The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

Lecture 6

Aryan Mokhtari Wednesday, September 18, 2019.

Goal: In this lecture, we first look at some special cases of primal-dual LP where one side is unbounded or infeasible. Then, we review different forms of primal-dual problems. Then, we study the dual problem of a piecewise-linear minimization problem as well as an ℓ_∞-norm approximation. In the last part of the lecture, we talk about complementary slackness.

1 Infeasible and unbounded cases

Simple cases:
If the primal problem is unbounded ($p^* = -\infty$) then by weak duality the dual problem should be infeasible. [Argument: If not, then the dual problem is feasible and by weak duality $p^* \geq d^* > -\infty$ which is a contradiction].

If the dual problem is unbounded ($d^* = +\infty$) then by weak duality the primal problem should be infeasible. [Argument: If not, then the primal problem is feasible and by weak duality $\infty > p^* \geq d^*$ which is a contradiction].

Theorem 1. *If the primal problem is infeasible ($p^* = +\infty$), then the dual problem is either unbounded ($d^* = +\infty$) or infeasible ($d^* = -\infty$).*

Proof: Note that according to the theorem of alternatives when primal is infeasible, i.e., $\mathbf{Ax} \leq \mathbf{b}$ has no solution, then there exists $\mathbf{w}$ such that

$$\mathbf{w} \geq \mathbf{0}, \qquad \mathbf{A}^\top \mathbf{w} = \mathbf{0}, \qquad \mathbf{b}^\top \mathbf{w} < 0.$$

Case I: If the dual problem is *feasible*, then any $\mathbf{z}$ point that is feasible for the dual problem satisfies

$$\mathbf{z} + t\mathbf{w} \geq \mathbf{0} \qquad \mathbf{A}^\top(\mathbf{z} + t\mathbf{w}) + \mathbf{c} = \mathbf{0}, \qquad \text{for all} \quad t \geq 0.$$

Therefore, $\mathbf{z} + t\mathbf{w}$ is dual feasible for all $t \geq 0$. Moreover, as $t \to \infty$ we have that

$$-\mathbf{b}^\top(\mathbf{z} + t\mathbf{w}) = -\mathbf{b}^\top\mathbf{z} - t\mathbf{b}^\top\mathbf{w} \to \infty$$

Hence, in this case, the dual problem is unbounded, i.e., ($d^* = +\infty$).

Case II: If the dual problem is *infeasible*, then we are done as ($d^* = -\infty$).

Theorem 2. *If the dual problem is infeasible ($d^* = -\infty$), then the primal problem is either unbounded ($p^* = -\infty$) or infeasible ($p^* = +\infty$).*

Proof: Similar idea.

2 Different forms of primal-dual problems

In the last lecture, we introduced the dual of an LP which has inequality constraints

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{maximize} & -\mathbf{b}^\top \mathbf{z} \\ \text{subject to} & \mathbf{A}^\top \mathbf{z} + \mathbf{c} = \mathbf{0} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

We can also generalize it to the case that the primal problem has both equality and inequality constraints (it can be shown by writing $\mathbf{C}\mathbf{x} = \mathbf{d}$ as two inequalities)

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{C}\mathbf{x} = \mathbf{d} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{maximize} & -\mathbf{b}^\top \mathbf{z} - \mathbf{d}^\top \mathbf{y} \\ \text{subject to} & \mathbf{A}^\top \mathbf{z} + \mathbf{C}^\top \mathbf{y} + \mathbf{c} = \mathbf{0} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

Note that in this case $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^p$ and $\mathbf{z} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$.

The dual problem for the case that the primal problem is written in a standard form is given by

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{maximize} & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} & \mathbf{A}^\top \mathbf{y} \leq \mathbf{c} \end{array}$$

3 Examples of primal-dual problems

We study two examples in this section.

3.1 Piecewise-linear minimization

A function $f : \mathbb{R}^n \to R$ is piecewise-linear if it can be expressed as

$$f(\mathbf{x}) = \max_{i=1,\ldots,m} (\mathbf{a}_i^\top \mathbf{x} + b_i)$$

Minimizing a piecewise-linear can be written as

$$\text{minimize} \quad f(\mathbf{x}) = \max_{i=1,\ldots,m} (\mathbf{a}_i^\top \mathbf{x} + b_i)$$

This problem can be written as an LP by introducing a new variable t which is an upper bound on the values of $\mathbf{a}_i^\top \mathbf{x} + b_i$ for $i = 1, \ldots, m$, i.e.,

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (\mathbf{a}_i^\top \mathbf{x} + b_i) \leq t, \qquad i = 1, \ldots, m \end{array}$$

Primal LP: This problem can also be written as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{A} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} \leq -\mathbf{b} \end{array}$$

Dual LP: The dual problem therefore is given by

$$\begin{array}{ll} \text{maximize} & \mathbf{b}^\top \mathbf{z} \\ \text{subject to} & \mathbf{A}^\top \mathbf{z} = \mathbf{0}, \qquad \mathbf{1}^\top \mathbf{z} = 1, \qquad \mathbf{z} \geq \mathbf{0} \end{array}$$

Note: By finding a feasible solution for the primal problem we can always find an upper bound for the the optimal solution p^*. But, to find a lower bound for the optimal value of primal problem we can find a feasible solution for the dual problem. For instance, if we look at the value of $\mathbf{b}^\top \hat{\mathbf{z}}$ for a point $\hat{\mathbf{z}}$ that is dual feasible ($\mathbf{A}^\top \hat{\mathbf{z}} = \mathbf{0}, \mathbf{1}^\top \hat{\mathbf{z}} = 1, \hat{\mathbf{z}} \geq \mathbf{0}$) we can find a lower bound for the optimal value of the primal problem for the minimization of a piecewise-linear function.

3.2 ℓ_∞-Norm approximation

Consider $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ as our problem data, and $\mathbf{x} \in \mathbb{R}^n$ as our variable. The goal in norm approximation is to find a solution that approximately satisfies the condition $\mathbf{A}\mathbf{x} = \mathbf{b}$, while we keep the norm of the residual $\mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b}$ small. (We will talk about this example later in the application part of the class.)
When we aim to minimize the infinity norm of the residual we should solve

$$\text{minimize} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty$$

Note that the ℓ_∞-norm (Chebyshev norm) of a vector $\mathbf{y} \in \mathbb{R}^m$ with elements y_i is defined as

$$\|\mathbf{y}\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

This problem can be written as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -t\mathbf{1} \leq \mathbf{A}\mathbf{x} - \mathbf{b} \leq t\mathbf{1} \end{array}$$

If we then write it in inequality form with variables $\mathbf{x}$ and t we obtain that

Primal LP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{A} & -\mathbf{1} \\ -\mathbf{A} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} \end{array}$$

Dual LP: If we define the dual variable as $\mathbf{z} = [\mathbf{u}; \mathbf{v}] \in \mathbb{R}^{2m}$ then we can write the dual problem as

$$\begin{array}{ll} \text{maximize} & -\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}^\top \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} \mathbf{A} & -\mathbf{1} \\ -\mathbf{A} & -\mathbf{1} \end{bmatrix}^\top \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \geq \mathbf{0} \end{array}$$

Dual LP: This problem can be simplified as

$$\begin{array}{ll} \text{maximize} & -\mathbf{b}^\top \mathbf{u} + \mathbf{b}^\top \mathbf{v} \\ \text{subject to} & \mathbf{A}^\top \mathbf{u} - \mathbf{A}^\top \mathbf{v} = \mathbf{0} \\ & \mathbf{1}^\top \mathbf{u} + \mathbf{1}^\top \mathbf{v} = 1 \\ & \mathbf{u} \geq \mathbf{0}, \quad \mathbf{v} \geq \mathbf{0} \end{array}$$

(Exercise): Show that the dual LP can also be written as

$$\begin{array}{ll} \text{maximize} & \mathbf{b}^\top \mathbf{z} \\ \text{subject to} & \mathbf{A}^\top \mathbf{z} = \mathbf{0}, \qquad \|\mathbf{z}\|_1 \leq 1 \end{array}$$

4 Complementary Slackness

For the following primal-dual LP

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{C}\mathbf{x} = \mathbf{d} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{maximize} & -\mathbf{b}^\top \mathbf{z} - \mathbf{d}^\top \mathbf{y} \\ \text{subject to} & \mathbf{A}^\top \mathbf{z} + \mathbf{C}^\top \mathbf{y} + \mathbf{c} = \mathbf{0} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

the optimality conditions are ($\mathbf{x}^*$ and $(\mathbf{y}^*, \mathbf{z}^*)$ are optimal iff)

$$\begin{aligned} &\mathbf{A}\mathbf{x}^* \leq \mathbf{b}, \qquad \mathbf{C}\mathbf{x}^* = \mathbf{d} \\ &\mathbf{A}^\top \mathbf{z}^* + \mathbf{C}^\top \mathbf{y}^* + \mathbf{c} = \mathbf{0}, \qquad \mathbf{z}^* \geq \mathbf{0} \\ &\mathbf{c}^\top \mathbf{x}^* = -\mathbf{b}^\top \mathbf{z}^* - \mathbf{d}^\top \mathbf{y}^* \end{aligned}$$

If we define $\Delta = p^* - d^*$ as the duality gap, it can be shown that

$$\begin{aligned} \Delta &= p^* - d^* \\ &= \mathbf{c}^\top \mathbf{x}^* + \mathbf{b}^\top \mathbf{z}^* + \mathbf{d}^\top \mathbf{y}^* \\ &= (\mathbf{b} - \mathbf{A}\mathbf{x}^*)^\top \mathbf{z}^* + (\mathbf{d} - \mathbf{C}\mathbf{x}^*)^\top \mathbf{y}^* \\ &= (\mathbf{b} - \mathbf{A}\mathbf{x}^*)^\top \mathbf{z}^* \\ &= \sum_{i=1}^{m} z_i^* (b_i - \mathbf{a}_i^\top \mathbf{x}^*) \end{aligned}$$

where in the third inequality we replace $\mathbf{c}$ by $-\mathbf{A}^\top \mathbf{z}^* - \mathbf{C}^\top \mathbf{y}^*$, in the fourth equality we used the fact that $\mathbf{C}\mathbf{x}^* = \mathbf{d}$, and in the last equality b_i is the i-th element of vector $\mathbf{b}$ and $\mathbf{a}_i^\top$ is the i-th row of matrix $\mathbf{A}$.
Note that by strong duality we know that for primal and dual feasible LPs the duality gap is zero, i.e., $\Delta = 0$. Hence, we can conclude that $\mathbf{x}^*$ and $(\mathbf{y}^*, \mathbf{z}^*)$ are primal-dual optimal iff

$$z_i^* (b_i - \mathbf{a}_i^\top \mathbf{x}^*) = 0, \qquad i = 1, \ldots, m$$

Therefore, we observe that $\mathbf{b} - \mathbf{A}\mathbf{x}^*$ and $\mathbf{z}^* \geq \mathbf{0}$ have a **complementary sparsity pattern**:

$$\begin{array}{llll} \text{if} & z_i^* > 0 & \Rightarrow & \mathbf{a}_i^\top \mathbf{x}^* = b_i \\ \text{if} & \mathbf{a}_i^\top \mathbf{x}^* < b_i & \Rightarrow & z_i^* = 0 \end{array}$$