1. Consider the following LP problem:

(Primal) $\quad$ (Dual)

$$\min\ c^T x \qquad \max\ -b^T z$$

$$\text{s.t. } Ax \le b \qquad \text{s.t. } A^T z + c = 0$$

$$z \ge 0$$

If the dual problem is infeasible ($d^* = -\infty$), then there are two cases for the primal:

- primal is ~~f~~ infeasible. Done
- primal is feasible.

I will now show that if primal is feasible, then it must be unbounded.

Since the dual problem is infeasible, then it means there does not exist $z \in \mathbb{R}^m$ such that $A^T z + c = 0$, $z \ge 0$. By applying Farkas' Lemma, we know that the alternative is alway true:

$$\exists\ x \in \mathbb{R}^n \text{ s.t. } Ax \ge 0,\ (-c)^T x < 0$$

we also know that primal is feasible, which means

$$\exists\ x' \in \mathbb{R}^n \text{ s.t. } Ax' \le b$$

Then we know that $x' - tx$ is also primal feasible since.

$$A(x' - tx) = Ax' - tAx \le b \quad \text{. for any } t \ge 0$$

Then the objective value of this new point will be

$$c^T(x' - tx) = c^T x' - t c^T x \xrightarrow{t \to +\infty} -\infty$$

Hence, the primal is unbounded.

Therefore, if the dual problem is infeasible, then primal is either infeasible or unbounded.

2. $\Longleftarrow$ direction: If x^*, p^* are primal and dual optimal solution, show that
$$\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*) \le \mathcal{L}(x, p^*), \text{ for any } x \ge 0, \text{ any } p.$$

proof: 1) show $\mathcal{L}(x^*, p^*) \le \mathcal{L}(x, p^*)$, for all $x \ge 0$. This is equivalent to show that
$c^T x^* + p^{*T}(b - Ax^*) \le c^T x + p^{*T}(b - Ax)$, for $\forall x \ge 0$, which is equivalent to show that
$c^T x^* \le c^T x + p^{*T}(b - Ax)$, for $\forall x \ge 0$. We know that $c^T x^* = b^T p^*$ (strong duality). This is equivalent to show that
$0 \le (c^T - p^{*T}A)x$, for $\forall x \ge 0$.
We know that p^* is dual optimal so $A^T p^* \le c$ which means $c^T - p^{*T}A \ge 0$.
Then the inequality holds since we're just taking the inner product of two non-negative vectors.
Hence, we know $\mathcal{L}(x^*, p^*) \le \mathcal{L}(x, p^*)$, $\forall x \ge 0$

2) show $\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*)$, $\forall p$. This is equivalent to show
$c^T x^* + p^T(b - Ax^*) \le c^T x^* + p^{*T}(b - Ax^*)$. We know that $Ax^* = b$ (x^* is primal optimal),
then we just to show that $p^T 0 \le p^{*T} 0$, which is obviously true.
Hence, we know $\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*)$, $\forall p$

Therefore, we have $\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*) \le \mathcal{L}(x, p^*)$ for all $x \ge 0$, for all p.

$\Longrightarrow$ direction: If $\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*) \le \mathcal{L}(x, p^*)$ for all $x \ge 0$, for all p, show that x^*, p^* are optimal solution to the primal and dual problem.

proof: 1) First, let me show that x^* ~~are~~ and p^* are primal and dual feasible.
$\mathcal{L}(x^*, p) \le \mathcal{L}(x^*, p^*)$ can be expanded to $c^T x^* + p^T(b - Ax^*) \le c^T x^* + p^{*T}(b - Ax^*)$, $\forall p \Rightarrow$
$p^T(b - Ax^*) \le p^{*T}(b - Ax^*)$ $\forall p \Rightarrow b - Ax^* = 0 \Rightarrow x^*$ is primal feasible. (Continued on next page)
$x^* \ge 0$

2.(continued) Similarly, expand $L(x^*,p^*) \le L(x,p^*)$ $\forall x \ge 0$ to $c^T x^* + p^{*T}(b - Ax^*) \le c^T x + p^{*T}(b - Ax)$. We know from (1) that x^* is primal feasible, so we simplify it to $c^T x^* \le (c^T - p^{*T}A)x + p^{*T}b$, $\forall x \ge 0$. In order to ensure the right-hand side is larger than or equal to the left-hand side for arbitrary large x, $c^T - p^{*T}A \ge 0$. This is just $A^T p^* \le c$, which means p^* is dual feasible.

Next, we will show that $c^T x^* \le b^T p^*$ and plus the fact $c^T x^* \ge b^T x^*$ (weak duality) we know that $c^T x^* = b^T p^*$, which means x^* and p^* are optimal solution.

Expand $L(x^*,p) \le L(x,p^*)$ for all $x \ge 0$, all p to $c^T x^* + p^T(b - Ax^*) \le c^T x + p^{*T}(b - Ax)$
$\Rightarrow c^T x^* \le b^T p^* + (c^T - p^{*T}A)x \Rightarrow c^T x^* \le b^T p^* + (c - A^T p^*)^T x$, for all $x \ge 0$
Simply set $x = 0$ and we have $c^T x^* \le b^T p^*$. Thus, the proof is complete.

3. Idea: Convert statement (a), (b) to LP primal/dual problem and show their equivalence with the help of those two LP problems.

(Primal)
$$\min \tilde{c}^T x \quad \text{s.t. } \tilde{A}x \le \tilde{b}$$
where $\tilde{c} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^n$, $x \in R^n$,
$\tilde{A} = \begin{bmatrix} -A \\ -I \end{bmatrix} \in R^{(m+n)\times n}$,
$\tilde{b} = 0 \in R^{m+n}$

(Dual)
$$\max 0 \quad \text{s.t. } A^T p \le \tilde{c}, \; p \ge 0$$
where $p \in R^m$

$(b) \Rightarrow (a)$: If $\exists p$ s.t. $A^T p \le 0$, $p \ge 0$, $p^T a_1 < 0$, we can also find $p' = tp$ $(t > 0)$ s.t. $A^T p' \le 0$, $p' \ge 0$, $p'^T a_1 \le -1$.
This means p' is dual feasible with optimal objective function value $d^* = 0$. This means for any feasible point x in the primal ($\tilde{A}x \le \tilde{b}$, or equivalently $Ax \ge 0$, $x \ge 0$), the objective value $\tilde{c}^T x = -x_1 \ge 0$. This, together with $x_1 \ge 0$ (from the constraint $x \ge 0$), means x_1 must be 0.

$(a) \Rightarrow (b)$: If statement (a) holds, which is translated to all feasible points of primal have objective value 0. Then, of course, the optimal value is also 0. By strong duality, we know there exists a feasible point p for dual that achieves 0, which is exactly statement (b).

4. Idea: show that for any feasible point in one problem, there is always another feasible point in the other problem with exactly the same objective function value.

problem I $\Rightarrow$ problem II: Let $u \in R^m$, $v \in R^m$ be a feasible point in problem I. Simply let $z = v - u$. Then we know z is also a feasible point for problem II because

$A^T z = A^T(v-u) = \cancel{A^T} 0$

$||z||_1 = \sum_{i=1}^{m} |v_i - u_i| \le \sum_{i=1}^{m} v_i + u_i = 1$

Also, the objective function for $(u,v) = -b^T u + b^T v = b^T(v-u) = b^T z =$ objective function of z.

problem II $\Rightarrow$ problem I: Let $z \in R^m$ be a feasible point in problem II, which means $\begin{cases} A^T z = 0 \\ ||z||_1 \le 1 \end{cases}$

Let $u \in R^m$, $v \in R^m$ be the solution of $\begin{cases} v - u = z \\ v + u = |z| \end{cases}$. But (v,u) is not feasible for problem I because $1^T u + 1^T v = \sum_{i=1}^{m} (u_i + v_i) = \sum_{i=1}^{m} |z_i| \le 1$, when we want to have strict equality.

So, what we can do is to construct $u' = \frac{a}{2m} + u$, $v' = \frac{a}{2m} + v$, where $\begin{cases} a = 1 - \sum_{i=1}^{m} |z_i| \\ a \ge 0 \end{cases}$

then we can show that (u', v') is feasible for problem I because

$$\begin{cases} A^T u' - A^T v' = A^T(u' - v') = A^T(u - v) = A^T(-z) = 0 \\ 1^T u' + 1^T v' = \sum_{i=1}^{m} (u_i' + v_i') = \sum_{i=1}^{m} (u_i + \frac{a}{2m} + v_i + \frac{a}{2m}) = a + \sum_{i=1}^{m} (u_i + v_i) = a + \sum_{i=1}^{m} |z_i| = 1 \\ u' = \frac{a}{2m} + u \ge 0, \quad v' = \frac{a}{2m} + v \ge 0 \end{cases}$$

The objective function for $z = b^T z = b^T(v + \frac{a}{2m} - (u + \frac{a}{2m})) = b^T(v' - u') =$ objective function of (u', v').

5. (1). $\|x-x_0\|_\infty = \max_{i=1,\dots,n} |(x-x_0)_i|$ Let $t \geq \|x-x_0\|_\infty = \max_{i=1,\dots,n} |(x-x_0)_i|$. This is equivalent to $t \geq |(x-x_0)_i|, \forall i=1,\dots,n$. Then the original optimization problem can be written as

$$\begin{aligned} \min\ & t \\ \text{s.t. } & t \geq |(x-x_0)_i|,\ i=1,\dots,n, \\ & Ax \leq b \end{aligned}$$

which can be further simplified to

$$\begin{aligned} \min_{\substack{x\in\mathbb{R}^n,\\ t\in\mathbb{R}}}\ & c^T \begin{bmatrix} x \\ t \end{bmatrix}, \\ \text{s.t. } & \tilde{A}\begin{bmatrix} x \\ t \end{bmatrix} \leq \tilde{b} \end{aligned}$$

where $c^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T \in \mathbb{R}^{n+1}$, $\tilde{A} = \begin{bmatrix} -I & -\mathbb{1} \\ I & -\mathbb{1} \\ A & 0 \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+1)}$, $\tilde{b} = \begin{bmatrix} -x_0 \\ x_0 \\ b \end{bmatrix} \in \mathbb{R}^{2n+m}$

(2) Its dual problem is

$$\begin{aligned} \max_{z\in\mathbb{R}^{2n+m}}\ & -\tilde{b}^T z \\ \text{s.t. } & \tilde{A}^T z + \tilde{c} = 0 \\ & z \geq 0 \end{aligned}$$

, which we can simplify with $z = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{matrix} \}n \\ \}n \\ \}m \end{matrix}$ to

$$\begin{aligned} \max_{\substack{u\in\mathbb{R}^n \\ v\in\mathbb{R}^n \\ w\in\mathbb{R}^n}}\ & x_0^T(u-v) - b^T w \\ \text{s.t. } & -u+v+A^T w = 0 \\ & -\mathbb{1}^T u - \mathbb{1}^T v + 1 = 0 \end{aligned}$$

We can further simplify it by letting $y = u - v \in \mathbb{R}^n$ to

$$\begin{aligned} \max_{\substack{y\in\mathbb{R}^n \\ w\in\mathbb{R}^m}}\ & x_0^T y - b^T w \\ \text{s.t. } & A^T w = y \\ & w \geq 0 \\ & \|y\|_1 \leq 1 \end{aligned}$$

which is ultimately

$$\begin{aligned} \max_{w\in\mathbb{R}^m}\ & w^T(Ax_0 - b) \\ \text{s.t. } & \|A^T w\|_1 \leq 1 \\ & w \geq 0 \end{aligned}$$

(3) I can't solve it, but here is my attempt: The optimal dual objective is $d^* = w^{*T}(Ax_0 - b)$. According to strong duality, $d^* = t^*$, which is $w^{*T}(Ax_0 - b) = t^* \Rightarrow (A^T w^*)^T x_0 = w^{*T} b + t^*$

But if we plug in $x_{feasible} \in P$ then $w^{*T}(Ax_{feasible} - b) \leq 0 < t^* \Rightarrow (A^T w^*)^T x_{feasible} < w^{*T} b + t^*$

Thus, the hyperplane $(A^T w^*)^T x = w^{*T} b + t^*$ separate all $x \in P$ and x_0, in the sense that x_0 lies on the hyperplane and $x \in P$ on one side of the hyperplane.