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1. $\frac{d\psi}{dt} = F(\psi, t) = \gamma\psi = (\lambda + i\omega)\psi$

a) Backward-Euler differencing order of accuracy:

① $\frac{d\psi}{dt}(t_n) \approx \frac{\psi(t_n) - \psi(t_n - \Delta t)}{\Delta t}$

now we can taylor expand $\psi(t_n - \Delta t)$:

② $\psi(t_n - \Delta t) = \psi(t_n) - \Delta t \frac{d\psi}{dt} + \underbrace{\frac{(\Delta t)^2}{2} \frac{d^2\psi}{dt^2}}_{\text{cancel out } \Delta t} - \text{h.o.t.}$

combine ①/②:

$$\frac{\psi(t_n) - \psi(t_n - \Delta t)}{\Delta t} - \frac{d\psi}{dt}(t_n) = \frac{\Delta t}{2} \frac{d^2\psi}{dt^2}$$

$\therefore$ $\boxed{\text{Backward-Euler differencing has order of Accuracy} = 1}$
($\Delta t \propto$ power of 1)

Trapezoidal Differencing:

Since our goal is to obtain the order of accuracy, we can look at the general trapezoidal method equation:

$$\frac{\phi_{n+1} - \phi_n}{\Delta t} = \overset{\gamma}{(\lambda + i\omega)} \frac{\phi_{n+1} + \phi_n}{2}$$

and if we taylor expand $\phi_{n+1}$:

$$\phi_{n+1} = \psi(t_n + \Delta t) = \psi(t_n) + \Delta t \frac{d\psi}{dt}(t_n) + \frac{(\Delta t)^2}{2} \frac{d^2\psi}{dt^2} + \text{h.o.t.}$$

now if we examine the 2 previous equations we see on the RHS that the term $\frac{(\Delta t)^2}{2} \frac{d^2\psi}{dt^2}$ survives $\therefore$ $\boxed{\text{order of accuracy} = 2}$

1.

b) $|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \leq 1 + \eta \Delta t$

For Backward Euler we will have to simplify:

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \qquad \phi_{n+1} \xrightarrow{n-1} \phi_n$$

$$\phi_n \xrightarrow{n-1} \phi_{n-1}$$

$$= \left|\frac{\phi_n}{\phi_{n-1}}\right| = \left|\frac{1}{1-(i\gamma\Delta t)}\right| \qquad \text{From } \phi_{n-1} \propto \phi_n - \underbrace{\Delta t \frac{d\phi}{dt}}_{} + \frac{(\Delta t)^2}{2}\frac{d^2\phi}{dt^2} + \dots$$

$$(1-\lambda\Delta t)^2 + w^2\Delta t^2 \leq 1$$

$$1 - 2\lambda\Delta t + (w^2+\lambda^2)\Delta t^2 \leq 1$$

$$\Delta t \leq \frac{2\lambda}{(w^2+\lambda^2)}$$

$$= \boxed{\frac{1}{\left((1-(\lambda\Delta t))^2 + (w\Delta t)^2\right)^{1/2}}} \qquad |1-\gamma\Delta t| = \sqrt{(1-\gamma\Delta t)^2} = \sqrt{1 + (\lambda\Delta t)^2 - 2\lambda\Delta t} \; w$$

Comparing to figure 2, we find that as long as $\lambda\Delta t \leq 0$ we have A-stability and can use finite $\Delta t$ without worry of model blowing up. As soon as $\lambda > 0$ we can only avoid model blowing up if we have a large $\Delta t$.

For trapezoidal we have:

$$|A| = \left|\frac{\phi_{n+1}}{\phi_n}\right| \Rightarrow \frac{\phi_{n+1} - \phi_n}{\Delta t} = \gamma\frac{\phi_{n+1}+\phi_n}{2}$$

$$\frac{2}{\phi_{n+1}+\phi_n} \cdot \frac{\phi_{n+1}-\phi_n}{\Delta t} = \gamma$$

$\therefore |A| = 1$

$$\downarrow$$

$$\frac{1 + \frac{\gamma\Delta t}{2}}{1 - \frac{\gamma\Delta t}{2}} = 1$$

this means for any $\lambda\Delta t < 0$ we have absolute stability. Figure 3 reinforces this claim, and tells us $\lambda\Delta t > 0$ does not have any absolute stability, which if we consider this makes sense because we would arrive at an impossible statement

$$\left[1 + (+) \neq 1 - (-)\right]$$

2. Starting with $A = 1 - \frac{1}{2}(\omega \Delta t)^2 \leq 1$

we can solve for $\Delta t$ and get

$\Delta t \leq \dfrac{2\lambda}{(\lambda^2 + \omega^2)}$ which corresponds to a parameter in the code.

# It seems after adjusting $\Delta t$ values that when $\lambda \Delta t$ is positive, the higher time difference ($\lambda \Delta t$) does a better modeling job which is expected because the numerical solution should be approaching 0 as the true solution increases exponentially, but the large $\Delta t$ slows this from happening. When $\lambda \Delta t < 0$, no true solution was found, and the smaller $\Delta t$ (0.5) did the best job. Overall, for positive $\lambda \Delta t$, we want high $\Delta t$ steps, and for negative $\lambda \Delta t$, we want lower (more accurate) $\Delta t$ steps.

The true solution for $\lambda = -0.5$ shouldn't be 0, but I couldn't get the correct graph to show.

| | $\lambda < 0$ | $\lambda > 0$ |
|---|---|---|
| 2: | worst | best |
| 1.25: | okay | okay |
| 0.5: | best | worst |

3. While I could not get graphical data to support this, I know that choosing a higher $\Delta t$ step will cause divergence from the true solution. In general, trapezoid method does a much better job at avoiding amplitude error for the case of $\lambda \Delta t < 0$. A phase error is still present, but being able to properly model essentially any frequencies for a given $\Delta t$ (assuming $\lambda < 0$) is powerful.

4. I think the largest differences between these 3 approaches can be identified by considering what they do the best. If we need as little amplification error as possible, Trapezoidal method is the easy choice. If instead we need to amplify our model due to some little signal, then forward time method is the best choice. Dampening of course comes from backward time method, and both FT/BT have areas that are good at minimizing amplitude error. FT when our $\lambda \Delta t$ is $< 1$ and our frequencies are relatively low. BT on the other hand does better at higher frequencies or time steps ($\Delta t$) and larger $\Delta t$ steps provided $\lambda \Delta t < 0$ (like FT).

BT and Trapezoidal are perfectly A-stable. FT is not.