t) \phi_i''(t) \mathrm{d}t, \end{aligned}$$

此式可看成为一个对称的线性代数方程组 $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$，其中

$$a_{ij} = \int_a^b \phi_j''(t) \phi_i''(t) \mathrm{d}t, b_i = \int_a^b f(t) \phi_i''(t) \mathrm{d}t,$$

方程组的解就是参数向量 $\boldsymbol{x}$，$\boldsymbol{A}$ 和 $\boldsymbol{b}$ 中的积分可以用解析或数值积分的方法

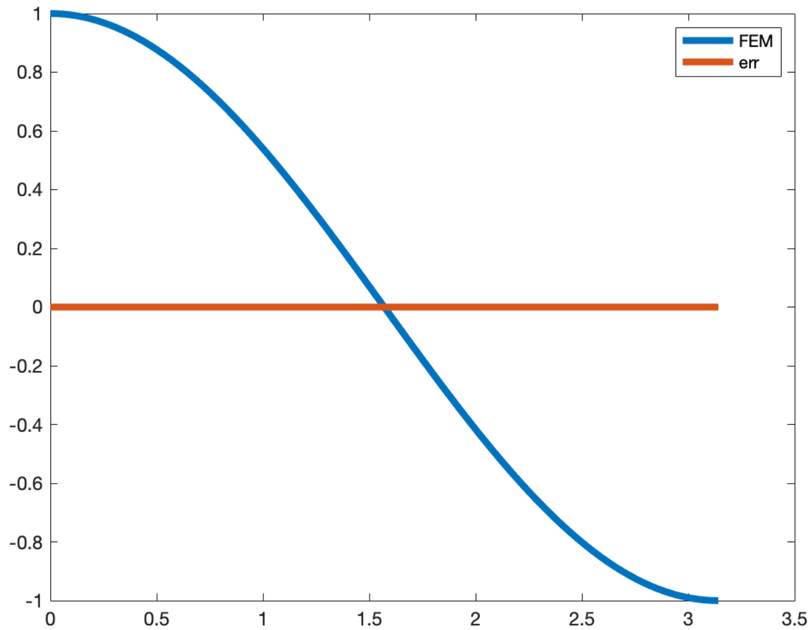


Figure 5.3: The FEM solution with its errors

稀疏Laplace算子-Matlab表示:上述稀疏Laplace算子在Matlab中的实现通常有两中不同的做法

- kron — 基于内部未知点($(n-1)\times(n-1)$个)；
- spLaplacian5.m — 基于全部点($(n+1)\times(n+1)$个)。

推荐采用后一种方案，代价是将边界条件也作为方程，优点是算子离散和边界条件的处理相对独立，因此数学上更直观。

例5.1.6. 考虑如下Poisson方程的Dirichlet边值问题

$$-\triangle u = f, \quad (x,y)\in[0,1]^2. \tag{5.2}$$

取解析函数

$$u(x,y)=\sin(\pi x)\sin(\pi y), \tag{5.3}$$

作为测试解，相应地取$f=2\pi^2 u(x,y)$。

实现细节参考testPoisson.m以及相关.m文件。

5.1.4 Semi-linear boundary value problem

考虑如下形式的"半"线性问题

$$-\triangle u + f(\mathbf{x},u) = g, \quad \mathbf{x}\in\Omega\subset\mathbf{R}^2 \tag{5.4}$$

仍取前例中的解析函数作为测试解，此外令

$$f(\mathbf{x},u)=u^3. \tag{5.5}$$

右端项g根据u和f的表达式可计算得到，参考fung.m

离散格式

可在所有内点$i,j=1,\cdots,n-1$上建立方程

$$\frac{1}{h^2}\begin{pmatrix} & -u_{i,j+1} & \\ -u_{i-1,j} & 4u_{i,j} & -u_{i+1,j} \\ & -u_{i,j-1} & \end{pmatrix} + f(x_i,y_j,u_{i,j}) = g_{i,j}.$$

如果f中不包含一阶微分算子(反应-扩散问题)，那么f(线性化后)只对$u_{i,j}$有贡献，若考虑带一阶微分算子项的f,称为对流-扩散问题。不妨将上述代数方程简记为

$$A\mathbf{u}+f(\mathbf{u})=\mathbf{g} \tag{5.6}$$

Newton迭代格式

令$F(\mathbf{u})=A\mathbf{u}+f(\mathbf{u})-\mathbf{g}$ 则有Newton迭代

$$\mathbf{u}^{new}=\mathbf{u}^{old}-F'(\mathbf{u}^{old})^{-1}F(\mathbf{u}^{old}) \tag{5.7}$$

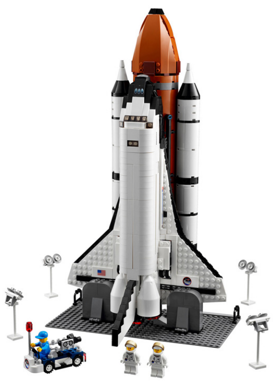
其中$F'(\mathbf{u})=A+f'(\mathbf{u})$. Matlab实现的时候会引进一些额外的变量以及向量化，请参考test_semi_newton.m

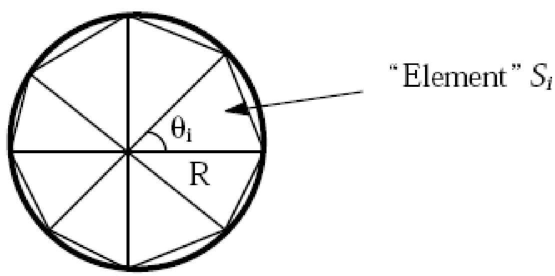
例5.1.7 (练习). • 修改算例一(用不同u)，记录$h = 10, 20, 40, 80, 160, 320$时数值解的$L_2$或$L_\infty$误差，计算收敛阶

- 修改半线性问题算例，计算Newton迭代法的收敛阶
- 尝试将上述方法扩展至x-和y-方向的区间和步长不一致的情形：$h_x \neq h_y$以及$(x, y) \in [a, b] \times [c, d]$
- 尝试变更方程编号的顺序（列优先或其他感兴趣的顺序）
- 进一步研究如何利用Newton迭代法求逆矩阵，以及其他可能加速求解线性方程足的方法

5.2 椭圆方程的有限元方法

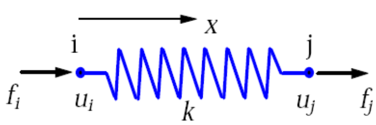
A more mathematical illustration of piecewise. Approximation of the area of a circle S with $N-$polygon where the area(S_i)= $\frac{1}{2}R^2\sin(\theta_i)$. A further formulation is $S = \lim_{N\to\infty}\sum_{i=1}^{N} S_i =$





,

$\frac{R^2}{2}\lim_{N\to\infty} N\sin\left(\frac{2\pi}{N}\right)$, which sum up to πR^2.



例5.2.1 (Formulation of a Spring System). To keep the equilibrium, additional fores is required at node i and j

$$
\begin{aligned}
f_i = -F = -k(u_j - u_i) &= ku_i - ku_j \\
f_j = F = k(u_j - u_i) &= -ku_i + ku_j
\end{aligned}
$$

In a matrix form(known as the stiff matrix of the spring)

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} f_i \\ f_j \end{bmatrix} \tag{5.8}$$

The finite element methods are widely applied in numerically solving partial differential equaitons. As an example, let us considering the simplest form of Poisson's equation

$$\begin{cases} -(u_{xx} + u_{yy}) &=& f(x,y), & (x,y) \in \Omega \\ u(x,y) &=& u_b(x,y), & (x,y) \in \partial\Omega \end{cases}$$

The purpose of this notes is Understand the fundamental ideas of the FEM， and Know the behavior and usage of classical finite elements.Solve elliptic boundary value problem with FEM routines, and interpret and evaluate the quality of the results. On the other hand, it is also essential to understand the limitations of the FEM, so that classical results on theoretical analysis are also presented.

Variation formulations

Find $u(x,y) \in H^1(\Omega)$, such that $u(x,y) = u_b(x,y)$ at $\partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Omega f v d\Omega, \quad \forall v(x,y) \in H_0^1(\Omega), \tag{5.9}$$

and also referred as $a(u,v) = F(v)$ briefly. Thed different variational frameworks for BVP is actually the Minimum of the energy variational. Other Variational methods are also popular for various applications, such as the Galerkin methods and the Ritz methods, which are all equivallent in the case of Poisson's e quaiton.

The finite element formulations

Find $u_h(x,y) \in H^1(\Omega) \cap V_h(\triangle)$, such that $u_h(x,y) = u_b(x,y)$ at the boundary of the domain and

$$\int_\triangle \nabla u_h \cdot \nabla v_h dxdy = \int_\triangle f v_h dxdy, \quad \forall v_h(x,y) \in H_0^1(\Omega) \cap V_h(\triangle). \tag{5.10}$$

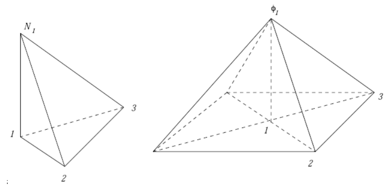
Remark: This is a discretized version of the weak formulations.

The Finite Element(FE) Space $V_h(\triangle)$

The key feature of finite element approximation is to construct the approximating space $V_h(\triangle)$ on the given mesh $\triangle$ with N nodes. A carefully chosen basis function set $\{\phi_i(x,y)\}_{i=1}^N$ is possible to approximate any desired function $u(x,y)$ by

$$u(x,y) \approx u_h(x,y) = \sum_{i=1}^N u_i \phi_i(x,y), \quad \forall (x,y) \in \triangle.$$

In practical calculation, the nodal basis function is preferred for simplicity.



例5.2.2 (P^1 - linear shape functions)**.** Denote $N_j(x,y), \forall j = 1,2,3.$. On element $\triangle^e \in \triangle$ with vertices $\langle v_i(x_i,y_i), v_j(x_j,y_j), v_k(x_k,y_k)\rangle$, The linear shape function $N_i(x,y) := \alpha_i + \beta_i x + \gamma_i y, (x,y) \in \triangle^e$ associated with vertex i satisfies

$$N_i(x_i,y_i) = 1, \quad N_i(x_j,y_j) = 0, \quad N_i(x_k,y_k) = 0.$$

The above 3×3 linear system gives

$$\alpha_i = \frac{x_j y_k - x_k y_j}{2|\triangle^e|}, \quad \beta_i = \frac{y_j - y_k}{2|\triangle^e|}, \quad \gamma_i = \frac{x_k - x_j}{2|\triangle^e|}$$

For example

$$N_2(x,y) = \frac{1}{2|\triangle^e|}\big(x_3y_1 - x_1y_3\big) + \big(y_3 - y_1\big)x + \big(x_1 - x_3\big)y.$$

例5.2.3 (P^2 - quadratic shape functions)**.** Consider a second order approximation on given element $\triangle^e$. Let

$$\lambda_1 = \frac{\begin{vmatrix} 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_2 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & \mathbf{x} & \mathbf{y} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}},$$

then it is straightforward to prove that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and

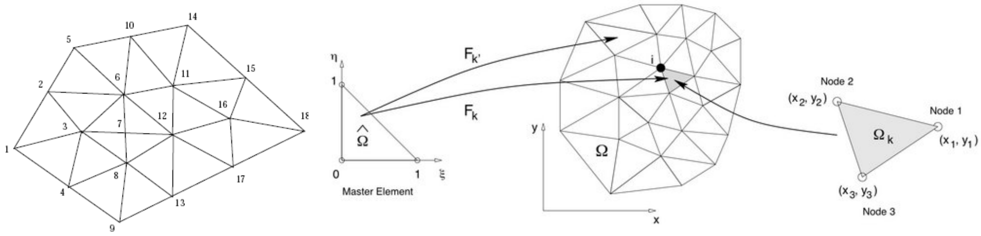
$$P^2(\triangle^e) = span\{\lambda_1^2, \lambda_2^2, \lambda_3^2, 4\lambda_1\lambda_2, 4\lambda_1\lambda_3, 4\lambda_2\lambda_3\},$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is referred as the *barycentric coordinate*. Let us leave the P^3 - cubic shape functions cases as the exercises.

Abstract definition for $V_h(\triangle)$

A Finite Element space can be described by

1. Finite Element Mesh $\triangle$
2. Degree Of Freedoms Σ
3. Master Element $\hat{\Omega}$
4. Affine Mapping F_k



Ciarlet's definition of a ***finite element***$(K, \mathcal{P}, \Sigma)$

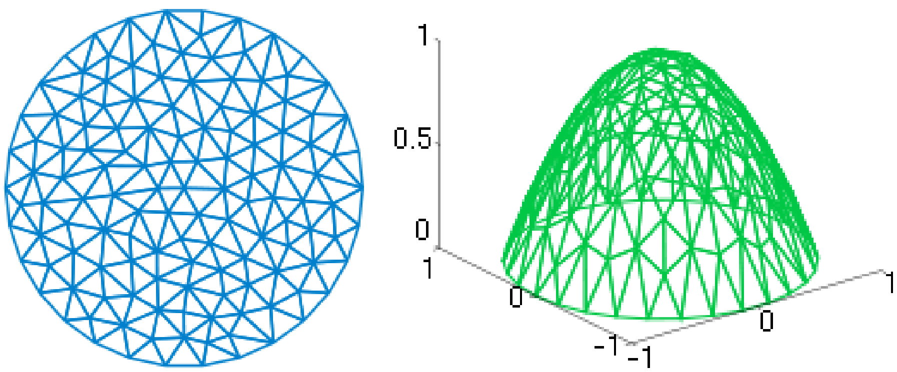
An alternative mathematical definition for $V_h(\triangle)$.

1. *element domain*: $K \subset \mathcal{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary
2. space of *shape functions*: $\mathcal{P}$ be a finite-dimensional space of functions on K
3. *degree of freedoms*: $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\}$ be a basis for $\mathcal{P}'$

P. G. Ciarlet: The Finite Element Method for Elliptic Equation, 1978.

例5.2.4. Interpolation

$$u(x,y) = 1 - x^2 - y^2, \quad (x,y) \in x^2 + y^2 < 1.$$



Apply Boundary Conditions

To close linear system, boundary should be applied. In general

稀疏性

The discrete variational problem results in a sparse linear system

$$K\mathbf{u} = \mathbf{f}$$

- Dirichlet BC on the entire boundary, *i.e.*, $u(x,y)|_{\partial\Omega} = u_0(x,y)$ is given.
- Neumann BC on the entire boundary, *i.e.*, $\partial u/\partial n|_{\partial\Omega} = g(x,y)$ is given.
  In this case, the solution to a Poisson equation may not be unique or even exist, depending upon whether a compatibility condition is satisfied. Integrating the Poisson equation over the domain, we have

$$\iint_\Omega f dxdy = \iint_\Omega \Delta u \, dxdy = \iint_\Omega \nabla \cdot \nabla u \, dxdy = \int_{\partial\Omega} u_n \, ds = \int_{\partial\Omega} g(x,y) \, ds = 0 \, ,$$

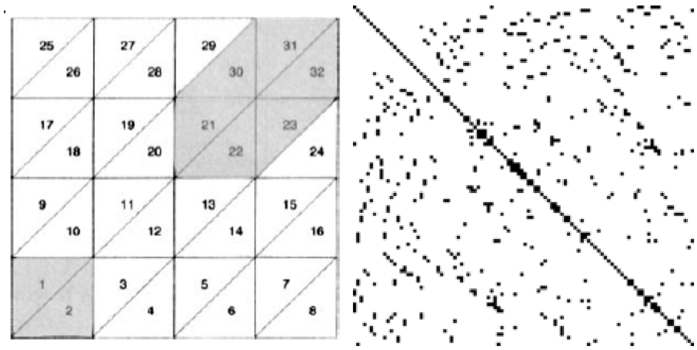
  which is the compatibility condition to be satisfied for the solution to exist. If a solution does exist, it is not unique as it is determined within an arbitrary constant.
- Mixed BC on the entire boundary, *i.e.*,

$$\alpha(x,y)u(x,y) + \beta(x,y)\frac{\partial u}{\partial n} = \gamma(x,y)$$

  is given, where $\alpha(x,y)$, $\beta(x,y)$, and $\gamma(x,y)$ are known functions.
- Dirichlet, Neumann, and Mixed BC on some parts of the boundary.

using P^1-FE space with N vertex(degree of freedoms), where the stiffness matrix K is $N \times N$ and its entries

$$K_{i,j} = a(\phi_i, \phi_j) := \int_\Omega \nabla\phi_i \cdot \nabla\phi_j, \quad \forall i,j = 1,2,\ldots,N.$$



Assemble

a element-by-element technique

例5.2.5. In this example, $u(x,y) = x/(x^2+y^2)$ is known at $\partial\Omega$ for the Dirichlet BVP. A sequence of meshes with different mesh size used.

It is easy to obtain the error of the FE approximation u_h

例5.2.6. In the second example, we consider more general form BVP

$$\nabla \cdot (\kappa(x,y)\nabla u) = f(x,y) \qquad in \quad \Omega,$$

where the coefficient $\kappa(x,y) = 1 + xy^2$ is nonconstant. The analytic solution $u(x,y) = xy(1-x)(1-y)$ is considered as a test problem. In this case, u is obviously zero at the boundary and

$$f(x,y) = -y^3 + y^4 + 4y^3x - 4y^4x + 2y - 2y^2 - 2x^2y + 6x^2y^2 + 2x^3y - 6x^3y^2 + 2x - 2x^2.$$

$$
\left[\begin{matrix} 1500 & 0 & -1500 & 0 \\ 0 & 0 & 0 & 0 \\ -1500 & 0 & 1500 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right]\begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}
\qquad
\left[\begin{matrix} 1500 & 0 & -1500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right]\begin{matrix} \\ \\ 3 \\ 4 \\ 5 \\ 6 \\ \\ \\ \end{matrix}
\quad (25.
$$

Element (3) joins nodes 1 and 4. Its EFT is { `1,2,7,8` }. Matrices $\mathbf{K}^{(3)}$ and $\mathbf{K}$ upon merge are

$$
\left[\begin{matrix} 768 & -576 & -768 & 576 \\ -576 & 432 & 576 & -432 \\ -768 & 576 & 768 & -576 \\ 576 & -432 & -576 & 432 \end{matrix}\right]\begin{matrix} 1 \\ 2 \\ 7 \\ 8 \end{matrix}
\qquad
\left[\begin{matrix} 2268 & -576 & -1500 & 0 & 0 & 0 & -768 & 576 \\ -576 & 432 & 0 & 0 & 0 & 0 & 576 & -432 \\ -1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & 576 & 0 & 0 & 0 & 0 & 768 & -576 \\ 576 & -432 & 0 & 0 & 0 & 0 & -576 & 432 \end{matrix}\right]\begin{matrix} 1 \\ 2 \\ \\ \\ \\ \\ 7 \\ 8 \end{matrix}
$$

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Parallel Realization of the Element-by-Element FEM Technique by CUDA

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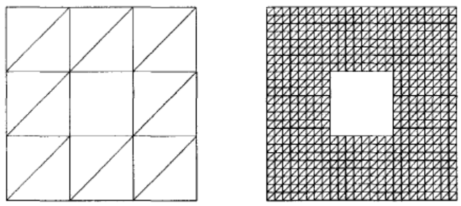
The utilization of Graphical Processing Units (GPUs) for the element-by-element (EbE) finite element method (FEM) is demonstrated. EbE FEM is a long known technique, by which a conjugate gradient (CG) type iterative solution scheme can be entirely decomposed into computations on the element level, i.e., without assembling the global system matrix. In our implementation NVIDIA's parallel computing solution, the Compute Unified Device Architecture (CUDA) is used to perform the required element-wise computations in parallel. Since element matrices need not be stored, the memory requirement can be kept extremely low. It is shown that this low-storage but computation-intensive technique is better suited for GPUs than those requiring the massive manipulation of large data sets.



5.2.1 Error estimation for Elliptic Equation

Existence of a weak solution

定理5.2.1 (Lax-Milgram)**.** Given a Hilbert space V, a continuous, coercive bilinear form $a(\cdot,\cdot)$ and a continuous linear functional $F \in V'$, there exists a unique $u \in V$ such that

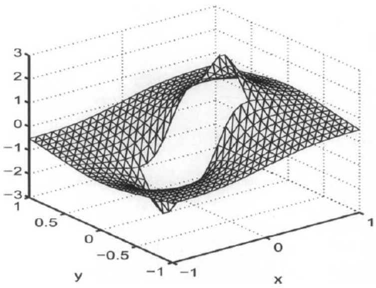
$$a(u,v) = F(v) \qquad \forall v \in V.$$

Convergence

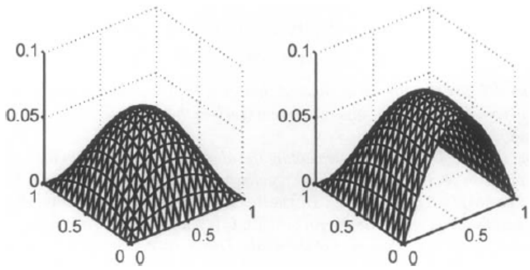
定理5.2.2 (Céa)**.** Suppose a Hilbert space V and V_h is a (closed) subspace of V, additionally, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive(not necessarily symmetric) on V. Then for the finite element variation problem (5.10) we have

$$\|u - u_h\|_V \le \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V,$$

where C is the continuity constant and α is the coercivity constant of $a(\cdot,\cdot)$.



h	$\|u - u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
0.9428	3.592	—
0.4714	2.513	1.429
0.2357	1.453	1.730
0.1179	0.7630	1.904



h	$\|u - u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
$\frac{\sqrt{2}}{2}$	0.1128	—
$\frac{\sqrt{2}}{4}$	0.06275	1.797
$\frac{\sqrt{2}}{8}$	0.03232	1.941
$\frac{\sqrt{2}}{16}$	0.01629	1.985
$\frac{\sqrt{2}}{32}$	0.008159	1.996

Figure 5.4: The solution with Dirichlet conditions(Left) and its errors when using different size meshes(Right). The solution when Neumann conditions $\frac{\partial u}{\partial n} = 0$ are applied at only one side(Middle)

It shows that u_h is quasi-optimal, and also a priori error estimation is derived.

5.2.2 Higher Order Finite Elements

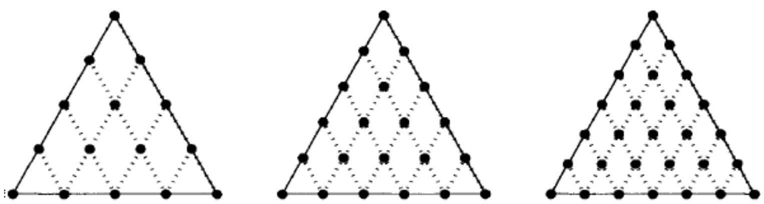
High Order Finite Elements (on Triangles) Continuous piecewise polynomial functions of degree d is straightforward on Lagrange triangles On each edge there must be $d+1$ nodes, and dimension of polynomial space in two variables with degree d:

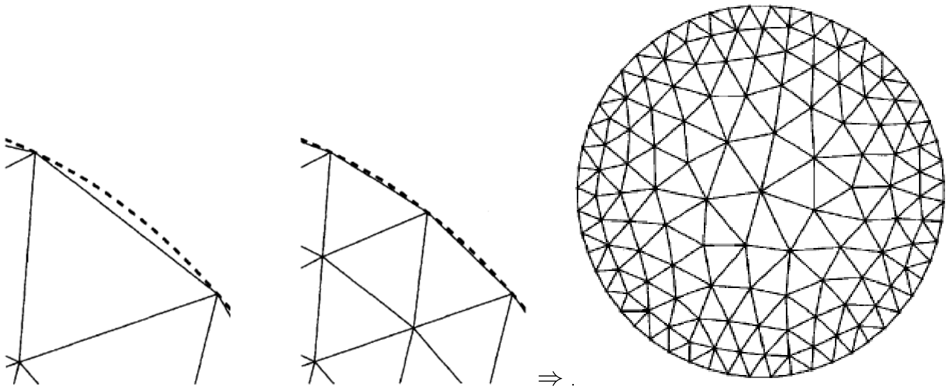
$$1 + 2 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2}$$

Domain with Curved Edges and Non-uniform Meshes

The errors of P^1 FE approximations for $u(x, y) = xy(1 - x^2 - y^2)$

5.3 Mesh Adaptivity





k	*Error on* Ω_k	*Error on* $\Omega \setminus \Omega_k$	*Error on* Ω
1	$7.737 \cdot 10^{-2}$	$1.927 \cdot 10^{-1}$	$2.076 \cdot 10^{-1}$
2	$2.573 \cdot 10^{-2}$	$9.941 \cdot 10^{-2}$	$1.027 \cdot 10^{-1}$
3	$8.597 \cdot 10^{-3}$	$5.010 \cdot 10^{-2}$	$5.049 \cdot 10^{-2}$
4	$2.179 \cdot 10^{-3}$	$2.510 \cdot 10^{-2}$	$2.519 \cdot 10^{-2}$

软件包与参考文献

FEM book

1. O .C. Zienkiewicz and K. Morgan: Finite Elements and Approximation, 1983; The Finite Element Method, 6th ed.

2. S. C. Brenner and L. R. Scott: The mathematical theory of finite element methods, 3rd ed.

3. Mark. S. Gockenbach: Understanding And Implementing the Finite Element Method

4. H. Elman, etc.: Finite Elements and Fast Iteraive Solvers - with applications in incompressible fluid dynamics

5. Jian-Ming Jin: The Finite Element Method in Electromagnetics, 2nd ed.

Open Source Packages

- deal.II - http://www.dealii.org
- libmesh - http://libmesh.sourceforge.net
- AFEPack - http://dsec.pku.edu.cn/ rli/software.php

- openDX/paraview (Visualize scientific data in 3D)
- gmsh (3D Mesh generator and visualize FE solution) - http://geuz.org/gmsh

- tetgen (Tetrahedron mesh generator) - http://wias-berlin.de/software/tetgen
- ...

Bibliography

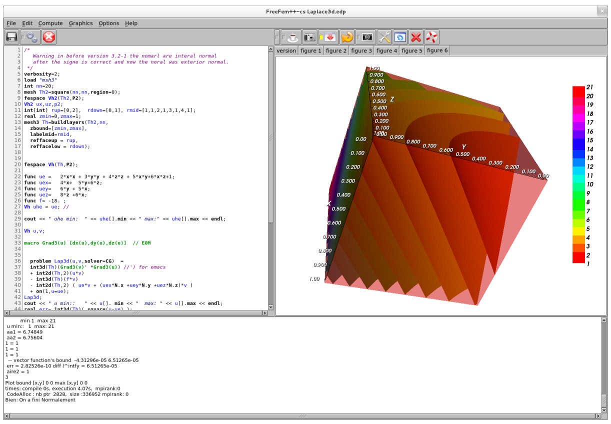
[1] Toader A Allaire G, Jouve F. Structural optimization using sensitivity analysis and a level-set method. *Journal of Computational Physics*, 2004.

[2] Francois Murat and Jacques Simon. *Etude de problemes d'optimal design*. Springer Berlin Heidelberg, 1976.

[3] J. Simon. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis Optimization*, 2(7):649–687, 2010.

FreeFEM++

FreeFem++是一个基于有限元方法，数值求解偏微分方程的免费软件，它是一个拥有自己高级编程语言的集成化产品。http://www.freefem.org. Hecht, F. New development in freefem++. J. Numer. Math. 20 (2012), no. 3-4, 251–265.



Exercises