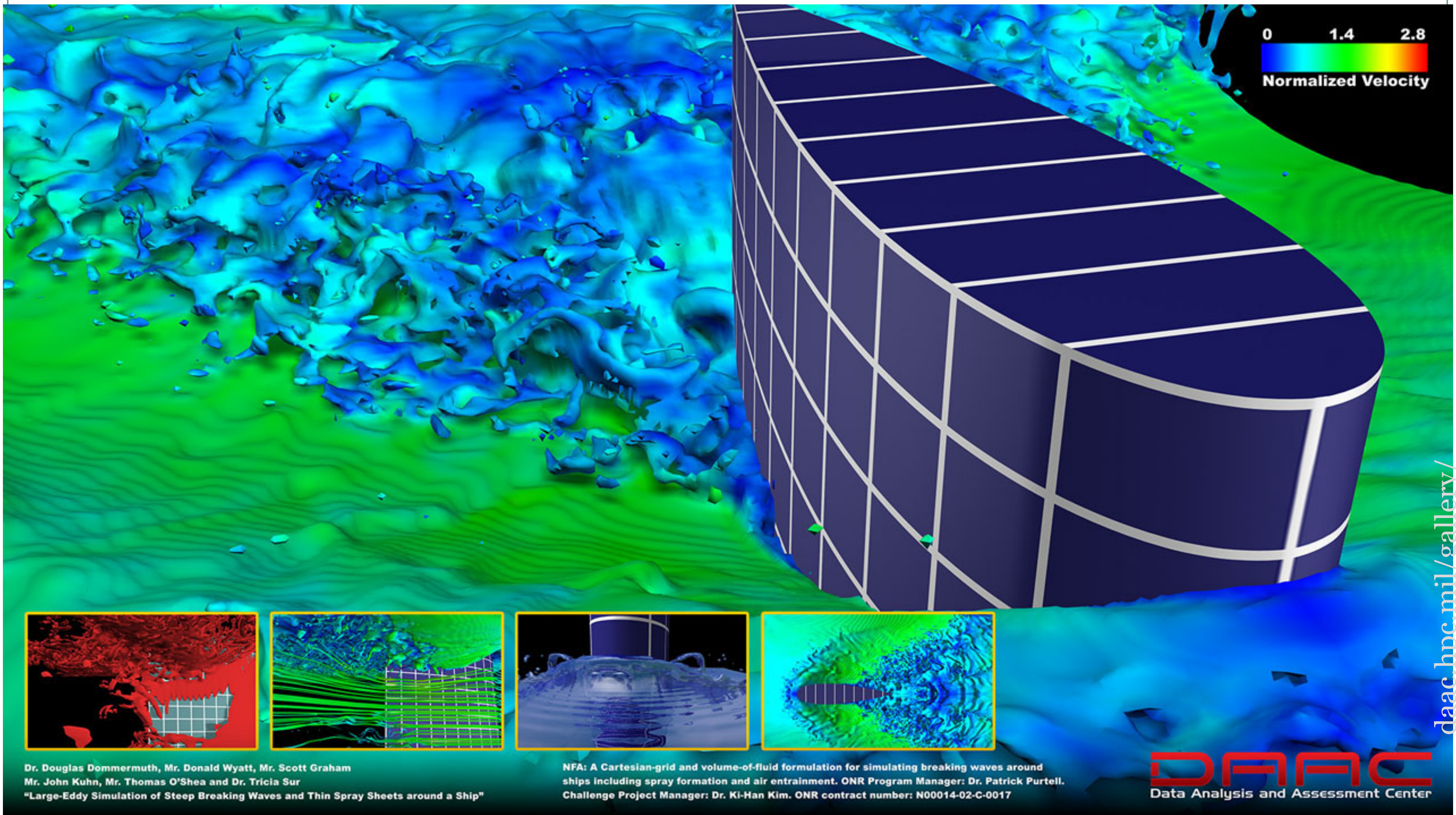
PHYS 8750
NUMERICAL FLUID DYNAMICS



PHYS 8750

Class #13 (Chapter 4.2)

1) PDE with two variables

2) Lax-Wendroff + CTU

Class #14
Summary of
Chapters 1-4

Outline

1. Summary

Temporal schemes for ODEs (Chapter 2)

Temporal filtering for ODEs (Chapter 2)

Spatial schemes for PDEs (Chapter 3)

Spatial filtering for PDEs (Chapter 3)

Staggered grids for 2 variables (Chapter 4)

Schemes (CTU) for 2 dependents (Chapter 4)

Summary of Chapter 1-4

Purpose:

Solve the systems/problems described by PDEs numerically (analytical form is too different to obtain)

Finite-Difference Scheme:

Use discretized form (grids) to approximate partial derivatives/ use discontinuous grids to represent continuous PDE.

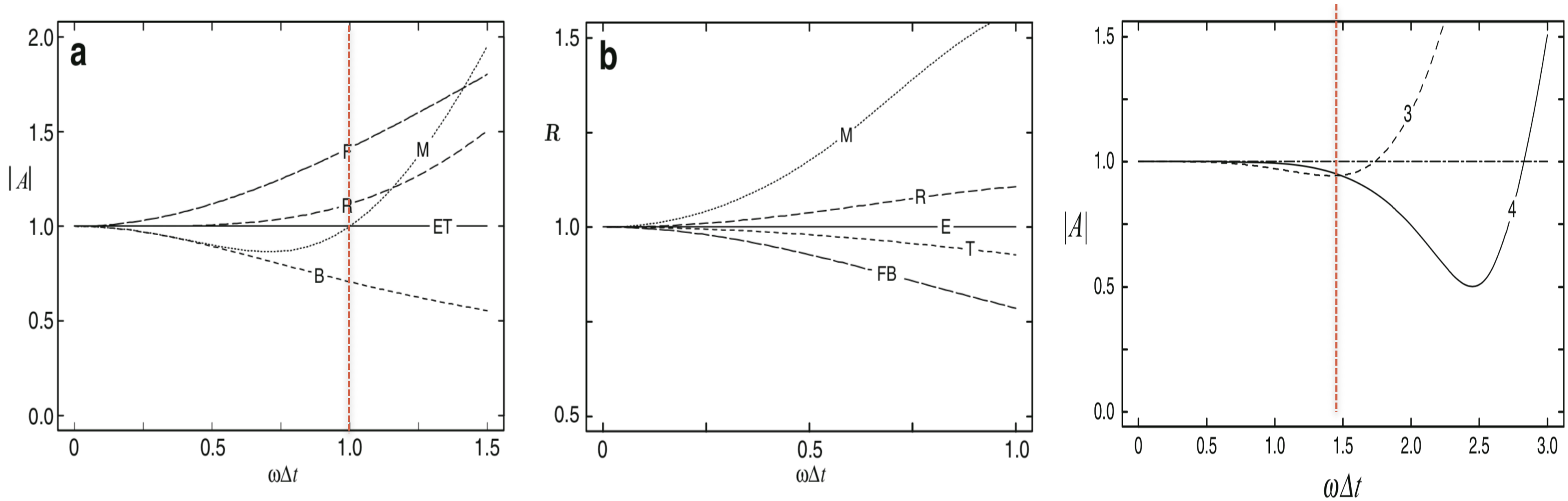
Criteria to judge a scheme:

1. Order of accuracy, convergence, stability (solutions are constrained)

2. Amplitude and phase errors $\longrightarrow$ damping/dissipation and dispersion

Key Issue: How to improve 1, and avoid 2.

Amplitude and Phase Behavior



E: exact solution T: trapezoidal F: Forward B: Backward R/3/4: 2/3/4-stage Runge-Kunta M: Matsuno

Multi-Stage Advantages

- High order of accuracy
- Small amp errors
- Small phase errors
- Efficient damping at high-frequency
- Stability
- Large Δt allowed

Disadvantages

- Evaluating derivatives multiple times
- Extensive storage
- More expensive computation

Multi-Step Scheme

- Multiple time steps are used to computer the next time step.
- Achieve high (usually second) order of accuracy, without using too much storage.
- Physical and computational modes both exist. Need to suppress the errors arising from computation modes usually by temporal filtering.
- Leapfrog & Adams-Bashforth

Multi-Step Scheme

- **Leapfrog**

$$\frac{\boxed{\phi_{n+1} - \phi_{n-1}}}{2\Delta t} = F(\phi_n, t_n) = i\omega\phi_n$$

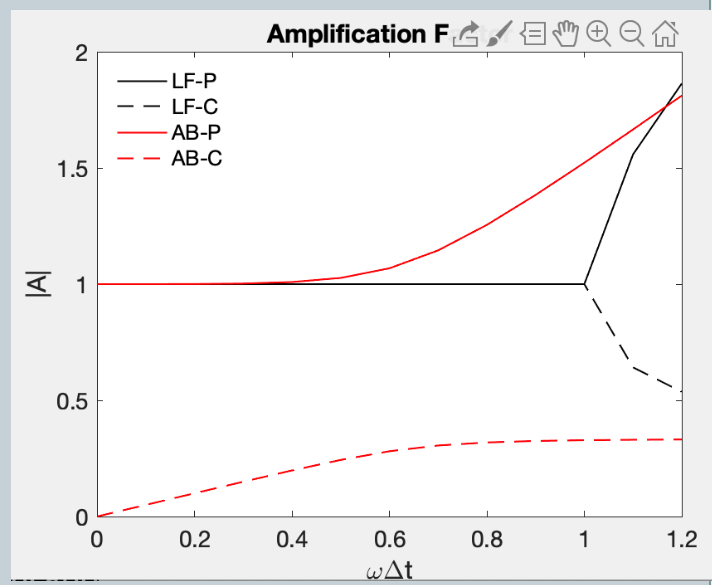
$$|A_\pm| = 1$$

Since amplification factors for both physical and computational modes are 1, use temporal filtering to suppress computational models

- **Adams-Bashforth**

$$\phi_{n+1} = \phi_n + \Delta t\left(\frac{3}{2}F(\phi_n, t_n) - \frac{1}{2}\boxed{F(\phi_{n-1}, t_{n-1})}\right)$$

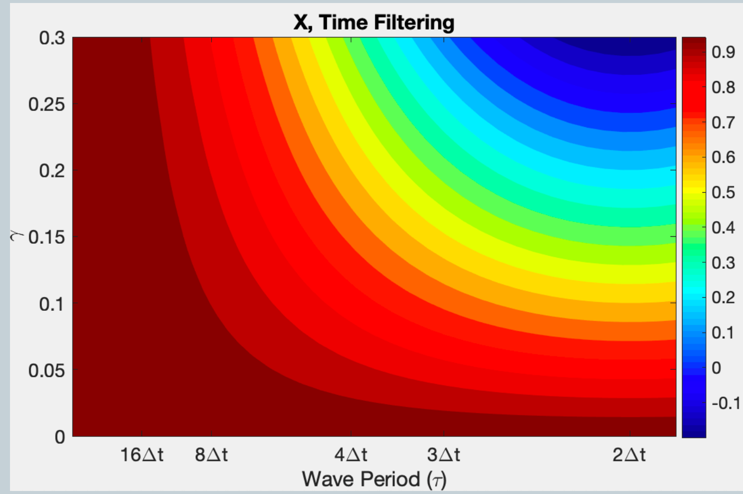
$$A_\pm = \frac{1}{2}\left(1 + \frac{3i\omega\Delta t}{2} \pm \left(1 - \frac{9}{4}\omega^2\Delta t^2 + i\omega\Delta t\right)^{1/2}\right)$$



Filtering is inherently built in scheme

Temporal Filtering

1. Postprocessing Way:



$$\overline{\phi_n} = \phi_n + \gamma(\phi_{n+1} - 2\phi_n + \phi_{n-1})$$

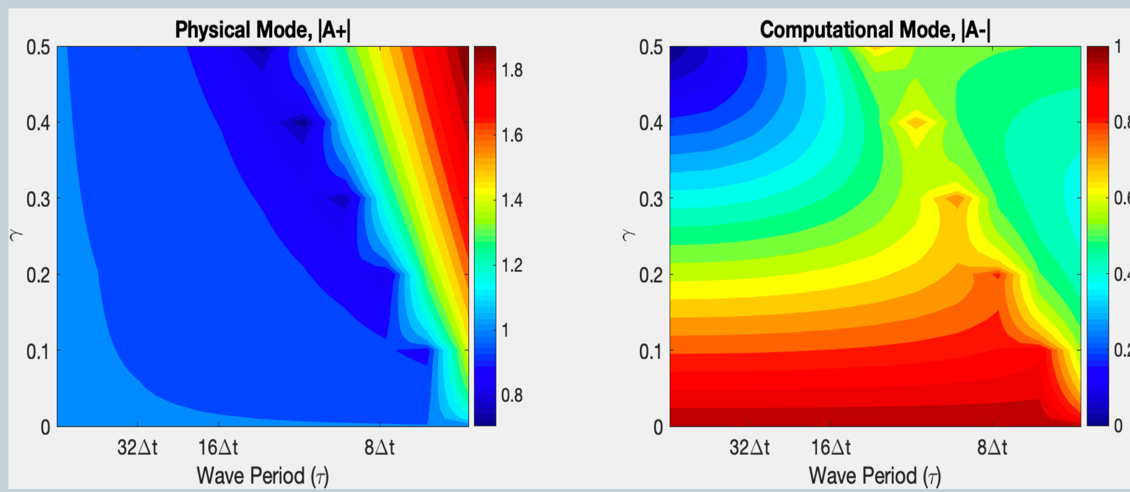
$$X = \frac{\overline{\phi_n}}{\phi_n} = 1 - 2\gamma(1 - cos\omega\Delta t)$$

2. Interactive Way:

Asselin-filtered

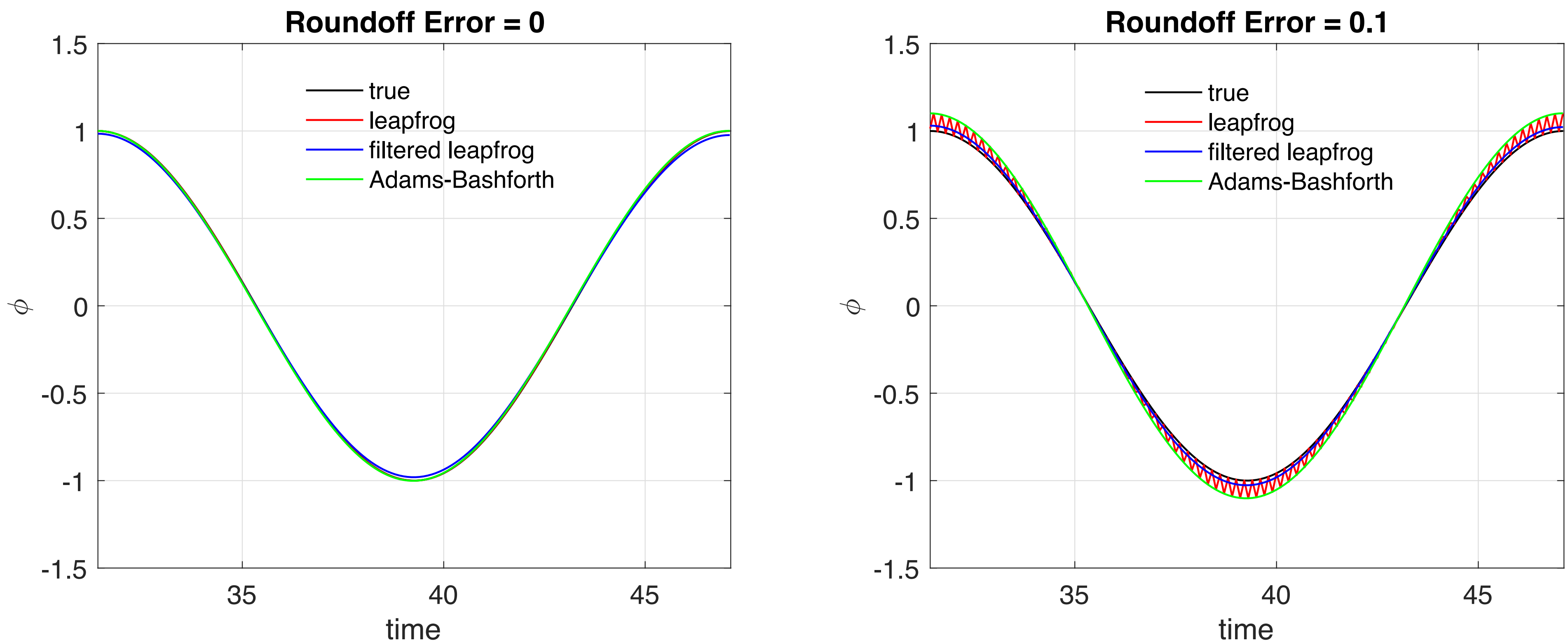
$$\phi_{n+1} = \overline{\phi_{n-1}} + 2\Delta t F(\phi_n)$$

$$\overline{\phi_n} = \phi_n + \gamma(\overline{\phi_{n-1}} - 2\phi_n + \phi_{n-1})$$



Oscillation Problem

Leapfrog w/wo filter, and Adams-Bashforth



Code: LF_AB_StabilityAmplification_ODE_OscillationProb_7.m

- LF: would suffer computational mode
- Asselin-LF: damp computational mode, accurate amplitude.
- AB: damp computational mode, errors contaminate

PDEs (when both time and space are present)

New Concepts:

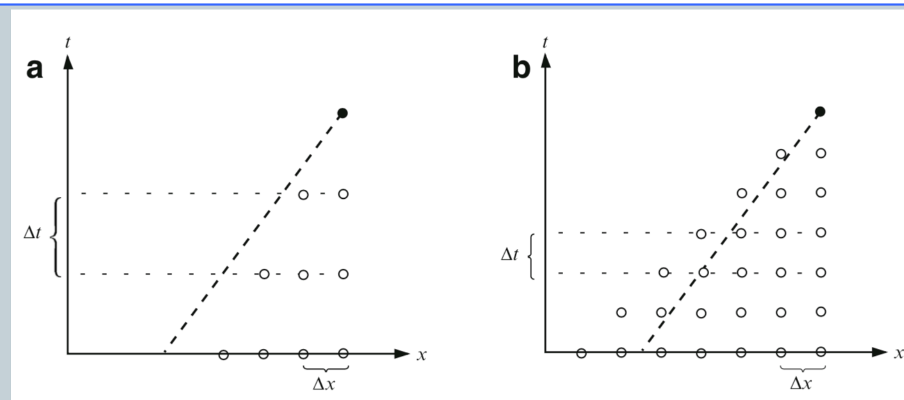
1. Courant number ($c\Delta t/\Delta x$) → put constraints on Δt
2. Von Neumann's method: decompose solution to different wave scales → Amplification factor depends on Δx.

Key Issue:

How to enable larger Δt for stability.

How to suppress short-scale waves.

- CFL condition: Domain 1 includes Domain 2.

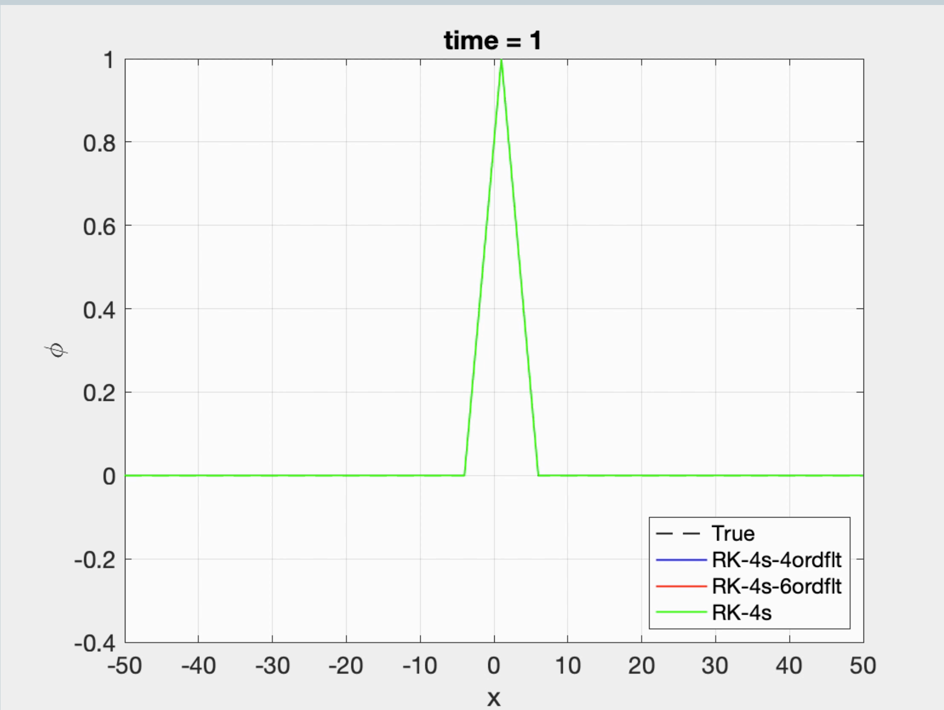


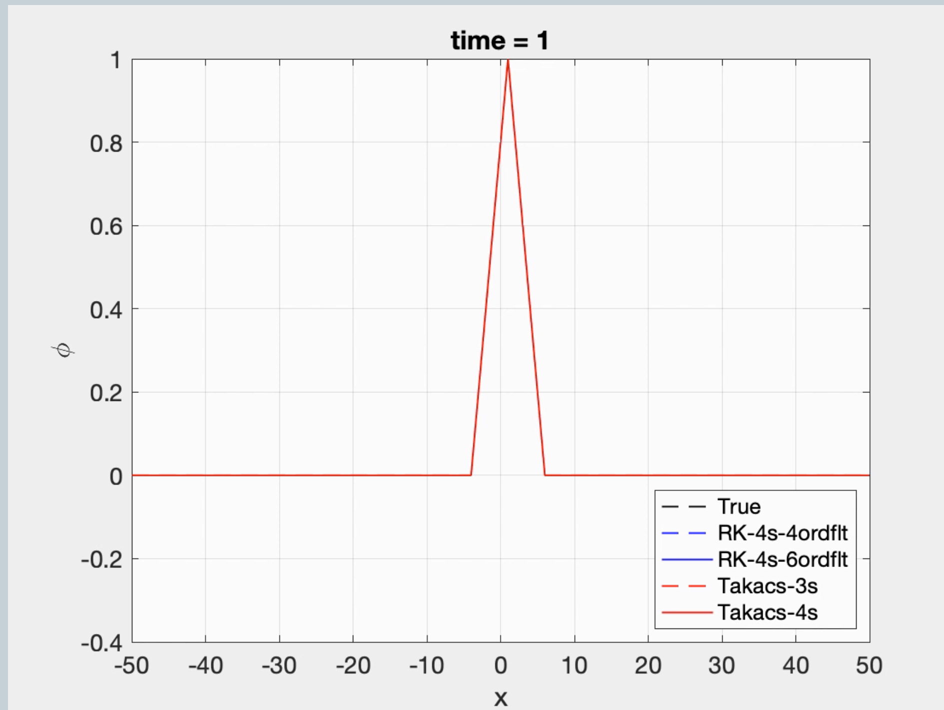
Strategies:

1. Higher-order of schemes by involving more spatial points.
2. Apply spatial filtering.

Takacs's Schemes (Lax-Wendroff)

- Higher-order of schemes by involving more spatial points.





Spatial Filtering

Second-derivative smoother

$$\frac{d\phi_j}{\partial t} = \gamma_2(\phi_{j+1} - 2\phi_j + \phi_{j-1})$$

4th-derivative smoother

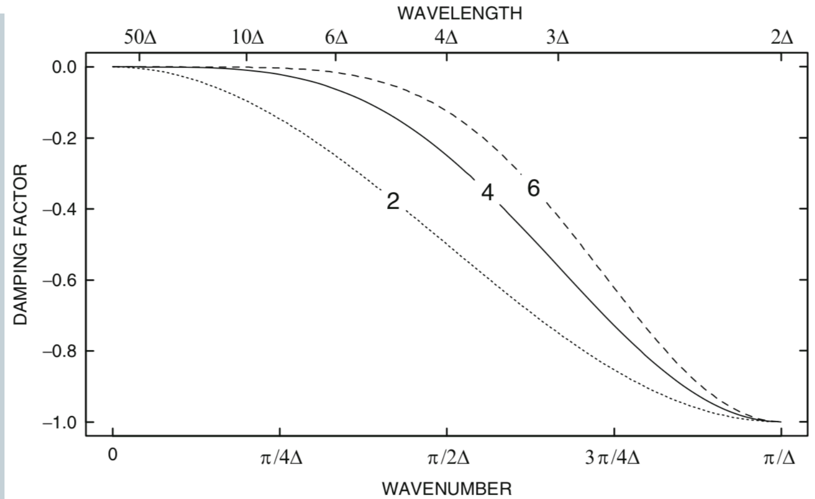
$$\frac{d\phi_j}{\partial t} = \gamma_4(-\phi_{j+2} + 4\phi_{j+1} - 6\phi_j + 4\phi_{j-1} - \phi_{j-2})$$

6th-derivative smoother

$$\frac{d\phi_j}{\partial t} = \gamma_6\begin{pmatrix}\phi_{j+3} - 6\phi_{j+2} + 15\phi_{j+1} - \\ 20\phi_j + 15\phi_{j-1} - 6\phi_{j-2} + \phi_{j-3}\end{pmatrix}$$

Damping factor nth-derivative smoother

$$\frac{db}{dt} = -\gamma_n[2(1 - cosk\Delta x)]^{n/2} b$$



Damping rate

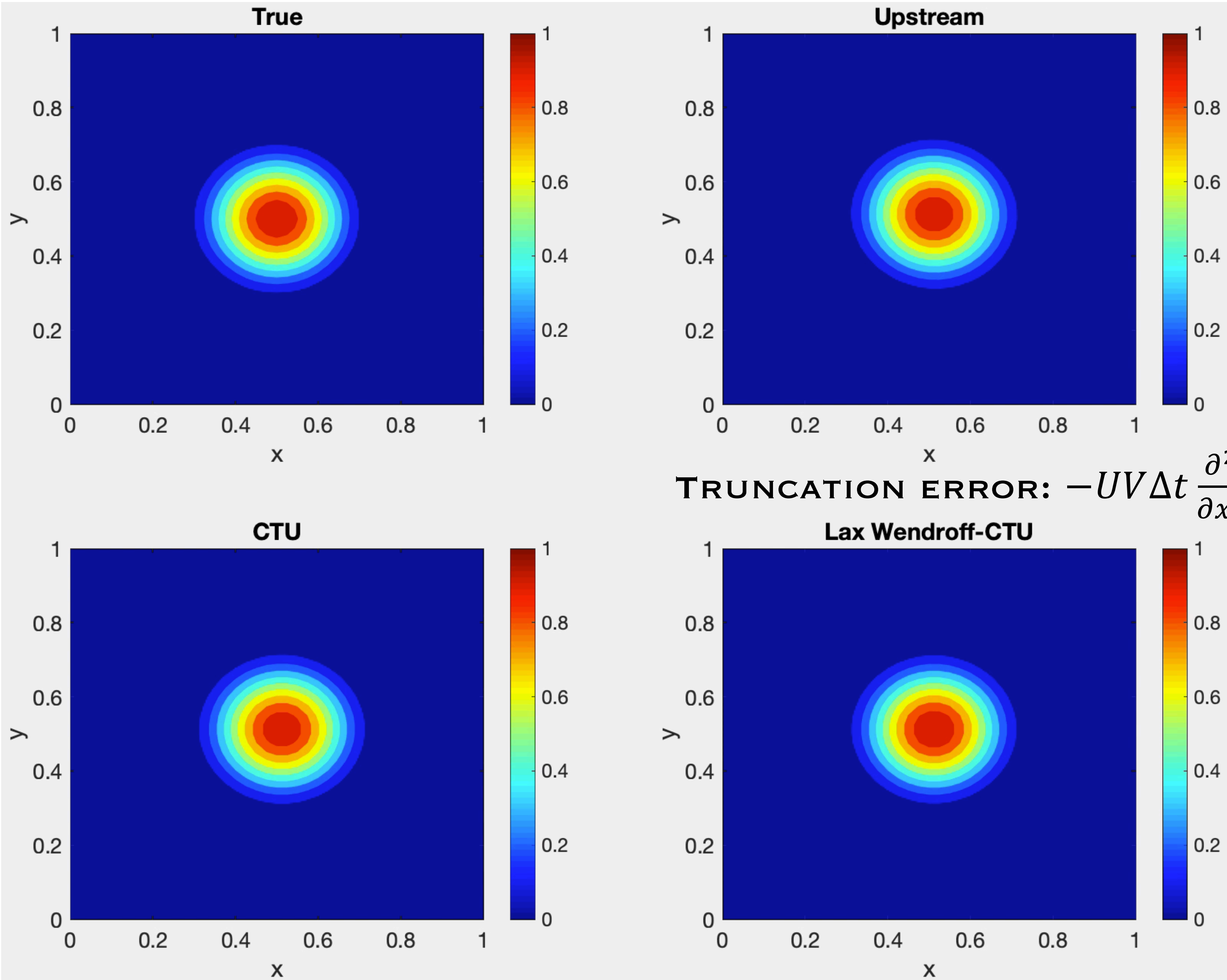
$$= -\gamma_n[2(1 - cosk\Delta x)]^{n/2}$$

Pros and Cons from low to high orders?

Multiple Dependent Variables/Staggered Grids

	Stability	Phase Performance	Storage	Mean flow limitation
Leapfrog time Center space	$\left\lvert \frac{c\Delta t}{\Delta x} \right\rvert \leq 1$	Most dispersive	More	No
Leapfrog time Staggered space (middle points)	$\left\lvert \frac{c\Delta t}{\Delta x} \right\rvert \leq \frac{1}{2}$	Least dispersive	More	No
Forward-Backward time Staggered space	$\left\lvert \frac{c\Delta t}{\Delta x} \right\rvert \leq 1$	Less dispersive	Less	U<<c U is very small

Multiple Dimensions (2D)



Truncation error: $-UV\Delta t\,\frac{\partial^2\psi}{\partial x\partial y}$

Summary

Temporal schemes for ODEs (Chapter 2)

2-time levels: Forward, backward, trapezoidal

2-time levels & multi-stage: Runge-Kunta, Mastuno

Multi-time levels: Leapfrog (centered), Adams-Bashforth

Temporal filtering for ODEs (Chapter 2)

Applied to multi-time levels: suppress computational modes

Spatial schemes for PDEs (Chapter 3)

Upstream downstream, centered,

Lax-Wendroff, Takacs (forward time, control order of accuracy by involving more spatial points)

Spatial filtering for PDEs (Chapter 3)

2-nd derivative to N–th derivative filters

suppress small scales