The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K-18: Convex Optimization — Fall 2019

Problem Set Zero

Due: Friday, September 6, 2019.

This problem set is intended to get the semester off to a good start and to help you refresh your memory about basic concepts of linear algebra.

Remark: *Any problems marked by a (?) do not need to be turned in, and are just additional review problems, or reading assignments.*

Reading Assignments

1. (?) Linear Algebra is probably the single most important technical tool used in this class. The course text by Boyd and Vandenberghe has a good review in Appendix A. An excellent and quite in depth review of the most relevant topics from Linear Algebra and Analysis can be found in Appendix A of the Lecture Notes by Ben-Tal and Nemirovski. These have been posted on Canvas.

   For example, concepts that will be used repeatedly in this class include:

   (a) Matrices, linear operators, vector spaces.
   (b) Independence, range, null space, etc.
   (c) Eigenvalues/eigenvectors, symmetric matrices, spectral theorem, singular values and singular value decomposition (SVD),
   (d) etc...

   If these topics are not fresh, spend some time learning/reminding yourself of the basic notions.

Linear Algebra Review

1. Vector Spaces: For the following examples, state whether or not they are in fact vector spaces.

   - The set of polynomials in one variable, of degree at most d.
   - The set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(0) = 0$.
   - The set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(1) = 1$.

2. Recall that a linear operator $T : V \to W$ is a map that satisfies:

$$T(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = aT\boldsymbol{v}_1 + bT\boldsymbol{v}_2,$$

   for every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $a, b \in \mathbb{R}$.

   Show which of the following maps are linear operators:

- $T : V \to V$ given by the identity map: $\boldsymbol{v} \mapsto \boldsymbol{v}$.
- $T : V \to W$ given by the constant map: $\boldsymbol{v} \mapsto \boldsymbol{w}_0$ for every $\boldsymbol{v} \in V$. Does your answer change depending on what $\boldsymbol{w}_0$ is?
- Let V be the vector space of polynomials of degree at most d. Let $T : V \to V$ be the map defined by the derivative: $p(x) \mapsto p'(x)$.
- For V as above, let T be given by:

$$T(p) = \int_0^1 p(x)\, dx.$$

- What about

$$T(p) = \int_0^1 p(x)x^3\, dx.$$

3. Independence:

   - If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in V$ are independent, and $T : V \to W$ is a linear operator, is it true that $T\boldsymbol{v}_1, \ldots, T\boldsymbol{v}_m \in W$ are independent?
   - If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in V$ are dependent, and $T : V \to W$ is a linear operator, is it true that $T\boldsymbol{v}_1, \ldots, T\boldsymbol{v}_m \in W$ are dependent?

4. (?) True or False: If vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are elements of a vector space V, and $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, $\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$, and $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ are independent, then the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is also linearly independent.

5. (?) Range and Nullspace of Matrices: Recall the definition of the null space and the range of a linear transformation, $T : V \to W$:

$$\begin{aligned} \text{null}(T) &= \{\boldsymbol{v} \in V \,:\, T\boldsymbol{v} = 0\} \\ \text{range}(T) &= \{T\boldsymbol{v} \in W \,:\, \boldsymbol{v} \in V\} \end{aligned}$$

   Remind yourselves of the Rank-Nullity Theorem.

6. More Range and Nullspace.

   - Suppose A is a 10-by-10 matrix of rank 5, and B is also a 10-by-10 matrix of rank 5. What is the **smallest** and **largest** the rank the matrix $C = AB$ could be?
   - Now suppose A is a 10-by-15 matrix of rank 7, and B is a 15-by-11 matrix of rank 8. What is the **largest** that the rank of matrix $C = AB$ can be?

7. Riesz Representation Theorem: Consider the standard basis for $\mathbb{R}^n$: $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$, etc. Recall that the inner-product of two vectors $\boldsymbol{w}_1 = (\alpha_1, \ldots, \alpha_n), \boldsymbol{w}_2 = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, is given by:

$$\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = \sum_{i=1}^{n} \alpha_i \beta_i.$$

   Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear map. Show that there exists a vector $\boldsymbol{x} \in \mathbb{R}^n$, such that

$$f(\boldsymbol{w}) = \langle \boldsymbol{x}, \boldsymbol{w} \rangle,$$

for any $\boldsymbol{w} \in \mathbb{R}^n$.

Remark: It turns out that this result is true in much more generality. For example, consider the vector space of square-integrable functions (something we will see much more later in the course). Let F denote a linear map from square integrable functions to $\mathbb{R}$. Then, as a consequence similar to the finite dimensional exercise here, there exists a square integrable function, g, such that:

$$F(f) = \int fg.$$

8. Let V be the vector space of (univariate) polynomials of degree at most d. Consider the mapping $T : V \to V$ given by:

$$Tp = a_0 p(t) + a_1 t p^{(1)}(t) + a_2 t^2 p^{(2)}(t) + \cdots + a_d t^d p^{(d)}(t),$$

where $p^{(r)}(t)$ denotes the r^{th} derivative of the polynomial p.

- True or False: if $Tp = 2p(t) - tp'(t)$, then for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.
- What about for T given by $Tp = 2p(t) - 3tp'(t)$?
- Provide a characterization of the set of coefficients $(a_0, a_1, \ldots, a_d)$, such that the operator T they define has the property that for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.

9. Recall the definition of rank, and show the following.

- For A an $m \times n$ matrix, $\mathrm{rank}(A) \leq \min\{m, n\}$.
- For A an $m \times k$ matrix and B a $k \times n$ matrix,

$$\mathrm{rank}(A) + \mathrm{rank}(B) - k \leq \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

- For A and B $m \times n$ matrices,

$$\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B).$$

- For A an $m \times k$ matrix, B a $k \times p$ matrix, and C a $p \times n$ matrix, then

$$\mathrm{rank}(AB) + \mathrm{rank}(BC) \leq \mathrm{rank}(B) + \mathrm{rank}(ABC)$$

10. Consider a mapping $T : V \to V$. If the vector space V is finite dimensional, then if $\mathrm{null}(T) = \{0\}$, T is surjective (also known as onto), that is, for any $\boldsymbol{v} \in V$, there exists $\hat{\boldsymbol{v}}$ such that $T\hat{\boldsymbol{v}} = \boldsymbol{v}$. Conversely, if T is surjective, then $\mathrm{null}(T) = \{0\}$, and $T\boldsymbol{v} = 0$ implies $\boldsymbol{v} = 0$.

- Give an example of an infinite dimensional vector space, V, and a linear operator $T : V \to V$, such that T is surjective, but $\mathrm{null}(T) \neq \{0\}$.
- Give an example of an infinite dimensional vector space, V, and a linear operator $T : V \to V$, such that $\mathrm{null}(T) = \{0\}$, but T is not surjective.

[Hint: consider the space of polynomials of arbitrary degree.]