Notes for Takacs (1985) paper ⇐ Lax-Wendroff is one special case of this paper's ideas.

"A Two-Step Scheme for the Advection Equation with Minimized Dissipation & Dissipation Errors"

start from
Transport-advection equation, $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$, $\underbrace{\psi = f(x-ct)}_{\text{wave equation}}$

general finite-difference scheme:

$$\phi_j^{n+1} = \sum_{j'} a_{j'}\,\phi_{j+j'}^{n} \qquad j' = 0, \pm 1, \pm 2, \pm 3, \ldots$$

First-order accuracy. $\cancel{\phi_j^{n+1} - \phi_j^{n}}$

$$\cancel{\sum_{j'} a_{j'}\,\phi_j^{n+1} = \sum_{j'} a_{j'}\,\phi_{j'}^{n}}$$

$$\sum_{j'} a_{j'} = 1, \quad \sum_{j'} j' a_{j'} = -\mu, \quad \mu = c\frac{\Delta t}{\Delta x}$$

second order: in addition to above $\sum_{j'} j'^2 a_{j'} = \mu^2$

m-th order, m+1 requirements, $\sum_{j'} j'^m a_{j'} = (-\mu)^m$ in addition to all above

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Error analysis, Von Neumann's method:

$$\phi_j^n = \sum_{k=-N}^{N} \hat{\phi}_k^n\, e^{i k \underbrace{j\Delta x}_{\text{distance}}}$$, diagnose individual wavenumber:

for k-th wavenumber
⇒

$$\phi_j^{n+1} = \sum_{j'} a_{j'}\,\hat{\phi}^n e^{ik(j+j')\Delta x} = \hat{\phi}^{n+1} e^{i(k)j\Delta x}$$

$$\Rightarrow A_k = \frac{\hat{\phi}_j^{n+1}}{\hat{\phi}_j^{n}} = \frac{\sum_{j'} a_{j'} e^{ik(j+j')\Delta x}}{e^{ikj\Delta x}} = \sum_{j'} a_{j'} e^{ikj'\Delta x}$$

$\theta = k\Delta x$

$$A_k = \sum_{j'} a_{j'} e^{ij'\theta}$$, amplification factor

⇒ wavenumber dependent.

① True amplification factor: Fourier series expansion

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0, \qquad \psi = \sum_{k=-N}^{N} a_k e^{ik(x-ct)}$$

$$A_r = \frac{\psi(t+\Delta t)}{\psi(t)} = \frac{\sum_{k=-N}^{N} a_k e^{i[k(x-c(t+\Delta t))]}}{\sum_{k=-N}^{N} a_k e^{i[k(x-ct)]}} = e^{-ic\cancel{\Delta t}k} \doteq -e^{ik\frac{c\Delta t}{\Delta x}\cdot\Delta x}$$

CFL number

$$= + e^{-i\theta\cdot\mu} \qquad \left(\theta = k\cdot\Delta x, \ \mu = c\frac{\Delta t}{\Delta x}\right)$$

$|A_r| = 1$

phase: $-\mu\theta$.

② numerical amplification factor:

$$A = \sum_{j'} a_j' e^{ikj'\Delta x} = \sum_{j'} a_{j'} \cdot e^{ij'\theta}$$

③ $$\frac{A}{A_r} = 1+\varepsilon = 1+\varepsilon_R + i\varepsilon_I = \frac{\sum_{j'} a_{j'} e^{ij'\theta}}{e^{-i\theta\mu}}$$

$$= \sum_{j'} a_{j'} e^{i(j'+\mu)\theta} = \sum_{j'} a_{j'} \cos[(j'+\mu)\theta] + i\sum_{j'} a_{j'} \sin[(j'+\mu)\theta]$$

$$\Rightarrow \varepsilon_R = \sum_{j'} a_{j'} \cos[(j'+\mu)\theta] - 1$$

$$\varepsilon_I = \sum_{j'} a_{j'} \sin[(j'+\mu)\theta]$$

assume: $$\frac{A}{A_r} = \frac{|A| e^{i\varphi_D}}{|A_r| e^{i\varphi_T}} = 1+\varepsilon_R + i\varepsilon_I = \cancel{|A_r| + \varepsilon_a}$$

$$\Rightarrow \frac{|A|^2 e^{2i\varphi_D}}{|A_r|^2 e^{2i\varphi_T}} = \left[(1+\varepsilon_R)^2 - \varepsilon_I^2\right] + 2\varepsilon_I(1+\varepsilon_R)\cdot i$$

if $|A_r|^2 = |A|^2 \cancel{\theta} = |A_r|^2 + \varepsilon_a$, $\quad \varphi_D = \varphi_T + \varepsilon_\varphi$,

ε_a: amp error

ε_φ: phase error

$$\frac{\left[|A_r|^2 + \mathcal{E}_a\right]}{(A_r)^2} \cdot e^{2 i \mathcal{E}_\varphi} = (1+\mathcal{E}_R)^2 - \mathcal{E}_I^2 + 2\mathcal{E}_I(1+\mathcal{E}_R)\, i$$

$\mathcal{E}_R$ determines the order of $\mathcal{E}_a$

$$\Rightarrow \quad \mathcal{E}_a = \mathcal{E}_R^2 + \mathcal{E}_I^2 + 2\mathcal{E}_R$$ amplitude error.

$$\mathcal{E}_\varphi = \tan^{-1}\left(\frac{\mathcal{E}_I}{\mathcal{E}_R + 1}\right) = \mathcal{E}_I\left(1 - \mathcal{E}_R + \mathcal{E}_R^2 - \cdots\right)$$ phase error

$\mathcal{E}_I$ determines the order of $\mathcal{E}_\varphi$.

This can be used to plot phase/amplitude errors.

$\theta = k\Delta x$

maximum possible θ:

$$\theta = \frac{2\pi}{2\Delta x} \cdot \Delta x = \pi$$

$\theta \sim [0, \pi]$

(3.8) $$\xi_R = \sum_{j'} a_{j'} \cos(j' + \mu)\theta - 1$$

(3.9) $$\xi_I = \sum_{j'} a_{j'} \sin(j' + \mu)\theta$$

Taylor expanding (3.8) and (3.9) along $\theta = 0$.

$$\xi_R = \sum_{j'} a_{j'} - 1 - \frac{(\Delta\theta)^2}{2!}\sum_{j'}(j'+\mu)^2 a_{j'} + \frac{(\Delta\theta)^4}{4}\sum_{j'}(j'+\mu)^4 a_{j'}$$

$$\xi_I = \Delta\theta \sum_{j'}(j'+\mu)a_{j'} - \frac{(\Delta\theta)^3}{3!}\sum_{j'}(j'+\mu)^3 a_{j'} + , \ldots\ldots$$

~~for/ on the order of~~

$$\mathcal{O}(\mathcal{E}_a) = \mathcal{O}(\xi_R), \quad \mathcal{O}(\mathcal{E}_\varphi) = \mathcal{O}(\mathcal{E}_I)$$

since first-order & ~~other~~ high-order scheme would all require $\sum_{j'} a_j - 1 = 0$

$$\xi_R = -\frac{(\Delta\theta)^2}{2!}\sum_{j'}(j'+\mu)^2 a_{j'} + \frac{(\Delta\theta)^4}{4}\sum_{j'}(j'+\mu)^4 a_{j'}$$: even-order scheme will help make zeros

$$\xi_I = \Delta\theta\sum_{j'}(j'+\mu)a_{j'} - \frac{(\Delta\theta)^3}{3!}\sum_{j'}(j'+\mu)^3 a_{j'} + \cdots$$ odd-number scheme will help make zeros

① even-order schemes will reduce the order of magnitude of real component of the order: amplitude error.

② odd-order schemes will reduce the order of magnitude associated with phase change.