The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

Lecture 5

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Goal: In this lecture, we introduce the dual of a linear program and explain weak duality. We further use Theorem of Alternatives and Farkas' Lemma to prove strong duality.

1 Dual of linear program in inequality form

Consider the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$ and the following LP (in inequality form)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \tag{1}$$

Note that in this problem the optimization variable $\mathbf{x}$ is in $\mathbb{R}^n$.
Then, the dual problem of the LP in (1) is defined as

$$\begin{aligned} \text{maximize} \quad & -\mathbf{b}^\top \mathbf{z} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{z} + \mathbf{c} = \mathbf{0} \\ & \mathbf{z} \geq \mathbf{0} \end{aligned} \tag{2}$$

Note the variable of the dual problem $\mathbf{z}$ is in $\mathbb{R}^m$.

Observations:

1. The primal problem is in inequality form but the dual problem is in standard form.
2. The number of constraints in the primal is equal to the number of variables in the dual.

How should we derive the dual problem? Later when we cover Lagrange multipliers you will learn how to derive the dual problem directly.

Notation:

1. p^* is the optimal value of the primal problem and d^* is the optimal value of the dual problem.
2. $p^* = +\infty$ if the primal is infeasible
3. $p^* = -\infty$ if the primal is unbounded
4. $d^* = +\infty$ if the dual is unbounded
5. $d^* = -\infty$ if the primal is infeasible

2 Weak Duality of LP

The weak duality in English words means that the optimal value of the primal problem is always larger than or equal to the optimal value of the dual problem. To be more precise, weak duality means that $p^* \geq d^*$.

Lower bound property: If $\mathbf{x}$ is primal feasible and $\mathbf{z}$ is dual feasible, then

$$\mathbf{c}^\top \mathbf{x} \;\geq\; -\mathbf{b}^\top \mathbf{z}$$

Proof: If $\mathbf{x}$ is primal feasible then $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. If $\mathbf{z}$ is dual feasible then $\mathbf{A}^\top \mathbf{z} + \mathbf{c} = \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$. Therefore, since $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{z} \geq \mathbf{0}$ we can write

$$0 \leq \mathbf{z}^\top (\mathbf{b} - \mathbf{A}\mathbf{x}) = \mathbf{b}^\top \mathbf{z} + \mathbf{c}^\top \mathbf{x}$$

where the equality follows from $\mathbf{A}^\top \mathbf{z} + \mathbf{c} = \mathbf{0}$. Hence $\mathbf{c}^\top \mathbf{x} \geq -\mathbf{b}^\top \mathbf{z}$ for any primal feasible $\mathbf{x}$ and dual feasible $\mathbf{z}$.

Weak duality: Since the lower bound property holds for any primal and dual feasible points, if $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{z}^*$ is an optimal solution for the dual problem we can write that

$$p^* \;=\; \mathbf{c}^\top \mathbf{x}^* \;\geq\; -\mathbf{b}^\top \mathbf{z}^* = d^*.$$

3 Strong Duality of LP

The strong duality in English words means that the optimal value of the primal problem is always equal to the optimal value of the dual problem. To be more precise, strong duality means that $p^* = d^*$.

We can also state the strong duality for LPs as the following:
If the primal and dual problems are feasible, then there exist $\mathbf{x}^*$ *and* $\mathbf{z}^*$ *such that*

$$\mathbf{c}^\top \mathbf{x}^* = -\mathbf{b}^\top \mathbf{z}^*, \qquad \mathbf{A}\mathbf{x}^* \leq \mathbf{b}, \qquad \mathbf{A}^\top \mathbf{z}^* + \mathbf{c} = \mathbf{0}, \qquad \mathbf{z}^* \geq \mathbf{0} \tag{3}$$

If these conditions are satisfied, then by the lower bound property we can show that (i) $\mathbf{x}^*$ is primal optimal and $\mathbf{z}^*$ is dual optimal and (i) the primal and dual optimal values are equal as we can show that $p^* = \mathbf{c}^\top \mathbf{x}^* = -\mathbf{b}^\top \mathbf{z}^* = d^*$. **Argument:** [According to the lower bound property $\mathbf{c}^\top \mathbf{x} \;\geq\; -\mathbf{b}^\top \mathbf{z}$ for any primal feasible $\mathbf{x}$ and dual feasible $\mathbf{z}$. Hence, for any primal feasible $\mathbf{x}$ we have $\mathbf{c}^\top \mathbf{x} \;\geq\; -\mathbf{b}^\top \mathbf{z}^*$. Since $\mathbf{c}^\top \mathbf{x}^* = -\mathbf{b}^\top \mathbf{z}^*$ then we have $\mathbf{c}^\top \mathbf{x} \;\geq\; \mathbf{c}^\top \mathbf{x}^*$. This result and the fact that $\mathbf{x}^*$ is primal feasible implies that $\mathbf{x}^*$ is an optimal primal solution. We can use the same argument for $\mathbf{z}^*$. Therefore, $p^* = d^*$.]

Hence, if we prove there exist $\mathbf{x}^*$ and $\mathbf{z}^*$ that satisfy the conditions in (3), then we have proven strong duality! We show this result by exploiting weak duality and Farkas' lemma.

Theorem 1. There exist $\mathbf{x}^* \in \mathbb{R}^n$ and $\mathbf{z}^* \in \mathbb{R}^m$ that satisfy

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \\ \mathbf{c}^\top & \mathbf{b}^\top \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{z}^* \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{0} & -\mathbf{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{z}^* \end{bmatrix} = \mathbf{c} \tag{4}$$

Side note: If we prove this theorem, then there exist $\mathbf{x}^*$ and $\mathbf{z}^*$ such that

$$\mathbf{c}^\top \mathbf{x}^* \leq -\mathbf{b}^\top \mathbf{z}^*, \qquad \mathbf{A}\mathbf{x}^* \leq \mathbf{b}, \qquad \mathbf{A}^\top \mathbf{z}^* + \mathbf{c} = 0, \qquad \mathbf{z}^* \geq \mathbf{0} \tag{5}$$

This along with weak duality $\mathbf{c}^\top \mathbf{x}^* \geq -\mathbf{b}^\top \mathbf{z}^*$ implies that $\mathbf{x}^*$ and $\mathbf{z}^*$ satisfy the conditions in (3).

Proof of Theorem 1: To prove the claim we first define the alternative of (5) using theorem of alternatives and then show that its alternative never holds.
According a side result of the theorem of alternatives exactly one of the following holds

(a) $\exists\ \mathbf{x}^* \in \mathbb{R}^n, \mathbf{z}^* \in \mathbb{R}^m$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \\ \mathbf{c}^\top & \mathbf{b}^\top \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{z}^* \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{0} & -\mathbf{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{z}^* \end{bmatrix} = \mathbf{c}$$

(b) $\exists\ \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^m, t \in R$ such that $\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, t \geq 0$ and

$$\begin{bmatrix} \mathbf{A}^\top & \mathbf{0} & \mathbf{c} \\ \mathbf{0} & -\mathbf{I} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ t \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{A} \end{bmatrix} \mathbf{w} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{b}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ t \end{bmatrix} + \mathbf{c}^\top \mathbf{w} < 0$$

SIDE RESULT that we used:
Exactly one of the following can hold
(i) There exists an $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{C}\mathbf{x} = \mathbf{d}$.
(ii) There exist $\mathbf{y}$ and $\mathbf{z}$ such that $\mathbf{z} \geq \mathbf{0}$, $\mathbf{A}^\top \mathbf{z} + \mathbf{C}^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{z} + \mathbf{d}^\top \mathbf{y} < 0$

The statement in (b) can be simplified as

$$\mathbf{u} \geq \mathbf{0}, \quad \mathbf{v} \geq \mathbf{0}, \quad t \geq 0, \qquad \mathbf{A}^\top \mathbf{u} + t\mathbf{c} = \mathbf{0}, \qquad -\mathbf{v} + t\mathbf{b} - \mathbf{A}\mathbf{w} = \mathbf{0}, \qquad \mathbf{b}^\top \mathbf{u} + \mathbf{c}^\top \mathbf{w} < 0$$

We can eliminate the variable $\mathbf{v}$ and write the equivalent of (b) as

(b') $\exists\ \mathbf{u} \in \mathbb{R}^m, t \in R$ such that

$$\mathbf{u} \geq \mathbf{0}, \quad t \geq 0, \qquad \mathbf{A}^\top \mathbf{u} + t\mathbf{c} = \mathbf{0}, \qquad \mathbf{A}\mathbf{w} \leq t\mathbf{b}, \qquad \mathbf{b}^\top \mathbf{u} + \mathbf{c}^\top \mathbf{w} < 0$$

Therefore, exactly one of the statements in (a) and (b') is true. Now we show that the system of equalities and inequalities in (b') does not have any solution, i.e., (b') never holds.

Case I: If $t > 0$, then defining $\tilde{\mathbf{x}} := \mathbf{w}/t$ and $\tilde{\mathbf{z}} := \mathbf{u}/t$ gives

$$\tilde{\mathbf{z}} \geq \mathbf{0}, \qquad \mathbf{A}^\top \tilde{\mathbf{z}} + \mathbf{c} = \mathbf{0}, \qquad \mathbf{A}\tilde{\mathbf{x}} \leq \mathbf{b}, \qquad \mathbf{c}^\top \tilde{\mathbf{x}} < -\mathbf{b}^\top \tilde{\mathbf{z}}$$

which contradicts with the lower bound property.

Case II: If $t = 0$ and $\mathbf{b}^\top \mathbf{u} < 0$, then $\mathbf{u}$ satisfies

$$\mathbf{u} \geq \mathbf{0}, \qquad \mathbf{A}^\top \mathbf{u} = \mathbf{0}, \qquad \mathbf{b}^\top \mathbf{u} < 0.$$

which contradicts the feasibility of the primal problem (since $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ is its alternate).

Case III: If $t = 0$ and $\mathbf{c}^\top \mathbf{w} < 0$, then $\mathbf{w}$ satisfies

$$\mathbf{A}\mathbf{w} \leq \mathbf{0}, \qquad \mathbf{c}^\top \mathbf{w} < 0.$$

which contradicts the feasibility of the dual problem (since $\mathbf{A}^\top \mathbf{z} + \mathbf{c} = \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$ is its alternate by Farkas' Lemma).

Hence, the statement in (b') never holds and therefore the statement in (b) never holds, which implies that the statement in (a) always holds. This result implies that there exist $\mathbf{x}^* \in \mathbb{R}^n$ and $\mathbf{z}^* \in \mathbb{R}^m$ that satisfy the condition in (4). The proof is complete.

Remark 1. *Strong duality only holds when both primal and dual problems are feasible.*