Lecture 2:

Face: for $J \subseteq \{1, \dots, m\}$, define

$$F_J = \{x \in P \mid a_i^T x = b_i \text{ for } i \in J\}.$$

If F_J is nonempty, then it is called a <u>face</u> of P

English words: face of a polyhedron is a subset points where some inequalities <u>are tight</u>.

example

$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_1 + x_2 + x_3 = 1$

F_J is a nonempty polyhedron, defined by

$$\begin{cases} a_i^T x \leq b_i & \text{for } i \notin J \\ a_i^T x = b_i & \text{for } i \in J \\ Cx = d \end{cases}$$

- faces of F_J are also faces of P
- all faces have the same lineality space as P
- The number of faces is finite and at least one $(P = F_\emptyset$

Minimal face:

A face of P is a <u>minimal face</u> if it doesn't contain another face of P. $(\dim F_J = 0)$

Extreme Point:

A minimal face of a pointed polyhedron is an extreme point.

Rank test: Given $\hat{x} \in P$, is $\hat{x}$ an extreme point?

Define active constraints at $\hat{x}$ as $J(\hat{x})$ where $J(\hat{x})$ shows the sets of inequalities that met equality at $\hat{x}$, i.e.,

- $\underbrace{J(\hat{x}) = \{i_1, \dots, i_k\}}_{\text{index of active constraint}}$ s.t. $\begin{cases} a_i^T \hat{x} = b_i & \text{for } i \in J(\hat{x}) \\ a_i^T x < b_i & \text{for } i \notin J(\hat{x}) \end{cases}$

- $\hat{x}$ is an extreme point if and only if
$$\operatorname{rank}\left(\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix}\right) = n \quad \text{where} \quad A_{J(\hat{x})} = \begin{bmatrix} a_{i_1}^T \\ \vdots \\ a_{i_k}^T \end{bmatrix}$$

a submatrix of A with rows indexed by $\hat{J}(x)$

Proof: The face $F_{J(\hat{x})}$ is defined as the set of points x that satisfy $a_i^T x = b_i \quad i \in J(\hat{x}), \quad a_i^T x \leq b_i \quad i \notin J(\hat{x}), \quad Cx = d$ ①

By definition $x = \hat{x}$ satisfies these conditions.

$\Leftarrow$ • If the rank condition is satisfied, $x = \hat{x}$ is the only point that satisfies ① therefore $F_{J(\hat{x})}$ is a minimal face ($\dim F_{J(\hat{x})} = 0$)

$\Rightarrow$ • If the rank condition doesn't hold, then there exits a $v \neq 0$ such that $A_{J(\hat{x})} v = 0 \quad Cv = 0$

this implies that $x = \hat{x} \pm tv$ satisfies for small positive and negative $t \Rightarrow$ the face $F_{J(\hat{x})}$ is not minimal ($\dim F_{J(\hat{x})} > 0$).

example:

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_1 + x_2 + x_3 = 1$

$\hat{x} = (1,0,0)$ is an extreme point

$$J(\hat{x}) = \{2,3\} \quad \text{rank}\left(\begin{bmatrix} A_{\{2,3\}} \\ C \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\right) = 3$$

Same argument for $(0,1,0)$ & $(0,0,1)$.

$\boxed{\text{Excercise:}}$ consider P as $x \geq 0, \; Cx = d$.

① Show that $\hat{x}$ is an extreme point if $\hat{x} \in P$ and
$\text{rank}([c_{i_1} \; c_{i_2} \; \dots \; c_{i_k}]) = k$ where c_j is column j of C
and $\{i_1, \dots, i_k\} = \{i \mid \hat{x}_i > 0\}$

② Show that an extreme point $\hat{x}$ has at most $\text{rank}(C)$ nonzero elements.

① Without loss of generality assume that constraints $1, \dots, k$ are not active and constraints $k+1, \dots, n$ are active. Then

$$\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}}^{k} & \overbrace{\begin{matrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \vdots \\ 0 & 0 & \ddots & -1 \end{matrix}}^{n-k} \\ \begin{matrix} c_1 & \cdots & c_k \end{matrix} & \begin{matrix} c_{k+1} & \cdots & c_n \end{matrix} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}}\right\} n-k \\ \end{matrix}$$

If rank of $[c_1, \dots c_k] = k$, then no matter what $c_{k+1}, \dots c_n$ are we can show that the last $n-k$ columns are linearly independent because of $-I$ structure. Moreover, it is easy to check that any column from the first k columns is LI from the last $n-k$ columns and vice versa.

Hence $\text{rank}\left(\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix}\right) = n$.

② If $\text{rank}(C) = k$, then k rows of C are LI.

Hence, if the number of nonzero elements of $\hat{x}$ is larger than $\text{rank}(C)$, then number of active constraints is smaller than $n-k$. Hence, the number of rows of $A_{J(\hat{x})}$ is strictly smaller than $n - \text{rank}(C) = n-k$.

Hence, in this case, $A_{J(\hat{x})}$ has $\phi < n-k$ rows that are LI and C has exactly k rows that are LI.

As a result the number of LI rows of $\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix}$ is strictly smaller than $n \Longrightarrow \text{rank}\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix} < n \Longrightarrow \hat{x}$ is not an extreme point

$\boxed{\text{HW: Prove Birkhoff's theorem.}}$

$x_{ij} \geq 0 \quad X\mathbf{1} = \mathbf{1} \quad X^T\mathbf{1} = \mathbf{1}$

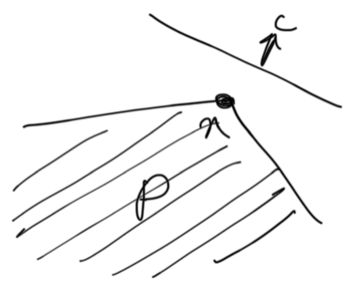
Show that the extreme points are the permutation matrices $\to$ (doubly st. with $\{0,1\}$)

Let us provide a new definition for an extreme point:

$\boxed{\text{Extreme Point definition 2}}$: A point $x \in P$ is an extreme point if $\nexists\, y, z \in P$ such that x can be written as $x = \lambda y + (1-\lambda) z$ for some $0 \leq \lambda \leq 1$

English words: x is not between any two points in P.

$\boxed{\text{Definition of vertex:}}$ A point $x \in P$ is a vertex if $\exists$ some c s.t. $c^T x < c^T y$ for all $y \in P$, $y \neq x$



Definition of Basic feasible solution:

Let $P = \{x \mid Ax \leq b,\ Cx = d\}$. Then, x is a basic feasible solution if the concatenation of its active constraints $(A_{\bar{J}(x)})$ and the equality matrix (C) has n Linearly independent Columns.

Theorem: The following statements are equivalent:

1. x^* is a vertex of P.
2. x^* is an extreme point of P.
3. x^* is a basic feasible solution.

Proof: homework

$\boxed{\text{Hint: Prove } ① \to ② \to ③ \to ①}$