1. $\left\|\begin{bmatrix}2x\\ y-z\end{bmatrix}\right\|_2 \le y+z \iff \begin{bmatrix}2x\\ y-z\end{bmatrix}^T\begin{bmatrix}2x\\ y-z\end{bmatrix} \le (y+z)^2 \iff x^Tx \le yz$

so, $\begin{cases} x^Tx \le yz \\ y \ge 0 \\ z \ge 0 \end{cases} \iff \begin{cases} \left\|\begin{bmatrix}2x\\ y-z\end{bmatrix}\right\|_2 \le y+z \\ y \ge 0 \\ z \ge 0 \end{cases}$

(a). $\max \left(\sum_{i=1}^{m} \frac{1}{a_i^Tx-b_i}\right)^{-1} \iff \min \sum_{i=1}^{m} \frac{1}{a_i^Tx-b_i} \xrightarrow{\text{let } t_i \ge \frac{1}{a_i^Tx-b_i}} \begin{array}{l} \min \sum_{i=1}^{m} t_i \\ \text{s.t. } 1 \le (t_i)(a_i^Tx-b_i),\ i=1,\dots,m \end{array}$

$\iff \begin{array}{l} \min \mathbb{1}^Tt \\ \text{s.t. } \left\|\begin{bmatrix}2\\ t_i-a_i^Tx+b_i\end{bmatrix}\right\|_2 \le t_i + a_i^Tx - b_i,\ i=1,\dots,m \end{array}$

(b). $\max \sqrt[m]{\prod_{i=1}^{m}(a_i^Tx-b_i)} \iff \max \prod_{i=1}^{m}(a_i^Tx-b_i) \xleftrightarrow{\text{let } y_i = a_i^Tx-b_i} \begin{array}{l} \max y_1 \cdots y_m \\ \text{s.t. } y_i = a_i^Tx-b_i,\ i=1,\dots,m \end{array}$

The idea for the following derivation is basically recursively applying the following trick to minimize the number of product terms in the objective function:

Assume $m=4$, then $\begin{array}{l} \max y_1y_2y_3y_4 \\ \text{s.t. } y_i = a_i^Tx-b_i,\ i=1,\dots,4 \end{array} \xrightarrow[\text{let } t_2^2 \le y_3y_4,\ t_2 \ge 0]{\text{let } t_1^2 \le y_1y_2,\ t_1 \ge 0} \begin{array}{l} \max t_1^2t_2^2 \\ \text{s.t. } y = Ax-b \\ \quad t_1 \ge 0,\ t_2 \ge 0 \\ \quad t_1^2 \le y_1y_2 \\ \quad t_2^2 \le y_3y_4 \end{array} \iff \begin{array}{l} \max t_1t_2 \\ \text{s.t. } y = Ax-b \\ \quad \left\|\begin{bmatrix}2t_1\\ y_1-y_2\end{bmatrix}\right\|_2 \le y_1+y_2 \\ \quad \left\|\begin{bmatrix}2t_2\\ y_3-y_4\end{bmatrix}\right\|_2 \le y_3+y_4 \\ \quad t_1 \ge 0\ \ t_2 \ge 0 \end{array}$

where we have reduced the number of variables appearing in the objective function from 4 to 2. We can recursively apply this trick and we will end up with a single variable in this objective function while all constraints are second order cone form.

2. $\lambda_{pf}(A) = \inf\{\lambda \mid Av \le \lambda v \text{ for some } v > 0\}$. For any $v>0$, λ's that satisfy $Av \le \lambda v$ will be in this set. This is saying, for any $v > 0$, $\frac{\sum_{j=1}^{n} A_{ij} v_j}{v_i} \le \lambda$, $i = 1, \dots, m$.

This is equivalent to, for any $v > 0$, $\max_{i=1,\dots,m} \frac{\sum_{j=1}^{n} A_{ij} v_j}{v_i} \le \lambda$.

Thus, we can rewrite $\lambda_{pf}(A) = \inf_{v>0} \max_{i=1,\dots,m} \frac{\sum_{j=1}^{n} A_{ij} v_j}{v_i}$, because those λ's that are larger than $\max_{i=1,\dots,m} \frac{\sum_{j=1}^{n} A_{ij} v_j}{v_i}$ will not be the infimum anyway, so, let's just consider those λ's that is equal to this quantity, and pick the smallest among them.

Then, let $\log A_{ij} = a_{ij}$ and take log of both sides: $\log \lambda_{pf}(A) = \inf_{v>0} \max_{i=1,\dots,m} \log\left(\frac{\sum_{j=1}^{n} e^{a_{ij}} v_j}{v_i}\right)$

Let $\log v_i = y_i$ and we can further simplify it to $\log \lambda_{pf}(A) = \inf_{y} \max_{i=1,\dots,m} \left(\log \sum_{j=1}^{n} e^{a_{ij}+y_j} - y_i\right)$

We know function $\log \sum_{j=1}^{n} e^{a_{ij}+y_j} - y_i$ is convex in a_{ij} and y.

Then $\max_{i=1,\dots,m} \left(\log \sum_{j=1}^{n} e^{a_{ij}+y_j} - y_i\right)$ is also a convex function in a_{ij} and y.

Taking the infimum over all possible y gives a convex function in a_{ij}.

Thus, $\log \lambda_{pf}(A)$ is a convex function in $\log A_{ij}$.

3. (c). Rewrite the optimization problem as

$$\begin{aligned} \min\ & t \\ \text{s.t. } & t - (Ax+b)^T F(x)^{-1} (Ax+b) \ge 0 \end{aligned}$$

Consider matrix $\begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix}$.

We know the Schur complement of $F(x)$ is $t - (Ax+b)^T F(x)^{-1} (Ax+b)$. Here $F(x)$ is invertible, so $t - (Ax+b)^T F(x)^{-1} (Ax+b) \ge 0 \iff \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0$. Hence, the original problem can be written as

$$\begin{aligned} \min\ & t \\ \text{s.t } & \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \end{aligned}$$

4. (a). • For $x \geq 0$, $x^TAx = x^T(B+C)x = \underbrace{x^TBx}_{\geq 0} + \underbrace{x^TCx}_{\geq 0} \geq 0$ since $B \succeq 0$ and $C_{ij} \geq 0, i,j=1,\cdots,n$.

• The feasibility problem can be written as

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & B+C=A \\ & B \succeq 0 \\ & C_{ij} \geq 0,\ i,j=1,\cdots,n \end{array} \iff \begin{array}{ll} \min & 0 \\ \text{s.t.} & tr(\widehat{A}_{ij}X) = A_{ij},\ i,j=1,\cdots,n \\ & X \succeq 0, \end{array}$$

where $X = \begin{bmatrix} B & & & \\ & diag(C_1) & & \\ & & \ddots & \\ & & & diag(C_n) \end{bmatrix}$,

C_i is the i^{th} row of C,

$\widehat{A}_{ij} = \begin{bmatrix} \mathbb{1}_{ij} & & & & \\ & 0 & & & \\ & & diag(e_j) & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$ diag(C_i) block

$\mathbb{1}_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots 1 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ has a 1 in i,j^{th} position and zeros everywhere.

$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ has a 1 in j^{th} position and zero everywhere else.

(b) • For $x \geq 0$, $x^T(DA + A^TD)x \geq 0 \iff 2x^TDAx \geq 0$

$\iff x^TDAx \geq 0 \iff \sum_{i=1}^{n} d_i x_i (Ax)_i \geq 0$

Since $d_i > 0$, the maximal $x_i(Ax)_i$ must be non-negative, because otherwise, the sum cannot be non-negative. Thus, it means $\max_{i=1,\cdots,n} x_i(Ax)_i \geq 0$.

• The feasibility problem can be written as

$$\cancel{\begin{array}{ll} \min & 0 \\ \text{s.t.} & D \succ 0 \\ & DA+A^TD \succeq 0 \end{array}} \iff \cancel{\begin{array}{ll} \min & 0 \\ \text{s.t.} & X \succeq 0, \text{ where } X = \begin{bmatrix} D & \\ & DA+A^TD \end{bmatrix} \end{array}}$$

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & D \text{ is diagonal} \\ & D \succ 0 \\ & DA + A^TD \succeq 0 \end{array} \iff \begin{array}{ll} \min & 0 \\ \text{s.t.} & tr(\widehat{A}_{ij}X) = 0,\ i,j=1,\cdots,n,\ i \neq j, \\ & X \succeq 0 \end{array}$$

where $X = \begin{bmatrix} D & \\ & DA+A^TD \end{bmatrix}$,

$\widehat{A}_{ij} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ with a 1 in i,j^{th} position and zero everywhere else

5\. (a) $\min \lambda_1(x) \xrightarrow{\text{let } t \geq \lambda_1(x)}$ $\min t$ s.t. $\lambda_1(x) \leq t$

If $\lambda_1(x) - t \leq 0$, then $\lambda_i(x) - t \leq 0$, $i = 1, \dots, m$

Thus $A(x) - tI \preceq 0$. The converse is also true.

$\Longleftrightarrow$ $\min t$ s.t. $A(x) - tI \preceq 0$

(b). Similarly, let $t_1 \geq \lambda_1(x)$, $t_2 \leq \lambda_m(x)$. The equivalent condition is $A(x) - t_1 I \preceq 0$, $-A(x) + t_2 I \preceq 0$

Hence, $\min \lambda_1(x) - \lambda_m(x) \Longrightarrow \min t_1 - t_2$

s.t. $A(x) - t_1 I \preceq 0$

$-A(x) + t_2 I \preceq 0$

(c). $\min \frac{\lambda_1(x)}{\lambda_m(x)}$ s.t. $A \succ 0$ $\xrightarrow[0 < \gamma \leq \lambda_m(x)]{\text{let } \lambda \geq \lambda_1(x)}$ $\min \frac{\lambda}{\gamma}$ s.t. $\lambda \geq \lambda_1(x)$, $\gamma \leq \lambda_m(x)$, $\gamma > 0$ $\Longleftrightarrow$ $\min \frac{\lambda}{\gamma}$ s.t. $A(x) - \lambda I \preceq 0$, $A(x) - \gamma I \succeq 0$, $\gamma > 0$

using the hint we can get $\min t$

s.t. $sA_0 + y_1 A_1 + \cdots + y_n A_n - tI \preceq 0$

$sA_0 + y_1 A_1 + \cdots + y_n A_n - I \succeq 0$

$s > 0$

(d) $\min \sum_{i=1}^{m} |\lambda_i|$ Using the hint we know we can write $A(x) = A^+ - A^-$, where $A^+ \succeq 0$, $A^- \succeq 0$

Hence we can write the problem as $\min tr(A^+) + tr(A^-)$

s.t. $A(x) = A^+ - A^-$

$A^+ \succeq 0$, $A^- \succeq 0$