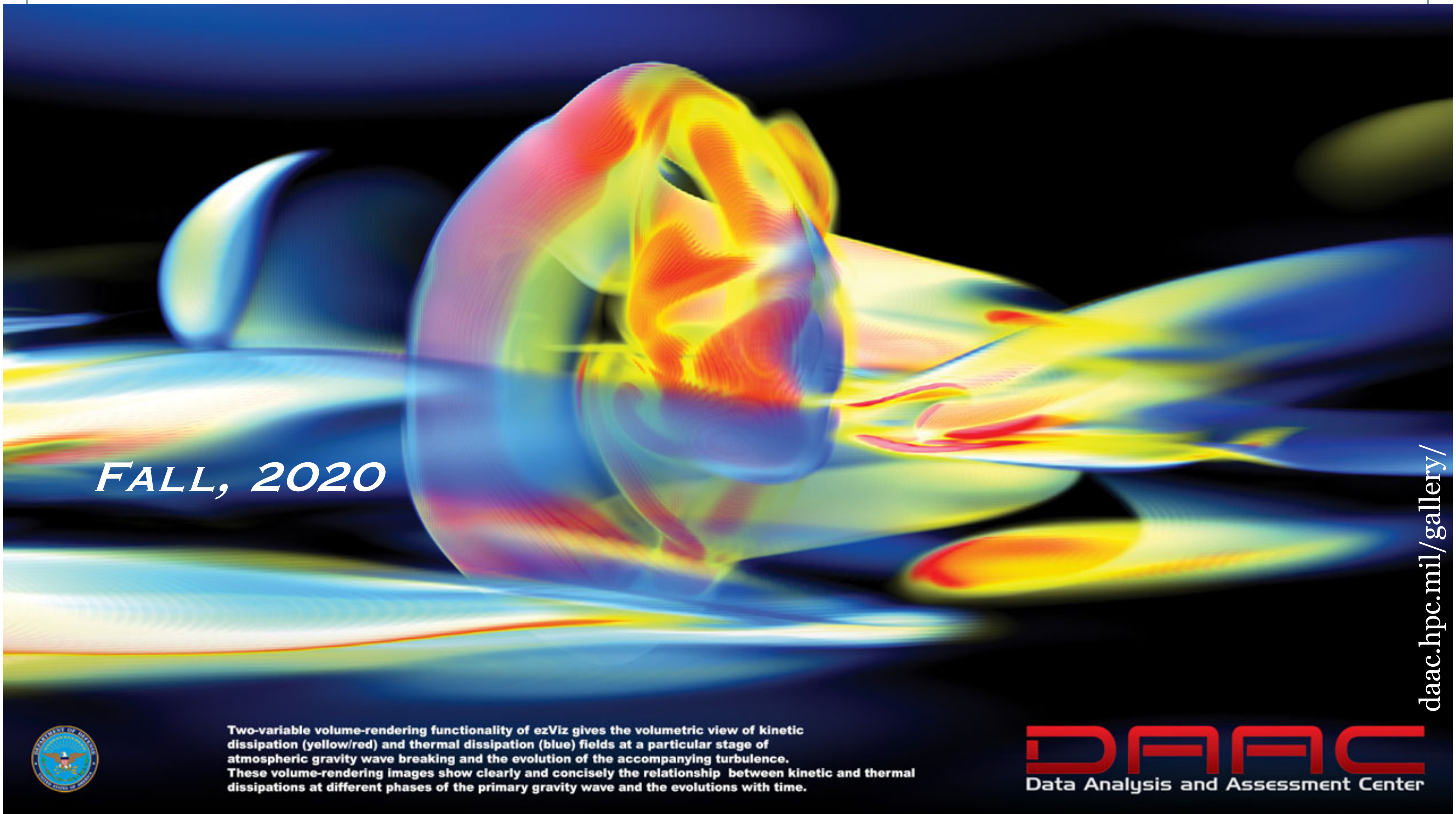
PHYS 8750
NUMERICAL FLUID DYNAMICS



PHYS 8750

Class #17 (Chapter 5.5)

Finite Volume Method

Flux limiters

CLASS #16
(CHAPTER 5.4)
FLUX CORRECTED
TRANSPORT

Outline

1. Review of FCT method

2. Flux limiter methods

2.1. Main ideas

2.2. Schemes: Superbee, Van Leer, minmod

3. Examples (coding)

Flux-Corrected Transport

- **General Idea:**

Flux Form

$$\phi_j^{n+1} = \phi_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

1. Compute low-order fluxes $F_{j+\frac{1}{2}}^1$ using a monotone scheme
2. Compute high-order fluxes $F_{j+\frac{1}{2}}^h$ using a high-order scheme
3. Transported & diffused (td) solution:

$$\phi_j^{td} = \phi_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^1 - F_{j-1/2}^1\right)$$

4. Compute anti-diffusive fluxes

$$A_{j+\frac{1}{2}} = F_{j+\frac{1}{2}}^h - F_{j+\frac{1}{2}}^1$$

Flux-Corrected Transport

5. Correct $A_{j+1/2}$ by weighting it, so "antidiffusion" step does not generate new minima or maxima:

$$A^c_{j+\frac{1}{2}} = C_{j+\frac{1}{2}} A_{j+\frac{1}{2}}$$

6. Correct td solutions:

$$\phi_j^{\mathrm{n}+1} = \phi_j^{td} - \frac{\Delta t}{\Delta x}\left(A^c_{j+\frac{1}{2}} - A^c_{j-\frac{1}{2}}\right)$$

$$\phi_j^{td} = \phi_j^n - \frac{\Delta t}{\Delta x}\left(F^1_{j+1/2} - F^1_{j-1/2}\right)$$

$$A_{j+\frac{1}{2}} = F^h_{j+\frac{1}{2}} - F^1_{j+\frac{1}{2}}$$

Flux-Corrected Transport

$$\phi_j^{n+1} = \phi_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^1 - F_{j-1/2}^1\right) - \frac{\Delta t}{\Delta x}\left(C_{j+\frac{1}{2}} A_{j+\frac{1}{2}} - C_{j-\frac{1}{2}} A_{j-\frac{1}{2}}\right)$$

$$= \phi_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^1 - F_{j-1/2}^1\right)$$

$$- \frac{\Delta t}{\Delta x}\left(C_{j+\frac{1}{2}}\left(F^h_{j+\frac{1}{2}} - F^1_{j+\frac{1}{2}}\right) - C_{j-\frac{1}{2}}\left(F^h_{j-\frac{1}{2}} - F^1_{j-\frac{1}{2}}\right)\right)$$

$$= \phi_j^n - \frac{\Delta t}{\Delta x}\left(\underline{\left[F^1_{j+\frac{1}{2}} + C_{j+\frac{1}{2}}\left(F^h_{j+\frac{1}{2}} - F^1_{j+\frac{1}{2}}\right)\right]} - \underline{\left[F^1_{j-\frac{1}{2}} + C_{j-\frac{1}{2}}\left(F^h_{j-\frac{1}{2}} - F^1_{j-\frac{1}{2}}\right)\right]}\right)$$

Corrected Flux at j+1/2

Corrected Flux at j-1/2

Flux-Limited Methods: do not calculate ϕ_j^{td}, but deal with $C_{j+\frac{1}{2}}$

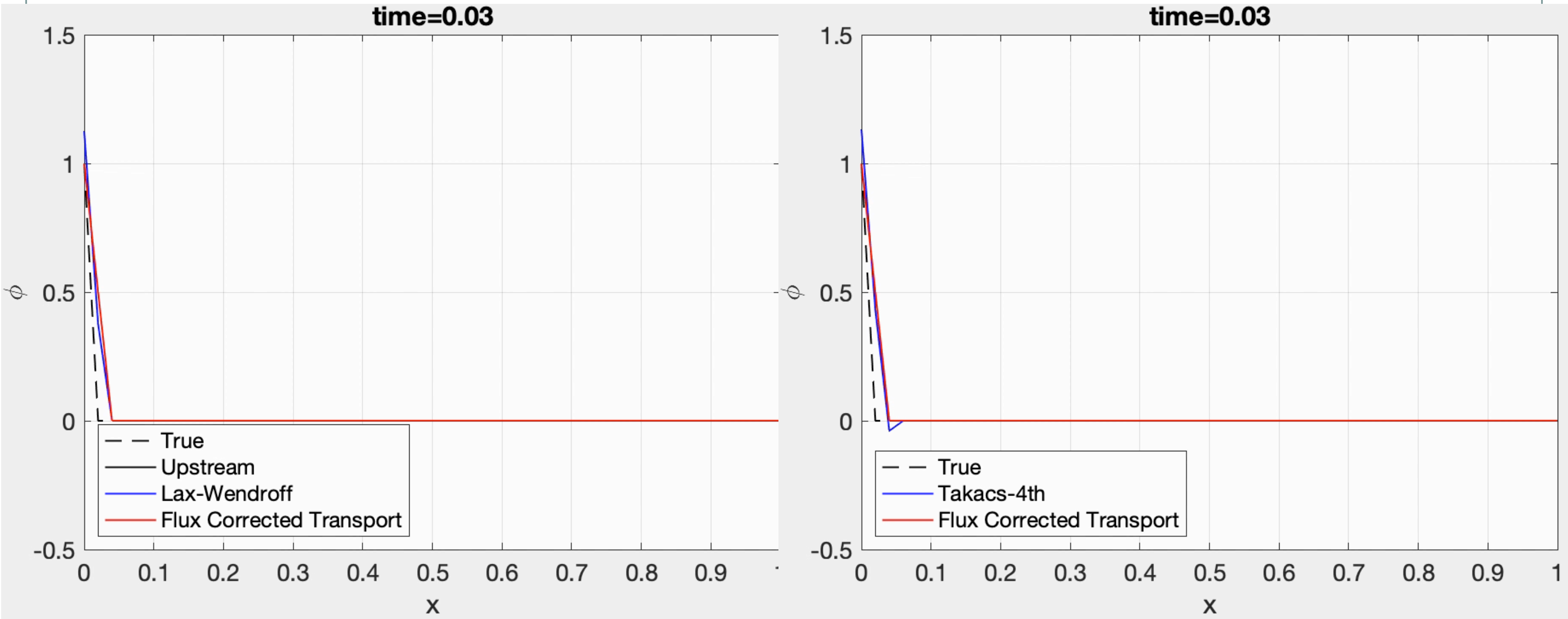
FCT Flux Correction

- Boris and Book (1973)

$$\phi_j^{n+1} = \phi_j^{td} - \frac{\Delta t}{\Delta x}\left(A^c_{j+\frac{1}{2}} - A^c_{j-\frac{1}{2}}\right)$$

$$A^c_{j+\frac{1}{2}} = \underbrace{\mathrm{sgn}\left(A_{j+\frac{1}{2}}\right) max}_{\text{Won't do anti-correction}} \left\{0, min \underbrace{\begin{bmatrix} \left|A_{j+\frac{1}{2}}\right| \\ \mathrm{sgn}\left(A_{j+\frac{1}{2}}\right)\left(\phi^{td}_{j+2} - \phi^{td}_{j+1}\right)\frac{\Delta x}{\Delta t} \\ \mathrm{sgn}\left(A_{j+\frac{1}{2}}\right)\left(\phi^{td}_{j} - \phi^{td}_{j-1}\right)\frac{\Delta x}{\Delta t} \end{bmatrix}}_{\text{Won't create new minima or maxima: retain the monotonicity of low-order scheme}}\right\}$$

Comparison w/wo FCT



FCT can efficiently remove ripples (avoid generating local new maxima/minima or make them extreme) and correct the high-order scheme.

Flux-Limited Methods

Flux-Limited Methods: do not calculate ϕ_j^{td}, but deal with $C_{j+\frac{1}{2}}$

$$F_{j+\frac{1}{2}} = F^1_{j+\frac{1}{2}} + C_{j+\frac{1}{2}}\left(F^h_{j+\frac{1}{2}} - F^1_{j+\frac{1}{2}}\right)$$

$$C_{j+\frac{1}{2}} = C(r_{j+\frac{1}{2}})$$

$$r_{j+\frac{1}{2}} = \frac{\phi_j - \phi_{j-1}}{\phi_{j+1} - \phi_j}$$

If solutions are smooth

$$r_{j+\frac{1}{2}} \approx 1$$

If local minima/maxima (ϕ_j) exist

$$r_{j+\frac{1}{2}} < 0$$

Lax-Wendroff Method

- Flux form for Lax-Wendroff method (Sweby, 1984):

$$F^{LW}_{j+\frac{1}{2}} = \underbrace{c\phi_j}_{\text{Upstream Flux}} + \frac{c}{2}(1-\mu)\left(\phi_{j+1}-\phi_j\right)$$

$$\boxed{F_{j+\frac{1}{2}} = c\phi_j + \frac{c}{2}(1-\mu)\left(\phi_{j+1}-\phi_j\right) C_{j+\frac{1}{2}}}$$

$$\boxed{\begin{aligned}\phi^{n+1}_{j+1} &= \phi^n_j - \left[\mu - \frac{\mu}{2}(1-\mu)C_{j-\frac{1}{2}}\right]\left(\phi^n_j - \phi^n_{j-1}\right) \\ &-\frac{\mu}{2}(1-\mu)C_{j+\frac{1}{2}}\left(\phi^n_{j+1}-\phi^n_j\right)\end{aligned}}$$

Use TVD to Constrain C

Section 5.5.1

$$\phi_{j+1}^{n+1} = \phi_j^n - \left[\mu - \frac{\mu}{2}(1-\mu)C_{j-\frac{1}{2}}\right]\left(\phi_j^n - \phi_{j-1}^n\right)$$
$$-\frac{\mu}{2}(1-\mu)C_{j+\frac{1}{2}}\left(\phi_{j+1}^n - \phi_j^n\right)$$

$$Apply\ TVD\ criterion\ and\ r_{j+\frac{1}{2}} = \frac{\phi_j - \phi_{j-1}}{\phi_{j+1} - \phi_j}$$

$$\sum_j \left|\phi_{j+1}^{n+1} - \phi_j^{n+1}\right| \leq \sum_j \left|\phi_{j+1}^n - \phi_j^n\right|\ to\ constrain\ coefficient\ C:$$

$$0 \leq \frac{C(r)}{r} \leq 2 \qquad \& \qquad 0 \leq C(r) \leq 2$$

Compare with FCT Method

$$r_{j+\frac{1}{2}} = \frac{\phi_j - \phi_{j-1}}{\phi_{j+1} - \phi_j}$$

If local minima/maxima exist

$$r_{j+\frac{1}{2}} < 0$$

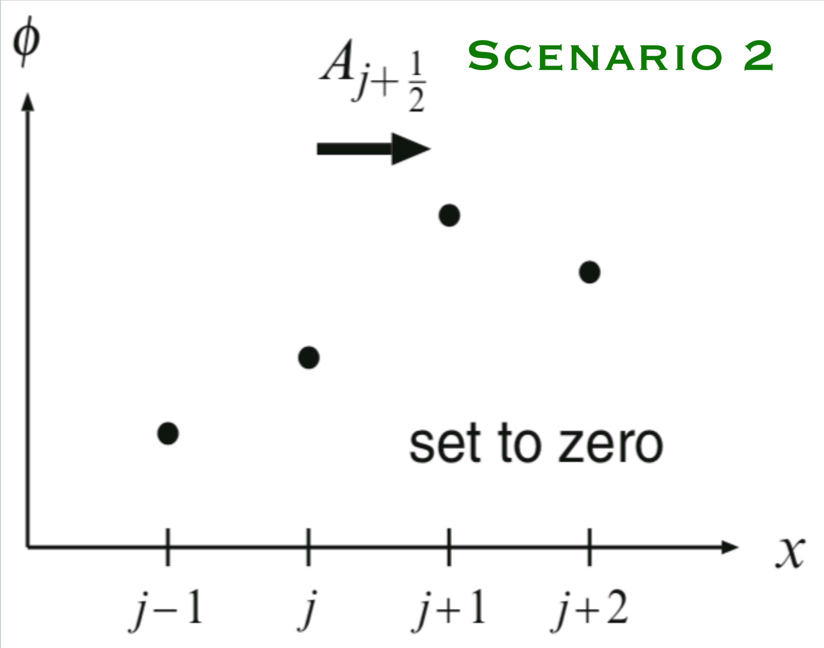
$+$

$$0 \le \frac{C(r)}{r} \le 2$$

$$0 \le C(r) \le 2$$

$$C(r) = 0$$

No correction



FCT: $\phi_{j+2}^{td} - \phi_{j+1}^{td} < 0$

$$min\left[A_{j+\frac{1}{2}}, \left(\phi_{j+2}^{td} - \phi_{j+1}^{td}\right)\frac{\Delta x}{\Delta t}\right] < 0$$

$$A_{j+\frac{1}{2}}^{c} = 0$$

No correction

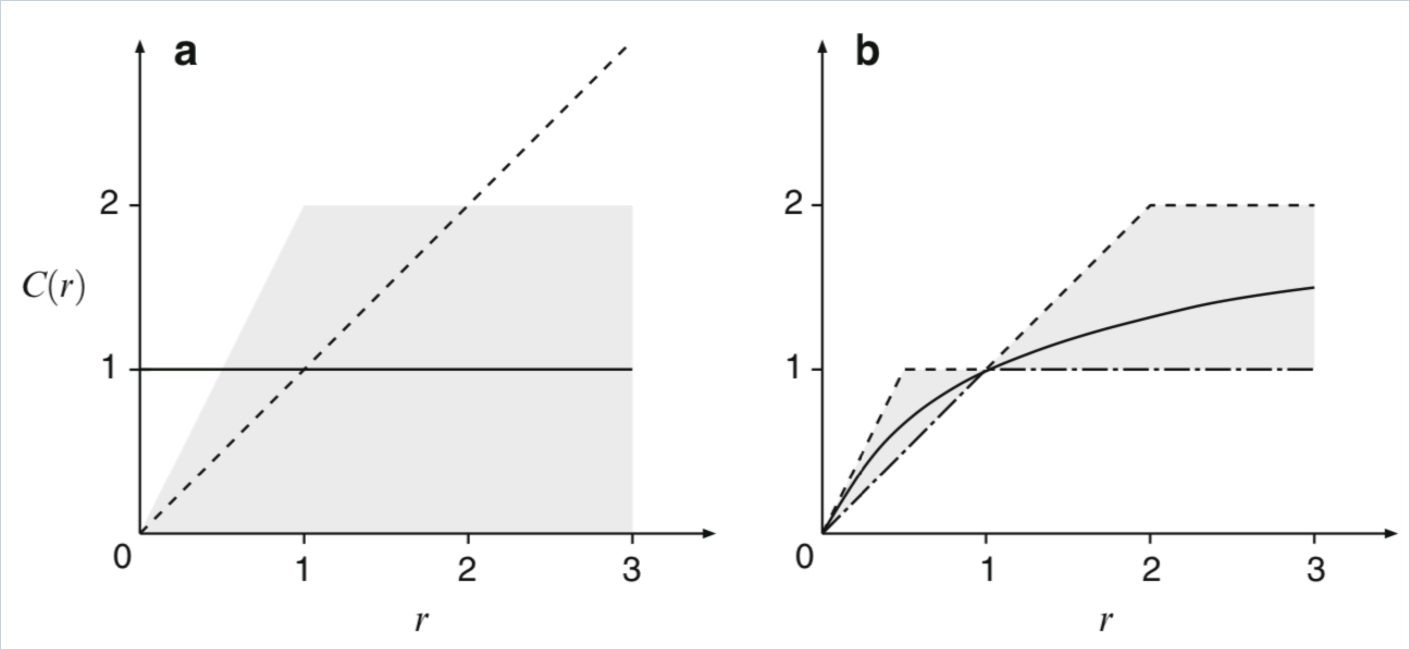
ϕ_{j+1}^{n+1} will not increase to amplify local maximum.

Selection of Coefficient C

- If no local minima/maxima exist: $r_{j+\frac{1}{2}} > 0$

$$0 \le \frac{C(r)}{r} \le 2 \qquad \& \qquad 0 \le C(r) \le 2$$

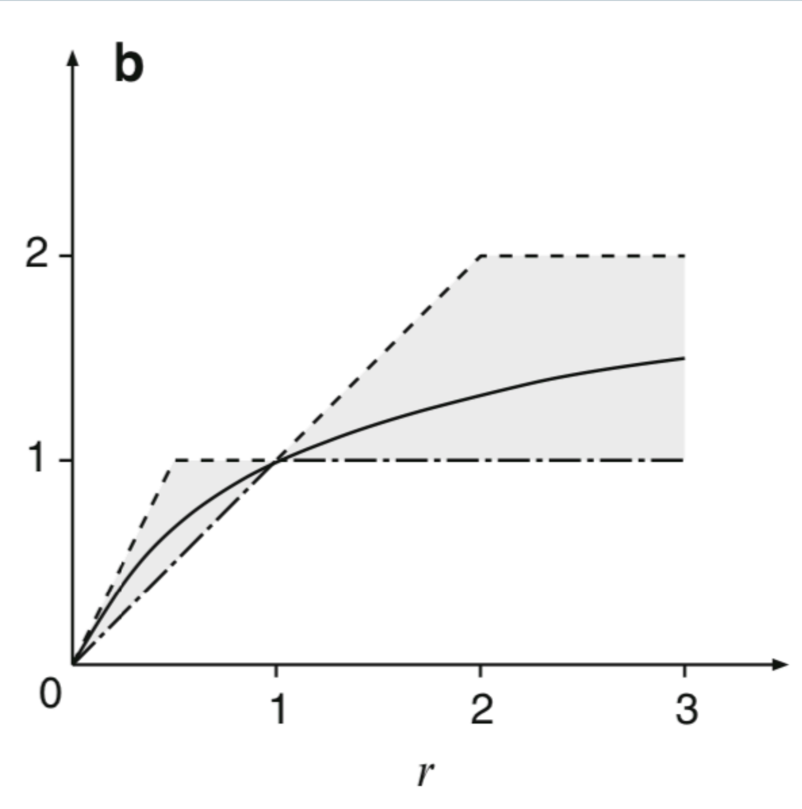
- To maintain second-order accuracy, weighted averages of Lax-Wendroff ($C(r) = 1$) and Warming and Beam ($C(r) = r$).



Durran's book

Flux-Limiters

C(r)



- Dashed line : Superbee

$$C(r) = \max[0, \min(1, 2r), \min(2, r)]$$

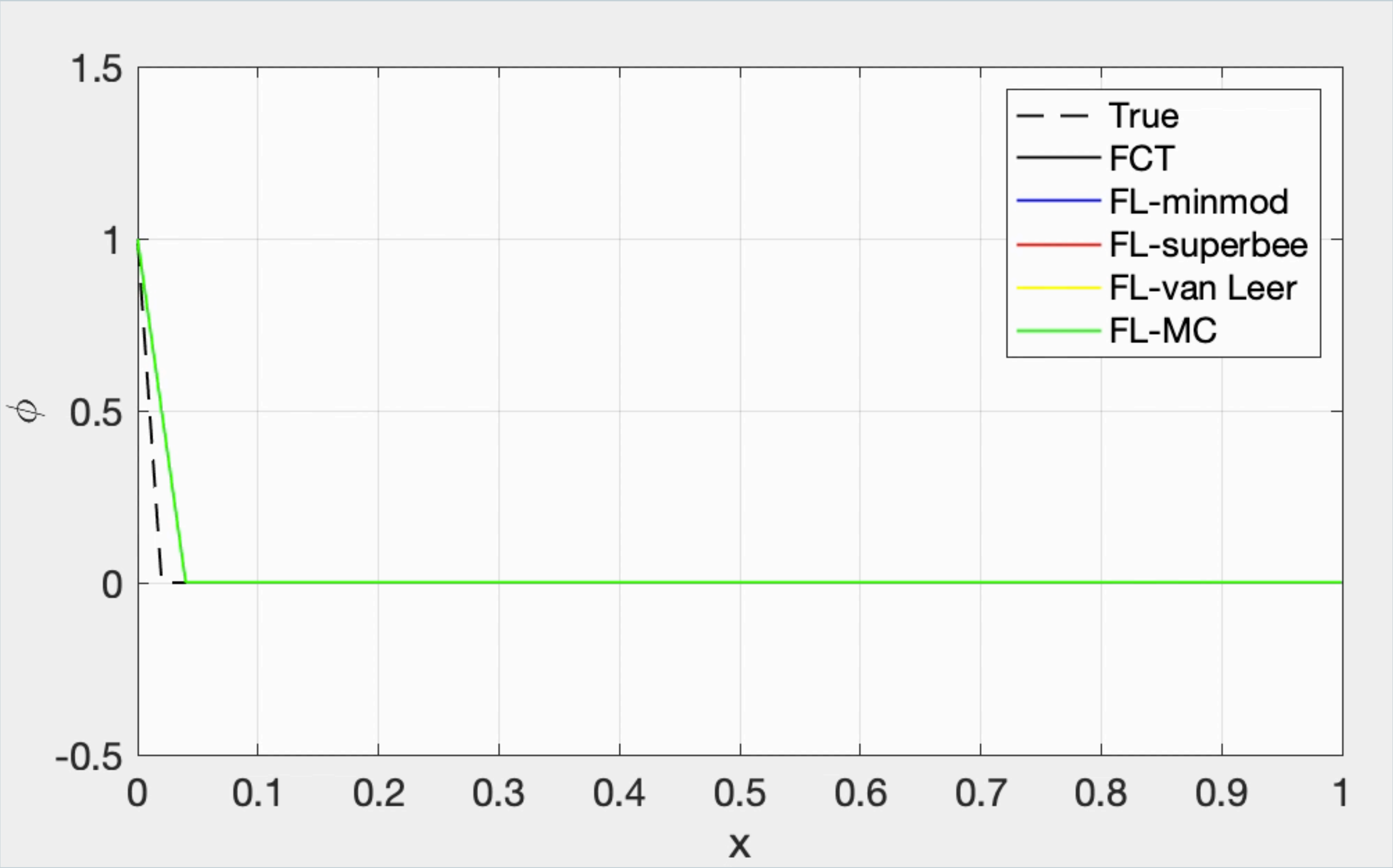
- Solid line: Van Leer

$$C(r) = \frac{r + |r|}{1 + |r|}$$

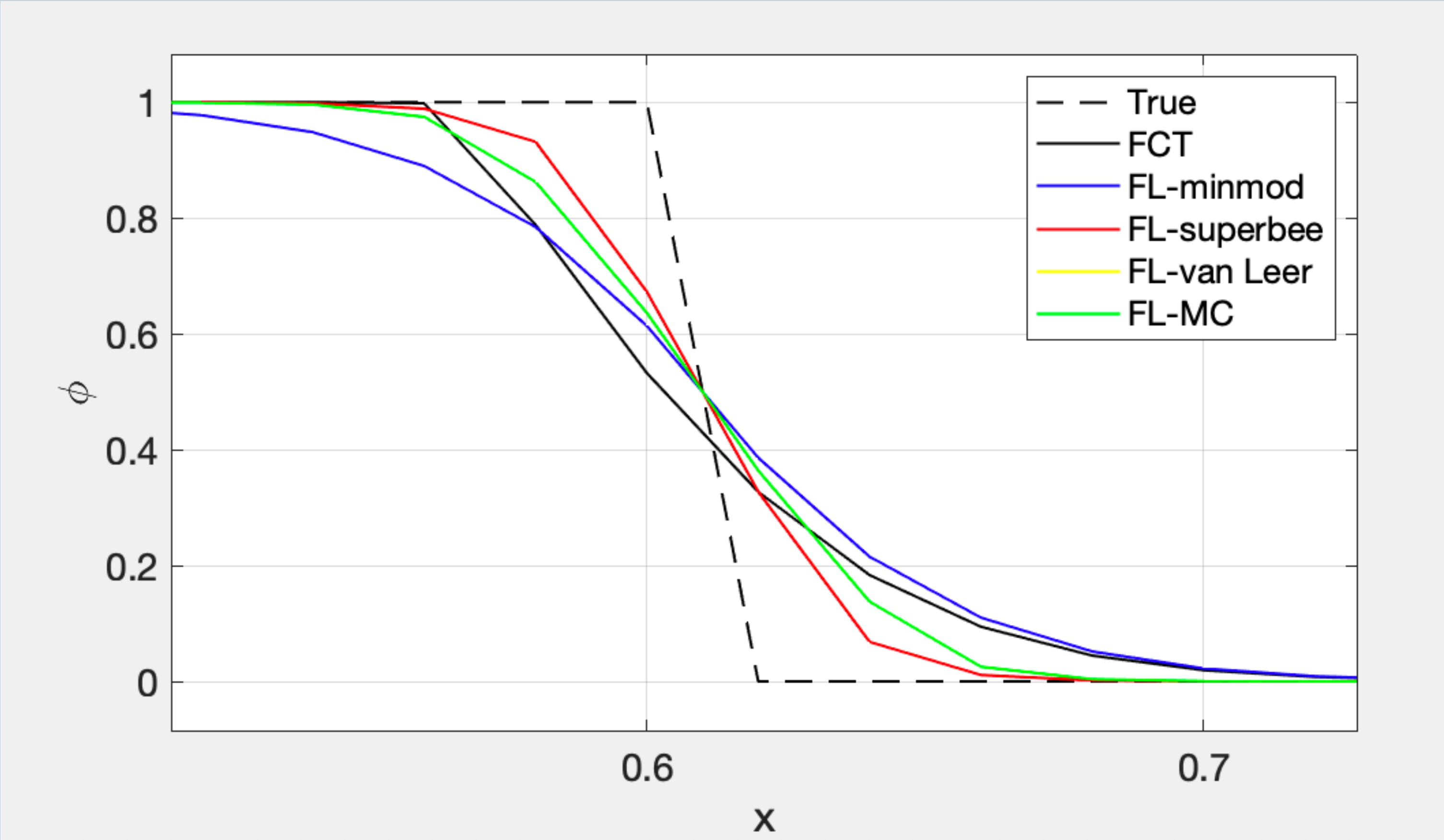
- Dot-dashed line: minmod

$$C(r) = \max[0, \min(1, r)]$$

Method Comparison



Method Comparison



Schedule

- Oct 27, 29
- Nov 3 (fall break), 5, 10, 12, 17, 19, 24,
  26 (Thanksgiving)
- Dec 1, 3

Proposal presentations: Nov 10, 12

Final project presentations: Nov 24, Dec 1, 3

Final Projects

Proposal & Final Projects:

1) Modify parameters in an existing model to study the science/engineering questions you are interested.

2) Modify an existing model (add new equations, new grids, and etc.)

3) Develop your own model.

Points to include in the report (subset of all of these):

1) Introduction of the model.
2) What is the numerical problem you want to solve?
3) What is the numerical scheme used?
4) What improvements/modifications you want to make and why?