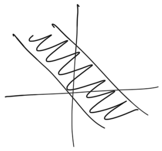
Lecture 3

Note: not all polyhedra have extreme point (when they contain a line)



P contains a line if $\exists x, d$ s.t. $x + \lambda d \in P$ for all $\lambda \in \mathbb{R}$

Theorem: For $P = \{x \in \mathbb{R}^n \mid a_i^T x \le b_i \ \forall i\}$ the following statements are equivalent.

(a) P has at least one extreme point.

(b) P does not contain a line

(c) $\exists$ n vectors out of $a_1, \dots, a_m$ that are linearly independent.

Proof: (a) $\Rightarrow$ (c)

If P has at least one extreme point $\hat{x}$ then the active set of that extreme point has n linearly independent vectors $\Rightarrow$ if a subset of $\{a_1, \dots, a_m\}$ has n LI " , the whole set $\{a_1, \dots, a_m\}$ also has n LI vectors.

(c) $\Rightarrow$ (b) Among $\{a_1, \dots, a_m\}$ consider $\{a_1', \dots, a_n'\}$ as the set of n LI vectors

Suppose there exits a line $\hat{x} + \lambda \hat{d} \in P$ in P.

$\Rightarrow a_i'^T (\hat{x} + \lambda \hat{d}) \le b_i'$ for $i = 1, \dots, n$

If $a_i'^T \hat{d}$ is nonzero, then by sending λ to $\pm\infty$ we can violate this inequality.

$\Rightarrow a_i^T \hat{d} = 0$ for $i = 1, \dots, n$

since $\{a_1', \dots, a_n'\}$ are LI, then $\hat{d} = 0$

$\Rightarrow \hat{x} + \lambda \hat{d}$ is not a line. $\times$

(b) $\Rightarrow$ (a) Intuition: keep growing the set of active constraints by moving in a direction orth. that is orthogonal to the current active constraints

Let $I(x_0) = \{j \mid a_j' x = b_j\}$ → active set of (x_0)

If $\text{span}(\{a_j \mid j \in I(x_0)\}) = \mathbb{R}^n$ → done! (since then x_0 is a basic feasible solution)

If not, assume $\exists d$ s.t. $d^T a_j = 0 \quad \forall j \in I(x_0)$

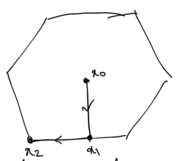
Consider the line $\{y : y = x_0 + \lambda d\}$.

At all these points, $a_j^T y = a_j^T x_0 = b_j$ for $j \in I(x_0)$ so active constraints remain active. However, since P contains no line $\exists \lambda^*$ s.t. $x_0 + \lambda^* d$ has to hit a new constraint $a_k^T x = b_k$ where $k \notin I(x_0)$.

it can be shown that $a_k \notin \text{span}\{a_j \mid j \in I(x_0)\}$ since if no then we have $d \perp a_k \Rightarrow a_k^T(x_0 + \lambda^* d) = b_k \Rightarrow k \in I($
contradiction

$\Rightarrow |I(x_1)| > I(x_0)$

$\Rightarrow \dim(\text{span}\{a_j \mid j \in I(x_1)\}) > \dim(\text{span}\{a_j \mid j \in I(x_0)\}) \Rightarrow$ do this until reach $\mathbb{R}^n$



extreme point.

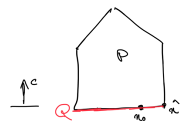
Optimality of Extreme Points:

LPs

- No optimal solution
  - P is infeasible, i.e., $P = \emptyset$
    e.g. $x \geq 0$, $x \leq -1$
  - optimal value is $-\infty$
    e.g. $\min x_1 + x_2$
    $x_1 \leq 0$
    $x_2 \leq 0$
    $x_1 - x_2 \leq 7$
- There exits an optimal solution. i.e., $\exists$ finite $x \in P$ such that $c^T x \leq c^T y \quad \forall y \in P$.

Theorem: Suppose the LP $\left\{\begin{array}{ll}\min & c^T x \\ \text{s.t.} & x \in P\end{array}\right\}$ has an optimal solution, and P has an extreme point. Then $\exists$ an extreme point of point that is an optimal solution.

Proof: If Lp has an optimal solution then we can consider σ as the optimal value (x_0 is optimal and $\sigma = c^T x_0$)



Now define the polytope Q as

$Q = \{x \in P \mid c^T x = \sigma\}$. It's clear that $Q \subseteq P$ and Q doesn't contain a line as P doesn't contain a line.
Therefore Q also has an extreme point.
Any point that belongs to Q is an optimal solution.

Now we proceed to show that an extreme point of Q is also an extreme point of P.

* $\boxed{\text{Consider } \hat{x} \text{ as an extreme point of } Q}$ Now suppose it is not an extreme point of P. This means that there are $y, z \in P$ such that $\hat{x} = (1-\lambda) y + \lambda z$ for some $0 < \lambda < 1$.

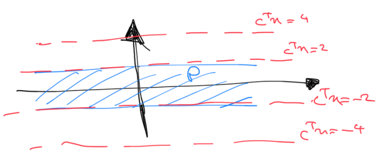
Therefore, $c^T \hat{x} = (1-\lambda) c^T y + \lambda c^T z = \sigma$

Note that $\left\{\begin{array}{l} c^T y \leq \sigma \\ c^T z \leq \sigma \end{array}\right\} \Rightarrow (1-\lambda) c^T y + \lambda c^T z = \sigma \iff c^T y = c^T z = \sigma$

Then therefore we obtain that both $\underline{y}$ and $\underline{z}$ also belong to Q.
Hence, $\hat{x}, y, z$ belong to Q and $\hat{x} = (1-\lambda) y + \lambda z$ which implies that $\hat{x}$ is not an extreme point of Q $\Longrightarrow$ Contradiction *.

Hence, $\hat{x}$ is an extreme point of P. Then, $\hat{x}$ is an extreme point of P that is an optimal solutio. Proof is complete.

Note: Does an optimal solution have to be an extreme point? No!! (when we don't have extreme points)



→ P is feasible
→ opt is not $-\infty$
→ No extreme point
⇒ optimal solution any point with $c^T x = 2 \Rightarrow \begin{bmatrix} x_1 \\ x_2 = 2 \end{bmatrix}$
⇒ optimal value $\underline{= 2}$.

Side effect of Previous theorem:

Every polyhedron is the convex hull of its extreme points.

Proof → Homework.

Let's go back to the example that we discussed in our first lecture.

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{N} x_{ij} = 1 \qquad j = 1, \dots, N \\ & \sum_{j=1}^{N} x_{ij} = 1 \qquad i = 1, \dots, N \\ & x_{ij} \in \{0,1\} \longrightarrow 0 \leq x_{ij} \leq 1 \end{aligned}$$

$\hookrightarrow$ solve $\rightarrow$ The answer satisfies $x_{ij} \in \{0,1\}$.

$N!$ feasible answers!!

According to Birkhoff's theorem the polyhedron P defined as $P = \{ X \mid 1 \geq x_{ij} \geq 0, X\mathbf{1} = \mathbf{1}, X^T\mathbf{1} = \mathbf{1} \}$ has extreme points and all extreme points are the permutation matrices, i.e., a doubly stochastic matrix with elements 0 & 1

Now based on the previous theorem one of the extreme points is going to be an optimal solution.

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Theorem: ~~*~~ Theorem of alternatives for linear inequalities:

For a given A, b exactly one of the following two statements is true

1. There exists an x that satisfies $Ax \leq b$
2. There exits a z that satisfies $z \geq 0, A^T z = 0, b^T z < 0$

→ It's clear that ① and ② cannot be both true.

$$\left.\begin{array}{l} Ax \leq b, \; z \geq 0 \\ A^T z = 0 \quad b^T z < 0 \end{array}\right\} \begin{array}{l} \Rightarrow z^T(Ax - b) \leq 0 \\ \Rightarrow z^T(Ax - b) > 0 \end{array}$$

→ Now we need to show that both ① and ② cannot be fal

To do so, we show that if ① is false, then ② is true. (induct

Basic Case: if A has 1 column.

$$\left.\begin{array}{l} ① \; Ax \leq b \\ ② \; A^T z = 0, z \geq 0, b^T z < 0 \end{array}\right\} \Rightarrow \left|\begin{array}{l} ①' \begin{cases} a_1 x \leq b_1 \\ \vdots \\ a_m x \leq b_m \end{cases} \\ ② \; A^T z = 0, z \geq 0, b^T z < 0 \end{array}\right.$$

Hence, $Ax \leq b$ is solvable if and only if

$\begin{cases} a_1 x \leq b_1 \\ \vdots \\ a_m x \leq b_m \end{cases}$ is solvable. This system is solvable

if and only if $\begin{cases} b_i \geq 0 & \text{for } a_i = 0 \\ x \leq b_i/a_i & \text{for } a_i > 0 \\ x \geq b_i/a_i & \text{for } a_i < 0 \end{cases}$

This is solvable if and only if

$$\begin{cases} b_i \geq 0 & \text{for } a_i = 0 \\ \min\limits_{a_i > 0} b_i/a_i \geq \max\limits_{a_i < 0} b_i/a_i \end{cases}$$

Hence, $Ax \leq b$ doesn't have any solution iff

$$\begin{cases} \text{case 1} \quad b_i < 0 & \text{for some } a_i = 0 \\ \text{or} & \\ \text{Case 2} \quad b_i/a_i < b_j/a_j & \text{for some } a_i > 0 \\ & \text{and some } a_j < 0 \end{cases}$$

Now we will show that under both cases the statement in (2) holds.

* in case 1, we can define $Z = e_i$ $\quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

we can check that $\begin{cases} Z \geq 0 \\ a^T Z = 0 \\ b^T Z = b_i < 0 \end{cases}$ ✓

* in case 2, we can define Z as

$$Z = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a_i} \\ 0 \\ -\frac{1}{a_j} \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} Z \geq 0 \\ a^T Z = \frac{1}{a_i} a_i + \frac{-1}{a_j} a_j = 0 \\ b^T Z = \frac{1}{a_i} b_i - \frac{1}{a_j} b_j < 0 \end{cases}$$ ✓

We showed, when $n = 1$, if $Ax \leq b$ doesn't hold then statement (2) ($Z \geq 0$, $A^T Z = 0$, $b^T Z < 0$) holds

Hence the base of induction is complete.

Induction step: Assume that the claim holds for sets of inequalities with $(n-1)$ variables.
Consider an inequality $Ax \leq b$ with $A \in R^{m \times n}$

- scale the inequalitie with $A_{in} \neq 0$ to get an equivalent system. I.e., Define $c_{ik} = \begin{cases} A_{ik}/A_{in} & i \in I_+ \\ -A_{ik}/A_{in} & i \in I_- \end{cases}$ $d_i = \begin{cases} b_i/A_{in} & i \in I_+ \\ -b_i/A_{in} & i \in I_- \end{cases}$

$$\sum_{k=1}^{n-1} c_{ik} x_k + x_n \leq d_i \quad \text{for } i \in I_+ \quad *$$

$$\sum_{k=1}^{n-1} c_{ik} x_k - x_n \leq d_i \quad \text{for } i \in I_- \quad **$$

$$\sum_{k=1}^{n} A_{ik} x_k \leq b_i \quad \text{for } i \in I_0$$

- $*$ and $**$ hold for some x_n iff

$$\max_{i \in I_-} \left(\sum_{k=1}^{n-1} c_{ik} x_k - d_i \right) \leq \min_{i \in I_+} \left(d_i - \sum_{k=1}^{n-1} c_{ik} x_k \right)$$

- Therefore $Ax \leq b$ is solvable iff there exists $(x_1, \ldots, x_{n-1})$ s.t.

$$\sum_{k=1}^{n-1} (c_{ik} + c_{jk}) x_k \leq d_i + d_j \quad \forall\, i \in I_-, j \in I_+$$

$$\sum_{k=1}^{n} A_{ik} x_k \leq b_i \quad \forall\, i \in I_0$$

This system of inequalities has $n-1$ variables.

- If this system is infeasible (1 doesn't hold)

  then there exists u_{ij} $(i \in I_-, j \in I_+)$, v_i $(i \in I_0)$, s.t.

induction for step $n-1$

$u_{ij} \geq 0$ for $i \in I_-$, $j \in I_+$, $v_i \geq 0$ for $i \in I_0$

$$\sum_{i \in I_-, j \in I_+} (C_{ik} + C_{jk}) u_{ij} + \sum_{i \in I_0} v_i A_{ik} = 0 \quad k = 1, \ldots$$

$$\sum_{i \in I_-, j \in I_+} (d_i + d_j) u_{ij} + \sum_{i \in I_0} b_i v_i < 0$$

- Now define

$$z_i = \frac{1}{-A_{in}} \sum_{j \in I_+} u_{ij} \quad \text{for } i \in I_-$$

$$z_j = \frac{1}{A_{jn}} \sum_{i \in I_-} u_{ij} \quad \text{for } j \in I_+$$

$$z_i = v_i \quad \text{for } i \in I_0$$

to get vector z that satisfies

$$z \geq 0, \; A^T z = 0, \; b^T z < 0.$$