The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

LECTURE 15

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Goal: In this lecture, we talk about quasi-convex optimization and quadratic programming.

1 Quasi-Convex Optimization

Standard form quasi-convex optimization problem is given by

$$\begin{aligned} \text{min}: \quad & f_0(\mathbf{x}) \\ \text{s.t.}: \quad & f_i(\mathbf{x}) \le 0, \qquad i = 1, \ldots, m. \\ & \mathbf{a}_i^\top \mathbf{x} + b_i = 0, \qquad i = 1, \ldots, p. \end{aligned}$$

f_0 is quasi-convex.
$f_1, \ldots, f_m$ are convex, equality constraints are affine. The constraint set is a convex set.

1.1 First-order sufficient condition

Let $\mathcal{X}$ denote the feasible set for the quasi-convex optimization problem. Then $\mathbf{x}^*$ is an optimal solution if $\mathbf{x}^* \in \mathcal{X}$ and

$$\nabla f_0(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) > 0 \quad \text{for all } \mathbf{x} \in \mathcal{X} - \{\mathbf{x}^*\}$$

Note that this is not a necessary condition. (There could exist an optimal solution $\mathbf{x}^*$ that does not satisfy this condition.)
–It can have locally optimal points that are not (globally) optimal
Proof of sufficient condition: Suppose it is not true. In other words, assume that $\mathbf{x}^* \in \mathcal{X}$ satisfies the condition

$$\nabla f_0(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) > 0 \quad \text{for all } \mathbf{x} \in \mathcal{X} - \{\mathbf{x}^*\}$$

but it is not a global optimum. Then, there exists a vector $\mathbf{y}^* \in \mathcal{C}$ such that $f_0(\mathbf{y}^*) < f_0(\mathbf{x}^*)$. Based on quasi-convex property we know that if $f_0(\mathbf{y}^*) \le f_0(\mathbf{x}^*)$ then we have

$$\nabla f_0(\mathbf{x}^*)^\top (\mathbf{y}^* - \mathbf{x}^*) \le 0,$$

but this contradicts we the assumption that $\nabla f_0(\mathbf{x}^*)^\top (\mathbf{y}^* - \mathbf{x}^*) > 0$. Hence, by contradiction we proved that if $\mathbf{x}^*$ is feasible and satisfies the above optimality condition it is an optimal solution.

1.2 Quasi-convex optimization via convex feasibility problems

A general approach to solve quasi-convex optimization is based on representing the sublevel sets of a quasi-convex function via a family of convex inequalities. Let, $\phi_t : \mathbb{R}^n \to \mathbb{R}$, $t \in \mathbb{R}$ be a family of convex functions that satisfy

$$f_0(\mathbf{x}) \le t \quad \Leftrightarrow \quad \phi_t(\mathbf{x}) \le 0$$

and also ϕ_t is a non-increasing function of t, i.e., $\phi_t(\mathbf{x}) \le \phi_s(\mathbf{x})$ for $s \le t$.

Now note that the quasi-convex problem can be written as

$$\begin{aligned} \min: \quad & t \\ \text{s.t.}: \quad & f_0(\mathbf{x}) \le t \\ & f_i(\mathbf{x}) \le 0, \qquad i = 1, \ldots, m. \\ & \mathbf{a}_i^\top \mathbf{x} + b_i = 0, \qquad i = 1, \ldots, p. \end{aligned}$$

Now for each fixed t, we can solve the following feasibility problem

$$\begin{aligned} \text{find}: \quad & \mathbf{x} \\ \text{s.t.}: \quad & \phi_t(\mathbf{x}) \le 0 \\ & f_i(\mathbf{x}) \le 0, \qquad i = 1, \ldots, m. \\ & \mathbf{a}_i^\top \mathbf{x} + b_i = 0, \qquad i = 1, \ldots, p. \end{aligned} \tag{1}$$

If this problem is feasible for a specific t, then we can show that $p^* \le t$. Conversely, if this problem is infeasible for a specific t, then we can conclude that $p^* \ge t$.

Using this logic we can apply a Bisection method to solve quasi-convex problems. To be more precise, assume that we know some upper bound u and lower bound l for the optimal value of the problem p^*. Then, we solve the feasibility problem for $t = (l+u)/2$. If the problem is feasible we conclude that $p^* \le (l+u)/2$ and therefore the new interval becomes $[l, (l+u)/2]$. But, if the problem is infeasible, we conclude that $p^* \ge (l+u)/2$ and therefore the new interval becomes $[(l+u)/2, u]$. To obtain an ϵ-accurate solution, we need to run this algorithm for $\log_2 \frac{(u-l)}{2}$ iterations.

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
1. $t := (l+u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

until $u - l \le \epsilon$.

To see that such a representation always exists, we can take

$$\phi_t(\mathbf{x}) = \begin{cases} 0 & f(\mathbf{x}) \le t \\ \infty & \text{otherwise} \end{cases}$$

In this case ϕ_t is the indicator function of the t-sublevel set of f_0.

A specific interesting instance of $\phi_t(\mathbf{x})$ function is for the case that $f_0(\mathbf{x})$ is defined as a convex over concave function. Suppose p is a convex function, q is a concave function, with $p(\mathbf{x}) \ge 0$ and $q(\mathbf{x}) > 0$ on a convex set $\mathcal{C}$. Then the function f defined by $f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x})$ is quasi-convex on $\mathcal{C}$. In particular, for this specific function we can show that

$$f(\mathbf{x}) \le t \quad \Leftrightarrow \quad p(\mathbf{x}) - tq(\mathbf{x}) \le 0.$$

It can be verified for a fixed value of t, the function $\phi_t(\mathbf{x}) := p(\mathbf{x}) - tq(\mathbf{x})$ is convex. Also, $\phi_t(\mathbf{x})$ is decreasing in t.

1.3 Linear-fractional program

The problem of minimizing a ratio of affine functions over a polyhedron is called a linear-fractional program

$$\begin{aligned} \text{min :} \quad & f_0(\mathbf{x}) \\ \text{s.t. :} \quad & \mathbf{G}\mathbf{x} \le \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

where the objective function is given by

$$f_0(\mathbf{x}) = \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + q} \qquad \textbf{dom } f_0 = \{\mathbf{x} \mid \mathbf{e}^\top \mathbf{x} + q > 0\}.$$

The objective function is quasi-convex (in fact, quasilinear) so linear-fractional programs are quasi-convex optimization problems.

Note: Since linear-fractional program is a quasi-convex optimization it can be solved using Bisection method.

Note: It can also be written as a Linear Program! (variables $\mathbf{y} \in \mathbb{R}^n$ and $z \in \mathbb{R}$)

$$\begin{aligned} \text{min :} \quad & \mathbf{c}^\top \mathbf{y} + dz \\ \text{s.t. :} \quad & \mathbf{G}\mathbf{y} \le \mathbf{h}z \\ & \mathbf{A}\mathbf{y} = \mathbf{b}z \\ & \mathbf{e}^\top \mathbf{y} + qz = 1 \\ & z \ge 0 \end{aligned}$$

To show the equivalence, use the following change of variables:

$$\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^\top \mathbf{x} + q}, \qquad z = \frac{1}{\mathbf{e}^\top \mathbf{x} + q}$$

More detail $--\!>$ check the textbook.

2 Quadratic Program (QP)

A convex optimization problem is called quadratic program, if
- The objective function is (convex) quadratic: $f_0(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x} + \mathbf{q}^\top \mathbf{x} + r$ with $\mathbf{P} \in \mathbf{S}^n_{++}$
- The constraint functions are affine: $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{G}\mathbf{x} \leq \mathbf{h}$ Hence, QP is given by

$$\begin{aligned} \text{min}: \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x} + \mathbf{q}^\top \mathbf{x} + r \\ \text{s.t.}: \quad & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

where $\mathbf{P} \in \mathbf{S}^n_{++}$, $\mathbf{G} \in \mathbb{R}^{m \times n}$, and $\mathbf{A} \in \mathbb{R}^{p \times n}$. The problem is called a quadratically constrained quadratic program (QCQP), if the inequality constraints functions are also (convex) quadratic.

$$\begin{aligned} \text{min}: \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{P}_0\mathbf{x} + \mathbf{q}_0^\top \mathbf{x} + r_0 \\ \text{s.t.}: \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{P}_i\mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i, \quad i = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

Note: feasible region is intersection of m ellipsoids and an affine set.

2.1 The Markowitz minimum variance portfolio

We would like to invest our money in n assets over a fixed period. The return r_i of each asset is a random variable; we only assume to know its first and second order moments. Denote this random return by

$$r_i = \frac{P_{i,end} - P_{i,begin}}{P_{i,begin}}$$

where $P_{i,begin}$ and $P_{i,end}$ are the prices of the asset at the beginning and end of the period. Let $\mathbf{r} \in \mathbb{R}^n$ be the random vector of all returns, which we assume has known mean $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$. If we decide to invest a portion x_i of our money in asset i, then the expected return of our portfolio would be

$$\mathbb{E}\left[\mathbf{x}^\top \mathbf{r}\right] = \mathbf{x}^\top \boldsymbol{\mu}$$

and its variance would be

$$\mathbb{E}\left[(\mathbf{x}^\top \mathbf{r} - \mathbf{x}^\top \boldsymbol{\mu})^2\right] = \mathbb{E}\left[(\mathbf{x}^\top (\mathbf{r} - \boldsymbol{\mu}))^2\right] = \mathbb{E}\left[\mathbf{x}^\top (\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^\top \mathbf{x}\right] = \mathbf{x}^\top \boldsymbol{\Sigma}\mathbf{x}$$

In practice, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ can be estimated from past data and be replaced with their empirical versions. The minimum variance portfolio optimization problem seeks to find a portfolio that meets a given desired level of return r_{min}, and has the lowest variance (or risk) possible:

$$\begin{aligned} \text{min}: \quad & \mathbf{x}^\top \boldsymbol{\Sigma}\mathbf{x} \\ \text{s.t.}: \quad & \mathbf{x}^\top \boldsymbol{\mu} \geq r_{min} \\ & \mathbf{x} \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1. \end{aligned}$$

This is a quadratic program. It can also be interesting to consider the problem of finding the maximum return portfolio that meets a given level of risk σ_{min}

$$\begin{aligned} \max: \quad & \mathbf{x}^\top \boldsymbol{\mu} \\ \text{s.t.}: \quad & \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq \sigma_{min} \\ & \mathbf{x} \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1. \end{aligned}$$

This is a convex QCQP.