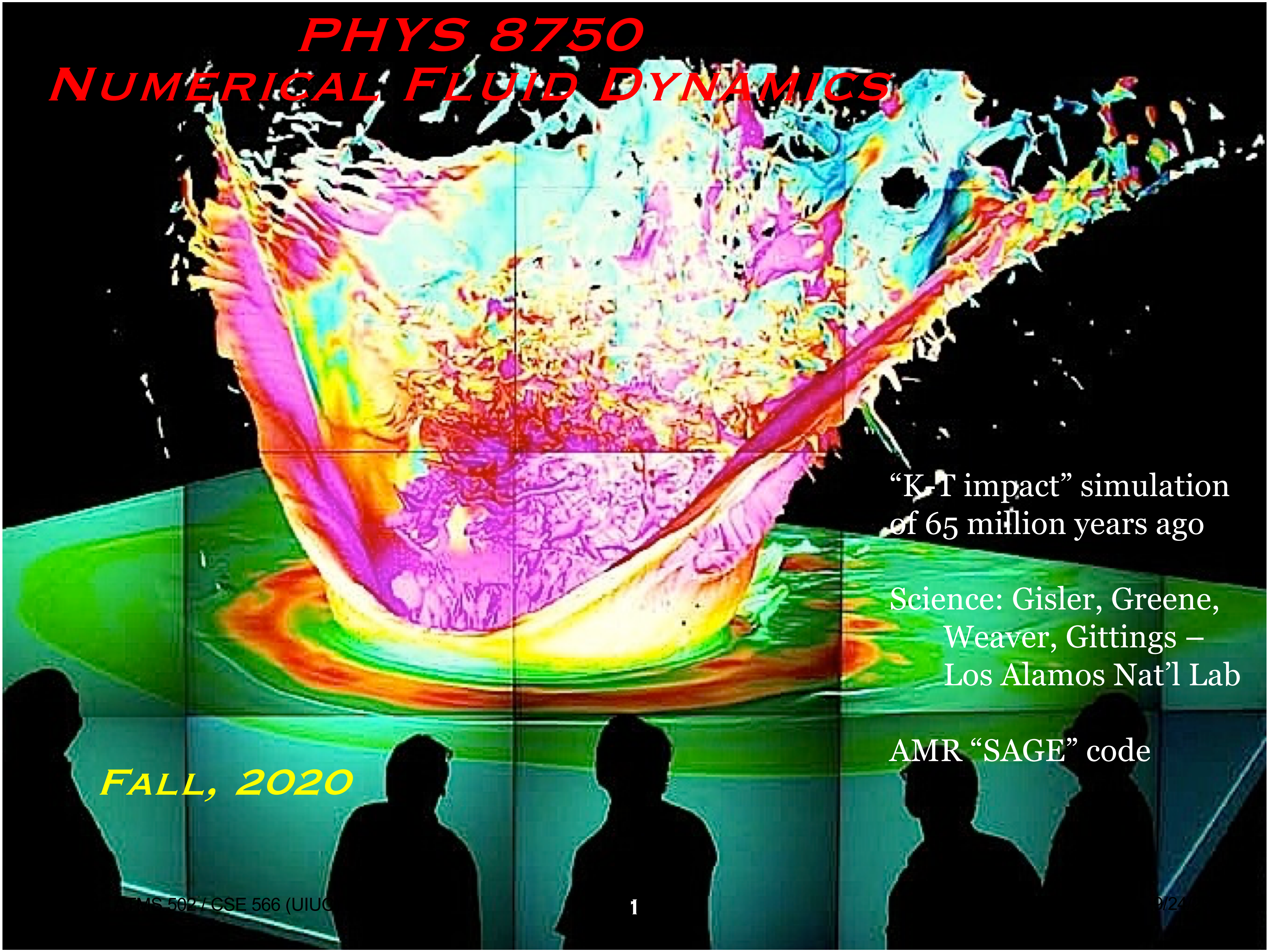

PHYS 8750
Numerical Fluid Dynamics
“K-T impact” simulation of 65 million years ago
Science: Gisler, Greene, Weaver, Gittings – Los Alamos Nat’l Lab
AMR “SAGE” code
Fall, 2020
1

PHYS 8750

Class #10
(Chapter 3.3)
Space Filtering
in TIEGCM

Class #11 (Chapter 4.1)

1) PDE with two variables

2) Shallow-water equations

3) Staggered grids

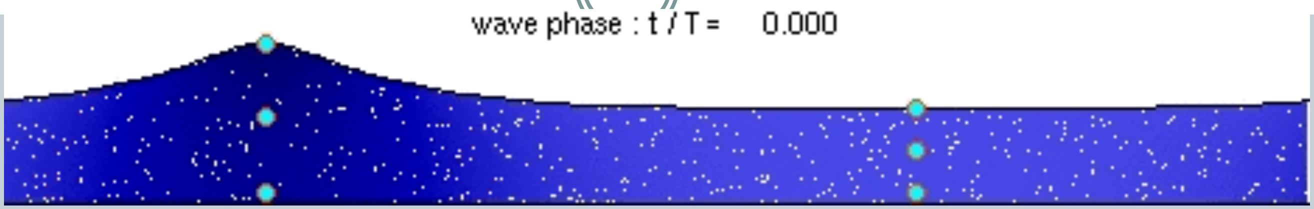
Outline

1. PDE with two variables
2. Shallow-water equations

   Von Neumann Method

   Discrete-dispersion method
3. Staggered grid for multi-variables PDEs linked to physics.
4. Staggered grid for shallow-water equations.
5. Examples (Codes)

Shallow-Water Equations



Shallow-Water Equations:

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

U(u): mean(perturbation) fluid velocity

H (h): mean(perturbation) flow velocity

g: gravity constant

Physics:

- Higher depth would be pulled down by gravity, which leads to horizontal motions.
- Building up of horizontal motions will lead to changes to water depths.
- Addition to this: advection of water depth and velocity by the mean flow.

$$if\ \frac{\partial h}{\partial x} < 0, u(t=0) = 0, u(\Delta t) > 0$$

$$if\ \frac{\partial u}{\partial x} < 0, \frac{\partial h}{\partial t} > 0, h\ will\ increase$$

Von Neumann Stability

For k-th wavenumber

$$\boxed{\boldsymbol{v}_k = \begin{pmatrix} u_k \\ h_k \end{pmatrix} e^{ikx}}$$

If true solution doesn't grow with time:

$$\boxed{\|\boldsymbol{v}_k^n\| = \left\|\boldsymbol{A}_k^n \boldsymbol{v}_k^0\right\| \le \left\|\boldsymbol{v}_k^0\right\|,\ \|\boldsymbol{A}_k^n\| \le 1}$$

If true solution grows with time:

$$\boxed{\|\boldsymbol{v}_k^n\| = \left\|\boldsymbol{A}_k^n \boldsymbol{v}_k^0\right\| \le C_T\left\|\boldsymbol{v}_k^0\right\|, \|\boldsymbol{A}_k^n\| \le C_T, for\ all\ n\Delta t \le T}$$

Apply to Leap-Frog Scheme and wave equation:

$$\frac{d\psi}{dt} = i\omega\psi \Longrightarrow \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} = i\omega\phi^n \Longrightarrow \phi^{n+1} = \phi^{n-1} + i2\omega\Delta t\phi^n$$

$$\chi^n = \phi^{n-1} \Longrightarrow \chi^{n+1} = \phi^n$$

$$\phi^{n+1} = \chi^n + i2\omega\Delta t\phi^n$$

$$\Longrightarrow$$

$$\phi^{n+1} = \chi^n + i2\omega\Delta t\phi^n$$

$$\chi^{n+1} = \phi^n$$

Von Neumann Stability
Multi-Variable PDEs

$$\phi^{n+1} = \chi^n + 2i\omega\Delta t\phi^n$$

$$\chi^{n+1} = \phi^n$$

$$\begin{pmatrix} 2i\omega\Delta t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi^n \\ \chi^n \end{pmatrix} = \begin{pmatrix} \phi^{n+1} \\ \chi^{n+1} \end{pmatrix} = \lambda \begin{pmatrix} \phi^n \\ \chi^n \end{pmatrix}$$

Eigenvalue λ for the amplification matrix satisfy:

$$\begin{pmatrix} 2i\omega\Delta t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi^n \\ \chi^n \end{pmatrix} = \lambda \begin{pmatrix} \phi^n \\ \chi^n \end{pmatrix} \longrightarrow \lambda^2 - 2i\omega\Delta t\lambda - 1 = 0$$

Similar to what we did before, solve for roots from the quadratic equation, and make sure $|\lambda_\pm| \leq 1$

$$\longrightarrow |\omega\Delta t| < 1$$

Key: find out the amplification coefficient matrix, the eigenvalue would be the amplification factor. Then solve for the quadratic equations and get constraints from bounded amplification factor

Discrete-Dispersion Relation

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

leapfrog

Center space

$$\delta_{2t}u + \mathrm{U}\delta_{2x}u + g\delta_{2x}h = 0$$

$$\delta_{2t}h + \mathrm{U}\delta_{2x}h + H\delta_{2x}u = 0$$

$$u_j^n = u_0 e^{i(kj\Delta x - \omega n\Delta t)}, h_j^n = h_0 e^{i(kj\Delta x - \omega n\Delta t)}$$

$$\left(-\frac{sin\omega\Delta t}{\Delta t} + \mathrm{U}\frac{sink\Delta x}{\Delta x}\right)u_0 + g\frac{sink\Delta x}{\Delta x}h_0 = 0$$

$$\left(-\frac{sin\omega\Delta t}{\Delta t} + \mathrm{U}\frac{sink\Delta x}{\Delta x}\right)h_0 + H\frac{sink\Delta x}{\Delta x}u_0 = 0$$

Discrete-Dispersion Relation

$$\left(-\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x}\right)u_0+g\frac{\sin k\Delta x}{\Delta x}h_0=0$$

$$\left(-\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x}\right)h_0+H\frac{\sin k\Delta x}{\Delta x}u_0=0$$

$$\begin{pmatrix} -\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x} & g\frac{\sin k\Delta x}{\Delta x} \\ H\frac{\sin k\Delta x}{\Delta x} & -\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x} \end{pmatrix}\begin{pmatrix} u_0 \\ h_0 \end{pmatrix}=0$$

$$\left(-\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x}\right)^2-gH\left(\frac{\sin k\Delta x}{\Delta x}\right)^2=0$$

Discrete-Dispersion Relation

$$\left(-\frac{\sin\omega\Delta t}{\Delta t}+\mathrm{U}\frac{\sin k\Delta x}{\Delta x}\right)^2-gH\left(\frac{\sin k\Delta x}{\Delta x}\right)^2=0$$

Define $\mathrm{c}=\sqrt{gH}$, discrete-dispersion relation:

$$\sin\omega\Delta t=\frac{\Delta t}{\Delta x}(U\pm c)\sin k\Delta x$$

If $\omega=\omega_r+i\omega_i$, then ω_r determines wave phase, ω_i gives growth rates

To avoid amplitude growth, ω cannot have imaginary part:

$$\left|\frac{\Delta t}{\Delta x}(U\pm c)\sin k\Delta x\right|=|\sin\omega\Delta t|$$

To satisfy this relation anytime: $\left|\frac{\Delta t}{\Delta x}(U\pm c)\right|\le 1,$ if $\mathrm{U}=0,\left|c\frac{\Delta t}{\Delta x}\right|\le 1$

Left side of "=" ... time rate of change of each variable.

Right side: *Pressure gradient force*

$$\frac{\partial u}{dt} = -\vec{V}\cdot\vec{\nabla}u - \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{dt} = -\vec{V}\cdot\vec{\nabla}v - \frac{1}{\overline{\rho}}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{dt} = -\vec{V}\cdot\vec{\nabla}w - \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} + g\frac{\theta}{\overline{\theta}} + \nu\nabla^2 w$$

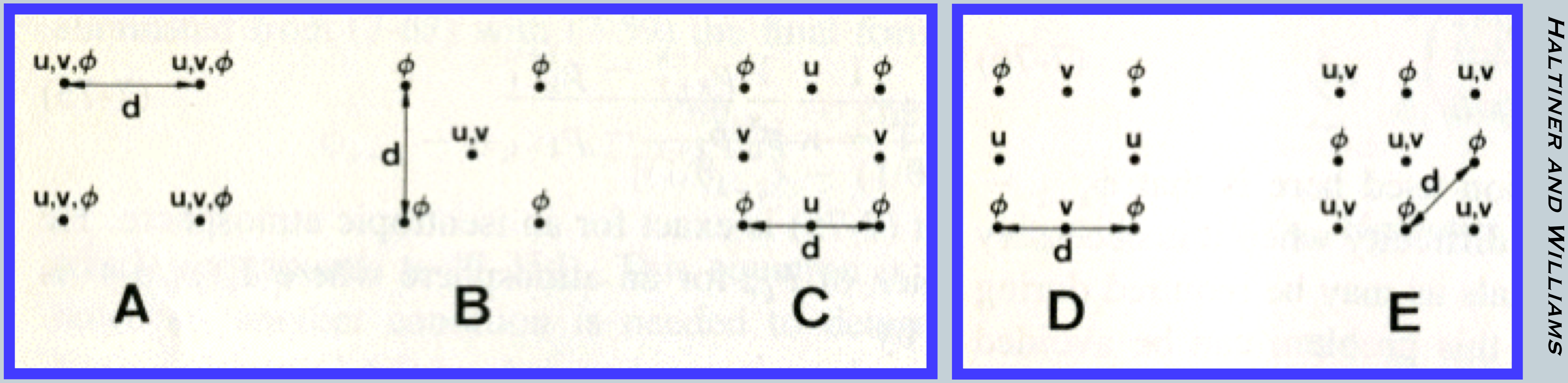
$$\frac{\partial \theta}{\partial t} = -\vec{V}\cdot\vec{\nabla}\theta + Q(x,y,z,t) + \nu\nabla^2\theta$$

$$\frac{\partial p}{\partial t} = -c_s^2\left[\overline{\rho}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial(\overline{\rho}z)}{\partial z}\right] + \nu\nabla^2 p$$

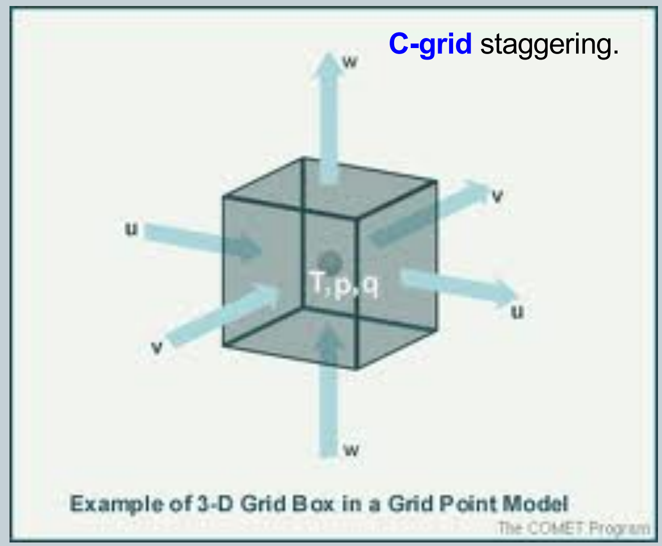
Variables:

- u: X-wind component
- v: Y-wind component
- w: Z-wind component
- θ: potential temperature
- p: pressure

Staggered grids

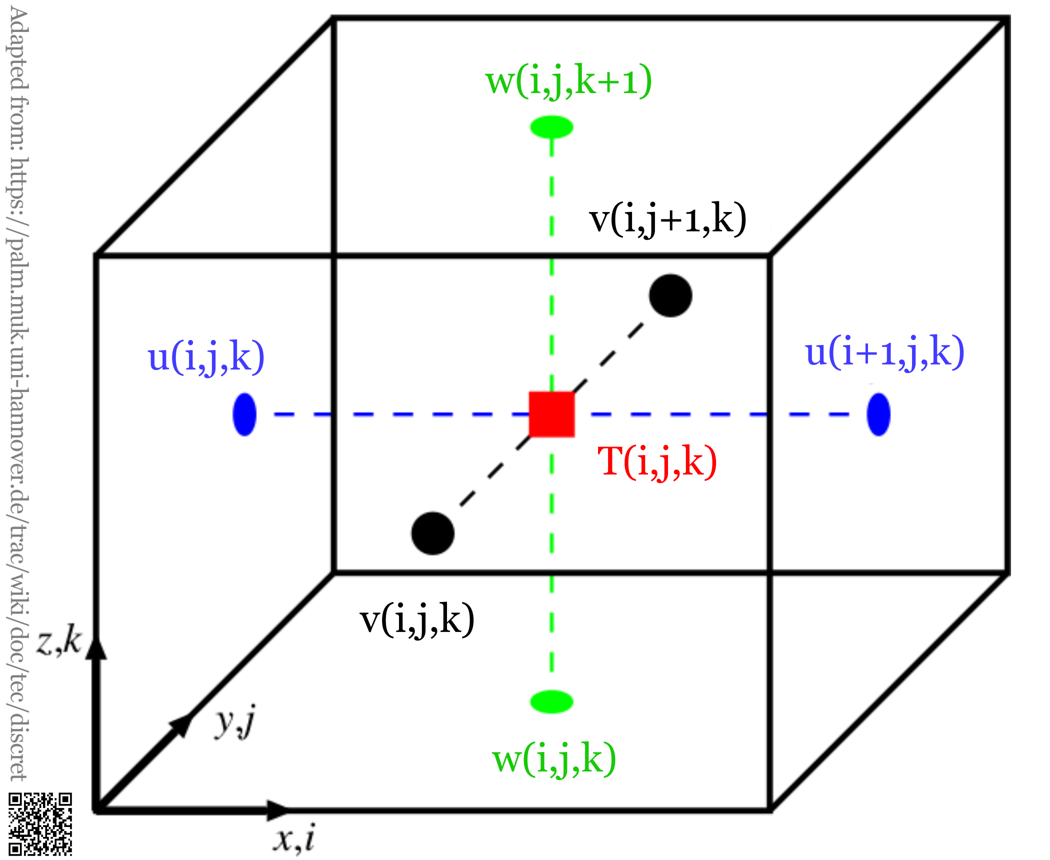


- A-grid: No staggering
- C-grid: velocities on normal faces of mass points.
  - ✓ Different u,v locations; truncation errors - Coriolis terms
  - Better at higher resolution (than B-grid)

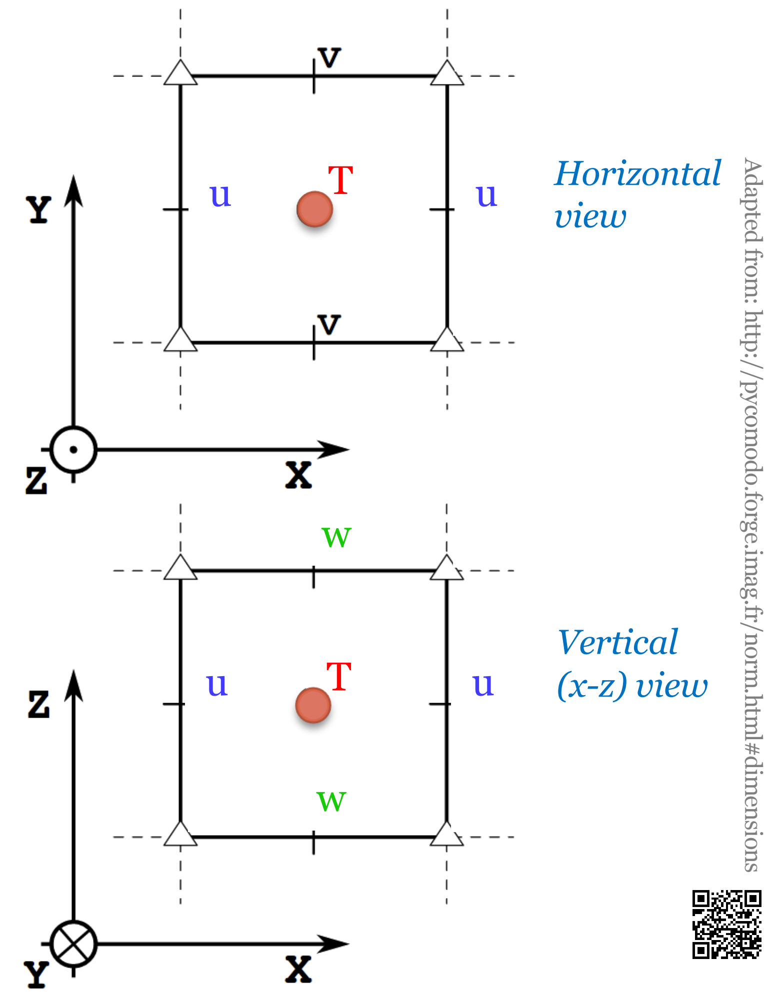


Staggered grids: C-grid for all variables

When coding a solver for multiple variables, you must consider the grid indexing centered on each variable.



"scalar quantities are defined at the center of each grid volume, whereas velocity components are shifted by half a grid width in their respective direction so that they are defined at the edges of the grid volumes"



Numerical methods: Staggered grids

KEY ADVANTAGE: *IMPROVED PHASE BEHAVIOR*

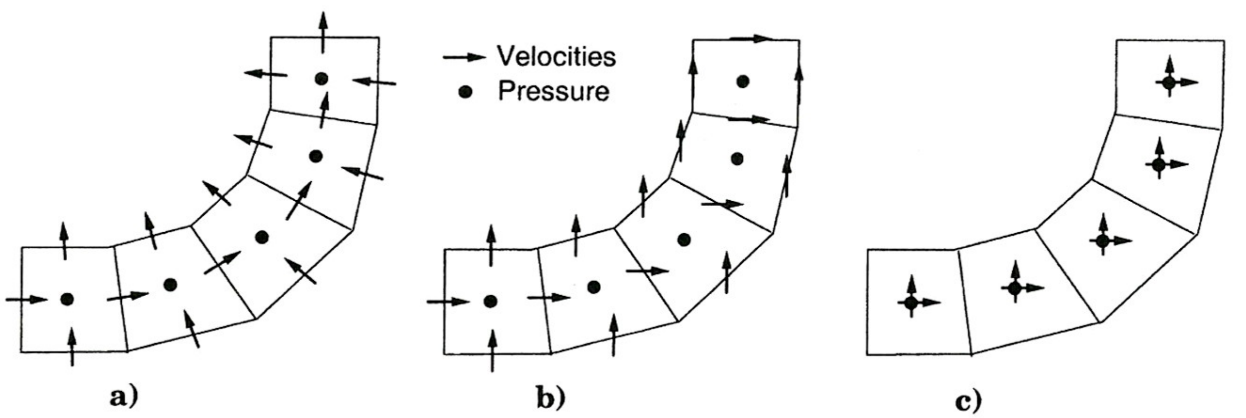


Fig. 8.5. Variable arrangements on a non-orthogonal grid: (a) – staggered arrangement with contravariant velocity components, (b) – staggered arrangement with Cartesian velocity components, (c) – colocated arrangement with Cartesian velocity components

Example: A *non-orthogonal* grid

Ferziger and Peric (2002), Chapter 8, *Complex Geometries*

Staggered grids: shallow-water eqs

$$\frac{\partial u}{\partial t} + \mathrm{U}\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \mathrm{U}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

$$\delta_{2t}u + g\delta_x h = 0$$

$$\delta_{2t}h_{j+\frac{1}{2}} + H\delta_x u_{j+\frac{1}{2}} = 0$$

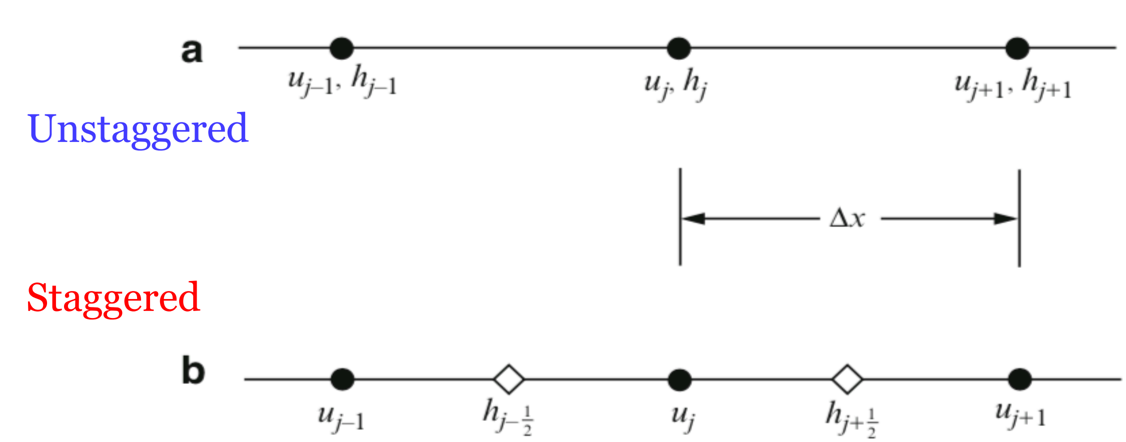


Fig. 4.1 Distribution of u and h on **a** an unstaggered and **b** a staggered mesh

Durran's book

Discrete-dispersion relation:

$$sin\omega\Delta t = \pm\frac{2c\Delta t}{\Delta x}sin\frac{k\Delta x}{2}$$

Stability criterion: $\left|\frac{c\Delta t}{\Delta x}\right| \leq \frac{1}{2}$, more strict than unstaggered grid scheme.

SUPERIOR PROPERTY OF STAGGERED GRID

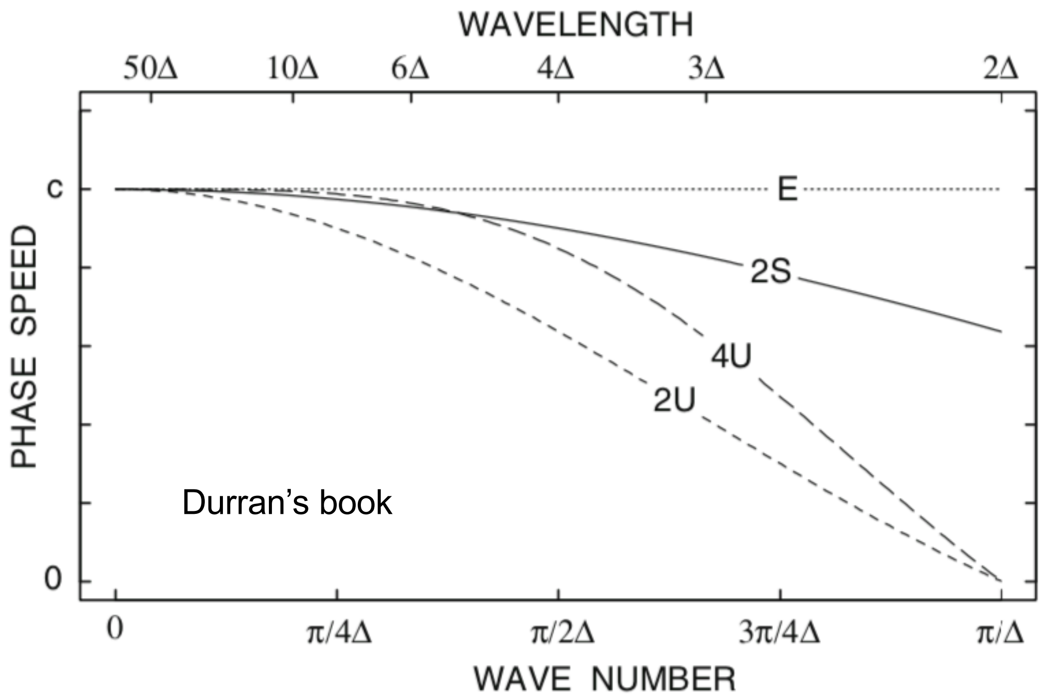


Fig. 4.2 Phase speed in the limit of good temporal resolution as a function of spatial resolution for the exact solution (*E*), for second-order approximations on a staggered mesh c_s (*2S*) and an unstaggered mesh c_u (*2U*), and for centered fourth-order spatial derivatives on an unstaggered mesh (*4U*)

Phase speeds:

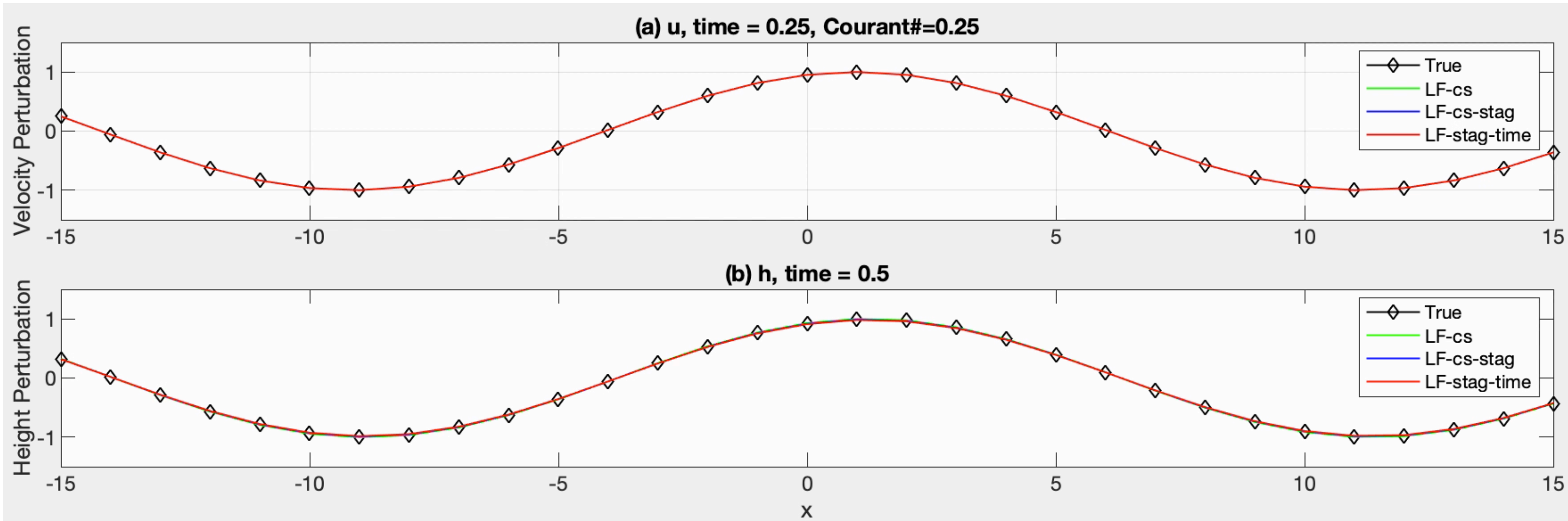
$$c_u = \frac{c}{k\Delta x} sin k\Delta x$$

$$c_s = \frac{2c}{k\Delta x} sin\left(\frac{k\Delta x}{2}\right)$$

- Staggered grid: smaller differences in phases & less dispersive.

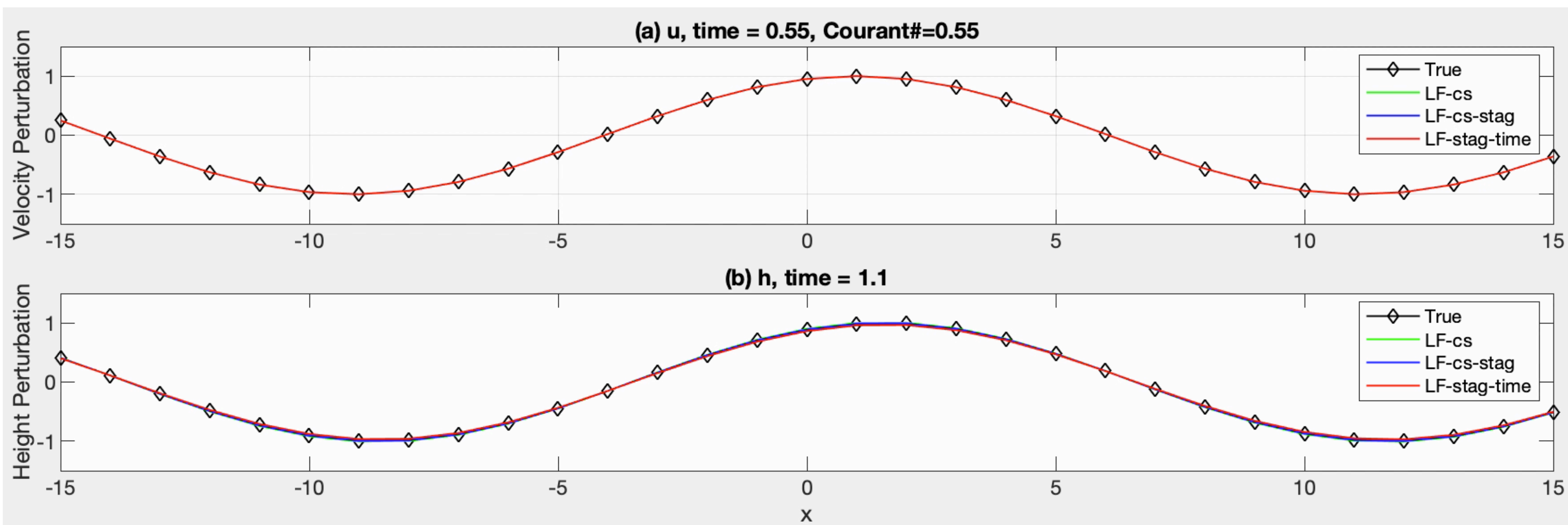
Programming & Comparison of Staggered and Unstaggered Schemes

U=0



$\lambda = 20\Delta x$

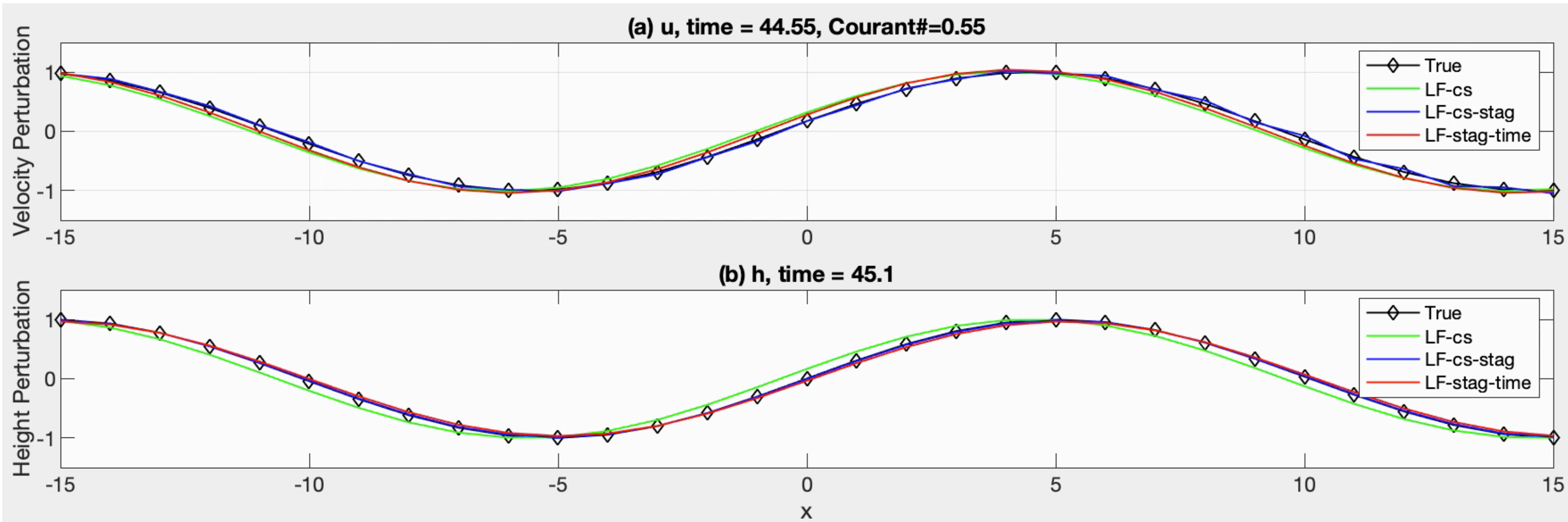
U=0



Programming & Comparison of Staggered and Unstaggered Schemes

$\lambda = 20\Delta x$

U=0



$\lambda = 20\Delta x$

U=1