The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

LECTURE 8

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Goal: In this lecture, we prove that max flow = min cut.

1 Total Unimodularity (TUM)

Definition: A matrix is totally unimodular if all its minors are −1, 0, 1. (A minor is the determinant of a square submatrix)

Example: The following matrix is unimodular

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{1}$$

Example: edge-node incidence matrix of a directed graph (Proof: Homework)

Important property of unimodular matrices: All the elements are −1, 0, or 1.

Totally unimodular matrices are very well behaved, because they always define polytopes with integer vertices, as long as the right-hand side is integer-valued

Theorem 1. *Consider the polyhedron $P = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. If $\mathbf{A}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then all the extreme points of P are integer vectors.*

Proof: Homework.

2 Maximum Flow = Minimum Cut

In this section, we aim to show that for a directed graph with edge capacities, the maximum possible flow from source to sink is equal to the minimum capacity possible for any $s - t$ cut of the graph. Recall that the maximum flow problem can be written as

$$\begin{aligned} \text{maximize} \quad & \phi \\ \text{subject to} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \\ & \mathbf{A}\mathbf{x} = \phi \mathbf{e} \end{aligned}$$

where $\mathbf{e} = [1; 0; \ldots; 0; -1] \in \mathbb{R}^m$. Here, $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the edge-node incidence matrix of the underlying graph, $\mathbf{c} \in \mathbb{R}^n$ is the concatenation of capacities, and each element of $\mathbf{x}$ indicates the flow assigned to each edge of the graph. Since the variables of this problem are ϕ and $\mathbf{x}$ we rewrite the problem as

Primal Problem (Max Flow)

$$\begin{array}{ll} \text{maximize} & \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ \phi \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} \mathbf{A} & -\mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \phi \end{bmatrix} = \mathbf{0}_m \\ \\ & \begin{bmatrix} \mathbf{I} & \mathbf{0}_n \\ -\mathbf{I} & \mathbf{0}_n \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \phi \end{bmatrix} \leq \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_n \end{bmatrix} \end{array}$$

If we consider the dual variables $\mathbf{z}_1 \in \mathbb{R}^n$, $\mathbf{z}_2 \in \mathbb{R}^n$, and $\mathbf{y} \in \mathbb{R}^m$, then the dual of this LP is given by

Dual Problem (Max Flow)

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_n \end{bmatrix}^\top \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \mathbf{0}_m^\top \mathbf{y} \\ \text{subject to} & \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{0}_n^\top & \mathbf{0}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{A}^\top \\ -\mathbf{e}^\top \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathbf{0}_n \\ 1 \end{bmatrix} \\ \\ & \mathbf{z}_1 \geq \mathbf{0}_n, \qquad \mathbf{z}_2 \geq \mathbf{0}_n \end{array}$$

which can be simplified as

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{z}_1 \\ \text{subject to} & \mathbf{z}_1 - \mathbf{z}_2 + \mathbf{A}^\top \mathbf{y} = \mathbf{0}_n \\ & y_m - y_1 = 1 \\ & \mathbf{z}_1 \geq \mathbf{0}_n, \qquad \mathbf{z}_2 \geq \mathbf{0}_n \end{array}$$

Simply replace $\mathbf{y}$ with $-\mathbf{y}$ to obtain

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{z}_1 \\ \text{subject to} & \mathbf{z}_1 - \mathbf{A}^\top \mathbf{y} = \mathbf{z}_2 \\ & y_1 - y_m = 1 \\ & \mathbf{z}_1 \geq \mathbf{0}_n, \qquad \mathbf{z}_2 \geq \mathbf{0}_n \end{array}$$

Eliminate $\mathbf{z}_2$ to obtain

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{z} \\ \text{subject to} & \mathbf{z} - \mathbf{A}^\top \mathbf{y} \geq \mathbf{0} \\ & y_1 - y_m = 1 \\ & \mathbf{z} \geq \mathbf{0}_n \end{array}$$

Note: If we write the constraint set of the above problem as $\mathbf{Az} \leq \mathbf{b}$ then $\mathbf{A}$ is TUM and $\mathbf{b}$ is an integer vector. Hence, all the extreme points of the dual problem of max flow problem are integer vectors.

This problem can be written as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(u,v)\in E} c_{u,v} z_{u,v} \\
\text{subject to} \quad & z_{u,v} \geq y_u - y_v \quad \text{for any } (u,v) \in E \\
& y_s - y_t = 1 \\
& z_{u,v} \geq 0 \qquad\quad \text{for any } (u,v) \in E
\end{aligned} \tag{2}
$$

Now we proceed to show that any $s-t$ cut can be considered as a feasible point of problem (2). To do so, note that for any $s-t$ denoted by $\mathcal{C}$ we can assign integer 1 to the elements of the cut $\mathcal{C}$ and 0 to the edges that do not belong to the cut $\mathcal{C}$. In other words, for a cut $\mathcal{C}$ define

$$
\begin{aligned}
& \hat{z}_{u,v} = 1 \qquad \text{if} \quad (u,v) \in \mathcal{C} \\
& \hat{z}_{u,v} = 0 \qquad \text{if} \quad (u,v) \notin \mathcal{C} \\
& \hat{y}_u = 1, \qquad \text{if } u \text{ can be reached from } s \\
& \hat{y}_u = 0, \qquad \text{if } u \text{ cannot be reached from } s \\
& \hat{y}_s = 1, \quad \hat{y}_t = 0.
\end{aligned}
$$

It can be shown that any $s-t$ cut can be represented uniquely based on the conditions above. Therefore, It is also easy to check that the above $\hat{\mathbf{z}}$ and $\hat{\mathbf{y}}$ are feasible in (2) [If $\hat{y_u} = 1$ and $\hat{y_v} = 0$, then $(u,v) \in \mathcal{C}$ and therefore $\hat{z}_{u,v} = 1$.] The cost of the dual problem for any $s-t$ cut is exactly equal to the capacity of the cut. Hence, we can conclude that the dual problem in (2) is a relaxed version of the min-cut problem, and, therefore,

$$
d^* \;\leq\; \text{minimum cut capacity}, \tag{3}
$$

where d^* is the optimal value of the dual problem in (2).

Now we proceed to show that the "minimum cut capacity" satisfies $d^* \geq$ minimum cut capacity. To do so, we use the fact that all extreme points of the dual problem are integer vectors.
Consider $\mathbf{z}^*$ and $\mathbf{y}^*$ as an optimal solution for the dual problem. It is clear that for such point we have $y_s^* = y_t^* + 1$. Now define the set of vertices $U = \{u \in V \mid y_u \geq y_s^*\}$. It is then clear that $s \in U$ and $t \notin U$. As a result, the set of edges $\mathcal{C} := \{(u,v) \in E \mid u \in U, v \notin U\}$ is an $s-t$ cut. Since, y_u^* are integers we can show that $z_{u,v}^* \geq y_u^* - y_v^* \geq y_s^* - y_v^* \geq y_s^* - (y_s^* - 1) \geq 1$ for each $(u,v) \in \mathcal{C}$. Hence, we can show that the optimal value of the dual problem is bounded below by

$$
\begin{aligned}
OPT = & \sum_{(u,v)\in E} c_{u,v} z_{u,v}^* \\
\geq & \sum_{(u,v)\in \mathcal{C}} c_{u,v} z_{u,v}^* \\
\geq & \sum_{(u,v)\in \mathcal{C}} c_{u,v} \\
= & \text{ capacity of cut } \mathcal{C}
\end{aligned}
$$

where the first inequality holds since all $c_{u,v}, z_{u,v}^*$ are nonnegative and $\mathcal{C} \subset E$ and the second inequality holds since $z_{u,v}^* \geq 1$ for each $(u,v) \in \mathcal{C}$. Therefore, the optimal value of the dual is greater than or equal to the capacity an $s-t$ cut, hence, it should be also greater than or equal to the capacity of the minimum $s-t$ cut, i.e.,

$$d^* \;\geq\; \text{minimum cut capacity} \tag{4}$$

Hence, we can conclude that

$$d^* \;=\; \text{minimum cut capacity} \tag{5}$$

Also, according to strong duality we know that max flow $= p^* = d^*$ and therefore

$$\text{max flow} \;=\; \text{minimum cut capacity} \tag{6}$$