The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

Lecture 16

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Goal: In this lecture, we talk about Second-Order Cone Programming (SOCP), Robust Linear Programming, and Geometric Programming (GP).

1 Second-Order Cone Programming (SOCP)

Definition 1. *A second-order cone program (SOCP) is an optimization problem of the form:*

$$\begin{aligned} \min: \quad & \mathbf{q}^\top \mathbf{x} \\ \text{s.t.}: \quad & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \le \mathbf{c}_i^\top \mathbf{x} + d_i, \qquad i = 1, \ldots, m. \\ & \mathbf{F}\mathbf{x} = \mathbf{g}. \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ *is the optimization variable, and* $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{A}_i \in \mathbb{R}^{n_i \times n}$, $\mathbf{b}_i \in \mathbb{R}^{n_i}$, $\mathbf{c}_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$, $\mathbf{F} \in \mathbb{R}^{p \times n}$, *and* $\mathbf{g} \in \mathbb{R}^p$.

- The inequalities are called second-order cone (SOC) constraints.
Note that we can also write the equality constraint $\mathbf{F}\mathbf{x} = \mathbf{g}$ as a second-order cone constraint by setting $\mathbf{A}_{m+1} = \mathbf{F}$, $\mathbf{b}_{m+1} = -\mathbf{g}$, $\mathbf{c}_{m+1} = \mathbf{0}$, and $d_{m+1} = 0$.

Remark 1. *As we discussed in Lecture 9, the second order cone in dimension* $n+1$ *is the set*

$$\mathcal{L}^{n+1} = \{(\mathbf{x}, t) \mid \|\mathbf{x}\|_2 \le t\}.$$

Hence, the above optimziation problem can also be written as

$$\begin{aligned} \min: \quad & \mathbf{q}^\top \mathbf{x} \\ \text{s.t.}: \quad & (\mathbf{A}_i \mathbf{x} + \mathbf{b}_i, \mathbf{c}_i^\top \mathbf{x} + d_i) \in \mathcal{L}^{n_i+1}, \qquad i = 1, \ldots, m. \end{aligned}$$

Observation: If $\mathbf{A}_i = \mathbf{0}$ for $i = 1, \ldots, m$, then we obtain an LP. (LP $\subset$ SOCP)

Observation: If we set $\mathbf{c}_i = \mathbf{0}$ for $i = 1, \ldots, m$, then we obtain a QCQP. (QCQP $\subset$ SOCP)

1.1 Robust linear programming

The parameters in optimization problems are often uncertain, e.g., in an LP:

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & \mathbf{a}_i^\top \mathbf{x} \le b_i, \qquad i = 1, \ldots, m. \end{aligned}$$

there can be uncertainty in $\mathbf{c}$, $\mathbf{a}_i$, and b_i.

There are two different ways to address uncertainty in LP. (i) We assume that $\mathbf{a}_i$ belong to a set of vectors $\mathbf{a}_i \in \mathcal{E}_i$ and we want to ensure for any of possible choices $\mathbf{a}_i$ our solution is valid. This approach is also known as *deterministic approach*.

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & \mathbf{a}_i^\top \mathbf{x} \le b_i, \text{ for all } \mathbf{a}_i \in \mathcal{E}_i, \quad i = 1, \ldots, m. \end{aligned}$$

(ii) The second approach for modeling uncertainty in LP is assuming that $\mathbf{a}_i$ is a random variable and constraints must hold with probability larger than a threshold η. This approach is also known as *stochastic approach*.

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & P(\mathbf{a}_i^\top \mathbf{x} \le b_i) \ge \eta, \quad i = 1, \ldots, m. \end{aligned}$$

1.1.1 Deterministic approach via SOCP

Consider deterministic robust linear programming when the set $\mathcal{E}_i$ is an ellipsoid.

$$\mathcal{E}_i = \{\bar{\mathbf{a}}_i + \mathbf{P}_i \mathbf{u} \mid \|\mathbf{u}\|_2 \le 1\}$$

where $\mathbf{P}_i \in \mathbb{R}^{n \times n}$ and $\mathbf{u} \in \mathbb{R}^n$. Here, $\mathbf{P}_i$ and $\bar{\mathbf{a}}_i$ for $i = 1, \ldots, m$ are given. For each constraint i, we want to ensure that the constraint $\mathbf{a}_i^\top \mathbf{x} \le b_i$ is satisfied for all possible $\mathbf{a}_i \in \mathcal{E}_i$. Hence, the i-th constraint can be written as

$$\mathbf{a}_i^\top \mathbf{x} \le b_i, \text{ for all } \mathbf{a}_i \in \mathcal{E}_i \quad \Leftrightarrow \quad (\sup_{\mathbf{a}_i \in \mathcal{E}_i} \mathbf{a}_i^\top \mathbf{x}) \le b_i$$

Further, note that

$$\begin{aligned} \sup_{\mathbf{a}_i \in \mathcal{E}_i} \mathbf{a}_i^\top \mathbf{x} &= \sup_{\|\mathbf{u}\| \le 1} (\bar{\mathbf{a}}_i + \mathbf{P}_i \mathbf{u})^\top \mathbf{x} \\ &= \bar{\mathbf{a}}_i^\top \mathbf{x} + \sup_{\|\mathbf{u}\|_2 \le 1} \mathbf{u}^\top \mathbf{P}_i^\top \mathbf{x} \\ &= \bar{\mathbf{a}}_i^\top \mathbf{x} + \|\mathbf{P}_i^\top \mathbf{x}\|_2 \end{aligned}$$

Thus, the robust linear constraint can be expressed as

$$\mathbf{a}_i^\top \mathbf{x} \le b_i, \text{ for all } \mathbf{a}_i \in \mathcal{E}_i \quad \Leftrightarrow \quad \bar{\mathbf{a}}_i^\top \mathbf{x} + \|\mathbf{P}_i^\top \mathbf{x}\|_2 \le b_i$$

which is evidently a second-order cone constraint. Hence, the Robust LP problem in this case can be written as

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & \bar{\mathbf{a}}_i^\top \mathbf{x} + \|\mathbf{P}_i^\top \mathbf{x}\|_2 \le b_i, \quad i = 1, \ldots, m. \end{aligned}$$

which is an SOCP.

1.1.2 Stochastic approach via SOCP

The robust LP described above can also be considered in a statistical framework. Here we suppose that the parameters $\mathbf{a}_i$ are independent Gaussian random vectors, with mean $\bar{\mathbf{a}}_i$ and covariance $\mathbf{\Sigma}_i$. In this case, the constraint of the following LP

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & P(\mathbf{a}_i^\top \mathbf{x} \le b_i) \ge \eta_i, \quad i = 1, \dots, m. \end{aligned}$$

Based on the probability distribution of $\mathbf{a}_i$, we know that $\mathbf{a}_i^\top \mathbf{x}$ is a Gaussian random variable with mean $\bar{\mathbf{a}}_i^\top \mathbf{x}$ and variance $\mathbf{x}^\top \mathbf{\Sigma}_i \mathbf{x}$. Hence,

$$P(\mathbf{a}_i^\top \mathbf{x} \le b_i) = \mathbf{\Phi}\left(\frac{b_i - \bar{\mathbf{a}}_i^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \mathbf{\Sigma}_i \mathbf{x}}}\right) \qquad \text{where} \quad \mathbf{\Phi}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} \, dt$$

Hence, the i-th constraint can be written as (assuming that $\eta \ge 1/2$)

$$\begin{aligned} P(\mathbf{a}_i^\top \mathbf{x} \le b_i) \ge \eta_i \quad & \Leftrightarrow \quad \mathbf{\Phi}\left(\frac{b_i - \bar{\mathbf{a}}_i^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \mathbf{\Sigma}_i \mathbf{x}}}\right) \ge \eta_i \\ & \Leftrightarrow \quad \frac{b_i - \bar{\mathbf{a}}_i^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \mathbf{\Sigma}_i \mathbf{x}}} \ge \mathbf{\Phi}^{-1}(\eta_i) \\ & \Leftrightarrow \quad \bar{\mathbf{a}}_i^\top \mathbf{x} + \mathbf{\Phi}^{-1}(\eta_i) \|\mathbf{\Sigma}_i^{1/2} \mathbf{x}\|_2 \le b_i \end{aligned}$$

Hence, the Robust LP problem in this case can be written as

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & \bar{\mathbf{a}}_i^\top \mathbf{x} + \mathbf{\Phi}^{-1}(\eta_i) \|\mathbf{\Sigma}_i^{1/2} \mathbf{x}\|_2 \le b_i, \quad i = 1, \dots, m. \end{aligned}$$

which is an SOCP.

2 Geometric Programming (GP)

We proceed to study a family of optimization problems that are not convex in their natural form, but can be transformed to convex optimization problems, by a change of variables and a transformation of the objective and constraint functions.

Definition 2. *A function $f : \mathbb{R}^n \to R$ with* ***dom****$f = \mathbb{R}^n_{++}$ is called monomial, if it can be expressed as*

$$f(\mathbf{x}) := c x_1^{a_1} \dots x_n^{a_n},$$

where $c > 0$ and $a_i \in \mathbb{R}$.

Definition 3. *A sum of monomials, i.e., a function of the form*

$$f(\mathbf{x}) := \sum_{k=1}^{K} c_k x_1^{a_{1k}} \dots x_n^{a_{nk}},$$

where $c_k > 0$ is called a posynomial function (with K terms), or simply, a posynomial.

We call an optimizaiton problem a **Geometric Program** if it can be written as

$$
\begin{aligned}
\min: \quad & f_0(\mathbf{x}) \\
\text{s.t.}: \quad & f_i(\mathbf{x}) \leq 1, \quad i = 1, \ldots, m. \\
& h_i(\mathbf{x}) = 1, \quad i = 1, \ldots, p.
\end{aligned}
$$

where $f_0, f_1, \ldots f_m$ are posynomial and $h_1, \ldots, h_p$ are monomial.

2.1 Geometric program in convex form

Change variables to $y_i = \log x_i$ so that $x_i = \exp y_i$
If f is the monomial function of x mentioned above then we have

$$
f(\mathbf{x}) := c x_1^{a_1} \ldots x_n^{a_n} = c(e^{y_1})^{a_1} \ldots (e^{y_n})^{a_n} = e^{\mathbf{a}^\top \mathbf{y} + b}
$$

where $b = \log(c)$. Similarly, for a posynomial fucntion we have

$$
f(\mathbf{x}) := \sum_{k=1}^{K} c_k x_1^{a_{1k}} \ldots x_n^{a_{nk}} = \sum_{k=1}^{K} e^{\mathbf{a}_k^\top \mathbf{y} + b_k}
$$

where $\mathbf{a}_k = (a_{1k}, \ldots, a_{nk})$ and $b_k = \log(c_k)$. The geometric program can be expressed in terms of the new variable $\mathbf{y}$ as

$$
\begin{aligned}
\min: \quad & \sum_{k=1}^{K_0} e^{\mathbf{a}_{0k}^\top \mathbf{y} + b_{0k}} \\
\text{s.t.}: \quad & \sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}^\top \mathbf{y} + b_{ik}} \leq 1, \quad i = 1, \ldots, m. \\
& e^{\mathbf{g}_i^\top \mathbf{y} + h_i} = 1, \quad i = 1, \ldots, p.
\end{aligned}
$$

Now we transform the objective and constraint functions, by taking the logarithm. This results in the problem

$$
\begin{aligned}
\min: \quad & \tilde{f}_0(\mathbf{y}) := \log \left(\sum_{k=1}^{K_0} e^{\mathbf{a}_{0k}^\top \mathbf{y} + b_{0k}} \right) \\
\text{s.t.}: \quad & \tilde{f}_i(\mathbf{y}) := \log \left(\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}^\top \mathbf{y} + b_{ik}} \right) \leq 0, \quad i = 1, \ldots, m. \\
& \tilde{h}_i(\mathbf{y}) := \mathbf{g}_i^\top \mathbf{y} + h_i = 0, \quad i = 1, \ldots, p.
\end{aligned}
$$

Since the functions $\tilde{f}_i$ are convex, and $\tilde{h}_i$ are affine, this problem is a convex optimization problem. If the posynomial objective and constraint functions all have only one term, i.e., are monomials, then the convex form geometric program reduces to a linear program!

Examples: Read the textbook.