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Similarity-Based Approach for Group Decision Making with Multi-Granularity Linguistic Information

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The aim of this article is to investigate the approach for multi-attribute group decision-making, in which the attribute values take the form of multi-granularity multiplicative linguistic information. Firstly, to process multiple sources of decision information assessed in different multiplicative linguistic label sets, a method for transforming multi-granularity multiplicative linguistic information into multiplicative trapezoidal fuzzy numbers is proposed. Then, a formula for ranking multiplicative trapezoidal fuzzy numbers is given based on geometric mean. Furthermore, the concept of similarity degree between two multiplicative trapezoidal fuzzy numbers is defined. The attribute weights are obtained by solving some optimization models. An effective approach for group decision making with multi-granularity multiplicative linguistic information is developed based on the ordered weighted geometric mean operator and proposed formulae. Finally, a practical example is provided to illustrate the practicality and validity of the proposed method.

Keywords: Multi-granularity multiplicative linguistic information; group decision-making; multiplicative trapezoidal fuzzy number; similarity degree.

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1. Introduction

Group decision-making problems under linguistic environment are an interesting topic of research, and have attracted more and more attention over the last decades.[1−5] It is intuitional for experts to express their preference information in natural linguistic labels, the use of linguistic labels makes expert judgment more reliable for decision-making. For example, when we talk about the speed of a car, linguistic labels like "very slow", "fast" and "extremely fast" may be used. Numerous methods have been proposed to solve group decision-making problems with linguistic information. Chakraborty[6] proposed a decision-aid tool, which provides homogeneousness from a set of heterogeneous opinions for a particular attribute. Wei[7] introduced some new aggregation operators, and applied these operators to multi-attribute decision-making problems with linguistic information of weight values and attribute values. By Xu and Da,[8] two methods for decision-making with linguistic information are proposed based on the deviation degree. Recently, group decision-making under linguistic environment has been a hot research topic in the world.

Two types of linguistic label sets are developed in the existing references. One is the balanced linguistic label set, in which linguistic labels are uniformly distributed. Herrera[9] and Cordon[10] introduced a finite and totally ordered discrete linguistic label set, and the limit of cardinality value is 11 or not more than 13. Then, Xu[11,12] defined a new discrete linguistic label set with odd cardinality, whose linguistic labels are uniformly symmetrically distributed. To preserve linguistic information in the operational process, the original (discrete) linguistic label set is extended to a continuous linguistic label set. Furthermore, Dong[13] pointed out that the main difference between the above two linguistic label sets is using different representation formats. The other type of linguistic label set is the unbalanced linguistic label set, which is very suit to deal with many practical problems, such as economic analysis,[14] information retrieval system[15] and educational evaluation,[16,17] etc. The appearance of unbalanced linguistic information springs from the nature of the linguistic variables used in the problems.[18] Herrera[19] made a review of the developments of computing with words in decision making. Some current trends and open questions of the unbalanced linguistic label set are introduced detailedly. Cabrerizo[20] presented a consensus model for group decision making problems where the opinions of experts provided by means of unbalanced fuzzy linguistic information. The computational operations of unbalanced fuzzy linguistic information are proposed based on the 2-tuple computational model. Moreover, Cabrerizo[21] developed a consensus model to help experts achieve consistency in each phase of group decision-making. This model supports the management of incomplete unbalanced linguistic information, and the consistent solutions are derived from a great level of agreement. Meng and Pei[22] generalized linguistic evaluation values and weights in group decision-making problems based on unbalanced linguistic labels. The weighted unbalanced linguistic aggregation operators with two types of weights are proposed, respectively. Xu[16] introduced an unbalanced linguistic label set named multiplicative linguistic label set, in which the middle linguistic label represents the meaning of "indifference", and with the rest of the linguistic labels being placed symmetrically. Since the concept of multiplicative linguistic label set is given, several approaches have been developed to solve decision-making problems with multiplicative linguistic information. To process multi-attribute group decision-making problems, Xu[14] proposed some new aggregation operators for aggregating multiplicative linguistic information. Zhang[23] generalized the continuous ordered weighted geometric operator to accommodate uncertain linguistic environment. Xu[24] defined the concept of incomplete multiplicative linguistic preference relation, and then an effective algorithm is given to group decision-making problems with multiplicative linguistic information. Recently, decision-making problems with multiplicative linguistic information have received more and more attention.

For linguistic label sets, an important decision parameter is the granularity of uncertainty. The granularity of a linguistic label set denotes the number of linguistic labels in this set. As the different backgrounds and levels of knowledge of experts, they may use linguistic labels in linguistic label sets with different cardinalities, which we call the multi-granularity linguistic label sets. Many methods have been proposed to solve the group decision-making problems with multi-granularity linguistic information. With respect to a multi-attribute decision-making problem, Herrera[25] developed a decision process as a view to obtain the solution set of alternatives by two steps. Chen and Ben-Arieh[26] presented a new method to combine the information assessed in different linguistic label sets. Chang[27] used the multi-granularity linguistic labels instead of numerical variable to eliminate the inaccuracy on qualification, and the fitting linguistic scale is employed in accordance with the characteristic of supply behavior. Cabrerizo[28] formulated the information granulation as an optimization problem in which a performance index is expressed as a weighted aggregation of the individual consistency. Massanet[29] proposed a linguistic computational model based on discrete fuzzy numbers. Morente-Molinera[30] introduced the evolution of multi-granular fuzzy linguistic modeling approaches and discussed their advantages and drawbacks. Jiang[31] transformed the multi-granularity linguistic information into the form of fuzzy numbers, and a linear programming model is developed to integrate the assessment information. Furthermore, to process the uncertain preference information in group decision-making, a formula for transforming multi-granularity uncertain linguistic labels into trapezoidal fuzzy numbers is proposed by Fan and Liu.[32] However, the group decision-making problems with multi-granularity multiplicative linguistic information are seldom discussed. The purpose of this paper is to develop an approach to solve the group decision-making problems with multi-granularity multiplicative linguistic information. Firstly, to process multiple sources of decision information assessed in different multiplicative linguistic label sets, the multi-granularity multiplicative linguistic labels are transformed into multiplicative trapezoidal fuzzy numbers based on multiplication operation. Then, a formula is proposed to rank multiplicative trapezoidal fuzzy numbers based on geometric mean. Furthermore, in order to obtain the attribute weights, the concept of similarity degree between two multiplicative trapezoidal fuzzy numbers is defined. In sum, an effective approach for group decision making with multi-granularity multiplicative linguistic information is developed based on the ordered weighted geometric mean operator and proposed formulae.

The rest of this paper is organized as follows. Section 2 introduces some preliminary knowledge. In Sec. 3, a method for transforming multi-granularity multiplicative linguistic information into multiplicative trapezoidal fuzzy numbers is proposed. A method for ranking multiplicative trapezoidal fuzzy numbers is proposed in Sec. 4. In Sec. 5, an effective approach for group decision making with multi-granularity multiplicative linguistic information is developed. Section 6 provides a numerical example to demonstrate the practicality and validity of the proposed approach. The paper is concluded in Sec. 7.

2. Preliminaries

In the following, some basic concepts on multiplicative linguistic information and trapezoidal fuzzy number are introduced.

2.1. *Multiplicative linguistic information*

Let $S = \{s_{\frac{1}{t}}, \ldots, s_{\frac{1}{2}}, s_1, s_2, \ldots, s_t\}$ be a multiplicative linguistic label set with odd cardinality,[14] where s_α denotes a possible value for a linguistic variable, $\alpha \in M = \{\frac{1}{t}, \ldots, \frac{1}{2}, 1, 2, \ldots, t\}$, and $2t-1$ is the cardinality of S. For example, a multiplicative linguistic label set with seven labels can be expressed as $S = \{s_{\frac{1}{4}} =$ Very Poor, $s_{\frac{1}{3}} =$ Poor, $s_{\frac{1}{2}} =$ Slightly Poor, $s_1 =$ Fair, $s_2 =$ Slightly Good, $s_3 =$ Good, $s_4 =$ Very Good$\}$. The above multiplicative linguistic label set S is graphically shown in Fig. 1 below.

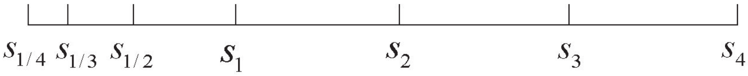

Fig. 1. Multiplicative linguistic label set with seven labels.

For any α_i, $\alpha_j \in M$, the multiplicative linguistic label set has the following characteristics:[14]

(1) $s_{\alpha_i} < s_{\alpha_j}$, if and only if $\alpha_i < \alpha_j$.
(2) There is the reciprocal operator: $\text{rec}(s_{\alpha_i}) = s_{\frac{1}{\alpha_j}}$, such that $\alpha_i\alpha_j = 1$.

For multiplicative linguistic label sets, an important decision parameter is the granularity of uncertainty. The granularity of a multiplicative linguistic label set denotes the number of linguistic labels in this set. Generally, the experts may have different backgrounds and levels of knowledge, so they often use multiplicative linguistic label sets with different granularities to express his/her preference on the alternative in a group decision making problem. For convenience of analysis, we give the definition of multi-granularity multiplicative linguistic label set as follows.

Definition 1. Let $S^k = \{s^k_{\frac{1}{t_k}}, \ldots, s^k_{\frac{1}{2}}, s^k_1, s^k_2, \ldots, s^k_{t_k}\}$ be a multiplicative linguistic label set with odd cardinalities, $k = 1, 2, \ldots, m$, then a multi-granularity multiplicative linguistic label set is expressed by $\overline{S} = \{S^k \mid k \in \{1, 2, \ldots, m\}\}$.

2.2. *Trapezoidal fuzzy number*

Definition 2. A trapezoidal fuzzy number $\widetilde{q} = (a, b, c, d)$ is a special fuzzy subset on the set $\mathbf{R}$ of real numbers. The membership function $\mu_{\widetilde{q}}$ of $\widetilde{q}$ is[33]

$$\mu_{\widetilde{q}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{d-x}{d-c}, & c \le x \le d \\ 0, & \text{others} \end{cases} \tag{1}$$

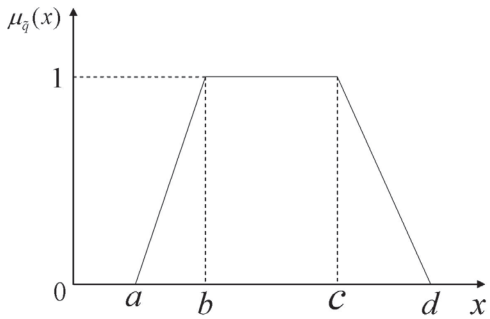

Fig. 2. A trapezoidal fuzzy number $\widetilde{q}$.

Trapezoidal fuzzy numbers are often used in practice to represent uncertain or incomplete information. Because its membership function includes both the horizontal form, increasing form and decreasing form. Ordinarily, a trapezoidal fuzzy number can be denoted briefly by $\widetilde{q} = (a, b, c, d)$, and we can get sketch map in Fig. 2. If $b = c$, then $\widetilde{q}$ reduces to a triangular fuzzy number (a, b, d) or (a, c, d). If $a \to b$ and $d \to c$, then $\widetilde{q}$ approximately reduces to an interval number $[b, c]$. Trapezoidal fuzzy number can be directly calculated by the following operational laws:

Definition 3. Let $\widetilde{q}_1 = (a_1, b_1, c_1, d_1)$ and $\widetilde{q}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, λ is a real number, and suppose that $a_1, a_2, \lambda \ge 0$. Four main operational laws of these two trapezoidal fuzzy numbers can be denoted as follows.[33]

$$\widetilde{q}_1 \oplus \widetilde{q}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) \tag{2}$$

$$\lambda \otimes \widetilde{q}_1 = (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1) \tag{3}$$

$$\widetilde{q}_1 \otimes \widetilde{q}_2 = (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2) \tag{4}$$

$$\widetilde{q}_1^{\lambda} = (a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}) \tag{5}$$

3. Transforming Multi-Granularity Multiplicative Linguistic Information into Fuzzy Numbers

It is difficult to process the multi-granularity multiplicative linguistic information in the direct way, because each level set of a multiplicative linguistic label set is non continuous, which may produce a loss of information and hence a lack of validity in the final results. In order to process multiplicative linguistic information in group decision-making problems, a method for transforming multiplicative linguistic information into trapezoidal fuzzy numbers is proposed below.

Definition 4. Let $S^k = \{s^k_{\frac{1}{t_k}}, \ldots, s^k_{\frac{1}{2}}, s^k_1, s^k_2, \ldots, s^k_{t_k}\}$ be a multiplicative linguistic label set with odd cardinalities, $k = 1, 2, \ldots, m$, and the cardinality of S^k is $2t_k - 1$, then the linguistic labels in S^k can be transformed into trapezoidal fuzzy numbers by function $\overline{G}$ or $\underline{G}$ as follows.

$$\overline{G}(s^k_i) = (x_i^{k1}, x_i^{k2}, x_i^{k3}, x_i^{k4}), \quad i = 1, 2, \ldots, t_k\,, \tag{6}$$

$$\underline{G}(s^k_{\frac{1}{i}}) = (x_{\frac{1}{i}}^{k1}, x_{\frac{1}{i}}^{k2}, x_{\frac{1}{i}}^{k3}, x_{\frac{1}{i}}^{k4}), \quad i = 1, 2, \ldots, t_k\,. \tag{7}$$

where

$$\begin{cases} x_i^{k1} = f(t_k)^{\frac{4i-7}{4t_k-3}} \\ x_i^{k2} = f(t_k)^{\frac{4i-5}{4t_k-3}} \\ x_i^{k3} = f(t_k)^{\frac{4i-3}{4t_k-3}} \\ x_i^{k4} = \min\{f(t_k)^{\frac{4i-1}{4t_k-3}}, f(t_k)\} \end{cases} \tag{8}$$

$$\begin{cases} x_{\frac{1}{i}}^{k1} = \max\{f(t_k)^{\frac{1-4i}{4t_k-3}}, \frac{1}{f(t_k)}\} \\ x_{\frac{1}{i}}^{k2} = f(t_k)^{\frac{3-4i}{4t_k-3}} \\ x_{\frac{1}{i}}^{k3} = f(t_k)^{\frac{5-4i}{4t_k-3}} \\ x_{\frac{1}{i}}^{k4} = f(t_k)^{\frac{7-4i}{4t_k-3}} \end{cases} \tag{9}$$

and f is a continuous and monotonic increasing function, $f : [1, \infty) \to [1, \infty)$ with (1) $f(1) = 1$, (2) If $t_k > 1$, then $f(t_k) > 1$.

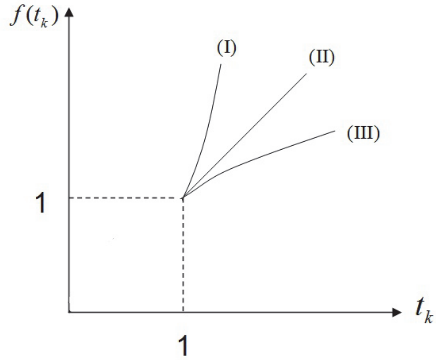

Fig. 3. Three curves corresponding to decision maker's preference of risk.

Table 1. Multiplicative linguistic label set with seven cardinalities and the corresponding trapezoidal fuzzy numbers.

Multiplicative linguistic labels	Multiplicative trapezoidal fuzzy numbers
$s^k_{\frac{1}{4}}$: Very Poor	$(0.250, 0.250, 0.309, 0.383)$
$s^k_{\frac{1}{3}}$: Poor	$(0.309, 0.383, 0.474, 0.587)$
$s^k_{\frac{1}{2}}$: Slightly Poor	$(0.474, 0.587, 0.726, 0.899)$
s^k_1: Fair	$(0.726, 0.899, 1.113, 1.377)$
s^k_2: Slightly Good	$(1.113, 1.377, 1.704, 2.110)$
s^k_3: Good	$(1.704, 2.110, 2.611, 3.232)$
s^k_4: Very Good	$(2.611, 3.232, 4.000, 4.000)$

Function f reflects the expert's tendency of risk, the curves of function f in accordance with three kinds of preference of experts are shown in Fig. 3, respectively. In the case (I), the expert is a risk lover. In the case (II), the attitude of the expert is neutral to the risk. In the case (III), the expert is averse to risk. Generally, function f can be given by experts in accordance with personal risk preference. For example, they may select $f(t_k) = (t_k)^{\theta}$, and using $\theta > 1$, $\theta = 1$ and $0 < \theta < 1$ corresponding to the case (I), (II) and (III), respectively. As such, we can transform multiplicative linguistic information into trapezoidal fuzzy numbers by Eqs. (8) and (9). Assume that an expert selects function $f(t_k) = t_k$, then seven labels in $S^k = \{s^k_{\frac{1}{4}}, s^k_{\frac{1}{3}}, s^k_{\frac{1}{2}}, s^k_1, s^k_2, s^k_3, s^k_4\}$ $(t_k = 4)$ can be expressed as the corresponding trapezoidal fuzzy numbers shown in Table 1. Moreover, we can get sketch map in Fig. 4 as follows.

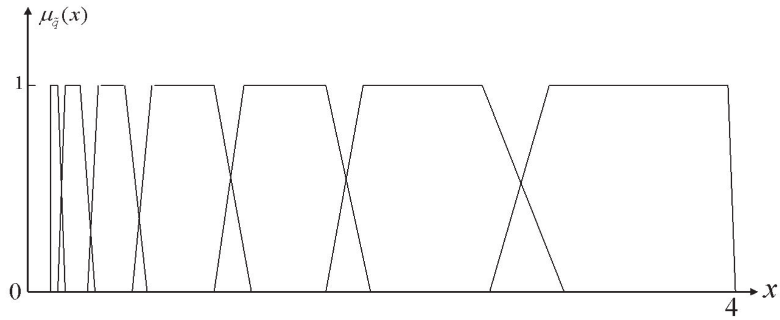

Fig. 4. Three curves corresponding to decision maker's preference of risk

Theorem 1. *Let $\overline{G}(s_i^k) = (x_i^{k1}, x_i^{k2}, x_i^{k3}, x_i^{k4})$ and $\underline{G}(s_{\frac{1}{i}}^k) = (x_{\frac{1}{i}}^{k1}, x_{\frac{1}{i}}^{k2}, x_{\frac{1}{i}}^{k3}, x_{\frac{1}{i}}^{k4})$ be two trapezoidal fuzzy numbers generated by Eqs.* (8) *and* (9), *$i = 1, 2, \ldots, t_k$, then*

$$x_i^{k1} x_{\frac{1}{i}}^{k4} = 1, \quad x_i^{k2} x_{\frac{1}{i}}^{k3} = 1, \quad x_i^{k3} x_{\frac{1}{i}}^{k2} = 1, \quad x_i^{k4} x_{\frac{1}{i}}^{k1} = 1\,. \tag{10}$$

Proof. By Eqs. (8) and (9), we have

$$x_i^{k1} x_{\frac{1}{i}}^{k4} = f(t_k)^{\frac{4i-7}{4t_k-3}} f(t_k)^{\frac{7-4i}{4t_k-3}} = 1\,,$$

$$x_i^{k2} x_{\frac{1}{i}}^{k3} = f(t_k)^{\frac{4i-5}{4t_k-3}} f(t_k)^{\frac{5-4i}{4t_k-3}} = 1\,,$$

$$x_i^{k3} x_{\frac{1}{i}}^{k2} = f(t_k)^{\frac{4i-3}{4t_k-3}} f(t_k)^{\frac{3-4i}{4t_k-3}} = 1\,,$$

$$x_i^{k4} x_{\frac{1}{i}}^{k1} = \min\{f(t_k)^{\frac{4i-1}{4t_k-3}}, f(t_k)\} \max\left\{f(t_k)^{\frac{1-4i}{4t_k-3}}, \frac{1}{f(t_k)}\right\}.$$

For $i = 1, 2, \ldots, t_k - 1$, we get $0 < \frac{4i-1}{4t_k-3} \le 1$, so we have

$$\min\{f(t_k)^{\frac{4i-1}{4t_k-3}}, f(t_k)\} = f(t_k)^{\frac{4i-1}{4t_k-3}}\,,$$

$$\max\left\{f(t_k)^{\frac{1-4i}{4t_k-3}}, \frac{1}{f(t_k)}\right\} = f(t_k)^{\frac{1-4i}{4t_k-3}}\,.$$

In this case, we have

$$x_i^{k4} x_{\frac{1}{i}}^{k1} = f(t_k)^{\frac{4i-1}{4t_k-3}} f(t_k)^{\frac{1-4i}{4t_k-3}} = 1\,.$$

If $i = t_k$, then we get $\frac{4i-1}{4t_k-3} > 1$, so we have

$$\min\{f(t_k)^{\frac{4i-1}{4t_k-3}}, f(t_k)\} = f(t_k)\,,$$

$$\max\left\{f(t_k)^{\frac{1-4i}{4t_k-3}}, \frac{1}{f(t_k)}\right\} = \frac{1}{f(t_k)}\,.$$

It follows that $x_i^{k4} x_{\frac{1}{i}}^{k1} = f(t_k)\frac{1}{f(t_k)} = 1$

In sum, Theorem 1 is proved. □

Since the scale of linguistic label set S^k is multiplicative, trapezoidal fuzzy numbers $\overline{G}(s_i^k)$ and $\underline{G}(s_{\frac{1}{i}}^k)$ are not additive. Theorem 1 indicates that the changes of $\overline{G}(s_i^k)$ and $\underline{G}(s_{\frac{1}{i}}^k)$ $(i = 1, 2, \ldots, t_k)$ are proportional. This is consistent with the multiplicative characteristics of linguistic label set S^k. In what follows, the results calculated from Eqs. (8) and (9) are called multiplicative trapezoidal fuzzy numbers, and the results operated by Eqs. (4) and (5) from multiplicative trapezoidal fuzzy numbers are also known as multiplicative trapezoidal fuzzy numbers.

4. Ranking of Multiplicative Trapezoidal Fuzzy Numbers

In multi-attribute decision-making problems with multi-granularity multiplicative linguistic information, the expert is asked to select alternatives in the best option. By using Eqs. (8) and (9) in Sec. 3, the expert can transform multi-granularity multiplicative linguistic information into multiplicative trapezoidal fuzzy numbers. Through Eqs. (4) and (5), the new attribute values can be aggregated into a collective value, which also is expressed as the form of multiplicative trapezoidal fuzzy numbers. In order to select the best alternative, the expert needs to rank these collective values in a certain order. Thus, the comparison of multiplicative trapezoidal fuzzy numbers is very important and necessary. Since the trapezoidal fuzzy numbers calculated by Eqs. (8) and (9) are multiplicative, the existing formulae for ranking additive fuzzy numbers are no longer applicable in the multiplicative setting. Lin *et al.*[34] developed a valid method for ranking multiplicative triangular fuzzy numbers based on multiplication operation. In what follows, the ranking formula is extended to the one with multiplicative trapezoidal fuzzy numbers.

Let $\widetilde{q} = (a, b, c, d)$ be multiplicative trapezoidal fuzzy number, the membership function of $\widetilde{q}$ is defined by Eq. (1). Based on the ranking method proposed by Lin *et al.*,[34] we extend their formula to the situation of multiplicative trapezoidal fuzzy numbers, and then the centroid of multiplicative trapezoidal fuzzy number $\widetilde{q}$ can be expressed as

$$C(\widetilde{q}) = e^{\frac{\int_0^{\theta_1} \frac{a^{1-t}d^t - a}{b-a}\lambda(t)dt + \int_{\theta_1}^{\theta_2} \lambda(t)dt + \int_{\theta_2}^{1} \frac{a^{1-t}d^t - d}{c-d}\lambda(t)dt}{\int_0^{\theta_1} \frac{a^{1-t}d^t - a}{b-a}dt + \int_{\theta_1}^{\theta_2} 1dt + + \int_{\theta_2}^{1} \frac{a^{1-t}d^t - d}{c-d}dt}}, \tag{11}$$

where $\lambda(t) = \ln a + t(\ln d - \ln a)$, $\theta_1 = \frac{\ln b - \ln a}{\ln d - \ln a}$ and $\theta_2 = \frac{\ln c - \ln a}{\ln d - \ln a}$.

Equation (11) indicates that $C(\widetilde{q})$ is the weighted average of numbers in closed interval $[a, d]$. The weight function is the membership function of trapezoidal fuzzy number $\widetilde{q}$. Based on the centroid of multiplicative trapezoidal fuzzy number, the ranking order is determined as follows.

Definition 5. Let $\widetilde{q}_1$ and $\widetilde{q}_2$ be two multiplicative trapezoidal fuzzy numbers, then we get

1. If $C(\widetilde{q}_1) > C(\widetilde{q}_2)$, then $\widetilde{q}_1$ is said to be superior to $\widetilde{q}_2$, denoted by $\widetilde{q}_1 \succ \widetilde{q}_2$.
2. If $C(\widetilde{q}_1) = C(\widetilde{q}_2)$, then $\widetilde{q}_1$ is said to be indifferent to $\widetilde{q}_2$, denoted by $\widetilde{q}_1 \sim \widetilde{q}_2$.
3. If $C(\widetilde{q}_1) < C(\widetilde{q}_2)$, then $\widetilde{q}_1$ is said to be inferior to $\widetilde{q}_2$, denoted by $\widetilde{q}_1 \prec \widetilde{q}_2$.

Theorem 2. *Let $\widetilde{q} = (a, b, c, d)$ be a multiplicative trapezoidal fuzzy number, then we have $a \le C(\widetilde{q}) \le d$.*

Proof. Since $\widetilde{q}$ is a multiplicative trapezoidal fuzzy number, we have $\mu_{\widetilde{q}}(x) \ge 0$ and $d > a$. It is clear that

$$\ln a \le \ln a + t(\ln d - \ln a) \le \ln d,\ 0 \le t \le 1\,, \tag{12}$$

then from Eq. (12), it follows that

$$\ln a \int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)dt \le \int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)[\ln a + t(\ln d - \ln a)]dt \le \ln d \int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)dt\,. \tag{13}$$

Equation (13) can be further written as

$$\ln a \le \frac{\int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)[\ln a + t(\ln d - \ln a)]dt}{\int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)dt} \le \ln d\,. \tag{14}$$

Eq. (14) can also be equivalently expressed as

$$a = e^{\ln a} \le e^{\frac{\int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)[\ln a + t(\ln d - \ln a)]dt}{\int_0^1 \mu_{\widetilde{q}}(a^{1-t}d^t)dt}} \le e^{\ln d} = d\,. \tag{15}$$

Thus, we get $a \le C(\widetilde{q}) \le d$. This completes the proof of Theorem 2. □

When trapezoidal fuzzy number (a, b, c, d) reduces to an interval number $[b, c]$, the usual methods for ranking fuzzy numbers reduce to compare the arithmetical mean of endpoints, i.e. $\frac{b+c}{2}$. The following Theorem 3 shows that the centroid of multiplicative interval number $[b, c]$ is equal to $\sqrt{bc}$, which is consistent with its multiplicative characteristics.

Theorem 3. *Let $\widetilde{q} = (a, b, c, d)$ be a multiplicative trapezoidal fuzzy number, then we have $\lim\limits_{\substack{a\to b\\ d\to c}} C(\widetilde{q}) = \sqrt{bc}$.*

Proof. Let $a \to b$ and $d \to c$, then we have $\lim\limits_{\substack{a\to b\\ d\to c}} \theta_1 = \lim\limits_{\substack{a\to b\\ d\to c}} \frac{\ln b - \ln a}{\ln d - \ln a} = 0$, $\lim\limits_{\substack{a\to b\\ d\to c}} \theta_2 = \lim\limits_{\substack{a\to b\\ d\to c}} \frac{\ln c - \ln a}{\ln d - \ln a} = 1$ From Eq. (11), we get

$$\begin{aligned}
\lim_{\substack{a\to b\\ d\to c}} C(\widetilde{q}) &= \lim_{\substack{a\to b\\ d\to c}} e^{\frac{\int_0^{\theta_1} \frac{a^{1-t}d^t - a}{b-a}\lambda(t)dt + \int_{\theta_1}^{\theta_2} \lambda(t)dt + \int_{\theta_2}^1 \frac{a^{1-t}d^t - d}{c-d}\lambda(t)dt}{\int_0^{\theta_1} \frac{a^{1-t}d^t - a}{b-a}dt + \int_{\theta_1}^{\theta_2} 1dt + + \int_{\theta_2}^1 \frac{a^{1-t}d^t - d}{c-d}dt}} \\
&= e^{\frac{\lim\limits_{\substack{a\to b\\ d\to c}} \int_0^1 \lambda(t)dt}{\int_0^1 1dt}} = e^{\int_0^1 [\ln b + t(\ln c - \ln b)]dt} = e^{\frac{\ln b + \ln c}{2}} = \sqrt{bc}\,.
\end{aligned}$$

Therefore, Theorem 3 is proved. □

In the above Theorem 3, it is clear that $C(\widetilde{q})$ is a general case of geometric mean by considering the weight of numbers in closed interval $[a, d]$.

5. The Proposed Approach

Based on the above analysis, we shall develop a new approach to deal with multi-attribute group decision-making problems with multi-granularity multiplicative linguistic information as follows.

For the multi-attribute group decision-making problems, in which the attribute values take the form of multi-granularity multiplicative linguistic information. Let $H = \{H_1, H_2, \ldots, H_l\}$ be a finite set of alternatives, let $A = \{A_1, A_2, \ldots, A_n\}$ be the set of attributes, and let $E = \{E_1, E_2, \ldots, E_m\}$ be a finite set of experts with $m > 1$. We assume that expert E_k uses multiplicative linguistic label set $S^k = \{s^k_{\frac{1}{t_k}}, \ldots, s^k_{\frac{1}{2}}, s^k_1, s^k_2, \ldots, s^k_{t_k}\}$ to express his/her preference. $\overline{S} = \{S^k \mid k \in \{1, 2, \ldots, m\}\}$ is the multi-granularity multiplicative linguistic label set. It is worthy to point out that experts may use the same multiplicative linguistic label set to express their preference. Namely, for any $k, q \in \{1, 2, \ldots, m\}$, $k \neq q$ is not necessarily equivalent to $S^k \neq S^q$. Suppose that $R^k = (r^k_{ij})_{l\times n}$ is the decision matrix, r^k_{ij} indicates the preference that the alternative H_i satisfies the attribute A_j by the expert E_k, where $r^k_{ij} \in S^k$, $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$. The problem concerned in this study is to select the most desirable alternative among a finite set $H = \{H_1, H_2, \ldots, H_l\}$ based on decision matrices $R^1, R^2, \ldots, R^m$.

It is difficult to process the multi-granularity multiplicative linguistic information in the direct way, because the multiplicative linguistic label set is not continuous, which may produce a loss of information and hence a lack of validity in the final results. In order to process multiplicative linguistic information in decision matrices $R^1, R^2, \ldots, R^m$, the elements of decision matrix R^k are expressed in the form of multiplicative trapezoidal fuzzy numbers by Eqs. (8) and (9), $k = 1, 2, \ldots, m$. Furthermore, we denote new decision matrix as $\widetilde{P}^k = (\widetilde{p}^k_{ij})_{l\times n}$ where $\widetilde{p}^k_{ij}$ represent trapezoidal fuzzy numbers with $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$. $\widetilde{P}^k$ is called a multiplicative trapezoidal fuzzy number decision matrix, $k = 1, 2, \ldots, m$.

In process of multi-attribute group decision-making, the individual decision information given by experts need to be aggregated into a collective opinion. The ordered weighted geometric mean (OWG) operator was introduced by Herrera,[35] which is a useful tool in aggregating a finite collection arguments, and can be defined as follows:

Definition 6. An OWG operator of dimension n is a mapping OWG: $\mathbf{R}^n \to \mathbf{R}$, defined by an associated weight vector $V = (v_1, v_2, \ldots, v_n)^T$, such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, according to the following formula:

$$\mathrm{OWG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} (a_{\eta(i)})^{v_i}\,, \tag{16}$$

where $\eta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation such that $a_{\eta(i)} \geq a_{\eta(i+1)}$, for all $i = 1, 2, \ldots, n-1$. Namely, $a_{\eta(i)}$ is the i-th largest element of the collection of real numbers $a_1, a_2, \ldots, a_n$ and $\mathbf{R}$ is the set of real numbers. Moreover,

$V = (v_1, v_2, \ldots, v_m)$ can be calculated by[35]

$$v_k = g\left(\frac{k}{m}\right) - g\left(\frac{k-1}{m}\right), \quad k = 1, 2, \ldots, m\,. \tag{17}$$

where g is a Basic Unit-interval Monotonic (BUM) function,[35] and can be expressed as

$$g(r) = \begin{cases} 0, & r \le a \\ \frac{r-a}{b-a}, & a \le r \le b \\ 1, & r \ge b \end{cases} \tag{18}$$

where $a, b \in \mathbf{R}$ (the set of real numbers), and (a, b) should be selected with respect to some fuzzy linguistic quantifier, such as "as many as possible", "at least half", and "more" are corresponding to two-dimensional arrays (0.5,1), $(0, 0.5)$ and $(0.3, 0.8)$, respectively.

Based on the OWG operator and operational laws Eqs. (4) and (5), all the multiplicative trapezoidal fuzzy numbers decision matrices $\widetilde{P}^1, \widetilde{P}^2, \ldots, \widetilde{P}^m$ are aggregated into the collective multiplicative trapezoidal fuzzy number decision matrix $\widetilde{P} = (\widetilde{p}_{ij})_{l\times n}$, shown as follows:

$$\widetilde{p}_{ij} = (\widetilde{p}_{ij}^1)^{v_{\eta^{-1}(1)}} \otimes (\widetilde{p}_{ij}^2)^{v_{\eta^{-1}(2)}} \otimes \cdots \otimes (\widetilde{p}_{ij}^m)^{v_{\eta^{-1}(m)}}, \quad i = 1, 2, \ldots, l; j = 1, 2, \ldots, n\,. \tag{19}$$

where η^{-1}: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the inverse permutation of η, and $\widetilde{p}_{ij}^k$ is the $\eta^{-1}(k)$-th largest element of the collection of multiplicative trapezoidal fuzzy numbers $\widetilde{p}_{ij}^1, \widetilde{p}_{ij}^2, \ldots, \widetilde{p}_{ij}^m$, $k = 1, 2, \ldots, m$. In the process, the order of $\widetilde{p}_{ij}^1, \widetilde{p}_{ij}^2, \ldots, \widetilde{p}_{ij}^m$ can be determined corresponding to the ranking of real numbers $C(\widetilde{p}_{ij}^1), C(\widetilde{p}_{ij}^2), \ldots, C(\widetilde{p}_{ij}^m)$ by Eq. (11).

In order to facilitate future discussions, the similarity degree between each two multiplicative trapezoidal fuzzy numbers is proposed as follows.

Definition 7. Let $\widetilde{q}_1 = (a_1, a_2, a_3, a_4)$ and $\widetilde{q}_2 = (b_1, b_2, b_3, b_4)$ be two multiplicative trapezoidal fuzzy numbers, then the similarity degree between $\widetilde{q}_1$ and $\widetilde{q}_2$ is defined as follows.

$$\rho(\widetilde{q}_1, \widetilde{q}_2) = \left(\prod_{i=1}^{4} \frac{\min\{a_i, b_i\}}{\max\{a_i, b_i\}}\right)^{\frac{1}{4}}. \tag{20}$$

Obviously, the greater the value of $\rho(\widetilde{q}_1, \widetilde{q}_2)$ is, the more similarity between $\widetilde{q}_1$ and $\widetilde{q}_2$ will be.

Theorem 4. *Let $\widetilde{q}_1$ and $\widetilde{q}_2$ be two multiplicative trapezoidal fuzzy numbers, then we have*

1. $0 < \rho(\widetilde{q}_1, \widetilde{q}_2) \le 1$;
2. $\widetilde{q}_1 = \widetilde{q}_2$ *if and only if* $\rho(\widetilde{q}_1, \widetilde{q}_2) = 1$;
3. $\rho(\widetilde{q}_1, \widetilde{q}_2) = \rho(\widetilde{q}_2, \widetilde{q}_1)$.

Proof. 1. Since $a_i, b_i > 0$, $i = 1, 2, 3, 4$, we have

$$0 < \min\{a_1, b_1\} \leq \max\{a_1, b_1\}\,. \tag{21}$$

Equation (21) can be further written as

$$0 < \frac{\min\{a_1, b_1\}}{\max\{a_1, b_1\}} \leq 1\,. \tag{22}$$

Similarly, we get $0 < \frac{\min\{a_2,b_2\}}{\max\{a_2,b_2\}} \leq 1$, $0 < \frac{\min\{a_3,b_3\}}{\max\{a_3,b_3\}} \leq 1$, $0 < \frac{\min\{a_4,b_4\}}{\max\{a_4,b_4\}} \leq 1$. In sum, we have

$$0 < \left(\prod_{i=1}^{4} \frac{\min\{a_i, b_i\}}{\max\{a_i, b_i\}}\right)^{\frac{1}{4}} \leq 1\,. \tag{23}$$

That is, $0 < \rho(\widetilde{q}_1, \widetilde{q}_2) \leq 1$.

2. If $\widetilde{q}_1 = \widetilde{q}_2$, then we get $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4$. It is clear that

$$\frac{\min\{a_1, b_1\}}{\max\{a_1, b_1\}} = \frac{\min\{a_2, b_2\}}{\max\{a_2, b_2\}} = \frac{\min\{a_3, b_3\}}{\max\{a_3, b_3\}} = \frac{\min\{a_4, b_4\}}{\max\{a_4, b_4\}} = 1\,. \tag{24}$$

So, we have $\rho(\widetilde{q}_1, \widetilde{q}_2) = 1$.

3. It is straightforward and thus omitted.

In sum, Theorem 4 is proved. □

The maximizing deviation method is developed to solve the multi-attribute decision-making problems with accurate information.[36] According to the information theory,[37] for a multi-attribute decision-making problem, if one attribute has greater different attribute values across alternatives, such an attribute may play an important role in selecting the most desirable alternative, so the attribute which makes greater deviation should be evaluated a bigger weight. Especially, if the values of an attribute among all the alternatives is equal, then such an attribute will be judged unimportant by experts, and such an attribute should be assigned a very small weight. The maximizing deviation method has been extended to solve decision-making problems under uncertain and fuzzy environment.[38,39] Motivated by the idea of the deviation method, we proposed a similarity-based method to obtain the weight vector of attributes based on multiplication operation and programming theorem. For the attribute $A_j \in A$, the similarity of alternative H_i to all the other alternatives can be expressed as

$$\rho_{ij}(W) = \prod_{k=1}^{l} (\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}))^{w_j}, \quad i = 1, 2, \ldots, l; j = 1, 2, \ldots, n\,. \tag{25}$$

where $\widetilde{p}_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})$ is a multiplicative trapezoidal fuzzy number, and $W = (w_1, w_2, \ldots, w_n)$ is the weight vector of the attributes. Moreover, let

$$\rho_j(W) = \prod_{i=1}^{l} \rho_{ij}(W) = \prod_{i=1}^{l} \prod_{k=1}^{l} (\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}))^{w_j}, \quad j = 1, 2, \ldots, n\,. \tag{26}$$

then $\rho_j(W)$ is the similarity value of all alternatives to other alternatives for the attribute A_j. The Eq. (26) can be further written as

$$\rho_j(W) = \prod_{i=1}^{l}\prod_{k=1}^{l}(\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}))^{w_j} = \prod_{i=1}^{l}\prod_{k=1}^{l}\left(\prod_{r=1}^{4}\frac{\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}{\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}\right)^{\frac{w_j}{4}}. \tag{27}$$

As to the analysis above, we should select the weight vector W to minimize similarity values for all the attributes. To do this, the following two cases are analyzed.

Case 1. If the information about attribute weights is partly known, let L be a set of the partly known weight information, namely, $W \in L$. Thus, we can establish the following multiple objective non-linear programming models to calculate the weight vector $W = (w_1, w_2, \ldots, w_n)$.

(M-1)

$$\min \rho_j(W) = \prod_{i=1}^{l}\prod_{k=1}^{l}\left(\prod_{r=1}^{4}\frac{\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}{\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}\right)^{\frac{w_j}{4}}$$

$$\text{S.t. } W \in L,\ \sum_{j=1}^{n} w_j = 1, w_j \geq 0, j = 1, 2, \ldots, n\,.$$

Since $\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}) > 0$ for all $i, k = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$, and each attribute is fair without any additional preference, then we can aggregate the above multiple objective non-linear programming models into a single objective non-linear programming model as follows.

(M-2)

$$\min Z = \prod_{j=1}^{n}\prod_{i=1}^{l}\prod_{k=1}^{l}\left(\prod_{r=1}^{4}\frac{\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}{\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}\right)^{\frac{w_j}{4}}$$

$$\text{S.t. } W \in L,\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq 0, j = 1, 2, \ldots, n\,.$$

It is well-known that programming model with non-linear objective is more difficult to be solved than the one with linear objective, so the above model (M-2) can be further expressed and simplified as

(M-3)

$$\min J = \sum_{j=1}^{n}\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})]$$

$$\text{S.t. } W \in L,\ \sum_{j=1}^{n} w_j = 1, w_j \geq 0, j = 1, 2, \ldots, n\,.$$

About the above models (M-2) and (M-3), there exist the following theorem.

Theorem 5. *Let $W^* = (w_1^*, w_2^*, \ldots, w_n^*)$ be the optimal solution of model (M-3), then W^* is also the optimal solution of model (M-2).*

Proof. Let J^* be the objective value of model (M-3), $W^* = (w_1^*, w_2^*, \ldots, w_n^*)$ be the optimal solution of model (M-3). Thus, W^* satisfies the constrains of model (M-3), namely, $W^* \in L$, $\sum_{j=1}^{n} w_j^* = 1$, $w_j^* \geq 0$, $j = 1, 2, \ldots, n$. That is, W^* is also a feasible solution of model (M-2). It is clear that

$$J = \sum_{j=1}^{n}\sum_{i=1}^{l}\sum_{k=1}^{l} w_j \ln \rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}) = \ln \prod_{j=1}^{n}\prod_{i=1}^{l}\prod_{k=1}^{l} (\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}))^{w_j} = \ln Z\,. \quad (28)$$

Moreover, $\ln x$ is strictly monotonic increasing for $x > 0$, so e^{J^*} is the objective value of model (M-2). Thus, W^* is the optimal solution of model (M-2). Therefore, Theorem 5 is proved. □

The optimal solution of model (M-2) can be obtained by solving the model (M-3) based on Theorem 1. Thus, the attribute weights are determined.

Case 2. If the information about weight is completely unknown, we can establish the following multiple objective non-linear programming models to calculate the weight vector $W = (w_1, w_2, \ldots, w_n)$.

(M-4)

$$\min \rho_j(W) = \prod_{i=1}^{l}\prod_{k=1}^{l} \left(\prod_{r=1}^{4} \frac{\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}{\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\}} \right)^{\frac{w_j}{4}}$$

$$\text{S.t.} \ \sum_{j=1}^{n} w_j^2 = 1, w_j \geq 0, j = 1, 2, \ldots, n\,.$$

Since $\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}) > 0$ for all $i, k = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$, and each attribute is fair without any additional preference, the above multiple objective non-linear programming models can be aggregated into a single objective non-linear programming model as follows.

(M-5)

$$\min Z = \prod_{j=1}^{n}\prod_{i=1}^{l}\prod_{k=1}^{l} \left(\prod_{r=1}^{4} \frac{\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}}{\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\}} \right)^{\frac{w_j}{4}}$$

$$\text{S.t.} \ \sum_{j=1}^{n} w_j^2 = 1, w_j \geq 0, j = 1, 2, \ldots, n\,.$$

To simplify the model (M-5), the above model can be further expressed as
(M-6)

$$\min J = \sum_{j=1}^{n}\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})]$$

$$\text{S.t. } \sum_{j=1}^{n} w_j^2 = 1,\ w_j \geq 0, j = 1, 2, \ldots, n\,.$$

About the above models (M-5) and (M-6), there exist the following theorem.

Theorem 6. *Let* $W^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})$ *be the optimal solution of model* (M-6), *then* W^{**} *is also the optimal solution of model* (M-5).

Proof. It is similar to the proof of Theorem 5 and thus omitted. □

The model (M-6) is easy to be solved, so we can get the optimal solution of model (M-5) by solving model (M-6) based on the Theorem 6. To solve the model (M-6), the Lagrange function is constructed as

$$L(W, \sigma) = \sum_{j=1}^{n}\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})] + \sigma\left(1 - \sum_{j=1}^{n} w_j^2\right) \tag{29}$$

where σ is the Lagrange multiplier, we differentiate Eq. (29) with respect to w_j, $j = 1, 2, \ldots, n$, and set these partial derivatives equal to zero, then we get a set of equations as follows.

$$\begin{cases} \dfrac{\partial L}{\partial w_j} = \displaystyle\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})] - 2\sigma w_j = 0\,, \\ \dfrac{\partial L}{\partial \sigma} = 1 - \displaystyle\sum_{j=1}^{n} w_j^2 = 0\,. \end{cases} \tag{30}$$

By solving the above set of equations, we obtain the optimal solution $W^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})$ of model (M-6), shown as follows.

$$w_j^{**} = \frac{-\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})]}{\sqrt{\sum_{j=1}^{n}(\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})])^2}}\,. \tag{31}$$

If $\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}) = 1$ for all $i, k = 1, 2, \ldots, l$; $j = 1, 2, \ldots, n$, that is, all alternatives have similar attribute values, then the weight vector of attributes should be assigned as

$W = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. In general, if we suppose $\rho(\widetilde{p}_{ij}, \widetilde{p}_{kj}) \neq 1$, then the corresponding weight $w_j^{**} > 0$. Accordingly, the normalized attribute weights can be obtained by normalizing Eq. (31), shown as follows.

$$w_j = \frac{\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})]}{\sum_{j=1}^{n}\sum_{i=1}^{l}\sum_{k=1}^{l}\sum_{r=1}^{4}\frac{w_j}{4}[\ln(\min\{a_{ij}^{(r)}, a_{kj}^{(r)}\}) - \ln(\max\{a_{ij}^{(r)}, a_{kj}^{(r)}\})]}\,. \tag{32}$$

In such case, we have $\sum_{j=1}^{n} w_j = 1$, $w_j \geq 0$, $j = 1, 2, \ldots, n$. Thus, we obtain a simple formula for determining the attribute weights.

Moreover, if we can use the decision information given in matrix $\widetilde{P}$, then the collective overall preference value $\widetilde{d}_i$ of alternative H_i is derived as

$$\widetilde{d}_i = (\widetilde{p}_{ij})^{w_1} \otimes (\widetilde{p}_{ij})^{w_2} \otimes \cdots \otimes (\widetilde{p}_{ij})^{w_n}, \quad i = 1, 2, \ldots, l\,. \tag{33}$$

Finally, the centroid of $\widetilde{d}_i$ is calculated by Eq. (11), then we rank these collective overall preference values $\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_l$ in descending order in accordance with the values of $C(\widetilde{d}_1), C(\widetilde{d}_2), \ldots, C(\widetilde{d}_l)$, and select the desirable alternative(s) among the finite set of alternatives H.

In sum, we develop an algorithm for multi-attribute group decision-making problems with multi-granularity multiplicative linguistic information, which can be stated as follows:

Step 1: Set up decision matrix R^k based on multiplicative linguistic label set S^k by expert E_k, $k = 1, 2, \ldots, m$.

Step 2: Choose the function f according to the experts' tendency of risk, respectively. Though Eqs. (8) and (9), decision matrices $R^1, R^2, \ldots, R^m$ are transformed into multiplicative trapezoidal fuzzy number decision matrices $\widetilde{P}^1, \widetilde{P}^2, \ldots, \widetilde{P}^m$, respectively.

Step 3: Utilize the Eq. (17) to derive the OWG associated weight vector V.

Step 4: According to Eq. (19), the collective multiplicative trapezoidal fuzzy number decision matrix $\widetilde{P}$ is calculated .

Step 5: If the information about the attribute weights is partly known, then the attribute weights are obtained by solving the model (M-3). If the information about the attribute weights is completely unknown, then we can calculate the attribute weights by using Eq. (32).

Step 6: The collective overall preference values $\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_l$ are calculated by Eq. (33).

Step 7: The centroid of $\widetilde{d}_i$ is calculated by Eq. (11), $i = 1, 2, \ldots, l$.

Step 8: Determine the ranking order of all alternatives in accordance with the values of $C(\widetilde{d}_1), C(\widetilde{d}_2), \ldots, C(\widetilde{d}_l)$.

6. Illustrative Example and Comparison

In this section, we use a numerical example to illustrate the application of the developed approach. The comparison of the proposed approach and Fan's approach is also investigated.

Let us suppose there is a ministry of personnel, which wants to select a accountant-general in the best option. After preliminary screening, there is a panel with three possible applicants to be chosen: H_1, H_2, H_3. The ministry must make a decision according to the following four attributes:

A_1 : Professional skills
A_2 : Educational background
A_3 : Work ethic
A_4 : Emotional intelligence

Three experts E_1, E_2 and E_3 have been invited to conduct the evaluation, and suppose that the expert E_k uses the following set S^k to express his individual preference information on alternatives under the above four attributes, $k = 1, 2, 3$.

S^1 = {Very Poor, Poor, Fair, Good, Very Good}
S^2 = {Very Poor, Poor, Slightly Poor, Fair, Slightly Good, Good, Very Good}
S^3 = {Extremely Poor, Very Poor, Poor, Slightly Poor, Fair, Slightly Good, Good, Very Good, Extremely Good}

Obviously, the cardinality of S^1, S^2 and S^3 are equal to 5, 7 and 9, respectively.

(1) Result calculated by the proposed method

Step 1: In the proposed method, we suppose that the experts use the multiplicative linguistic label sets to express his individual preference information. Accordingly, we have

$$S^1 = \{s^1_{\frac{1}{3}}, s^1_{\frac{1}{2}}, s^1_1, s^1_2, s^1_3\}\,,$$

$$S^2 = \{s^2_{\frac{1}{4}}, s^2_{\frac{1}{3}}, s^2_{\frac{1}{2}}, s^2_1, s^2_2, s^2_3, s^2_4\}\,,$$

$$S^3 = \{s^3_{\frac{1}{5}}, s^3_{\frac{1}{4}}, s^3_{\frac{1}{3}}, s^3_{\frac{1}{2}}, s^3_1, s^3_2, s^3_3, s^3_4, s^3_5\}\,.$$

Three experts express their individual preference information on alternatives under four attributes using multiplicative linguistic label sets S^1, S^2 and S^3 respectively. Then, decision matrices R^1, R^2 and R^3 are established (shown in Tables 2, 3 and 4, respectively).

Step 2: For the purpose of comparing to Fan's method, we assume that the attitude of the experts is neutral to the risk. Then $f(t_k) = t_k$ is selected here. Through Eqs. (8) and (9), decision matrices R^1, R^2 and R^3 are transformed into multiplicative trapezoidal fuzzy number decision matrices $\widetilde{P}^1, \widetilde{P}^2$ and $\widetilde{P}^3$ (shown in Tables 5, 6 and 7, respectively).

Table 2. Decision matrix R^1 assessed from multiplicative linguistic label set S^1.

	A_1	A_2	A_3	A_4
H_1	s^1_2	s^1_1	$s^1_{\frac{1}{2}}$	s^1_3
H_2	s^1_1	$s^1_{\frac{1}{2}}$	s^1_1	s^1_1
H_3	s^1_3	$s^1_{\frac{1}{2}}$	s^1_2	$s^1_{\frac{1}{2}}$

Table 3. Decision matrix R^2 assessed from multiplicative linguistic label set S^2.

	A_1	A_2	A_3	A_4
H_1	s^2_3	$s^2_{\frac{1}{2}}$	s^2_1	s^2_4
H_2	s^2_2	s^2_3	s^2_2	$s^2_{\frac{1}{2}}$
H_3	s^2_2	s^2_1	s^2_3	$s^2_{\frac{1}{4}}$

Table 4. Decision matrix R^3 assessed from multiplicative linguistic label set S^3.

	A_1	A_2	A_3	A_4
H_1	s^3_4	$s^3_{\frac{1}{5}}$	s^3_2	s^3_2
H_2	s^3_2	s^3_5	s^3_1	$s^3_{\frac{1}{3}}$
H_3	s^3_3	s^3_2	s^3_2	$s^3_{\frac{1}{4}}$

Table 5. Multiplicative trapezoidal fuzzy number decision matrices $\widetilde{P}^1$.

	A_1	A_2	A_3	A_4
H_1	(1.130, 1.442, 1.841, 2.350)	(0.693, 0.885, 1.130, 1.442)	(0.426, 0.543, 0.693, 0.885)	(1.841, 2.350, 3.000, 3.000)
H_2	(0.693, 0.885, 1.130, 1.442)	(1.130, 1.442, 1.841, 2.350)	(0.693, 0.885, 1.130, 1.442)	(0.693, 0.885, 1.130, 1.442)
H_3	(1.841, 2.350, 3.000, 3.000)	(0.426, 0.543, 0.693, 0.885)	(1.130, 1.442, 1.841, 2.350)	(0.426, 0.543, 0.693, 0.885)

Table 6. Multiplicative trapezoidal fuzzy number decision matrices $\widetilde{P}^2$.

	A_1	A_2	A_3	A_4
H_1	(1.704, 2.110, 2.611, 3.232)	(0.474, 0.587, 0.726, 0.899)	(0.726, 0.899, 1.113, 1.377)	(2.611, 3.232, 4.000, 4.000)
H_2	(1.113, 1.377, 1.704, 2.110)	(1.704, 2.110, 2.611, 3.232)	(1.113, 1.377, 1.704, 2.110)	(0.474, 0.587, 0.726, 0.899)
H_3	(1.113, 1.377, 1.704, 2.110)	(0.726, 0.899, 1.113, 1.377)	(1.704, 2.110, 2.611, 3.232)	(0.250, 0.250, 0.309, 0.383)

Step 3: Assume $(a, b) = (0.3, 0.8)$ is selected by experts. Using Eq. (17), the OWG associated weight vector $V = (v_1, v_2, v_3)$ is derived as

$$v_1 = g(\tfrac{1}{3}) - g(0) = 0.067; \quad v_2 = g(\tfrac{2}{3}) - g(\tfrac{1}{3}) = 0.667; \quad v_3 = g(1) - g(\tfrac{2}{3}) = 0.266.$$

Table 7. Multiplicative trapezoidal fuzzy number decision matrices $\widetilde{P}^3$.

	A_1	A_2	A_3	A_4
H_1	(2.344, 2.833, 3.424, 4.138)	(0.200, 0.200, 0.242, 0.292)	(1.099, 1.328, 1.605, 1.940)	(1.099, 1.328, 1.605, 1.940)
H_2	(1.099, 1.328, 1.605, 1.940)	(3.424, 4.138, 5.000, 5.000)	(0.753, 0.910, 1.099, 1.328)	(0.353, 0.427, 0.515, 0.623)
H_3	(1.605, 1.940, 2.344, 2.833)	(1.099, 1.328, 1.605, 1.940)	(1.099, 1.328, 1.605, 1.940)	(0.242, 0.292, 0.353, 0.427)

Step 4: According to Eq. (19), the collective multiplicative trapezoidal fuzzy number decision matrix $\widetilde{P}$ is calculated as Table 8.

Table 8. The collective multiplicative trapezoidal fuzzy number decision matrix $\widetilde{P}$.

	A_1	A_2	A_3	A_4
H_1	(1.561, 1.945, 2.423, 3.019)	(0.386, 0.453, 0.558, 0.688)	(0.648, 0.807, 1.006, 1.253)	(1.643, 2.063, 2.590, 2.724)
H_2	(0.973, 1.195, 1.468, 1.803)	(1.601, 1.995, 2.485, 3.057)	(0.756, 0.929, 1.140, 1.400)	(0.450, 0.554, 0.683, 0.842)
H_3	(1.470, 1.794, 2.189, 2.630)	(0.648, 0.807, 1.006, 1.253)	(1.153, 1.447, 1.817, 2.281)	(0.254, 0.292, 0.356, 0.436)

Step 5: Based on Eq. (20), the similarity degrees are calculated below:

$$\begin{aligned}
&\rho(p_{11},p_{11})=1, && \rho(p_{11},p_{21})=0.610, && \rho(p_{11},p_{31})=0.909,\\
&\rho(p_{21},p_{21})=1, && \rho(p_{21},p_{31})=0.671, && \rho(p_{31},p_{31})=1,\\
&\rho(p_{12},p_{12})=1, && \rho(p_{12},p_{22})=0.229, && \rho(p_{12},p_{32})=0.565,\\
&\rho(p_{22},p_{22})=1, && \rho(p_{22},p_{32})=0.406, && \rho(p_{32},p_{32})=1,\\
&\rho(p_{13},p_{13})=1, && \rho(p_{13},p_{23})=0.876, && \rho(p_{13},p_{33})=0.556,\\
&\rho(p_{23},p_{23})=1, && \rho(p_{23},p_{33})=0.635, && \rho(p_{33},p_{33})=1,\\
&\rho(p_{14},p_{14})=1, && \rho(p_{14},p_{24})=0.279, && \rho(p_{14},p_{34})=0.148,\\
&\rho(p_{24},p_{24})=1, && \rho(p_{24},p_{34})=0.531, && \rho(p_{34},p_{34})=1.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\ln\rho(p_{11},p_{11})=0, && \ln\rho(p_{11},p_{21})=-0.494, && \ln\rho(p_{11},p_{31})=-0.095,\\
&\ln\rho(p_{21},p_{21})=0, && \ln\rho(p_{21},p_{31})=-0.399, && \ln\rho(p_{31},p_{31})=0,\\
&\ln\rho(p_{12},p_{12})=0, && \ln\rho(p_{12},p_{22})=-1.474, && \ln\rho(p_{12},p_{32})=-0.571,\\
&\ln\rho(p_{22},p_{22})=0, && \ln\rho(p_{22},p_{32})=-0.901, && \ln\rho(p_{32},p_{32})=0,\\
&\ln\rho(p_{13},p_{13})=0, && \ln\rho(p_{13},p_{23})=-0.132, && \ln\rho(p_{13},p_{33})=-0.587,\\
&\ln\rho(p_{23},p_{23})=0, && \ln\rho(p_{23},p_{33})=-0.454, && \ln\rho(p_{33},p_{33})=0,\\
&\ln\rho(p_{14},p_{14})=0, && \ln\rho(p_{14},p_{24})=-1.277, && \ln\rho(p_{14},p_{34})=-1.911,\\
&\ln\rho(p_{24},p_{24})=0, && \ln\rho(p_{24},p_{34})=-0.633, && \ln\rho(p_{34},p_{34})=0.
\end{aligned}$$

To consider different types of attribute weights given by experts, the following two cases are analyzed.

Case 1. If the information about the attribute weights is partly known and the known weight information is given as follows:

$$L=\{0.1\le w_1\le 0.2, 0.25\le w_2\le 0.35, 0.12\le w_3\le 0.28, 0.22\le w_4\le 0.45\}.$$

Accordingly, the following model is established as

$$\min J = -0.988w_1 - 2.946w_2 - 1.173w_3 - 3.821w_4$$
$$\text{S.t. } W \in L, w_1 + w_2 + w_3 + w_4 = 1, w_i \geq 0, i = 1,2,3,4\,.$$

Then the attribute weights are obtained by solving the above model as:

$$w_1 = 0.150,\ w_2 = 0.350,\ w_3 = 0.280,\ w_4 = 0.220\,.$$

By Eq. (33), $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ are calculated as

$$\widetilde{d}_1 = (\widetilde{p}_{11})^{w_1} \otimes (\widetilde{p}_{12})^{w_2} \otimes (\widetilde{p}_{13})^{w_3} \otimes (\widetilde{p}_{14})^{w_4} = (0.757, 0.925, 1.150, 1.375)\,,$$
$$\widetilde{d}_2 = (\widetilde{p}_{21})^{w_1} \otimes (\widetilde{p}_{22})^{w_2} \otimes (\widetilde{p}_{23})^{w_3} \otimes (\widetilde{p}_{24})^{w_4} = (0.911, 0.977, 1.204, 1.479)\,,$$
$$\widetilde{d}_3 = (\widetilde{p}_{31})^{w_1} \otimes (\widetilde{p}_{32})^{w_2} \otimes (\widetilde{p}_{33})^{w_3} \otimes (\widetilde{p}_{34})^{w_4} = (0.701, 0.857, 1.061, 1.313)\,.$$

Through Eq. (11), the centroid of $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ are calculated as

$$C(\widetilde{d}_1) = 1.028, \quad C(\widetilde{d}_2) = 1.129, \quad C(\widetilde{d}_3) = 0.960\,.$$

It follows that $C(\widetilde{d}_2) > C(\widetilde{d}_1) > C(\widetilde{d}_3)$. The ranking of $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ is $\widetilde{d}_2 \succ \widetilde{d}_1 \succ \widetilde{d}_3$. Namely, the most desirable applicant is H_2.

Case 2. If the information about the attribute weights is completely unknown, then we utilize Eq. (32) to calculate attribute weights as

$$w_1 = 0.111,\ w_2 = 0.330,\ w_3 = 0.131,\ w_4 = 0.428\,.$$

Step 6: By Eq. (33), $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ are calculated as

$$\widetilde{d}_1 = (\widetilde{p}_{11})^{w_1} \otimes (\widetilde{p}_{12})^{w_2} \otimes (\widetilde{p}_{13})^{w_3} \otimes (\widetilde{p}_{14})^{w_4} = (0.897, 1.099, 1.369, 1.580)\,,$$
$$\widetilde{d}_2 = (\widetilde{p}_{21})^{w_1} \otimes (\widetilde{p}_{22})^{w_2} \otimes (\widetilde{p}_{23})^{w_3} \otimes (\widetilde{p}_{24})^{w_4} = (0.798, 0.985, 1.218, 1.499)\,,$$
$$\widetilde{d}_3 = (\widetilde{p}_{31})^{w_1} \otimes (\widetilde{p}_{32})^{w_2} \otimes (\widetilde{p}_{33})^{w_3} \otimes (\widetilde{p}_{34})^{w_4} = (0.513, 0.616, 0.760, 0.937)\,.$$

Step 7: Through Eq. (11), the centroid of $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ are calculated as

$$C(\widetilde{d}_1) = 1.209, \quad C(\widetilde{d}_2) = 1.098, \quad C(\widetilde{d}_3) = 0.692\,.$$

Step 8: Thus, we have $C(\widetilde{d}_1) > C(\widetilde{d}_2) > C(\widetilde{d}_3)$. The ranking of $\widetilde{d}_1, \widetilde{d}_2$ and $\widetilde{d}_3$ is $\widetilde{d}_1 \succ \widetilde{d}_2 \succ \widetilde{d}_3$. Namely, the most desirable applicant is H_1.

(2) Result calculated by Fan's method

Fan and Liu[32] used additive linguistic label sets, and transformed the additive linguistic information into trapezoidal fuzzy numbers. According to their method, additive linguistic label sets S^1, S^2 and S^3 can be expressed as follows:

$$S^1 = \{s_0^1, s_1^1, s_2^1, s_3^1, s_4^1\}$$
$$S^2 = \{s_0^2, s_1^2, s_2^2, s_3^2, s_4^2, s_5^2, s_6^2\}$$
$$S^3 = \{s_0^3, s_1^3, s_2^3, s_3^3, s_4^3, s_5^3, s_6^3, s_7^3, s_8^3\}$$

Table 9. Decision matrix U^1 assessed from additive linguistic label set S^1.

	A_1	A_2	A_3	A_4
H_1	s_3^1	s_2^1	s_1^1	s_4^1
H_2	s_2^1	s_1^1	s_2^1	s_2^1
H_3	s_4^1	s_1^1	s_3^1	s_1^1

Table 10. Decision matrix U^2 assessed from additive linguistic label set S^2.

	A_1	A_2	A_3	A_4
H_1	s_5^2	s_2^2	s_3^2	s_6^2
H_2	s_4^2	s_5^2	s_4^2	s_2^2
H_3	s_4^2	s_3^2	s_5^2	s_0^2

Table 11. Decision matrix U^3 assessed from additive linguistic label set S^3.

	A_1	A_2	A_3	A_4
H_1	s_7^3	s_0^3	s_5^3	s_5^3
H_2	s_5^3	s_8^3	s_4^3	s_2^3
H_3	s_6^3	s_5^3	s_5^3	s_1^3

Three experts express their individual preference information on alternatives under four attributes using additive linguistic label sets S^1, S^2, S^3, respectively. Then, decision matrices U^1, U^2, U^3 are established (shown in Tables 9, 10 and 11, respectively).

Fan[32] changed U^1, U^2 and U^3 into another form, shown in Tables 12, 13 and 14.

Moreover, decision matrices B^1, B^2 and B^3 are expressed in the form of trapezoidal fuzzy number, shown in Tables 15, 16 and 17.

Suppose experts have no preference among two attributes and the information about the attribute weights is completely unknown, then expert E_i should select attribute weight $V^i = (v_1^i, v_2^i, v_3^i, v_4^i) = (s_{i+1}^i, s_{i+1}^i, s_{i+1}^i, s_{i+1}^i)$, $i = 1, 2, 3$. Furthermore, V^1, V^2 and V^3 are also transform into the form of trapezoidal fuzzy number as follows:

$$\widetilde{W}^1 = ((0.333, 0.444, 0.556, 0.667), (0.333, 0.444, 0.556, 0.667), (0.333, 0.444, 0.556, 0.667), (0.333, 0.444, 0.556, 0.667));$$

$$\widetilde{W}^2 = ((0.385, 0.462, 0.538, 0.615), (0.385, 0.462, 0.538, 0.615), (0.385, 0.462, 0.538, 0.615), (0.385, 0.462, 0.538, 0.615));$$

$$\widetilde{W}^3 = ((0.412, 0.471, 0.529, 0.588), (0.412, 0.471, 0.529, 0.588), (0.412, 0.471, 0.529, 0.588), (0.412, 0.471, 0.529, 0.588)).$$

Table 12. Decision matrix B^1 corresponding to alternative H_1.

	A_1	A_2	A_3	A_4
E_1	s_3^1	s_2^1	s_1^1	s_4^1
E_2	s_5^2	s_2^2	s_3^2	s_6^2
E_3	s_7^3	s_0^3	s_5^3	s_5^3

Table 13. Decision matrix B^2 corresponding to alternative H_2.

	A_1	A_2	A_3	A_4
E_1	s_2^1	s_1^1	s_2^1	s_2^1
E_2	s_4^2	s_5^2	s_4^2	s_2^2
E_3	s_5^3	s_8^3	s_4^3	s_2^3

Table 14. Decision matrix B^3 corresponding to alternative H_3.

	A_1	A_2	A_3	A_4
E_1	s_4^1	s_1^1	s_3^1	s_1^1
E_2	s_4^2	s_3^2	s_5^2	s_0^2
E_3	s_6^3	s_5^3	s_5^3	s_1^3

Table 15. Trapezoidal fuzzy number decision matrix $\widetilde{F}^1$ corresponding to alternative H_1.

	A_1	A_2	A_3	A_4
E_1	$(0.556, 0.667, 0.778, 0.889)$	$(0.333, 0.444, 0.556, 0.667)$	$(0.111, 0.222, 0.333, 0.444)$	$(0.778, 0.889, 1.000, 1.000)$
E_2	$(0.692, 0.769, 0.846, 0.923)$	$(0.231, 0.308, 0.385, 0.462)$	$(0.385, 0.462, 0.538, 0.615)$	$(0.846, 0.923, 1.000, 1.000)$
E_3	$(0.765, 0.824, 0.882, 0.941)$	$(0.000, 0.000, 0.059, 0.118)$	$(0.529, 0.588, 0.647, 0.706)$	$(0.529, 0.588, 0.647, 0.706)$

Table 16. Trapezoidal fuzzy number decision matrix $\widetilde{F}^2$ corresponding to alternative H_2.

	A_1	A_2	A_3	A_4
E_1	$(0.333, 0.444, 0.556, 0.667)$	$(0.111, 0.222, 0.333, 0.444)$	$(0.333, 0.444, 0.556, 0.667)$	$(0.333, 0.444, 0.556, 0.667)$
E_2	$(0.538, 0.615, 0.692, 0.769)$	$(0.692, 0.769, 0.846, 0.923)$	$(0.538, 0.615, 0.692, 0.769)$	$(0.231, 0.308, 0.385, 0.462)$
E_3	$(0.529, 0.588, 0.647, 0.706)$	$(0.882, 0.941, 1.000, 1.000)$	$(0.412, 0.471, 0.529, 0.588)$	$(0.176, 0.235, 0.294, 0.353)$

Table 17. Trapezoidal fuzzy number decision matrix $\widetilde{F}^3$ corresponding to alternative H_3.

	A_1	A_2	A_3	A_4
E_1	$(0.778, 0.889, 1.000, 1.000)$	$(0.111, 0.222, 0.333, 0.444)$	$(0.556, 0.667, 0.778, 0.889)$	$(0.111, 0.222, 0.333, 0.444)$
E_2	$(0.538, 0.615, 0.692, 0.769)$	$(0.385, 0.462, 0.538, 0.615)$	$(0.692, 0.769, 0.846, 0.923)$	$(0.000, 0.000, 0.077, 0.154)$
E_3	$(0.647, 0.706, 0.765, 0.824)$	$(0.529, 0.588, 0.647, 0.706)$	$(0.529, 0.588, 0.647, 0.706)$	$(0.059, 0.118, 0.176, 0.235)$

Table 18. The weighted trapezoidal fuzzy matrix $\widetilde{Q}^1$ on alternative H_1.

	A_1	A_2	A_3	A_4
E_1	(0.185, 0.296, 0.433, 0.593)	(0.111, 0.197, 0.309, 0.445)	(0.037, 0.099, 0.185, 0.296)	(0.259, 0.395, 0.556, 0.667)
E_2	(0.266, 0.355, 0.455, 0.568)	(0.089, 0.142, 0.207, 0.284)	(0.148, 0.213, 0.289, 0.378)	(0.326, 0.426, 0.538, 0.615)
E_3	(0.315, 0.388, 0.467, 0.553)	(0.000, 0.000, 0.031, 0.069)	(0.218, 0.277, 0.342, 0.415)	(0.218, 0.277, 0.342, 0.415)

Table 19. The weighted trapezoidal fuzzy matrix $\widetilde{Q}^2$ on alternative H_2.

	A_1	A_2	A_3	A_4
E_1	(0.111, 0.197, 0.309, 0.445)	(0.037, 0.099, 0.185, 0.296)	(0.111, 0.197, 0.309, 0.445)	(0.111, 0.197, 0.309, 0.445)
E_2	(0.207, 0.284, 0.372, 0.473)	(0.266, 0.355, 0.455, 0.568)	(0.207, 0.284, 0.372, 0.473)	(0.089, 0.142, 0.207, 0.284)
E_3	(0.218, 0.277, 0.342, 0.415)	(0.363, 0.443, 0.529, 0.588)	(0.170, 0.222, 0.280, 0.346)	(0.073, 0.111, 0.156, 0.208)

Table 20. The weighted trapezoidal fuzzy matrix $\widetilde{Q}^3$ on alternative H_3.

	A_1	A_2	A_3	A_4
E_1	(0.259, 0.395, 0.556, 0.0667)	(0.037, 0.099, 0.185, 0.296)	(0.185, 0.296, 0.433, 0.593)	(0.037, 0.099, 0.185, 0.296)
E_2	(0.207, 0.284, 0.372, 0.473)	(0.148, 0.213, 0.289, 0.378)	(0.266, 0.355, 0.455, 0.568)	(0.000, 0.000, 0.041, 0.095)
E_3	(0.267, 0.333, 0.405, 0.485)	(0.218, 0.277, 0.342, 0.415)	(0.218, 0.277, 0.342, 0.415)	(0.024, 0.056, 0.093, 0.138)

Therefore, the weighted trapezoidal fuzzy matrices $\widetilde{Q}^1$, $\widetilde{Q}^2$ and $\widetilde{Q}^3$ are derived as shown in Tables 18, 19 and 20, respectively.

Then, fuzzy positive-ideal decision matrix and the fuzzy negative-ideal decision matrix are defined as follows:

$$\widetilde{Q}^+ = (\widetilde{q}_{ij}^+)_{3\times 4}, \quad \widetilde{Q}^- = (\widetilde{q}_{ij}^-)_{3\times 4}\,,$$

where $\widetilde{q}_{ij}^+ = (1,1,1,1)$ and $\widetilde{q}_{ij}^- = (0,0,0,0)$ are trapezoidal fuzzy numbers, $i = 1,2,3; j = 1,2,3,4$. For the sake of convenience, we denote weighted trapezoidal fuzzy matrix as

$$\widetilde{Q}^k = (\widetilde{q}_{ij}^k)_{3\times 4} = ((\widetilde{q}_{ij}^{k(1)}, \widetilde{q}_{ij}^{k(2)}, \widetilde{q}_{ij}^{k(3)}, \widetilde{q}_{ij}^{k(4)}))_{3\times 4}, \quad k = 1,2,3\,.$$

The distance of each alternative from matrices $\widetilde{Q}^+$ and $\widetilde{Q}^-$ can be calculated by the following formulae[31]

$$D_k^+ = \sum_{i=1}^{3}\sum_{j=1}^{4}\left\{\frac{1}{6}[(1-\widetilde{q}_{ij}^{k(1)})^2 + 2(1-\widetilde{q}_{ij}^{k(2)})^2 + 2(1-\widetilde{q}_{ij}^{k(3)})^2 + (1-\widetilde{q}_{ij}^{k(4)})^2]\right\},$$

$$D_k^- = \sum_{i=1}^{3}\sum_{j=1}^{4}\left\{\frac{1}{6}[(\widetilde{q}_{ij}^{k(1)})^2 + 2(\widetilde{q}_{ij}^{k(2)})^2 + 2(\widetilde{q}_{ij}^{k(3)})^2 + (\widetilde{q}_{ij}^{k(4)})^2]\right\}.$$

Thus, we can get

$$D_1^+ = 8.425,\ D_2^+ = 8.690,\ D_3^+ = 8.825\,,$$
$$D_1^- = 3.808,\ D_2^- = 3.522,\ D_3^- = 3.401\,.$$

The closeness coefficients of alternatives are calculated by

$$CC_1 = \frac{D_1^-}{D_1^+ + D_1^-} = 0.311,\ CC_2 = \frac{D_2^-}{D_2^+ + D_2^-} = 0.288,\ CC_3 = \frac{D_3^-}{D_3^+ + D_3^-} = 0.278\,.$$

Table 21. Comparative results of two methods.

	Case 1	Case 2
Fan's method	*None*	$H_1 \succ H_2 \succ H_3$
The proposed method	$H_2 \succ H_1 \succ H_3$	$H_1 \succ H_2 \succ H_3$

The greater CC_K is, the better alternative H_k will be. Therefore, the ranking of closeness coefficients is $CC_1 > CC_2 > CC_3$. Namely, the most desirable applicant is H_1.

In sum, the ranking orders calculated by two methods are shown in Table 21.

In Case 1, the information about the attribute weights is partly known. In Case 2, the information about the attribute weights is completely unknown. When attribute weights is partly known, the advantage of the proposed method is that the attribute weights can be derived from some optimal models. Moreover, the choice of function f in Eqs. (8) and (9) reflects the expert's personal risk preference and hence make the decision-making process more rational. Obviously, when the information about the attribute weights is completely unknown, the ranking orders derived from two methods are consistent. It is worth to point out that there are still some difference among these results. Using $\frac{C(\widetilde{d}_1)}{C(\widetilde{d}_2)}$ and $\frac{CC_1}{CC_2}$ to denote the distinguishability of computational results. Thus, the distinguishability of computational results corresponding to Fan's method and the proposed method are 1.101 and 1.080, respectively. Although the ranking order is all the same, the distinguishability of computational results are changed.

7. Conclusions and Discussion

This paper proposed an approach to multi-attribute group decision-making, in which the attribute values take the form of multi-granularity multiplicative linguistic information. In the approach, multi-granularity multiplicative linguistic labels are transformed into multiplicative trapezoidal fuzzy numbers based on multiplication operation. Then the group decision-making problem with multi-granularity multiplicative linguistic information is changed into the one with multiplicative trapezoidal fuzzy numbers. A similarity-based algorithm is developed to solve the group decision-making problems. Recently, there are many research papers investigating how to transform the linguistic label sets to trapezoidal fuzzy numbers (or triangular fuzzy number).[25,26,31,32,40–44] The main advantages of this paper are shown as follows:

(1) If the multiplicative linguistic labels are processed directly, some information will be lost, because the multiplicative linguistic label set is not continuous. Therefore, we propose two formulae to transform the multiplicative linguistic information into multiplicative trapezoidal fuzzy numbers.

(2) The choice of function f in Eqs. (8) and (9) reflects the expert's personal risk preference and the suitable choice of function f can make the decision-making process more rational.
(3) A formula for ranking multiplicative trapezoidal fuzzy numbers is given based on geometric mean, which is consistent with the characteristics of multiplicative linguistic label set.
(4) Converting the goal of the model (M-2), (M-5) into a linear function can simplify the process of solving the model (M-2), (M-5). Then an exact expression of attribute weights can be solved easily. The distinction of decision-making information can be effectively enhanced by the attribute weights, which is obtained from the similarity-based method.
(5) The risk preference of experts affect the distinguishability of computational results, and thus make some potential variation in decision making.

We will continue working in the extension and application of the developed method to uncertain linguistic environment.

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