The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

LECTURE 13

Aryan Mokhtari Monday, October 14, 2019.

Goal: In this lecture we talk about quasi-convex (quasi-concave and quasi-linear) functions and log-concave (log-convex) functions.

1 Quasi-convex functions

Definition 1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* ***quasi-convex*** *if its domain and all its* ***sub-level sets*** *are $\mathcal{S}_\alpha = \{\mathbf{x} \in \textbf{dom} f \mid f(\mathbf{x}) \leq \alpha\}$ for any $\alpha \in \mathbb{R}$, are convex.*

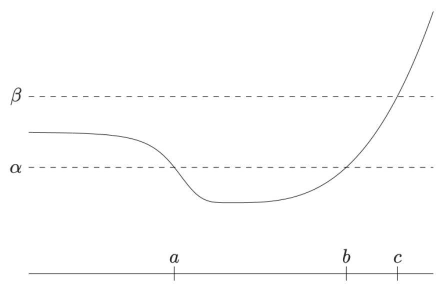

Figure 3.9 A quasiconvex function on $\mathbf{R}$. For each α, the α-sublevel set S_α is convex, *i.e.*, an interval. The sublevel set S_α is the interval $[a, b]$. The sublevel set S_β is the interval $(-\infty, c]$.

Definition 2. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* ***quasi-concave*** *if its domain and all its* ***super-level sets*** *are $\mathcal{S}'_\alpha = \{\mathbf{x} \in \textbf{dom} f \mid f(\mathbf{x}) \geq \alpha\}$ for any $\alpha \in \mathbb{R}$, are convex.*

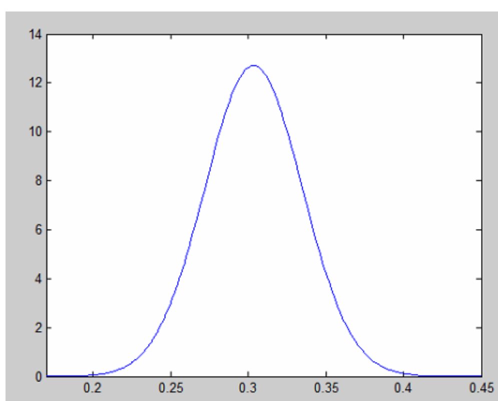

Figure 1: The probability density function of the normal distribution is quasi-concave but not concave.

Definition 3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is is called* ***quasi-linear*** *if it is both quasi-convex and quasi-concave.*

Note: If a function f is quasi-linear, then its domain, and every level set $\mathcal{L}_\alpha = \{\mathbf{x} \in \mathbf{dom} f \mid f(\mathbf{x}) = \alpha\}$ are convex.

1.1 A few examples

- The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt{|x|}$ is quasi-convex.
Let's prove it. For any $\alpha < 0$, we can see that $\mathcal{S}_\alpha = \varnothing$. For any $\alpha \in R_+$, consider the sub-level set $\mathcal{S}_\alpha = \{x \in \mathbb{R} \mid \sqrt{|x|} \leq \alpha\}$. This sub-level set can also be written as

$$\begin{aligned}\mathcal{S}_\alpha &= \{x \in \mathbb{R} \mid \sqrt{|x|} \leq \alpha\} \\ &= \{x \in \mathbb{R} \mid |x| \leq \alpha^2\} \\ &= \{x \in \mathbb{R} \mid -\alpha^2 \leq x \leq \alpha^2\} \\ &= [-\alpha^2, \alpha^2]\end{aligned}$$

Is it also quasi-concave? No, since its super-level set is $\mathcal{S}'_\alpha = \{x \in \mathbb{R} \mid \sqrt{|x|} \geq \alpha\} = (-\infty, -\alpha^2] \cup [\alpha^2, \infty)$

- The function $f(x) = \log x$ is quasi-convex on $\mathbb{R}_{++}$. Why? Since we have

$$\begin{aligned}\mathcal{S}_\alpha &= \{x \in \mathbb{R}_{++} \mid \log x \leq \alpha\} \\ &= \{x \in \mathbb{R}_{++} \mid x \leq e^\alpha\} \\ &= (0, e^\alpha]\end{aligned}$$

It is also quasi-concave, since

$$\begin{aligned}\mathcal{S}_\alpha &= \{x \in \mathbb{R}_{++} \mid \log x \geq \alpha\} \\ &= \{x \in \mathbb{R}_{++} \mid x \geq e^\alpha\} \\ &= [e^\alpha, \infty)\end{aligned}$$

Hence, $f(x) = \log x$ is quasi-linear on $\mathbb{R}_{++}$.

- Linear-fractional function: The function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f = \frac{\mathbf{a}^\top \mathbf{x} + b}{\mathbf{c}^\top \mathbf{x} + d}$ with $\mathbf{dom} f = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{c}^\top \mathbf{x} + d > 0\}$ is quasi-convex, and quasi-concave, i.e., quasi-linear.

$$\begin{aligned}\mathcal{S}_\alpha &= \{x \in \mathbb{R}^n \mid \mathbf{c}^\top \mathbf{x} + d > 0\ ,\ f(\mathbf{x}) \leq \alpha\} \\ &= \left\{x \in \mathbb{R}^n \mid \mathbf{c}^\top \mathbf{x} + d > 0\ ,\ \frac{\mathbf{a}^\top \mathbf{x} + b}{\mathbf{c}^\top \mathbf{x} + d} \leq \alpha\right\} \\ &= \left\{x \in \mathbb{R}^n \mid \mathbf{c}^\top \mathbf{x} + d > 0\ ,\ \mathbf{a}^\top \mathbf{x} + b \leq \alpha \mathbf{c}^\top \mathbf{x} + \alpha d\right\} \\ &= \left\{x \in \mathbb{R}^n \mid \mathbf{c}^\top \mathbf{x} + d > 0\ ,\ (\mathbf{a} - \alpha \mathbf{c})^\top \mathbf{x} + (b - \alpha d) \leq 0\right\}\end{aligned}$$

which is a convex set, since it is the intersection of an open half-space and a closed half-space. (The same method can be used to show its super-level sets are convex.)

1.2 Properties of quasi-convex functions

Zeroth-order condition

- A function f is quasi-convex **if and only if** $\mathbf{dom} f$ is convex and for any $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ and $\theta \in [0, 1]$ we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}$$

Note: The value of the function on a segment does not exceed the maximum of its values at the endpoints.

- A function f is quasi-concave **if and only if** $\mathbf{dom} f$ is convex and for any $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ and $\theta \in [0, 1]$ we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$$

Using this definition it is easy to see that any convex function is quasi-convex since for a convex function f we can write

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}$$

Example: The function rank $\mathbf{X}$ is quasi-concave on $\mathbf{S}^n_{++}$ since for any $\theta \in [0, 1]$ and $\mathbf{X}, \mathbf{Y} \in \mathbf{S}^n_{++}$ we have

$$rank(\theta\mathbf{X} + (1-\theta)\mathbf{Y}) \geq \min\{rank(\mathbf{X}), rank(\mathbf{Y})\}$$

First-order condition

Differentiable function f with convex domain is quasi-convex if and only if

$$f(\mathbf{y}) \leq f(\mathbf{x}) \quad \Rightarrow \quad \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x}) \leq 0$$

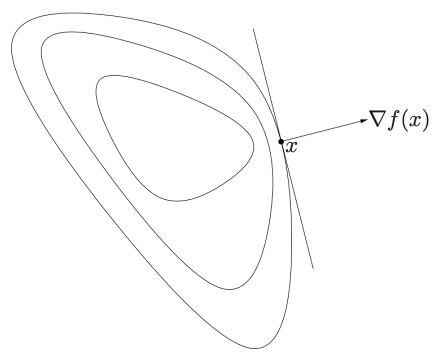

Figure 3.12 Three level curves of a quasiconvex function f are shown. The vector $\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{z \mid f(z) \leq f(x)\}$ at x.

Second-order condition

Twice differentiable function f with convex domain is quasi-convex if and only if

$$\mathbf{y}^\top \nabla f(\mathbf{x}) = 0 \quad \Rightarrow \quad \mathbf{y}^\top \nabla^2 f(\mathbf{x})^\top \mathbf{y} \geq 0$$

Proof of second-order conditions for quasi-convexity: Check the textbook.

2 Log-concave and log-convex functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is logarithmically concave or log-concave if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{dom} f$ and $\log f$ is a concave function. In other words, f is log-concave if for any $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ and $\theta \in [0, 1]$ we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq f(\mathbf{x})^\theta f(\mathbf{y})^{(1-\theta)}$$

In particular, the value of a log-concave function at the average of two points is at least the geometric mean of the values at the two points.

Further, f is log-convex if for any $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ and $\theta \in [0, 1]$ we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq f(\mathbf{x})^\theta f(\mathbf{y})^{(1-\theta)}$$

2.1 Examples

- Powers: x^α on $\mathbb{R}_{++}$ is log-concave for $\alpha \geq 0$ and log-convex for $\alpha \leq 0$

- Cumulative Gaussian distribution function Φ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

- Determinant. $\det(\mathbf{x})$ is log-concave on $\mathbf{S}^n_{++}$.

2.2 Properties

- A twice differentiable function f with convex domain is log-concave if and only if

$$f(\mathbf{x})\nabla^2 f(\mathbf{x}) \preceq \nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top$$

for all $\mathbf{x} \in \mathbf{dom} f$.
Proof: The Hessian of $\log(f(\mathbf{x}))$ is given by

$$\nabla^2 \log(f(\mathbf{x})) = \frac{1}{f(\mathbf{x})}\nabla^2 f(\mathbf{x}) - \frac{1}{f(\mathbf{x})^2}\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top$$

- Product of log-concave functions is log-concave.

- Convolution of convolution log-concave functions is log-concave.

$$(f * g)(x) = \int f(x-y)g(y)dy$$

Proof: Homework.