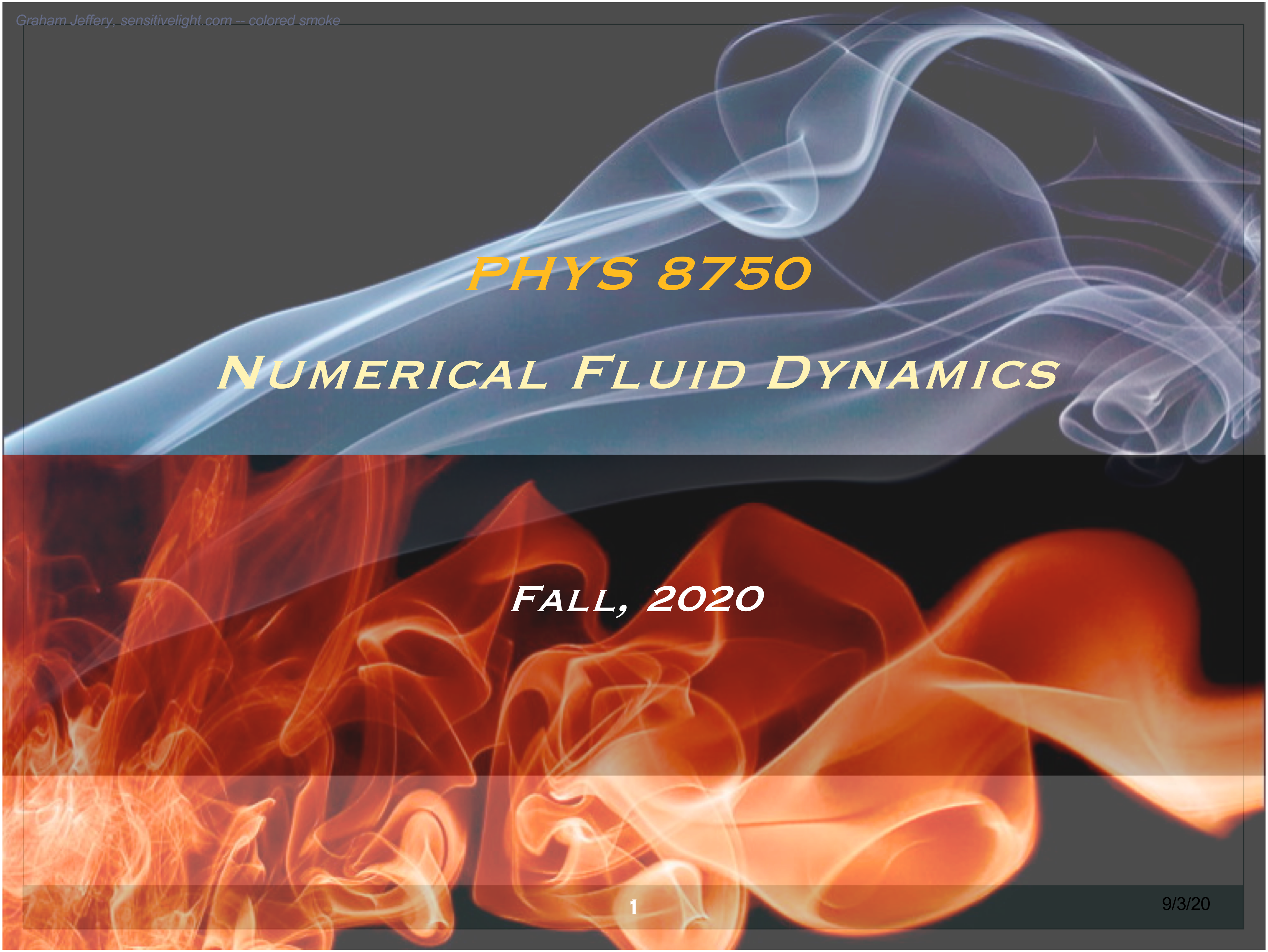


Graham Jeffery, sensitivelight.com -- colored smoke

PHYS 8750

Numerical Fluid Dynamics

Fall, 2020

PHYS 8750

Class #3 (Chapter 2.3)
1) Multi-stage methods

Runge-Kutta 2-stage and 4-stage

2) Stability, amplitude and phase diagram

3) Amplitude and phase error

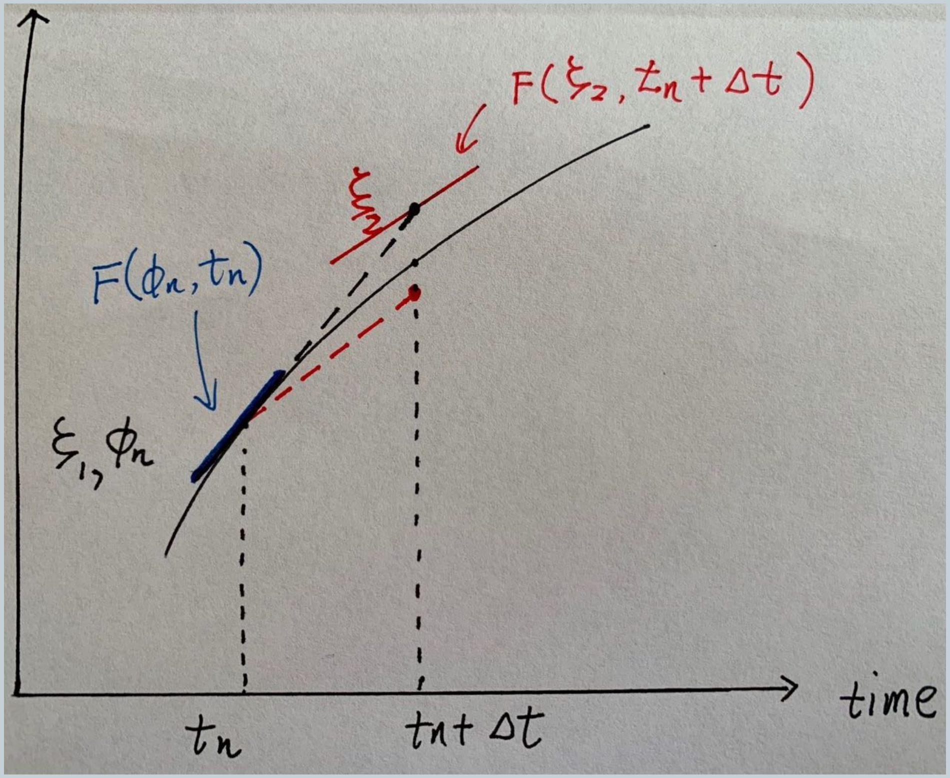
CLASS #4
(CHAPTER 2.4)

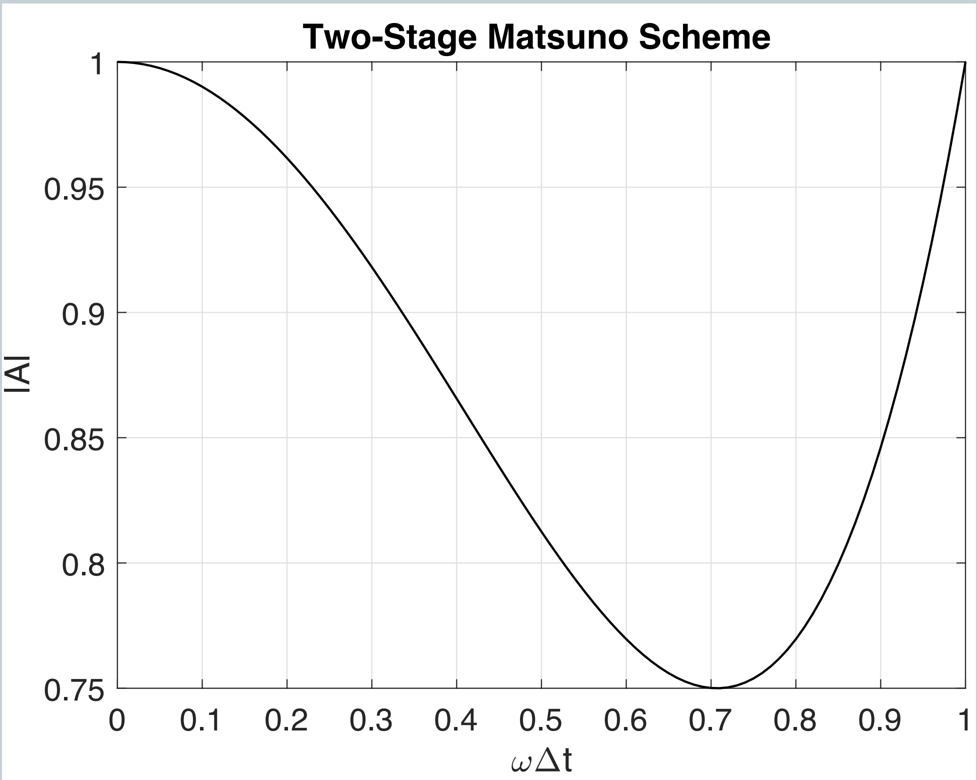
Outline

- Multi-stage methods
  1) Runge-Kutta scheme
  2) Matsuno scheme
- Multi-step methods
  1) Leap-frog scheme
  2) Adams-Bashforth scheme
- Amplitude and phase behavior of multi-stage(step) methods

Two-Stage Matsuno Scheme

- $\xi_1 = \phi_n, \xi_2 = \phi_n + \Delta t F(\phi_n, t_n),\ \phi_{n+1} = \phi_n + \Delta t F(\xi_2, t_n + \Delta t)$





$$|A|^2_{\text{Matsuno}} = 1 - (\omega\Delta t)^2 + (\omega\Delta t)^4$$

Damp high-frequency gravity waves in meteorological models

Amplitude and Phase Behavior

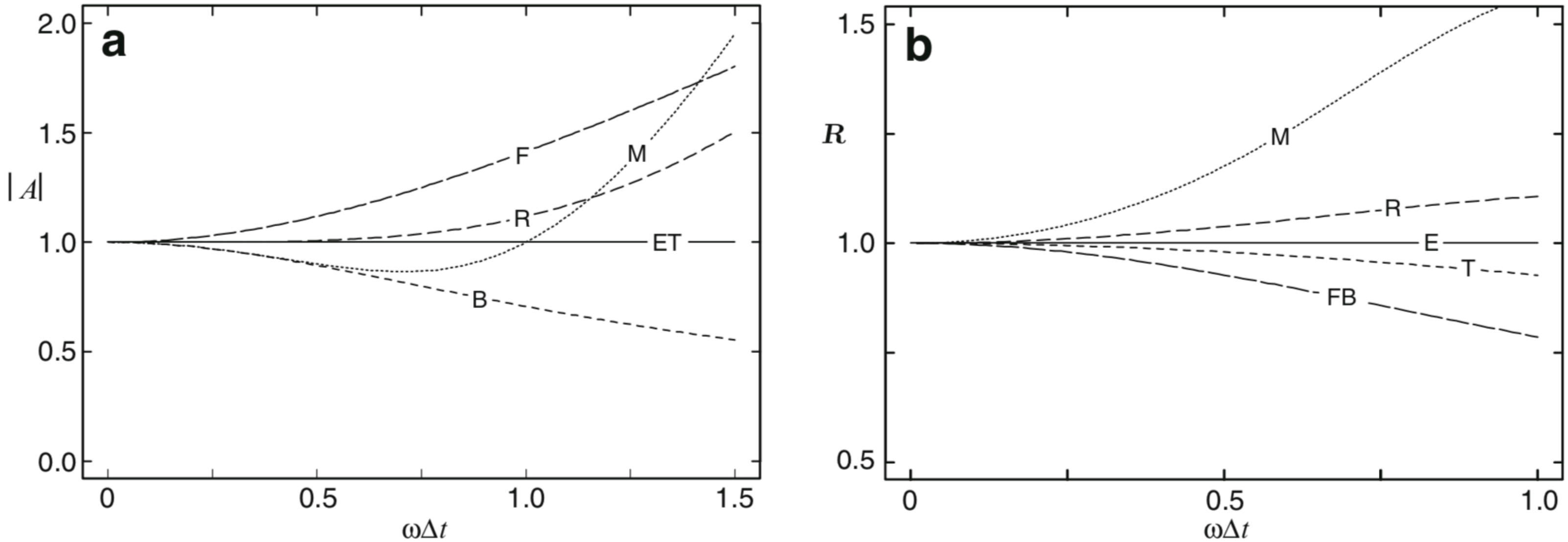


Fig. 2.2 The modulus of the amplification factor (**a**) and the relative phase change (**b**) as a function of temporal resolution $\omega\Delta t$ for the true solution and five two-level schemes: exact solution (E) and trapezoidal method (T), forward-Euler scheme (F), backward-Euler scheme (B), two-stage second-order Runge–Kutta scheme (R), and Matsuno scheme (M)

$$|A|_{\text{forward}} \approx 1 + \tfrac{1}{2}(\omega\Delta t)^2$$

$$|A|_{\text{backward}} \approx 1 - \tfrac{1}{2}(\omega\Delta t)^2$$

$$|A|^2_{\text{Matsuno}} = 1 - (\omega\Delta t)^2 + (\omega\Delta t)^4$$

$$|A|_{\text{RKe2}} \approx 1 + \tfrac{1}{8}(\omega\Delta t)^4$$

$$R_{\text{trapezoidal}} \approx \frac{1}{\omega\Delta t}\arctan\left(\omega\Delta t\left(1 + \frac{\omega^2\Delta t^2}{4}\right)\right) \approx 1 - \frac{\omega^2\Delta t^2}{12}$$

$$R_{\text{forward}} = R_{\text{backward}} \approx 1 - \frac{(\omega\Delta t)^2}{3}$$

$$R_{\text{RKe2}} \approx 1 + \tfrac{1}{6}(\omega\Delta t)^2$$

$$R_{\text{Matsuno}} \approx 1 + \tfrac{2}{3}(\omega\Delta t)^2$$

Multi-Stage Runge-Kutta Method

Carl Runge

Carl David Tolmé Runge

Born	30 August 1856 Bremen, German Confederation
Died	3 January 1927 (aged 70) Göttingen, Weimar Republic
Citizenship	German

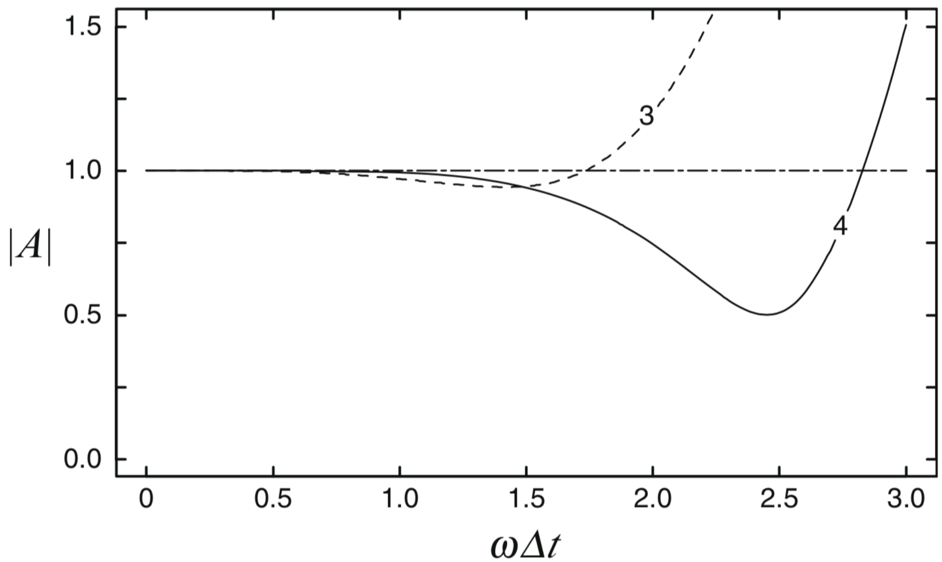


Fig. 2.3 Modulus of the amplification factor as a function of temporal resolution $\omega\Delta t$ for third-order three-stage (*dashed line*) and fourth-order four-stage (*solid line*) explicit Runge–Kutta solutions to the oscillation equation

Martin Kutta

Martin Kutta (1867–1944)

Born	3 November 1867 Pitschen, Upper Silesia
Died	25 December 1944 (aged 77) Fürstenfeldbruck
Nationality	German

Disadvantage & Advantage

Advantages

- High order of accuracy
- Small amp errors
- Small phase errors
- Efficient damping at high-frequency
- Stability
- Large Δt allowed

Disadvantages

- Evaluating derivatives multiple times
- Extensive storage
- More expensive computation

Multi-Step Scheme

- Multiple time steps are used to computer the next time step.
- Achieve high order of accuracy, without using too much storage.
- Physical and computational modes both exist. Need to suppress the errors arising from computation modes.
- Leapfrog & Adams-Bashforth

Leapfrog

A_+: PHYSICAL MODE

A_-: COMPUTATIONAL MODE

$$\frac{\phi_{n+1} - \phi_{n-1}}{2\Delta t} = F(\phi_n, t_n) = i\omega\phi_n$$

$$A^2 - 2i\omega\Delta t - 1 = 0$$

$$A_\pm = i\omega\Delta t \pm (1 - \omega^2\Delta t^2)^{1/2}$$

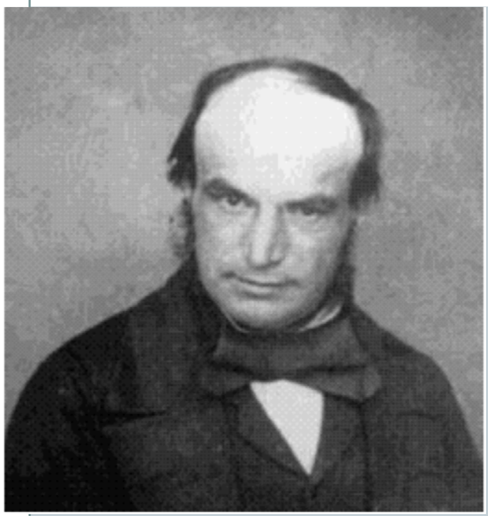
$|\omega\Delta t| \leq 1$

$$|A_\pm| = 1$$

$|\omega\Delta t| \geq 1$

$$|A_+| > |\omega\Delta t| > 1$$

Adams-Bashforth



John Couch Adams

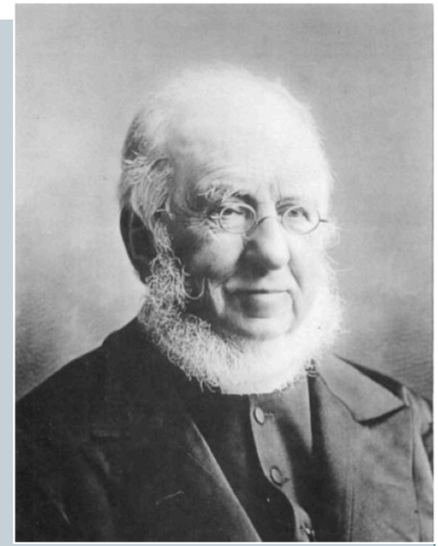
1819-1892

British astronomer and mathematician; predicted existence of Neptune

Francis Bashforth

1819-1912

British mathematician; influential ballistics expert.

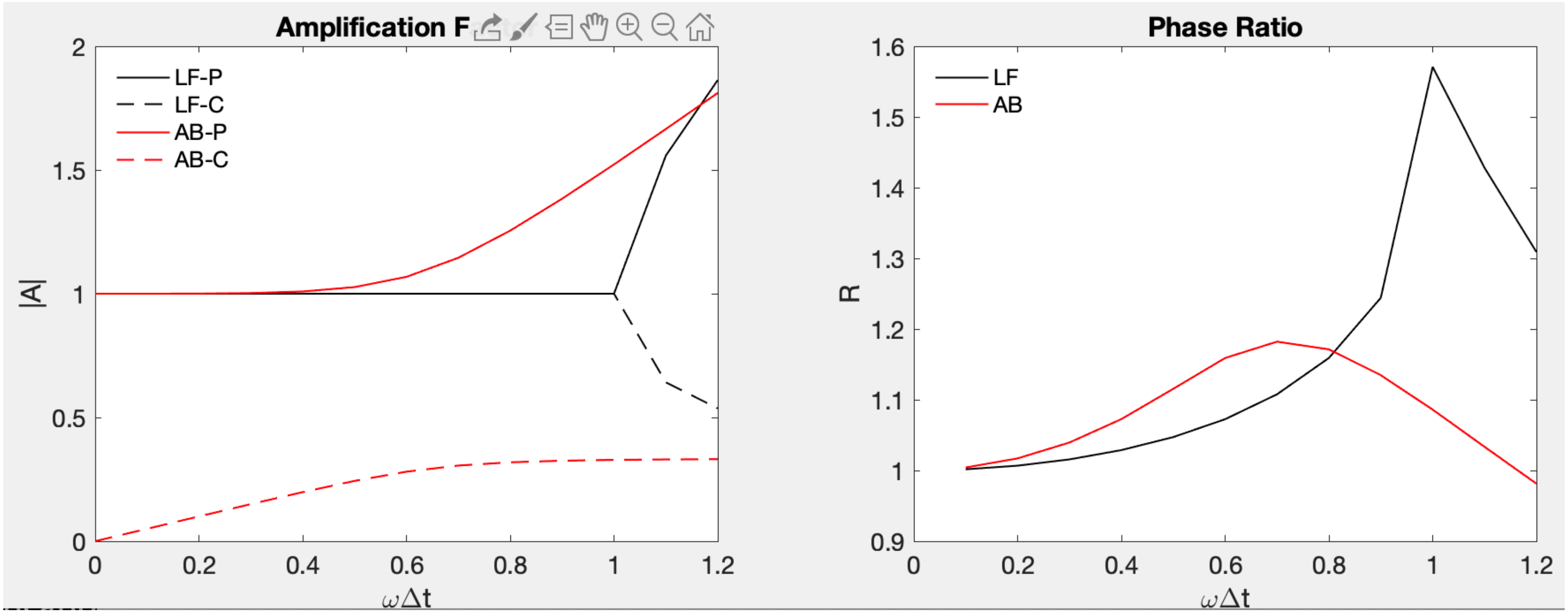


$$\phi_{n+1} = \phi_n + \Delta t\left(\frac{3}{2}F(\phi_n, t_n) - \frac{1}{2}F(\phi_{n-1}, t_{n-1})\right)$$

$$\Downarrow$$

$$A_{\pm} = \frac{1}{2}\left(1 + \frac{3i\omega\Delta t}{2} \pm \left(1 - \frac{9}{4}\omega^2\Delta t^2 + i\omega\Delta t\right)^{1/2}\right)$$

Amplification & Phase Ratio



Leapfrog

$$A_{\pm} = i\omega\Delta t \pm (1 - \omega^2\Delta t^2)^{1/2}$$

Adams-Bashforth

$$A_{\pm} = \frac{1}{2}\left(1 + \frac{3i\omega\Delta t}{2} \pm \left(1 - \frac{9}{4}\omega^2\Delta t^2 + i\omega\Delta t\right)^{1/2}\right)$$

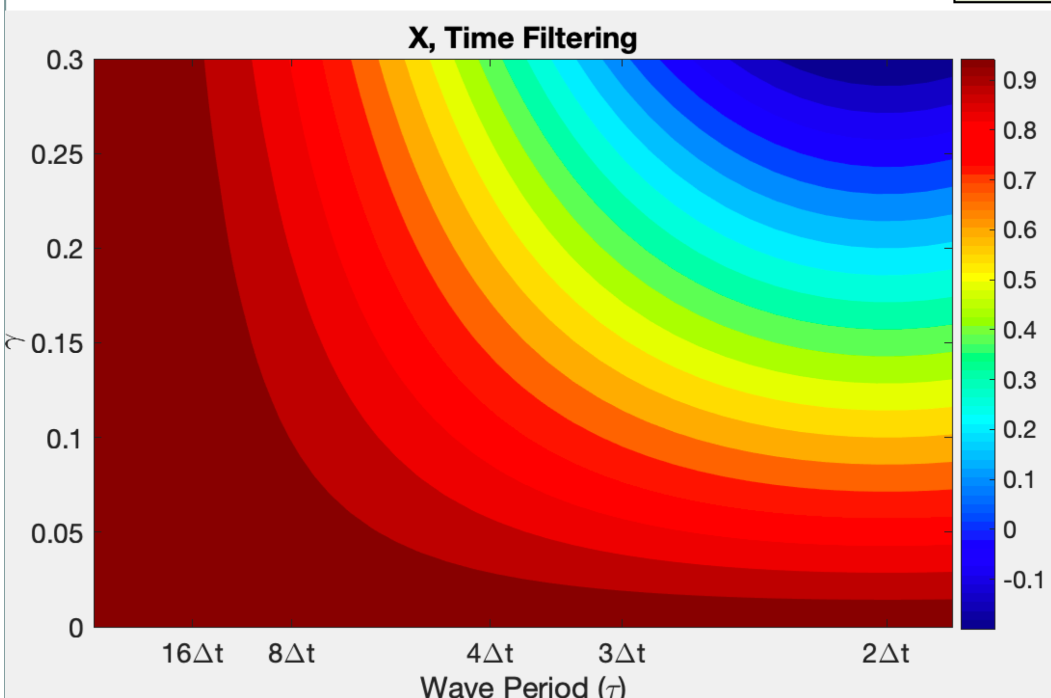
Leapfrog + Time Filtering

- Time filtering: Damp the poorly resolved oscillation ($2\Delta t$) and/or suppress computational mode

1) Post-processing way

$$\overline{\phi_n} = \phi_n + \gamma(\phi_{n+1} - 2\phi_n + \phi_{n-1})$$

$$X = \frac{\overline{\phi_n}}{\phi_n} = 1 - 2\gamma(1 - cos\omega\Delta t)$$



Damp high-frequency waves such as those with periods of $2\Delta t - 4\Delta t$ without scarifying low-frequency waves by prospering choosing γ

Leapfrog + Time Filtering

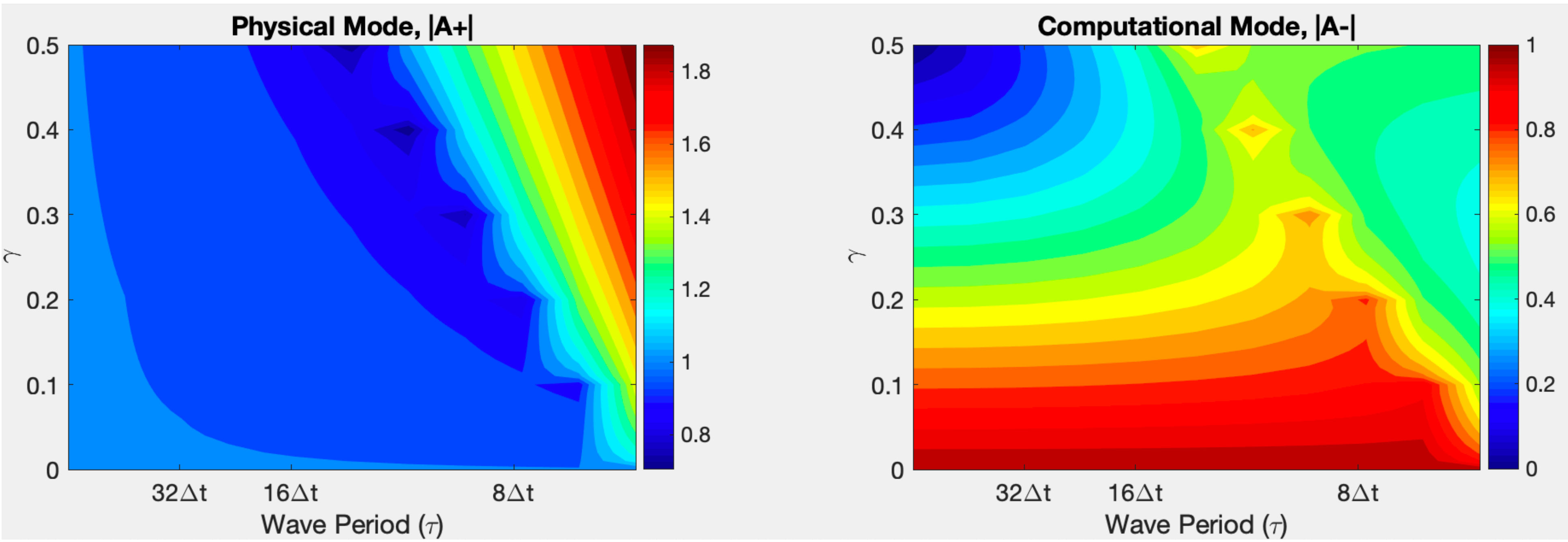
2) Interactive Way: e.g., incorporate second-derivative time filter into the time integration cycle

$$\phi_{n+1} = \overline{\phi_{n-1}} + 2\Delta t F(\phi_n)$$

$$\overline{\phi_n} = \phi_n + \gamma(\overline{\phi_{n-1}} - 2\phi_n + \phi_{n-1})$$

Asselin-filtered

$$A_{\pm} = \gamma + i\omega\Delta t \pm \left((1-\gamma)^2 - \omega^2\Delta t^2\right)^{1/2}$$

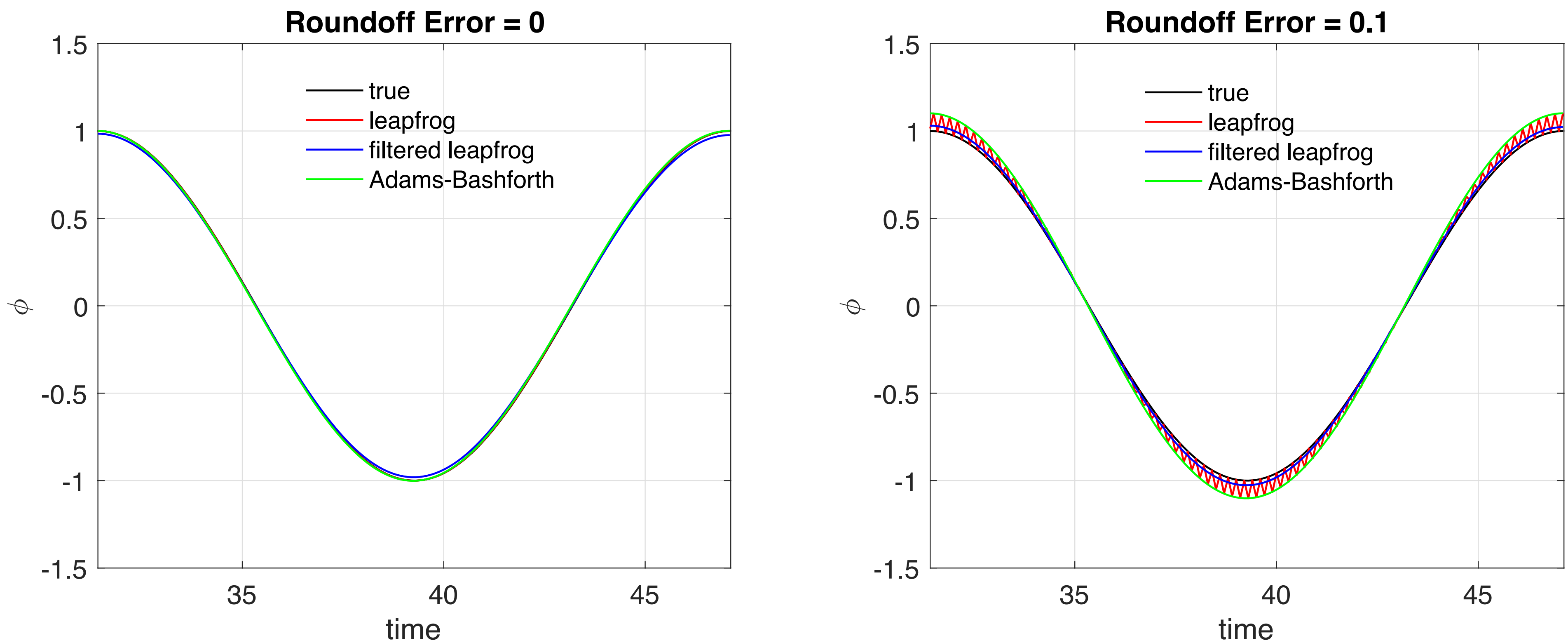


Choose γ: $A_+ \approx 1$, & $A_- < 1$ and $A_- \rightarrow 0$ for low-freq waves.

Limitation: unstable for high-frequency components.

Oscillation Problem

Leapfrog w/wo filter, and Adams-Bashforth



Code: LF_AB_StabilityAmplification_ODE_OscillationProb_7.m

- LF: would suffer computational mode
- Asselin-LF: damp computational mode, accurate amplitude.
- AB: damp computational mode, errors contaminate