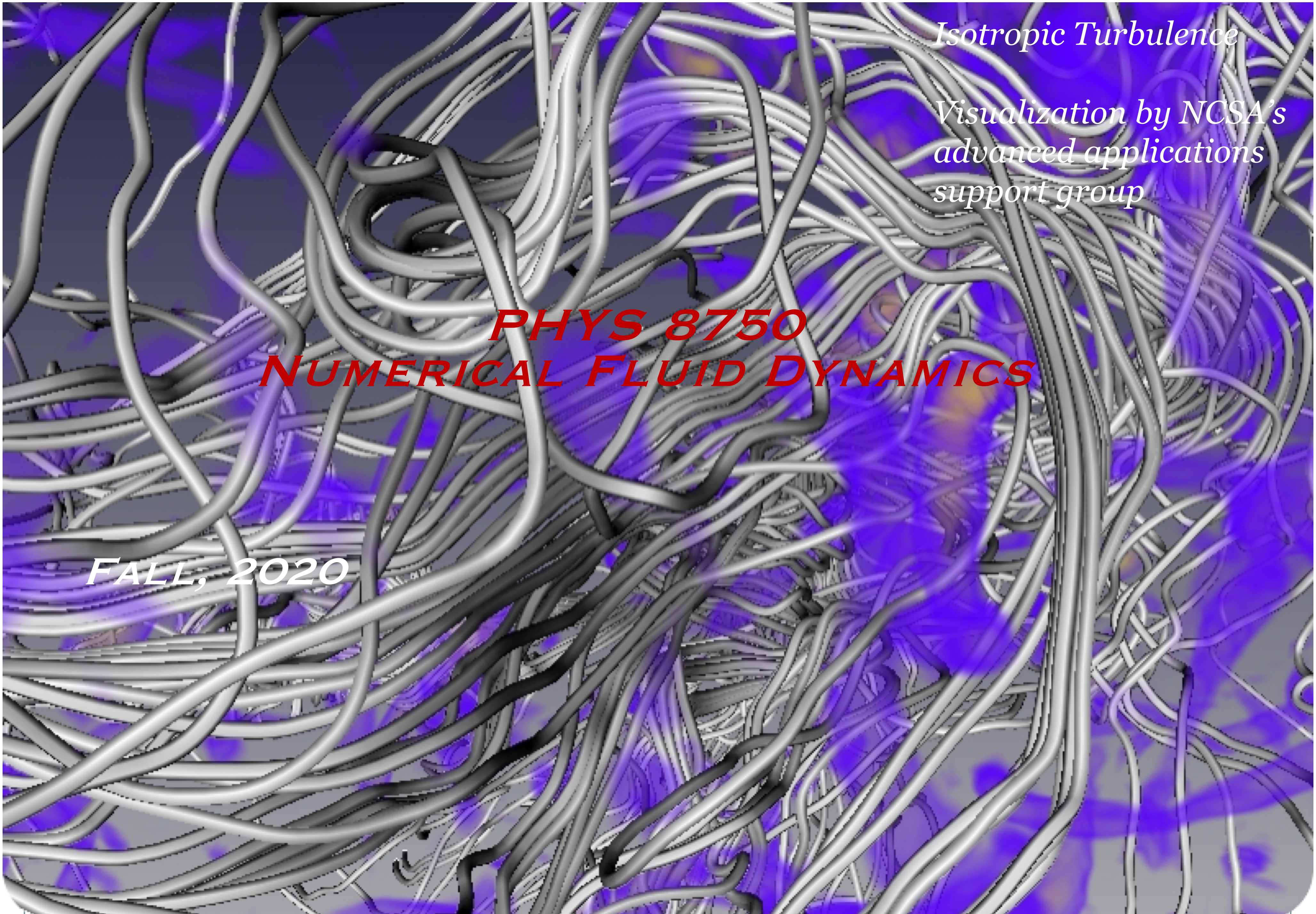


PHYS 8750
Numerical Fluid Dynamics

Fall, 2020

Isotropic Turbulence

Visualization by NCSA's advanced applications support group

vis.ncsa.illinois.edu/gallery.html

PHYS 8750

CLASS #14

SUMMARY OF

CHAPTERS 1-4

Class #15 (Chapter 5.1)

Finite Volume Method

Outline

1. Fundamental concept of finite volume method
2. Flux form of FV method (3D and 1D)
3. Flux form (high-order vs low-order) for different schemes.
4. Flux limited/corrected methods
5. Monotonic and total variances diminishing (TVD) schemes

Finite Volume Method

Flux concept:

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0 \;\Rightarrow\; \frac{\partial \psi}{\partial t} + \frac{\partial (c\psi)}{\partial x} = 0$$

If we define $f(\psi) = c\psi$ as flux term, then

$$\frac{\partial \psi}{\partial t} + \frac{\partial f(\psi)}{\partial x} = 0 \;\xrightarrow{\text{3D}}\; \frac{\partial \psi}{\partial t} = -\nabla \cdot \vec{f}$$

Flux form

$$\int \frac{\partial \psi}{\partial t} d\tau = -\int \nabla \cdot \vec{f} d\tau = -\oint \vec{f} \cdot d\vec{S}$$

$$\frac{d}{dt}\int \psi \, d\tau = -\int \nabla \cdot \vec{f} d\tau = -\oint \vec{f} \cdot d\vec{S}$$

Changing rate of ψ in a certain volume equals to the flux coming in (out) of the surface that encloses the volume

Example (continuity):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial x} = 0$$

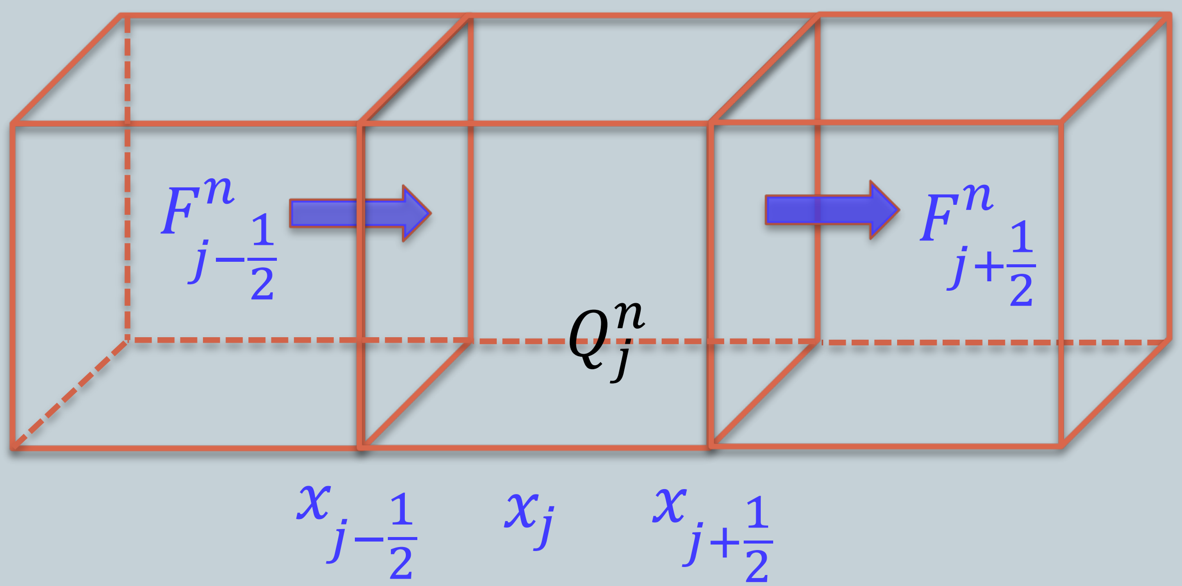
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Only evaluating $f(\psi)$ instead of its derivative, useful when the solution contains discontinuity, and the derivatives are not defined at the discontinuity.

Flux Form 3D

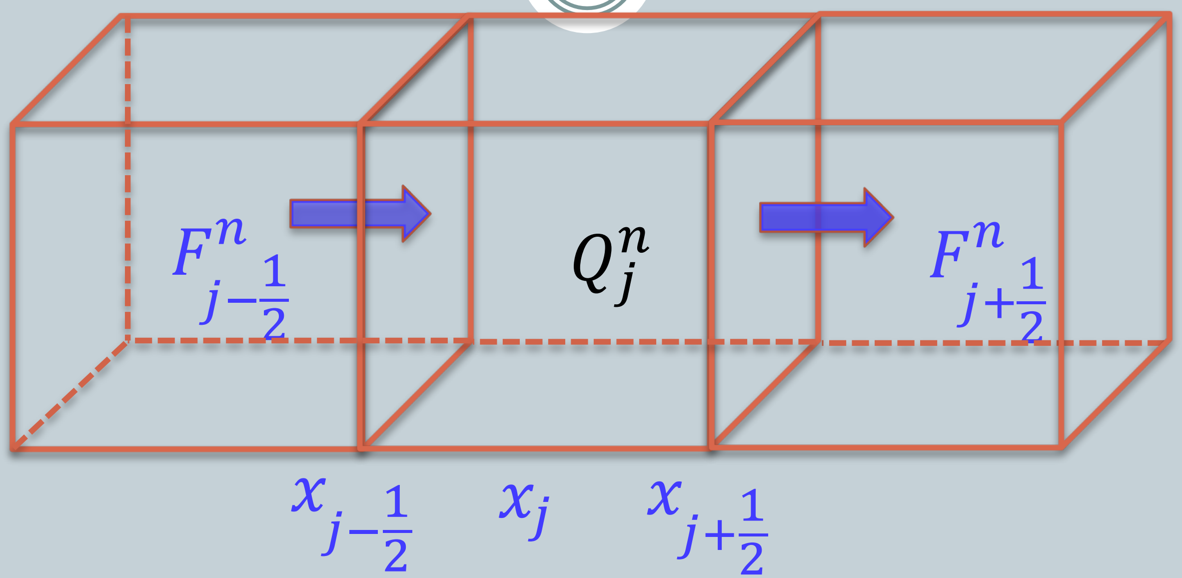
$$\frac{d}{dt}\int \psi\, d\tau = -\int \nabla \cdot \vec{f}\, d\tau = -\oint \vec{f} \cdot d\vec{S}$$

$Q = \frac{\int \psi d\tau}{\Delta\tau}$ *averaged form of* ψ *in a certain volume*



$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} \times A \times \Delta x = -\left(F^n_{j+1/2} - F^n_{j-1/2}\right) \times A \quad \Rightarrow \quad Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+1/2} - F^n_{j-1/2}\right)$$

Finite Volume Principle



Flux Form

$$\boxed{Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+1/2} - F^n_{j-1/2}\right)}$$

FV methods seek to obtain an appropriate flux term "F", to approximate an averaged flux at the cell interfaces occurring between certain time steps.

$$F^n_{j+1/2} \approx \frac{1}{\Delta t}\int_{t_1}^{t_2} f\left(t, x_{j+1/2}\right) dt \quad \& \quad F^n_{j-1/2} \approx \frac{1}{\Delta t}\int_{t_1}^{t_2} f\left(t, x_{j-1/2}\right) dt$$

1D Flux Form

Integrate over time and space which can avoid discontinuity problem:

$$\frac{\partial\psi}{\partial t} + \frac{\partial f(\psi)}{\partial x} = 0 \quad \Rightarrow \quad \int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial\psi}{\partial t}dxdt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial f(\psi)}{\partial x}dx\,dt$$

$$\int_{x_1}^{x_2}\psi(x,t_2)dx - \int_{x_1}^{x_2}\psi(x,t_1)dx = -\left(\int_{t_1}^{t_2} f\big(\psi(x_2,t)\big)\,dt - \int_{t_1}^{t_2} f\big(\psi(x_1,t)\big)\,dt\right)$$

$$[Q(t_2) - Q(t_1)]\Delta x = -[F(x_2) - F(x_1)]\Delta t$$

- $Q(t_2), Q(t_1)$: cell (volume) average quantify ψ at time t_2 and t_1.
- $F(x_2), F(x_1)$: time average flux at right and left surface

Physical Meaning

Flux Form

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

- Q_j^{n+1}, Q_j^n: cell (volume) average ψ at current and next time steps.
- $F_{j+1/2}^n$,$F_{j-1/2}^n$: time average flux at right and left surface

Change of averaged ψ in a certain volume equals to the flux coming in minus the flux coming out (net flux coming in) of the surface that encloses the volume.

- Different schemes use different flux forms for the finite volume methods.
- Key issue: identify the most efficient/appropriate flux terms.

Forward Time/Upstream Method

UPSTREAM SOLUTION FOR CONSTANT COEFFICIENT ADVECTION EQUATION:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = c\,\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(cQ_j^n - cQ_{j-1}^n\right)$$

FLUX FORM

$$F_{j+1/2}^n = cQ_j^n,\ F_{j-1/2}^n = cQ_{j-1}^n$$

Lax-Wendroff

Upstream solution for constant coefficient advection equation:

$$\phi_j^{n+1} = \phi_j^n - \frac{c\Delta t}{2\Delta x}\left(\phi_{j+1}^n - \phi_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2 \left(\phi_{j-1}^n - 2\phi_j^n + \phi_{j+1}^n\right)$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

Flux form

$$F_{j+1/2}^n = \frac{1}{2}c\left(Q_{j+1}^n + Q_j^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}c^2\left(Q_{j+1}^n - Q_j^n\right)$$

$$F_{j-1/2}^n = \frac{1}{2}c\left(Q_{j-1}^n + Q_j^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}c^2\left(Q_j^n - Q_{j-1}^n\right)$$

Flux-Limited Methods

DIFFERENT SCHEMES CAN DEFINE DIFFERENT FLUX FORMS (FROM LOW TO HIGH ORDERS), FLUX LIMITED & FLUX CORRECTED METHODS COMBINE THE BEST FEATURES OF LOW ORDER AND HIGH ORDER METHODS:

FLUX LIMITED

$$F_{j+\frac{1}{2}} = F^L_{j+\frac{1}{2}} + \gamma_{j+\frac{1}{2}}\left(F^H_{j+\frac{1}{2}} - F^L_{j+\frac{1}{2}}\right)$$

$$F_{j-\frac{1}{2}} = F^L_{j-\frac{1}{2}} + \gamma_{j-\frac{1}{2}}\left(F^H_{j-\frac{1}{2}} - F^L_{j-\frac{1}{2}}\right)$$

$$if\ \gamma = 0, totally\ low - order\ flux, highly\ diffusive$$

$$if\ \gamma = 1, totally\ high - order\ flux, doesnot\ necessarily\ satisfy\ TVD$$

$$if\ 0 < \gamma < 1, mix\ of\ low\ and\ \ high - order\ fluxes$$

Total Variation Diminishing (TVD)

- TVD is a criteria to judge whether a FVM is practical. Independent of scheme stability, not a sufficient condition for a "satisfactory" solution.
- If the system is not to develop nonphysical maxima or minima (e.g., oscillation problem), the "total variation" of the solution must remain constant or decreasing over each time step:

$$\boxed{TV(\psi) = \sum_j \left|\psi_{j+1} - \psi_j\right|}$$

$$\boxed{TV(\psi^{n+1}) \leq TV(\psi^n)}$$

- Useful higher order methods do not satisfy TVD alone, and many TVD low-order methods are too diffusive for practical application. Flux limiters: anti-diffusion.
- "High resolution" TVD methods are of great utility for systems with initially discontinuous or steep solutions, or those apt to produce shocks or steep solutions.

Monotonicity of a Method

- Methods which satisfy TVD also preserve monotonicity of solutions. A scheme is monotone if:

Given: $$\phi_j^n \geq \phi_{j+1}^n$$

Then: $$\phi_j^{n+1} \geq \phi_{j+1}^{n+1}$$

- Won't develop new local maxima or minima.

$$\phi_j^{n+1} = H(\phi_{j-p,\ldots\ldots}^n \phi_{j+q+1}^n)$$

$$\frac{\partial H(\phi_{j-p,\ldots\ldots}^n \phi_{j+q+1}^n)}{\partial \phi_i} \geq 0$$

- First-Order Upstrem/Forward-Time

$$\phi_j^{n+1} = (1-\mu)\phi_j^n + \mu\phi_{j-1}^n$$

$$\phi_j^n \geq \phi_{j+1}^n,\ \phi_{j-1}^n \geq \phi_j^n$$

$$\phi_j^{n+1} = (1-\mu)\phi_j^n + \mu\phi_{j-1}^n \geq (1-\mu)\phi_{j+1}^n + \mu\phi_j^n = \phi_{j+1}^{n+1}$$

Goals of FVM

- Pure second order methods cannot be monotone; any linear monotone scheme is at best first-order accurate.

- We week to use flux limiters to control the inclusion of high-order terms, creating nonlinear schemes with greater than first order accuracy, while preserving TVD criteria.