The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

PROBLEM SET III

Due: Sunday, September 29, 2019.

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1. Show that for a linear program, if the dual problem is infeasible ($d^* = -\infty$), then the primal problem is either unbounded ($p^* = -\infty$) or infeasible ($p^* = +\infty$).

2. **(Saddle points of the Lagrangian)** Consider the standard form problem of minimizing $c^T x$ subject to $Ax = b$ and $x \geq 0$. We define Lagrangian by

$$L(x, p) = c^T x + p^T(b - Ax).$$

Consider the following "game": player 1 chooses some $x \geq 0$, and player 2 chooses some p; then, player 1 pays to player 2 the amount $L(x, p)$. Player 1 would like to minimize $L(x, p)$, while player 2 would like to maximize it.

A pair (x^*, p^*), with $x^* \geq 0$, is called an equilibrium point (or a saddle point, or a Nash equilibrium) if

$$L(x^*, p) \leq L(x^*, p^*) \leq L(x, p^*), \qquad \text{for all } x \geq 0, \text{for all } p.$$

(Thus, we have an equilibrium if no player is able to improve her performance by unilaterally modifying her choice.)

Show that a pair (x^*, p^*) is an equilibrium if and only if x^* and p^* are optimal solutions to the standard form problem under consideration and its dual, respectively.

3. Let A be a given matrix. Use duality to show that the following two statements are equivalent.

   (a) Every vector such that $Ax \geq 0$ and $x \geq 0$ must satisfy $x_1 = 0$.

   (b) There exists some p such that $A^T p \leq 0$, $p \geq 0$, and $p^T a_1 < 0$, where a_1 is the first column of A.

4. Show that the following problems are equivalent.

$$\begin{aligned} \text{maximize} \quad & -\mathbf{b}^\top \mathbf{u} + \mathbf{b}^\top \mathbf{v} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{u} - \mathbf{A}^\top \mathbf{v} = \mathbf{0} \\ & \mathbf{1}^\top \mathbf{u} + \mathbf{1}^\top \mathbf{v} = 1 \\ & \mathbf{u} \geq \mathbf{0}, \ \mathbf{v} \geq \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \mathbf{b}^\top \mathbf{z} \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{z} = \mathbf{0} \\ & \|\mathbf{z}\|_1 \leq 1 \end{aligned}$$

5. Solve Exercise 59 from LLP.

   Hint: Show that the dual problem can be written as

$$\begin{array}{rl} \text{maximize} & \mathbf{w}^\top(\mathbf{A}\mathbf{x}_0 - \mathbf{b}) \\ \text{subject to} & \|\mathbf{A}^\top \mathbf{w}\|_1 \leq 1 \\ & \mathbf{w} \geq \mathbf{0} \end{array}$$