The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Convex Optimization — Fall 2019

Lecture 9

Aryan Mokhtari Monday, September 30, 2019.

Goal: In this lecture we give the basic definition of a convex set and convex optimization.

1 Formulation of Convex Optimization Problems

A typical **formulation of a convex optimization problem** is described by

$$\begin{aligned} \text{min}: &\quad f_0(\mathbf{x}) \\ \text{s.t.}: &\quad f_i(\mathbf{x}) \le 0, \qquad i = 1, \dots, m. \end{aligned}$$

where each f_j is convex for $j = 0, \dots, m$. We also often write the problem with an abstract (convex) set constraint,

$$\begin{aligned} \text{min}: &\quad f_0(\mathbf{x}) \\ \text{s.t.}: &\quad \mathbf{x} \in \mathcal{X}, \end{aligned}$$

where f_0 is a convex function and $\mathcal{X}$ is a convex set.

2 Convex Sets

Before we are able to formulate a convex optimization problem, we must understand what constitutes a convex set. We will begin by providing the general definition of a convex set and look at many different sets that are convex.

2.1 Definition of a Convex Set

Definition 1. *A* ***convex combination*** *of points* $\mathbf{x}_1, \dots, \mathbf{x}_k$ *is described by*

$$\sum_{i=0}^{k} \theta_i \mathbf{x}_i, \tag{1}$$

where $\theta_1 + \cdots + \theta_k = 1$ *and* $\theta_i \ge 0$.

Definition 2. *A set,* $\mathcal{X}$*, is called a* ***convex set*** *if and only if the convex combination of any two points in the set belongs to the set, i.e.*

$$\mathcal{X} \subseteq \mathbb{R}^n \textit{ is convex } \Leftrightarrow \textit{ if} \text{ for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X} \textit{ and} \text{ for all } \lambda \in [0,1],\ \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \mathcal{X}\ . \tag{2}$$

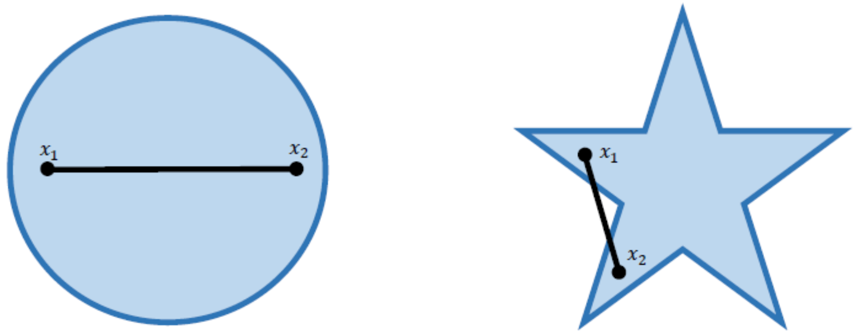

Figure 1: A convex set can be easily determined by examining whether the line segment between any two points in the set are in the set. Thus the figure on the left (circle) is in the set whereas the figure on the right (star) is not.

2.2 Examples of Convex Sets

Definition 3. *An* ***affine combination*** *of points* $\mathbf{x}_1, \ldots, \mathbf{x}_k$ *is described by*

$$\sum_{i=0}^{k} \theta_i \mathbf{x}_i, \tag{3}$$

where $\theta_1 + \cdots + \theta_k = 1$*. (Note: Affine combination lacks the nonnegative constraint on* θ_i*).*

Definition 4. *A set,* $\mathcal{A}$*, is called an* ***affine set*** *if and only if the affine combination of any two points in the set belongs to the set. In symbols:*

$$\mathcal{A} \subseteq \mathbb{R}^n \textit{ is affine } \Leftrightarrow \textit{ if } \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{A},\ \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \mathcal{A}\ . \tag{4a}$$

An equivalent definition, using the solution set of a system of linear equations is

$$\mathcal{A} \subseteq \mathbb{R}^n \textit{ is affine } \Leftrightarrow \left\{ x \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^n \right\}. \tag{4b}$$

Remark 1. *Affine sets are convex.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{A}$ and $\lambda \in [0,1]$. Taking the convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$\begin{aligned} \mathbf{A}(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &= \lambda \mathbf{A}\mathbf{x}_1 + (1-\lambda)\mathbf{A}\mathbf{x}_2 \\ &= \lambda \mathbf{b} + (1-\lambda)\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

Since the convex combination of points in $\mathcal{A}$ is also in the set, $\mathcal{A}$ is a convex set. □

Definition 5. ***Conic (non-negative) combination*** *of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*: Any point* $\mathbf{x}$ *that can be written as* $\mathbf{x} = \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2$ *with* $\theta_1 \geq 0$ *and* $\theta_2 \geq 0$*.*

Definition 6. ***Convex cone:*** *A set that contains all conic combinations of points in the set.*

Definition 7. *A **hyperplane**, $\mathcal{H}$, is a set defined by*

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{s}^T\mathbf{x} = \mathbf{b}\}(\mathbf{s} \neq \emptyset), \tag{5}$$

where b is the offset and s is the normal vector.

Remark 2. *Hyperplanes are convex.*

Proof. The proof follows the same line of reasoning as that of ***Remark 1***. □

Definition 8. *A **halfspace**, $\mathcal{H}_+$, is a set defined as*

$$\mathcal{H}_+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{s}^T\mathbf{x} \leq \mathbf{b}\}(\mathbf{s} \neq \emptyset), \tag{6}$$

where b is the offset and s is the normal vector.

Remark 3. *Halfspaces are convex.*

Proof. The proof follows the same line of reasoning as that of ***Remark 1***. □

Definition 9. *A **polyhedron** is an intersection of finite number of halfspaces and hyperplanes. A **polytope** is a polyhedron that is also bounded.*

Remark 4. *Polyhedra are convex.*

Proof. Covex sets are closed under intersections. This is readily seen when taking two points $x_1, x_2 \in A \cap B$, where both A and B are convex sets, and applying the definition of convex sets as in the proof of ***Remark 1***. Since halfspaces and hyperplanes are convex, polyhedra must also be convex. □

Definition 10. $\mathbf{S}^n$ *is the set of symmetric $n \times n$ matrices.* $\mathbf{S}^n_+ = \{\mathbf{X} \in \mathbf{S}^n \mid \mathbf{X} \succeq \mathbf{0}\}$ *is the set of positive semi-definite $n \times n$ matrices.* $\mathbf{S}^n_{++} = \{\mathbf{X} \in \mathbf{S}^n \mid \mathbf{X} \succ \mathbf{0}\}$ *is the set of positive definite $n \times n$ matrices.*

A very interesting set that turns out to be useful for many applications, and in general comes up frequently in many applications of convex optimization which we consider, is the set $\mathbf{S}^n_+$, of symmetric $n \times n$ matrices with non-negative eigenvalues. As a simple exercise, we can show that this set is in fact convex.

Proof. We use the fact that a symmetric matrix $\mathbf{M}$ is in $\mathbf{S}^n_+$ iff $\mathbf{x}^\top\mathbf{M}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Using this, checking convexity becomes straightforward. For a convex combination of any two matrices $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{S}^n_+$ we have

$$\begin{aligned}\mathbf{x}^\top(\lambda\mathbf{M}_1 + (1-\lambda)\mathbf{M}_2)\mathbf{x} &= \lambda\mathbf{x}^\top\mathbf{M}_1\mathbf{x} + (1-\lambda)\mathbf{x}^\top\mathbf{M}_2\mathbf{x} \\ &\geq 0\end{aligned}$$

Hence, $\lambda\mathbf{M}_1 + (1-\lambda)\mathbf{M}_2 \in \mathcal{S}^n_+$ for all $\lambda \in [0, 1]$. □

Definition 11. ***(Euclidean) ball*** *with center* $\mathbf{x}_c$ *and radius* r *is*

$$B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\}$$

Indeed, a Euclidean ball is a convex set.

Definition 12. ***Ellipsoid*** *with center* $\mathbf{x}_c$ *is defined as*

$$\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}(\mathbf{x} - \mathbf{x}_c) \leq 1\}$$

with $\mathbf{P} \in \mathbf{S}^n_{++}$*, (*$\mathbf{P}$ *is symmetric positive definite).*

It is easy to verify that an ellipsoid is a convex set.

Definition 13. ***Norm:*** *a function* $\|.\|$*that satisfies the following properties:*

- $\|\mathbf{x}\| \geq 0$*;* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$*.*
- $t\|\mathbf{x}\| = |t|\|\mathbf{x}\|$ *for* $t \in \mathbb{R}$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Definition 14. ***Norm ball*** *with center* $\mathbf{x}_c$ *and radius* r*:* $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$

Norm ball is a convex set!

Definition 15. *The* ***Norm Cone*** *associated with the norm* $\|.\|$ *is the set* $\{(\mathbf{x}, t) \mid \|\mathbf{x}\| \leq t\} \in \mathbb{R}^{n+1}$*.*

Norm cone is a convex cone. (it is a cone and also it is a convex set.)

Example: *The second-order cone* is the norm cone for the Euclidean norm, i.e.,

$$\begin{aligned}
\mathcal{C} &= \{(\mathbf{x}, t) \mid \|\mathbf{x}\|_2 \leq t\} \in \mathbb{R}^{n+1} \\
&= \left\{ \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} \;\middle|\; \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}^\top \begin{bmatrix} \mathbf{I} & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} \leq 0,\ t \geq 0 \right\}
\end{aligned}$$

Definition 16. *A convex hull of a set* $\mathcal{C}$ *is the set of all convex combinations of points in* $\mathcal{C}$*. As the name implies, convex hulls are convex. See Fig. 2 for an illustration.*

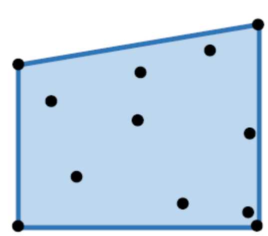

Figure 2: Convex hull