The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

LECTURE 7

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Goal: In this lecture, we first study strict complementary and a geometric interpretation of complementary slackness. Then, we focus on maximum flow and min cut problems.

1 Strict complementary

Recall that the feasible points $\mathbf{x}^*$ and $(\mathbf{y}^*, \mathbf{z}^*)$ are primal-dual optimal iff

$$z_i^*(b_i - \mathbf{a}_i^\top \mathbf{x}^*) = 0, \qquad i = 1, \ldots, m,$$

which implies that for all $i = 1, \ldots, m$ we must have

$$z_i^* > 0, \quad \mathbf{a}_i^\top \mathbf{x}^* = b_i \qquad \text{or} \qquad z_i^* = 0, \quad \mathbf{a}_i^\top \mathbf{x}^* < b_i \qquad \text{or} \quad z_i^* = 0, \quad \mathbf{a}_i^\top \mathbf{x}^* = b_i$$

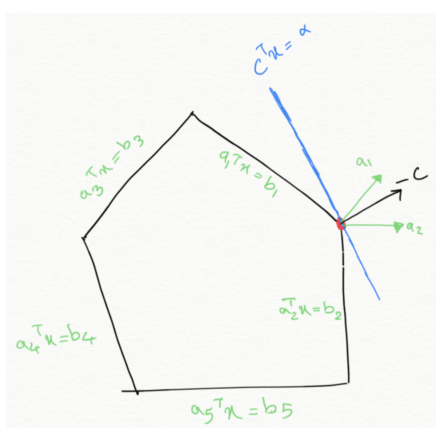
(we might have a case that both of the terms z_i and $b_i - \mathbf{a}_i^\top \mathbf{x}^*$ are zero).

Primal and dual optimal points are **strictly complementary** if for all $i = 1, \ldots, m$ we have

$$z_i^* > 0, \quad \mathbf{a}_i^\top \mathbf{x}^* = b_i \qquad \text{or} \qquad z_i^* = 0, \quad \mathbf{a}_i^\top \mathbf{x}^* < b_i$$

Remark: For any LP with a finite optimal value, it can be shown that strictly complementary solutions exist. (Proof: Homework)

2 Geometric interpretation of complementary slackness



In this problem we consider a linear program in $\mathbb{R}^2$ where we have 5 constraints.

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{a}_1^\top \mathbf{x} \leq b_1, \quad \mathbf{a}_2^\top \mathbf{x} \leq b_2, \quad \mathbf{a}_3^\top \mathbf{x} \leq b_3, \quad \mathbf{a}_4^\top \mathbf{x} \leq b_4, \quad \mathbf{a}_5^\top \mathbf{x} \leq b_5 \end{array}$$

At the optimal solution two of the constraints (constraints 1 and 2) are active. In other words, $\mathbf{a}_1^\top \mathbf{x}^* = b_1$ and $\mathbf{a}_2^\top \mathbf{x}^* = b_2$.

For the optimal dual solution we know that

$$\mathbf{A}^\top \mathbf{z}^* + \mathbf{c} = \mathbf{0}, \qquad \mathbf{z}^* \geq \mathbf{0}$$

and in addition we know that

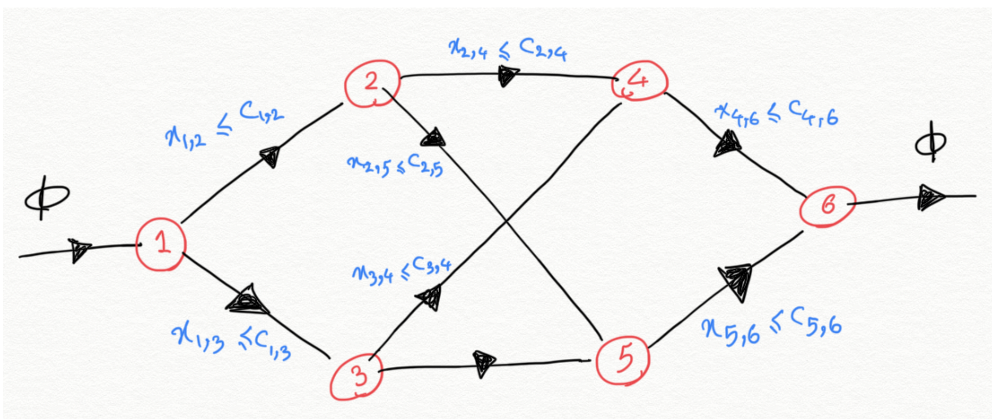
$$z_3^* = z_4^* = z_5^* = 0$$

Hence, by combining these conditions we obtain that

$$-\mathbf{c} = z_1^* \mathbf{a}_1 + z_2^* \mathbf{a}_2 \qquad \text{with} \quad z_1^* \geq 0, z_2^* \geq 0$$

Observation: $-\mathbf{c}$ lies in the cone generated by $\mathbf{a}_1$ and $\mathbf{a}_2$.

3 Maximum Flow Problem



Consider a directed graph $G = (V, E)$ where V is the set of nodes and E is the set of edges. An edge (u, v) belongs to set E if there is a link from u to v. We use s and t to refer to the source and the sink (terminal) in the graph G. The capacity $c_{u,v}$ of edge (u, v) represents the maximum amount of flow that can pass through (u, v). We use $x_{u,v}$ to show the amount of flow that passes through the link (u, v) from u to v. Indeed, $x_{u,v}$ should satisfy the following conditions

$$\begin{array}{ll} 0 \leq x_{u,v} \leq c_{u,v} & \text{for any edge } (u,v) \in E \\ \sum_v x_{u,v} = \sum_v x_{v,u} & \text{for any vertex } u \in V \setminus \{s,t\} \end{array}$$

Therefore, the maximum flow problem can be written as

$$\begin{array}{lll} \text{maximize} & \sum_{v:(s,v)\in E} x_{s,v} & \\ \text{subject to} & 0 \leq x_{u,v} \leq c_{u,v} & \text{for any edge } (u,v) \in E \\ & \sum_v x_{u,v} = \sum_v x_{v,u} & \text{for any vertex } u \in V \setminus \{s,t\} \end{array}$$

To simplify the notation we can define ϕ as $\phi = \sum_v x_{s,v} = \sum_v x_{v,t}$ and write

$$\begin{array}{lll} \text{maximize} & \phi & \\ \text{subject to} & 0 \leq x_{u,v} \leq c_{u,v} & \text{for any edge } (u,v) \in E \\ & \sum_v x_{u,v} = \sum_v x_{v,u} & \text{for any vertex } u \in V \setminus \{s,t\} \\ & \sum_v x_{s,v} = \phi & \\ & \sum_v x_{v,t} = \phi & \end{array}$$

If we assign an ordering to the edges, then we can define $\mathbf{x} \in \mathbb{R}^n$ as a vector that contains the flow information of all the edges. Similarly we can define the vector $\mathbf{c} \in R^n$ which contains the capacity of edges in the network. Now consider the edge-node incidence matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ where each component A_{ij} is given by

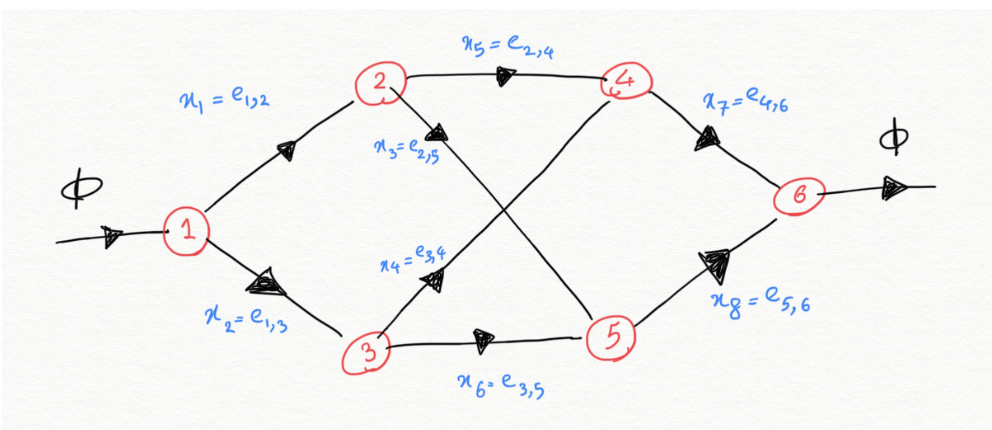
$$A_{ij} = \begin{cases} 1 & \text{if edge } j \text{ starts at node } i \\ -1 & \text{if edge } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Each column of $\mathbf{A}$ has exactly one 1 and one -1 and $m-2$ zeros. Given these definitions we can rewrite the constraints of the maximum flow problem as

$$\begin{array}{ll} \text{maximize} & \phi \\ \text{subject to} & \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \\ & \mathbf{A}\mathbf{x} = \phi \mathbf{e} \end{array}$$

where $\mathbf{e} = [1; 0; \ldots; 0, -1] \in \mathbb{R}^m$. It is easy to see that this problem has $n+1 = |E|+1$ variables ($\mathbf{x}$ and ϕ), $m = |V|$ equality constraints, and $2n = 2|E|$ inequality constraints.

To ensure that $\mathbf{A}\mathbf{x} = \phi\mathbf{e}$ is right consider the following example with the specified ordering for the edges:



We can check that the incidence matrix is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Now if we simplify the constraint $\mathbf{Ax} = \phi\mathbf{e}$ given by

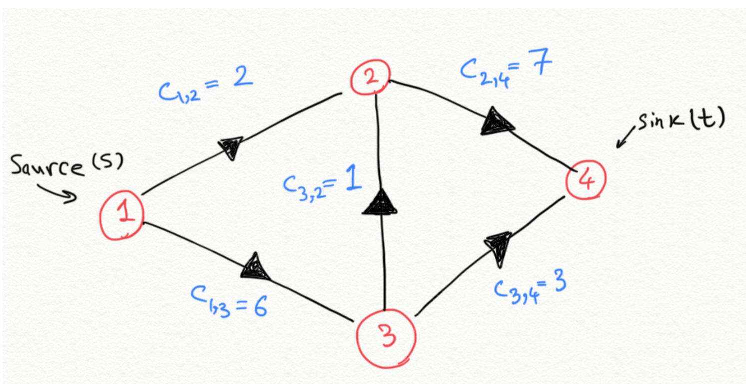
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \\ 0 \\ 0 \\ 0 \\ -\phi \end{bmatrix}$$

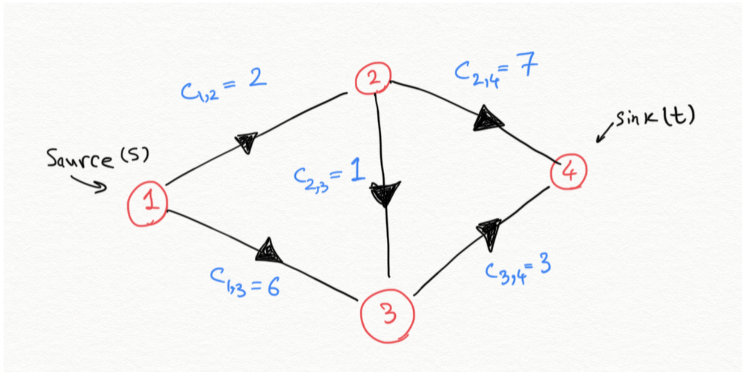
we obtain the following set of conditions:

$$\begin{aligned} x_1 + x_2 &= \phi \\ -x_1 + x_3 + x_5 &= 0 \\ -x_2 + x_4 + x_6 &= 0 \\ -x_4 - x_5 + x_7 &= 0 \\ -x_3 - x_6 + x_8 &= 0 \\ -x_7 - x_8 &= -\phi \end{aligned}$$

4 Minimum Cut Problem

Definition: For a directed graph $G = (V, E)$ an $s - t$ cut is a set of edges $\mathcal{C}$ such that by removing them there is no directed path from the source s to the sink t. The capacity of cut $\mathcal{C}$ is defined as $\sum_{(u,v)\in\mathcal{C}} c_{u,v}$.
For example, in the following graph, $\mathcal{C} = \{c_{1,2}, c_{1,3}\}$ is an $s - t$ cut with capacity 8, and the $s - t$ minimum cut is $\mathcal{C} = \{c_{1,2}, c_{3,2}, c_{3,4}\}$ with capacity 6. (The maximum flow is also 6.)





Now we consider a slightly different example. In the above graph, the $s - t$ minimum cut is $\mathcal{C} = \{c_{1,2}, c_{3,4}\}$ with capacity 5. (The maximum flow is also 5.)

In minimum cut problem we aim to find the cut that has the smallest capacity. Solving this problem directly is costly as we need to explore all combinations of edges that disconnect the graph from source to sink, and in the worst case the cost can be exponential in the number of edges.

We can avoid this issue by exploiting duality!