Chapter 1

Homework 2

1.1

Consider the advection problem

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$$

Using von Neumann stability analysis, derive the stability condition for the following difference scheme:

$$\delta_{2t}\phi + c\left(\frac{4}{3}\delta_{2x}\phi - \frac{1}{3}\delta_{4x}\phi\right) = 0$$

Here

$$\delta_{nt}f(t) = \frac{f(t+n\Delta t/2) - f(t-n\Delta t/2)}{n\Delta t} \quad \delta_{nx}f(x) = \frac{f(x+n\Delta x/2) - f(x-n\Delta x/2)}{n\Delta x}$$

1.1.1

The finite difference form is

$$\frac{\phi(t+\Delta t, x) - \phi(t-\Delta t, x)}{2\Delta t} + c\left(\frac{4}{3}\frac{\phi(t, x+\Delta x) - \phi(t, x-\Delta x)}{2\Delta x} - \frac{1}{3}\frac{\phi(t, x+2\Delta x) - \phi(t, x-2\Delta x)}{4\Delta x}\right) = 0$$

Denote ϕ_j^n as the solution at time $n\Delta t$ and position $j\Delta x$, the above equation can be rewritten as

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c\left(\frac{4}{3}\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} - \frac{1}{3}\frac{\phi_{j+2}^n - \phi_{j-2}^n}{4\Delta x}\right) = 0$$

Use $\mu = c\Delta t/\Delta x$

$$6\left(\phi_j^{n+1} - \phi_j^{n-1}\right) + \mu\left[8\left(\phi_{j+1}^n - \phi_{j-1}^n\right) - \left(\phi_{j+2}^n - \phi_{j-2}^n\right)\right] = 0$$

Decompose the solution into wave components

$$\phi_j^n = \sum_{k=-N}^{N} a_k^n \, \mathrm{e}^{\mathrm{i}\, kj\Delta x}$$

For k-th component, the above equation can be simplified as

$$6\left(a_k^{n+1}\,\mathrm{e}^{\mathrm{i}\,kj\Delta x} - a_k^{n-1}\,\mathrm{e}^{\mathrm{i}\,kj\Delta x}\right) + \mu\left[8\left(a_k^n\,\mathrm{e}^{\mathrm{i}\,k(j+1)\Delta x} - a_k^n\,\mathrm{e}^{\mathrm{i}\,k(j-1)\Delta x}\right) - \left(a_k^n\,\mathrm{e}^{\mathrm{i}\,k(j+2)\Delta x} - a_k^n\,\mathrm{e}^{\mathrm{i}\,k(j-2)\Delta x}\right)\right] = 0$$

Then

$$6\left(a_k^{n+1} - a_k^{n-1}\right) + \mu a_k^n\left[8\left(\mathrm{e}^{\mathrm{i}\,k\Delta x} - \mathrm{e}^{-\mathrm{i}\,k\Delta x}\right) - \left(\mathrm{e}^{2\,\mathrm{i}\,k\Delta x} - \mathrm{e}^{-2\,\mathrm{i}\,k\Delta x}\right)\right] = 0$$

$$6\left(a_k^{n+1} - a_k^{n-1}\right) + \mu a_k^n\left(8 \cdot 2\,\mathrm{i}\sin k\Delta x - 2\,\mathrm{i}\sin 2k\Delta x\right) = 0$$

The recursion relation of a_k^n is

$$a_k^{n+1} = \mathrm{i}\,\frac{\mu}{3}\left(8\sin k\Delta x - \sin 2k\Delta x\right)a_k^n + a_k^{n-1}$$

The characteristic equation of the recursion relation is

$$p^2 = \mathrm{i}\,\frac{\mu}{3}(8\sin k\Delta x - \sin 2k\Delta x)p + 1$$

The roots of the characteristic equation are

$$p_{1,2} = \mathrm{i}\,\frac{\mu}{6}(8\sin k\Delta x - \sin 2k\Delta x) \pm \sqrt{1 - \frac{\mu^2}{36}(8\sin k\Delta x - \sin 2k\Delta x)^2}$$

The general term of a_k^n is

$$a_k^n = P_1(p_1)^n + P_2(p_2)^n$$

where $P_{1,2}$ are determined from initial conditions. To stay stable, the amplification factor of a_k^n needs to be smaller than 1, then amplitudes of both roots need to be smaller than 1. If $1 - \mu^2/36(8\sin k\Delta x - \sin 2k\Delta x)^2 < 0$, then

$$|p_{1,2}| = \left|\frac{\mu}{6}(8\sin k\Delta x - \sin 2k\Delta x) \pm \sqrt{\frac{\mu^2}{36}(8\sin k\Delta x - \sin 2k\Delta x)^2 - 1}\right|$$

In this case, if $8\sin k\Delta x - \sin 2k\Delta x > 0$,

$$\left|\frac{\mu}{6}(8\sin k\Delta x - \sin 2k\Delta x) + \sqrt{\frac{\mu^2}{36}(8\sin k\Delta x - \sin 2k\Delta x)^2 - 1}\right| > 1$$

And if $8\sin k\Delta x - \sin 2k\Delta x < 0$,

$$\left|\frac{\mu}{6}(8\sin k\Delta x - \sin 2k\Delta x) - \sqrt{\frac{\mu^2}{36}(8\sin k\Delta x - \sin 2k\Delta x)^2 - 1}\right| > 1$$

Thus for both situations ($8\sin k\Delta x - \sin 2k\Delta x$ greater or smaller than 0) when $\mu^2/36(8\sin k\Delta x - \sin 2k\Delta x)^2 > 1$, one of the roots will have an amplitude larger than 1, therefore the amplification factor of a_k^n will be larger than 1, and the scheme is unstable.

To make the amplification factor of a_k^n smaller than 1, $\mu^2/36(8\sin k\Delta x - \sin 2k\Delta x)^2 \leq 1$, then

$$|p_{1,2}| = \sqrt{\left[\frac{\mu}{6}(8\sin k\Delta x - \sin 2k\Delta x)\right]^2 + 1 - \frac{\mu^2}{36}(8\sin k\Delta x - \sin 2k\Delta x)^2} = 1$$

Both roots have amplitude equal to 1. In this case, a_k^n will stay bounded, and the scheme is stable. So the requirement for μ is

$$\mu \leq \frac{6}{|8\sin k\Delta x - \sin 2k\Delta x|}$$

Define $f(x) = 8\sin x - \sin 2x$, then

$$f'(x) = 8\cos x - 2\cos 2x$$

Let $f'(x_0) = 0$, then

$$2\cos^2 x_0 - 4\cos x_0 - 1 = 0$$

Solve for $\cos x_0$, discard the root which is larger than 1, we get the criteria of $f(x)$'s extreme points

$$\cos x_0 = 1 - \frac{\sqrt{6}}{2}$$

Then the extreme values of $f(x)$ are

$$f(x_0) = 8\sin x_0 - \sin 2x_0 = (8 - 2\cos x_0)\sin x_0 = (6+\sqrt{6})\sin x_0 = \pm(6+\sqrt{6})\sqrt{\sqrt{6} - \frac{3}{2}}$$

Therefore the range of $f(x)$ is

$$-(6+\sqrt{6})\sqrt{\sqrt{6} - \frac{3}{2}} \leq f(x) \leq (6+\sqrt{6})\sqrt{\sqrt{6} - \frac{3}{2}}$$

The range of $6/|8\sin k\Delta x - \sin 2k\Delta x|$ is

$$\frac{6}{|8\sin k\Delta x - \sin 2k\Delta x|} \geq \frac{6}{(6+\sqrt{6})\sqrt{\sqrt{6} - \frac{3}{2}}} = \frac{\sqrt{6}}{(\sqrt{6}+1)\sqrt{\sqrt{6} - \frac{3}{2}}}$$

To satisfy the stability conditions for all wave numbers, the range of μ is

$$\mu \leq \frac{\sqrt{6}}{(\sqrt{6}+1)\sqrt{\sqrt{6} - \frac{3}{2}}} \approx 0.73$$

1.2

Read Takacs (1985) paper, derive the five coefficients (a_{-2}, a_{-1}, a_0, a_1, and a_2) for the fourth order Takacs method. Examine and understand the function `F_phin_4s_Takacs.m`. Are the coefficients correctly set up?

1.2.1

The equation set associated with the 4th order Takacs method is

$$\sum_{j'=-2}^{2} j'^m a_{j'} = (-\mu)^m \quad m = 0, \dots, 4$$

Denote A as the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{pmatrix}$$

The augmented matrix of the linear system is

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 & -\mu \\ 4 & 1 & 0 & 1 & 4 & \mu^2 \\ -8 & -1 & 0 & 1 & 8 & -\mu^3 \\ 16 & 1 & 0 & 1 & 16 & \mu^4 \end{pmatrix}$$

Apply a series of elementary operations to each row of A', A' can be reduced to upper triangular form

$$A' \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 & -\mu + 2 \\ & & 2 & 6 & 12 & \mu^2 - 3\mu + 2 \\ & & & 6 & 24 & -\mu^3 + 3\mu^2 - 2\mu \\ & & & & 24 & \mu^4 - 2\mu^3 - \mu^2 + 2\mu \end{pmatrix}$$

Further reduce A' to triangular form

$$A' \to \begin{pmatrix} 24 & & & & & \mu^4 + 2\mu^3 - \mu^2 - 2\mu \\ & 6 & & & & -\mu^4 - \mu^3 + 4\mu^2 + 4\mu \\ & & 4 & & & \mu^4 - 5\mu^2 + 4 \\ & & & 6 & & -\mu^4 + \mu^3 + 4\mu^2 - 4\mu \\ & & & & 24 & \mu^4 - 2\mu^3 - \mu^2 + 2\mu \end{pmatrix}$$

The coefficients are

$$\begin{cases} a_{-2} & = \frac{1}{24}\left(\mu^4 + 2\mu^3 - \mu^2 - 2\mu\right) \\ a_{-1} & = \frac{1}{6}\left(-\mu^4 - \mu^3 + 4\mu^2 + 4\mu\right) \\ a_0 & = \frac{1}{4}\left(\mu^4 - 5\mu^2 + 4\right) \\ a_1 & = \frac{1}{6}\left(-\mu^4 + \mu^3 + 4\mu^2 - 4\mu\right) \\ a_2 & = \frac{1}{24}\left(\mu^4 - 2\mu^3 - \mu^2 + 2\mu\right) \end{cases}$$

1.3

Use the code "`Advection_PDE_RK_2rdSpace_LaxWen_Takacs_stability_2.m`" and answer the following questions:

1. For "waveform = 3 (select one wave component to start)", try CFL $= 0.2, 0.5, 0.9, 1.01, 1.05, 1.1$, run the model for the same amount of time (not the same amount of time step), comment on the CFL stability condition. Mark the maximum amplitudes and compare these among different schemes and different CFL numbers for each individual scheme.
   Note: you can name the time you will stop the model, as long as you can make up your story and put forward reasonings. (stopping time is flexible).

2. For “waveform = 3”, turn on two waves. Comment on dispersion errors of different schemes.
3. Which scheme leads to the best amplitude performance? Comment on amplitude and phase errors, respectively?
4. Replace the wave function to triangle and step functions, refer to questions 1 and 3, and comment on what you find. Discuss shared principles among these three functions, and differences.

Some of the answers will be quantitative with respect to the settings you choose. The comments from you will learn from these exercises can be qualitative.

1.3.1

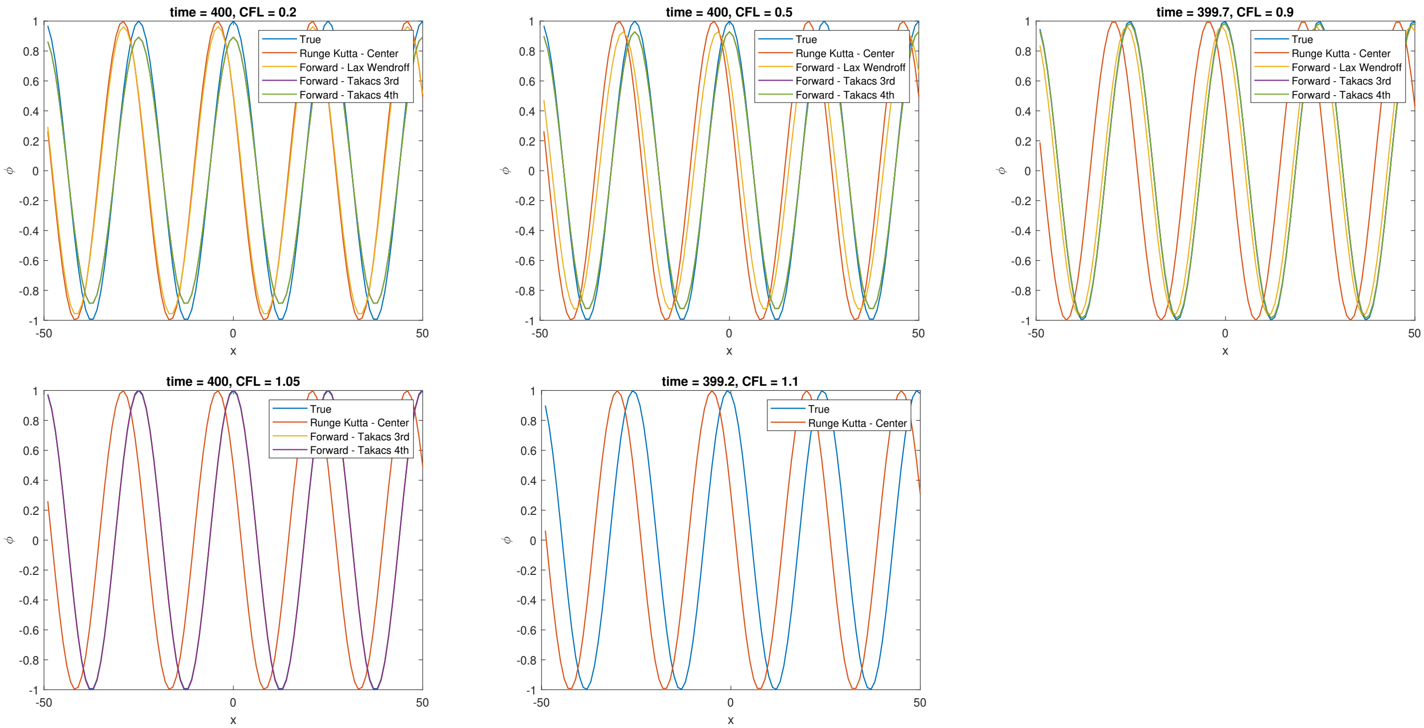

Figure 1.1: Comparison of different numerical schemes with varying CFL numbers, only stable solutions are shown.

1. In figure 1.1, for CFL < 1, all numerical schemes are stable. But for CFL > 1, only scheme with Runge-Kutta in time and centered difference in space is stable.
2. In figure 1.2, schemes with forward difference in time and Takacs methods in space are in phase with the true solution, scheme with Runge-Kutta in time, centered difference in space and scheme with forward difference in time, Lax-Wendroff method in space don’t resolve the correct phase.
3. Schemes with forward difference in time and Takacs methods in space have both the best amplitude and phase performance. In the situation of two trigonometric waves, during the time of simulation, two Takacs methods don’t show distinguishable differences.
4. In figure 1.3, schemes with forward difference in time and Takacs methods in space perform best among four difference numerical schemes, two Takacs schemes are correct in phase, but they still underestimated amplitudes in the situation of triangle wave.

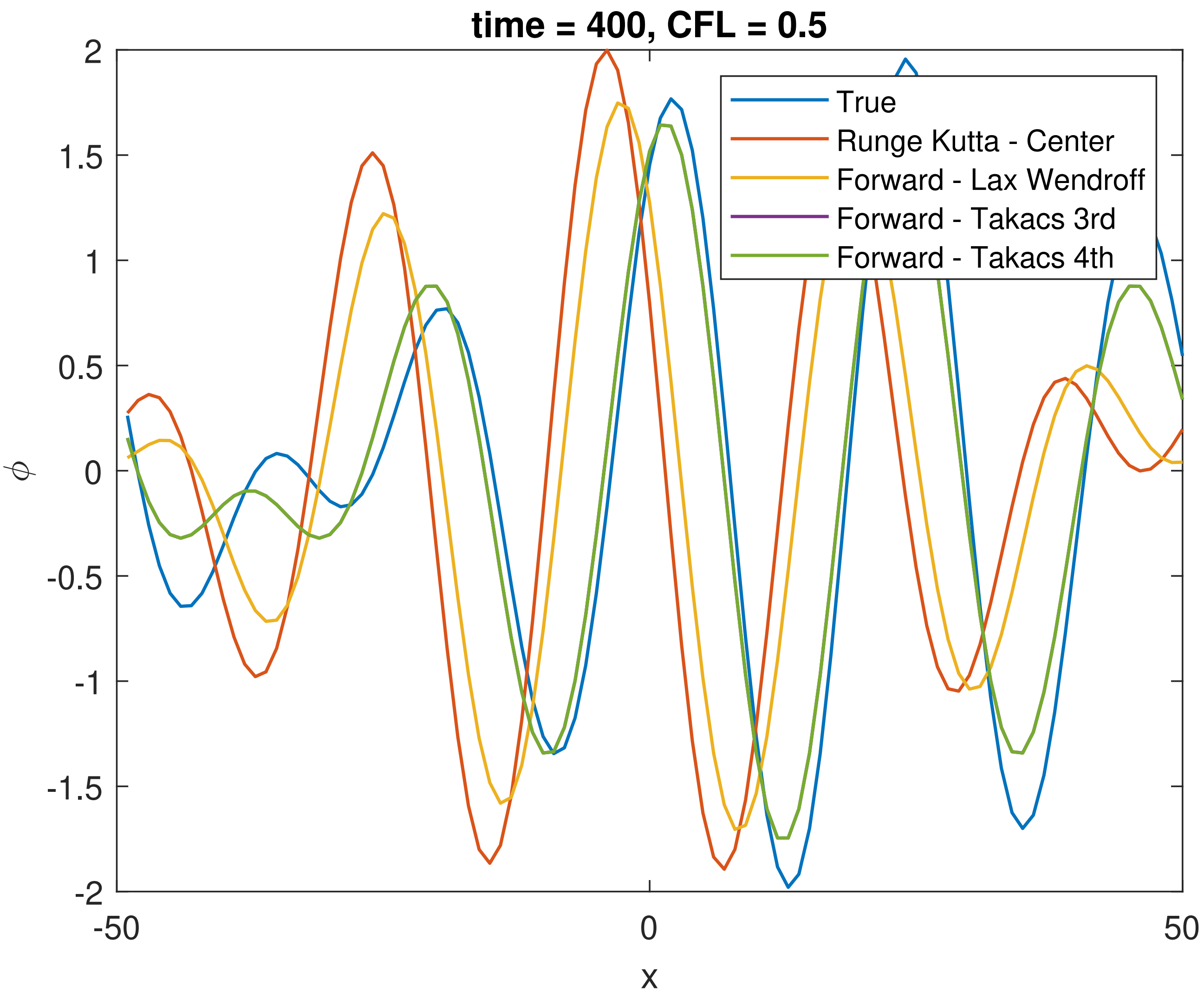

Figure 1.2: Comparison of different numerical schemes for two trigonometric waves.

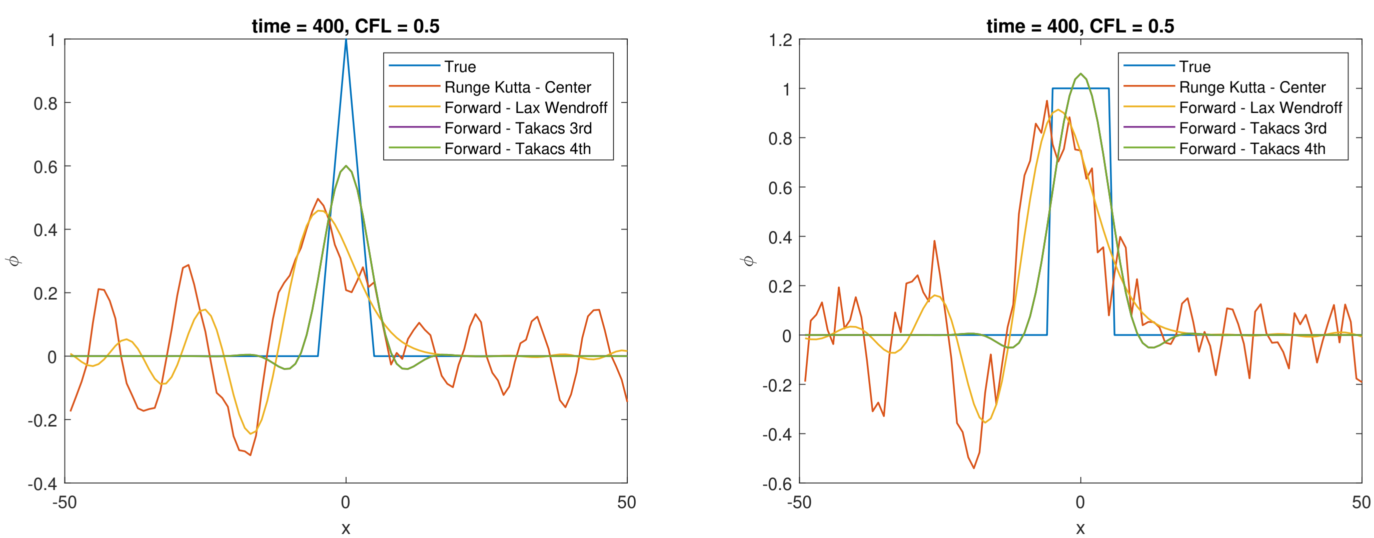

Figure 1.3: Comparison of different numerical schemes for triangle and rectangle waves.