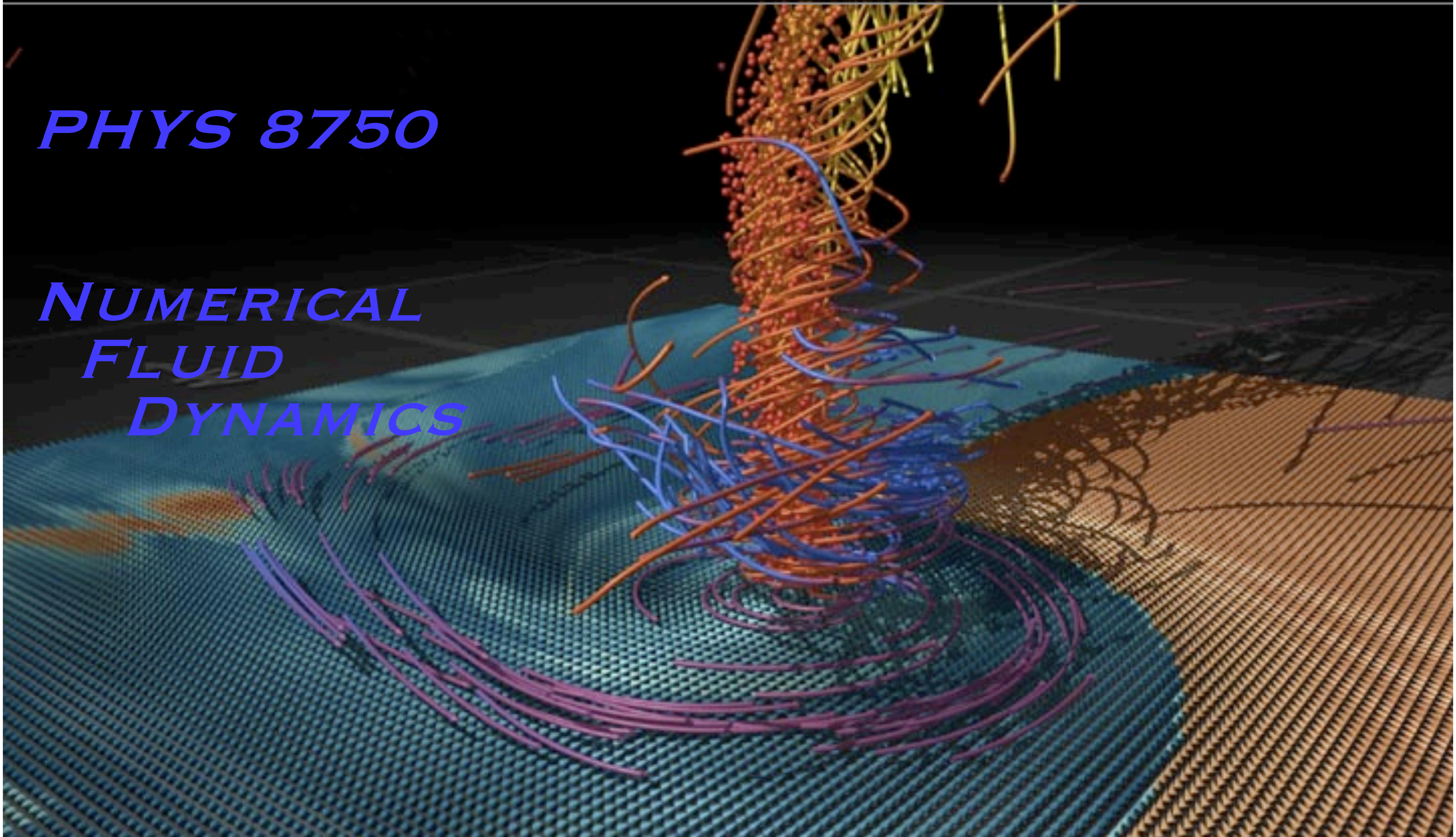
PHYS 8750

Numerical Fluid Dynamics



Hunt for the supertwister – pbs.org/nova/tornado

Fall 2020

PHYS 8750

Class #1 (Chapter 1.1)
1) Terminology
ODE vs. PDE

Order of PDE

Linear vs. nonlinear

Types of equations

2) Numerical Scheme
Forward/backward/leapfrog
Upstream/downstream/center space

CLASS #2 (CHAPTER 2.1, 2.2)

Outline

- Criteria to evaluate
  Stability; accuracy; convergence; consistency
- 1) Truncation errors & order of accuracy
- 2) A-stability and stability diagram
- 3) Amplitude and phase error

Truncation Error & Order of Accuracy

1-D TRANSPORT ODE

$$\frac{d\psi}{dt} = F(\psi, t) = \lambda\, \psi$$

FORWARD TIME

$$\frac{d\psi}{dt}(t_n) \approx \frac{\psi(t_n + \Delta t) - \psi(t_n)}{\Delta t}$$

CENTERED TIME

$$\frac{d\psi}{dt}(t_n) \approx \frac{\psi(t_n + \Delta t) - \psi(t_n - \Delta t)}{2\Delta t}$$

TAYLOR EXPANSION:

$$\psi(t_n + \Delta t) = \psi(t_n) + \Delta t\, \frac{d\psi}{dt}(t_n) + \frac{(\Delta t)^2}{2}\frac{d^2\psi}{dt^2}(t_n) + \frac{(\Delta t)^3}{6}\frac{d^3\psi}{dt^3}(t_n) + \cdots$$

$$\psi(t_n - \Delta t) = \psi(t_n) - \Delta t\, \frac{d\psi}{dt}(t_n) + \frac{(\Delta t)^2}{2}\frac{d^2\psi}{dt^2}(t_n) - \frac{(\Delta t)^3}{6}\frac{d^3\psi}{dt^3}(t_n) + \cdots$$

TRUNCATION ERROR

$$\frac{\psi(t_n + \Delta t) - \psi(t_n)}{\Delta t} - \frac{d\psi}{dt}(t_n) = \underline{\frac{\Delta t}{2}\frac{d^2\psi}{dt^2}(t_n) + \frac{(\Delta t)^2}{6}\frac{d^3\psi}{dt^3}(t_n) + \cdots}$$

$$\frac{\psi(t_n + \Delta t) - \psi(t_n - \Delta t)}{2\Delta t} - \frac{d\psi}{dt}(t_n) = \underline{\frac{(\Delta t)^2}{3}\frac{d^3\psi}{dt^3}(t_n) + \cdots}$$

The lowest order of Δt determines the order of accuracy of the finite difference scheme:

Forward time: first-order accurate; Centered time: second-order accurate

Consistency and Convergence

1. Consistency: truncation error $\tau_n \to 0$ as $\Delta t \to 0$

Forward time

$$\tau_n = \frac{\Delta t}{2}\frac{d^2\psi}{dt^2}(t_n) + \frac{(\Delta t)^2}{6}\frac{d^3\psi}{dt^3}(t_n) + \cdots = \frac{\Delta t}{2}\frac{d^2\psi}{dt^2}(t_n) + O[(\Delta t)^2]$$

Centered time

$$\tau_n = \frac{(\Delta t)^2}{3}\frac{d^3\psi}{dt^3}(t_n) + O[(\Delta t)^3]$$

2. Convergence: global error at time T $E_N \to 0$ as $\Delta t \to 0$

accumulate local errors over every time step.

Forward time

$$E_N \le N\Delta t(1 + |\lambda|\Delta t)^N \,\tau_{max} \le Te^{|\lambda|T}\tau_{max}$$

$\boxed{\tau_{max} \to 0 \text{ as } \Delta t \to 0}$ → $\boxed{E_N \to 0 \text{ as } \Delta t \to 0}$

The order of accuracy determines the rate at which the solution of a stable finite-difference method converges to the true solution as $\Delta t \to 0$.
Higher order schemes converge to true solution faster.

Examples of instability in nature

- Thunderstorms

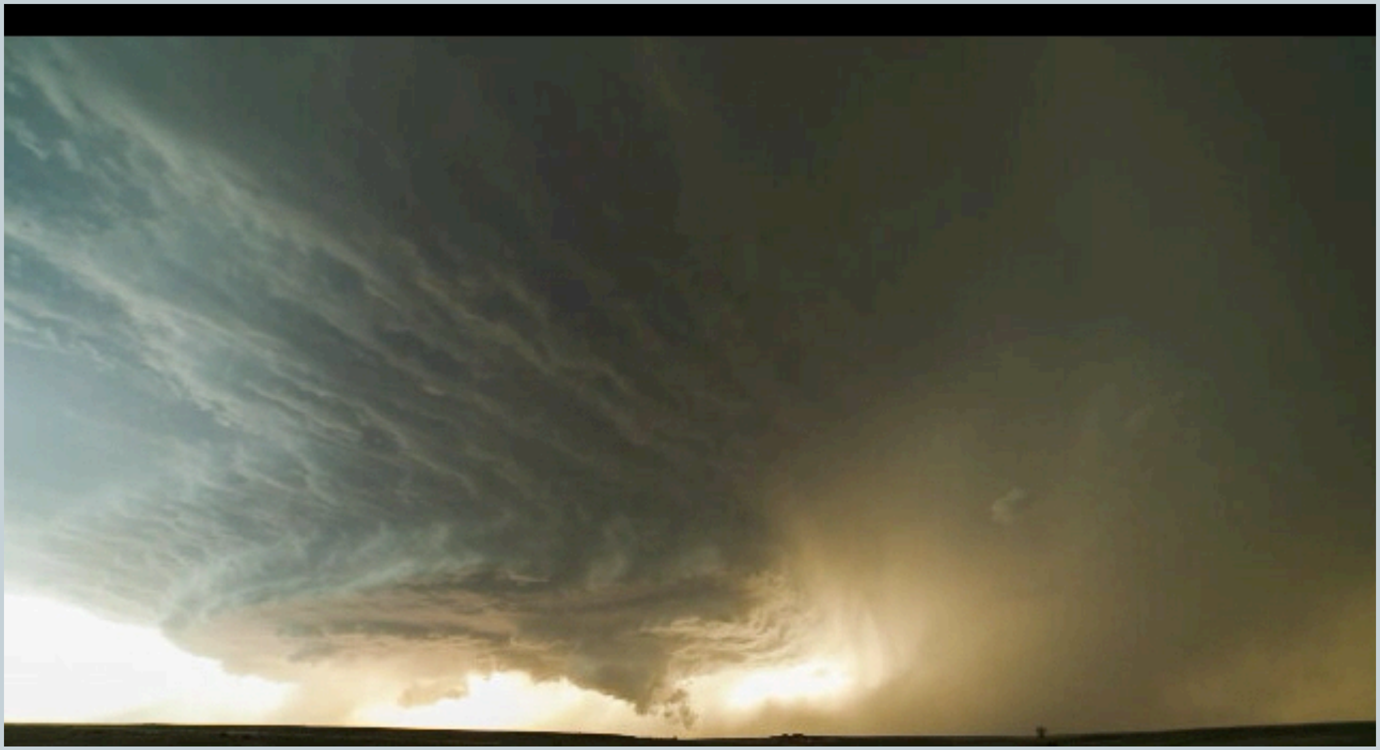
earthobservatory.nasa.gov/IOTD/view.php?id=78497

smoke from Colorado fire

- From
  - Aircraft
- Location
  - Wyoming
- Duration
  - hours
- Date
  - 7/11/2012
- Credit
  - DC3 project

Examples of instability in nature

- Thunderstorms

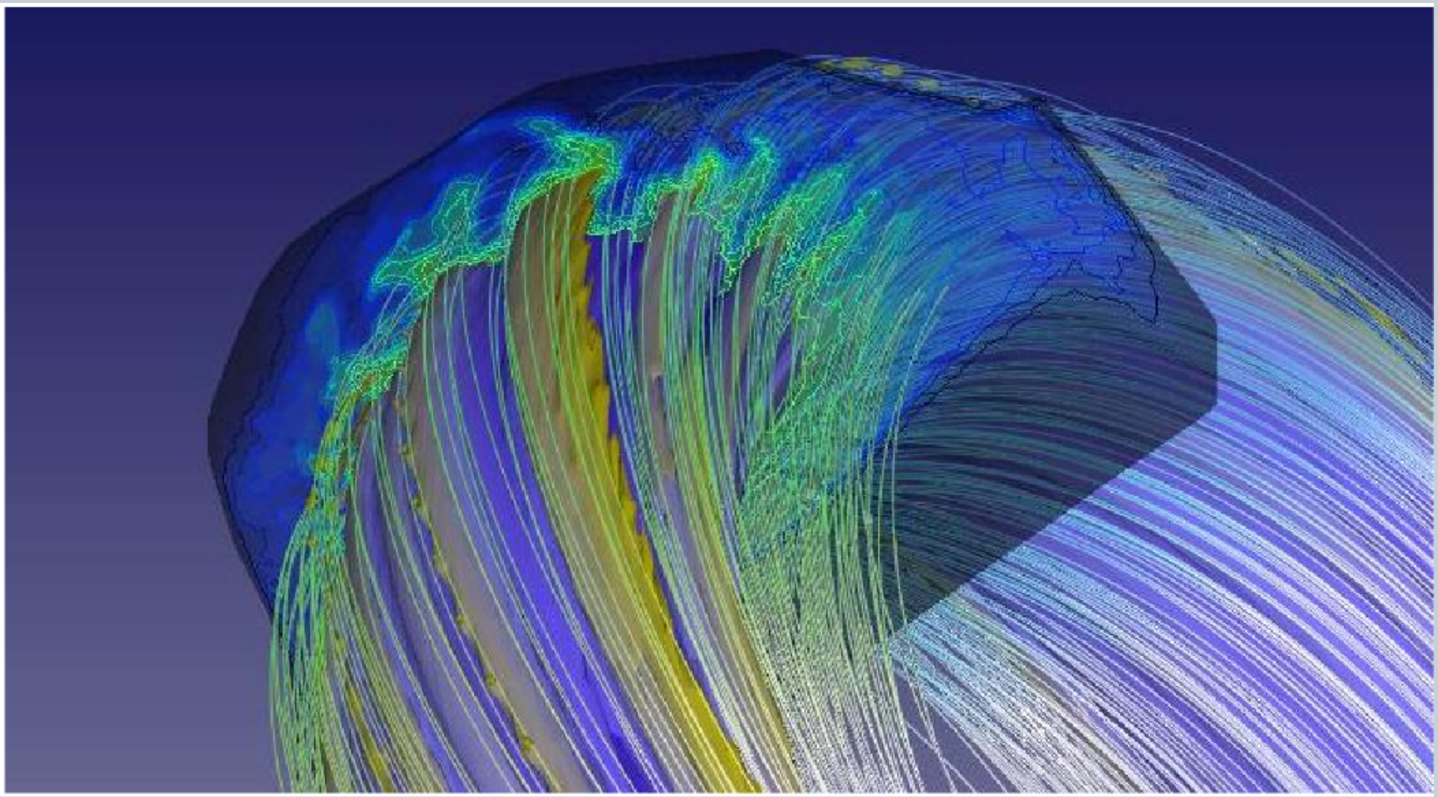


Instability has been defined as "outputs of internal states growing without bounds" or "if small perturbations cause changes that reinforce the original perturbation"

- From
  - Ground
- Location
  - Booker TX
- Duration
  - hours
- Date
  - 6/3/2013
- Credit
  - Mike Oblinski

Examples of instability in nature

- Nonlinear instability in toroidal fusion plasma



- The *extended magnetohydrodynamics* (MHD) code M3D was used to study magnetic confinement and stability properties of fusion plasma in a Tokamak. *Edge Localized Modes* were noted, a new class of plasma instability.

Ref: www.nersc.gov/science/fusion-science/a-new-class-of-tokamak-nonlinear-plasma-instability/

- From:
  - Simulation
- Location:
  - Nat'l Energy Research Scientific Computing Center *(NERSC)*
- Credit
  - MIT: Linda Sugiyama

Stability: computational perspective

- What is stability?
  - Let's work backwards. What is *instability?*
- Instability
  - *Unstable numerical* scheme: numerical solution grows <u>much more rapidly</u> than the true one
- What is "more rapidly?"
  - We *must* be knowledgeable of the PDE properties (and thus of the physical phenomenon) to assess *reasonable* behavior.
  - If amplitude should *not* change –
    - any continued growth in the numerical solution is *unstable*
  - If exponential growth in amplitude is possible
    - than any growth *beyond* that is considered a numerical instability.

Stability

To make model really run reasonably, consistency and convergence are not enough, because $\Delta t \to 0$ is an idealized case.

- Stability: prevent model to "blow up" with finite Δt
  - General definition:

$$\boxed{\phi_n: numerical\ solution\ of\ \psi\ at\ time\ step\ n+1}$$

Amplification factor: $$\boxed{|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \le 1 + \eta\Delta t}$$

  - A-stability (absolute stable): Solution that doesn't increase with time

$$\frac{d\psi}{dt} = F(\psi, t) = \gamma\psi = (\lambda + i\omega)\psi$$

$$\psi = \psi_0 e^{(\lambda+i\omega)t} = \psi_0 e^{\lambda t} e^{i\omega t}$$

λ: *amplitude*, ω: *phase*

$\lambda<0$, *damping system*

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \le 1$$

A-Stability

$$\frac{d\psi}{dt} = F(\psi, t) = \gamma\psi = (\lambda + i\omega)\psi$$

$$\psi = \psi_0 e^{(\lambda+i\omega)t} = \psi_0 e^{\lambda t} e^{i\omega t}$$

λ: *amplitude*, ω: *phase*

λ<0, damping system

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \le 1$$

- Forward time difference

$$\frac{\phi_{n+1} - \phi_n}{\Delta t} = (\lambda + i\omega)\phi_n \Longrightarrow \phi_{n+1} = [1 + (\lambda + i\omega)\Delta t]\phi_n$$

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| = |1 + (\lambda + i\omega)\Delta t| \le 1 \Longrightarrow (1 + \lambda\Delta t)^2 + (\omega\Delta t)^2 \le 1$$

$$1 + 2\lambda\Delta t + (\lambda^2 + \omega^2)(\Delta t)^2 \le 1 \Longrightarrow \Delta t \le \frac{-2\lambda}{(\lambda^2+\omega^2)}$$

A-Stability

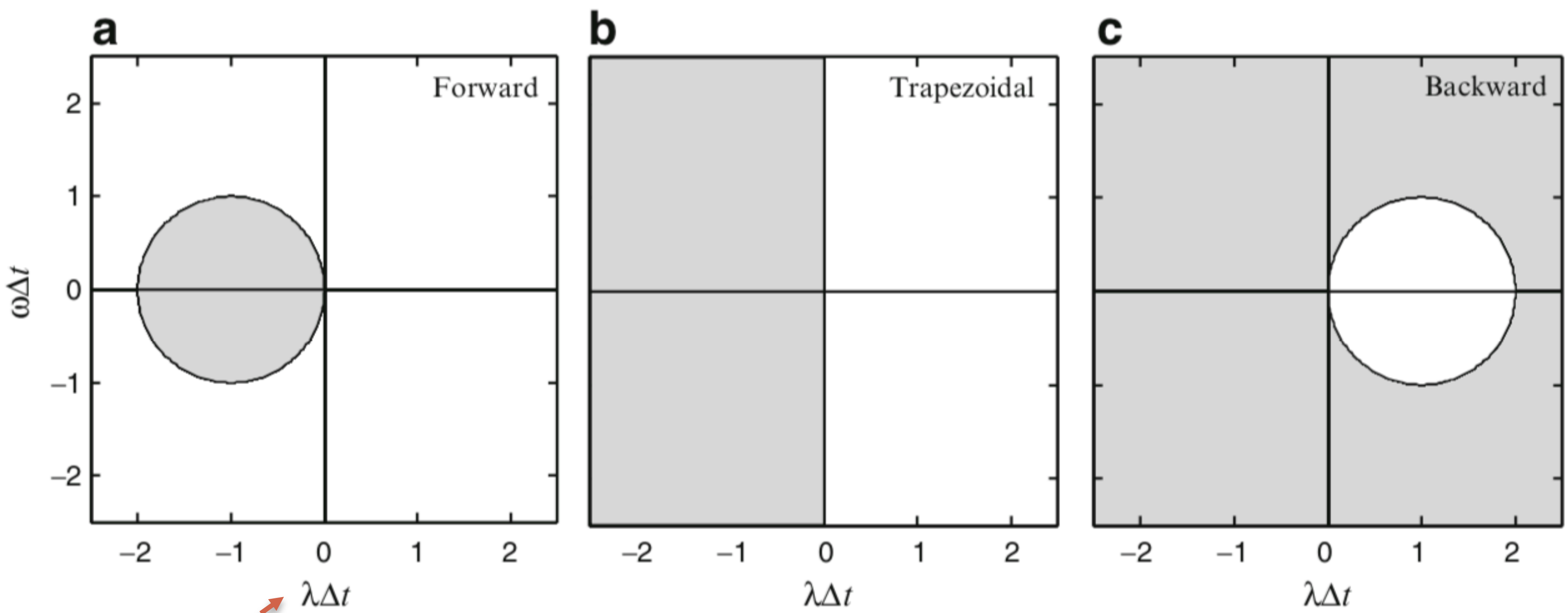


Fig. 2.1 Absolute stability regions (shaded) for **a** forward-Euler differencing, **b** trapezoidal differencing, and **c** backward-Euler differencing

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| = [1 + (\lambda + i\omega)] \leq 1 \Longrightarrow (1 + \lambda\Delta t)^2 + (\omega\Delta t)^2 \leq 1$$

A-Stability

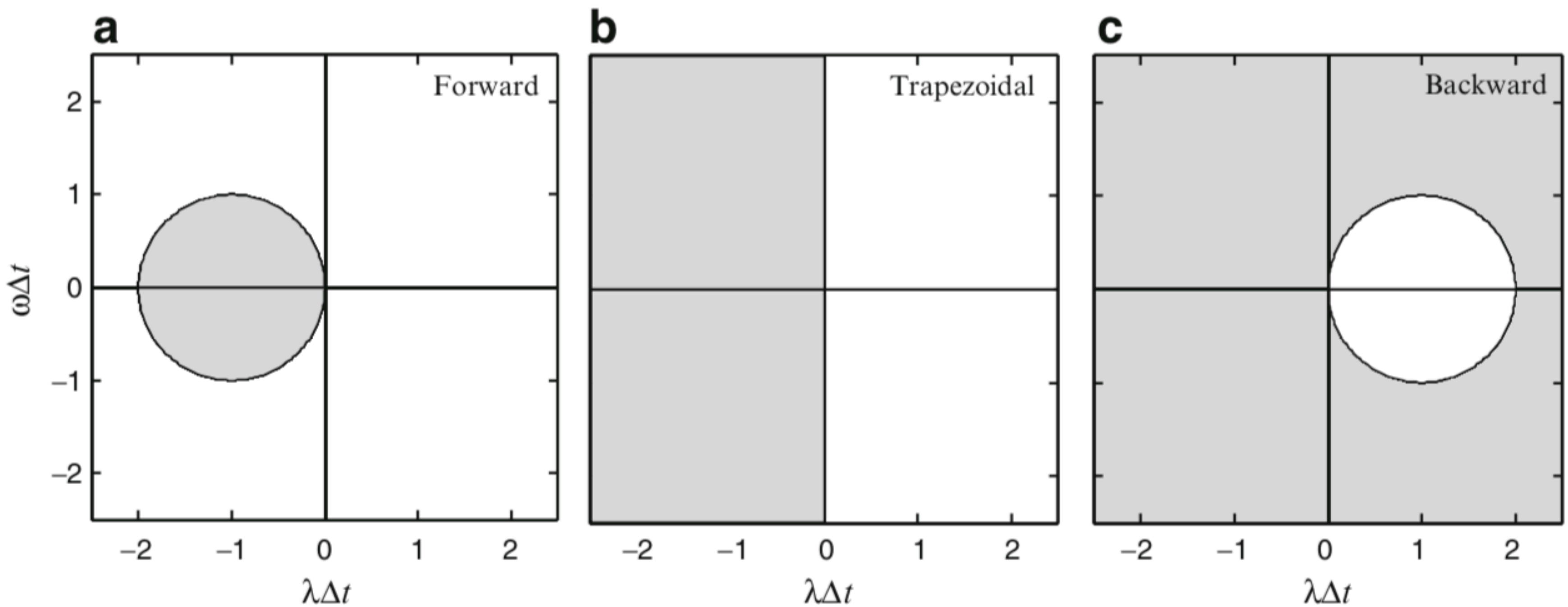


Fig. 2.1 Absolute stability regions (shaded) for **a** forward-Euler differencing, **b** trapezoidal differencing, and **c** backward-Euler differencing

Backward time

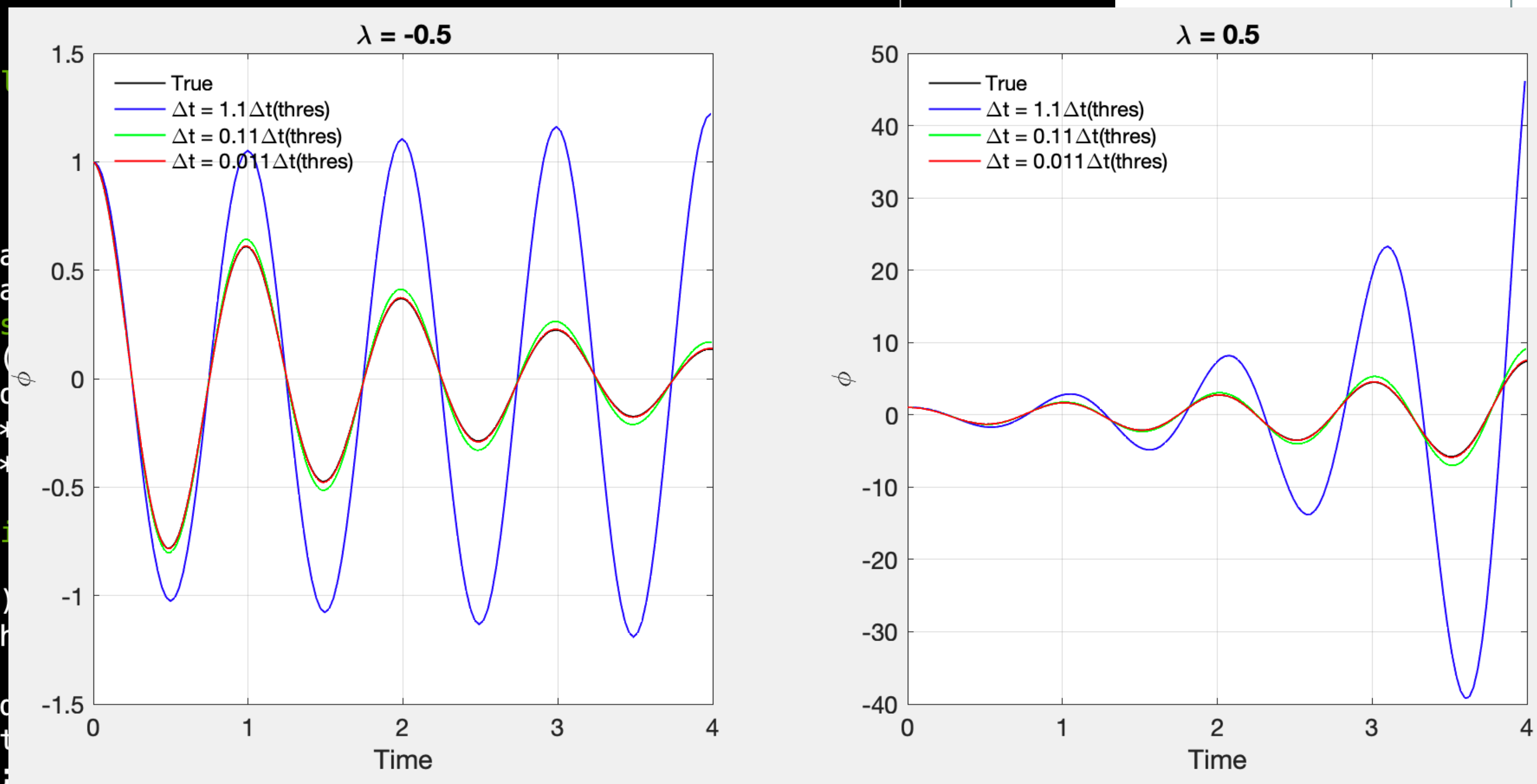
$$\frac{\phi_n - \phi_{n-1}}{\Delta t} = (\lambda + i\omega)\phi_n$$

Trapezoidal

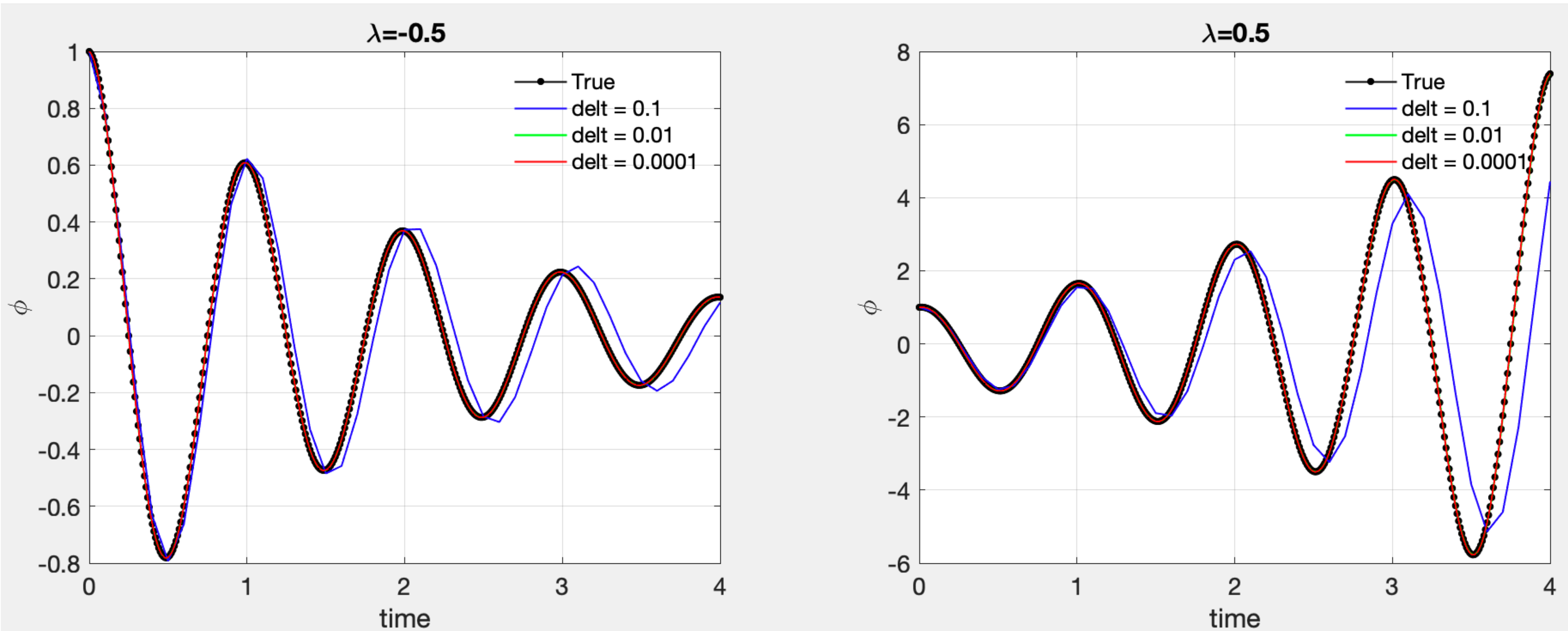
$$\frac{\phi_{n+1} - \phi_n}{\Delta t} = (\lambda + i\omega)\frac{\phi_{n+1} + \phi_n}{2}$$

Matlab Code-Forward Time

```
%%% define constants %%%
tau = 1;                                              % wave period
omega = 2*pi/(tau);                                   % wave frequency
%%% define true solution parameters %%%
phi0 = 1;
deltt = 0.01;
t0  = 0; t1 = 4*tau;
tt_true = t0:deltt:t1;
%%% select different del
col = {'b','g','r'};

lambda0 = [-0.5,0.5];
figure(1);clf;
for ilambda = 1:numel(la
    lambda = lambda0(ila
    %%% construct true s
    phi_true = phi0*exp(
    gamma = (lambda+1i*c
    delt_shred = abs(-2*
    delt0 = [delt_shred*

    %%% plot true soluti
    figure(1);
    subplot(1,2,ilambda)
    plot(tt_true,real(ph
    %%% solve ODE %%%
    for idtt = 1:numel(d
        delt = delt0(idt
        tt = t0:delt:t1;

        phi = nan(size(tt));
        phi(1) = phitrue(1);
        for itt = 2:numel(tt)
            phi(itt) = phi(itt-1)*(1+gamma*delt);
        end
        plot(tt,real(phi),col{idtt},'linewidth',1);
```



Trapezoidal



Trapezoidal method is less critical to Δt. But still, all show amplitude and phase errors compared with true solutions. How to quantify these errors?

Amplitude Error

Oscillation Equation

$$\frac{d\psi}{dt} = F(\psi, t) = i\omega\psi$$

$$\psi = \psi_0 e^{i\omega t}$$

$$|A|_{real} = 1$$

Forward time

$$A = \frac{\phi_{n+1}}{\phi_n} = [1 + (i\omega\Delta t)] \Longrightarrow |A| = \sqrt{1 + (\omega\Delta t)^2} \approx 1 + \frac{1}{2}(\omega\Delta t)^2 \geq 1$$

Backward time

$$\frac{\phi_n - \phi_{n-1}}{\Delta t} = i\omega\phi_n$$

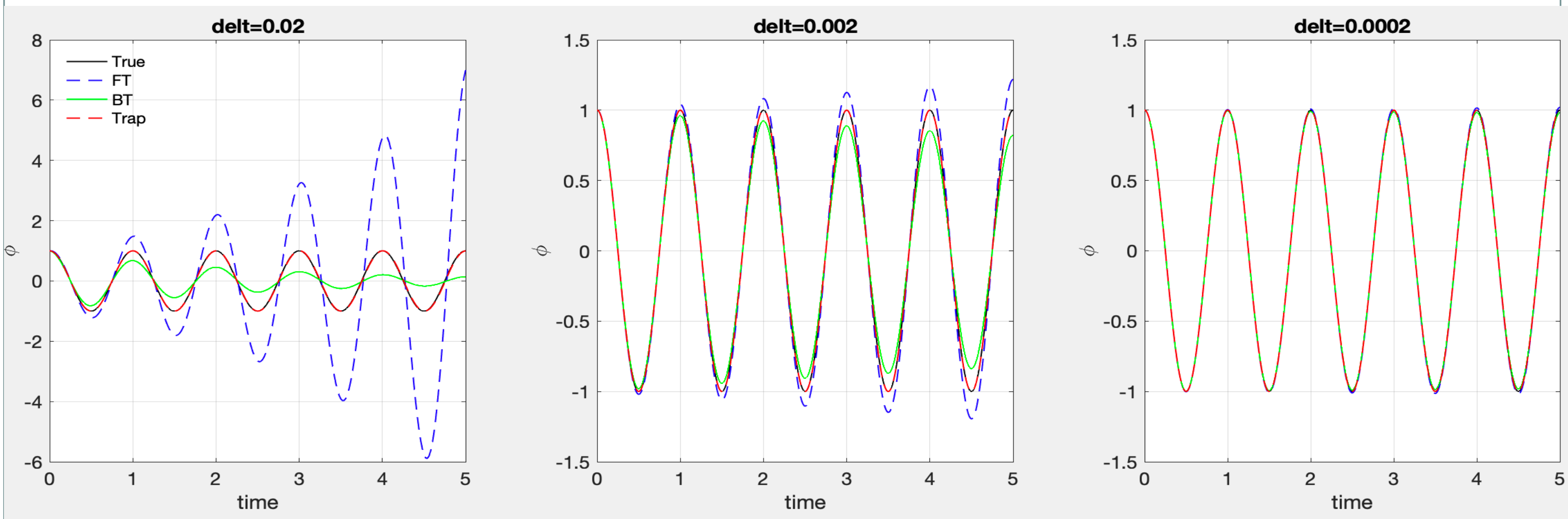
$$A = \frac{\phi_n}{\phi_{n-1}} = \frac{1}{1 - (i\omega\Delta t)} \Longrightarrow |A| = (1 + (\omega\Delta t)^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(\omega\Delta t)^2 \leq 1$$

Amplitude Error

Trapezoidal

$$\frac{\phi_{n+1}-\phi_n}{\Delta t} = i\omega \frac{\phi_{n+1}+\phi_n}{2}$$

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| = \frac{1+(i\omega\Delta t)/2}{1-(i\omega\Delta t)/2} \Longrightarrow |A| = 1$$



Forward scheme amplifies, while backward scheme damps solution.
Amp errors increase with larger Δt. Trapezoidal scheme is free of amp error.

Phase Error

Real Phase Evolution

$$\frac{d\psi}{dt} = i\omega\psi, and\ \psi = \psi_0 e^{i\omega t} \quad \Rightarrow \quad \psi(t+\Delta t) = \psi(t)\boxed{e^{i\omega\Delta t}}$$

Forward time

$$A = \frac{\phi_{n+1}}{\phi_n} = 1 + (i\omega\Delta t) = |A|e^{i\theta} \Longrightarrow \theta = \arctan(\omega\Delta t)$$

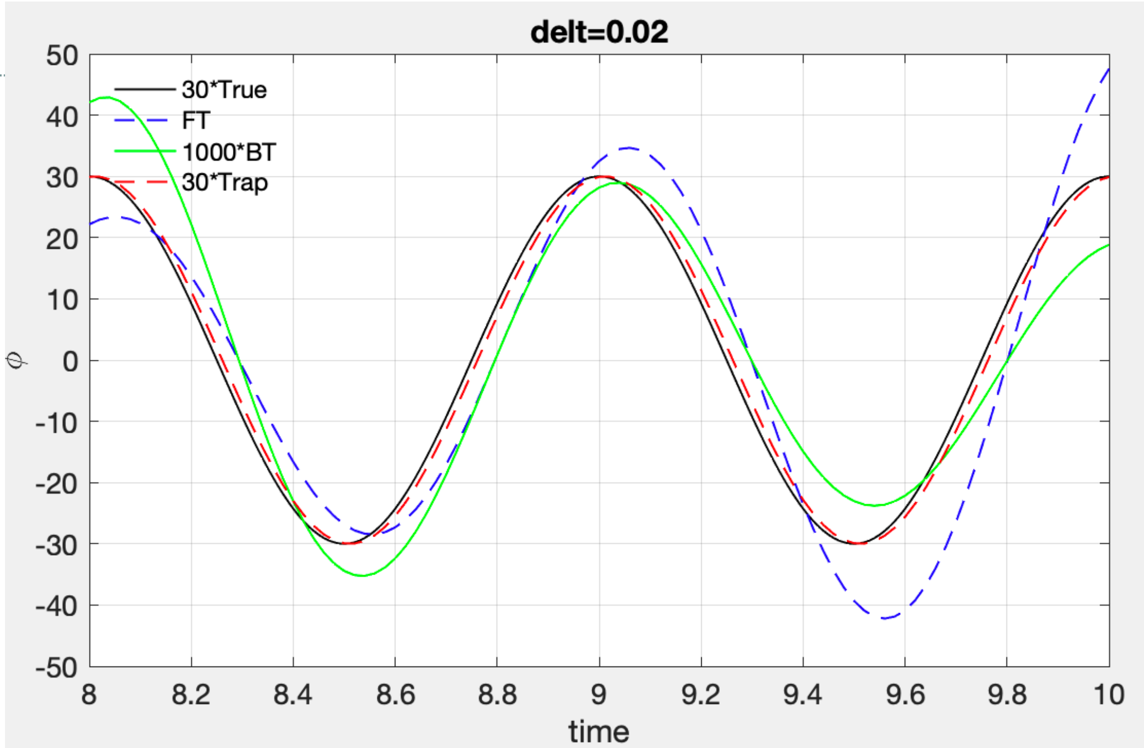
$$R = \frac{\theta}{real\ phase\ change} = \frac{\arctan(\omega\Delta t)}{\omega\Delta t} \approx 1 - \frac{(\omega\Delta t)^2}{3} < 1$$

Trapezoidal

$$A = \frac{1 + (i\omega\Delta t)/2}{1 - (i\omega\Delta t)/2} \Longrightarrow \theta = \arctan\left(\frac{\omega\Delta t}{1 - (\omega\Delta t)^2/4}\right) \quad \Rightarrow \quad R \approx 1 - \frac{(\omega\Delta t)^2}{12} < 1$$

All two-level schemes will delay the phase of true solution. The deceleration by trapezoidal is one quarter of that by forward and backward schemes.

Phase Error



Forward time

$$R = \frac{\theta}{real\ phase\ change} 1 - \frac{(\omega\Delta t)^2}{3} < 1$$

Trapezoidal

$$R \approx 1 - \frac{(\omega\Delta t)^2}{12} < 1$$

All two-level schemes will delay the phase of true solution. The deceleration by trapezoidal is less than that by forward and backward schemes.