1. • Yes, each element in this set can be written as $a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$. We can easily verify that ~~this set satisfies~~ addition and scalar multiplication satisfy the eight axioms.

• No, let $f = x$ $(0 \le x \le 1)$, then obviously f is in the set but $2f$ is not.

• No, let $f = x$ $(0 \le x \le 1)$, then obviously f is in the set but $2f$ is not.

2. • Yes, $T(av_1 + bv_2) = av_1 + bv_2 = aT(v_1) + bT(v_2)$, for all $v_1, v_2 \in V$, $a, b \in \mathbb{R}$

• No except when $w_0 = 0$. $\begin{cases} \text{if } w_0 \neq 0,\ T(av_1+bv_2) = w_0 \neq aw_0 + bw_0 = aT(v_1) + bT(v_2), \\ \qquad \text{for all } v_1, v_2 \in V, a, b \in \mathbb{R} \\ \text{if } w_0 = 0,\ T(av_1+bv_2) = 0 = a \cdot 0 + b \cdot 0 = aT(v_1) + bT(v_2), \\ \qquad \text{for all } v_1, v_2 \in V, a, b \in \mathbb{R} \end{cases}$

• Yes, $T(av_1 + bv_2) = (av_1 + bv_2)' = av_1' + bv_2' = aT(v_1) + bT(v_2)$, for all $v_1, v_2 \in V$, $a, b \in \mathbb{R}$

• Yes, $T(av_1 + bv_2) = \int_0^1 (av_1 + bv_2)\,dx = a\int_0^1 v_1\,dx + b\int_0^1 v_2\,dx = aT(v_1) + bT(v_2)$, for all $v_1, v_2 \in V$, $a, b \in \mathbb{R}$

• Yes, $T(av_1 + bv_2) = \int_0^1 (av_1 + bv_2)x^3\,dx = a\int_0^1 v_1 x^3\,dx + b\int_0^1 v_2 x^3\,dx$
$= aT(v_1) + bT(v_2)$, for all $v_1, v_2 \in V$, $a, b \in \mathbb{R}$

3. • No, consider the linear operator that maps all inputs to 0 as a simple counterexample

• Yes. let $a_1 v_1 + \cdots + a_m v_m = 0$ for some non-trivial a. Then $a_1 T(v_1) + \cdots + a_m T(v_m) = 0$ with the same (non-trivial) coefficients.

6. • Using the theorem : $A_{m\times k}, B_{k\times n}$, $rank(A)+rank(B)-k \le rank(AB) \le \min\{rank(A), rank(B)\}$
we have $0 \le rank(AB) \le 5$

• similarly, the largest rank AB can be is 7.

7. We can write $w \in \mathbb{R}^n$ as $w = w_1 e_1 + \cdots + w_n e_n$, where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, $w_1, w_2, \dots, w_n \in \mathbb{R}$

$$f(w) = f(w_1 e_1 + \cdots + w_n e_n) = w_1 f(e_1) + \cdots + w_n f(e_n)$$
$$= \sum_{i=1}^{n} w_i f(e_i) = \langle w, x \rangle, \text{ where } x = \begin{bmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{bmatrix} \in \mathbb{R}^n.$$

8. • ~~True~~ False , $T(p)$ always missed the t^2 term for all $p \in V$

• True

• $\sum_{j=0}^{i} \frac{i!}{(i-j)!} a_j \neq 0, \forall i \in \{0, 1, \dots, d\}$

9. • Define column rank of A to be the number of independent columns of A. Column rank of A is then obviously less than or equal to n (# of columns). Similarly, row rank of A is less than or equal to m (# of rows).
A fundamental result in linear algebra is the column rank is equal to row rank, and it's referred to as just rank of A. Hence $\begin{cases} \text{rank of } A \le m \\ \text{rank of } A \le n \end{cases} \Rightarrow \{ \text{rank of } A \le \min(m, n)$

• Columns of AB can be thought of linear combination of columns of A. so, column space $(AB) \subseteq$ column space (A). Hence $\text{rank}(AB) \le \text{rank}(A)$;
Similarly, rows of AB can be thought of linear combination of rows of B. so, row space $(AB) \subseteq$ row space (B). Hence, $\text{rank}(AB) \le \text{rank}(B)$.
Therefore, we have $\text{rank}(AB) \le \min\{\cancel{\text{rank}} \text{rank}(A), \text{rank}(B)\}$.
$\text{rank}(AB) \ge \text{rank}(A) + \text{rank}(B) - n$ is known as Sylvester's rank inequality. There are many proof out there. An elegant proof can be found in https://math.stackexchange.com/q/1438762

• Let a_i, b_j, $1 \le i, j \le n$ be columns of A and B respectively. The column space of $A+B$ is $\text{span}(a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$ $\cancel{\subseteq \text{span}(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n) = \text{span}(a_1, \cdots, a_{\text{rank}(A)}, b_1, \cdots, b_{\text{rank}(B)})}$
~~Since a_i might be dependent to b_j so $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$~~
so $\text{rank}(A+B) = \dim(\text{span}(a_1+b_1, \cdots, a_n+b_n))$. We claim $\text{span}(a_1+b_1, \cdots, a_n+b_n) \subseteq \text{span}(a_1, \cdots, a_n) + \text{span}(b_1, \cdots, b_n)$
so $\text{rank}(A+B) \le \dim(\text{Span}(a_1, \cdots, a_n) + \text{Span}(b_1, \cdots, b_n)) \le \dim(\text{span}(a_1, \cdots, a_n)) + \dim(\text{span}(b_1, \cdots, b_n)) = \text{rank}(A) + \text{rank}(B)$

• I don't know. I just memorize it as the Frobenius inequality.

10. • Consider $V = \{$all polynomials of arbitrary degree$\}$, which is a infinite dimensional vector space.

Define a linear mapping T : $T(a_0 + a_1 x + a_2 x^2 + \cdots) = a_1 + a_2 x + a_3 x^2 + \cdots$

(It's easy to verify this mapping is linear)

T is also surjective, because for any $v = a_0 + a_1 x + \cdots$, we can always find $\hat{v} = a_{-1} + a_0 x + a_1 x^2 + \cdots$ $(a_{-1} \in \mathbb{R})$ such that $T(\hat{v}) = v$.

However, the null space of T $null(T) \neq \{0\}$ since a ~~vector~~ polynomial like $a_{-1} + 0 \cdot x + 0 \cdot x^2 + \cdots$ is in the null space.

• Consider the same infinite dimensional vector space $V = \{$all polynomials of arbitrary degree$\}$.

Define a linear mapping T : $T(a_0 + a_1 x + \cdots) = 0 + a_0 x + a_1 x^2 + \cdots$

(It's easy to verify T is linear)

The null space of T $null(T) = \{0\}$, but obviously T is not surjective since the outcome polynomial of T has always zero in the constant term.