1. The Lagrangian is $L(t,x,Z,z,\nu) = \frac{1}{t} - tr(Z(\sum_{i=1}^{p} x_i v_i v_i^T - tI)) - z^T x + \nu(\mathbb{1}^T x - 1)$

$$= \frac{1}{t} + t\, tr(Z) - tr(Z \sum_{i=1}^{p} x_i v_i v_i^T) - z^T x + \nu \mathbb{1}^T x - \nu$$

$$= \frac{1}{t} + t\, tr(Z) - \sum_{i=1}^{p} tr(Z x_i v_i v_i^T) - z^T x + \nu \mathbb{1}^T x - \nu$$

$$= \frac{1}{t} + t\, tr(Z) - \sum_{i=1}^{p} x_i tr(v_i^T Z v_i) - z^T x + \nu \mathbb{1}^T x - \nu$$

$$= \frac{1}{t} + t\, tr(Z) + \sum_{i=1}^{p} x_i(-v_i^T Z v_i - z_i + \nu) - \nu$$

The dual function is thus $g(Z,z,\nu) = \inf_{t,x} L(t,x,Z,z,\nu)$

$$= \begin{cases} 2\sqrt{tr Z} - \nu, & \text{if } v_i^T Z v_i + z_i = \nu,\ Z \succeq 0 \\ -\infty & \text{o.w.} \end{cases}$$

The dual problem is thus

$$\begin{aligned} &\max \quad 2\sqrt{tr Z} - \nu \\ &\text{s.t.} \quad v_i^T Z v_i + z_i = \nu \\ &\qquad\ Z \succeq 0,\ z \succeq 0 \end{aligned}$$

It can be simplified to

$$\begin{aligned} &\max \quad 2\sqrt{tr Z} - \nu \\ &\text{s.t.} \quad v_i^T Z v_i \le \nu \\ &\qquad\ Z \succeq 0 \end{aligned}$$

2 (a) We know $\hat{x}$ minimizes $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$, so the gradient of $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$ evaluated at $\hat{x}$ is zero: $\sum_{i=1}^{m} \hat{r}_i (\hat{r}_i + \varepsilon)^{-\frac{1}{2}} a_i = 0$

Now we consider the dual problem of l_1-norm approximation problem:

$$\max \sum_{i=1}^{m} b_i \lambda_i$$

$$\text{s.t. } |\lambda_i| \le 1, \; i=1,\dots,m$$

$$\sum_{i=1}^{m} \lambda_i a_i = 0$$

Thus, we see $\lambda_i = -\frac{\hat{r}_i}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}}$, $i=1,\dots,m$ is dual feasible. Thus, the dual objective ~~of~~ function value at this λ is $\sum_{i=1}^{m} b_i \lambda_i = \sum -\frac{b_i \hat{r}_i}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}}$, which is a lower bound for p^*: $p^* \ge \sum_{i=1}^{m} -\frac{b_i \hat{r}_i}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}} = \sum_{i=1}^{m} (a_i^T \hat{x} - b_i) \frac{\hat{r}_i}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}} = \sum_{i=1}^{m} \frac{\hat{r}_i^2}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}}$

(b). Starting from the result above, subtract $\|A\hat{x} - b\|_1$ from both sides:

$$p^* - \|A\hat{x} - b\|_1 \ge \sum_{i=1}^{m} \left(\frac{\hat{r}_i^2}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}} - |\hat{r}_i| \right)$$

Rearrange this inequality:

$$\|A\hat{x} - b\|_1 \le p^* + \sum_{i=1}^{m} |\hat{r}_i| \left(1 - \frac{|\hat{r}_i|}{(\hat{r}_i^2 + \varepsilon)^{\frac{1}{2}}} \right)$$

3. The Lagrangian is $\mathcal{L}(x, r, v) = \sum_{i=1}^{m} \phi(r_i) + \sum_{i=1}^{m} v_i (a_i^T x - b_i - r_i)$

$$= \sum_{i=1}^{m} (\phi(r_i) - v_i r_i) + x^T A^T v - b^T v$$

The dual is thus $g(v) = \inf_{r \in R^m} \left(\sum_{i=1}^{m} (\phi(r_i) - v_i r_i) \right) + \inf_{x \in R^n} (x^T A^T v) - b^T v$

$$= \sum_{i=1}^{m} \inf_{r_i \in R} (\phi(r_i) - v_i r_i) - b^T v, \text{ if } A^T v = 0$$

$$= -\sum_{i=1}^{m} \sup_{r_i \in R} (v_i r_i - \phi(r_i)) - b^T v, \text{ if } A^T v = 0$$

$$= -\sum_{i=1}^{m} \phi^*(v_i) - b^T v, \text{ if } A^T v = 0$$

where $\phi^*(v_i)$ is the ~~congu~~ conjugate function of penalty function $\phi(u)$.

The Lagrangian dual is thus

$$\max_{v \in R^m} \ -\sum_{i=1}^{m} \phi^*(v_i) - b^T v$$
$$\text{s.t.} \quad A^T v = 0$$

The problem thus becomes finding the conjugate function of each penalty function.

(a). Deadzone-linear: The conjugate function is $\phi^*(v_i) = \begin{cases} |v_i|, & \text{if } |v_i| \le 1 \\ \infty, & \text{if } |v_i| > 1 \end{cases}$

Hence, the dual problem is

$$\max \ -\sum_{i=1}^{m} |v_i| - b^T v$$
$$\text{s.t.} \quad A^T v = 0$$
$$|v_i| \le 1, \ i = 1, \dots, m$$

which can be simplified to

$$\max \ -\|v\|_1 - b^T v$$
$$\text{s.t.} \quad A^T v = 0$$
$$\|v\|_\infty \le 1, \ i = 1, \dots, m$$

(b). Huber penalty: the conjugate function is $\phi^*(\nu_i) = \begin{cases} \frac{\nu_i^2}{4}, & \text{if } |\nu_i| \le 2 \\ \infty, & \text{otherwise} \end{cases}$

Hence, the dual problem is

$$\begin{aligned} \max \quad & -\sum_{i=1}^{m} \frac{\nu_i^2}{4} - b^T\nu \\ \text{s.t.} \quad & A^T\nu = 0 \\ & |\nu_i| \le 2, \quad i=1,\dots,m \end{aligned}$$

which can be simplified to

$$\begin{aligned} \max \quad & -\frac{1}{4}\|\nu\|_2^2 - b^T\nu \\ \text{s.t.} \quad & A^T\nu = 0 \\ & \|\nu\|_\infty \le 2 \end{aligned}$$

(c). Log-barrier: the conjugate function is $\phi^*(\nu_i) = \sup_{|x|<1} (\nu_i x + \log(1-x^2)) = -1 + \sqrt{1+\nu_i^2} + \log(-1+\sqrt{1+\nu_i^2}) - 2\log|\nu_i| + \log 2$

Hence, the dual problem is

$$\begin{aligned} \max \quad & -\sum_{i=1}^{m}\left(-1+\sqrt{1+\nu_i^2} + \log(-1+\sqrt{1+\nu_i^2}) - 2\log|\nu_i| + \log 2\right) - b^T\nu \\ \text{s.t.} \quad & A^T\nu = 0 \end{aligned}$$

(d). Relative deviation from one: the conjugate function is $\phi^*(\nu_i) = \begin{cases} -2\sqrt{-\nu_i}, & \text{if } \nu_i \le -1 \\ \nu_i - 1, & \text{if } -1 \le \nu_i \le 1 \\ -\infty, & \text{if } \nu_i \ge 1 \end{cases}$

If $\nu_i > 1$ for any feasible ν_i, the dual objective function value becomes unbounded, which leads to infeasibility in the primal. Hence, here, ~~for~~ to avoid that, we make $\nu_i \le 1$ a constraint:

Dual is

$$\begin{aligned} \max \quad & -\sum_{i=1}^{m} \phi^*(\nu_i) - b^T\nu \\ \text{s.t.} \quad & A^T\nu = 0 \\ & \nu \le 1 \end{aligned}$$

where $\phi^*(\nu_i) = \begin{cases} -2\sqrt{-\nu_i}, & \nu_i \le -1 \\ \nu_i - 1, & \nu_i \ge -1 \end{cases}$

4. (a) Let $t_i \geq \|A_i x - b\|$, for $i = 1, \dots, k$

Then we can rewrite the problem as
$$\min \; p^T t$$
$$\text{s.t.} \; \|A_i x - b\| \leq t_i, \; i = 1, \dots, k$$

1). l_1-norm:
$$\min \sum_{i=1}^{k} p_i \mathbb{1}^T t_i$$
$$\text{s.t.} \; -t_i \leq A_i x - b \leq t_i, \; i = 1, \dots, k$$
which is an LP with $x \in R^n$, $t_i \in R^m$, $i = 1, \dots, k$

2). l_2-norm:
$$\min \; p^T t$$
$$\text{s.t.} \; \|A_i x - b\|_2 \leq t_i, \; i = 1, \dots, k$$
which is a SOCP with $x \in R^n$, $t \in R^k$

3). l_∞-norm:
$$\min \; p^T t$$
$$\text{s.t.} \; -t_i \mathbb{1} \leq A_i x - b \leq t_i \mathbb{1}, \; i = 1, \dots, k$$
which is an LP with $x \in R^n$, $t \in R^k$

(b). Let's take a look at $\sup_{l_{ij} < a_{ij} \leq u_{ij}} |a_i^T x - b_i|$, which is the supremum of the absolute value of one component in $Ax - b$ over all possible choice of a_i^T, a vector where each component is bounded by l_{ij} and u_{ij}.

$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} |a_i^T x - b_i| = \sup_{l_{ij} \leq a_{ij} \leq u_{ij}} (\max(a_i^T x - b_i, -a_i^T x + b_i)) = \max\Big(\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} (a_i^T x - b_i), \sup_{l_{ij} \leq a_{ij} \leq u_{ij}} (-a_i^T x + b_i)\Big)$$

Now, consider
$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} (a_i^T x - b_i) = \Big(\sum_{j=1}^{n} \sup_{l_{ij} \leq a_{ij} \leq u_{ij}} (a_{ij} x_j)\Big) - b_i = \sum_{j=1}^{n} \left(\bar{a}_{ij} x_j + v_{ij} |x_j|\right) - b_i,$$
$$= \bar{a}_i^T x + v_i^T |x| - b_i$$

where $\bar{a}_{ij} = \frac{l_{ij} + u_{ij}}{2}$, $v_{ij} = \frac{u_{ij} - l_{ij}}{2}$.

Similarly, $\sup_{l_{ij} \le a_{ij} \le u_{ij}} (-a_i^T x + b_i) = -\bar{a}_i^T x + v_i^T |x| + b_i$

Therefore, $\sup_{l_{ij} \le a_{ij} \le u_{ij}} |a_i^T x - b_i| = |\bar{a}_i^T x - b_i| + v_i^T |x|$

1). l_1-norm: $\min \sum_{i=1}^{m} (|\bar{a}_i^T x - b_i| + v_i^T |x|)$, which can be expressed as

$$\begin{aligned} \min\ & \mathbb{1}^T (y + Vw) \\ \text{s.t. } & -y \le \bar{A}x - b \le y \\ & -w \le x \le w, \end{aligned}$$

which is an LP with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$.

2). l_2-norm: $\min \sum_{i=1}^{m} (|\bar{a}_i^T x - b_i| + v_i^T |x|)^2$, which can be expressed as

$$\begin{aligned} \min\ & t \\ \text{s.t. } & -y \le \bar{A}x - b \le y \\ & -w \le x \le w \\ & \|y + Vw\|_2 \le t, \end{aligned}$$

which is a SOCP with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, $t \in \mathbb{R}$

3) l_∞-norm: $\min \max_{i=1,\dots,m} (|\bar{a}_i^T x - b_i| + v_i^T |x|)$, which can be expressed as

$$\begin{aligned} \min\ & t \\ \text{s.t. } & -y \le \bar{A}x - b \le y \\ & -w \le x \le w \\ & -t\mathbb{1} \le \underbrace{\cancel{Ax-b}}_{y} + Vw \le t\mathbb{1}, \end{aligned}$$

which is an LP with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, $t \in \mathbb{R}$

(c). $P_i = \{a \mid C_i a \le d_i\}$.

$$\sup_{a_i \in P_i} |a_i^T x - b_i| = \sup_{a_i \in P_i} \max(a_i^T x - b_i, -a_i^T x + b_i) = \max(\sup_{a_i \in P_i}(a_i^T x - b_i), \sup_{a_i \in P_i}(-a_i^T x + b_i))$$

By LP strong duality

$$\sup_{a_i \in P_i} a_i^T x = \inf\{d_i^T v \mid C_i^T v = x, v \succeq 0\} \qquad \sup_{a_i \in P_i} -a_i^T x = \inf\{d_i^T w \mid C_i^T w = -x, w \succeq 0\}$$

Therefore, $t_i \ge \sup_{a_i \in P_i} |a_i^T x - b_i|$ iff there exist v, w, such that

$$v, w \succeq 0, \quad x = C_i^T v = -C_i^T w, \quad d_i^T v \le t_i, \quad d_i^T w \le t_i$$

This allows us to pose the original problem as

$$\begin{aligned} \min\ & \|t\| \\ \text{s.t.}\ & x = C_i^T v_i,\ x = -C_i^T w_i,\ i = 1, \dots, m \\ & d_i^T v_i \le t_i,\ d_i^T w_i \le t_i,\ i = 1, \dots, m \\ & v_i \succeq 0,\ w_i \succeq 0,\ i = 1, \dots, m \end{aligned}$$

1). l_1-norm:

$$\begin{aligned} \min\ & \mathbf{1}^T t \\ \text{s.t.}\ & x = C_i^T v_i,\ x = -C_i^T w_i,\ i = 1, \dots, m \\ & d_i^T v_i \le t_i,\ d_i^T w_i \le t_i,\ i = 1, \dots, m \\ & v_i \succeq 0,\ w_i \succeq 0,\ i = 1, \dots, m \end{aligned}$$

which is an LP with $t \in R^m$, $x \in R^n$, $v_i \in R^m$, $w_i \in R^m$, $i = 1, \dots, m$

2). l_2-norm:

$$
\begin{aligned}
\min\ & u \\
\text{s.t.}\ & x = C_i^T v_i,\ x = -C_i^T w_i,\ i=1,\dots,m \\
& d_i^T v_i \le t_i,\ d_i^T w_i \le t_i,\ i=1,\dots,m \\
& v_i \succeq 0,\ w_i \succeq 0,\ i=1,\dots,m \\
& \|t\|_2 \le u,
\end{aligned}
$$

which is a SOCP with $u \in R^1$, $t \in R^m$, $x \in R^n$, $v_i \in R^m$, $w_i \in R^m$, $i=1,\dots,m$

3) l_∞-norm:

$$
\begin{aligned}
\min\ & t \\
\text{s.t.}\ & x = C_i^T v_i,\ x = -C_i^T w_i,\ i=1,\dots,m \\
& d_i^T v_i \le t,\ d_i^T w_i \le t,\ i=1,\dots,m \\
& v_i \succeq 0,\ w_i \succeq 0,\ i=1,\dots,m
\end{aligned}
$$

which is an LP with $t \in R$, $x \in R^n$, $v_i \in R^m$, $w_i \in R^m$, $i=1,\dots,m$