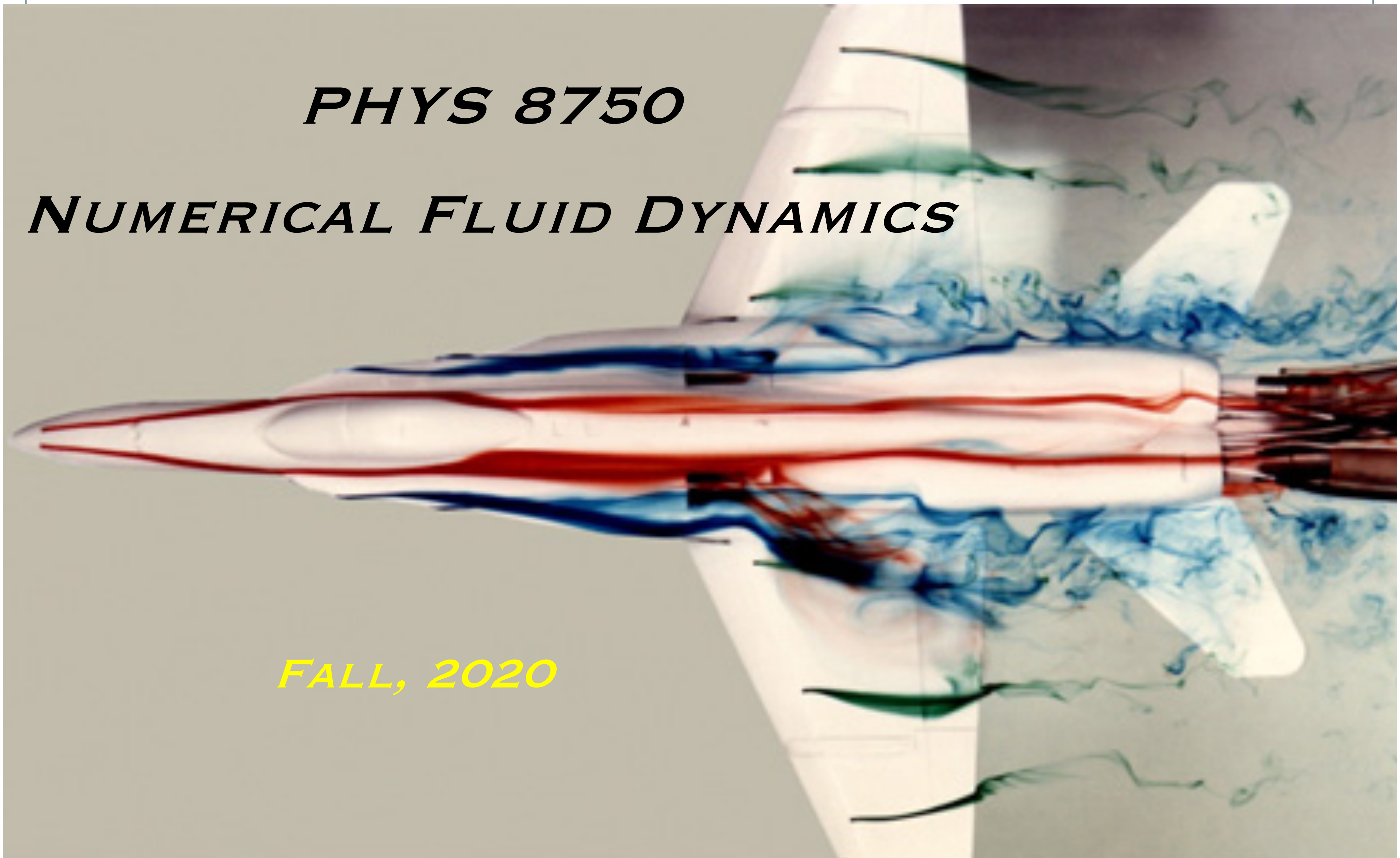
NASA/DRYDEN flow visualization facility

PHYS 8750

Numerical Fluid Dynamics



Fall, 2020

PHYS 8750

Class #4 (Chapter 2.4)
1) Multi-step schemes

Leapfrog

Adams-Bashforth

2) Physical and computational modes

3) Time filtering for multi-step schemes

CLASS #5 (CHAPTER 3.1, 3.2)

Outline

- Partial Differential Equations
  - 1) Truncation errors
  - 2) Stability, Convergence, Consistency
- Von Neumann's methods (Stability)
- Dispersion and Dissipation errors
- 1-st, 2-nd, 3-rh, and 4-th order space schemes (Runge-Kutta scheme).
- Lax-Wendroff Scheme (two time step)
- Takacs Scheme (two time step)

Truncation Error & Order of Accuracy

TRANSPORT PDE

$$\frac{\partial \psi}{\partial t} + \mathrm{c}\frac{\partial \psi}{\partial x} = 0$$

ANALYTICAL SOLUTION

$$\psi = f(x - ct), c \text{ is speed}$$

FORWARD TIME UPSTREAM SPACE

$$\frac{\partial \psi}{\partial t} \approx \frac{\psi(t_n + \Delta t, x_n) - \psi(t_n, x_n)}{\Delta t}$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(t_n, x_n) - \psi(t_n, x_n - \Delta x)}{\Delta x}$$

TAYLOR EXPANSION:

$$\psi(t_n + \Delta t, x_n) = \psi(t_n, x_n) + \Delta t \frac{\partial \psi}{\partial t}(t_n, x_n) + \frac{(\Delta t)^2}{2}\frac{\partial^2 \psi}{\partial t^2}(t_n, x_n) + \frac{(\Delta t)^3}{6}\frac{\partial^3 \psi}{\partial t^3}(t_n, x_n) + \cdots$$

$$\psi(t_n, x_n - \Delta x) = \psi(t_n, x_n) - \Delta x \frac{\partial \psi}{\partial x}(t_n, x_n) + \frac{(-\Delta x)^2}{2}\frac{\partial^2 \psi}{\partial x^2}(t_n) + \frac{(-\Delta x)^3}{6}\frac{\partial^3 \psi}{\partial x^3}(t_n) + \cdots$$

TRUNCATION ERROR

$$\frac{\psi(t_n + \Delta t, x_n) - \psi(t_n, x_n)}{\Delta t} - \frac{\partial \psi}{\partial t} + c\left(\frac{\psi(t_n, x_n) - \psi(t_n, x_n - \Delta x)}{\Delta x} - \frac{\partial \psi}{\partial x}\right) = \underline{\frac{\Delta t}{2}\frac{\partial^2 \psi}{\partial t^2}(t_n, x_n) - c\,\frac{\Delta x}{2}\frac{\partial^2 \psi}{\partial x^2}}(t_n, x_n) + \cdots$$

The lowest orders of Δt and Δx determines the order of accuracy of the finite difference scheme: first order of accuracy in both time and space

Consistency and Convergence

- A scheme is consistent if truncation error of the finite-difference scheme approaches zero as $\Delta t \to 0, \Delta x \to 0$

  e.g., Forward-time and Upstream-space scheme is consistent.

- A finite scheme is convergent of order of (p, q) if at any time:

$$\|\psi^n - \phi^n\| = O[(\Delta t)^p] + O[(\Delta x)^p]$$
$$as\ \Delta t \to 0, \Delta x \to 0$$

$$Maximum\ norm:\ \|\phi\|_\infty = \max_{1 \le j \le N} |\phi_j|$$

$$L2\ norm:\ \|\phi\|_2 = \left(\sum_{j=1}^{N} |\phi_j|^2\ \Delta x\right)^{1/2}$$

Lax equivalent theorem: if a finite-difference scheme is linear, stable, and accurate of order of (p, q) , then it is convergent of order of (p, q) [Lax and Richtmyer, 1956].

Stability

- A consistent linear finite-difference scheme is convergent, satisfy Lax equivalent theorem, provided that for any time T, there exists such as

$$\|\phi^n\| \leq C_T\|\phi^0\| \quad for\ as\ n\Delta t \leq T$$
$$C_T could\ be\ function\ of\ time, but\ not\ for\ \Delta t, \text{and} \Delta x$$

Solutions can grow, but will grow with a bound: won't blow up.

- A-Stability for Non-increasing problems (wave, advection, transport, diffusion)

$$\|\phi^n\| \leq \|\phi^0\| \quad for\ n$$

- Two popular methods to judge: Energy method & Von Neumann's method.

Energy method

- Compared with Von Neumann's method, used to nonlinear questions and problems without periodic boundaries.

$$\sum_{j=1}^{N} \left(\phi_j^n\right)^2 \; is\ bounded\ for\ any\ n$$

Forward time Upstream space

$$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t} + \mathrm{c}\,\frac{\phi_j^n-\phi_{j-1}^n}{\Delta x} = 0 \;\Rightarrow\; \phi_j^{n+1} = \left(1-\mathrm{c}\frac{\Delta t}{\Delta x}\right)\phi_j^n + \mathrm{c}\frac{\Delta t}{\Delta x}\phi_{j-1}^n$$

$$if\ \mu\ (1-\mu) \geq 0 \qquad \mu = \mathrm{c}\frac{\Delta t}{\Delta x}$$

$$\sum_j \left(\phi_j^{n+1}\right)^2 \leq \left[(1-\mu)^2 + 2\mu(1-\mu) + \mu^2\right]\sum_j \left(\phi_j^n\right)^2 = \sum_j \left(\phi_j^n\right)^2$$

$$Sufficient\ condition\ for\ stability: 0 < \mu \leq 1, \qquad i.e., \left(0 < \mathrm{c}\frac{\Delta t}{\Delta x} \leq 1\right)$$

Stability condition

- The *von Neumann stability condition*:
  - The (numerical) amplification factor A_k of every resolvable Fourier component must be bounded such that: $$\boxed{|A_k| \leq 1 + \gamma \Delta t \;\; ; \; \gamma \text{ independent of } k, \Delta t, \Delta x}$$
  - We'll go with the more restrictive criteria: $\boxed{|A_k| \leq 1}$
    - This "≤1" is satisfactory for *constant-speed advection* …for which there should be *no* distortion and *no* amplitude change w/time.
    - $|A_k| \leq 1$ means the numerical solution results, over time, remain bounded by their initial values. Appropriate if the *norm* of the *true* solution is constant with time.

John von Neumann

Wikipedia

von Neumann's method: limitations

- *Small print:* $\boxed{|A_k| \le 1}$
  - This is appropriate when the true solution is bounded by the norm of the initial data
  - If the stability criteria is met, every Fourier component is stable, and the full solution is, too.
  - The Von Neumann condition is a necessary and sufficient condition for stability.
    - The Von Neumann method is strictly speaking only applicable to linear, constant-coefficient problems
    - *Sufficiency* only for *single equations* in *one unknown*
    - *Periodic* boundary conditions are implied here.

Von Neumann's method

- Only applicable for linear F-D equations with constant coefficients.
- Decompose the solution (in analogue to Fourier transformation)

$$\phi_j^n = \sum_{k=-N}^{N} a_k^n e^{ikj\Delta x},$$

a_k^n: amplitude for kth wavenumber at n timestep and $x = j\Delta x$

Von Neumann's method: for every wavenumber, amplification factor is smaller than 1.

For kth wavenumber

$$\phi_j^{n+1} = A_k^n \phi_j^n = (1-\mu)\phi_j^n + \mu\phi_j^n e^{-ik\Delta x}$$

$$A_k^n = (1-\mu) + \mu e^{-ik\Delta x}$$

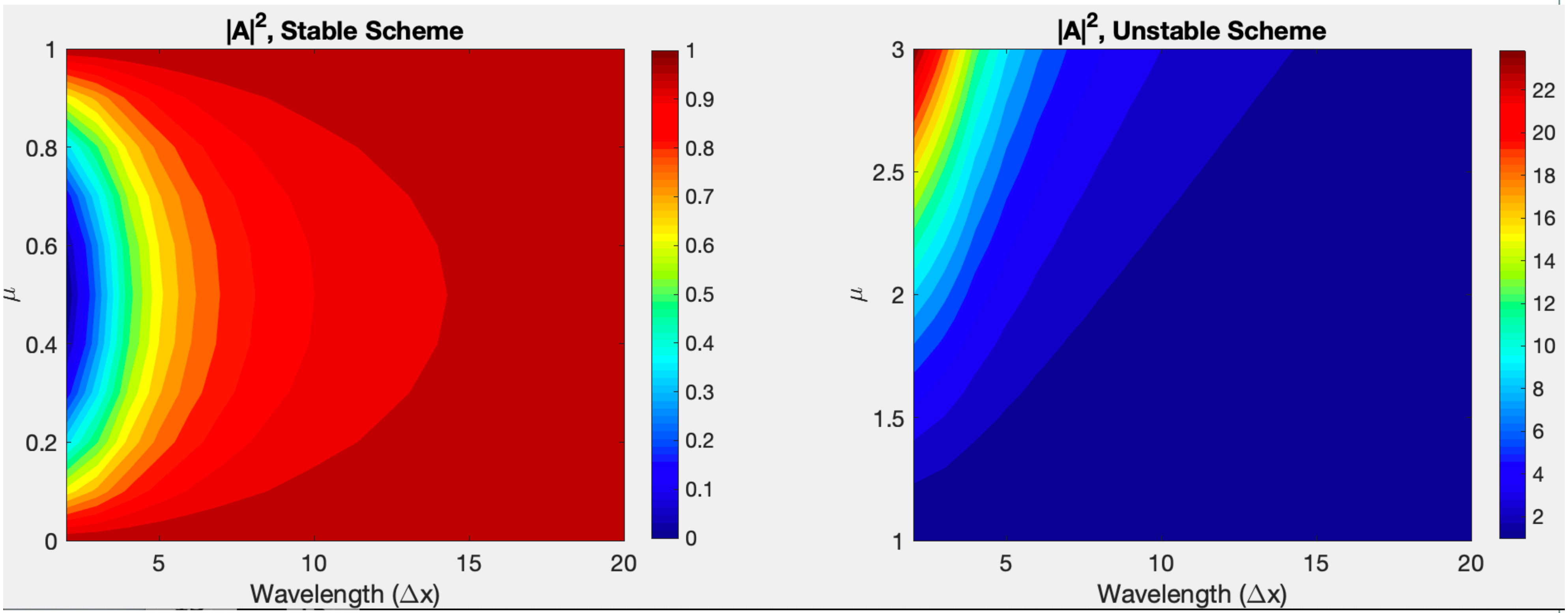
$$|A_k^n|^2 = 1 - 2\mu(1-\mu)(1 - cos k\Delta x) \leq 1$$

$$0 < \mu \leq 1$$

Stability & Amplification

$$|A_k^n|^2 = 1 - 2\mu(1-\mu)(1 - cos k\Delta x) \leq 1$$

$$0 < \mu \leq 1$$



- For stable scheme, high-wavenumber (small-scale) waves are damped more efficiently.
- For unstable scheme, high-wavenumber (small-scale) waves grow most rapidly.

Resolution: extremes

- An infinitely long wave
  - … is a straight line.
- A well-resolved wave
  - … has 7-10 (or more) grid points over a wavelength
- A poorly-resolved wave
  - … has 3-5 grid points spanning a wavelength
- A $2\Delta x$ (*"two delta-x"*) wave
  - … is basically unresolved at all (see Durran figure), though we say $2\Delta x$ is the "minimum" wavelength we could describe.
  - … similarly, a $2\Delta t$ wave appears over the span of two time steps. We say $2\Delta t$ is the minimum period we could describe.

Courant-Friedrichs-Lewy Condition

$$\phi_j^{n+1} = (1-\mu)\phi_j^n + \mu\phi_{j-1}^n$$

$$0 < \mu\left(= c\frac{\Delta t}{\Delta x}\right) \leq 1$$

$\psi = f(x - ct), \Delta x / \Delta t = c$

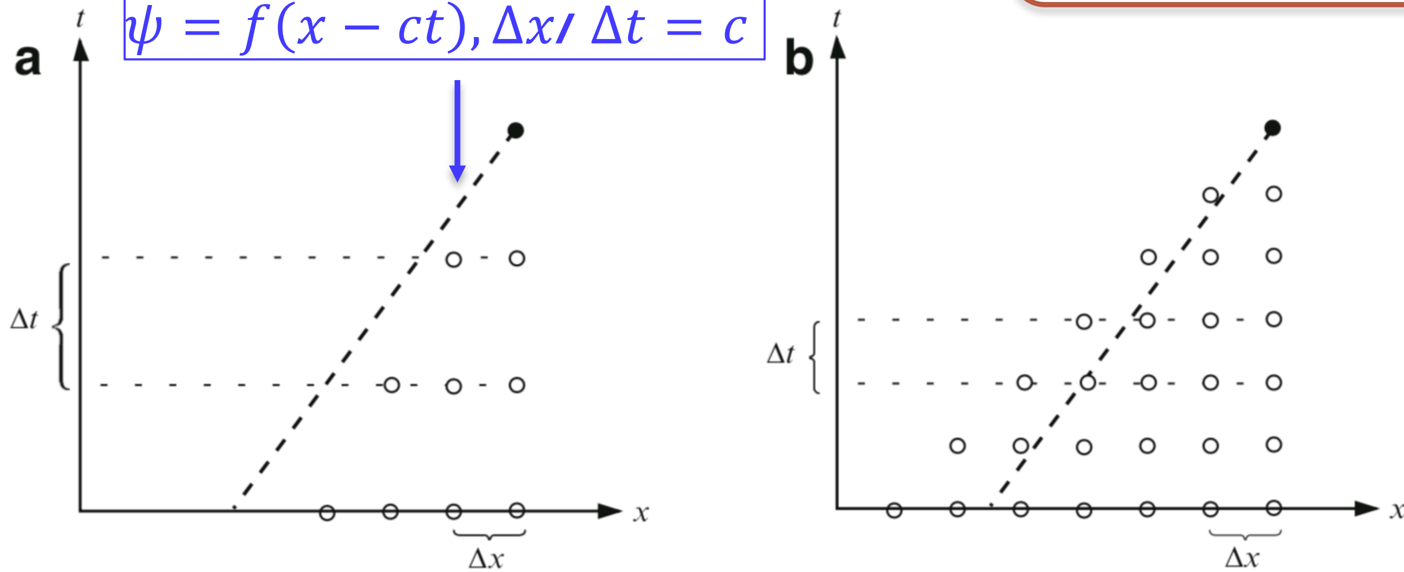


Fig. 3.1 The influence of the time step on the relationship between the numerical domain of dependence of the upstream scheme (*open circles*) and the true domain of dependence of the advection equation (*dashed line*): **a** unstable Δt, **b** stable Δt

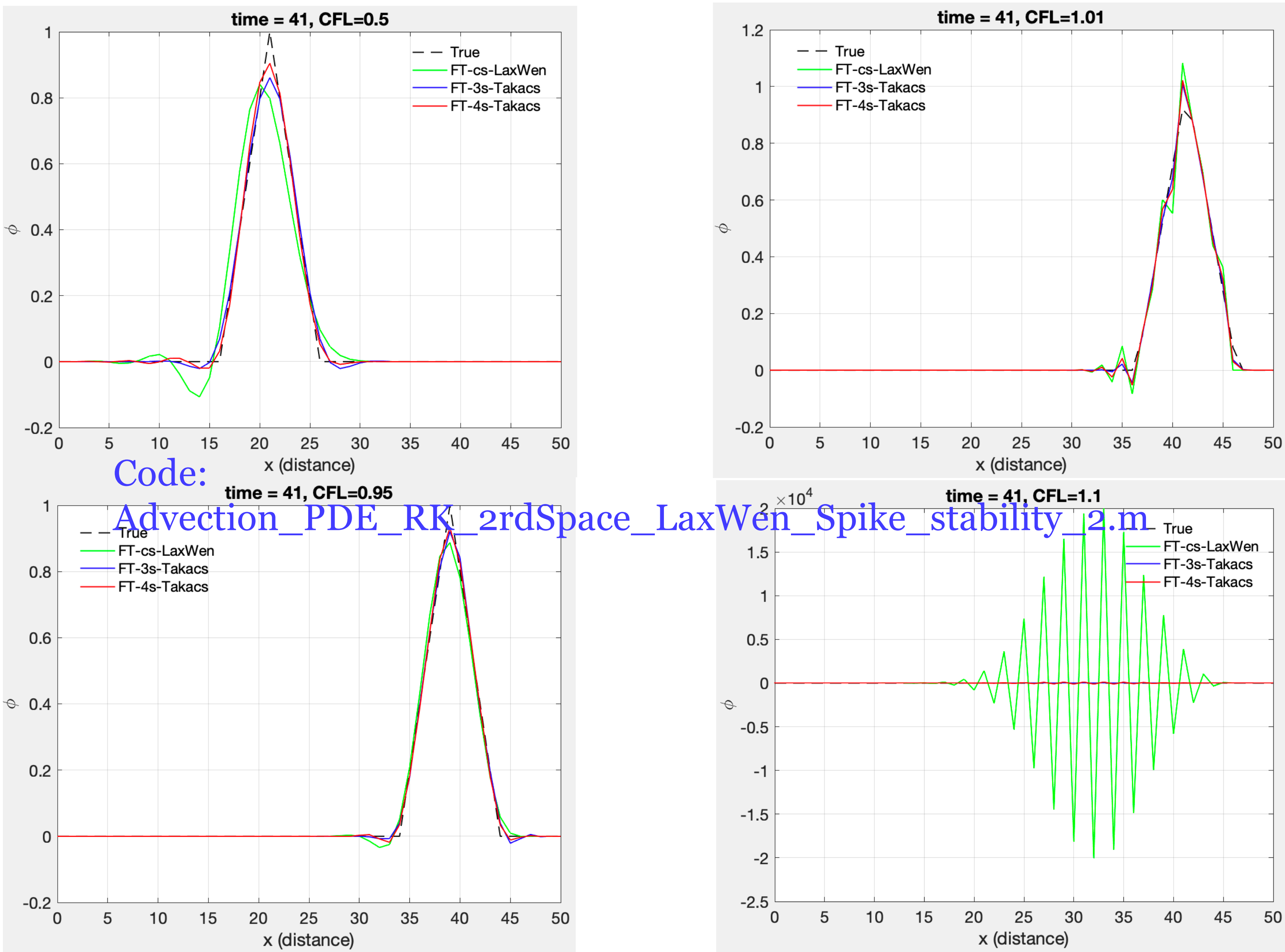
To reach solid circle:

FD scheme: open circles are domain of dependence of (Domain 1).

Associated PDE: domain of dependence is dashed line (Domain 2).

➢ CFL condition: Domain 1 includes Domain 2.

Stability & CFL Condition



Code: _Advection_PDE_RK_2rdSpace_LaxWen_Spike_stability_2.m