The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

PROBLEM SET II

Due: Sunday, September 22, 2019.

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1. **(Extreme points of Isomorphic polyhedra)** A mapping f is called affine if it is of the form $f(x) = Ax + b$, where A is a matrix and b is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \mapsto Q$ and $g : Q \mapsto P$ such that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

   (a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are as above, show that x is an extreme point of P if and only if $f(x)$ is an extreme point of Q.

   (b) **(Introducing slack variables leads to an isomorphic polyhedron)** Let $P = \{x \in \mathbb{R}^n | Ax \geq b, x \geq 0\}$, where A is a matrix of dimensions $k \times n$. Let $Q = \{(x, z) \in \mathbb{R}^{n+k} | Ax - z = b, x \geq 0, z \geq 0\}$. Show that P and Q are isomorphic.

2. Let P be a bounded polyhedron in $\mathbb{R}^n$, let a be a vector in $\mathbb{R}^n$ and let b be some scalar. We define

$$Q = \{x \in P | a^T x = b\}.$$

   Show that every extreme point of Q is either an extreme point of P or a convex combination of two extreme points of P.

3. Let A be a symmetric square matrix. Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq c \\ & x \geq 0. \end{array}$$

   Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

4. Use the Theorem of Alternatives to prove the following:

   For any $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, and $\mathbf{d} \in \mathbb{R}^p$ exactly one of the following statements holds:

   (a) There exists an $\mathbf{x} \in \mathbb{R}^n$ that satisfies $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{Cx} = \mathbf{d}$.

   (b) There exist $\mathbf{z} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^p$ that satisfy $\mathbf{z} \geq \mathbf{0}$, $\mathbf{A}^\top \mathbf{z} + \mathbf{C}^\top \mathbf{y} = \mathbf{0}$, $\mathbf{b}^\top \mathbf{z} + \mathbf{d}^\top \mathbf{y} < 0$.

5. Solve Exercise 41 in LLP.