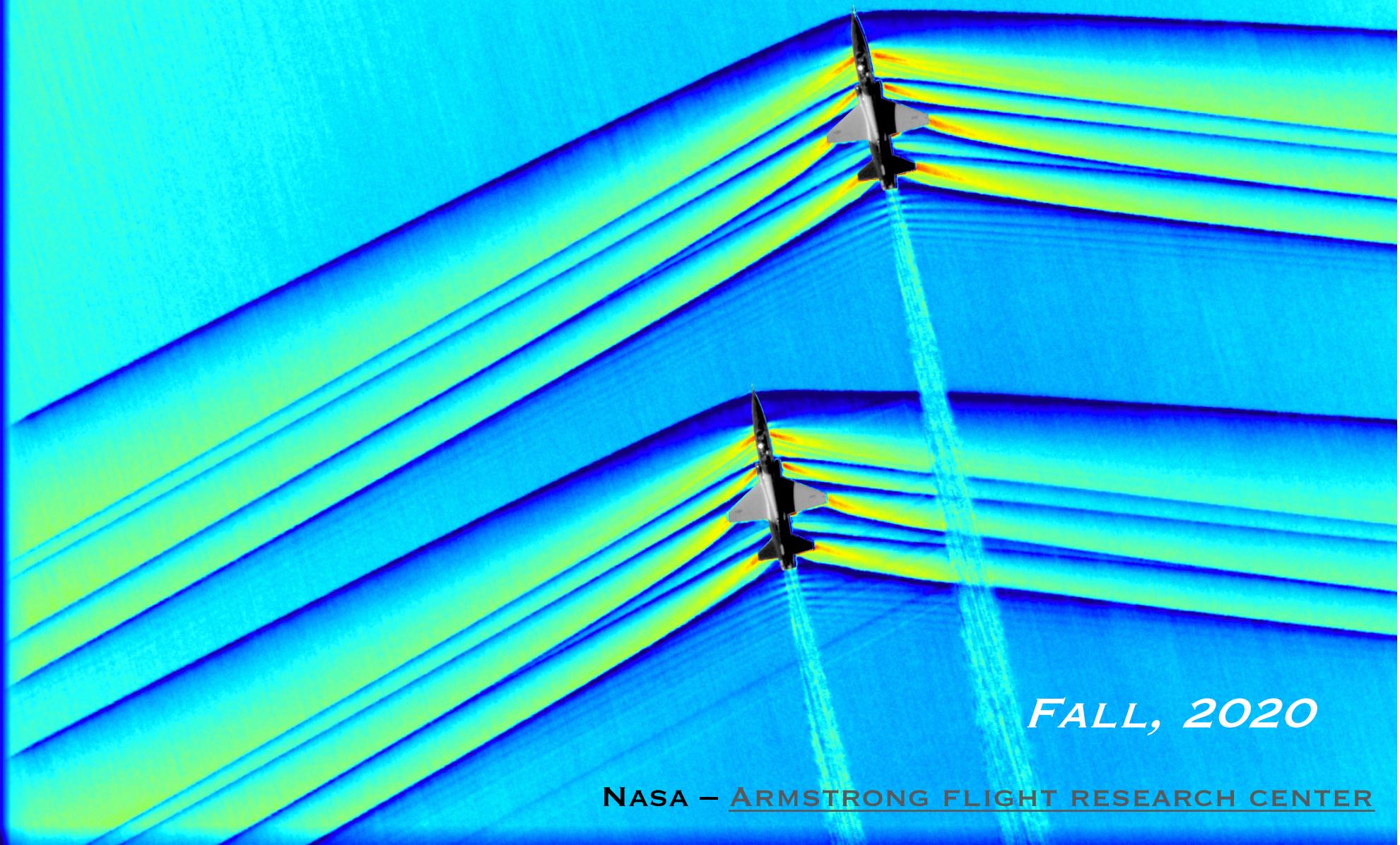


PHYS 8750
NUMERICAL FLUID DYNAMICS



FALL, 2020

NASA – ARMSTRONG FLIGHT RESEARCH CENTER

Outline

1. Stationary Planetary Wave Model
 - Governing equations
 - Numerical schemes
 - Lower boundary condition and mean winds
 - Modeling results
 - Role of gravity waves for MLT SPWs
2. Traveling Planetary Wave Model
 - Governing equations
 - Initial condition
 - Mean wind and instability
 - Modeling results

GOVERNING EQUATIONS

The vorticity and thermodynamic equations are:

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \zeta' + \frac{\partial \bar{Z}}{a \partial \theta} - 2\Omega \sin \theta \frac{\partial}{p \partial z} (pw') = 0 \quad (1)$$

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \frac{\partial \phi'}{\partial z} - 2\Omega \sin \theta (a \cos \theta) \frac{\partial \bar{\omega}}{\partial z} v' + N^2 w' = 0 \quad (2)$$

$z = -H_0 \ln \left(\frac{p}{p_0} \right)$: pressure height

a : radius of Earth

Ω : rotation rate of the earth

N^2 : Brunt – Väisälä frequency squared

$\bar{\omega}$: angular speed of the basic zonal flow

\bar{Z} : absolute vorticity of the basic state

u', v', w' : zonal, meridional, vertical wind perturbations

ζ' : perturbation vorticity

ϕ' : perturbation geopotential

NO
EXTERNAL
FORCING

Eliminating w' in the governing equations, the potential vorticity equation can be written as:

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \left[\zeta' + \frac{1}{p} \frac{\partial}{\partial z} \left(\frac{p}{N^2} \frac{\partial \phi'}{\partial z} \right) 2\Omega \sin \theta \right] + \frac{4\Omega^2 \sin^2 \theta}{N^2} a \cos \theta \frac{\partial \bar{\omega}}{\partial z} \left(\frac{\partial v_g}{\partial z} - \frac{\partial v'}{\partial z} \right) + \frac{\partial \bar{q}}{a \partial \theta} v' = 0 \quad (3)$$

Unknown variables: ζ' , ϕ' , v' , perturbations associated with waves
 Knowns ($\bar{-}$): associated with mean state (background winds)

Quasi-geostrophic (QG) model
 (often used for large-scale wave dynamics)

Assume geostrophic wind balance

Coriolis force = pressure gradient force

Eliminating w' in the governing equations, the potential vorticity equation can be written as:

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \left[\zeta' + \frac{1}{p} \frac{\partial}{\partial z} \left(\frac{p}{N^2} \frac{\partial \phi'}{\partial z} \right) 2\Omega \sin \theta \right] + \frac{4\Omega^2 \sin^2 \theta}{N^2} a \cos \theta \frac{\partial \bar{\omega}}{\partial z} \left(\frac{\partial v_g}{\partial z} - \frac{\partial v'}{\partial z} \right) + \frac{\partial \bar{q}}{a \partial \theta} v' = 0 \quad (3)$$

Unknown variables: ζ' , ϕ' , v' , perturbations associated with waves
 Knowns ($\bar{-}$): associated with mean state (background winds)

$$\zeta' = \frac{1}{2\Omega \sin \theta} \frac{1}{a^2} \left[\frac{\sin \theta}{\cos \theta} \frac{\partial}{\partial \theta} \left(\frac{\cos \theta}{\sin \theta} \frac{\partial \phi'}{\partial \theta} \right) + \frac{1}{\cos^2 \theta} \frac{\partial^2 \phi'}{\partial \lambda^2} \right]$$

$$v' = \frac{1}{2\Omega \sin \theta} \frac{1}{a \cos \theta} \frac{\partial \phi'}{\partial \lambda}$$

Applying the QG assumption, leading to a single equation (ϕ'):

$$\left(\cancel{\frac{\partial}{\partial t}} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \left[\frac{\sin^2 \theta}{\cos \theta} \frac{\partial}{\partial \theta} \left(\frac{\cos \theta}{\sin^2 \theta} \frac{\partial \phi'}{\partial \theta} \right) + \frac{1}{\cos^2 \theta} \frac{\partial^2 \phi'}{\partial \lambda^2} \right. \\ \left. + 4\Omega^2 a^2 \sin^2 \theta \frac{\partial}{p \partial z} \left(\frac{p}{N^2} \frac{\partial \phi'}{\partial z} \right) \right] + \frac{\partial \bar{q}}{\partial \theta} \frac{1}{\cos \theta} \frac{\partial \phi'}{\partial \lambda} = 0$$

im

im

$-m^2$

Simplifications:

$$1. \frac{\partial}{\partial t} = 0$$

$$2. \phi'(\lambda, \theta, z) = \sum_{m=1}^{\infty} \phi_m(\theta, z) e^{im\lambda}$$

$$3. \psi(\theta, z) = e^{-z/(2H_0)} \phi(\theta, z)$$

Numerical scheme is based on:

$$\frac{\sin^2 \theta}{\cos \theta} \frac{\partial}{\partial \theta} \left(\frac{\cos \theta}{\sin^2 \theta} \frac{\partial \psi_m}{\partial \theta} \right) + l^2 \sin^2 \theta \frac{\partial^2 \psi_m}{\partial z^2} + Q_m \psi_m = 0$$

$$Q_m = \left\{ \left[2(\Omega + \bar{\omega}) - \frac{\partial^2 \bar{\omega}}{\partial \theta^2} + 3 \tan \theta \frac{\partial \bar{\omega}}{\partial \theta} - l^2 \sin^2 \theta \left(\frac{\partial^2 \bar{\omega}}{\partial z^2} - \frac{1}{H_0} \frac{\partial \bar{\omega}}{\partial z} \right) \right] \right\} \bar{\omega} \\ - \sin^2 \theta \frac{l^2}{4H_0^2} - \frac{m^2}{\cos^2 \theta}$$

refractive index square, determined by background atmospheric status such as wind and temperature.

Modeling results are very sensitive to refractive index square.
Waves stop propagating when $Q_m < 0$

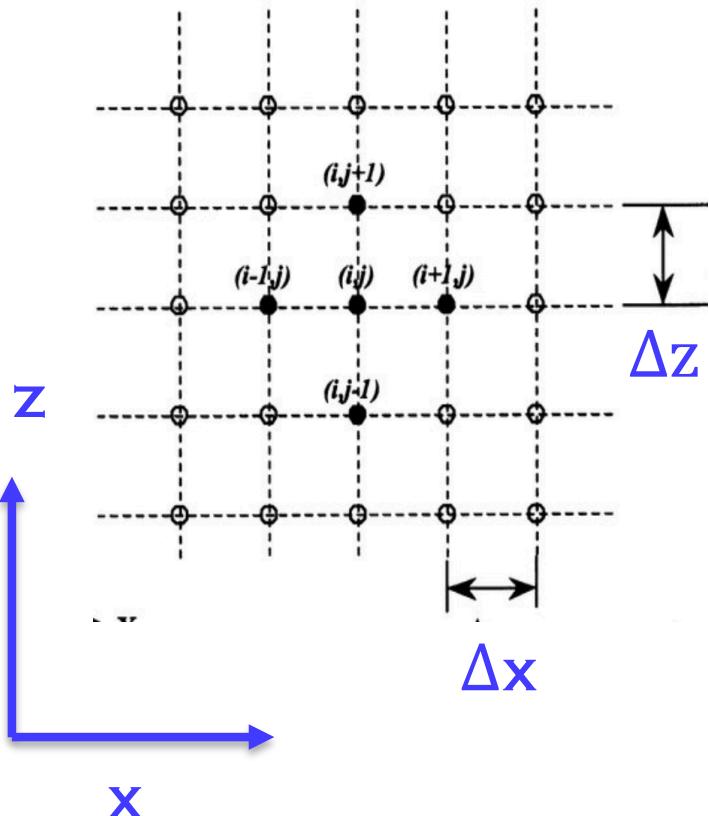
NUMERICAL SCHEME

Space Difference second order differential equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2}$$

$$\Delta z = 2 \text{ km}$$

$$\Delta \phi = 5^\circ$$



Matrix Method

$$\begin{bmatrix}
 A_{1,1}^1 & A_{2,1}^1 & 0 & \dots & 0 & A_{1,2}^1 & 0 & \dots & 0 \\
 A_{1,1}^2 & A_{2,1}^2 & A_{3,1}^2 & 0 & \dots & 0 & A_{2,2}^2 & 0 & \dots & 0 \\
 0 & A_{2,1}^3 & A_{3,1}^3 & A_{4,1}^3 & 0 & \dots & 0 & A_{3,2}^3 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 \vdots & 0 & A_{3,1}^4 & A_{4,1}^4 & A_{5,1}^4 & 0 & \dots & 0 & A_{4,2}^4 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & & & & 0 & & & & & & & & & & \vdots \\
 & & & & & & 0 & & & & & & & & & \vdots \\
 & & & & & & & 0 & & & & & & & & \vdots \\
 & & & & & & & & 0 & & & & & & & \vdots \\
 & & & & & & & & & 0 & & & & & & \vdots \\
 & & & & & & & & & & 0 & & & & & \vdots \\
 & & & & & & & & & & & 0 & & & & \vdots \\
 & & & & & & & & & & & & \psi_{1,1} & B_{1,1} \\
 & & & & & & & & & & & & \psi_{2,1} & B_{2,1} \\
 & & & & & & & & & & & & \psi_{3,1} & B_{3,1} \\
 & & & & & & & & & & & & \vdots & \vdots \\
 & & & & & & & & & & & & \psi_{n,1} & B_{n,1} \\
 & & & & & & & & & & & & & \psi_{1,2} & B_{1,2} \\
 & & & & & & & & & & & & & \psi_{2,2} & B_{2,2} \\
 & & & & & & & & & & & & & \psi_{3,2} & B_{3,2} \\
 & & & & & & & & & & & & & \vdots & \vdots \\
 & & & & & & & & & & & & & \psi_{n,2} & B_{n,2} \\
 & & & & & & & & & & & & & & \psi_{1,m} & B_{1,m} \\
 & & & & & & & & & & & & & & \psi_{2,m} & B_{2,m} \\
 & & & & & & & & & & & & & & \psi_{3,m} & B_{3,m} \\
 & & & & & & & & & & & & & & \vdots & \vdots \\
 & & & & & & & & & & & & & & \psi_{n,m} & B_{n,m}
 \end{bmatrix} =
 \begin{bmatrix}
 A : [n \times m] \times [n \times m] \\
 \psi : [n \times m] \times 1 \\
 B : [n \times m] \times 1
 \end{bmatrix}$$

NUMERICAL SCHEME

Matrix Method

$$\begin{bmatrix}
 A_{1,1}^1 & A_{2,1}^1 & 0 & \cdots & 0 & A_{1,2}^1 & 0 & \cdots & 0 \\
 A_{1,1}^2 & A_{2,1}^2 & A_{3,1}^2 & 0 & \cdots & 0 & A_{2,2}^2 & 0 & \cdots & 0 \\
 0 & A_{2,1}^3 & A_{3,1}^3 & A_{4,1}^3 & 0 & \cdots & 0 & A_{3,2}^3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 \vdots & 0 & A_{3,1}^4 & A_{4,1}^4 & A_{5,1}^4 & 0 & \cdots & 0 & A_{4,2}^4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 \vdots & & 0 & & & & & & & & & & & & & \vdots \\
 \vdots & & & 0 & & & & & & & & & & & & \vdots \\
 \vdots & & & & 0 & & & & & & & & & & & \vdots \\
 \vdots & & & & & 0 & & & & & & & & & & \vdots \\
 \vdots & & & & & & 0 & & & & & & & & & \vdots \\
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 \vdots & & & & & & & & & & & 0 & & & & \vdots \\
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 \vdots & & & & & & & & & & & & & 0 & & \vdots \\
 \vdots & & & & & & & & & & & & & & 0 & = \\
 & & & & & & & & & & & & & & & \left[\begin{array}{c|c}
 \psi_{1,1} & B_{1,1} \\
 \psi_{2,1} & B_{2,1} \\
 \psi_{3,1} & B_{3,1} \\
 \vdots & \vdots \\
 \psi_{n,1} & B_{n,1} \\
 \vdots & \vdots \\
 \psi_{1,2} & B_{1,2} \\
 \psi_{2,2} & B_{2,2} \\
 \psi_{3,2} & B_{3,2} \\
 \vdots & \vdots \\
 \psi_{n,2} & B_{n,2} \\
 \vdots & \vdots \\
 \psi_{1,m} & B_{1,m} \\
 \psi_{2,m} & B_{2,m} \\
 \psi_{3,m} & B_{3,m} \\
 \vdots & \vdots \\
 \psi_{n,m} & B_{n,m}
 \end{array} \right] & A : [n \times m] \times [n \times m] \\
 & & & & & & & & & & & & & & & \psi : [n \times m] \times 1 \\
 & & & & & & & & & & & & & & & B : [n \times m] \times 1 \\
 & & & & & & & & & & & & & & & B : zero matrix
 \end{bmatrix}$$

Use lower boundary condition to determine the internal field:

$$[\psi_{1,1}, \psi_{2,1}, \psi_{3,1}, \psi_{4,1}, \psi_{5,1}, \dots \dots] \Rightarrow [\psi_{i,j}]$$

SOLVE QUASI-TRIDIAGONAL MATRICES

NO EXTERNAL FORCING: FORCED BY LOWER BOUNDARY

Sparse matrix (a lot of zeros)

QUASI-TRIDIAGONAL MATRICES AND TYPE-INSENSITIVE DIFFERENCE EQUATIONS*

BY

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1. Introduction. In solving linear partial differential equations by finite difference methods a boundary value problem is reduced to solving a set of linear equations. In such instances the matrix involved usually takes a special form and consists mainly of zeros. Many of these matrices fall into the class to be considered here which may be called *quasi-tridiagonal* matrices. That is, we consider partitioned matrices of the form

$$Q = \begin{bmatrix} M_1 & E_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ D_2 & M_2 & E_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & D_3 & M_3 & E_3 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & 0 & D_{q-1} & M_{q-1} & E_{q-1} & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & D_q & M_q & \end{bmatrix} = [D_n, M_n, E_n]_1^q, \quad (1.1)$$

*g: zero
matrix*

where the D_n, M_n, E_n are matrices with the same number of rows, E_n, M_{n+1}, D_{n+2} have the same number of columns, and the M_n are square. We propose to solve

$$Qv = g \quad (1.2)$$

by direct methods.

FORWARD/BACKWARD ITERATION

$$Q = \begin{bmatrix} M_1 & E_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ D_2 & M_2 & E_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & D_3 & M_3 & E_3 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & D_{q-1} & M_{q-1} & E_{q-1} & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & D_q & M_q & \end{bmatrix} = [D_n, M_n, E_n]_1^q$$

$$Qv = g$$

Solving for v

LU Method: decompose the original matrix to one lower triangular matrix and one upper triangular matrix

$$Q = LU, \quad (2.1)$$

where L and U are square matrices, partitioned in the same manner as Q , of the form

$$L = [C_n, I_n, 0]_1^q, \quad (2.2)$$

$$U = [0, A_n, E_n]_1^q, \quad (2.3)$$

Equation converts to:

$$LUv = g \implies \begin{cases} Ly = g \\ Uv = y \end{cases}$$

Coefficient Matrix Correlation

$$[D_n, M_n, E_n] \Leftrightarrow [C_n, A_n, E_n]$$



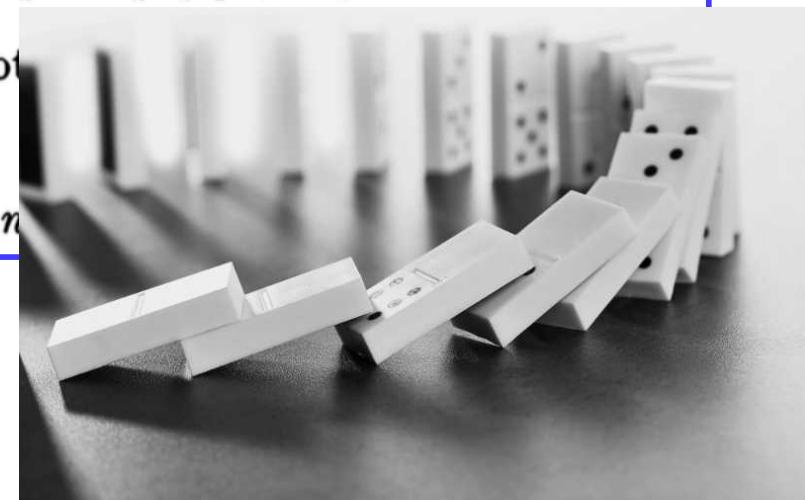
Comparing right and left hand sides of (2.1) we set, for $1 < n \leq q$,

$$A_1 = M_1, \quad D_n = C_n A_{n-1}, \quad M_n = C_n E_{n-1} + A_n.$$

If the A_n are non-singular, the A_n and C_n may be obtained by

$$A_n = M_n - D_n A_{n-1}^{-1} E_{n-1},$$

$$C_n = D_n A_{n-1}^{-1}, \quad 1 < n \leq q.$$



Solution $LUv = g \Rightarrow \begin{cases} Ly = g \\ Uv = y \end{cases}$

To solve for v let $Uv = y$, then $Ly = g$; y and v may then be obtained recursively from

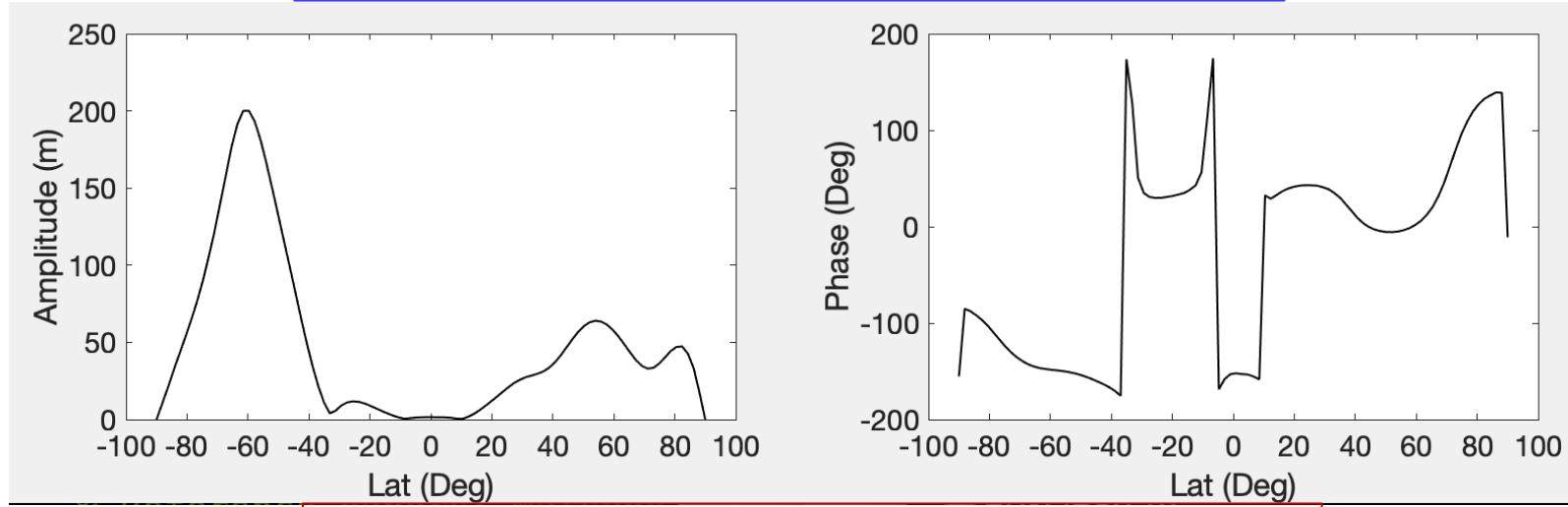
$$y_n = g_n - C_n y_{n-1} \quad 1 < n \leq q, \tag{2.6}$$

where

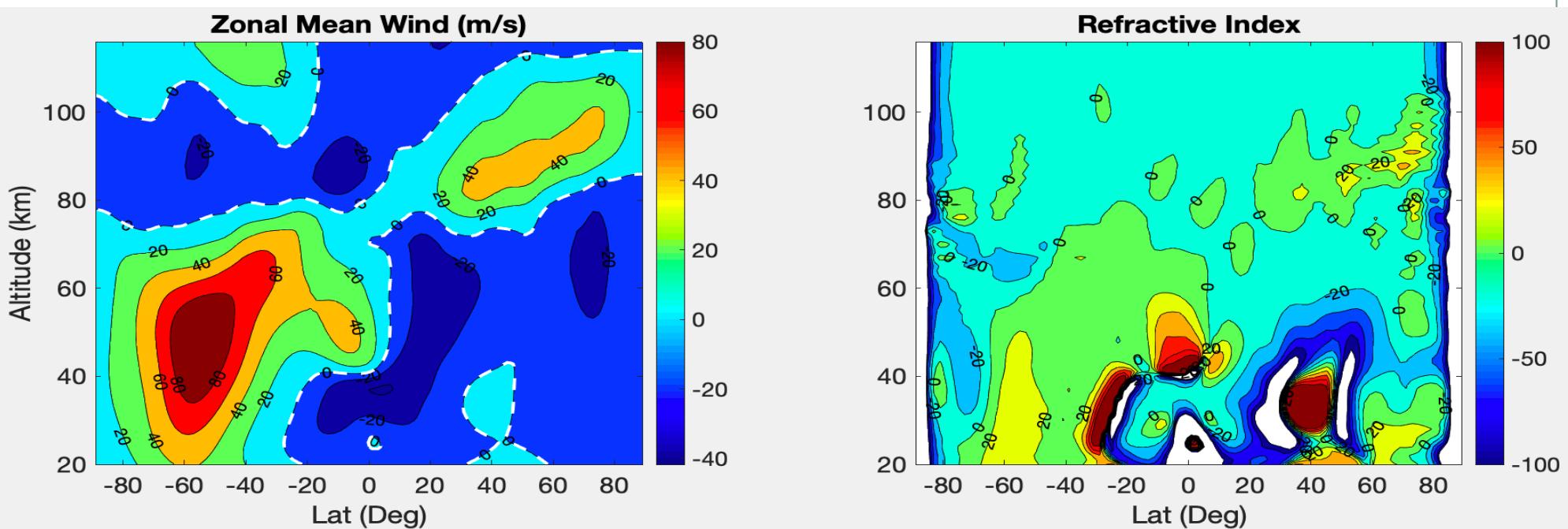
$$y_1 = g_1, \quad \text{and}$$

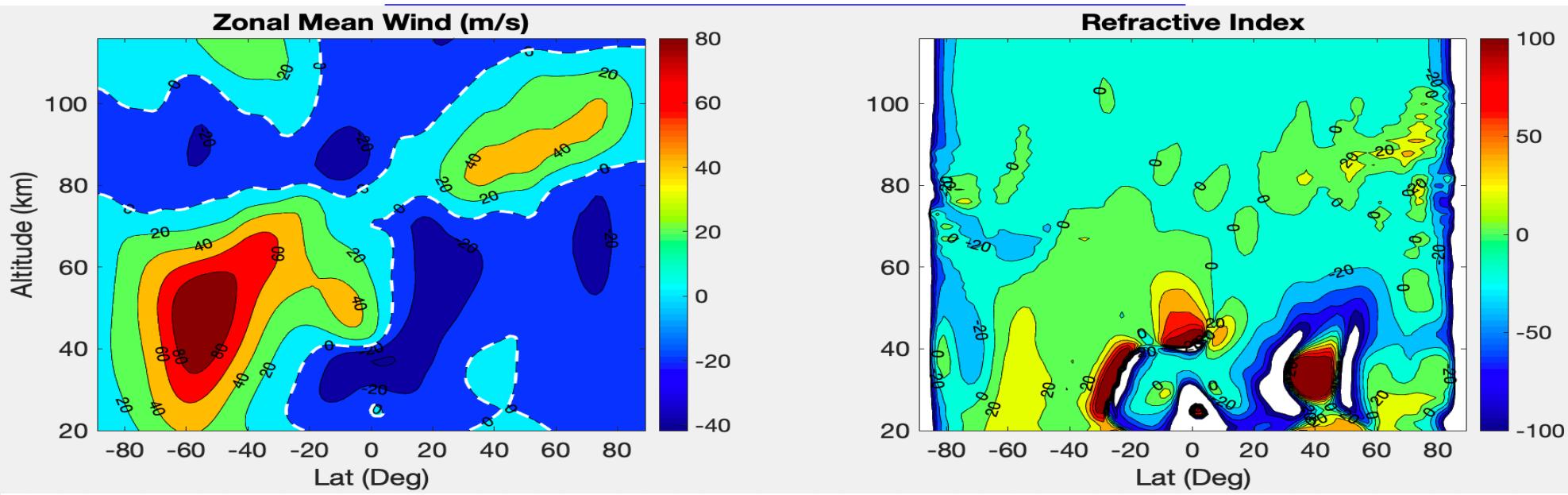
$$v_n = A_n^{-1}(y_n - E_n v_{n+1}) \quad 1 \leq n < q, \tag{2.7}$$

LOWER BOUNDARY CONDITION

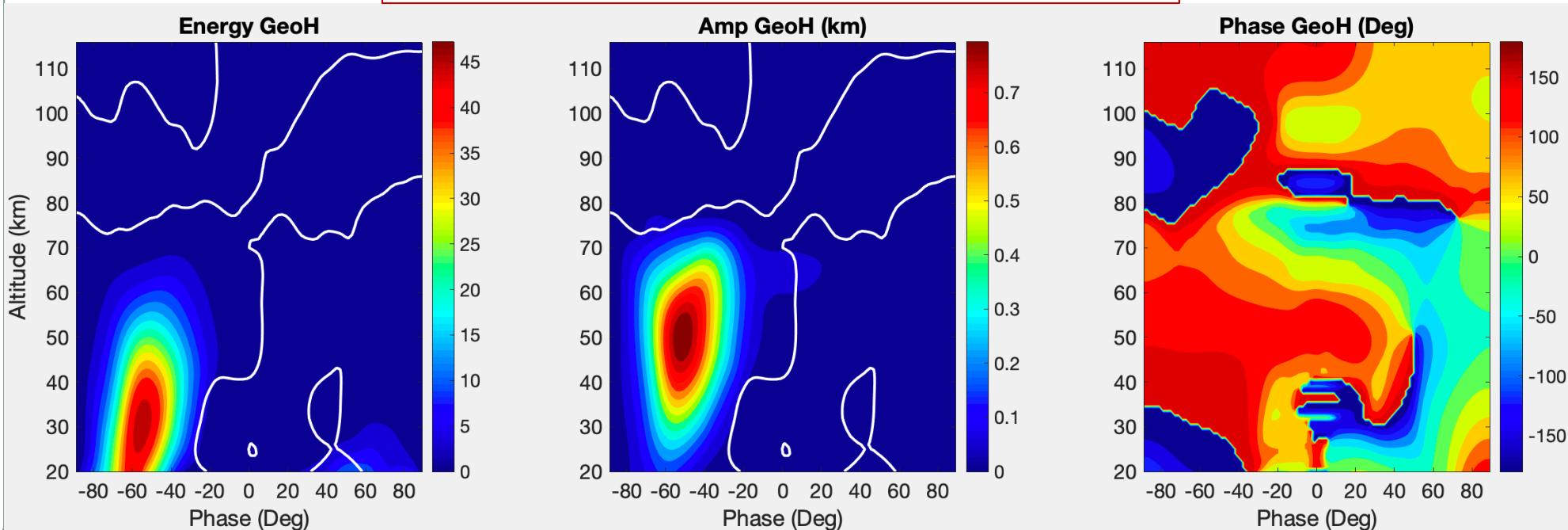


BACKGROUND WIND CONDITION

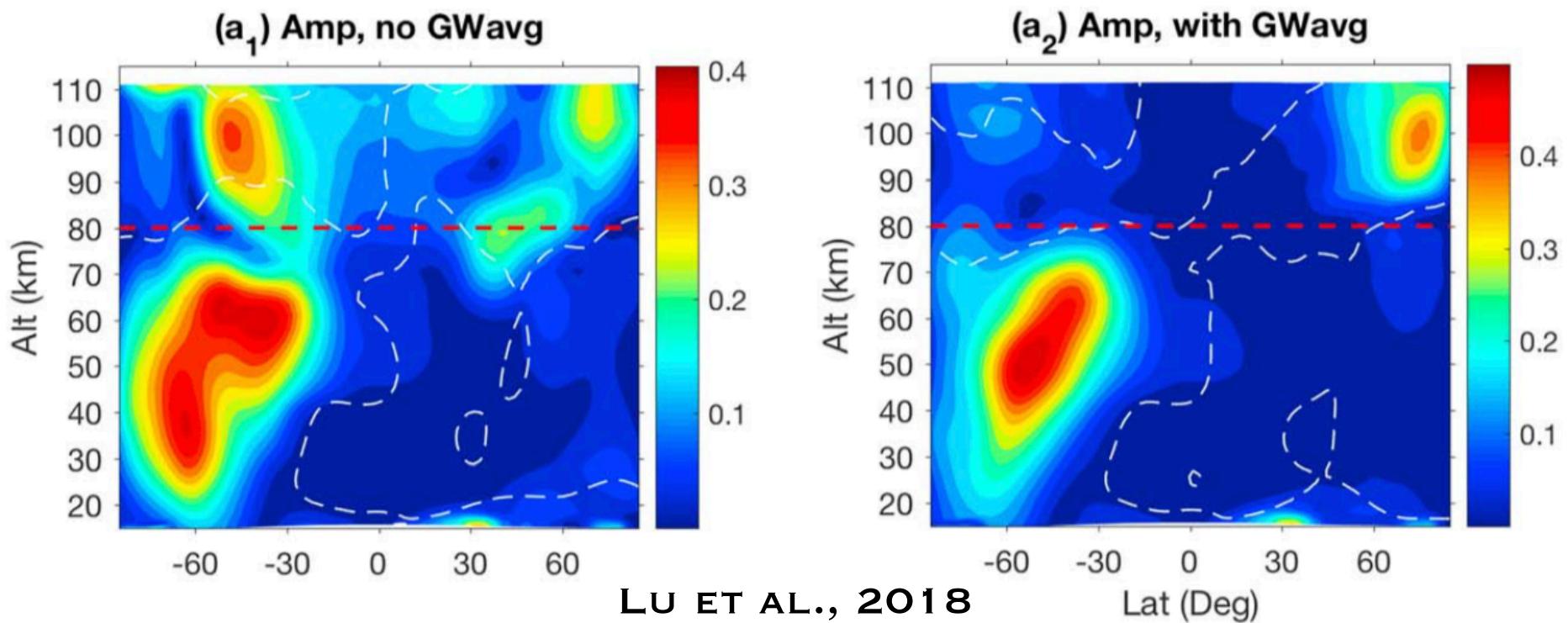




WAVE AMPLITUDE AND PHASE



- From governing equations without forcing and gravity wave effects, internal fields of SPWs are fully determined by lbc and mean background winds.
→ SPWs can't pass zero-wind lines and propagate into the upper atmosphere (being filtered): **different from observations.**



- Stratospheric SPWs modulate GWs, which propagate upward and bring SPW signatures to upper atmosphere where they break.

GOVERNING EQUATIONS FOR TPWS

We start from the vorticity equation of the QG model [Hartmann, 1979]:

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a \cos \phi} \right) \left[\frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left(\frac{\cos \phi}{\sin^2 \phi} \frac{\partial \Phi'}{\partial \phi} \right) + \frac{1}{\sin^2 \phi \cos^2 \phi} \frac{\partial^2 \Phi'}{\partial \lambda^2} + (2\Omega a)^2 \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\rho}{N^2} \frac{\partial \Phi'}{\partial z} \right) \right] + \frac{1}{\cos \phi \sin^2 \phi} \frac{\partial \bar{q}}{\partial \phi} \frac{\partial \Phi'}{\partial \lambda} = -(2\Omega a)^2 \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\alpha \rho}{N^2} \frac{\partial \Phi'}{\partial z} \right)$$

$\Phi' = \psi(\phi, z, t) \exp(z/2H + im\lambda)$ is geopotential having a wave form along longitude λ .

$$\frac{\partial \bar{q}}{\partial \phi} = 2\Omega \cos \phi - \frac{\partial}{\partial \phi} \left[\frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left(\frac{\bar{u} \cos \phi}{a} \right) \right] - (2\Omega \sin \phi)^2 \frac{a}{\rho} \frac{\partial}{\partial z} \left(\frac{\rho}{N^2} \frac{\partial \bar{u}}{\partial z} \right) \text{ is vorticity gradient.}$$

ϕ is latitude and α is newton cooling coefficient.

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + im\bar{\omega} \right) L(\psi) \\ & + \frac{im}{\cos \phi \sin^2 \phi} \frac{\partial \bar{q}}{\partial \phi} \psi = -n^2 K(\psi), \end{aligned}$$

$$\begin{aligned} L(\psi) &= \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left(\frac{\cos \phi}{\sin^2 \phi} \frac{\partial \psi}{\partial \phi} \right) - \frac{m^2}{\sin^2 \phi \cos^2 \phi} \psi \\ &+ \frac{n^2}{N^2} \left[\frac{\partial^2 \psi}{\partial z^2} + N^2 \frac{\partial N^{-2}}{\partial z} \left(\frac{\partial \psi}{\partial z} + \frac{\psi}{2H} - \frac{\psi}{4H^2} \right) \right], \end{aligned}$$

$$\begin{aligned} K(\psi) &= \frac{\alpha}{N^2} \frac{\partial^2 \psi}{\partial z^2} + \left(\frac{1}{N^2} \frac{\partial \alpha}{\partial z} + \alpha \frac{\partial N^{-2}}{\partial z} \right) \frac{\partial \psi}{\partial z} \\ &+ \frac{1}{2H} \left(\frac{1}{N^2} \frac{\partial \alpha}{\partial z} + \alpha \frac{\partial N^{-2}}{\partial z} - \frac{\alpha}{2HN^2} \right) \psi. \end{aligned}$$

Because it is a wave form along longitude, it becomes a 3-D problem as a function of latitude, altitude and time.

NUMERICAL SCHEME

Forward Time

$$\frac{\partial \psi}{\partial t} = \frac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{\Delta t}$$

$$\Delta t = 1.5h$$

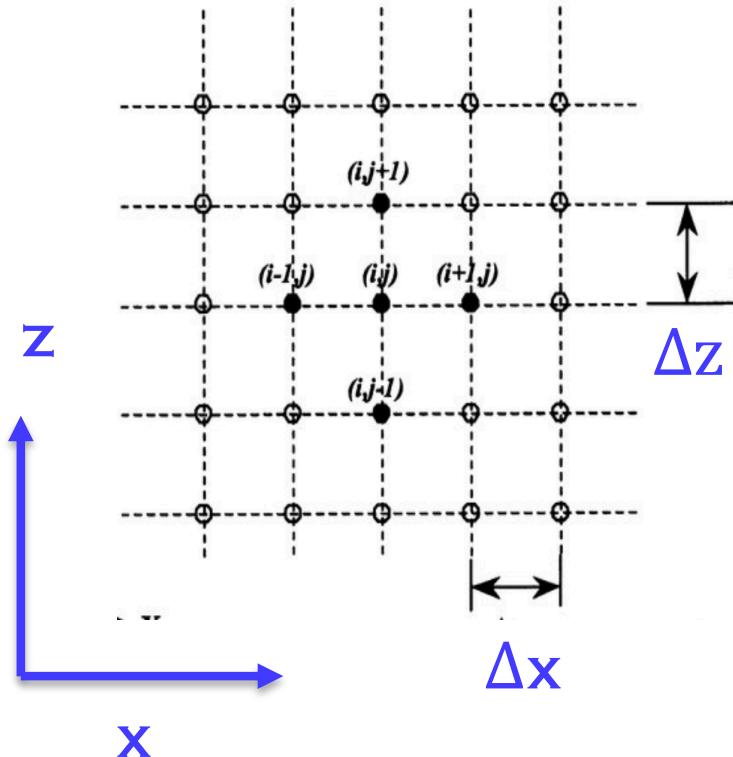
Space Difference second order differential equation

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta z^2}$$

$$\Delta z = 2km$$

$$\Delta\phi = 5^\circ$$

Matrix Method



$$A : [n \times m] \times [n \times m]$$

$$\psi : [n \times m] \times 1$$

$$A = A(\psi^n, \Delta t, \Delta z)$$

$$B : [n \times m] \times 1$$

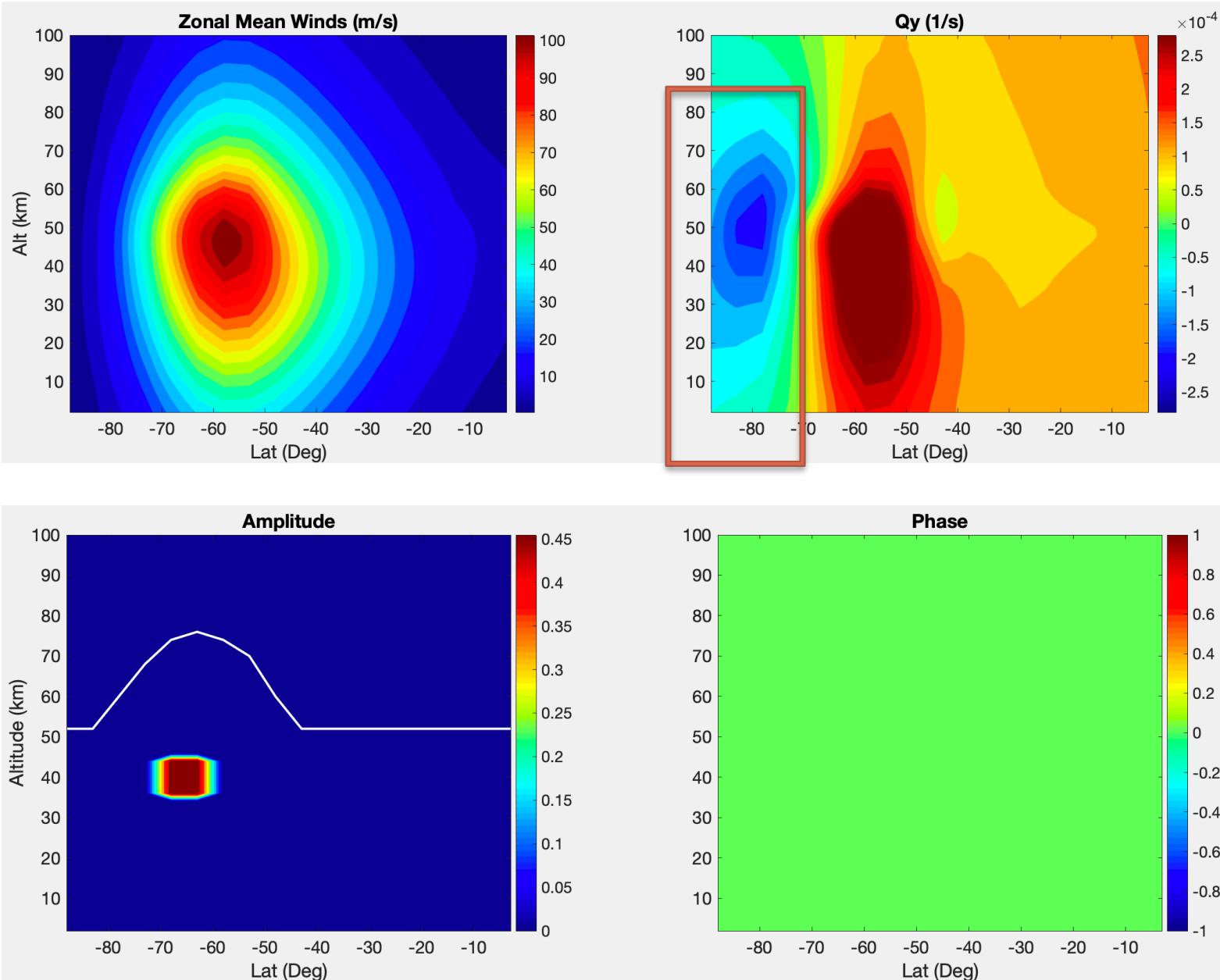
$$B = B(\psi^n, \Delta t, \Delta z)$$

INITIAL CONDITION PROBLEM

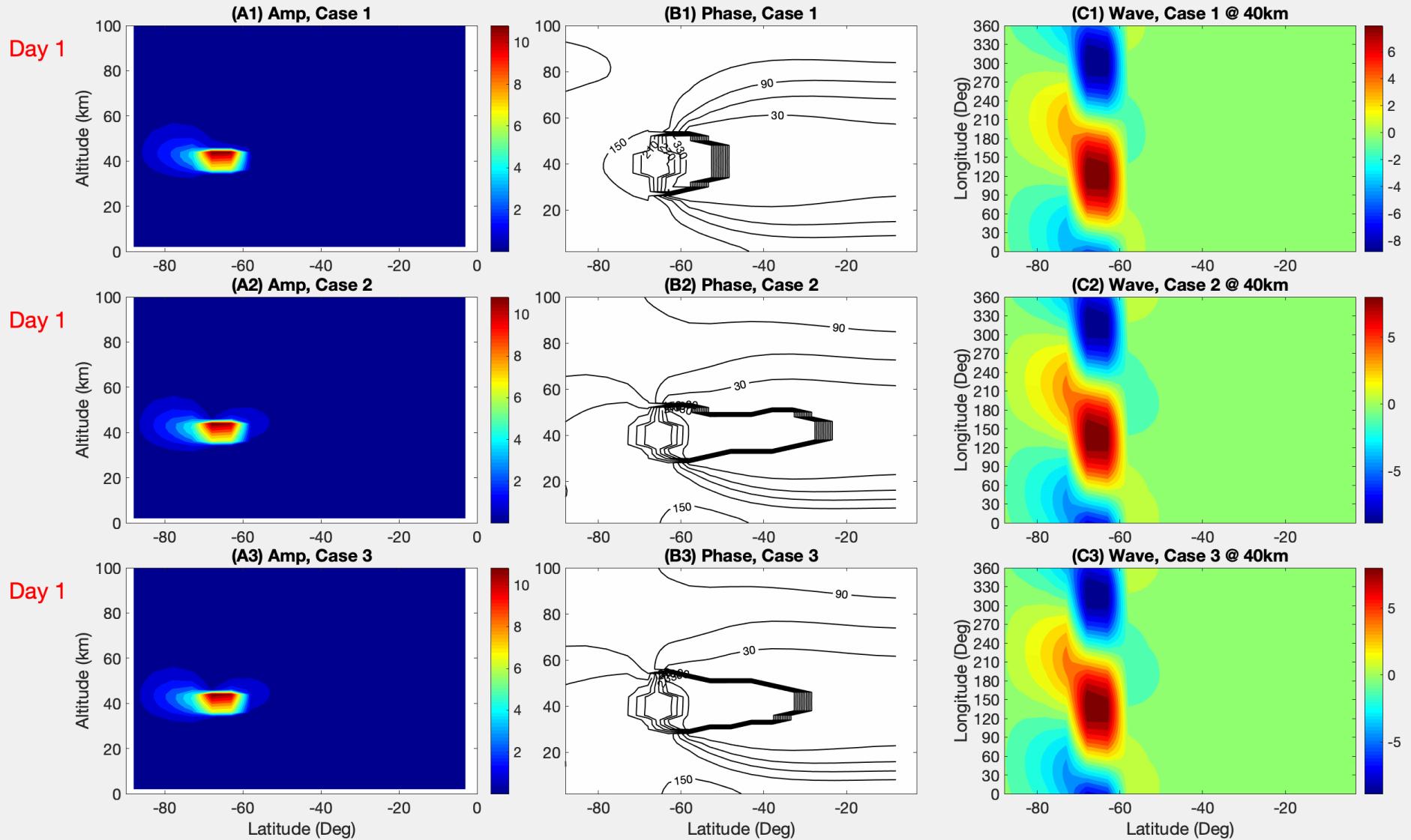
BACKGROUND
WIND
& INSTABILITY

INITIAL
CONDITION

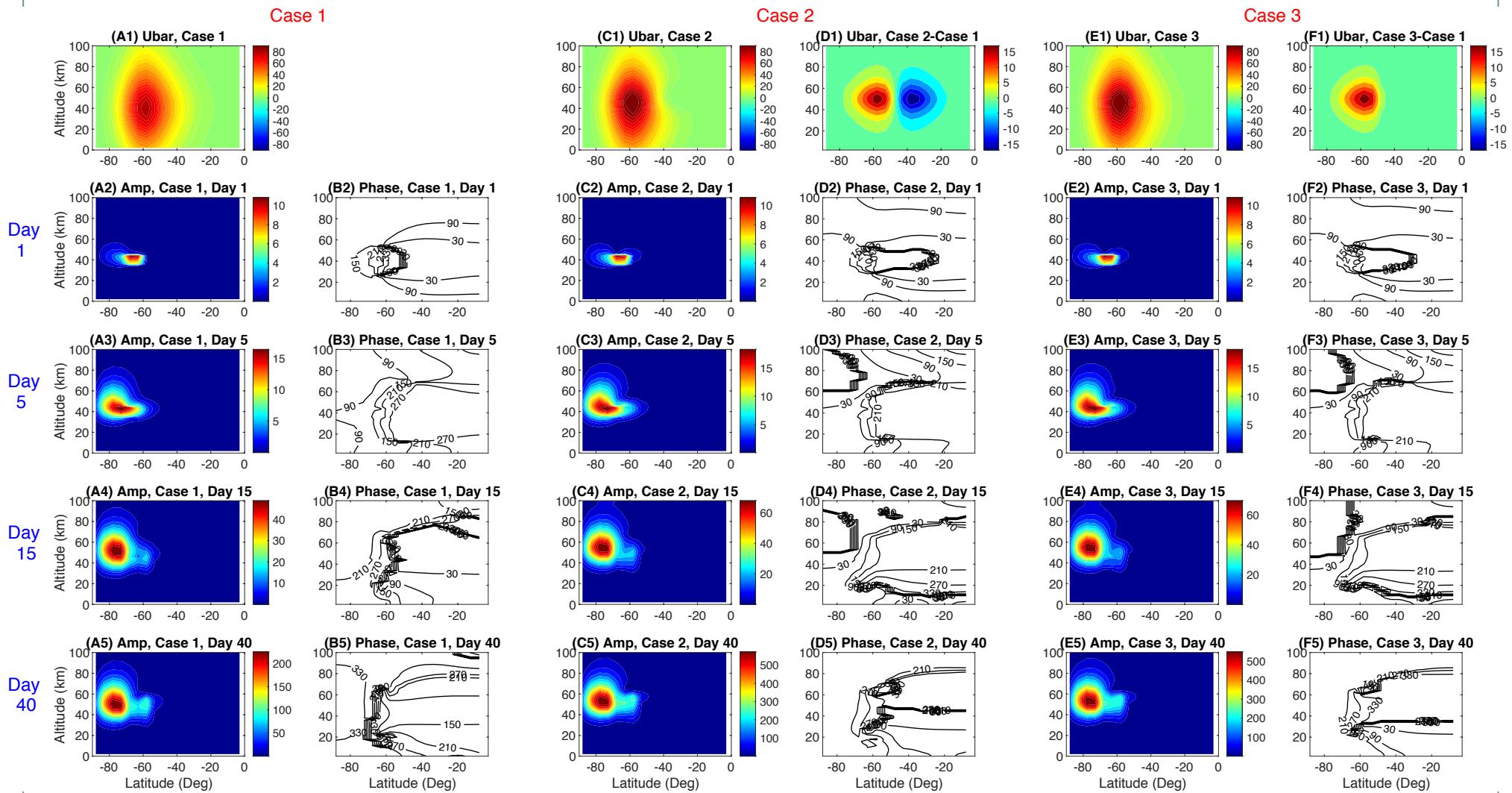
Random
disturbances



INITIAL CONDITION PROBLEM



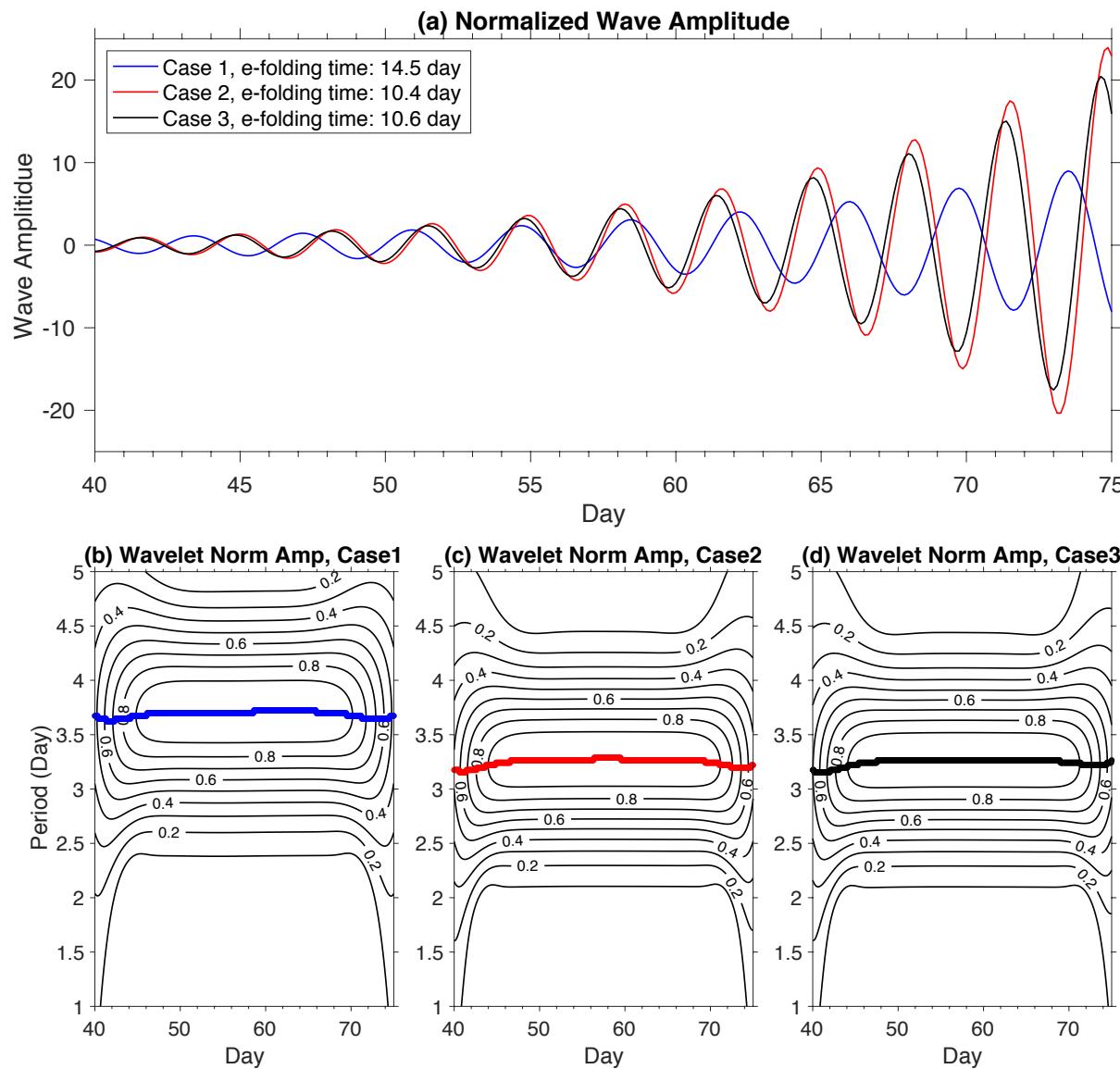
QBO IMPACTS ON TPWS



Double cell = QBOe-QBOW

LU ET AL., 2019

QBO IMPACTS ON TPWS



QBOE:

Stronger Polar Vortex



Great instability



Larger growth rate
Shorter wave period



Larger amplitudes of
planetary waves

LU ET AL., 2019