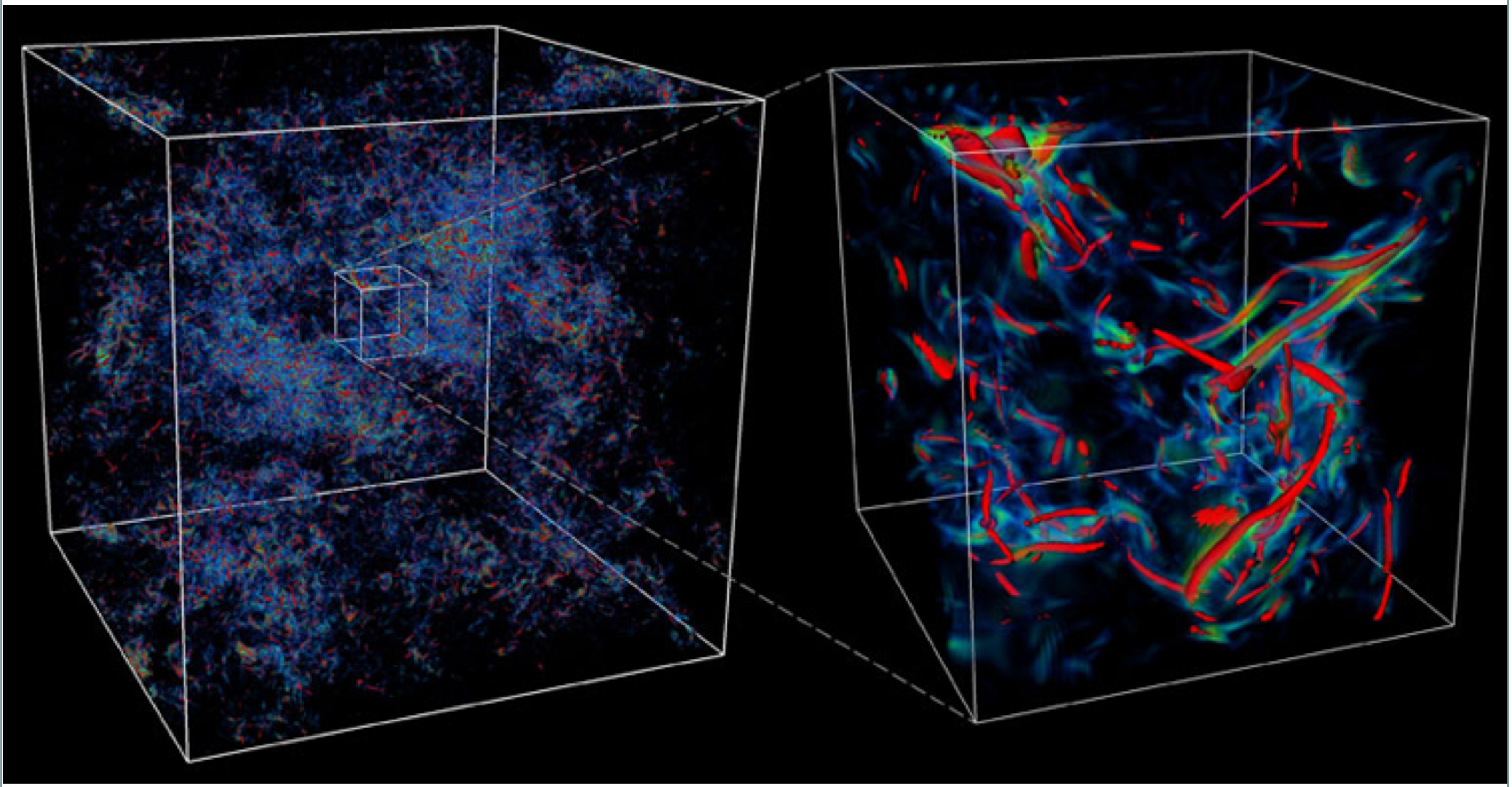
PHYS 8750
Numerical Fluid Dynamics

Fall, 2020



Isotropic turbulence, Donzis and Yeung. http://www.tacc.utexas.edu/scivis-gallery/isotropic-turbulence

PHYS 8750

Class #11 (Chapter 4.1)

1) PDE with two variables

2) Shallow-water equations

3) Staggered grids in time

CLASS #12

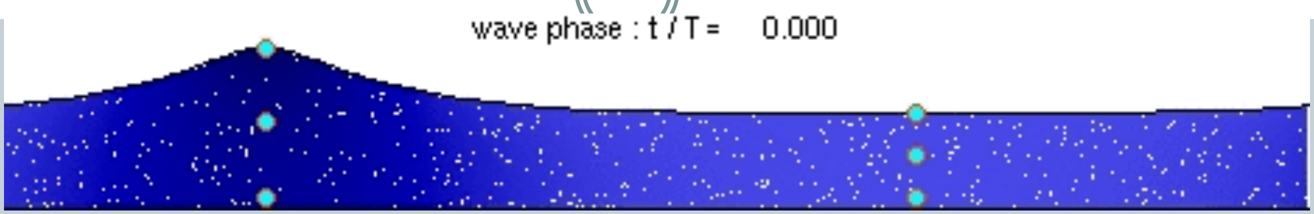
(CHAPTER 4.1, 4.2)

STAGGERED GRIDS IN TIME AND SPACE

Outline

1. Shallow-water equations
    - Discrete-dispersion method
2. Staggered grid for shallow-water equations.
    - Staggered in space
    - Staggered in space & time
    - Forward-backward differencing
3. Examples (Codes)
4. Advantages/disadvantages

Shallow-Water Equations



Shallow-Water Equations:

Momentum Equation: $$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

Continuity Equation: $$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

Physics:

- Higher depth would be pulled down by gravity, which leads to horizontal motions.
- Building up of horizontal motions will lead to changes to water depths.
- Addition to this: advection of water depth and velocity by the mean flow.

Discrete-Dispersion Relation

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

LEAPFROG

CENTER SPACE

$$\delta_{2t}u + \mathrm{U}\delta_{2x}u + g\delta_{2x}h = 0$$

$$\delta_{2t}h + \mathrm{U}\delta_{2x}h + H\delta_{2x}u = 0$$

$$u_j^n = u_0 e^{i(kj\Delta x - \omega n\Delta t)}, h_j^n = h_0 e^{i(kj\Delta x - \omega n\Delta t)}$$

$$\left(-\frac{sin\omega\Delta t}{\Delta t} + \mathrm{U}\frac{sink\Delta x}{\Delta x}\right)u_0 + g\frac{sink\Delta x}{\Delta x}h_0 = 0$$

$$\left(-\frac{sin\omega\Delta t}{\Delta t} + \mathrm{U}\frac{sink\Delta x}{\Delta x}\right)h_0 + H\frac{sink\Delta x}{\Delta x}u_0 = 0$$

Discrete-Dispersion Relation

$$\boxed{\left(-\frac{sin\omega\Delta t}{\Delta t}+\mathrm{U}\,\frac{sink\Delta x}{\Delta x}\right)^2-gH\left(\frac{sink\Delta x}{\Delta x}\right)^2=0}$$

Define c $=\sqrt{gH}$, discrete-dispersion relation:

$$sin\omega\Delta t=\frac{\Delta t}{\Delta x}(U\pm c)sink\Delta x$$

If $\omega=\omega_r+i\omega_i$, then ω_r determines wave phase, ω_i gives growth rates

To avoid amplitude growth, ω cannot have imaginary part:

$$\left|\frac{\Delta t}{\Delta x}(U\pm c)sink\Delta x\right|=|sin\omega\Delta t|$$

To satisfy this relation anytime: $\left|\frac{\Delta t}{\Delta x}(U\pm c)\right|\leq 1,$ if U $=0,\left|c\,\frac{\Delta t}{\Delta x}\right|\leq 1$

Staggered grids: shallow-water eqs

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

$$\delta_{2t}u + g\delta_x h = 0$$

$$\delta_{2t}h_{j+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}} = 0$$

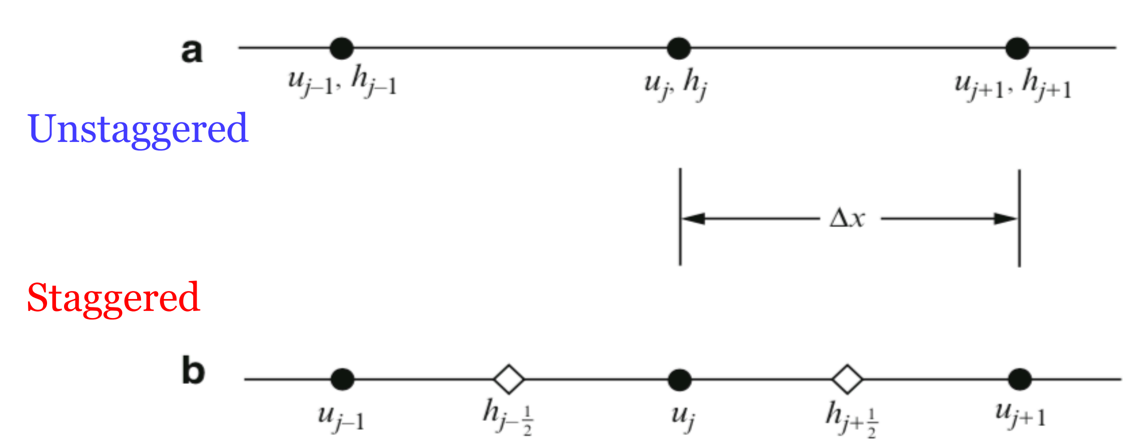


Fig. 4.1 Distribution of u and h on **a** an unstaggered and **b** a staggered mesh

Durran's book

Discrete-dispersion relation:

$$\sin\omega\Delta t = \pm\frac{2c\Delta t}{\Delta x}\sin\frac{k\Delta x}{2}$$

Stability criterion: $\left|\frac{c\Delta t}{\Delta x}\right| \leq \frac{1}{2}$, more strict than unstaggered grid scheme.

Superior Property of Staggered Grid

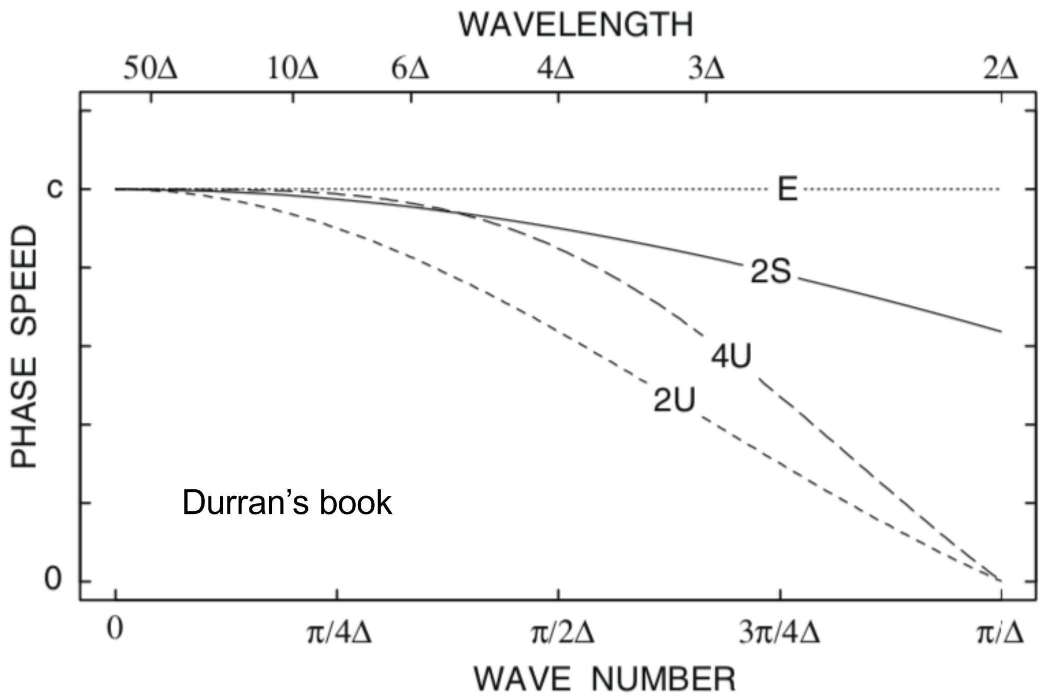


Fig. 4.2 Phase speed in the limit of good temporal resolution as a function of spatial resolution for the exact solution (*E*), for second-order approximations on a staggered mesh c_s (*2S*) and an unstaggered mesh c_u (*2U*), and for centered fourth-order spatial derivatives on an unstaggered mesh (*4U*)

Phase speeds:

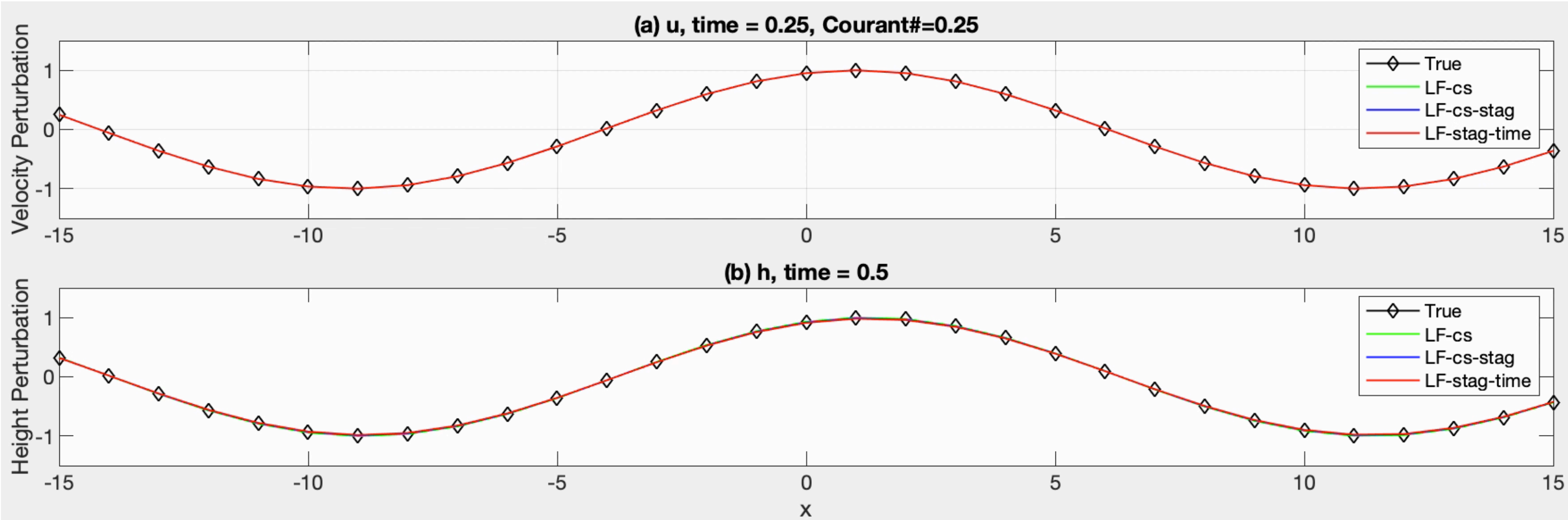
$$c_u = \frac{c}{k\Delta x} sin k\Delta x$$

$$c_s = \frac{2c}{k\Delta x} sin\left(\frac{k\Delta x}{2}\right)$$

- Staggered grid: smaller differences in phases & less dispersive.

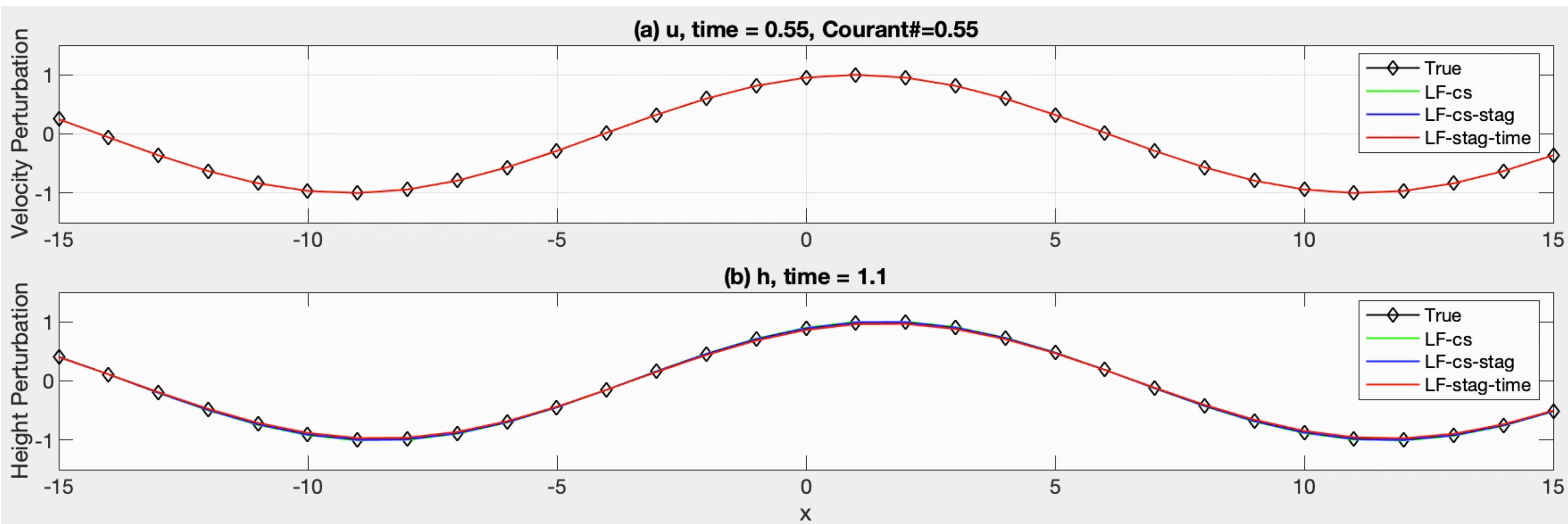
Programming & Comparison of Staggered and Unstaggered Schemes

U=0



$\lambda = 20\Delta x$

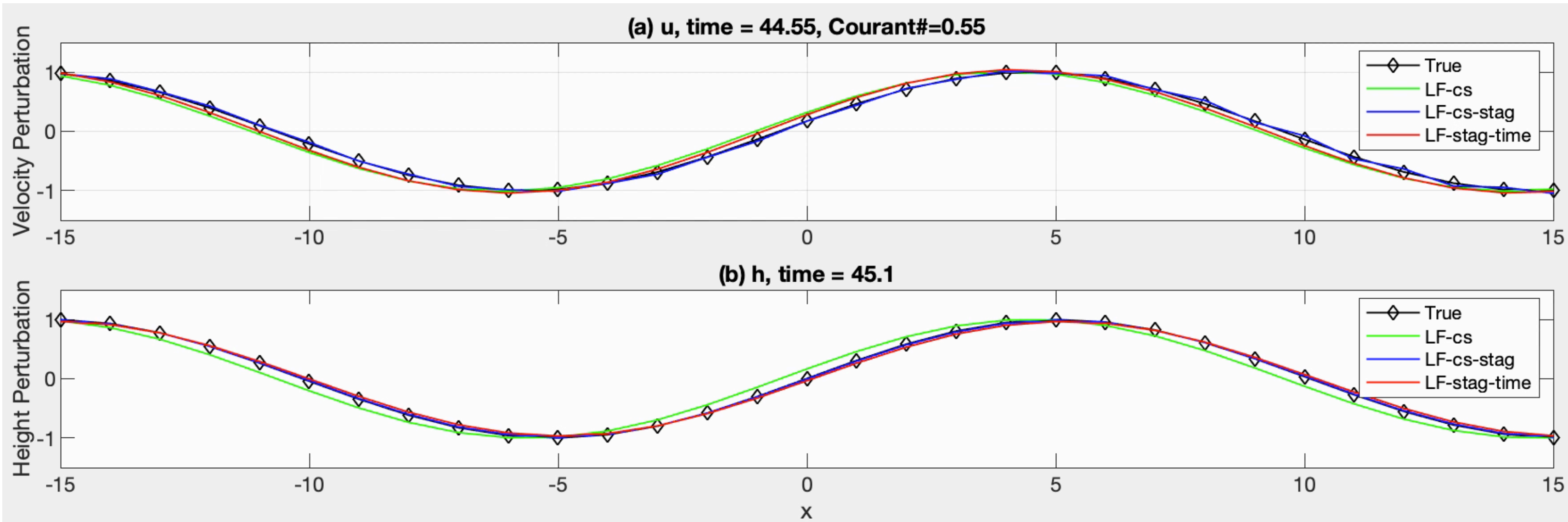
U=0



Programming & Comparison of Staggered and Unstaggered Schemes

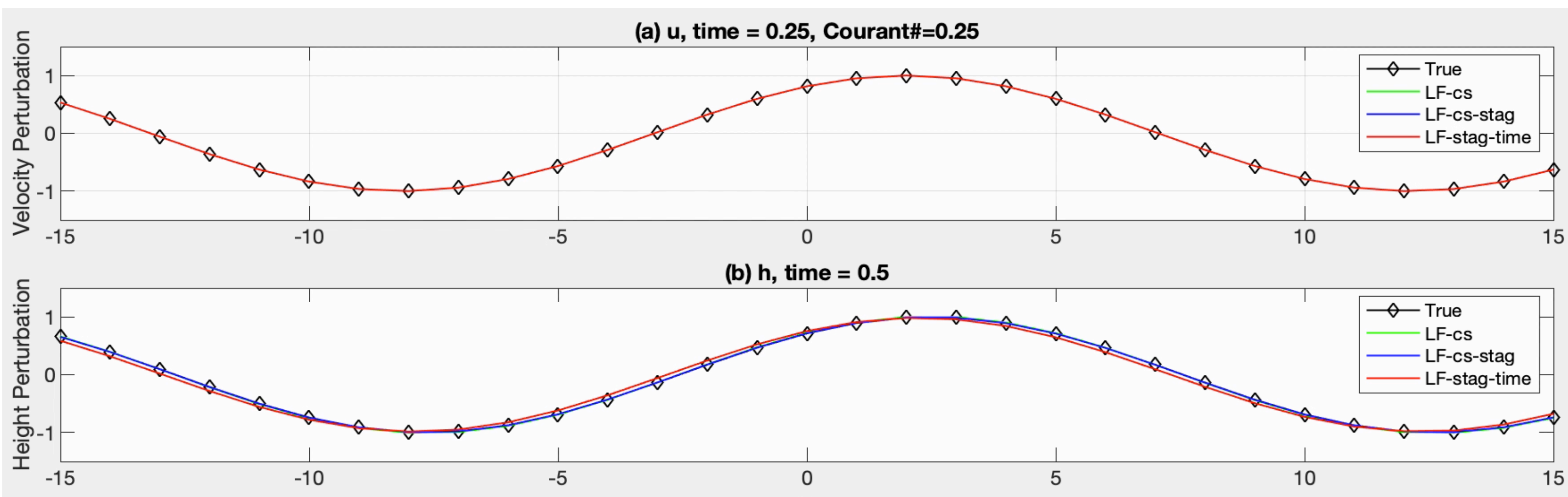
$\lambda = 20\Delta x$

U=0



$\lambda = 20\Delta x$

U=1



Temporal Staggering

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

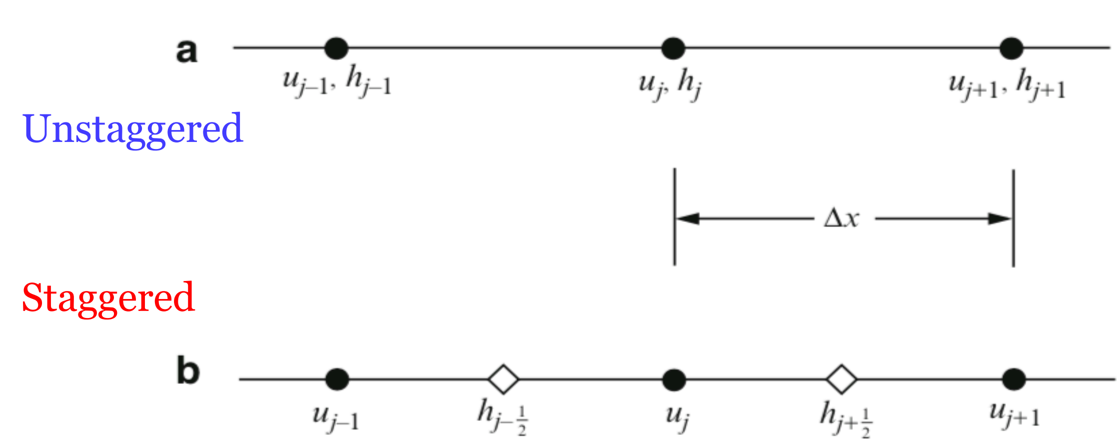


Fig. 4.1 Distribution of u and h on **a** an unstaggered and **b** a staggered mesh

Durran's book

Stagger in space

$$\delta_{2t} u + g\delta_x h = 0$$

$$\delta_{2t} h_{j+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}} = 0$$

Stagger in time

Time step is divided by 2

$$\delta_t u_j^{n+\frac{1}{2}} + g\delta_x h_j^n = 0$$

$$\delta_t h_{j+\frac{1}{2}}^{n+\frac{1}{2}} + H\delta_x h_{j+\frac{1}{2}}^{n+1} = 0$$

Forward-Backward Differencing

$$\delta_t u_j^{n+\frac{1}{2}} + g\delta_x h_j^n = 0$$

$$\delta_t h_{j+\frac{1}{2}}^{n+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}}^{n+1} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + g\,\frac{h_{j+\frac{1}{2}}^n - h_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

$$\frac{h_{j+\frac{1}{2}}^{n+1} - h_{j+\frac{1}{2}}^n}{\Delta t} + H\,\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} = 0$$

Space staggering: u is evaluated at integral grid point (u_j), while h is evaluated at half point ($h_{j+\frac{1}{2}}$).

Time staggering: u is evaluated at odd time levels ($[2\mathrm{n}+1]\Delta t$), while h is evaluated at even time levels ($[2\mathrm{n}]\Delta t$).

Discrete-Dispersion Relation

Stagger in time

$$\delta_{2t} u + g\delta_x h = 0$$

$$\delta_{2t} h_{j+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}} = 0$$

$$\sin\omega\Delta t = \pm\frac{2c\Delta t}{\Delta x}\sin\frac{k\Delta x}{2}$$

$$\left|\frac{c\Delta t}{\Delta x}\right| \le \frac{1}{2}$$

Stagger in time and space

$$\delta_t u_j^{n+\frac{1}{2}} + g\delta_x h_j^n = 0$$

$$\delta_t h_{j+\frac{1}{2}}^{n+\frac{1}{2}} + H\delta_x h_{j+\frac{1}{2}}^{n+1} = 0$$

$$\sin\frac{\omega\Delta t}{2} = \pm\frac{c\Delta t}{\Delta x}\sin\frac{k\Delta x}{2}$$

$$\left|\frac{c\Delta t}{\Delta x}\right| \le 1$$

Stability criterion is relaxed in staggered grids in both time and space

Forward-Backward Differencing

$$\delta_t u_j^{n+\frac{1}{2}} + g\delta_x h_j^n = 0$$

$$\delta_t h_{j+\frac{1}{2}}^{n+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}}^{n+1} = 0$$

$$\Rightarrow$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + g\,\frac{h_{j+\frac{1}{2}}^n - h_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

$$\frac{h_{j+\frac{1}{2}}^{n+1} - h_{j+\frac{1}{2}}^n}{\Delta t} + H\,\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} = 0$$

Time staggering: u is evaluated at odd time levels ($[2n+1]\Delta t$), while h is evaluated at even time levels ($[2n]\Delta t$).

$$u_1, h_1 \xRightarrow{\text{update momentum eq}} u_2 \xrightarrow[h_1]{} h_2 \xRightarrow{u_2} u_3 \xrightarrow[h_2]{} h_3 \xRightarrow{u_3} u_4 \xrightarrow[h_3]{} h_4 \xRightarrow{u_4} u_5$$

	u_2	h_2	u_3	h_3	u_4	h_4	u_5
Storage:	×	√	√	×	×	√	√
	√	×	×	√	√	×	×

u at even time levels, h at odd time levels are all intermediate values:
SAVE STORAGE. (go to code)

Advantage/Disadvantage

	Stability	Phase Performance	Storage	Mean flow limitation
Leapfrog time Center space	$\left\|\frac{c\Delta t}{\Delta x}\right\| \leq 1$	Most dispersive	More	No
Leapfrog time Staggered space (middle points)	$\left\|\frac{c\Delta t}{\Delta x}\right\| \leq \frac{1}{2}$	Least dispersive	More	No
Forward-Backward time Staggered space	$\left\|\frac{c\Delta t}{\Delta x}\right\| \leq 1$	Less dispersive	Less	U<<c U is very small