Chapter 1

Homework 1

1.1

Consider backward-Euler differencing and trapezoidal differencing schemes for the damping oscillation problem:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = F(\psi, t) = \gamma\psi = (\lambda + \mathrm{i}\,\omega)\psi$$

1.1.1

Estimate the order of accuracy for these two difference schemes.

For backward difference

$$\frac{\psi_n - \psi_{n-1}}{\Delta t} - \psi'(t_n) = \left\{\psi_n - \left[\psi_n - \psi'(t_n)\Delta t + \frac{1}{2}\psi''(t_n)\Delta t^2 + \mathcal{O}\big(\Delta t^3\big)\right]\right\}\Big/\Delta t - \psi'(t_n) = -\frac{1}{2}\psi''(t_n)\Delta t + \mathcal{O}\big(\Delta t^2\big)$$

This is on the first order of accuracy.

For trapezoidal difference, the differential problem is

$$\frac{\psi_{n+1} - \psi_n}{\Delta t} = (\lambda + \mathrm{i}\,\omega)\frac{\psi_{n+1} + \psi_n}{2}$$

Rearrange terms

$$\frac{\psi_{n+1} - \psi_n}{\Delta t}\frac{2\psi_n}{\psi_{n+1} + \psi_n} = (\lambda + \mathrm{i}\,\omega)\psi_n$$

Equivalently, it uses $\frac{\psi_{n+1}-\psi_n}{\Delta t}\frac{2\psi_n}{\psi_{n+1}+\psi_n}$ to approximate $\psi'(t_n)$

$$\begin{aligned}
&\frac{\psi_{n+1} - \psi_n}{\Delta t}\frac{2\psi_n}{\psi_{n+1} + \psi_n} - \psi'(t_n) \\
&= \left[\psi_n + \psi'(t_n)\Delta t + \frac{1}{2}\psi''(t_n)\Delta t^2 + \mathcal{O}\big(\Delta t^3\big) - \psi_n\right]2\psi_n\Big/\Delta t\left[\psi_n + \psi'(t_n)\Delta t + \frac{1}{2}\psi''(t_n)\Delta t^2 + \mathcal{O}\big(\Delta t^3\big) + \psi_n\right] - \psi'(t_n) \\
&= \left[\psi'(t_n)\Delta t + \frac{1}{2}\psi''(t_n)\Delta t^2 + \mathcal{O}\big(\Delta t^3\big)\right]2\psi_n\Big/\Delta t\left[2\psi_n + \psi'(t_n)\Delta t + \frac{1}{2}\psi''(t_n)\Delta t^2 + \mathcal{O}\big(\Delta t^3\big)\right] - \psi'(t_n) \\
&= \left[\psi'(t_n) + \frac{1}{2}\psi''(t_n)\Delta t + \mathcal{O}\big(\Delta t^2\big)\right]\Big/\big[1 + \psi'(t_n)/2\psi_n\Delta t + \psi''(t_n)/4\psi_n\Delta t^2 + \mathcal{O}\big(\Delta t^3\big)\big] - \psi'(t_n) \\
&= \left[\psi'(t_n) + \frac{1}{2}\psi''(t_n)\Delta t + \mathcal{O}\big(\Delta t^2\big)\right]\big\{1 - \psi'(t_n)/2\psi_n\Delta t + \big[(\psi'(t_n)/2\psi_n)^2 - \psi''(t_n)/4\psi_n\big]\Delta t^2 + \mathcal{O}\big(\Delta t^3\big)\big\} - \psi'(t_n) \\
&= \psi'(t_n) + [\psi''(t_n)/2 - \psi'(t_n)/2\psi_n]\Delta t + \mathcal{O}\big(\Delta t^2\big) - \psi'(t_n) \\
&= [\psi''(t_n)/2 - \psi'(t_n)/2\psi_n]\Delta t + \mathcal{O}\big(\Delta t^2\big)
\end{aligned}$$

This is also on the first order of accuracy.

1.1.2

Derive A-stability criteria for these two difference schemes, compare your results with the figure below.

For backward difference, the differential problem is

$$\frac{\psi_n - \psi_{n-1}}{\Delta t} = (\lambda + \mathrm{i}\,\omega)\psi_n$$

Solve for ψ_{n-1}

$$\psi_{n-1} = \psi_n[1 - (\lambda + \mathrm{i}\,\omega)\Delta t]$$

Absolute stability criteria

$$\left|\frac{\psi_n}{\psi_{n-1}}\right| = \frac{1}{(1-\lambda\Delta t)^2 + (\omega\Delta t)^2} \leq 1$$

Then

$$(\lambda\Delta t - 1)^2 + (\omega\Delta t)^2 \geq 1$$

which is the outside region of a circle centered at 1 with radius 1.

For trapezoidal difference, solve for ψ_{n+1}

$$\psi_{n+1} = \psi_n \frac{2 + (\lambda + \mathrm{i}\,\omega)\Delta t}{2 - (\lambda + \mathrm{i}\,\omega)\Delta t}$$

Absolute stability criteria

$$\left|\frac{\psi_{n+1}}{\psi_n}\right| = \frac{(2+\lambda\Delta t)^2 + (\omega\Delta t)^2}{(2-\lambda\Delta t)^2 + (\omega\Delta t)^2} \leq 1$$

Then

$$\lambda\Delta t \leq 0$$

which is the left half of the plane.

1.2

Modify the code (`AStabilityConvergence_ODE_FT.m`) and make it work for backward differencing schemes. Choose different Δt and compare which one leads to the best convergence, which ones leads to unstable solution (you may refer to the matlab code provided).

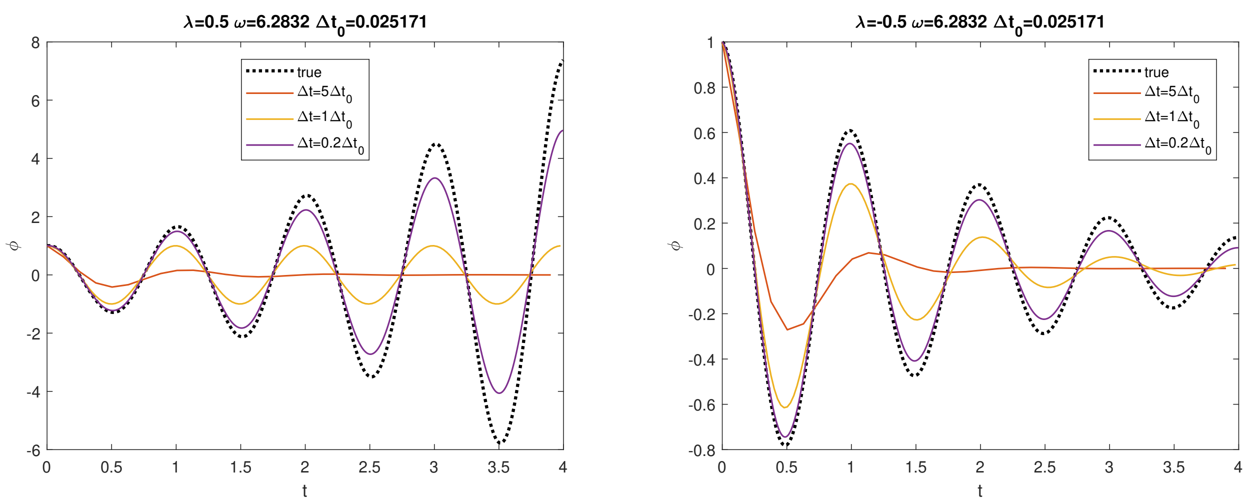

Figure 1.1: Different settings of backward difference scheme

In figure 1.1, smaller amplitudes are present in all three schemes. When step is too large, the solution diverges from true solution.

1.3

Modify the code (`AStabilityConvergence_ODE_FT.m` and make it work for trapezoidal differencing schemes. Choose different Δt and compare which one leads to the best convergence, which ones leads to unstable solution (you may refer to the matlab code provided). In this case, try positive λ.

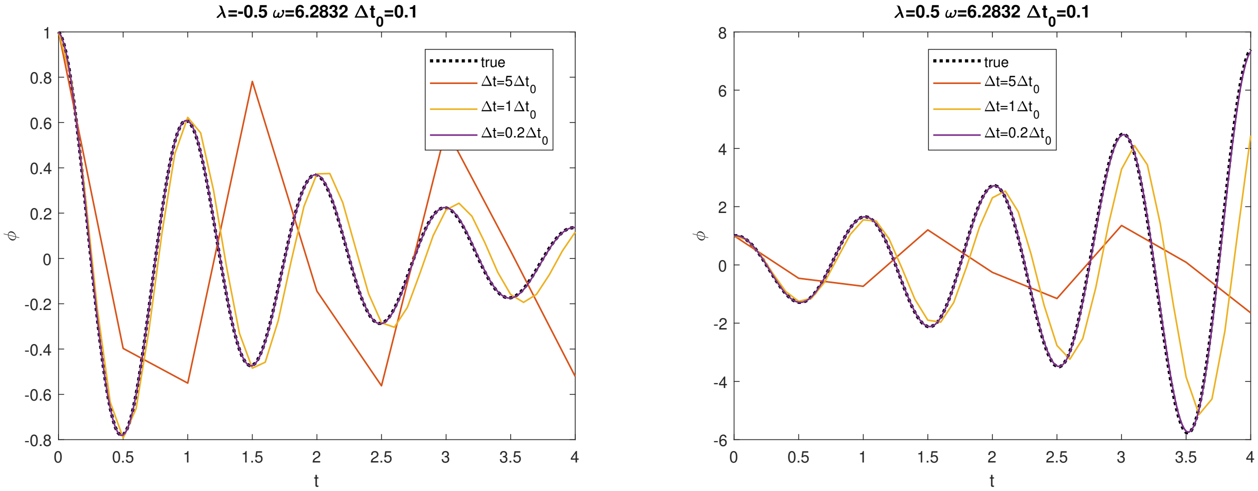

Figure 1.2: Different settings of trapezoidal difference scheme

In figure 1.2, phase delays are present in all three schemes. When step is too large, the solution diverges from true solution.

1.4

Compare with forward, backward and trapezoidal schemes, and discuss what you find.

In all three schemes, large step all leads to divergence. A small step is typically necessary for all difference schemes.