The University of Texas at Austin
Department of Electrical and Computer Engineering

**EE381K: Convex Optimization — Fall 2019**

Lecture 17

Aryan Mokhtari Wednesday, October 30, 2019.

---

**Goal:** In this lecture, we talk about Semidefinite Programming (SOCP), Schur complement, and the connection between SOCP and SDP.

# 1 Semidefinite Programming (SDP)

The general form of a Semidefinite Program is given by

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & x_1\mathbf{F}_1 + x_2\mathbf{F}_2 + \cdots + x_n\mathbf{F}_n + \mathbf{G} \preceq \mathbf{0} \\ & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{G}, \mathbf{F}_1, \ldots, \mathbf{F}_n \in \mathbf{S}^k$ and $\mathbf{A} \in \mathbb{R}^{p\times n}$.

**Note:** The inequality constraint is called linear matrix inequality (LMI).
We might have multiple LMI constraints: for example

$$x_1\hat{\mathbf{F}}_1 + x_2\hat{\mathbf{F}}_2 + \cdots + x_n\hat{\mathbf{F}}_n + \hat{\mathbf{G}} \preceq \mathbf{0}, \qquad x_1\tilde{\mathbf{F}}_1 + x_2\tilde{\mathbf{F}}_2 + \cdots + x_n\tilde{\mathbf{F}}_n + \tilde{\mathbf{G}} \preceq \mathbf{0}$$

we can write them as

$$x_1 \begin{bmatrix} \hat{\mathbf{F}}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{F}}_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{\mathbf{F}}_n & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{F}}_n \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{bmatrix} \preceq \mathbf{0}$$

## 1.1 Inequality form semidefinite programs

An inequality form SDP, analogous to an inequality form LP, has no equality constraints, and one LMI

$$\begin{aligned} \min: \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.}: \quad & x_1\mathbf{F}_1 + x_2\mathbf{F}_2 + \cdots + x_n\mathbf{F}_n + \mathbf{G} \preceq \mathbf{0} \end{aligned}$$

-If we have multiple LMI we can combine them
-We can also write $\mathbf{Q}\mathbf{x} \leq \mathbf{h}$ as an LMI. Note that this constraint is equivalent to

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + \cdots + x_n\mathbf{q}_n \leq \mathbf{h}$$

where $\mathbf{q}_i$ is the $i$-th column of $\mathbf{Q}$. This condition is equivalent to

$$x_1\mathbf{diag}(\mathbf{q}_1) + x_2\mathbf{diag}(\mathbf{q}_2) + \cdots + x_n\mathbf{diag}(\mathbf{q}_n) + \mathbf{diag}(\mathbf{h}) \preceq \mathbf{0}$$

- Hence, we can also write $\mathbf{A}\mathbf{x} = \mathbf{b}$ as an LMI.
**Observation:** Any linear equality and inequality constraint can be written as LMI. Hence, we can conclude that any LP can be written as an SDP (LP $\subset$ SDP).

**Remark 1.** *Whenever we have a matrix $F(\mathbf{x})$ that its components are affine functions of $x_1, \dots, x_n$, we can express $F(\mathbf{x})$ as an LMI. To be more specific, consider the general form of $F(\mathbf{x})$*

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} a_{11}^1 x_1 + \cdots + a_{11}^n x_n + b_{11} & \dots & a_{1n}^1 x_1 + \cdots + a_{1n}^n x_n + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}^1 x_1 + \cdots + a_{n1}^n x_n + b_{n1} & \dots & a_{nn}^1 x_1 + \cdots + a_{nn}^n x_n + b_{nn} \end{bmatrix}$$

*Then, we can write*

$$\mathbf{F}(\mathbf{x}) = x_1 \mathbf{F}_1 + x_2 \mathbf{F}_2 + \cdots + x_n \mathbf{F}_n + \mathbf{G}$$

*where*

$$\mathbf{F}_i = \begin{bmatrix} a_{11}^i & \dots & a_{1n}^i \\ \vdots & \ddots & \vdots \\ a_{n1}^i & \dots & a_{nn}^i \end{bmatrix} \quad \text{for } i = 1, \dots, n, \qquad \mathbf{G} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}.$$

### 1.2 Standard form semidefinite programs

Following the analogy to LP, a standard form SDP has linear equality constraints, and a (matrix) nonnegativity constraint on the variable $\mathbf{X} \in \mathbf{S}^n$

$$\begin{aligned} \min: \quad & \mathbf{tr}(\mathbf{C}\mathbf{X}) \\ \text{s.t.}: \quad & \mathbf{tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad \text{for } i = 1, \dots, p \\ & \mathbf{X} \succeq \mathbf{0}, \end{aligned}$$

where $\mathbf{C}, \mathbf{A}_1, \dots, \mathbf{A}_p \in \mathbf{S}^n$.

Note that when $\mathbf{A}$ and $\mathbf{B}$ are symmetric we have

$$\mathbf{tr}(\mathbf{A}\mathbf{B}) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij}$$

Hence, trace can be considered as an inner product of symmetric matrices.

A semidefinite program is an optimization problem over the space of symmetric matrices. It has two types of constraints

1. **Affine constraints** in the entries of the decision matrix $\mathbf{X}$
2. A constraint forcing some matrix to be **positive semidefinite**.

It is easy again to see an LP is a special case of SDP with diagonal matrices.

**Exercise:** Show that a linear matrix inequality (LMI) constraint can be written in standard form. As a result, show that an SDP in inequality form can be transferred to standard form.

## 2 Schur complement

Consider a symmetric matrix $\mathbf{M}$ given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$$

If matrix $\mathbf{C}$ is invertible, then the matrix $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top$ is called the Schur Complement of $\mathbf{C}$ in $\mathbf{M}$. Futher, if $\mathbf{A}$ is invertible the Schur complement of $\mathbf{A}$ in $\mathbf{M}$ is $\mathbf{C} - \mathbf{B}^\top\mathbf{A}^{-1}\mathbf{B}$.

**Lemma 1.** *For any symmetric matrix,* $\mathbf{M}$*, of the form* $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$ *if* $\mathbf{C}$ *is invertible then*
*(i)* $\mathbf{M} \succ \mathbf{0} \;\Leftrightarrow\; \mathbf{C} \succ \mathbf{0}$ *and* $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top \succ \mathbf{0}$.
*(ii) If* $\mathbf{C} \succ \mathbf{0}$*, then* $\mathbf{M} \succeq \mathbf{0} \;\Leftrightarrow\; \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top \succeq \mathbf{0}$.

*Proof.* Observe that

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^\top$$

Further, note that

$$\begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

We know that for any symmetric matrix, $\mathbf{T}$, and any invertible matrix, $\mathbf{N}$, the matrix $\mathbf{T}$ is positive definite iff $\mathbf{N}\mathbf{T}\mathbf{N}^\top$ (which is obviously symmetric) is positive definite. Further, a block diagonal matrix is positive definite iff each diagonal block is positive definite, which concludes the proof of first claim.
The second result also holds since any symmetric matrix, $\mathbf{T}$, and any invertible matrix, $\mathbf{N}$, we have $\mathbf{T} \succeq \mathbf{0}$ iff $\mathbf{N}\mathbf{T}\mathbf{N}^\top \succeq \mathbf{0}$ □

**Lemma 2.** *For any symmetric matrix,* $\mathbf{M}$*, of the form* $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$ *if* $\mathbf{A}$ *is invertible then*
*(i)* $\mathbf{M} \succ \mathbf{0} \;\Leftrightarrow\; \mathbf{A} \succ \mathbf{0}$ *and* $\mathbf{C} - \mathbf{B}^\top\mathbf{A}^{-1}\mathbf{B} \succ \mathbf{0}$.
*(ii) If* $\mathbf{A} \succ \mathbf{0}$*, then* $\mathbf{M} \succeq \mathbf{0} \;\Leftrightarrow\; \mathbf{C} - \mathbf{B}^\top\mathbf{A}^{-1}\mathbf{B} \succeq \mathbf{0}$.

### 2.1 Application of Schur complement for SDP

Consider the following optimization problem:

$$\begin{aligned} \min: \quad & \frac{(\mathbf{c}^\top\mathbf{x})^2}{\mathbf{d}^\top\mathbf{x}} \\ \text{s.t.}: \quad & \mathbf{d}^\top\mathbf{x} \geq 1, \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{aligned}$$

Then, we can write this problem as

$$\begin{aligned} \min: \quad & t \\ \text{s.t.}: \quad & (\mathbf{c}^\top\mathbf{x})^2 \leq t\mathbf{d}^\top\mathbf{x} \\ & \mathbf{d}^\top\mathbf{x} \geq 1, \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{aligned}$$

This problem can also be written as

$$
\begin{aligned}
\min: \quad & t \\
\text{s.t.}: \quad & \begin{bmatrix} t & \mathbf{c}^\top\mathbf{x} & \mathbf{0} & \mathbf{0} \\ \mathbf{c}^\top\mathbf{x} & \mathbf{d}^\top\mathbf{x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}^\top\mathbf{x} - 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{diag}(\mathbf{b} - \mathbf{A}\mathbf{x}) \end{bmatrix} \succeq \mathbf{0}
\end{aligned}
$$

which is an SDP. (Each component of the constraint matrix is an affine function of $x_1, \ldots, x_n, t$ and therefore it can be written as an LMI)

### 2.2 Matrix norm minimization

Let $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n$, where $\mathbf{A}_i \in \mathbb{R}^{p\times q}$. Consider the problem

$$
\min_{\mathbf{x}} \quad \|\mathbf{A}(\mathbf{x})\|_2
$$

which is the problem of minimizing $l_2$ norm (maximum singular value) of matrix $\mathbf{A}(\mathbf{x})$. This problem can also be written as

$$
\begin{aligned}
\min: \quad & t \\
\text{s.t.}: \quad & \|\mathbf{A}(\mathbf{x})\|_2 \leq t
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\min: \quad & t \\
\text{s.t.}: \quad & \mathbf{A}(\mathbf{x})^\top\mathbf{A}(\mathbf{x}) \preceq t^2\mathbf{I}
\end{aligned}
$$

We can write this problem as an SDP by using Schur complement

$$
\begin{aligned}
\min: \quad & t \\
\text{s.t.}: \quad & \begin{bmatrix} t\mathbf{I} & \mathbf{A}(\mathbf{x}) \\ \mathbf{A}(\mathbf{x})^\top & t\mathbf{I} \end{bmatrix} \succeq \mathbf{0}
\end{aligned}
$$

with variable $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. [All the components of the constraint matrix are affine functions of $x_1, \ldots, x_n$ and $t$.]

## 3 SOCP as a special case of SDP

Recall that SOCP has a general form of

$$
\begin{aligned}
\min: \quad & \mathbf{q}^\top\mathbf{x} \\
\text{s.t.}: \quad & \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^\top\mathbf{x} + d_i, \qquad i = 1, \ldots, m.
\end{aligned}
$$

(Note: The affine equality constraints can also be written as an SOC constraint). It can be shown that this problem is equivalent to the following SDP:

$$
\begin{aligned}
\min: \quad & \mathbf{q}^\top\mathbf{x} \\
\text{s.t.}: \quad & \begin{bmatrix} (\mathbf{c}_i^\top\mathbf{x} + d_i)\mathbf{I}_{n_i} & \mathbf{A}_i\mathbf{x} + \mathbf{b}_i \\ (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top & (\mathbf{c}_i^\top\mathbf{x} + d_i) \end{bmatrix} \succeq \mathbf{0}, \qquad i = 1, \ldots, m.
\end{aligned}
$$

Proof: We can assume $\mathbf{c}_i^\top \mathbf{x} + d_i > 0$. In this case, according to Schur complement we have

$$\begin{aligned}
\begin{bmatrix} (\mathbf{c}_i^\top \mathbf{x} + d_i)\mathbf{I}_{n_i} & \mathbf{A}_i\mathbf{x} + \mathbf{b}_i \\ (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top & (\mathbf{c}_i^\top \mathbf{x} + d_i) \end{bmatrix} \succeq \mathbf{0} \;&\Leftrightarrow\; \mathbf{c}_i^\top \mathbf{x} + d_i - (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top (\frac{1}{\mathbf{c}_i^\top \mathbf{x} + d_i})\mathbf{I}(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \geq 0 \\
&\Leftrightarrow\; \mathbf{c}_i^\top \mathbf{x} + d_i - (\frac{1}{\mathbf{c}_i^\top \mathbf{x} + d_i})(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \geq 0 \\
&\Leftrightarrow\; (\mathbf{c}_i^\top \mathbf{x} + d_i)^2 - \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2^2 \geq 0 \\
&\Leftrightarrow\; (\mathbf{c}_i^\top \mathbf{x} + d_i) \geq \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2
\end{aligned}$$

For the case that $\mathbf{c}_i^\top \mathbf{x} + d_i = 0$, one can easily show that

$$\begin{bmatrix} \mathbf{0}_{n_i} & \mathbf{A}_i\mathbf{x} + \mathbf{b}_i \\ (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top & 0 \end{bmatrix} \succeq \mathbf{0} \;\Leftrightarrow\; \mathbf{A}_i\mathbf{x} + \mathbf{b}_i = \mathbf{0}$$

(Consider vectors $[v_1, \dots, v_{n_i}, 1]$ and $[-v_1, \dots, -v_{n_i}, 1]$).

Therefore, SOCP $\subset$ SDP.

**Important observation:**

LP $\subset$ (convex) QP $\subset$ (convex) QCQP $\subset$ SOCP $\subset$ SDP.