1. For any two ~~sets~~ points in the set $\|Ax_1+b\|_2 \le c^Tx_1+d$ and $\lambda \in [0,1]$,

$\|Ax_2+b\|_2 \le c^Tx_2+d$

$$\|A(\lambda x_1+(1-\lambda)x_2)+b\|_2 = \|\lambda(Ax_1+b)+(1-\lambda)(Ax_2+b)\|_2 \le \lambda\|Ax_1+b\|_2+(1-\lambda)\|Ax_2+b\|_2$$
$$\le \lambda(c^Tx_1+d)+(1-\lambda)(c^Tx_2+d) = c^T(\lambda x_1+(1-\lambda)x_2)+d,$$

~~Thus~~ which means the convex combination of those two points is also in the set, so, the set is a convex ~~point~~ set.

2. (a) No, $M=\begin{bmatrix}1&2\\0&1\end{bmatrix}$. It's positive-semidefinite since $[v_1\ v_2]\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=(v_1+v_2)^2 \ge 0$, but $M_{11}<|M_{12}|$

(b) No, $M=\begin{bmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{bmatrix}$. $M_{ii}\ge|M_{ij}|$ for all i,j and M is symmetric, but M is not positive-semidefinit because $\begin{bmatrix}1\\1\\1\end{bmatrix}^TM\begin{bmatrix}1\\1\\1\end{bmatrix}=-3<0$

(c) Yes. $v^TMv=v^T\sum_i a_ia_i^Tv=\sum_i v^Ta_ia_i^Tv=\sum_i(a_i^Tv)^2 \ge 0$, for any v.

(d) Yes, let $v=\begin{bmatrix}x\\0\end{bmatrix}$, then $v^TMv=x^TM_1x\ge0$. Let $v=\begin{bmatrix}0\\y\end{bmatrix}$, $v^TMv=y^TM_3y\ge0$.

For any $v=\begin{bmatrix}x\\y\end{bmatrix}$, $v^T\begin{bmatrix}M_1&0\\0&M_3\end{bmatrix}v=x^TM_1x+y^TM_3y\ge0$.

3. $\Longrightarrow$ direction: We know $M\succeq0$, which means $v^TMv=v^TM^Tv=\mathrm{Tr}(v^TM^Tv)=\mathrm{Tr}(M^Tvv^T)\ge0$

Then for any $Z\succeq0$, factorize it to $Z=\sum_{i=1}^n\lambda_iv_iv_i^T$, where λ_i, v_i are the eigenvalue and eigenvector of Z.

$$\mathrm{Tr}(M^TZ)=\mathrm{Tr}(M^T(\lambda_1v_1v_1^T+\cdots+\lambda_nv_nv_n^T))=\underbrace{\lambda_1}_{\ge0}\underbrace{\mathrm{Tr}(M^Tv_1v_1^T)}_{\ge0}+\cdots+\underbrace{\lambda_n}_{\ge0}\underbrace{\mathrm{Tr}(M^Tv_nv_n^T)}_{\ge0}\ge0$$

3. (continued). $\Longleftarrow$ direction: We know $Tr(M^TZ) \geq 0$ for any $Z \succeq 0$. Then for any v,

$$v^TMv = Tr(v^TMv) = Tr(M^Tvv^T) = Tr(M^TZ) \geq 0, \text{ where } Z = vv^T \succeq 0.$$

4. (a). $\begin{matrix} x_1 \in C \\ x_2 \in C \end{matrix} \Rightarrow \begin{matrix} x_1^TAx_1 + b^Tx_1 + c \leq 0 \\ x_2^TAx_2 + b^Tx_2 + c \leq 0 \end{matrix}$

$$(\lambda x_1 + (1-\lambda)x_2)^T A(\lambda x_1 + (1-\lambda)x_2) + b^T(\lambda x_1 + (1-\lambda)x_2) + c = \lambda x_1^T A \lambda x_1 + (1-\lambda)x_2^T A(1-\lambda)x_2 +$$

$$\lambda x_1^T A(1-\lambda)x_2 + (1-\lambda)x_2^T A \lambda x_1 + \lambda b^T x_1 + (1-\lambda)b^T x_2 + c =$$

$$\underbrace{\lambda^2 x_1^T A x_1 + \lambda^2 b^T x_1 + \lambda^2 c}_{\leq 0} + \underbrace{(1-\lambda)^2 x_2^T A x_2 + (1-\lambda)^2 b^T x_2 + (1-\lambda)^2 c}_{\leq 0} + \lambda(1-\lambda)(\underbrace{x_1^TAx_2 + x_2^TAx_1 + b^Tx_1 + b^Tx_2 + 2c}))$$

$$\leq \lambda(1-\lambda)(2x_1^TAx_2 + b^T(x_1+x_2) + 2c)$$

Since $A \succeq 0$, then $(x_1 - x_2)^T A(x_1 - x_2) = x_1^TAx_1 - x_1^TAx_2 + x_2^TAx_2 - x_2^TAx_1 \geq 0$

hence, continuing the previous calculation, we have

$$\lambda(1-\lambda)(2x_1^TAx_2 + b^T(x_1+x_2) + 2c) \leq \lambda(1-\lambda)(x_1^TAx_1 + x_2^TAx_2 + b^T(x_1+x_2) + 2c)$$

$$= \underbrace{\lambda(1-\lambda)}_{\geq 0}(\underbrace{x_1^TAx_1 + b^Tx_1 + c}_{\leq 0} + \underbrace{x_2^TAx_2 + b^Tx_2 + c}_{\leq 0}) \leq 0$$

In other words, we just showed a convex combination of x_1, x_2 is also in C if $A \succeq 0$.

Hence, C is a convex set if $A \succeq 0$.

4. (continued) (b). No, consider a 1D counter-example where $C = \{x \mid -x^2 \le 0\} = \mathbb{R}$.
C is a convex set but $A = -1 < 0$ is obviously not positive-semidefinite

5. For any two points $\begin{bmatrix} x^{(1)} \\ y_1^{(1)} + y_2^{(1)} \end{bmatrix}, \begin{bmatrix} x^{(2)} \\ y_1^{(2)} + y_2^{(2)} \end{bmatrix} \in S$, we know $\begin{bmatrix} x^{(1)} \\ y_1^{(1)} \end{bmatrix} \in S_1$, $\begin{bmatrix} x^{(1)} \\ y_2^{(1)} \end{bmatrix} \in S_2$

$\begin{bmatrix} x^{(2)} \\ y_1^{(2)} \end{bmatrix} \in S_1$, $\begin{bmatrix} x^{(2)} \\ y_2^{(2)} \end{bmatrix} \in S_2$

We know S_1, S_2 are convex sets so $\lambda \begin{bmatrix} x^{(1)} \\ y_1^{(1)} \end{bmatrix} + (1-\lambda) \begin{bmatrix} x^{(2)} \\ y_1^{(2)} \end{bmatrix} \in S_1$

$\lambda \begin{bmatrix} x^{(1)} \\ y_2^{(1)} \end{bmatrix} + (1-\lambda) \begin{bmatrix} x^{(2)} \\ y_2^{(2)} \end{bmatrix} \in S_2$

Hence, $\begin{bmatrix} \lambda x^{(1)} + (1-\lambda) x^{(2)} \\ \lambda y_1^{(1)} + (1-\lambda) y_1^{(2)} + \lambda y_2^{(1)} + (1-\lambda) y_2^{(2)} \end{bmatrix} \in S$.

In other words, we showed $\lambda \begin{bmatrix} x^{(1)} \\ y_1^{(1)} + y_2^{(1)} \end{bmatrix} + (1-\lambda) \begin{bmatrix} x^{(2)} \\ y_1^{(2)} + y_2^{(2)} \end{bmatrix} \in S$. for $\lambda \in [0,1]$

Therefore, S is a convex set.