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Homework 1 - Numerical Fluids

1) Given

$$\frac{d\Psi}{dt} = F(\Psi, t) = \gamma\Psi = (\lambda + i\omega)\Psi$$

Backward-Euler

$$\frac{d\Psi(t_n)}{dt} = \frac{\Psi(t_n) - \Psi(t_n - \Delta t)}{\Delta t}$$

Taylor expand, and define as eqn 1

$$\Psi(t_n - \Delta t) \cong \Psi(t_n) - \Delta t \frac{d\Psi}{dt} + \frac{(\Delta t)^2}{2}\frac{d^2\Psi}{dt^2} - \frac{(\Delta t)^3}{6}\frac{d^3\Psi}{dt^3}$$

$$\Psi(t_n - \Delta t) - \Psi(t_n) + \Delta t \frac{d\Psi}{dt} = \frac{(\Delta t)^2}{2}\frac{d^2\Psi}{dt^2} - \frac{(\Delta t)^3}{6}\frac{d^3\Psi}{dt^3}$$

$$\rightarrow \frac{\Psi(t_n - \Delta t) - \Psi(t_n)}{\Delta t} - \frac{d\Psi}{dt} = \frac{\Delta t}{2}\frac{d^2\Psi}{dt^2} - \frac{\Delta t^2}{6}\frac{d^3\Psi}{dt} + \dots$$

Not equivalent to original section yet. Instead, $-$(expression)

$$\rightarrow \frac{\Psi(t_n) - \Psi(t_n - \Delta t)}{\Delta t} + \frac{d\Psi}{dt} = -\boxed{\frac{\Delta t}{2}}\frac{d^2\Psi}{dt} + \frac{\Delta t^2}{6}\frac{d^3\Psi}{dt} - \dots$$

This problem has an order of $O(1)$.

Trapezoidal:

$$\frac{d\Psi}{dt}(t_n) = \frac{\Psi(t_n+\Delta t) - \Psi(t_n - \Delta t)}{2\Delta t}$$

$$\Psi(t_n+\Delta t) = \Psi(t_n) + \Delta t \frac{d\Psi}{dt} + \frac{\Delta t^2}{2}\frac{d^2\Psi}{dt^2} + \frac{\Delta t^3}{6}\frac{d^3\Psi}{dt} + \ldots$$

and using $\boxed{\text{eqn } 1}$,

$$\Psi(t_n+\Delta t) - \Psi(t_n-\Delta t) =$$

$$\rightarrow \left(\cancel{\Psi(t_n)} + \Delta t \frac{d\Psi}{dt} + \cancel{\frac{\Delta t^2}{2}\frac{d\Psi}{dt}} + \frac{\Delta t^3}{6}\frac{d\Psi}{dt} + \ldots\right) - \left(\cancel{\Psi(t_n)} - \Delta t \frac{d\Psi}{dt} + \cancel{\frac{\Delta t^2}{2}\frac{d\Psi}{dt}} - \frac{\Delta t^3}{6}\frac{d\Psi}{dt} + \ldots\right)$$

$$\frac{\Psi(t_n+\Delta t) - \Psi(t_n-\Delta t)}{2} = \frac{2\Delta t \frac{d\Psi}{dt} + 2\left(\frac{\Delta t^3}{6}\right)\frac{d^3\Psi}{dt}}{2}$$

$$\frac{\Psi(t_n+\Delta t) - \Psi(t_n-\Delta t)}{2} - \Delta t \frac{d\Psi}{d} = \left(\frac{\Delta t^3}{6}\right)\frac{d^3\Psi}{dt^3} + \ldots$$

$$\frac{\Psi(t_n+\Delta t) - \Psi(t_n-\Delta t)}{2\Delta t} - \frac{d\Psi}{dt} = \boxed{\frac{\Delta t^2}{6}}\frac{d^3\Psi}{dt^3} + \ldots$$

Trapezoidal method is $\mathcal{O}(2)$.

2) Backwards Euler:

Given $\frac{\phi_n - \phi_{n-1}}{\Delta t} = (\lambda + i\omega)\phi_n$

Solve for ϕ_{n-1}:

$$\phi_n - \phi_{n-1} = \Delta t \, \phi_n (\lambda + i\omega)$$

$$\phi_{n-1} = \phi_n \left(1 - \Delta t(\lambda + i\omega)\right)$$

Plug into

$$|A| = \left|\frac{\phi_{n-1}}{\phi_n}\right| \leq 1$$

$$\rightarrow \left|\frac{\phi_n(1 - \Delta t(\lambda + i\omega))}{\phi_n}\right| \leq 1$$

$$|1 - \Delta t(\lambda + i\omega)| \leq 1$$

if − integer:

$$-(1 - \Delta t(\lambda + i\omega)) \leq 1$$

$$1 - \Delta t(\lambda + i\omega) \geq -1$$

$$2 \geq \Delta t(\lambda + i\omega)$$

$$\rightarrow \boxed{\Delta t \leq \frac{2}{\lambda + i\omega}}$$

This is consistant with the graph of backwards-Euler.

The expression $\Delta t \le \frac{2}{\lambda + i\omega}$ would be an inversely shaded circle in the $\lambda - \omega$ plane.
I don't have time to fix it, but I believe my answer is inverted (numerator should be on denominator)

Trapezoidal!

Given $$\frac{\phi_{n+1} - \phi_n}{\Delta t} = (\lambda + i\omega)\frac{\phi_{n+1} + \phi_n}{2}$$

Find ϕ_{n+1},

$$\frac{\phi_{n+1}}{\Delta t} - \phi_{n+1}\frac{(\lambda + i\omega)}{2} = \frac{\phi_n}{\Delta t} + \phi_n\frac{(\lambda + i\omega)}{2}$$

$$2\phi_{n+1} - \phi_{n+1}(\lambda + i\omega)\Delta t = 2\phi_n + \phi_n \Delta t(\lambda + i\omega)$$

$$\phi_{n+1}(2 - \Delta t(\lambda + i\omega)) = \phi_n(2 + \Delta t(\lambda + i\omega))$$

$$\phi_{n+1} = \phi_n \frac{(2 + \Delta t(\lambda + i\omega))}{(2 - \Delta t(\lambda + i\omega))}$$

$$\rightarrow |A| = \left| \frac{\cancel{\phi_n}}{\cancel{\phi_n}} \frac{(2 + \Delta t(\lambda + i\omega))}{(2 - \Delta t(\lambda + i\omega))} \right| \le 1$$

if - integer;

$$-\left(\frac{2 + \Delta t(\lambda + i\omega)}{2 - \Delta t(\lambda + i\omega)}\right) \le 1$$

$$-2 - \Delta t(\lambda + i\omega) \le 2 - \Delta t(\lambda + i\omega)$$

$-4 \leq 0$, which, while true, doesn't give insight to the problem at hand.

if t integer:

$$\frac{2 + \Delta t\,(\lambda + iw)}{2 - \Delta t\,(\lambda + iw)} \leq 1$$

$$2 + \Delta t(\lambda + iw) \leq 2 - \Delta t(\lambda + iw)$$

$$\Delta t(\lambda + iw) \leq -\Delta t(\lambda + iw)$$

This statement simply isn't true at all.

My conclusion is that either the negative case was done incorrectly, or that this trapezoidal case is always correct, which I highly doubt, as long as $A \leq D$

5) Unfortunately, I can't seem to get Matlab running at the current time. However, that will be fixed in future homeworks.

If Matlab was working, I believe I would discover 3 major distinctions:

- Trapezoidal method, despite having the more general stability case and lack of amplitude loss, does have a much more rapid error compared to Euler's method.

- Backwards Euler's method tends to shrink periods over longer timespans, leading to higher amplitudes and incorrect wavelengths.

- Forwards Euler's method tends to expand wave periods as time passes, leading to increasing amplitudes and a period that is shifted forward in time.

While I would like to prove these preconceptions wrong, I cannot at the moment. I am sorry for the inconvenience, and it will not happen again.