PHYS 8750 | HW #1 | TEDI GODFREY

1. Consider backward-Euler differencing and trapezoidal differencing for the damping-oscillator problem:

$$\frac{d\psi}{dt} = F(\psi, t) = \gamma\psi = (\lambda + i\omega)\gamma$$

(A) Estimate the order of accuracy for both schemes.

$\boxed{\text{Backwards}}$: (Implicit method)

$$\frac{d\psi}{dt} = F(\psi, t) \Rightarrow \frac{d\psi_{n+1}}{dt} = F(\psi_{n+1}, t_{n+1})$$

→ Taylor series expansion of ψ around $t_n + \Delta t$:

$$\psi(t_n) \approx \psi(t_n + \Delta t) - \Delta t \cdot \psi'(t_n + \Delta t) + \frac{(\Delta t)^2}{2}\psi''(t_n + \Delta t) + \dots$$

$$\Rightarrow \psi_n \approx \psi_{n+1} - \Delta t\,\psi'_{n+1} + (\Delta t)^2 C$$

$$\Rightarrow \psi_n \approx \psi_{n+1} - \Delta t \cdot F(\psi_{n+1}, t_{n+1}) + (\Delta t)^2 C$$

$$\Rightarrow \psi_{n+1} \approx \psi_n + \Delta t \cdot F(\psi_{n+1}, t_{n+1})$$

$\boxed{\text{First order accurate}}$ (lowest order of Δt)

$\boxed{\text{Trapezoidal}}$: (Implicit Method)

truncation error $\uparrow$

$$\psi(t_{n+1}) \approx \psi_n + \frac{1}{2}\Delta t\left[F(t_n, \psi_n) + F(t_{n+1}, \psi_{n+1})\right] + TE$$

$$\Rightarrow TE = \psi_{n+1} - \psi_n - \frac{1}{2}\Delta t\left[F(t_n, \psi_n) + F(t_{n+1}, \psi_{n+1})\right]$$

$$\Rightarrow TE = \psi_{n+1} - \psi_n - \frac{1}{2}\Delta t\left[\psi_n' + \psi_{n+1}'\right]$$

Now, replace terms involving t_{n+1} with Taylor series expansion around t_n.

$$\Rightarrow TE = \psi_n + \Delta t\,\psi_n' + \frac{\Delta t^2}{2}\psi_n'' + O(\Delta t^3)$$
$$- \psi_n - \frac{1}{2}\Delta t\left[\psi_n' + (\psi_n' + \Delta t\,\psi_n'' + O(\Delta t^2))\right]$$

So, the TE is $O(\Delta t^3)$ and the Trapezoidal Rule is a $\boxed{\text{2nd Order}}$ Method.

(B) Derive the A-Stability criteria for the two schemes. Compare results with figures.

A-Stability Criteria: $\quad |A| = \left|\dfrac{\Psi_{n+1}}{\Psi_n}\right| \leq 1$

Backwards: $\quad \Psi_{n+1} = \Psi_n + \Delta t \cdot F(\Psi_{n+1}, t_{n+1})$ ①

$$\frac{d\Psi}{dt} = F(\Psi_n, t_n) = \gamma \Psi_n = (\lambda + i\omega)\Psi_n$$

$$\Rightarrow \frac{d\Psi_{n+1}}{dt} = F(\Psi_{n+1}, t_{n+1}) = \gamma \Psi_{n+1} = (\lambda + i\omega)\Psi_{n+1}$$

Plug back in to ①

$$\Rightarrow \Psi_{n+1} = \Psi_n + \Delta t \cdot (\gamma \Psi_{n+1})$$

$$\Rightarrow \Psi_{n+1} - \Delta t(\gamma \Psi_{n+1}) = \Psi_n$$

$$\Rightarrow \Psi_{n+1}(1 - \Delta t \gamma) = \Psi_n$$

Applying A-Stability Criteria:

$$\left|\frac{\Psi_{n+1}}{\Psi_n}\right| = \left|\frac{1}{1 - \Delta t \gamma}\right| \leq 1$$

$$\Rightarrow 1 \leq |1 - \Delta t \gamma|$$

$$\Rightarrow 1 \leq |1 - (\lambda + i\omega)\Delta t|$$

$$\Rightarrow 1 \leq |(1 - \lambda\Delta t) - i\omega\Delta t|$$

$$\Rightarrow \boxed{1 \leq (1 - \lambda\Delta t)^2 + (\omega\Delta t)^2}$$

Comparing this result with the given figure, it makes sense.

→ The result is centered at $(1,0)$

→ with a radius of 1

→ And the inequality determines that the shaded region (A-stability region) is outside of the circle.

$$\psi(t_{n+1}) \approx \psi_n + \frac{1}{2}\Delta t \left[F(t_n, \psi_n) + F(t_{n+1}, \psi_{n+1}) \right] \quad (1)$$

where

$$\frac{d\psi}{dt} = F(t_n, \psi_n) = \gamma \psi_n = (\lambda + i\omega)\psi_n$$

Plug back into (1)

and

$$\frac{d\psi_{n+1}}{dt} = F(t_{n+1}, \psi_{n+1}) = \gamma \psi_{n+1} = (\lambda + i\omega)\psi_{n+1}$$

$$\Rightarrow \psi(t_{n+1}) \approx \psi_n + \frac{1}{2}\Delta t \left[\gamma \psi_n + \gamma \psi_{n+1} \right]$$

$$\Rightarrow \psi_{n+1} \approx \psi_n + \frac{1}{2}\Delta t \gamma (\psi_n + \psi_{n+1})$$

$$\Rightarrow \psi_{n+1} \approx \psi_n + \frac{1}{2}\Delta t (\lambda + i\omega)(\psi_n + \psi_{n+1})$$

$$\Rightarrow \psi_{n+1} \approx \psi_n + \frac{1}{2}\Delta t \left[\psi_n \lambda + \psi_n i\omega + \psi_{n+1}\lambda + \psi_{n+1} i\omega \right]$$

$$\Rightarrow \psi_{n+1} \approx \psi_n + \frac{1}{2}\Delta t \psi_n \lambda + \frac{1}{2}\Delta t \psi_n i\omega + \frac{1}{2}\Delta t \psi_{n+1}\lambda + \frac{1}{2}\Delta t \psi_{n+1} i\omega$$

$$\Rightarrow \psi_{n+1} - \frac{1}{2}\Delta t \lambda \psi_{n+1} - \frac{1}{2}\Delta t \psi_{n+1} i\omega \approx \psi_n + \frac{1}{2}\Delta t \psi_n \lambda + \frac{1}{2}\Delta t \psi_n i\omega$$

$$\Rightarrow \psi_{n+1}\left(1 - \frac{1}{2}\Delta t \lambda - \frac{1}{2}\Delta t i\omega\right) \approx \psi_n \left(1 + \frac{1}{2}\Delta t \lambda + \frac{1}{2}\Delta t i\omega\right)$$

$$\Rightarrow \frac{\psi_{n+1}}{\psi_n} \approx \frac{\left(1 + \frac{1}{2}\Delta t \lambda + \frac{1}{2}\Delta t i\omega\right)}{\left(1 - \frac{1}{2}\Delta t \lambda - \frac{1}{2}\Delta t i\omega\right)}$$

$$\Rightarrow \left| \frac{\psi_{n+1}}{\psi_n} \right| = \left| \frac{\left(1 + \frac{1}{2}\Delta t \lambda + \frac{1}{2}\Delta t i\omega\right)}{\left(1 - \frac{1}{2}\Delta t \lambda - \frac{1}{2}\Delta t i\omega\right)} \right| \leq 1$$

$$\Rightarrow \boxed{\left| \frac{\sqrt{(1+\frac{1}{2}\Delta t\lambda)^2+(\frac{1}{2}tw)^2}}{\sqrt{(1-\frac{1}{2}\Delta t\lambda)^2+(\frac{1}{2}tw)^2}} \right| \leq 1}$$

This is only true if $\boxed{\Delta t\lambda \leq 0}$,

since the denominator will be greater than the numerator.

Comparing this result with the given figure, it makes sense.

→ The inequality is satisfied for any $\Delta t\lambda \leq 0$.

$\boxed{2}$ Backward.

$\Delta t = 0.011$ is the smallest time step, so it is the most accurate, and converges.

For $\lambda = 0.5$:

$\Delta t > 0.11$ gives an unstable solution

$\boxed{3}$ Trapezoidal.

$\Delta t = 0.011$ is the smallest time step, so it is the most accurate, and converges.

For $\lambda > 0$:

All Δt's give an unstable solution.

$\boxed{4}$ Compare Forward, Backward, Trapezoidal.

As expected, the trapezoidal scheme is the most accurate, but both values of λ that I used show an unstable solution. Which makes since, as for any $\lambda > 0$, the trapezoidal method should be unstable.

For backward scheme, $\lambda = -0.5$, all Δt values where found to be stable. This matches with the given figures.

For $\lambda = 0.5$, $\Delta t < 0.11$, the solution is unstable, which makes since, as a smaller Δt value forces $\lambda \Delta t$ to be inside the unstable circle.

For forward scheme, unstable for $\Delta t > 1.1$, when $\lambda = -0.5$, as it is forced out of the stable circle.

For $\lambda = 0.5$, all values of Δt are unstable.