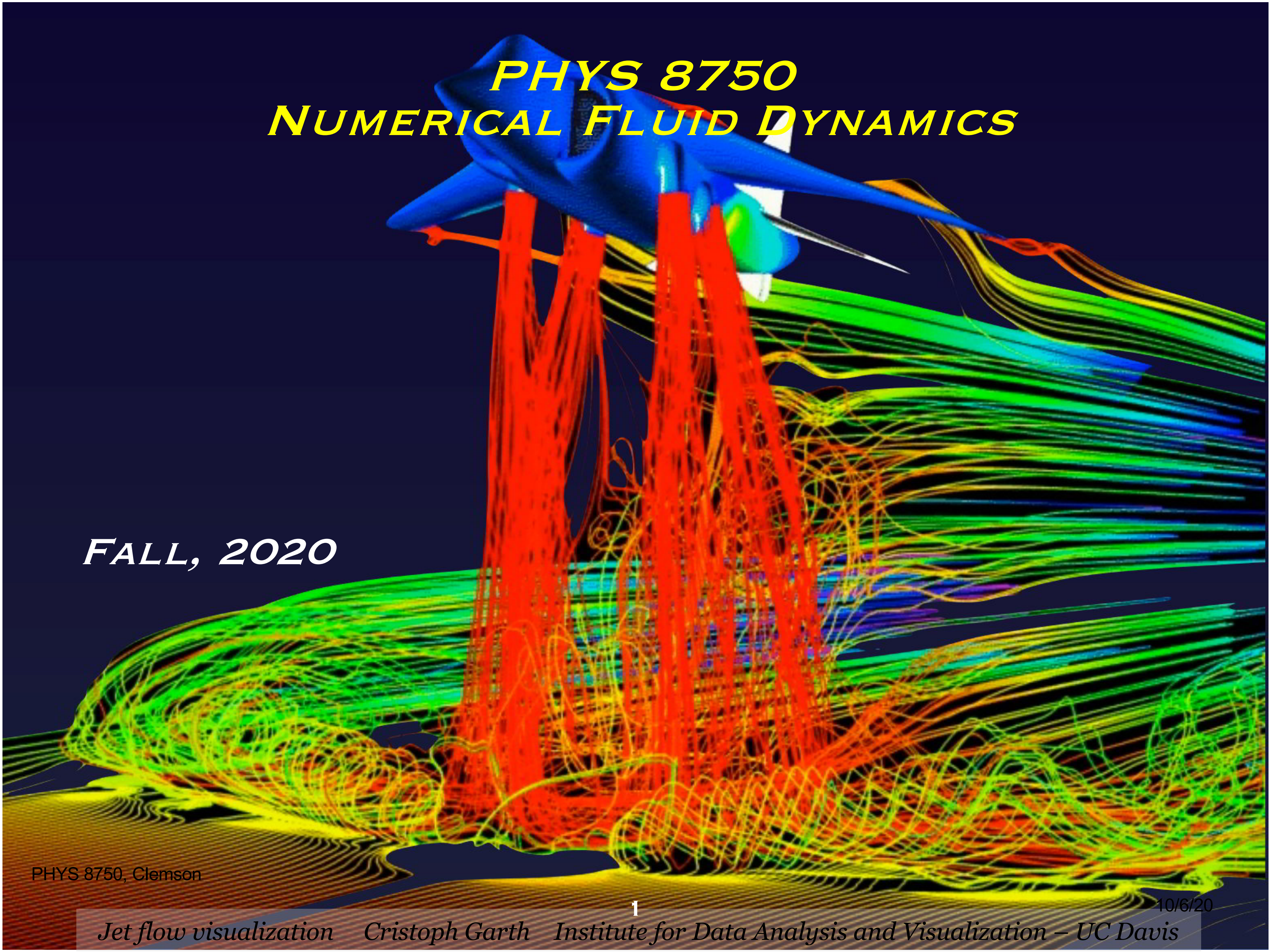
PHYS 8750
Numerical Fluid Dynamics

Fall, 2020



Jet flow visualization Cristoph Garth Institute for Data Analysis and Visualization – UC Davis

PHYS 8750

CLASS #12
(CHAPTER 4.1)
STAGGERED GRIDS
IN TIME AND SPACE

Class #13 (Chapter 4.2)

1) PDE with two variables

2) Lax-Wendroff + CTU

Outline

1. 2D-Problems (x and y)
   - Discrete-dispersion method
   - Stability condition
   - Difference with 1-D problems
2. Directionality in 2D advection
   - Using upstream scheme
3. Corner Transport Upstream (CTU)
4. Lax-Wendroff + CTU
5. Examples (Codes)

2D Advection Problem

$$1D\ Advection: \quad \frac{\partial\psi}{\partial t} + c\frac{\partial\psi}{\partial x} = 0$$

$$2D\ Advection: \quad \frac{\partial\psi}{\partial t} + U\frac{\partial\psi}{\partial x} + V\frac{\partial\psi}{\partial y} = 0$$

Discrete-dispersion Relation to evaluate stability of various schemes

Leapfrog/Center space

$$\begin{aligned} \delta_{2t}u + \mathrm{U}\delta_{2x}u + g\delta_{2x}h &= 0 \\ \delta_{2t}h + \mathrm{U}\delta_{2x}h + H\delta_{2x}u &= 0 \end{aligned} \xrightarrow{u_j^n = u_0 e^{i(kj\Delta x - \omega n\Delta t)},\, h_j^n = h_0 e^{i(kj\Delta x - \omega n\Delta t)}}$$

$$sin\omega\Delta t = \frac{\Delta t}{\Delta x}(U \pm c)sink\Delta x$$

$$\left|\frac{\Delta t}{\Delta x}(U \pm c)\right| \leq 1,$$

$$\text{if } \mathrm{U} = 0, \left|c\frac{\Delta t}{\Delta x}\right| \leq 1$$

Discrete-Dispersion Relation

$$\frac{\partial\psi}{\partial t} + U\frac{\partial\psi}{\partial x} + V\frac{\partial\psi}{\partial x} = 0$$

leapfrog $\longrightarrow$ Center space

$$\delta_{2t}\psi + \mathrm{U}\delta_{2x}\psi + V\delta_{2y}\psi = 0$$

$$\phi^{j}_{m,n} = e^{i(km\Delta x + ln\Delta y - \omega j\Delta t)}$$

discrete-dispersion relation:

$$sin\omega\Delta t = \mu\sin(k\Delta x) + \nu\sin(l\Delta y) \Rightarrow |\mu| + |\nu| < 1$$

$$\mu = U\frac{\Delta t}{\Delta x}, \nu = V\frac{\Delta t}{\Delta y} : courant\ numbers\ in\ x\ and\ y\ directions$$

$$U = c\times cos\theta, V = c\times sin\theta, and\ suppose\ \Delta x = \Delta y = \Delta s \Rightarrow$$

$$c(|cos\theta| + |sin\theta|)\frac{\Delta t}{\Delta s} < 1 \Rightarrow c\frac{\Delta t}{\Delta s} < 1/\sqrt{2}$$

Stability Criterion: From 1D to 2D

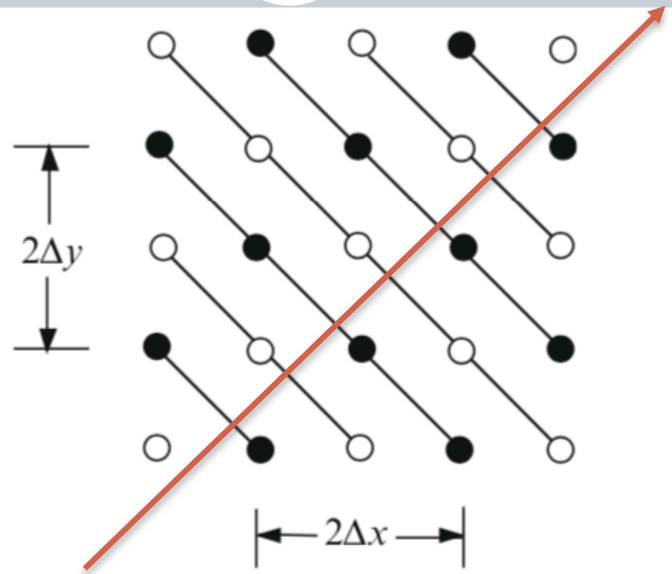


Fig. 4.3 Distribution of wave crest (*solid circles*) and wave troughs (*open circles*) in the shortest-wavelength disturbance resolvable on a square mesh in which $\Delta x = \Delta y$

If $\Delta x = \Delta y = \Delta s$, the shortest resolvable wavelength is the shortest wavelength resolved would be along the diagonal direction: $\frac{\Delta s}{\sqrt{2}}$, this is where the factor of $\sqrt{2}$ comes from.

Average Scheme

Way to loosen the constraint: take average of the spatial derivative

$$\boxed{\delta_{2t}\phi + \mathrm{U}\langle\delta_{2x}\phi\rangle^{2y} + V\langle\delta_{2y}\psi\rangle^{2x} = 0}$$

$$\langle f(x)\rangle^{nx} = \left[\frac{f(x+n\Delta x/2) + f(x-n\Delta x/2)}{2}\right]$$

Discrete-dispersion relation:

$$\begin{aligned} sin\omega\Delta t &= \mu sin(k\Delta x)\cos(l\Delta y) + \nu cos(k\Delta x)\sin(l\Delta y) \\ &\leq \max(|\mu|,|\nu|)\,(sin(k\Delta x)\cos(l\Delta y) + cos(k\Delta x)\sin(l\Delta y)) \end{aligned}$$

Stability criterion:

$$\max(|\mu|,|\nu|) \leq 1 \Leftrightarrow \left|U\frac{\Delta t}{\Delta x}\right| \leq 1, \left|V\frac{\Delta t}{\Delta y}\right| \leq 1$$

$U = c\times cos\theta, V = c\times sin\theta, and\ suppose\ \Delta x = \Delta y = \Delta s$

$$c\frac{\Delta t}{\Delta s} < \frac{1}{\sqrt{2}},\ \frac{U}{cos\theta}\frac{\Delta t}{\Delta s} < 1/\sqrt{2},\ U\ \frac{\Delta t}{\Delta s} = U\ \frac{\Delta t}{\Delta x} < cos\theta/\sqrt{2}$$

Phase Speed Behaviors

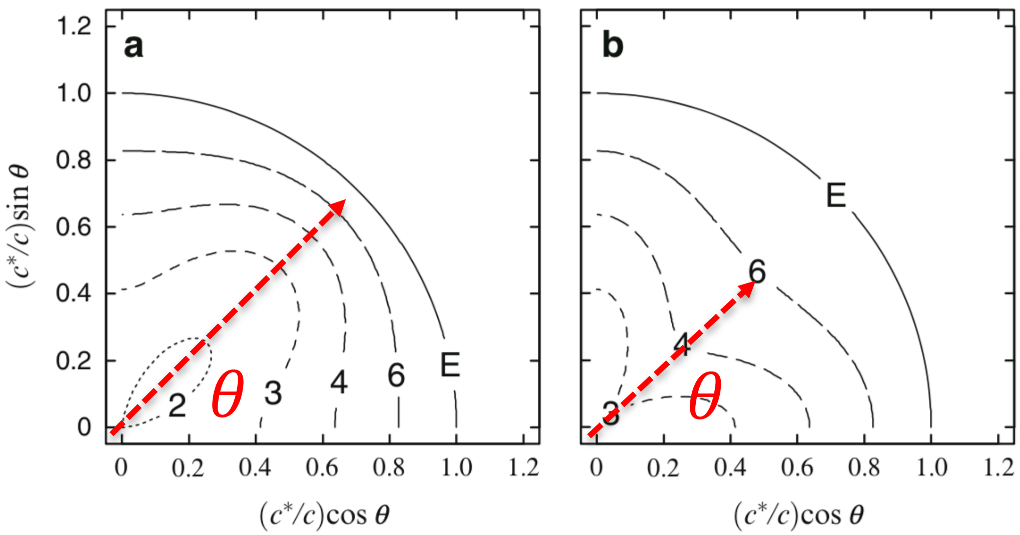


Fig. 4.4 Polar plot of the relative phase speeds of $2\Delta s$ (*shortest dashed line*), $3\Delta s$, $4\Delta s$, and $6\Delta s$ (*longest dashed line*) waves generated by **a** the nonaveraged finite-difference formula and **b** the averaging scheme. Also plotted is the curve for perfect propagation (E), which is independent of the wavelength and appears as a circular arc of radius unity

The radius of the curve is the normalized phase speed : Depend on wind directions & wave wavelengths (dispersive)

$$\frac{c_{na}^*}{c} = \frac{cos\theta\,\sin(\beta cos\theta) + sin\theta\,\sin(\beta sin\theta)}{\beta}$$

$$\frac{c_a^*}{c} = \frac{cos\theta\,\sin(\beta cos\theta)\cos(\beta sin\theta) + sin\theta\,\sin(\beta sin\theta)\cos(\beta sin\theta)}{\beta}$$

$U = c\times cos\theta, V = c\times sin\theta, \theta$ *is the angle of mean wind*

$\beta = K\Delta s, K$ *is magnitude of wavenumber*

Forward-in-time Schemes

Forward-in-time and upstream scheme:

$$\delta_t \phi_{m,n}^{j+1/2} + \mathrm{U}\delta_x \phi_{m-1/2,n}^{j} + V\delta_y \phi_{m,n-1/2}^{j} = 0$$

Stability criterion:

$$\mu \geq 0, \nu \geq 0, and\ \mu + \nu \leq 1$$

Taylor expansion, the truncation error is:

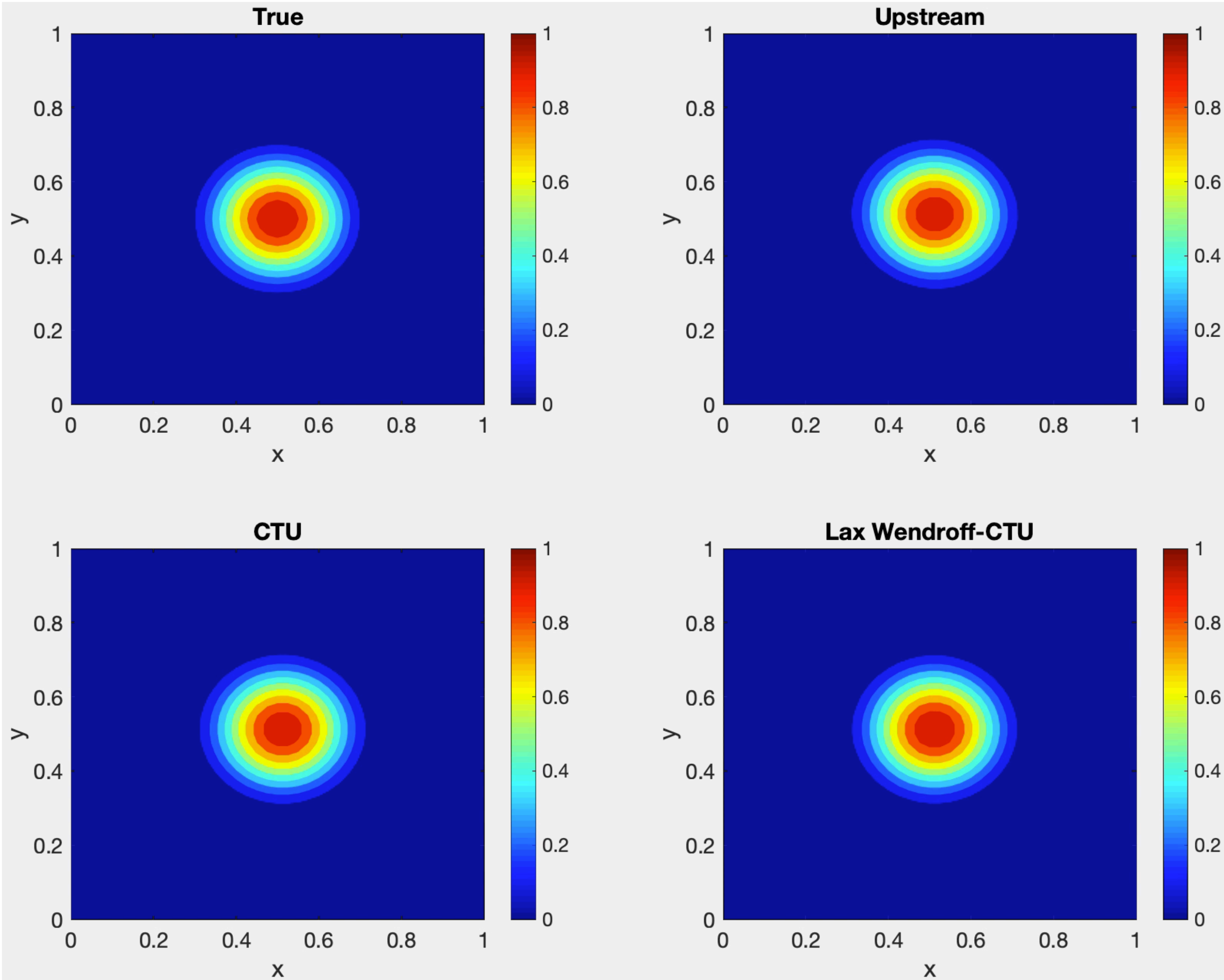
$$\frac{\partial \psi}{\partial t} + U\frac{\partial \psi}{\partial x} + V\frac{\partial \psi}{\partial y} = \frac{U\Delta x}{2}(1-\mu)\frac{\partial^2 \psi}{\partial x^2} + \frac{U\Delta y}{2}(1-\nu)\frac{\partial^2 \psi}{\partial y^2} \boxed{- UV\Delta t\frac{\partial^2 \psi}{\partial x \partial y}}$$

The coefficient $-UV\frac{\partial^2 \psi}{\partial x \partial y}$ on Δt error term has directional preference

$$if\ we\ write\ r = x + y, s = x - y, then\ \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial s^2} - \frac{\partial^2 \psi}{\partial r^2}$$

"Diffusive" along r and "antidiffusive" along s, which tends to distort solution, elongated in one direction, and shortening in the other.

Problem of 2-D Upstream Scheme



Corner Transport Upstream (CTU) Method

Truncation error after Taylor expansion:

$$\frac{\partial\psi}{\partial t} + U\frac{\partial\psi}{\partial x} + V\frac{\partial\psi}{\partial y} = \frac{U\Delta x}{2}(1-\mu)\frac{\partial^2\psi}{\partial x^2} + \frac{U\Delta y}{2}(1-\nu)\frac{\partial^2\psi}{\partial y^2} - UV\Delta t\frac{\partial^2\psi}{\partial x\partial y}$$

Forward-in-time and upstream scheme:

$$\boxed{\delta_t\phi_{m,n}^{j+1/2} + \mathrm{U}\delta_x\phi_{m-1/2,n}^{j} + V\delta_y\phi_{m,n-1/2}^{j} = 0}$$

Modification

$$\boxed{\delta_t\phi_{m,n}^{j+1/2} + \mathrm{U}\delta_x\phi_{m-1/2,n}^{j} + V\delta_y\phi_{m,n-1/2}^{j} = UV\Delta t\frac{\partial^2\psi}{\partial x\partial y}}$$

Corner Transport Upstream (CTU) method

$$\boxed{\delta_t\phi_{m,n}^{j+1/2} + \mathrm{U}\delta_x\phi_{m-1/2,n}^{j} + V\delta_y\phi_{m,n-1/2}^{j} = UV\delta_x\delta_y\phi_{m-1/2,n-1/2}^{j}}$$

Lax-Wendroff-CTU Method

Forward-in-time and upstream scheme, stability condition:

$$C\frac{\Delta t}{\Delta s} < 1/\sqrt{2}$$

Corner Transport Upstream (CTU) method

$$0 \le \mu \le 1, 0 \le \nu \le 1, C\frac{\Delta t}{\Delta s} \le 1$$

Truncation Error of CTU:

$$\frac{\partial \psi}{\partial t} + U\frac{\partial \psi}{\partial x} + V\frac{\partial \psi}{\partial y} = \frac{U\Delta x}{2}(1-\mu)\frac{\partial^2 \psi}{\partial x^2} + \frac{U\Delta y}{2}(1-\nu)\frac{\partial^2 \psi}{\partial y^2}$$

First-order of accuracy: Δx and Δy, highly damping

Lax-Wendroff + CTU:

lessen damping + correct directionality

$$\delta_t \phi^{j+\frac{1}{2}} + \mathrm{U}\delta_{2x}\phi^j + V\delta_{2y}\phi^j = \frac{U^2\Delta t}{2}\delta_x^2\phi^j + \frac{V^2\Delta t}{2}\delta_y^2\phi^j + UV\ \Delta t\ \delta_{2x}\delta_{2y}\phi^j$$

Lax-Wendroff-CTU Method

Lax-Wendroff + CTU:

lessen damping + correct directionality

$$\delta_t \phi^{j+\frac{1}{2}} + \mathrm{U}\delta_{2x}\phi^j + V\delta_{2y}\phi^j = \frac{U^2\Delta t}{2}\delta_x^2\phi^j + \frac{V^2\Delta t}{2}\delta_y^2\phi^j + UV\ \Delta t\ \delta_{2x}\delta_{2y}\phi^j$$

Necessary and sufficient stability condition:

$$C\frac{\Delta t}{\Delta s} < 1/\sqrt{2}$$

Stability-improved:

$$\delta_t \phi_{m,n}^{j+\frac{1}{2}} + \mathrm{U}\delta_{2x}\phi_{m,n}^j + V\delta_{2y}\phi_{m,n}^j$$
$$= \frac{U^2\Delta t}{2}\phi_{m,n}^j + \frac{V^2\Delta t}{2}\delta_y^2\phi_{m,n}^j + UV\ \Delta t\ \delta_x\delta_y\phi_{m-\frac{1}{2},n-\frac{1}{2}}^j$$

Necessary and sufficient stability condition:

$$0 \le \mu \le 1, 0 \le \nu \le 1, C\frac{\Delta t}{\Delta s} \le 1$$

Problem of 2-D Upstream Scheme