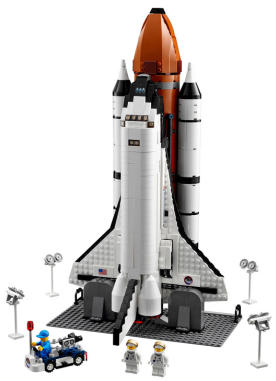
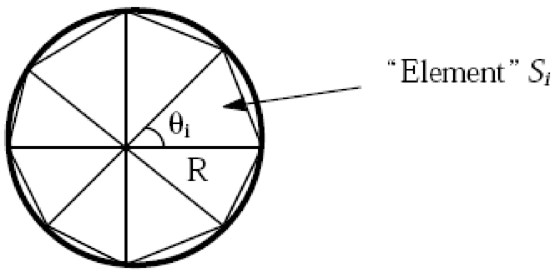
Chapter 5

有限元方法

5.1 椭圆方程的有限元方法

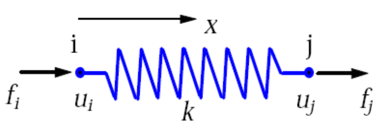
A more mathematical illustration of piecewise. Approximation of the area of a circle S with $N-$polygon where the area(S_i)= $\frac{1}{2}R^2\sin(\theta_i)$. A further formulation is $S = \lim_{N\to\infty}\sum_{i=1}^{N} S_i =$





,

$\frac{R^2}{2}\lim_{N\to\infty} N\sin\left(\frac{2\pi}{N}\right)$, which sum up to πR^2.



例5.1.1 (Formulation of a Spring System)**.** To keep the equilibrium, additional fores is required at node i and j

$$\begin{aligned} f_i = -F = -k(u_j - u_i) &= ku_i - ku_j \\ f_j = F = k(u_j - u_i) &= -ku_i + ku_j \end{aligned}$$

In a matrix form(known as the stiff matrix of the spring)

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} f_i \\ f_j \end{bmatrix} \tag{5.1}$$

The finite element methods are widely applied in numerically solving partial differential equaitons. As an example, let us considering the simplest form of Poisson's equation

$$\begin{cases} -(u_{xx} + u_{yy}) = f(x,y), & (x,y) \in \Omega \\ u(x,y) = u_b(x,y), & (x,y) \in \partial\Omega \end{cases}$$

The purpose of this notes is Understand the fundamental ideas of the FEM, and Know the behavior and usage of classical finite elements.Solve elliptic boundary value problem with FEM routines, and interpret and evaluate the quality of the results. On the other hand, it is also essential to understand the limitations of the FEM, so that classical results on theoretical analysis are also presented.

Variation formulations

Find $u(x,y) \in H^1(\Omega)$, such that $u(x,y) = u_b(x,y)$ at $\partial\Omega$ and

$$\int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Omega f v d\Omega, \quad \forall v(x,y) \in H_0^1(\Omega), \tag{5.2}$$

and also referred as $a(u,v) = F(v)$ briefly. Thed different variational frameworks for BVP is actually the Minimum of the energy variational. Other Variational methods are also popular for various applications, such as the Galerkin methods and the Ritz methods, which are all equivallent in the case of Poisson's e quaiton.

The finite element formulations

Find $u_h(x,y) \in H^1(\Omega) \cap V_h(\triangle)$, such that $u_h(x,y) = u_b(x,y)$ at the boundary of the domain and

$$\int_\triangle \nabla u_h \cdot \nabla v_h dxdy = \int_\triangle f v_h dxdy, \quad \forall v_h(x,y) \in H_0^1(\Omega) \cap V_h(\triangle). \tag{5.3}$$

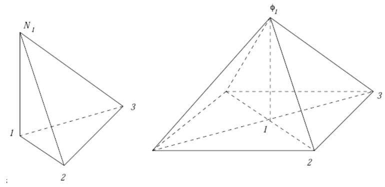
Remark: This is a discretized version of the weak formulations.

The Finite Element(FE) Space $V_h(\triangle)$

The key feature of finite element approximation is to construct the approximating space $V_h(\triangle)$ on the given mesh $\triangle$ with N nodes. A carefully chosen basis function set $\{\phi_i(x,y)\}_{i=1}^N$ is possible to approximate any desired function $u(x,y)$ by

$$u(x,y) \approx u_h(x,y) = \sum_{i=1}^N u_i \phi_i(x,y), \quad \forall (x,y) \in \triangle.$$

In practical calculation, the nodal basis function is preferred for simplicity.



例5.1.2 (P^1 - linear shape functions)**.** Denote $N_j(x,y), \forall j = 1,2,3.$. On element $\triangle^e \in \triangle$ with vertices $\langle v_i(x_i,y_i), v_j(x_j,y_j), v_k(x_k,y_k)\rangle$, The linear shape function $N_i(x,y) := \alpha_i + \beta_i x + \gamma_i y, (x,y) \in \triangle^e$ associated with vertex i satisfies

$$N_i(x_i,y_i) = 1, \quad N_i(x_j,y_j) = 0, \quad N_i(x_k,y_k) = 0.$$

The above 3×3 linear system gives

$$\alpha_i = \frac{x_j y_k - x_k y_j}{2|\triangle^e|}, \quad \beta_i = \frac{y_j - y_k}{2|\triangle^e|}, \quad \gamma_i = \frac{x_k - x_j}{2|\triangle^e|}$$

For example

$$N_2(x,y) = \frac{1}{2|\triangle^e|}\big(x_3 y_1 - x_1 y_3\big) + \big(y_3 - y_1\big)x + \big(x_1 - x_3\big)y.$$

例5.1.3 (P^2 - quadratic shape functions)**.** Consider a second order approximation on given element $\triangle^e$. Let

$$\lambda_1 = \frac{\begin{vmatrix} 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_2 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & \mathbf{x} & \mathbf{y} \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}, \lambda_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & \mathbf{x} & \mathbf{y} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}},$$

then it is straightforward to prove that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and

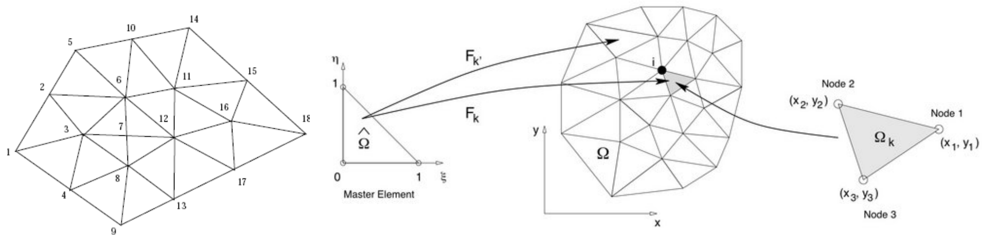
$$P^2(\triangle^e) = span\{\lambda_1^2, \lambda_2^2, \lambda_3^2, 4\lambda_1\lambda_2, 4\lambda_1\lambda_3, 4\lambda_2\lambda_3\},$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is referred as the *barycentric coordinate*. Let us leave the P^3 - cubic shape functions cases as the exercises.

Abstract definition for $V_h(\triangle)$

A Finite Element space can be described by

1. Finite Element Mesh $\triangle$
2. Degree Of Freedoms Σ
3. Master Element $\hat{\Omega}$
4. Affine Mapping F_k



Ciarlet's definition of a ***finite element*** $(K, \mathcal{P}, \Sigma)$

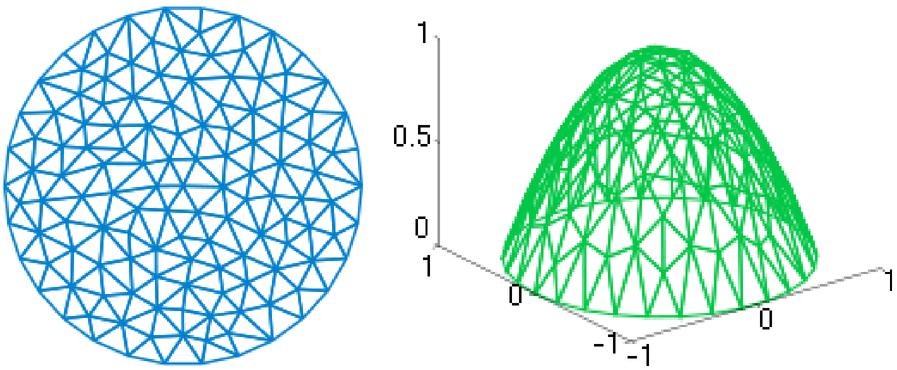
An alternative mathematical definition for $V_h(\triangle)$.

1. *element domain*: $K \subset \mathcal{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary
2. space of *shape functions*: $\mathcal{P}$ be a finite-dimensional space of functions on K
3. *degree of freedoms*: $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\}$ be a basis for $\mathcal{P}'$

P. G. Ciarlet: The Finite Element Method for Elliptic Equation, 1978.

例5.1.4. Interpolation

$$u(x, y) = 1 - x^2 - y^2, \quad (x, y) \in x^2 + y^2 < 1.$$



Apply Boundary Conditions

To close linear system, boundary should be applied. In general

稀疏性

The discrete variational problem results in a sparse linear system

$$K\mathbf{u} = \mathbf{f}$$

- Dirichlet BC on the entire boundary, *i.e.*, $u(x,y)|_{\partial\Omega} = u_0(x,y)$ is given.
- Neumann BC on the entire boundary, *i.e.*, $\partial u/\partial n|_{\partial\Omega} = g(x,y)$ is given.
  In this case, the solution to a Poisson equation may not be unique or even exist, depending upon whether a compatibility condition is satisfied. Integrating the Poisson equation over the domain, we have

  $$\iint_{\Omega} f dxdy = \iint_{\Omega} \Delta u \, dxdy = \iint_{\Omega} \nabla \cdot \nabla u \, dxdy = \int_{\partial\Omega} u_n \, ds = \int_{\partial\Omega} g(x,y) \, ds = 0 \,,$$

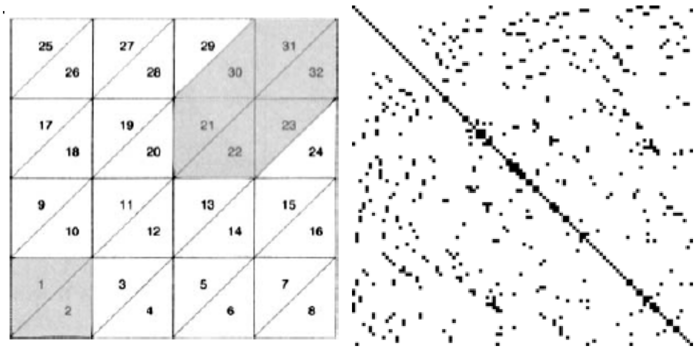
  which is the compatibility condition to be satisfied for the solution to exist. If a solution does exist, it is not unique as it is determined within an arbitrary constant.
- Mixed BC on the entire boundary, *i.e.*,

  $$\alpha(x,y)u(x,y) + \beta(x,y)\frac{\partial u}{\partial n} = \gamma(x,y)$$

  is given, where $\alpha(x,y)$, $\beta(x,y)$, and $\gamma(x,y)$ are known functions.
- Dirichlet, Neumann, and Mixed BC on some parts of the boundary.

using P^1-FE space with N vertex(degree of freedoms), where the stiffness matrix K is $N \times N$ and its entries

$$K_{i,j} = a(\phi_i, \phi_j) := \int_{\Omega} \nabla\phi_i \cdot \nabla\phi_j, \quad \forall i, j = 1, 2, \ldots, N.$$



Assemble

a element-by-element technique

例5.1.5. In this example, $u(x,y) = x/(x^2+y^2)$ is known at $\partial\Omega$ for the Dirichlet BVP. A sequence of meshes with different mesh size used.

It is easy to obtain the error of the FE approximation u_h

例5.1.6. In the second example, we consider more general form BVP

$$\nabla \cdot (\kappa(x,y)\nabla u) = f(x,y) \qquad in \quad \Omega,$$

$$
\begin{array}{c}
\begin{bmatrix}
1500 & 0 & -1500 & 0 \\
0 & 0 & 0 & 0 \\
-1500 & 0 & 1500 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}
\qquad
\begin{bmatrix}
1500 & 0 & -1500 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{array}{c} \\ \\ 3 \\ 4 \\ 5 \\ 6 \\ \\ \\ \end{array}
\quad (25.
$$

Element (3) joins nodes 1 and 4. Its EFT is { 1,2,7,8 }. Matrices $\mathbf{K}^{(3)}$ and $\mathbf{K}$ upon merge are

$$
\begin{bmatrix}
768 & -576 & -768 & 576 \\
-576 & 432 & 576 & -432 \\
-768 & 576 & 768 & -576 \\
576 & -432 & -576 & 432
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}
\qquad
\begin{bmatrix}
2268 & -576 & -1500 & 0 & 0 & 0 & -768 & 576 \\
-576 & 432 & 0 & 0 & 0 & 0 & 576 & -432 \\
-1500 & 0 & 3000 & 0 & -1500 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1500 & 0 & 1500 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-768 & 576 & 0 & 0 & 0 & 0 & 768 & -576 \\
576 & -432 & 0 & 0 & 0 & 0 & -576 & 432
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ \\ \\ \\ \\ 7 \\ 8 \end{array}
$$

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Parallel Realization of the Element-by-Element FEM Technique by CUDA

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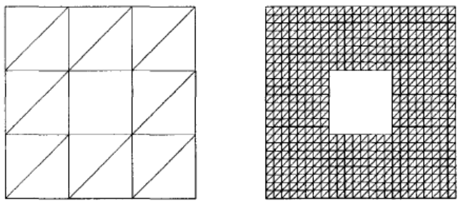
The utilization of Graphical Processing Units (GPUs) for the element-by-element (EbE) finite element method (FEM) is demonstrated. EbE FEM is a long known technique, by which a conjugate gradient (CG) type iterative solution scheme can be entirely decomposed into computations on the element level, i.e., without assembling the global system matrix. In our implementation NVIDIA's parallel computing solution, the Compute Unified Device Architecture (CUDA) is used to perform the required element-wise computations in parallel. Since element matrices need not be stored, the memory requirement can be kept extremely low. It is shown that this low-storage but computation-intensive technique is better suited for GPUs than those requiring the massive manipulation of large data sets.



where the coefficient $\kappa(x, y) = 1 + xy^2$ is nonconstant. The analytic solution $u(x, y) = xy(1-x)(1-y)$ is considered as a test problem. In this case, u is obviously zero at the boundary and

$$f(x, y) = -y^3 + y^4 + 4y^3x - 4y^4x + 2y - 2y^2 - 2x^2y + 6x^2y^2 + 2x^3y - 6x^3y^2 + 2x - 2x^2.$$

5.1.1 Analysis

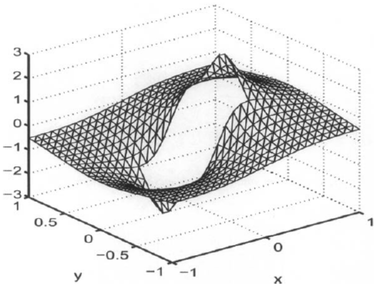
Existence of a weak solution

定理5.1.1 (Lax-Milgram)**.** Given a Hilbert space V, a continuous, coercive bilinear form $a(\cdot, \cdot)$ and a continuous linear functional $F \in V'$, there exists a unique $u \in V$ such that

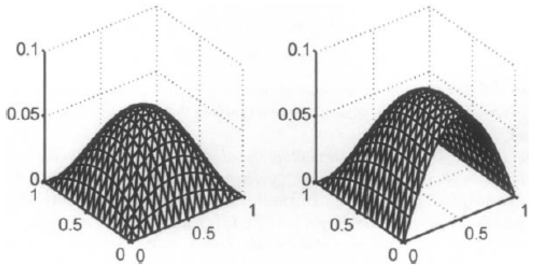
$$a(u, v) = F(v) \qquad \forall v \in V.$$

收敛性

定理5.1.2 (Céa)**.** Suppose a Hilbert space V and V_h is a (closed) subspace of V, additionally, the bilinear form $a(\cdot, \cdot)$ is continuous and coercive(not necessarily symmetric) on V. Then



h	$\|u-u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
0.9428	3.592	—
0.4714	2.513	1.429
0.2357	1.453	1.730
0.1179	0.7630	1.904



h	$\|u-u_h\|_E$	$\frac{\|u-u_{2h}\|_E}{\|u-u_h\|_E}$
$\frac{\sqrt{2}}{2}$	0.1128	—
$\frac{\sqrt{2}}{4}$	0.06275	1.797
$\frac{\sqrt{2}}{8}$	0.03232	1.941
$\frac{\sqrt{2}}{16}$	0.01629	1.985
$\frac{\sqrt{2}}{32}$	0.008159	1.996

Figure 5.1: The solution with Dirichlet conditions(Left) and its errors when using different size meshes(Right). The solution when Neumann conditions $\frac{\partial u}{\partial n} = 0$ are applied at only one side(Middle)

for the finite element variation problem (5.3) we have

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V,$$

where C is the continuity constant and α is the coercivity constant of $a(\cdot, \cdot)$.

It shows that u_h is quasi-optimal, and also a priori error estimation is derived.

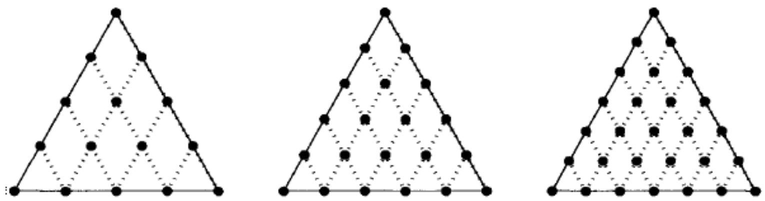
高阶方法

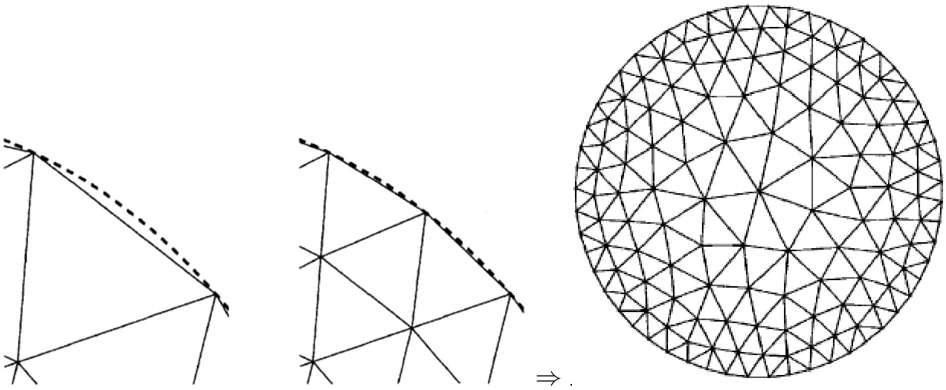
High Order Finite Elements (on Triangles) Continuous piecewise polynomial functions of degree d is straightforward on Lagrange triangles On each edge there must be $d+1$ nodes, and dimension of polynomial space in two variables with degree d:

$$1 + 2 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2}$$

Domain with Curved Edges and Non-uniform Meshes

The errors of P^1 FE approximations for $u(x,y) = xy(1 - x^2 - y^2)$





k	*Error on* Ω_k	*Error on* $\Omega \setminus \Omega_k$	*Error on* Ω
1	$7.737 \cdot 10^{-2}$	$1.927 \cdot 10^{-1}$	$2.076 \cdot 10^{-1}$
2	$2.573 \cdot 10^{-2}$	$9.941 \cdot 10^{-2}$	$1.027 \cdot 10^{-1}$
3	$8.597 \cdot 10^{-3}$	$5.010 \cdot 10^{-2}$	$5.049 \cdot 10^{-2}$
4	$2.179 \cdot 10^{-3}$	$2.510 \cdot 10^{-2}$	$2.519 \cdot 10^{-2}$

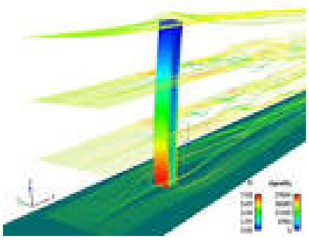
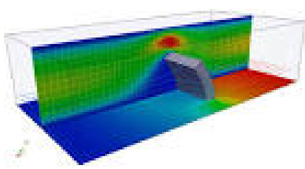
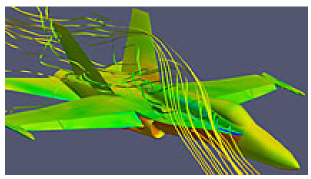
5.2 自适应方法

5.3 多物理耦合

基本物理过程的有限元建模和计算已相对成熟,然而多物理、多尺度现象的建模和计算仍是难点，有限元方法仍面临的挑战和机遇。"未来的几年内，**多物理场(Multiphysics)**分析工具(基于有限元/有限体积法)将会给学术界和工程界带来震惊。单调的“设计－校验”的设计方法将会慢慢被淘汰，虚拟造型技术将让你的思想走得更远，通过模拟仿真将会点燃创新的火花"，Dr. Louis Komzsik说道，他是Chief Numerical Analyst at Siemens PLM Software，被世人誉为'the father of algorithms for Nastran'。

5.3.1 Fluid Structure Interaction

考察可变形/移动的物体与流体环境的相互影响 主要有 stable 和 oscillatory 两种情形：固

体结构的应变产生应力，以抵消环境流体的作用力；或由固体结构在多个空间位置往复运动。让我们看一个有趣的FSI数学模型 **Reference**: Andrew T. Barker and Xiao-Chuan Cai,

$$\rho_s \frac{\partial^2}{\partial t^2}\mathbf{x}_s = \nabla\cdot\sigma_s + \beta\frac{\partial}{\partial t}(\Delta\mathbf{x}_s) - \gamma\mathbf{x}_s, \quad \begin{array}{c} \dfrac{\partial \mathbf{u}_f}{\partial t} + (\mathbf{u}_f\cdot\nabla)\mathbf{u}_f + \dfrac{1}{\rho_f}\nabla p_f = \nu_f\Delta\mathbf{u}_f, \\ \nabla\cdot\mathbf{u}_f = 0. \end{array} \Rightarrow$$

$$(\tilde{M} - (\Delta t/2)\tilde{K})y^{n+1} = (\tilde{M} + (\Delta t/2)\tilde{K})y^n,$$

$$y^n = \begin{pmatrix} u_f \\ p_f \\ x_s \\ \dot{x}_s \\ p_s \\ x_f \end{pmatrix}^n, \quad \tilde{M} = \begin{pmatrix} M_f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_s M_s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

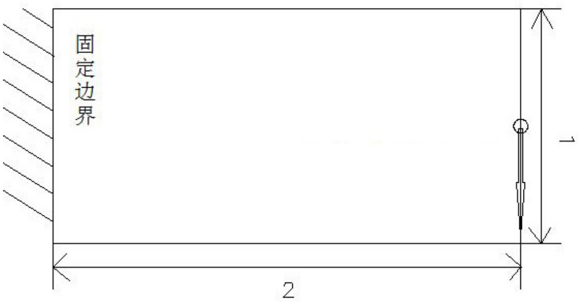
$$\tilde{K} = \begin{pmatrix} N(u_f) - \nu_f K_f & -Q_f^T & 0 & 0 & 0 & 0 \\ Q_f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ A_u & A_p & K_s + \gamma M_s & \beta K_s & -Q_s^T & 0 \\ 0 & 0 & Q_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_m \end{pmatrix}$$

SIAM J. Sci. Comput., 32(4), 2395-2417. 流动问题de数值方法，参考lecturenoteforNS.pdf

求解耦合问题FSI 常用数值方案：couple 与 decouple。利用牛顿（Newton-Raphson）迭代法，同时在流体和固体区域求解耦合方程组（One Shot方法）或者利用不动点(Fixed-point)迭代: 分别求解流动问题（the flow problem）和结构问题（structural problem），直至变化量趋向于零。适用情形：Interaction between the fluid and the structure is weak, only one fixed-point iteration is required within each time step.

5.3.2 Shape Optimization with PDE Constrains

形状优化 目标函数为柔度(极小化)



$$J(\Omega) = \int_\Omega f * u dx + \int_{\Gamma_N} g * u ds = \int_\Omega Ae(u) * e(u) dx \tag{5.4}$$

约束条件:弹性平衡条件。 应用实例：桥梁设计、发动机支架设计等

例5.3.1 (线弹性模型的有限元方法). 弹性平衡方程

$$\begin{cases} -div(Ae(u) = f) & in \quad \Omega, \\ u = 0 & on \quad \Gamma_D, \\ (Ae(u))n = g & on \quad \Gamma_N, \end{cases} \tag{5.5}$$

有限元方法（参考前一讲）离散得到线性代数方程组

$$F=\begin{pmatrix}F_1\\F_2\\\vdots\\F_n\end{pmatrix}=\begin{pmatrix}k_{11}&k_{12}&\cdots&k_{1n}\\k_{21}&k_{22}&\cdots&k_{2n}\\\vdots&\ddots&&\vdots\\\vdots&&\ddots&\vdots\\k_{n1}&k_{n2}&\cdots&k_{nn}\end{pmatrix}\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}=K\cdot u$$

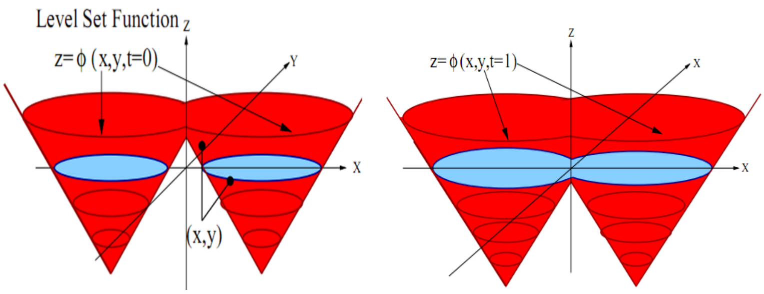
5.3.3 水平集方法

Motivation： 平面封闭曲线可隐式表示为二维函数的水平集线（一般使用零水平集，即c=0）

$$C=\{(x,y),u(x,y)=c\}$$

若平面封闭曲线随时间t变化则（如下图所示）

$$C(t)=\{(x,y),u(x,y,t)=c\}$$



水平集方法的控制方程

$$\begin{cases}\psi(x)=0 & \Leftrightarrow x\in\partial\Omega,\\ \psi(x)<0 & \Leftrightarrow x\in\Omega\\ \psi(x)>0 & \Leftrightarrow x\in(D\backslash\bar{\Omega})\end{cases}$$

零水平集满足：$\psi(t,x(t))=0,\forall x(t)\in\partial\Omega(t)$,则关于时间求导

$$\frac{\partial\psi}{\partial t}+\nabla\psi*\frac{\partial\vec{X}}{\partial t}=\frac{\partial\psi}{\partial t}+Vn*\nabla\psi=0,$$

其中，外法向量$n=\nabla\psi/|\nabla\psi|$。可得Hamilton-Jacobi方程

$$\frac{\partial\psi}{\partial t}+V|\nabla\psi|=0, \tag{5.6}$$

H-J方程的数值解法

$$\frac{\psi_i^{n+1}-\psi_i^n}{\Delta t}+min(V_i^n,0)g^-(D_x^+\psi_i^n,D_x^-\psi_i^n)$$
$$+max(V_i^n,0)g^+(D_x^+\psi_i^n,D_x^-\psi_i^n)=0,$$

其中

$$D_x^+\psi_i^n=\frac{\psi_{i+1}^n-\psi_i^n}{\Delta x},D_x^-\psi_i^n=\frac{\psi_i^n-\psi_{i-1}^n}{\Delta x}$$
$$g^+(d^+,d^-)=\sqrt{min(d^+,0)^2+max(d^-,0)^2},$$
$$g^-(d^+,d^-)=\sqrt{max(d^+,0)^2+min(d^-,0)^2}.$$

重新初始化

求解水平集函数，更新设计域后要重新初始化,使函数为符号距离函数，满足$|\nabla\phi|=1$，方法求解下述方程直至收敛

$$\frac{\partial\phi}{\partial t}+s(\phi_0)(|\nabla\phi|-1)=0, \tag{5.7}$$

其中，

$$\begin{cases} s(\phi_0)= & 1 \quad \phi>0\\ s(\phi_0)= & 0 \quad \phi=0\\ s(\phi_0)= & -1 \quad \phi<0\end{cases}$$

关于形状导数

若定义$\Omega_\theta=(Id+\theta)(\Omega)$为区域$\Omega$的一个扰动，那么

$$J((Id+\theta)(\Omega))=J(\Omega)+J'(\Omega)(\theta)+o(\theta)$$

一些事实: 若考虑$J(\Omega)=\int_\Omega\phi(x)dx$，则

$$J'(\Omega)(\theta)=\int_\Omega div(\theta\phi(x))dx=\int_{\partial\Omega}\theta(x)\mathbf{n}(x)\phi(x)ds \tag{5.8}$$

若考虑$J(\Omega)=\int_{\partial\Omega}\phi(x)ds$，则

$$J'(\Omega)(\theta)=\int_{\partial\Omega}\theta(x)\mathbf{n}(x)\big(\frac{\partial\phi}{\partial n}+H\phi\big)ds \tag{5.9}$$

证明参考文献[2]或[3]

优化模型（5.4）的形状导数

对于目标函数

$$J(\Omega)=\int_\Omega fudx+\int_{\partial\Omega}guds$$

$\int_\Omega fudx$借助公式（5.8）可得形状导数

$$\int_\Omega\theta(x)\cdot n(x)fuds$$

另一方面， $\int_{\partial\Omega}guds$借助公式（5.9）可得形状导数

$$\int_{\partial\Omega}\theta(x)\cdot n(x)\big(\frac{\partial(gu)}{\partial n}+Hgu\big)ds$$

终于

$$J'(\Omega)=\int_{\partial\Omega}\theta(x)\cdot n(x)\big(\frac{\partial(gu)}{\partial n}+Hgu+fu\big)ds$$

水平集方程的速度场- 形状导数

形状优化：取方程(5.6)中的速度场为

$$V = \frac{\partial(gu)}{\partial n} + Hgu + fu \tag{5.10}$$

分析：依上述V的取法

$$J'(\Omega) = \int_{\partial\Omega} V\theta \cdot \mathbf{n} ds$$

定义向量场$\theta = -V\mathbf{n}$

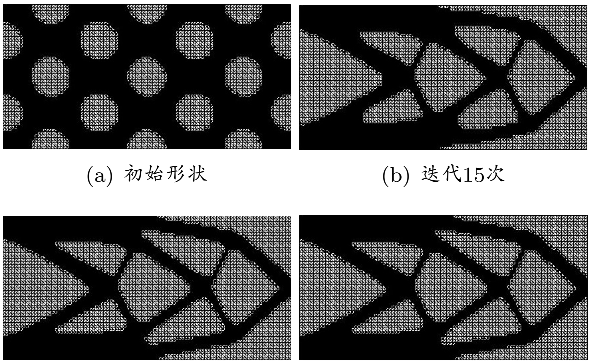
$$J((Id+\theta)(\Omega)) = J(\Omega) - \int_{\partial\Omega} V^2 ds + o(\theta),$$

设置时间步长t，将形状Ω更新到$\Omega_t = (Id + t\theta)\Omega,$

$$J(\Omega_t) = J(\Omega) - t\int_{\partial\Omega} V^2 ds + 0(t^2),$$

则$V = \frac{\partial(gu)}{\partial n} + Hgu + fu$即为各点速度标量。

数值结果



(a) 初始形状 (b) 迭代15次

(c) 迭代30次 (d) 最优形状参考

Figure 5.2: 一阶精度水平集方法

初始条件取杨氏模量E=1.0，泊松比$\mu = 0.3$,网格数量为100×50,采用一阶精度的迎风格式，时间步长dt=1.0。

初始形状的依赖性:多孔更优

作为本章的总结。 多物理/多尺度现象是自然科学和工程领域问题的理论研究所关心的广泛性问题。科学计算工作者则更关心相应问题的数学模型与数值算法的研究：

- Assume scientific computing be a secondary tool beside the practical experiments/第三种科学方法
- Interdisciplinary(交叉) topics, arising from material/nuclear science and bio-mechanics.
- Modeling meso-scale(介观尺度) problems.

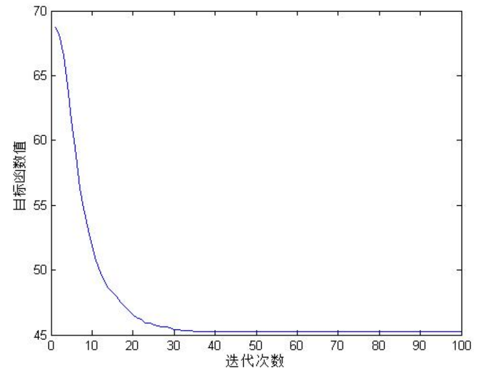
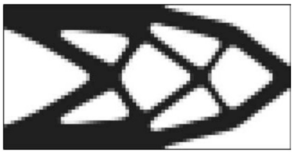
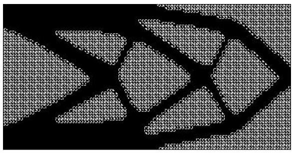


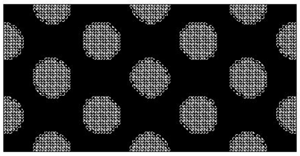
Figure 5.3: 目标函数变化

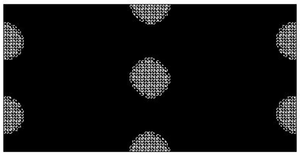


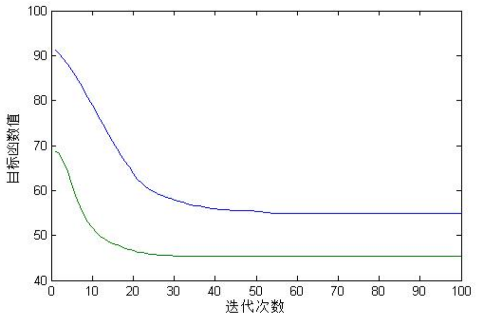
(a) 来自文献[1]



(b) 最优形状







软件包与参考文献

FEM book

1. O .C. Zienkiewicz and K. Morgan: Finite Elements and Approximation, 1983; The Finite Element Method, 6th ed.

2. S. C. Brenner and L. R. Scott: The mathematical theory of finite element methods, 3rd ed.

3. Mark. S. Gockenbach: Understanding And Implementing the Finite Element Method

4. H. Elman, etc.: Finite Elements and Fast Iteraive Solvers - with applications in incompressible fluid dynamics

5. Jian-Ming Jin: The Finite Element Method in Electromagnetics, 2nd ed.

Open Source Packages

- deal.II - http://www.dealii.org
- libmesh - http://libmesh.sourceforge.net
- AFEPack - http://dsec.pku.edu.cn/ rli/software.php

- openDX/paraview (Visualize scientific data in 3D)
- gmsh (3D Mesh generator and visualize FE solution) - http://geuz.org/gmsh
- tetgen (Tetrahedron mesh generator) - http://wias-berlin.de/software/tetgen
- ...

Bibliography

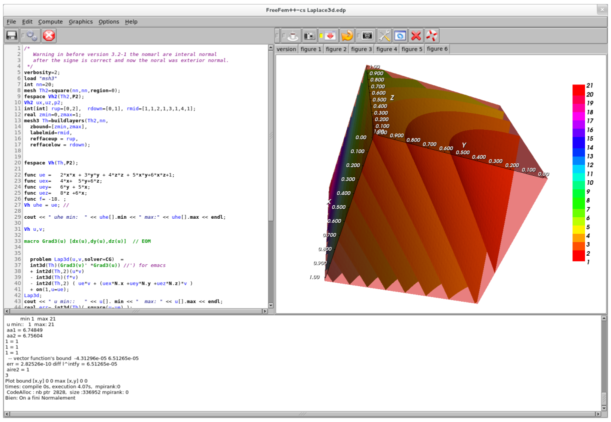
[1] Toader A Allaire G, Jouve F. Structural optimization using sensitivity analysis and a level-set method. *Journal of Computational Physics*, 2004.

[2] Francois Murat and Jacques Simon. *Etude de problemes d'optimal design*. Springer Berlin Heidelberg, 1976.

[3] J. Simon. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis Optimization*, 2(7):649–687, 2010.

FreeFEM++

FreeFem++是一个基于有限元方法，数值求解偏微分方程的免费软件，它是一个拥有自己高级编程语言的集成化产品。http://www.freefem.org. Hecht, F. New development in freefem++. J. Numer. Math. 20 (2012), no. 3-4, 251–265.



Exercises