第8章 数值积分与数值微分

本章目标：近似计算$I=\int_a^b f(x)\mathrm{d}x$和$f'(x)$

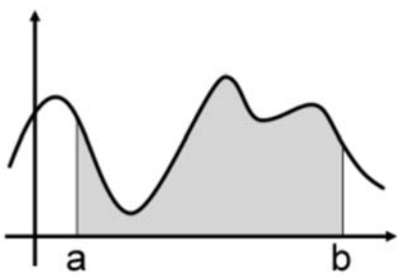
8.1 Newton-Cotes（牛顿-科特斯）公式
8.2 复化求积公式
8.3 自适应步长求积方法：龙贝格求积
8.4 Gauss求积方法

解决两类问题

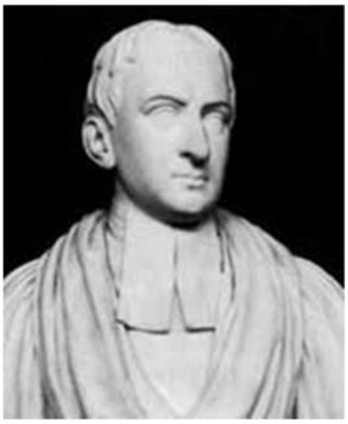
❖ 数据表格

	A	B	C	D
1				
2				
3	Time (s)	Acceleration (in/sec^2)	Velocity (in/s)	Position (in)
4	0	0	0	0
5	0.01	1		
6	0.02	2		
7	0.03	3		
8	0.04	4		
9	0.05	5		
10	0.06	6		
11	0.07	7		
12	0.08	8		

❖ 复杂函数

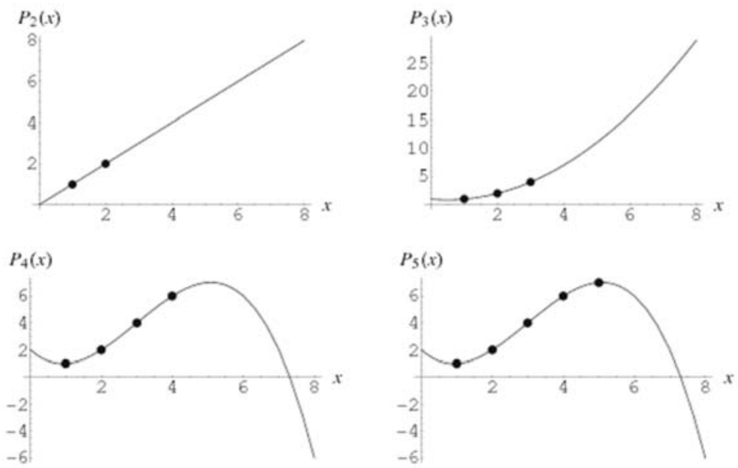


8.1 Newton-Cotes（牛顿-科特斯）公式



Roger Cotes FRS (10 July 1682 – 5 June 1716) was an English mathematician, known for working closely with Isaac Newton by proofreading the second edition of his famous book, the Principia, before publication. He also invented the quadrature formulas known as Newton–Cotes formulas and first introduced what is known today as Euler's formula. He was the first Plumian Professor at Cambridge University from 1707 until his death.

主要思想：拉格朗日插值



$$L(x)=\sum_{j=0}^{k} y_j \ell_j(x)$$

$$\ell_j(x)=\prod_{\substack{0\le m\le k\\ m\ne j}} \frac{x-x_m}{x_j-x_m}=\frac{(x-x_0)}{(x_j-x_0)}\cdots\frac{(x-x_{j-1})}{(x_j-x_{j-1})}\frac{(x-x_{j+1})}{(x_j-x_{j+1})}\cdots\frac{(x-x_k)}{(x_j-x_k)}$$

8.1 Newton-Cotes公式

思路 利用插值多项式 $P_n(x) \approx f(x)$ 则积分易算。

插值型积分公式

☞ 在 $[a, b]$ 上取 $a \le x_0 < x_1 < \ldots < x_n \le b$，做 f 的 n 次插值多项式 $L_n(x) = \sum_{k=0}^{n} f(x_k) l_k(x)$，即得到

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b l_k(x)dx}_{A_k}$$

$$A_k = \int_a^b \prod_{j \ne k} \frac{(x - x_j)}{(x_k - x_j)} \mathrm{d}x$$

由节点 决定，与 $f(x)$ 无关。

误差 $R[f]$

$$= \int_a^b f(x)dx - \sum_{k=0}^{n} A_k f(x_k)$$

$$= \int_a^b [f(x) - L_n(x)]dx = \int_a^b R_n(x)dx$$

$$= \int_a^b \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{k=0}^{n} (x - x_k)\, dx$$

一、梯形公式

用直线代替 $y=f(x)$：

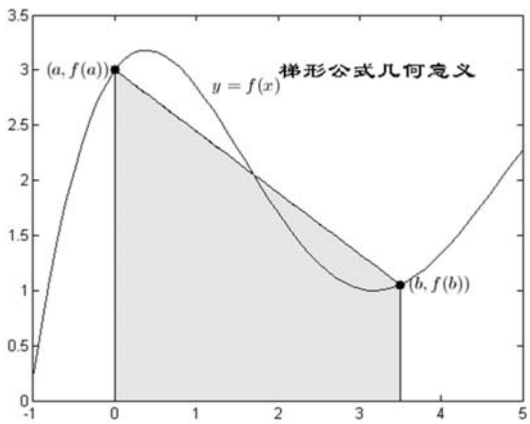
$$A_0 = \int_a^b \frac{x-b}{a-b} dx = \frac{b-a}{2}$$

$$A_1 = \int_a^b \frac{x-a}{b-a} dx = \frac{b-a}{2}$$

$$I = A_0 f(x_0) + A_1 f(x_1)$$

$$= \frac{b-a}{2} f(x_0) + \frac{b-a}{2} f(x_1)$$

$$= \frac{b-a}{2}[f(a) + f(b)]$$

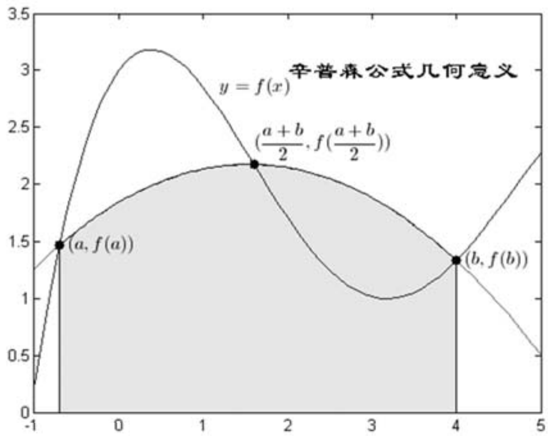


二、Simpson公式

Thomas Simpson FRS (20 August 1710 – 14 May 1761) was a British mathematician and inventor known for the eponymous Simpson's rule to approximate definite integrals. The attribution, as often in mathematics, can be debated: this rule had been found 100 years earlier by Johannes Kepler, and in German it is called Keplersche Fassregel.

二、Simpson公式

利用二次插值多项式近似代替$f(x)$。



$$A_0=\int_a^b \frac{(x-m)(x-b)}{(a-m)(a-b)}dx=\frac{b-a}{6}$$

$$A_1=\int_a^b \frac{(x-a)(x-b)}{(m-a)(m-b)}dx=\frac{4(b-a)}{6}$$

$$A_2=\int_a^b \frac{(x-a)(x-m)}{(b-a)(b-m)}dx=\frac{b-a}{6}$$

$$I=\frac{b-a}{6}f(x_0)+\frac{4(b-a)}{6}f(x_1)+\frac{b-a}{6}f(x_2)$$

$$=\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]$$

三、Cotes公式

在**Newton-Cotes**公式中取

$$n=4, x_0=a, x_1=x_0+h, x_2=x_0+2h, x_3=x_0+3h, x_4=b$$

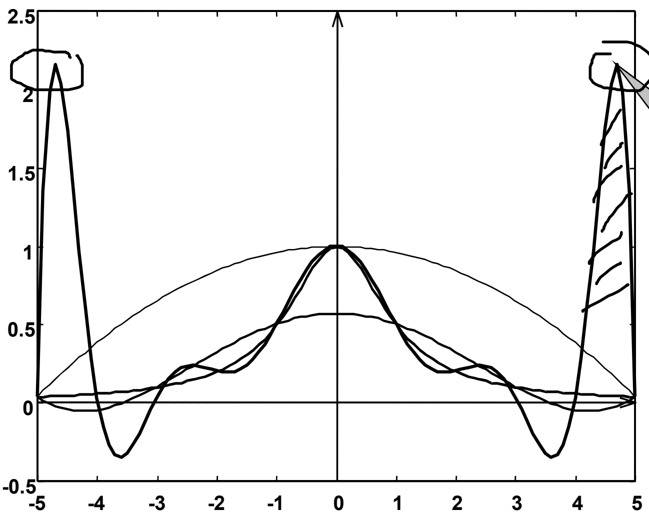
则有

$$A_0=A_4=\frac{7}{90}(b-a),\quad A_1=A_3=\frac{32}{90}(b-a),\quad A_2=\frac{12}{90}(b-a)$$

$$I=\frac{b-a}{90}[7f(a)+32f(\frac{3a+b}{4})+12f(\frac{a+b}{2})+32f(\frac{a+3b}{4})+7f(b)]$$

6.4复化求积公式

在$[-5, 5]$上考察 $f(x)=\dfrac{1}{1+x^2}$的$L_n(x)$。取 $x_i=-5+\dfrac{10}{n}i\quad (i=0,\dots,n)$



$\boxed{L_n(x) \not\to f(x)}$

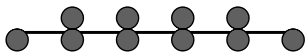
n 越大，
端点附近抖动
越大，称为
Runge 现象

故采用分段低次插值

$\Rightarrow$ 分段低次合成的 ***Newton-Cotes*** 复合求积公式。

一、复化梯形公式： $h=\dfrac{b-a}{n},\ x_k=a+k\,h\quad(k=0,\dots,n)$

在每个 $[x_{k-1},x_k]$ 上用梯形公式：



$$\int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{x_k-x_{k-1}}{2}[f(x_{k-1})+f(x_k)],\quad k=1,\dots,n$$

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n}\frac{h}{2}[f(x_{k-1})+f(x_k)] = \frac{h}{2}\left[f(a)+2\sum_{k=1}^{n-1}f(x_k)+f(b)\right] = T_n$$

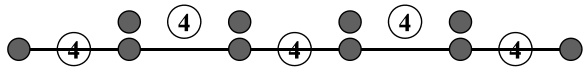
$$R[f]=\sum_{k=1}^{n}[-\frac{h^3}{12}f''(\xi_k)] = -\frac{h^2}{12}(b-a)\frac{\sum_{k=1}^{n}f''(\xi_k)}{n}$$ /*中值定理*/

$$=-\frac{h^2}{12}(b-a)f''(\xi),\quad \xi\in(a,b)$$

二、复化辛普森公式： $h=\dfrac{b-a}{n},\ x_k=a+k\,h\quad(k=0,\dots,n)$

$$\int_{x_k}^{x_{k+1}} f(x)\,dx \approx \frac{h}{6}[f(x_k)+4f(x_{k+\frac{1}{2}})+f(x_{k+1})]$$

x_k $\quad x_{k+\frac{1}{2}}$ $\quad x_{k+1}$



$$\int_a^b f(x)dx \approx \frac{h}{6}[f(a)+4\sum_{k=0}^{n-1}f(x_{k+\frac{1}{2}})+2\sum_{k=0}^{n-1}f(x_{k+1})+f(b)] = S_n$$

$$R[f]=-\frac{b-a}{180}\left(\frac{h}{2}\right)^4 f^{(4)}(\xi)$$

注：为方便编程，可采用另一记法：令 $n'=2n$ 为偶数，

这时 $h'=\dfrac{b-a}{n'}=\dfrac{h}{2},\ x_k=a+k\,h'$，有

$$S_n=\frac{h'}{3}[f(a)+4\sum_{odd\ k}f(x_k)+2\sum_{even\ k}f(x_k)+f(b)]$$

三、复化Cotes公式:

类似于复化梯形公式和复化辛普森公式的推导过程，可以得到复化Cotes公式:

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \frac{h}{90}\left[7f(x_k)+32f(x_{k+\frac{1}{4}})+12f(x_{k+\frac{1}{2}})+32f(x_{k+\frac{3}{4}})+7f(x_{k+1})\right]$$

$$=\frac{h}{90}[7f(a)+32\sum_{k=0}^{n-1} f(x_{k+\frac{1}{4}})+12\sum_{k=0}^{n-1} f(x_{k+\frac{2}{4}})+32\sum_{k=0}^{n-1} f(x_{k+\frac{3}{4}})$$

$$+14\sum_{k=1}^{n-1} f(x_k)+7f(b)]$$

注：为方便编程，可采用另一记法：令 $n' = 4n$ 为偶数，

这时 $h'=\dfrac{b-a}{n'}=\dfrac{h}{4}$, $x_k = a+kh'$，有

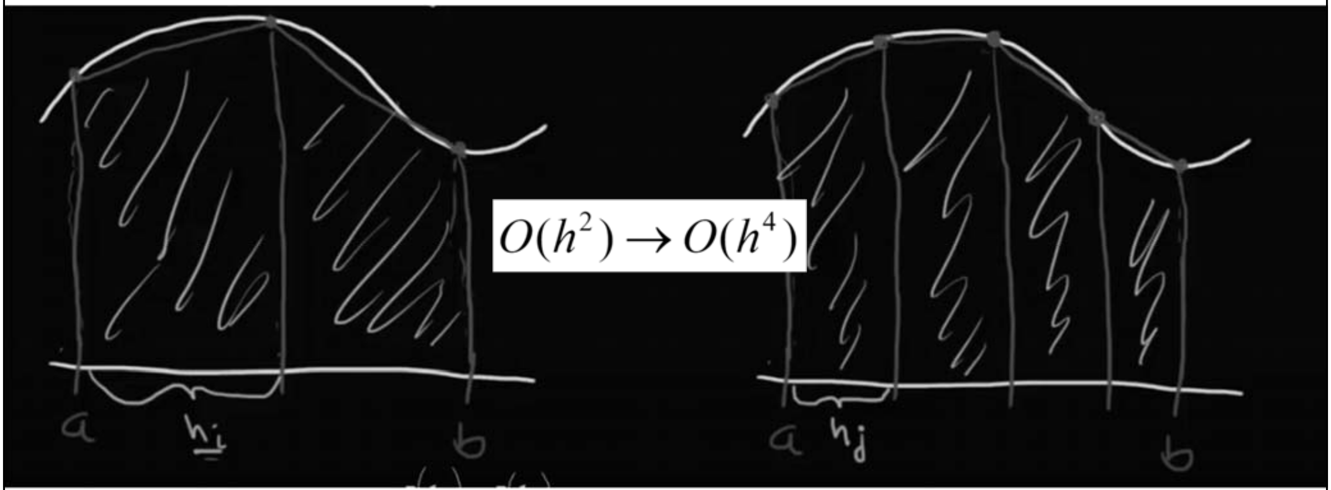
$$I=\frac{h}{90}[7f(a)+32\sum_{k=0}^{n-1} f(x_{4k+1})+12\sum_{k=0}^{n-1} f(x_{4k+2})+32\sum_{k=0}^{n-1} f(x_{4k+3})$$

$$+14\sum_{k=1}^{n-1} f(x_{4k})+7f(b)]$$

8.3 自适应步长求积方法-龙贝格（Romberg）求积

Werner Romberg (born 16 May 1909 in Berlin; died 5 February 2003 in Heidelberg) was a German mathematician and physicist.

8.3 自适应步长求积方法-龙贝格（Romberg）求积



$$I = I(h_j) + E(h_j) \approx I(h_j) + \frac{I(h_j) - I(h_i)}{(\frac{h_i}{h_j})^2 - 1}$$

如果$h_i = 2h_j, I = 4/3I(h_j) - 1/3I(h_i)$

8.3 自适应步长求积方法-龙贝格（Romberg）求积

通用龙贝格积分算法：

$$I_{j,k} \approx \frac{4^{k-1} I_{j+1,k+1} - I_{j,k-1}}{4^{k-1} - 1} \qquad I_{1,2} \approx \frac{4I_{2,1} - I_{1,1}}{3}$$

	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
(a)	0.172800	1.367467		
	1.068800			
(b)	0.172800	1.367467	1.640533	
	1.068800	1.623467		
	1.484800			
(c)	0.172800	1.367467	1.640533	1.640533
	1.068800	1.623467	1.640533	
	1.484800	1.639467		
	1.600800			

8.3 自适应步长求积方法-龙贝格（Romberg）求积

$[a,b]n$等分时：

$$E(h_i)=-\frac{h^2}{12}(b-a)f''(\xi)=-\frac{f''(\xi)}{12n^2}(b-a)^3$$

$[a,b]2n$等分时：

$$E(h_j)=-\frac{f''(\xi)}{12(2n)^2}(b-a)^3$$

$$I=T_n+E(h_i)$$

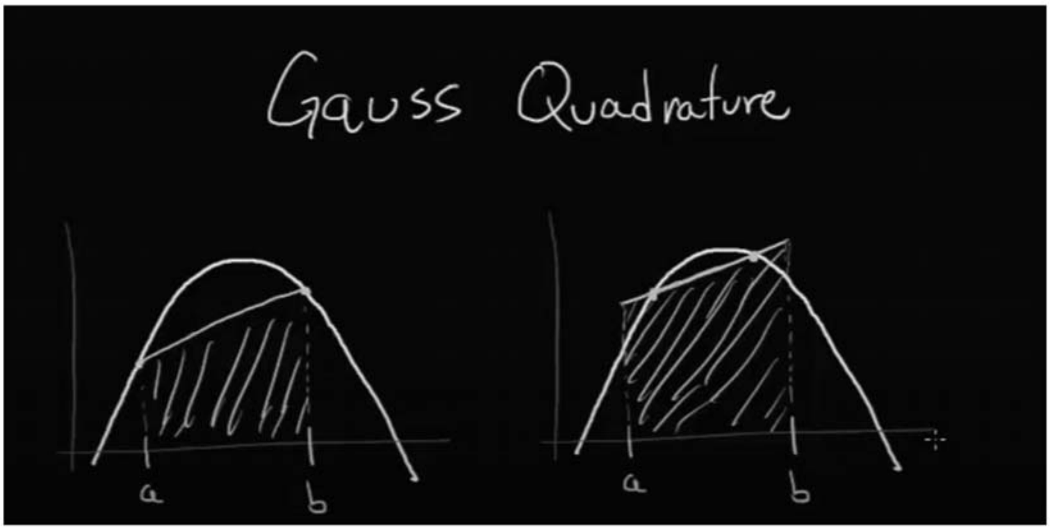
$$I=T_{2n}+E(h_j)$$

则：

$$\frac{I-T_{2n}}{I-T_n}\approx\frac{1}{4}\to 4\left(I-T_{2n}\right)\approx I-T_n$$

$$\to 3I-3T_{2n}\approx T_{2n}-T_n\to I-T_{2n}\approx\frac{1}{3}(T_{2n}-T_n)$$

8.4 Gauss求积方法



$$\int_a^b f(t)dt = w_1 f(t_1) + w_2 f(t_2)$$

$$\int_{-1}^1 f(t)dt = w_1 f(t_1) + w_2 f(t_2)$$

$$\int_{-1}^1 1dt = 2 = w_1 + w_2 \quad \int_{-1}^1 tdt = 0 = w_1 t_1 + w_2 t_2$$

$$\int_{-1}^1 t^2 dt = \frac{2}{3} = w_1 t_1^2 + w_2 t_2^2$$

$$\int_{-1}^1 t^3 dt = 0 = w_1 t_1^3 + w_2 t_2^3$$

$$w_1 = w_2 = 1;$$

$$t_1 = \frac{1}{\sqrt{3}}$$

$$t_2 = -\frac{1}{\sqrt{3}}$$

$$\int_{-1}^1 f(t)dt = f(\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}})$$

$$\int_a^b f(x)dx$$

$$x = \frac{(b-a)t+b+a}{2}, dx = \frac{b-a}{2}dt$$

$$\int_a^b f(x)dt = \frac{b-a}{2}\int_{-1}^{1} f(\frac{(b-a)t+b+a}{2})\frac{b-a}{2}dt$$

$$= \frac{b-a}{2}\int_{-1}^{1} f(\frac{(b-a)t+b+a}{2})dt$$

$$\int_0^{1.2} \sin x dx = 0.637642$$

$$a = 0, b = 1.2 \rightarrow$$

$$x = \frac{1.2t+1.2}{2} = 0.6t + 0.6$$

$$dx = 0.6dt$$

$$0.6\int_{-1}^{1} \sin(0.6t+0.6)dt$$

$$= 0.6\left(\sin(\frac{0.6}{\sqrt{3}}+0.6) + \sin(-\frac{0.6}{\sqrt{3}}+0.6) \right)$$

$$= 0.6373216$$

问：为什么要用$1,t,t^2,t^3$

答：$1,t,t^2,t^3$为线性代数中的三阶标准幂基