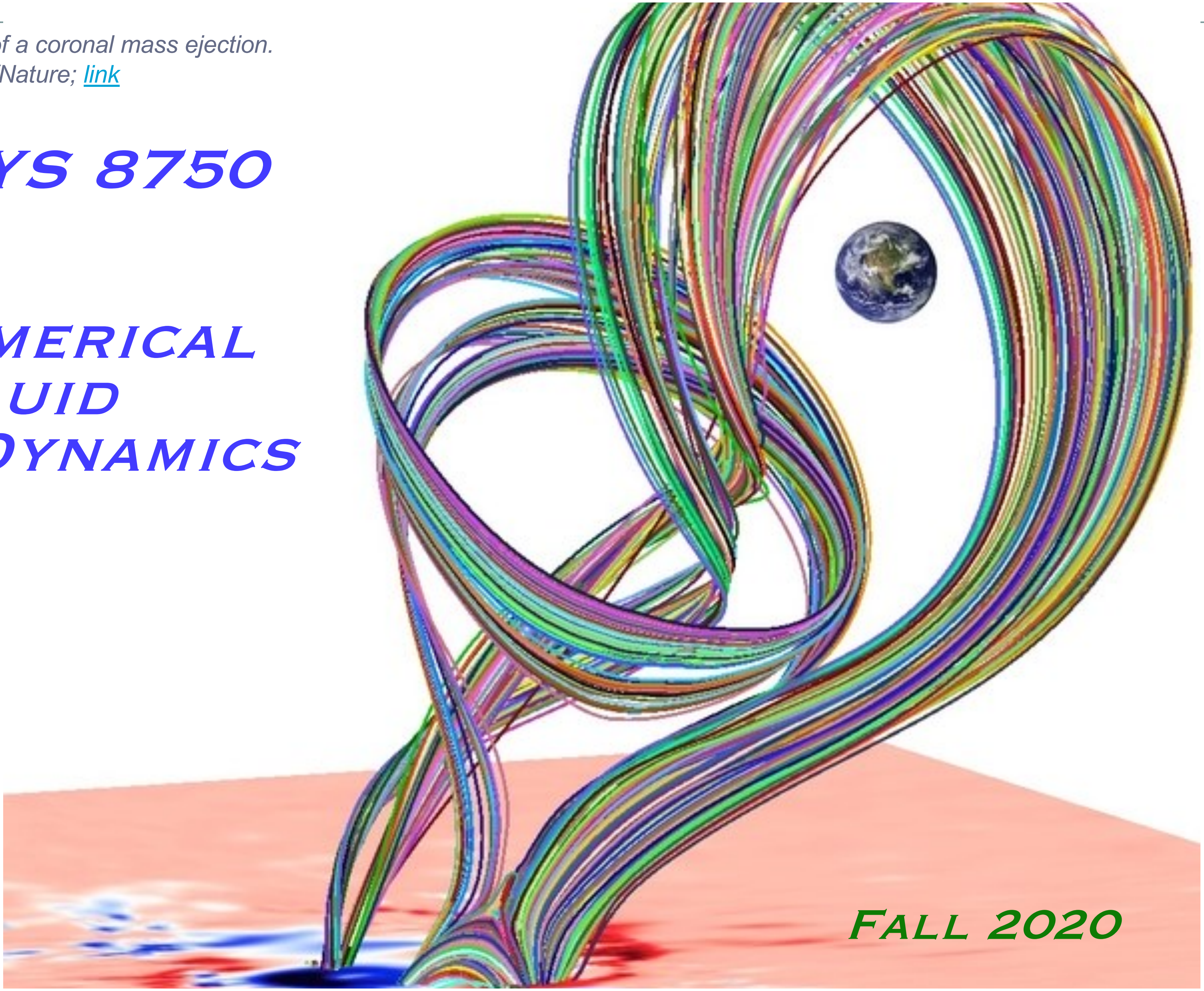
Simulation of a coronal mass ejection.
Amari et al./Nature; link

PHYS 8750

NUMERICAL FLUID DYNAMICS



FALL 2020

PHYS 8750

9:30 – 10:45 AM

Tuesday,Thursday 2020

Class #1

Outline

- 1) Introduction
- 2) Terminology and basics
  ODE vs. PDE
  Order of PDE
  Linear vs. nonlinear
  Types of equations
- 3) Numerical Scheme
  Forward/backward/leapfrog
  Upstream/downstream/center space
- 4) Criteria to evaluate
  Stability; accuracy; convergence; consistency

Our world consists of ordinary and partial differential equations

Wave equations:

$$\frac{\partial \psi}{\partial t} + \mathrm{c}\frac{\partial \psi}{\partial x} = 0 \qquad \text{Solution:} \psi = f(x - ct)$$

Heat conduction/diffusion equations:

$$\frac{\partial f}{\partial t} = k\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

Laplace Equation:

$$\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) = 0$$

Maxwell's Equation:

E and B	E, B, D, and H
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu\, \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Left side of "=" ... time rate of change of each variable.

Right side: *advective* terms for each variable.

$$\frac{\partial u}{dt} = -\vec{V}\cdot\vec{\nabla}u - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{dt} = -\vec{V}\cdot\vec{\nabla}v - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{dt} = -\vec{V}\cdot\vec{\nabla}w - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} + g\frac{\theta}{\bar{\theta}} + \nu\nabla^2 w$$

$$\frac{\partial \theta}{\partial t} = -\vec{V}\cdot\vec{\nabla}\theta + Q(x,y,z,t) + \nu\nabla^2\theta$$

$$\frac{\partial p}{\partial t} = -c_s^2\left[\bar{\rho}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial(\bar{\rho}z)}{\partial z}\right] + \nu\nabla^2 p$$

VARIABLES:

u: X-wind component
v: Y-wind component
w: Z-wind component
θ: potential temperature
p: pressure

Sometimes, the solutions to these PDEs are analytical; but more than often, they are not! So we need to solve the equations numerically.

Finite-Difference

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$$

Forward time

Upstream space

$$\frac{\partial \psi}{\partial t} = \lim_{\Delta t \to 0} \frac{\psi(t+\Delta t, x) - \psi(t,x)}{\Delta t}$$

$$\frac{\partial \psi}{\partial x} = \lim_{\Delta x \to 0} \frac{\psi(t, x+\Delta x) - \psi(t,x)}{\Delta x}$$

$$\psi(t+\Delta t, x) = \psi(t,x) - c\frac{\Delta t}{\Delta x}\left(\psi(t, x+\Delta x) - \psi(t,x)\right)$$

Δt, Δx can approach zero, but will never be zero. And due to computational limitation and expenses, we aim to obtain the numerical solutions that approach the true solution with finite but small Δt, Δx.

1. For different physical system/equations, what are the options to do finite-difference?
2. How do we evaluate the different numerical schemes? Advantages and limitations?
3. Typical strategies?

Terminology and Basics

- ODE vs. PDE
  - ODE: contains only one independent variable, functions of this variable, and the derivatives of those functions.

$\boxed{\psi = \psi(\mathrm{t})}$ $\boxed{F(t,\psi',\psi'',\ldots,\psi^{(n-1)},\psi^{(n)}) = 0}$ $\boxed{e.g., \frac{d\psi}{dt} = \lambda\psi}$

  - PDE: more than one independent variables are involved.

$\boxed{\psi = \psi(\mathrm{t},\mathrm{x})}$ $\boxed{\frac{\partial\psi}{\partial t} + \mathrm{c}\frac{\partial\psi}{\partial x} = 0}$

$$\boxed{F(t,\psi_t',\psi_t'',\ldots,\psi_t^{(n)},\mathrm{x},\psi_x',\psi_x'',\ldots,\psi_x^{(n)},\psi_{tx}'',\psi_{ttx}''',\psi_{txx}''',\ldots) = 0}$$

Order of ODE and PDE

- Determined by highest order of derivatives

$$F(t, \psi', \psi'', \dots, \psi^{(n-1)}, \psi^{(n)}) = 0$$

Order of n

$$F(t, \psi_t', \psi_t'', \dots, \psi_t^{(n)}, \mathrm{x}, \psi_x', \psi_x'', \dots, \psi_x^{(n)}, \psi_{tx}'', \psi_{ttx}''', \psi_{txx}''', \dots) = 0$$

$$\frac{\partial \psi}{\partial t} + \mathrm{c}\frac{\partial \psi}{\partial x} = \mathrm{a}\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial t^2} + b\frac{\partial^2 \psi}{\partial x^2}\frac{\partial \psi}{\partial y}$$

Second Order

$$\frac{\partial^4 \psi}{\partial t^4} + \left(\frac{\partial \psi}{\partial t}\right)^3 - \psi^3 = \frac{\partial^2 \psi}{\partial x^2}\left(\frac{\partial \psi}{\partial y}\right)^2$$

Fourth

$$\left(\frac{\partial \psi}{\partial t}\right)^5 + \sin(\psi)^5 = \frac{\partial \psi}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)^2$$

First

Linearity

- A *linear* PDE is of first degree (linear) in the unknown functions (ψ) and their derivatives.
- thus the coefficients depend *only* on *independent* variables.

linear

$$\frac{\partial \psi}{\partial t} + \mathrm{c}\frac{\partial \psi}{\partial x} = 0$$

$$a\frac{\partial^2 \psi}{\partial x^2} + b\frac{\partial^2 \psi}{\partial y^2} = \psi$$

quasi-linear

$$\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi}{\partial x} = 0$$

nonlinear

$$\left(\frac{\partial \psi}{\partial t}\right)^2 + \mathrm{c}\frac{\partial \psi}{\partial x} = 0$$

$$\frac{\partial \psi}{\partial t} + \sin\left(\frac{\partial \psi}{\partial x}\right) = 0$$

$$a\frac{\partial^2 \psi}{\partial x^2} + b\frac{\partial^2 \psi}{\partial y^2} = \psi^2$$

Type of PDE

- Suppose we have a second-order PDE of the form:

$$a(x_1,x_2)\frac{\partial\psi^2}{\partial {x_1}^2}+b(x_1,x_2)\frac{\partial\psi^2}{\partial x_1\partial x_2}+c(x_1,x_2)\frac{\partial\psi^2}{\partial {x_2}^2}+$$
$$d(x_1,x_2)\frac{\partial\psi}{\partial x_1}+e(x_1,x_2)\frac{\partial\psi}{\partial x_1}+f(x_1,x_2)\psi=g(x_1,x_2)$$

Elliptical: $b^2-4ac<0, e.g., 2D\ Laplace\ equation$

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0, a=1, b=0, c=1, b^2-4ac=-4$$

Hyperbolic $b^2-4ac>0, e.g., wave\ (advection, transport)\ equation,$

$$\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}=0\rightarrow\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^2\psi}{\partial x\partial t}, a=1, b=1, c=0, b^2-4ac=1$$

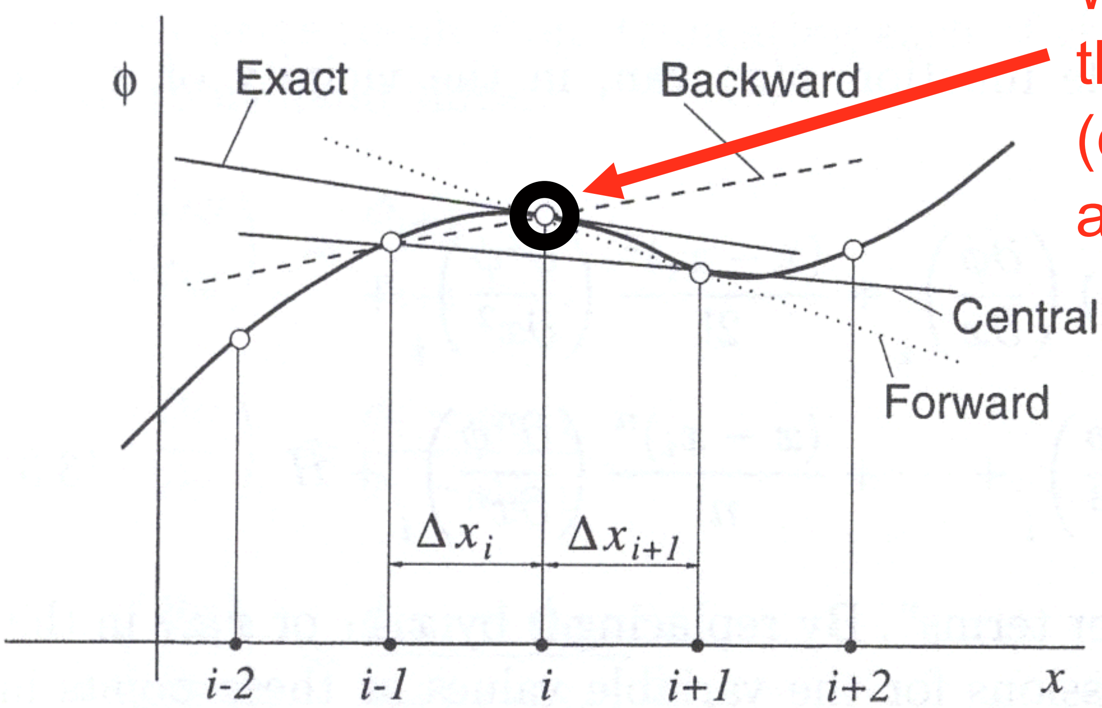
Parabolic: $b^2-4ac=0, e.g., heat\ diffusion$

$$\frac{\partial f}{\partial t}=k\left(\frac{\partial^2 f}{\partial x^2}\right), a=0, b=0, c=k, b^2-4ac=0$$

Finite difference approximations to derivatives

$$\frac{\partial \psi}{\partial x} = \lim_{\Delta x \to 0} \frac{\psi(x + \Delta x) - \psi(x)}{\Delta x}$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(x_i) - \psi(x_{i-1})}{\Delta x}$$



We want the slope (derivative) at this point

Fig. 3.2. On the definition of a derivative and its approximations

Approximations Forward

$$\frac{\partial \psi}{\partial x} = \lim_{\Delta x \to 0} \frac{\psi(x + \Delta x) - \psi(x)}{\Delta x}$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(x_{i+1}) - \psi(x_i)}{\Delta x}$$

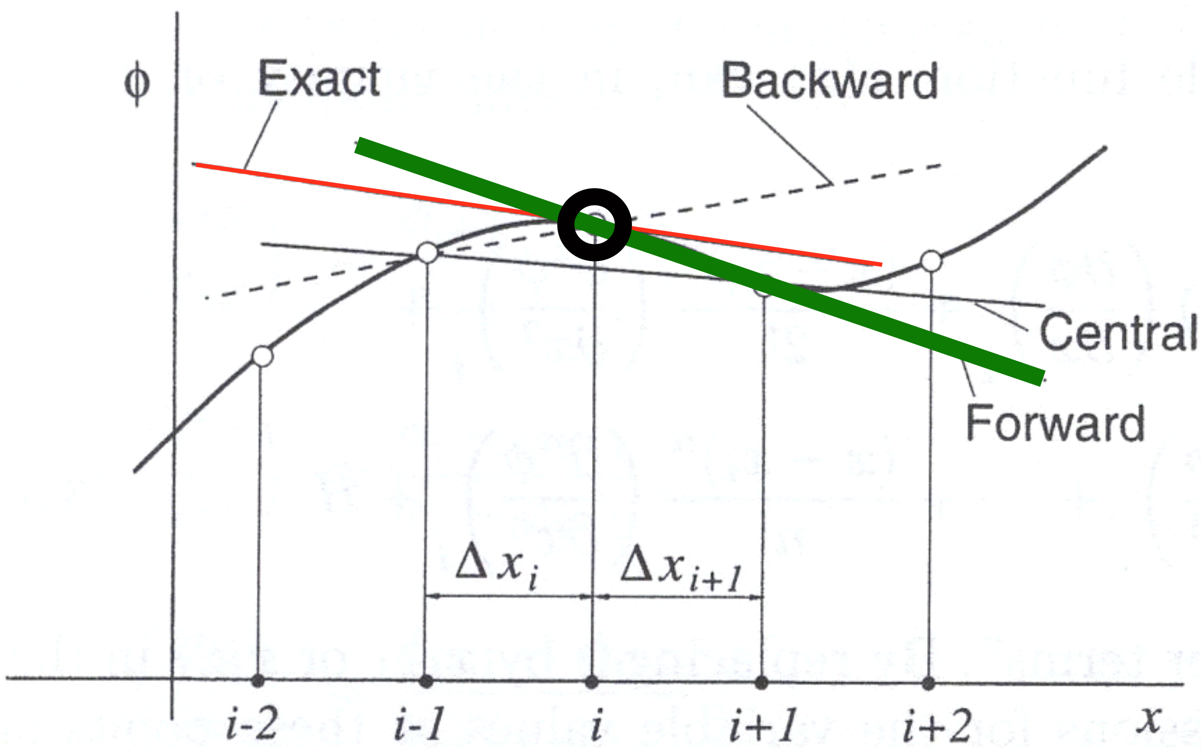


Fig. 3.2. On the definition of a derivative and its approximations

Approximations Central

$$\frac{\partial \psi}{\partial x} = \lim_{\Delta x \to 0} \frac{\psi(x+\Delta x) - \psi(x)}{\Delta x}$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(x_{i+1}) - \psi(x_{i-1})}{2\Delta x}$$

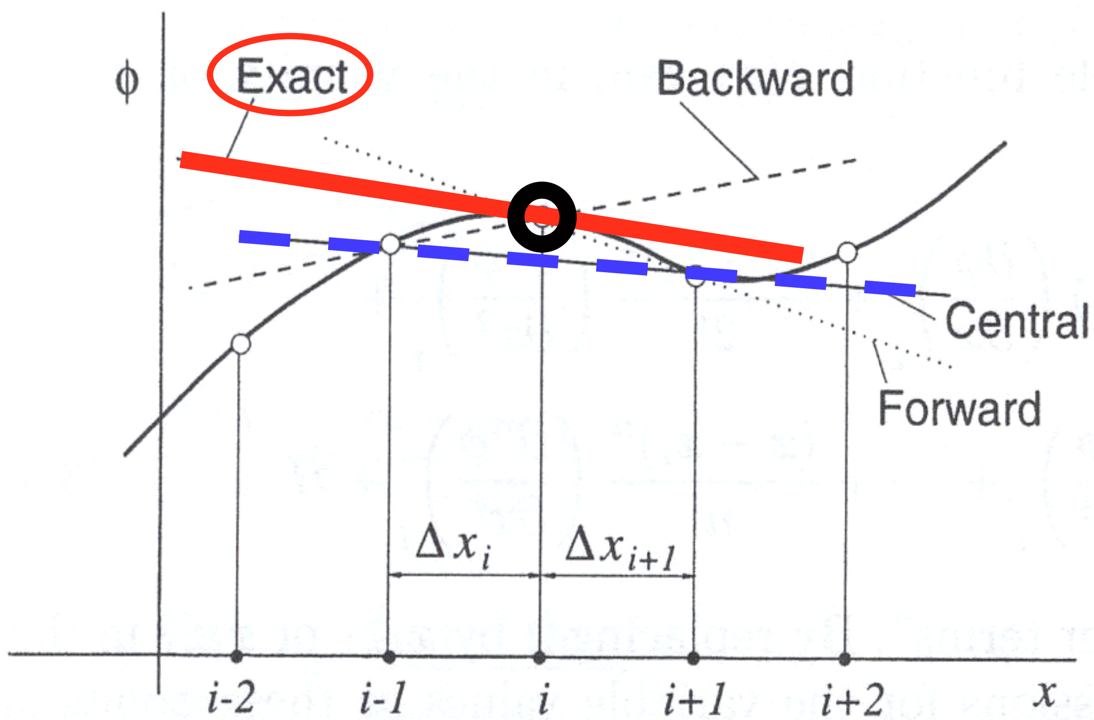


Fig. 3.2. On the definition of a derivative and its approximations