Chapter 1

PHYS8750, Homework 1

1.1

1.1.1

For backward difference, the finite difference scheme is:

$$\frac{d\psi}{dt} \simeq \frac{\psi_n - \psi_{n-1}}{\Delta t}$$

by Taylor expansion:

$$\psi_{n-1} = \psi_n - \Delta t \psi'(t_n) + \frac{1}{2}\psi''(t_n)(\Delta t)^2 + \mathcal{O}(\Delta t)^3$$

thus:

$$\frac{\psi_n - \psi_{n-1}}{\Delta t} - \psi'(t_n) = -\frac{1}{2}\psi''(t_n)\Delta t + \mathcal{O}(\Delta t)^2$$

So the backward scheme has the first order of accuracy.

1.1.2

For trapezoidal difference, the finite difference scheme is

$$\frac{\psi_{n+1} - \psi_n}{\Delta t} = \frac{\psi'(t_n) + \psi'(t_{n+1})}{2}$$

by Taylor expansion:

$$\psi'(t_{n+1}) = \psi'(t_n) + \psi''(t_n)\Delta t + \frac{1}{2}\psi'''(t_n)\Delta t^2 + \mathcal{O}(\Delta t)^3$$

$$\psi_{n+1} = \psi_n + \Delta t \psi'(t_n) + \frac{1}{2}\psi''(t_n)(\Delta t)^2 + \frac{1}{6}\psi'''(t_n)(\Delta t)^3 + \mathcal{O}(\Delta t)^4$$

thus:

$$\frac{\psi_{n+1} - \psi_n}{\Delta t} - \frac{\psi'(t_n) + \psi'(t_{n+1})}{2}$$

$$= \psi'(t_n) + \frac{1}{2}\psi''(t_n)(\Delta t) + \frac{1}{6}\psi'''(t_n)(\Delta t)^2 + \mathcal{O}(\Delta t)^3 - \left[\psi'(t_n) + \frac{1}{2}\psi''(t_n)\Delta t + \frac{1}{4}\psi'''(t_n)(\Delta t)^2 + \mathcal{O}(\Delta t)^3\right]$$

$$= -\frac{1}{12}\psi'''(t_n)(\Delta t)^2 + \mathcal{O}(\Delta t)^3$$

So the trapezoidal scheme has the second order of accuracy.

1.1.3

For oscillation-diffusion problem ($\lambda < 0$), the difference scheme is

$$\frac{\psi_n - \psi_{n-1}}{\Delta t} = (\lambda + \mathrm{i}\,\omega)\psi_n$$

Solve for ψ_{n-1}

$$\psi_{n-1} = \psi_n[1 - (\lambda + \mathrm{i}\,\omega)\Delta t]$$

Absolute stability criteria

$$\left|\frac{\psi_n}{\psi_{n-1}}\right| = \frac{1}{(1-\lambda\Delta t)^2 + (\omega\Delta t)^2} \leq 1$$

Then

$$(\lambda\Delta t - 1)^2 + (\omega\Delta t)^2 \geq 1$$

which is the outside region of a circle centered at 1 with radius 1.

For oscillation-amplification problem ($\lambda > 0$),

$$\left|\frac{\psi_{n-1}}{\psi_n}\right| = (1-\lambda\Delta t)^2 + (\omega\Delta t)^2 \leq 1$$

which is the inside of the circle.

For trapezoidal difference scheme, solve for ψ_{n+1}

$$\psi_{n+1} = \psi_n\frac{2 + (\lambda + \mathrm{i}\,\omega)\Delta t}{2 - (\lambda + \mathrm{i}\,\omega)\Delta t}$$

For oscillation-diffusion problem, absolute stability criterion

$$\left|\frac{\psi_{n+1}}{\psi_n}\right| = \frac{(2+\lambda\Delta t)^2 + (\omega\Delta t)^2}{(2-\lambda\Delta t)^2 + (\omega\Delta t)^2} \leq 1$$

Then

$$\lambda\Delta t \leq 0$$

which is the left half of the plane. Since λ is negative, the scheme is stable for all positive Δt.

For oscillation-amplification problem, absolute stability criterion

$$\left|\frac{\psi_n}{\psi_{n+1}}\right| = \frac{(2-\lambda\Delta t)^2 + (\omega\Delta t)^2}{(2+\lambda\Delta t)^2 + (\omega\Delta t)^2} \leq 1$$

Then

$$\lambda\Delta t \geq 0$$

which is the right half of the plane. Since λ is positive, the scheme is stable for all positive Δt. Overall, trapezoidal difference scheme would be stable for all positive Δt.

1.2

For backward time scheme (Figure 1.1), when λ is negative, the schemes using all Δt would be stable, i.e., if true solution decreases with time, the numerical solution would not increase with time.

For λ is positive, blue and red curves correspond to unstable situation since they fall into the unstable region, because

$$(\lambda\Delta t - 1)^2 + (\omega\Delta t)^2 = [1.2973, 1.0189, 0.9968, 0.9996]$$

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For stable schemes, smaller Δt results in smaller amplitude and phase errors.

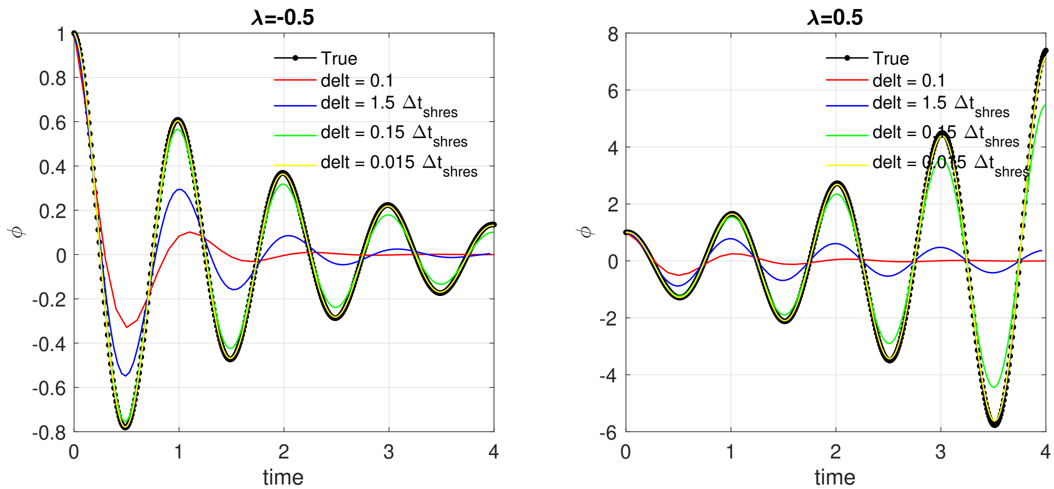

Figure 1.1: Different settings of backward difference scheme

1.3

For trapezoidal scheme (Figure 1.2), all Δt correspond to stable situation, and errors are smaller than those from backward schemes. Phase delay becomes more apparent as Δt increases.

1.4

In all three schemes, trapezoidal scheme which has second order of accuracy shows the best performance of stability, also less strict criterion for the time step Δt. Larger time step Δt usually lead to bigger errors.

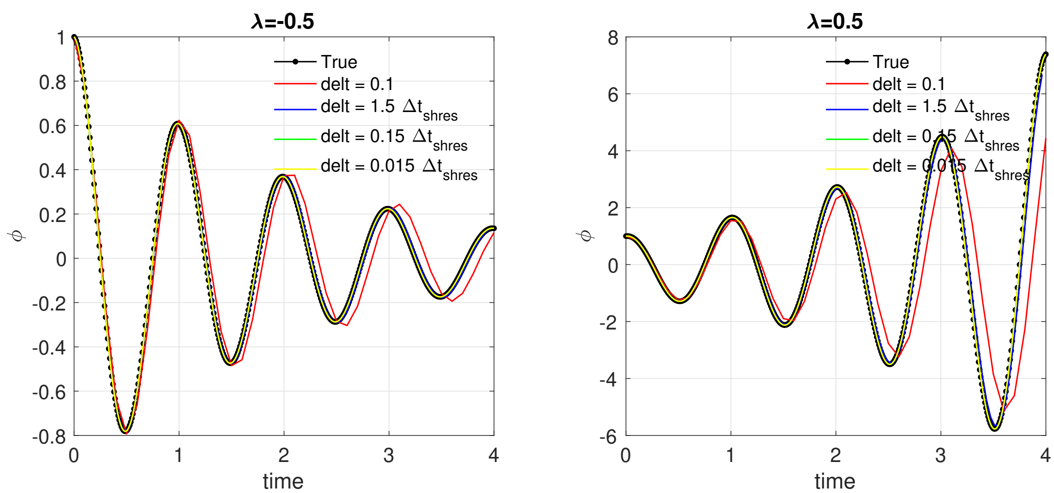

Figure 1.2: Different settings of trapezoidal difference scheme