1. (a) x is an EP of $P \Longleftrightarrow f(x)$ is an EP of Q

$\Longrightarrow$ direction: Assume $f(x)$ is not an EP of Q. Then $f(x) = \lambda y + (1-\lambda) z$ for some $y, z \in Q$, $\lambda \in [0,1]$. We apply g to both sides and have $x = g(f(x)) = g(\lambda y + (1-\lambda) z) = \lambda g(y) + (1-\lambda) g(z)$, which implies that x can be written as a convex combination of another two points $g(y), g(z) \in P$. Hence, x is not an EP. Contradiction $\lightning$

$\Longleftarrow$ direction: Similarly, we first assume x is not an EP of P, which means $x = \lambda y + (1-\lambda) z$ for some $y, z \in P$, $\lambda \in [0,1]$. We apply f to both sides and we have $f(x) = f(\lambda y + (1-\lambda) z) = \lambda f(y) + (1-\lambda) f(z)$, which implies $f(x)$ is not an EP in Q. Contradiction $\lightning$

(b). Consider the following affine mapping

$f(x) = \hat{A} x + \hat{b}$, where $\hat{A} = \begin{bmatrix} I \\ A \end{bmatrix}_{(n+k)\times n}$, $\hat{b} = \begin{bmatrix} 0 \\ -b \end{bmatrix}_{(n+k)\times 1}$, $x \in R^n$

$g(y) = A' y + b'$, where $A' = [I \ \ 0]_{n\times(n+k)}$, $b' = 0_{n\times 1}$, $y \in R^{n+k}$

It can be easily shown that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$. Thus, P and Q are isomorphic.

2. ~~Let $\hat{x}$ be an EP of Q. If $\hat{x}$ is also an extreme point of P, good.~~
~~If $\hat{x}$ is not an EP of P, then $\hat{x}$ we know~~

Let $\hat{x}$ be an EP of Q. If $\hat{x}$ happens to be the EP of P, great!
Suppose $\hat{x}$ is not an EP of P.
We know that $\hat{x}$ has n active constraints since $\hat{x}$ is an EP. One of them is the $a^Tx=b$ constraint and this means $\hat{x}$ has another $n-1$ active constriants that are associated with P.
This means $\hat{x}$ lies on the segment defined by these $n-1$ active constraints, which has two extreme points on each side. Hence, $\hat{x}$ can be written as a convex combination of these two extreme points.

3. Proof by contradiction: Assume ~~there~~ the optimal solution $x' \in P$ does not satisfy $Ax'=c$. That is, there exists x' s.t. $Ax'>c$, $x' \geq 0$, and $c^Tx' < c^Tx^*$.
$Ax'>c \Rightarrow (Ax')^T > c^T \Rightarrow (Ax')^T x^* > c^T x^* > c^T x'$. We can see that the left-hand is $(Ax')^T x^* = x'^T A^T x^* = x'^T A x^* = x'^T c = c^T x'$. Hence, we get $c^Tx' > c^Tx'$, contradiction.
Thus, every optimal solution x^* should satisfy $Ax'=c$, $x' \geq 0$.

4. Idea: Write an equivalent statement (a') to (a). Apply theorem of alternative to get (b'), prove (b') is equivalent to b.
(a) can be rewritten as the following statement:
(a'): $\exists x \in R^n$ s.t. $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} = \begin{bmatrix} A \\ C \\ -C \end{bmatrix} \in R^{(m+2p)\times n}$, $\tilde{b} = \begin{bmatrix} b \\ d \\ -d \end{bmatrix} \in R^{m+2p}$
Apply the theorem of alternative we can have (a')'s alternative:
(b'): $\exists \begin{bmatrix} z \\ v \\ w \end{bmatrix} \in R^{m+2p}$ s.t. $\begin{bmatrix} z \\ v \\ w \end{bmatrix} \geq 0$, $\tilde{A}^T \begin{bmatrix} z \\ v \\ w \end{bmatrix} = 0$, $\tilde{b}^T \begin{bmatrix} z \\ v \\ w \end{bmatrix} < 0$. We can expand these and we get $z \geq 0$, $v \geq 0$, $w \geq 0$, $A^T z + C^T(v-w) = 0$, $b^T z + d^T(v-w) < 0$. (see next page)

Continue on 4: Let $y = v - w$, we can further simplify the statement to

$\exists z \in \mathbb{R}^m, y \in \mathbb{R}^p$ s.t. $z \geq 0$, $A^T z + C^T y = 0$, $b^T z + d^T y < 0$,

which is exactly (b). Proof is complete.

5. Idea: write down the statement in standard form, apply Farkas' Lemma to get the alternative, and show the alternative never hold.

Rewrite the statement as:

(I) $\exists y \in \mathbb{R}^n$ s.t. $Ay = b$ and $y \geq 0$, where $A = \begin{bmatrix} P-I \\ \mathbb{1}^T \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$

$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Its alternative is

(II) $\exists x \in \mathbb{R}^{n+1}$ s.t. $A^T x \geq 0$, $b^T x < 0$.

Let's expand (II) and we have

$$x = \begin{bmatrix} x_{1,n} \\ x_{n+1} \end{bmatrix}, \quad \begin{bmatrix} (P-I)^T & \mathbb{1} \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{n+1} \end{bmatrix} \geq 0, \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{n+1} \end{bmatrix} < 0$$

which is

$$(P-I)^T x_{1,n} + \mathbb{1} x_{n+1} \geq 0, \quad x_{n+1} < 0$$

This means $\exists x_{1,n} \in \mathbb{R}^n$ s.t. $(P-I)^T x_{1,n} > 0$. Recall that P is a matrix with non-negative entries in $[0,1]$, and each column of P sums up to 1, so, this will never be satisfied. ↯

Thus, (I) always holds.