Overbuilding and Underinvestment over Housing Boom-Bust Cycles

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Abstract

In this paper, I unveil a novel mechanism through which a housing market boom can lead to a deep recession by decreasing the physical investment and rendering capital to be scarce. This inefficiency arises from a crowd-out effect: the available liquidity, which could otherwise be channeled into firms' capital investments (e.g., factories, equipment, R&D), is redirected toward the residential sector. The crowded-out physical investment subsequently amplifies the losses of the bust and prolongs the duration of the recession. Employing a new identification method of a shock that generates housing boom-bust cycles via a structure vector regression model, this paper empirically verifies the crowd-out effect, and find that every 2% jump in housing prices can crowd out 1% physical investment at the peak. Then, I develop a heterogeneous household model to quantify this welfare effects of this novel mechanism, It documents that the crowd-out effect can account for up to 13% of the welfare losses during the recession period. Finally, I show that a macroprudential policy upon the overheated housing market can alleviate the crowd-out effect and welfare losses significantly.

JEL classification: E21, E22, E30, E51, E58

Keywords: Heterogeneous Household, Consumption, Expectations, Great Recession, Business Cycle, VAR

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1 Introduction

The Great Recession in the United States, starting in December 2007, constituted the most severe economic downturn since the Great Depression. This recession precipitated a significant upswing in unemployment rates and a downturn in output, consumption, and investment, as examined by Mian and Sufi (2010) and Grusky et al. (2011). Numerous researchers have endeavored to comprehend the genesis of this recession, exploring questions regarding its origin and the mechanisms through which it permeated the entire economy. The consensus among many scholars posits that the boom and subsequent bust in the housing market exacerbated the collapse of the financial markets, leading to a recession, yet people debate in how this boom-bust cycle led to the recession. The Great Recession lingered for an extended period, a phenomenon some attribute to behavior inefficiency such as self-fulfilling equilibrium and "animal spirits"[1], liquid trap[2], and zero lower boundary(ZLB).[3] These channels typically suggest that the fallout from the housing market bust had tangible economic impacts, mainly through financial friction in supply side by influencing the production. Moreover, in demand side, real estate served as collateral enabling households to borrow money and smooth consumption patterns[4], but after the recession, the fall in price of real estate significantly eroded household wealth, adversely affecting the real economy. In this paper I propose a new mechanism, predicated on the intricate interplay between the supply and demand sides, that can precipitate a severe economic downturn and contribute to the economic malaise after the Great Recession.

The focus of this study is instead on the mechanism through which the overbuilding arises the capital scarcity (herein, overbuilding s defined as an increase in residential construction that lacks support from underlying economic fundamentals). Limited theoretical frameworks[5] have been employed to elucidate how a housing market boom might absorb substantial liquidity. When this boom is inefficient and becomes a bubble[6] which is caused by imperfect information[7] rather

[1] Islam and Verick (2011) and Cochrane (2011) discuss this problem.

[2] Brunnermeier (2009), Ivashina and Scharfstein (2010) and Jermann and Quadrini (2012)argue that the lack of liquidity of financial institution, mostly referring to the commercial bank, helps the crisis diffuse around and induce large recession.

[3] Christiano et al. (2015) and Fisher (2015) conduct an extension to the liquidity trap happened in great recession and argued that the prolonged trap caused the ZLB later. Recent works such as Guerrieri and Lorenzoni (2017) and Bayer et al. (2019) focused on the heterogeneous agent model and drew the conclusion that idiosyncratic shock and distribution channel are also important to explain the lack of liquidity.

[4] Eggertsson and Krugman (2012), Mian and Sufi (2010), Mian and Sufi (2014) and Qian (2023) discuss this problem. Household extracted their equity via collateral during the boom period which increased the consumption a lot. This constructed a mirage through general equilibrium. When the bust came, people struggled against the rapid constraint tightening and led to the Great Recession.

[5] except Beaudry et al. (2018), Rognlie et al. (2018) and J Caballero and Farhi (2018) and Chakraborty et al. (2018) recently.

[6] Throughout this paper I define the bubble as the inefficient boom, i.e. a boom that is not supported by the fundamental.

[7] In this paper I introduce imperfect information (misguided household beliefs) to blow the bubble up because it is the most convenient and suitable way to generate crowd-out effect. Yet the crowd-out effect is not unique to the imperfect information and other factors, such as real friction (financial accelerator, shadow bank, search and matching, moral hazard, etc.) and behavioural friction (sentiment shock, irrational expectation) can also produce the

than some shifts in economic fundamentals, the available liquidity, which could otherwise be channeled into firms' capital investments (e.g., factories, equipment, R&D), is redirected toward the residential sector. This suboptimal reallocation of liquidity within financial institutions results in inefficiencies when compared to a first-best allocation scenario. During a housing market boom, financial institutions exhibit a proclivity for directing loans toward the household sector, at the expense of directing to the firm sector. Owing to the increased influx of liquidity into the residential real estate market, there is a concomitant reduction in the allocation of liquidity to the supply side of the economy. This effect is especially pronounced when the liquidity supply is inelastic and resistant to expansion. Furthermore, the positive correlation between housing prices and nondurable consumption elucidates a reduction in investments in the nondurable production sector due to general equilibrium effects. This phenomenon can be comprehended through the goods market-clearing condition in a frictionless model, which apportions output into three categories: durable consumption, nondurable consumption, and physical investment. Assuming fixed labor supply and predetermined capital, increased investment in residential assets (i.e., durable consumption) and its corresponding rise in nondurable consumption yield a diminished investment in nondurable assets—a situation I call *crowd-out effect.*

This paper first shows that the crowd-out effect empirically exists and is significant in explaining the shortfall in physical investment subsequent to a housing market boom. To probe the intricacies of the housing market's boom-and-bust cycle, I introduce a novel identification strategy aimed at exploring the effect of housing price news shock in the context of imperfect information. Within this context, household are suspicious to a news shock when they observe it, as it may not realize when it were supposed to realize, and I call it the fake news shock: a news shock but will not realize when it should be. Because the fake news shock ultimately does not come to fruition, people's response to it is both suboptimal and inefficient. The subsequent market bust ensues when households eventually discern the fallacious nature of the original news, thereby prompting a recalibration of market dynamics. The empirical result reveals that any 2% increase in housing price can bring a 1% drop in physical investment at the peak. After the market busts, a 1% drop in housing price correlates with a 0.1% drop in nondurable consumption, which implies moderate welfare loss.

This paper then employs an Aiyagari-Bewley-Huggett model to offer rigorous analytical results explicating the emergence of physical capital scarcity, and complements this analysis with a comprehensive heterogeneous agent model to quantitatively examine the overbuilding phenomenon. A salient contribution of this analytical study is the theoretical investigation of this novel mechanism—the crowd-out effect—whereby overbuilding amplifies physical capital scarcity, thereby exacerbating the recession. The analysis further identifies three pivotal factors that modulate the crowd-out effect, each corresponding to a distinct functional role that residential assets play in the economy: utilitarian function (provides utility to household),

inefficiency.

proprietary function (works as a type of asset in budget constraint) and inequality (is related to wealth distribution in economy). The first characteristic of the residential asset, utilitarian feature, corresponds to relative intratemporal elasticity of substitution (IAS) to intertemporal elasticity of substitution (IES). This aspect has been extensively scrutinized within the housing literature, albeit typically within the framework of partial equilibrium analyses confined to the housing sector, for a considerable period of time.[8]

However, the interplay between intratemporal and intertemporal elasticity of substitution should not be overlooked within the framework of general equilibrium and the importance of the pass-through mechanism between durable and nondurable consumption in determining the crowd-out effect. In the standard Ramsey model, any fluctuation in housing prices will have a null wealth effect on households, as all profits accrued by producers are subsequently rebated to households as lump-sum transfers. In such an equilibrium setting, variations in housing prices are solely driven by the substitution effect. Yet there are two substitution effects, intratemporal substitution and intertemporal substitution, that determine the total substitution effect and the relative elasticity between them together governs the immediate response of nondurable consumption to changes in housing prices. When the relative intratemporal elasticity of substitution exceeds one[9] and continues to grow, the demand for intratemporal consumption smoothing supersedes that for intertemporal consumption smoothing. Consequently, there is either a modest increase or even a decline in nondurable consumption, as the complementarity between nondurable goods and housing services weakens—put differently, the substitution effect becomes increasingly pronounced. Therefore, the crowd-out effect is attenuated, as, in accordance with the market-clearing condition, an increase in residential investment is associated with a more modest rise in nondurable consumption.

In addition to the relative intratemporal elasticity of substitution, the financial friction also exerts the crowd-out effect, a concept well-embedded within the literature.[10] A housing market bubble driven by demand shocks elevates housing prices and triggers overbuilding, on top of a shift in demand function. As a result, the boom in the residential property market alleviates credit constraints. Households previously constrained by liquidity can then more readily meet their nondurable consumption requirements—a phenomenon termed "equity extraction" that Bhutta and Keys (2016) first proposed. Therefore, the rise of financial friction enhances the influence of the marginal propensity to consume (MPC), thereby accentuating the stimulation of

[8]For instance, when the utility function is separable in durable and nondurable goods, the relative intratemporal elasticity is always one, i.e. $\frac{\text{IAS}}{\text{IES}} = 1$. Iacoviello (2005), Liu et al. (2019) and Greenwald (2018) used the separable utility function to analyze thier problems. However because their models lack of intratemporal channel they can only put weight on other elements such as bubbles, self-fulling and multiple credit constraints to generate enough consumption response to house price. On the contrary Berger et al. (2018) and Kaplan et al. (2020) used the nonseparable utility function to discuss the housing problem and they focus on the consumption response more, which requires the intratemporal effect.

[9]Khorunzhina (2021) provides empirical evidence to $\frac{\text{IAS}}{\text{IES}} > 1$ in housing market.

[10]Garriga and Hedlund (2020),Hurst et al. (2016),Bailey et al. (2019), Garriga et al. (2017), Gorea and Midrigin (2017) and Chen et al. (2020) contribute to this literature and investigate how financial frictions influence the cross effect between the house and nondurable consumption.

nondurable consumption via the wealth effect, particularly during the housing market boom.

Moreover, household heterogeneity further amplifies the crowd-out effect through idiosyncratic income shocks and wealth distribution. Unlike representative agent models, households with uninsured income—subject to idiosyncratic shocks—exhibit a precautionary saving motive and consequently maintain a higher saving rate. During periods when income risk is countercyclical[11], overbuilding tends to coincide with economic upswings. Lower risk encourages households to reduce capital accumulation, thereby intensifying the crowding out of investment. Beyond the uncertainty channel, the distribution of wealth and MPC across households also holds considerable influence. Although the uncertainty operates in a second-order manner, wealth inequality exerts its effects in the first order. Household with greater disposable income, who is also the primary driver of overbuilding, contribute significantly to the aggregate-level crowd-out effect. Conversely, household facing tighter budget constraints tends to have a higher MPC and therefore exhibits more substantial increases in nondurable consumption in response to elevated housing prices, facilitated by equity extraction. Consequently, the more the wealth distribution skews to the right and the MPC distribution to the left[12], the more pronounced the crowd-out effect becomes.

Meanwhile, overbuilding can exert a significant influence on the depth of a recession through its ramifications on the labor market and general equilibrium. An initial shortfall in physical investment can set the stage for a severe recession, as the available total capital stock may prove inadequate for sustaining optimal production levels. Moreover, the presence of hand-to-mouth households, characterized by a high MPC and low labor income, can exacerbate the recession via a smaller demand, particularly given the complementarity between labor and capital. It is also crucial to consider the inherent durability and irreversibility of residential properties, which are underscored by high transaction costs. These features can give rise to a transitions from underinvestment to overinvestment during the bust phase, leading to economic losses due to an overshooting response in investment.[13] As a result, the duration of the recession may be prolonged, and its overall impact further intensified.

Finally, using a full-fledged heterogeneous agent model with financial frictions, I integrate the theory with the real world via the full-information Bayesian estimation and show that the crowd-out effect can explain up to 13% of the welfare losses during the recession period. Furthermore, after implementing a countercyclical macroprudential policy on controlling the credit expansion capacity and overheated housing market, the policy maker could calm the boom-bust cycles and

[11] Debortoli and Galí (2017), Acharya and Dogra (2020) and Bilbiie and Ragot (2021) analyzed this problem linked with monetary policy theoretically. Storesletten et al. (2004), Schulhofer-Wohl (2011) and Guvenen et al. (2014) analyzed the countercyclical idiosyncratic shock empirically.

[12] "In 2019, the top 10% of U.S. households controlled more than 70 percent of total household wealth" argued by Batty et al. (2020) and related data can be found in Distributional Financial Accounts in federal reserve web. Orchard et al. (2022) demonstrates that the MPC distribution is heavily right-skewed.

[13] McKay and Wieland (2019) refined this channel penetratingly and argued that this channel is important to explain the persistent ZLB and negative real interest rate after the Great Recession. This channel can also explain the low interest rate after the implementing of unconventional monetary policy, as Sterk and Tenreyro (2018) did.

decrease the welfare loss rendered by crowd-out effect significantly, from 13% to 6%.

This paper offers several noteworthy contributions to the existing literature. Firstly, it establishes a novel link between the housing market boom (overbuilding) preceding the recession and the recession itself. A substantial body of research has suggested that the surge in both the housing market and the market for nondurable goods before the Great Recession was largely an illusion, driven by expectation and speculation as opposed to sustainable growth. This is evidenced by the work of Landvoigt (2017), McQuinn et al. (2021) and Kaplan et al. (2020), among others. Other studies have posited that the credit supply also played a significant role, a perspective supported by Campbell and Cocco (2007), Favara and Imbs (2015), Favilukis et al. (2017), Justiniano et al. (2019), Mian and Sufi (2022) and Martínez (2023). However real estate only functioned as an asset in the context of collateral constraints among these studies and the inherent recession comes from the demand side that is initiated by the collapse in housing market. They omit the supply-side effect of the recession as a lot of researches contend that the capital misallocation contributed significantly to the Great Recession, such as Justiniano et al. (2010) and Justiniano et al. (2011), with supply-side effects accounting for nearly 40% of the economic downturn. When the major companies could undertake extensive margin investments through self-finance, as outlined in Bachmann et al. (2013) and Winberry (2016), the housing market boom not only impacted investment in the construction sector (Boldrin et al. (2013)) but also diverted physical investment from other sectors, through which I investigate in this paper, the crowd-our effect, to which Chakraborty et al. (2018) provides the evidence via micro data.

Some literature also employs the term "crowd-out" to describe the investment trade-off and capital misallocation between housing and non-housing sectors (labor, physical assets, and intangible assets), such as Dong et al. (2022) and Dong et al. (2023). However, their conceptualization of "crowd-out" aligns more closely with firms' balance sheet portfolio adjustments in partial equilibrium. This view deviates from reality[14], given that enterprises do not hold the majority of residential assets, nor do these assets play a pivotal role in production activity. These studies merely substituted a asset type in the asset misallocation literature of firms' problem with residential assets perfunctorily, like new wine in old bottles. The paper that bears the closest resemblance to this paper is that of Rognlie et al. (2018). They employed their partial equilibrium model (in the financial market) to explain the investment hangover via higher real interest rates. They proposed that an exogenous investment hangover at the outset precipitated a demand-driven recession due to high real interest rates, nominal rigidity, and the ZLB on monetary policy. Oppositely, my paper argues that the real interest rate and demand contraction are not the sole reason of the severity of recession, and even in the absence of nominal rigidity, overbuilding can also catalyze a supply-driven recession with significant welfare loss.

Secondly, this paper not only provides a new explanation for the severity of the Great Recession but also sheds light on elements of policy failure as discussed by Mitman (2016) and

[14] Kaplan et al. (2014) shows that "Housing equity forms the majority of illiquid wealth for households in every country with the exception of Germany".

Antunes et al. (2020). Accordingly, it also makes a valuable contribution to the literature on protracted recessions and the macroprudential policy. Since the recession is propelled by both supply and demand dynamics, singular stimulus efforts in the demand sector fail to effectively counteract the economic decline. Both of the aforementioned studies do not take into account the supply of housing services, even though Khan and Thomas (2008) demonstrated that a general equilibrium framework could yield entirely distinct results. My research extends the findings of Chodorow-Reich et al. (2021), Chahrour and Gaballo (2021) and Beaudry et al. (2018) and emphasizes the investment in the nondurable sector, arguing that overbuilding exacerbated the crowd-out effect and incited a more profound recession, which can be attenuate by macroprodential policy on supply side dramatically.

Furthermore, this research contributes to a methodological advancement in the literature: a new implementation of the SVAR identification strategy for distinctly identifying news and fake news shocks with endogenous contemporaneous effect, predicated on the theoretical insights of Wolf and McKay (2022). Almost all the previous identification methods to news shock, such as Barsky and Sims (2012), Blanchard et al. (2013), Barsky et al. (2015) and Sims (2016), identified the TFP shock which is observable and exogenous and its news does not have any contemporaneous effect on itself. However, there are enormous number of unobservable shocks with observation on endogenous variables such as news to inflation or monetary policy news shock.[15] All the existing literature aforementioned fail to identify this type of news shock, let alone the fake news shock.

Numerous studies emphasize the importance of household heterogeneity in explaining the housing boom-and-bust cycle, either empirically, such as Etheridge (2019), Mian et al. (2013), Li et al. (2016) and Díaz and Luengo-Prado (2010), or theoretically, such as Kaplan et al. (2020), Favilukis et al. (2017) and Garriga and Hedlund (2020). This paper builds a model that demonstrates the distribution of wealth and income is pivotal in determining the strength of overbuilding and supplements the literature on how expectations and animal spirits can fuel a boom. To achieve this, I also propose an enhancement in the numerical solution approach for handling intricate heterogeneous agent model with imperfect information on both first and second order. Earlier research either employed a guess-and-verify approach, as in Lorenzoni (2009) and Barsky and Sims (2012), or a reconstruction methodology, as demonstrated by Baxter et al. (2011), Blanchard et al. (2013) and Hürtgen (2014), to solve imperfect information DSGE models. The latter necessitates specific analytical equations to regulate the unobserved state variable with other state variables, which is unfeasible to derive from a heterogeneous agent model due to its extensive number of state variables.

In section 2 I use two identification strategy separately lay out the crowd-out effect generated by a contemporaneous and news shock to housing price. Later in section 3 I analytically demon-

[15] For instance, a news that indicating a drop of federal fund rate in the future will persuade household to increase their consumption this period, yet this contemporaneous economic boom will increase the federal fund rate right now.

strate the crowd-out effect is driven by relative intratemporal elasticity of substitution, financial friction, income inequality and wealth distribution. In section 4 I quantitatively investigate the drawback of crowd-out effect spawned by a fake news shock through the lens of a full fledged heterogeneous agent model. In the last section I conclude the result.

2 Empirical evidence

Firstly, we turn our attention to the statistical characteristics of the data, which provide insights into the mechanisms this paper seeks to discuss. Figure 1 displays the quantity of nonresidential investment (expressed as a percentage of GDP) spanning the period from 1960 to 2016. The data indicate that total investment typically grew during economic expansions and decreased during downturns. However, the upward trend that commenced in 2003 was interrupted by the Great Recession that struck towards the end of 2007 and the third spike does not reach as high as preceding two peaks. The trajectory following this event, although ostensibly similar to those observed during the 1970s and 1990s, is distinct due to the influence of crowd-out effect.

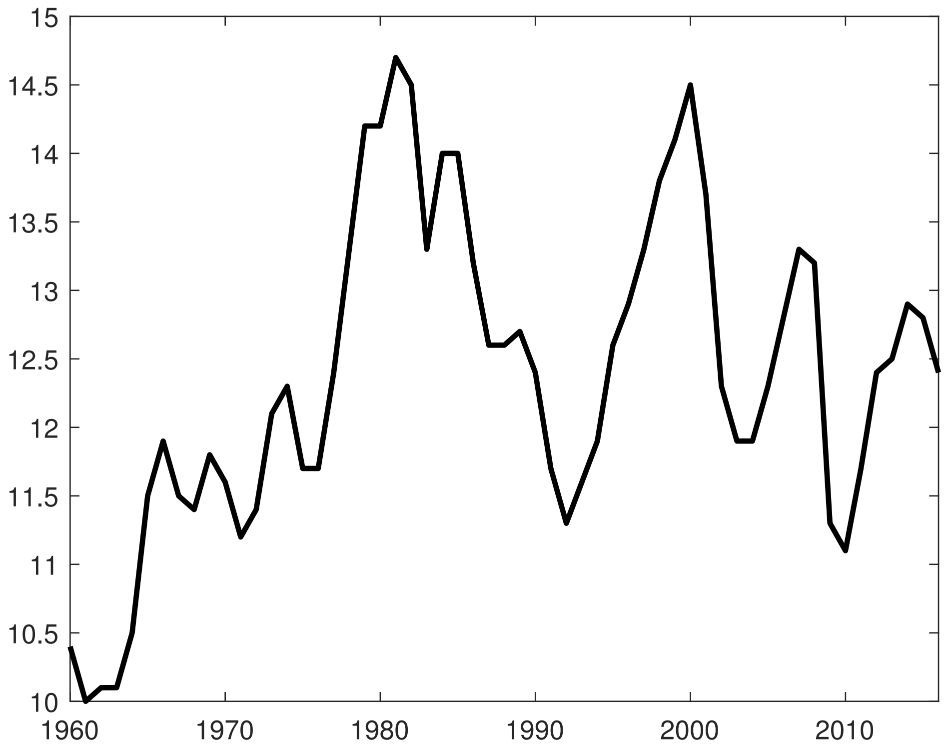


Figure 1: Nonresidential Investment (share of GDP)

The upward trend of the 1970s, spanning from 1975 to 1981, elevated from 11.7% of GDP in 1975 to 14.7% of GDP in 1981. This translates to an average annual growth of 4.27%. The trend experienced during the 1990s, from 1992 to 2000, resulted in an average annual increase of 3.53% in investment. and the most recent trend, from 2010 to 2014, spurred an average yearly increase of 4.05%. Contrarily, the trend observed prior to the Great Recession, from 2003 to

2007, only produced an average annual increase of 2.94%, the lowest one in history, as outlined in Table 1.

Table 1: Extent of increased investment

Trend range	1975-1981	1992-2000	2003-2007	2010-2014
Increased investment	4.27%	3.53%	2.94%	4.05%

Enlightened by these statistical disparities, one must consider the reasons that prompted the distinctive drop in investment prior to the Great Recession, thereby contributing to a portion of the output loss during the recession. My investigation implicates the housing market boom as a significant factor responsible for the reduction in investment. Specifically, the housing market boom led to the crowding out of investment in the demand sector. Financial institutions, in this scenario, might have prioritized households, favoring lending to households for real estate acquisitions over funding businesses for investment purposes. Simultaneously, households might have chosen to expend more resources on durable goods rather than depositing their savings in banks, which in conjunction with businesses, could eventually channel these funds into investments and physical capital. Due to the enduring nature of durable goods and the impetus for precaution, households would tend to secure these goods when prices are escalating or show a propensity to rise, a scenario evident from 2005 to 2007. This dynamic contributes to the crowding out of a portion of investment on the supply side. In summary, both demand and supply sides collaborate to squeeze out investment, with general equilibrium serving to magnify this effect. The importance of general equilibrium in explaining investment activities is widely recognized, as substantiated by Khan and Thomas (2008), who demonstrated that previous partial analyses such as that of Caballero et al. (1995) could be misleading. Given the equation $Y = C_{nd} + I + C_d$, an increase in C_{nd} and C_d will impact I since Y is concave at predetermined capital and labor, which cannot increase excessively as they are complementary to capital and constrained by technology.[16] In this context, this paper can also be viewed as a complement to the work of Berger and Vavra (2015).

2.1 Contemporaneous real price shock

Figure 16 in appendix sheds light on the crowd-out effect engendered by a housing market boom. However, given the speed at which the Impulse Response Function (IRF) reverts to the steady-state, it may not generate a significant scarcity in physical capital, thereby rendering the crowd-out effect less consequential in this rudimentary identification test. Moreover, the

[16]Upon detrending the growth elements in per capita real GDP, real nonresidential investment, and new constructed housing units, the data reveals a significant negative correlation between relative physical investment and residential estate investment. The relative correlation between relative physical investment and residential estate investment, $\mathrm{corr}(\frac{I_{t,c}}{y_{t,c}}, \frac{I^H_{t,c}}{y_{t,c}})$ is -0.873 and $\mathrm{corr}(\frac{I_t}{y_t}, \frac{I^H_t}{y_t}) = -0.17764$.(The subscript c denotes the cyclical data detrended from HP filter)

identification method I employed, namely Sims et al. (1986), has been critiqued for its potential overemphasis on identifying the underlying shocks, occasionally leading to artificial unreliability. To surmount these limitations, I utilize an alternative canonical workhorse identification method, the Cholesky decomposition, to identify the effect of contemporaneous housing price shocks. Following the method of Bernanke and Mihov (1998), Cholesky decomposition ensures that the shock can only impact the last variable at first, while the variables that precede it will not be contemporaneously influenced by the shock. Throughout this section, I am planning to argue the implications of the crowd-out effect incited by a housing market boom devoid of fundamental support. Therefore, I place the housing price at the end to simulate a non-fundamental housing price boom, where only the housing price is stimulated initially. As a result, a single unit housing price shock triggers the movement of other variables, following the inherent relationship and mechanism (Φ in equation 2). Inspired by existing literature, I order the economic variables in the data vector Y_t as

$$Y_t = [\Upsilon_t, y_t, c_t, i_t, r_t, r_t^d, q_t, h_t^s, p_t^h]' \tag{1}$$

where Υ_t is the NAHB/Wells Fargo Housing Market Index; y_t is real GDP; c_t is real non-durable consumption plus services; i_t is real investment in non-residential sector; r_t is the real interest rate; r_t^d is the real mortgage debt rate; q_t is the real stock price index; h_t^s is the real housing supply; p_t^h is the real housing price. I pick the time interval between 1985Q1 and 2007Q2 when the housing market boom reached its peak before the Great Recession.[17] I add housing market index Υ_t in estimation for comparative purpose as it is further used in section 2.2 and 2.3. All the variable are in logarithm form and are detrended by hybrid specification, a method through which I use all non-stationary variables as growth rate and all the variables in Y_t pass the unit-root test.

Figure 2 presents the impulse responses to a one-unit housing price shock, encased within a 90% confidence band. It reveals that a 10 basis points (bp) initial increase in housing price p_t^h instigates a housing market boom, escalating the housing price to a peak of 60bps four quarters later. This is approximately six times larger than the original increase. Individuals without enough residential asset holdings display optimism and a strong desire to acquire more houses. This, in turn, shifts the demand curve of residential assets upward as both price and quantity increase simultaneously. However, these individuals make only a partial down payment for the asset value, borrowing the remainder from commercial banks as mortgage debt. In parallel, those who already possess housing leverage the increased housing price to extract equity and liberate their liquidity, particularly if they are financially constrained and require more liquidity to meet their consumption needs. Nevertheless, the initial impacts on the consumption of non-durable goods and output are insignificant or negligible , potentially due to identification problems or data issues as argued by Sims (1998), Christiano et al. (1999) and Romer and Romer (2004). Investment in the non-durable sector declines throughout the entire period, stabilizing after two years at

[17] In appendix I do some robustness tests to this span selection by extending the data to 2019Q4 with shadow rate or 1-year treasury bonds rate that is proposed by Gertler and Karadi (2015). The crowd-out effect exists in all these robustness test.

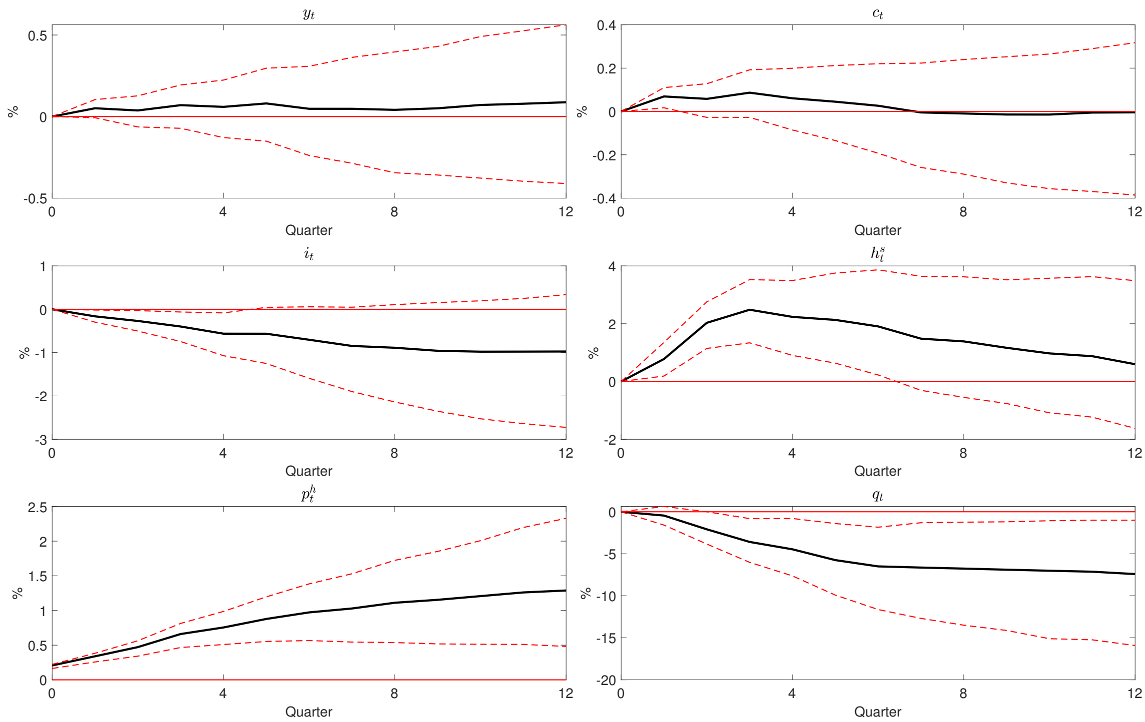


Figure 2: IRF to one unit house price jump

approximately 1% annualized. This clearly uncovers the crowd-out effect. It demonstrates that the crowd-out effect is potent and sensitive to housing price - a 10bps increase in housing price engenders a 100bps decrease in investment. This over reaction suggests an underlying conduit that transmits and amplifies the flow from housing price to physical investment and the drop in capital demand decreases the capital price up to 7%. Observations reveal that an increase in housing price corresponds with an increased housing supply in the same direction, affirming the two key arguments discussed previously: overbuilding and crowd-out effect spurred by a non-fundamental housing price demand shock. Furthermore, the non-exponential expansion in housing supply sheds light on the shape of the supply function in the housing market, which is not fully inelastic, contradicting the assumption made in literature.

2.2 Real price news shock

Although the previous section achieved successful identification of the housing market boom, overbuilding, and the crowd-out effect, a consequential query remains: what is the source of this "contemporaneous real price shock"? Through empirical testing, I have found that an exogenous surge in housing price could instigate a housing market boom, consequently predicting the overbuilding and crowd-out effects. However, the authenticity of this shock naturally invites skepticism. While the mechanisms I have proposed in this paper may be theoretically valid, they might not accurately portray the realities leading up to the Great Recession. The source for the pre-recession housing market boom extends beyond merely an exogenous contemporaneous real housing price shock. Other variables such as optimistic expectations, excess credit supply, and a secular decline in interest rates also contributed to this boom. To delve deeper into this issue, this section employs a SVAR model to identify the effect of a news shock on housing demand. My objective is to answer the following question: given future expectations of housing price inflation, how would other economic components respond to this anticipatory shock? I adopt, with minor modifications, the method put forward by Barsky and Sims (2011) (henceforth referred to as 'BS'). Through this approach, the news shock is identified as the component that can account for the largest forecast-error variance of housing price while remaining some orthogonal restrictions to rule out the effect of other contemporaneous shocks. This orthogonal restriction procedure is designed to mitigate any risk that an unexpected contemporaneous shock realized in the future could influence the forecast error. Furthermore, rather than adopting the level specification used by BS, I process the data using a hybrid specification or detrending method as mentioned in the previous section. This alternative approach was necessitated due to the data I utilized failing the unit-root test within the level specification.

Firstly I propose the reduce-form VAR system as

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Phi_3 y_{t-3} + ... + u_t \tag{2}$$

where the residual follows $u_t = Q\varepsilon_t$, $\varepsilon \sim N(0, I)$ and $\Omega = \text{var}(u_t) = QQ'$. Moreover I assume P is the Cholesky decomposition to the covariance matrix of residual u_t so $P = \text{chol}(\Omega)$ will hold. I further define the "news" vector $R = [r_1, r_2, ..., r_{N-1}, r_N]'$ where r_i is the unknown parameters of the vector R which need to be estimated. It measures the effect of housing-price-change news. The response to the news will be PR and by introducing this "vector shock" R I can directly solve the response to news shock and avoid drawing difference alternative orthogonal matrix.

It is worth to notice that solving the response vector R, instead of solving the response matrix Q, is more convenient and can provide analytical solution argued by Uhlig et al. (2004). As long as the orthogonal assumption 4 holds, we can find an orthogonal matrix $\mathcal{Q}$ which satisfies $\mathcal{Q}'R = e_i$ where $i \in [1, N] \cap \mathbb{N}$. Multiple $\mathcal{Q}$ on LHS to yield $R = \mathcal{Q}e_i$ and hence R is just the ith column of $\mathcal{Q}$. Throughout this paper I will mix these two definitions 1). Response vector R; 2) A shock R, because they represent the same thing in identification problem.

After proposing the VAR formula, I define the forecast error decomposition along the horizontal up to time h as

$$\text{fevd}^i_{n,h} = \frac{e'_n \text{var}(y^i_{t+h} - E_{t-1}y^i_{t+h})e_n}{e'_n \text{var}(y_{t+h} - E_{t-1}y_{t+h})e_n}$$

whose economic meaning is that the proposition of variance of variable n's expectation error that can be explained by shock i across time 0 to time h. Respectively the total forecast error from 0 to period H with unit weight should be $\text{fevd}_n = \sum_{h=0}^{H} \text{fevd}_{n,h}$ where $H = 12$.[18] The superscript

[18] Uhlig et al. (2004) and Barsky and Sims (2011) argued the weight-selection problem and arbitrary maximized

i in vector y_t denotes the impulse response spurred by shock i and the subscript n in vector y_t (equivalent to $e'_n y_t$) denotes the nth variable in vector y_t. Therefore $y^i_{n,t}$ denotes the response of variable n at time t to shock i and I will use this notation throughout the discussion in this section.

To identify the news shock, I solve the problem 8 below which finds a shock R^* that can explains the variance of expectation error of housing price most.

$$R^* = \text{argmaxfevd}_n = \text{argmax} \sum_{h=0}^{H} \frac{e'_n \left(\sum_{s=0}^{h} \Phi^s P R R' P' \Phi'^s \right) e_n}{e'_n \left(\sum_{s=0}^{h} \Phi^s P P' \Phi'^s \right) e_n} \tag{3}$$

s.t

$$R'R = 1 \tag{4}$$

$$e'_j P R = 0 \tag{5}$$

The first constraint 4 guarantees the orthogonality of response R^* and insures the unit realization of news shock which pertains to corresponding column of orthogonal matrix Q, otherwise there always exists an infinitely large shock $e'_n R = \infty$ which makes the identification meaningless. Additionally it renders the existence of maximization problem 3 as the Hessian of the objective function is semi-positive definite where the maximized point is not on the saddle point. The second constraint 5 rules out any contemporaneous shock in the future that influences the expectation error. Basically there are two type of shocks that can affect the expectation error $y_{t+h} - E_{t-1} y_{t+h}$: one is the news shock that arrives at time t yet realizes at a future time throughout $t+1$ to $t+h$ (based on the type of news and how informative it is); another one is the contemporaneous shock that arrives at any time from t to $t+h$ $\varepsilon_{t+i}, \forall i \in [0, h]$. It would be inappropriate to posit that the news shock accounts for more variation in the expectation error than what the contemporaneous shock does. Sims (2016) asserts that, more often than not, this proposition does not hold correct in reality. As such, I necessitate this secondary constraint 5 to segregate the effects of the contemporaneous shock from the identified R^*. The objective of the above problem 3 is to pinpoint a shock, apart from any contemporaneous shock that influences variable j, which can explain the expectation error to the greatest extent. Appendix C.1 discusses the requirement of orthogonal restriction in detail.

While the method of identification employed here is not exclusive to news about housing price—news about endogenous variables such as commodity prices, marginal costs or inflation could also fit—I limit the focus to the housing market in this paper. Here, i denotes the housing price news shock and $y_{n,t}$ represents the housing price. Given that the identified news shock R^* is subject to sign, I further impose a sign restriction on the impulse response $y_{n,t}$ to generate a

horizontal problem. Based on their argument I choose the unit weight and 3 years forecasting as the baseline cases which is reasonable and robust in the range from 5 quarters to 40 quarters.

positive demand shock on the housing price. The final issue in identification 3 involves finding a variable j in constraint 5 that aids in eliminating the possibility of contemporaneous shock during identification.

Before elaborating on the construction method for variable j which has zero contemporaneous effect of the news shock i, it is worth discussing proposition 1. This proposition highlights that canonical identification techniques, such as zero restriction, sign restriction, and long-run restriction, are ineffective for identifying the news shock in this context without constructing or finding variable j.

Proposition 1. *The identification to a news shock R^* through equation 3 is unique to covariance of the residual $\Omega = PP'$ from VAR's DGP 2.*

Proof. Give the covariance matrix of the residual from the DGP 2, the Cholesky P is unique to the covariance matrix Ω. Following Rubio-Ramirez et al. (2010), we know that any identification to the DGP is unique to PQ where Q is an orthogonal matrix. To identify the news shock I solve the maximization problem 3 to get the news shock R^* that maximizes fevd$_n$ subjecting to two constraints 4 and 5 and the rotation Q is identity $Q = I$. However when the rotation Q is not identity, i.e. for any different response matrix $P\widetilde{Q}$, the optimization problem that helps to find $\widetilde{R}^*$ from $g\left(\widetilde{R}\right) = 0$ is equivalent to that helps to find R^* from $g\left(f\left(R\right)\right) = 0$ as long as $f\left(R\right) = \widetilde{R}$ holds. If the mapping $f(\cdot)$ and its inverse $f^{-1}(\cdot)$ are all bijections, for any $\widetilde{R} \in \mathbb{R}^N$ there will exist an unique $R \in \mathbb{R}^N$ which satisfies $f\left(R\right) = \widetilde{R}$. It is easy to set $f^{-1}(\widetilde{R}) = \widetilde{Q}\widetilde{R}$ and $f(R) = \widetilde{Q}'R$. Therefore corresponding identified news shock $\widetilde{R}$ must satisfy $\widetilde{R}^* = \widetilde{Q}'R^*$ because of equation 3 and the impulse response of news shock is same to the Cholesky identification $P\widetilde{Q}\widetilde{Q}'R^* = PR^*$. □

Proposition 1 intuitively suggests that news or information is neutral to the fundamentals, and individuals respond to it based on their perception or belief about the news's reliability. Whether the news is genuine or false can only be discerned after the fundamental shock is realized and observed by economic agents several periods later. Therefore, the initial response to the news at time zero is unique to the covariance matrix, and the authenticity of the news, along with the corresponding response, cannot be determined by any rotation method on Cholesky P.

Above proposition 1 raises the question: Why should we construct variable j rather than seek one which is observable in reality? This deviation from standard news literature, where scholars typically focus on TFP shock and the underlying exogenous TFP is observable or calculable from data, arises due to the unobservable nature of the demand shock and the exogenous fundamental variation path. As such, our task is to unearth a variable j that is correlated with the contemporaneous variation of housing demand within the demand function, which I denote as the direct fundamental impact. The term "fundamental impact" refers to an index of the core elements that drive the demand function of housing, i.e., the preference ϕ_t in the Cobb-Douglas utility function $U(c_t, h_t, l_t) = \frac{\left(c_t^{\phi_t} h_t^{1-\phi_t}\right)^{1-\sigma}}{1-\sigma} + \kappa\frac{l_t^{1+\psi}}{1+\psi}$, following $\phi_t = (1-\rho_\phi)\overline{\phi} + \rho_\phi\phi_{t-1} + w_{t-\tau} + w_t^\tau$

where $w_{t-\tau}$ as the news shock to housing demand. The modifier "direct" indicates that variable y_t^j reflects the contemporaneous impact ϕ_t, rather than ϕ_{t+i}. Moreover, when imperfect information exists and households cannot precisely observe the fundamentals, as discussed in section 2.3, the fundamental impact y_t^j should serve as an indicator of the perceived fundamentals $\phi_{t|t}$, rather than the true fundamentals. Consequently, survey data appears to be the most suitable variable for ruling out contemporaneous shocks via constraint 5. However, neither the true fundamentals nor the perceived fundamentals are observable, and all observations in the survey relating to fundamental impact are endogenous, tainted by macro variables and the endogenous response of news shocks. Therefore, this paper proposes a method to cleanse the endogenous perception data and eliminate the contemporaneous endogenous news effect.

Before discussing the purification process, I first broach the ammunition, the data that I can use to purify the endogenous perception of the status of housing market. In this paper I use the NAHB/Wells Fargo Housing Market Index (HMI) which is a monthly survey on NAHB members about their perception about the status of housing market right now Υ_t(in equation 6), as well as their expectation over the next six month $E_t\Upsilon_{t+6}$(in equation 7).

To elucidate the purification process intuitively, let's consider a model with perfect information. Assume that Υ_t represents survey data about people's perception of the housing market, and follows the relationship

$$\Upsilon_t = \rho\Upsilon_{t-1} + \alpha_1 x_t + w_{t-\tau} + u_t + \alpha_2 w_t \tag{6}$$

where x_t stands for any macroeconomic variable such as interest rate, GDP, unemployment rate, etc. The coefficient α_1 quantifies the cross-linkages between macroeconomics and perceptions about fundamentals. For instance, a monetary policy shock may initially affect the interest rate and output, leading to a commensurate change in Υ_t. In this context, $w_{t-\tau}$ represents a news shock announced τ periods ahead. Meanwhile, u_t denotes a contemporaneous shock, and α_2 captures the endogenous contemporaneous effect induced by the news shock w_t. If households anticipate the realization of the shock three periods ahead, they would react in the present time. Because of this contemporaneous response α_2, the news shock w_t will exert an endogenous effect at the time of its arrival, in addition to the direct effect occurring three periods later when the shock materializes. Under the rational expectation, the expectation about housing-market status τ period ahead will follow

$$E_t\Upsilon_{t+6} = \begin{cases} \rho^6\Upsilon_t + \alpha_3 x_t + \sum_{n=1}^{n=\tau} \rho^{6-n} w_{t-\tau+n} & \tau \leq 6 \\ \rho^6\Upsilon_t + \alpha_3 x_t + \sum_{n=1}^{n=6} \rho^{6-n} w_{t-\tau+n} & \tau > 6 \end{cases} \tag{7}$$

To ensure simplicity in our discussion, I have deliberately omitted terms with additional lags, such as Υ_{t-2}, x_{t-1}, in equations 6 and 7. These terms may indeed manifest in these models, and as such, I have performed a range of robustness tests to investigate these independent variables in

Appendix C.6 and better understand the underlying models. However, it's worth noting that these equations make an implicit assumption: any other macroeconomic shocks, such as monetary policy shock, TFP shock, or marginal cost shock, will influence the status of the housing market solely through macro variables x_t, without any direct effects. This assumption parallels the notion that Υ_t occupies the first row of y_t in equation 2, corresponding to the first column of the Cholesky P.

The basic idea of this purification process is to identify the parameter ρ, α_1, α_2 and w_{-1}, w_{-2}... which yield the purified housing market status, denoted as $\widehat{\Upsilon}_t = \Upsilon_t - \alpha_2 w_t$. However the canonical regression based method cannot be used here because of endogeneity and imperfect identification problem. For instance, even for the simplest model of 6 (or 7) without aggregate effect, the regression of Υ_t on Υ_{t-1} ($E_t\Upsilon_{t+6}$ on Υ_t) will yield biased result as $w_{t-\tau}$ already embeds into $\Upsilon_{t-1}(\Upsilon_t)$. Further, the residual of this regression represents a "near" moving average process that contains several components instead of $w_{t-\tau}$ itself. Thus, the second regression, a regression of Υ_t on the residual or its lagged and lead term will not be exactly α_2 and the $\widehat{\Upsilon}_t$ will still encompass some amount of contemporaneous effect, w_t. In addition to addressing the standard issues of endogeneity and heteroscedasticity that are common in OLS regression, another crucial challenge must be overcome: understanding the informative power of the news w_t, specifically how far in advance households become aware of it. This challenge will directly impact the structure of the expectation 7, and subsequently, the structure of the residual, which I use to extract the w_t term from Υ_t. Given that the only observable expectation linked to a six-period lead, the form of the expectation would alter to different form when the news arrives at different periods prior to realization. On that account, I use the maximum likelihood estimation method to estimate and purify the contemporaneous endogenous effect α_2 and the likelihood for different informative power of the news can be used to determine how many periods ahead that the news is announced to household. In the appendix C.3 and C.3, I provide a range of numerical and empirical tests demonstrating that this purification method can effectively eliminate the endogenous news effect w_t from the perception of housing market status Υ_t, albeit to a certain scale. Additionally, I also conduct a series of robustness check by using the instrument variable to purify Υ_t through 2SLS regression analysis.

Figure 3 presents the IRFs of a one-unit news shock that delivers information about future housing price to agents, with the red dashed lines indicating the 90% confidence band which visibly confirms the significant crowd-out effect. The pattern of housing price response closely mirrors that observed in the contemporaneous shock, although the boom in the housing market is nearly twice as significant. Housing price progressively rise from 30bps to a peak of 200bps, approximately five times larger than that under the contemporaneous shock. This marked expansion in the housing market, driven by expectations and news shocks, triggers a drop in capital price five times larger than the surge in housing price. Households considerably decrease their capital holdings, even entering into negative positions (in debt), thereby depressing capital price due to reduced demand. This reveals the crowd-out effect as a manifestation of capital

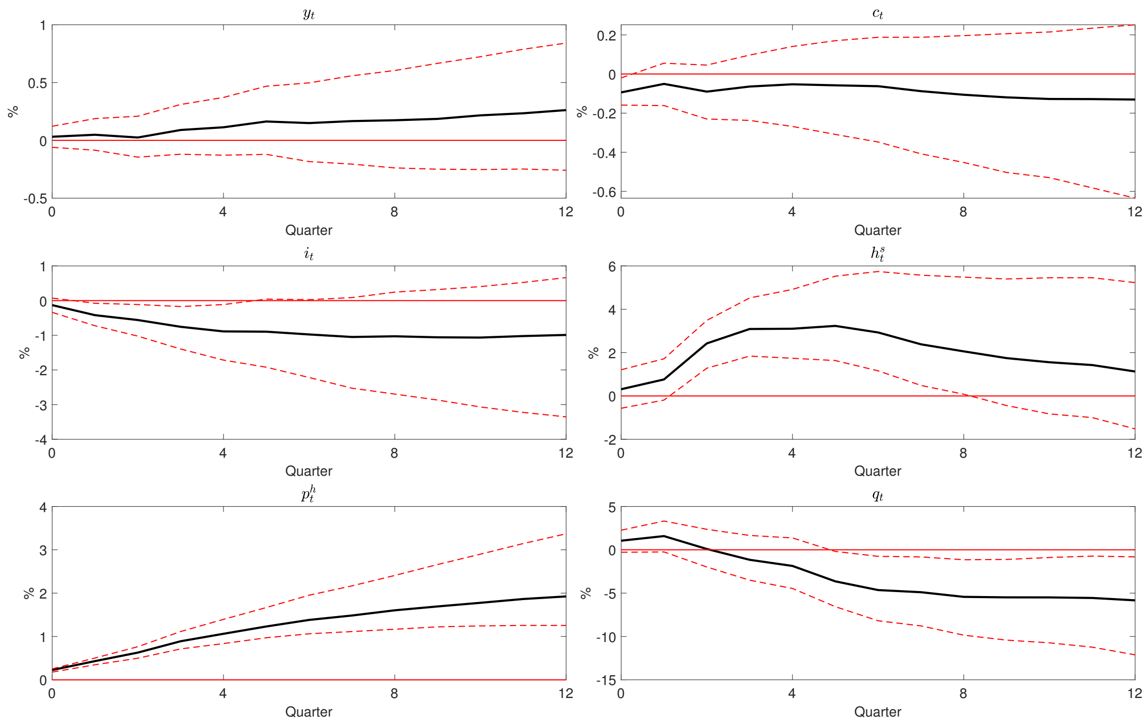


Figure 3: IRF to one unit housing price news shock at 90% confidence band

misallocation at the micro level. These observations underscore the effective identification of news shocks and demonstrate the reliability and transparency of the results. The study aligns with existing literature that attributes housing market booms primarily to expectations and slackness in the credit market. Additionally, it highlights the sensitivity and fragility of the housing market during the pre-recession period, as the market could be triggered into a boom merely by initial expectations, culminating in a considerable peak without any observed hesitations or declines. This housing market boom also coincides with significant overbuilding, which is five times greater than that observed under contemporaneous shocks, culminating at 300bps. Concurrently, the output experiences a slight yet insignificant increase due to general equilibrium effects, with the economy overheating. The contemporaneous response of non-durable consumption exhibits a small but insignificant decrease (10bps), potentially resulting from a stronger substitution effect than the wealth effect, which is revealed in new evidence from survey data (Kuang et al. (2023)). As previously observed, substantial physical investment is crowded out during periods of housing market boom and overbuilding. In comparison to the response of physical investment to a contemporaneous housing price shock, investment is crowded out to a greater but milder extent (compared to the difference in housing price and stock price), reaching up to 100bps. This is reasonable, as the crowd-out effect arises from the general equilibrium among investment, output, and consumption, which remains mild change.

2.3 Real price fake news shock

Due to Proposition 1, canonical identification methodologies such as sign restriction (Uhlig (2005)) and short-run restriction ((Sims (1980) and Basu et al. (2006)) are insufficient to differentiate true news from fake news within the previously identified news shock. As an alternative, I introduce a novel identification strategy through which the effect of true news about future housing price is refined by a contemporaneous shock during the realization of news shock, so isolating the effect of fake news.

In section 2.2, I previously presented the concept of the news shock R^*, representing the shock that most adequately accounts for the expectation error over the next H periods from period 0. However, this shock is agnostic to its own status and does not yield any insights regarding whether it is a true or fake news. This is because it is identified based on expectation error, devoid of any proxy for the underlying "fundamental situation", and both fake and true news can elicit identical responses before the news' type is realized. Despite the neutrality of the news shock R^* and our inability to directly identify fake news prior to its realization, I design the strategy to differentiate between fake news and true news by adjusting the combined news with contemporaneous shock and refining the preceding impulse response. Before introducing this identification strategy, which allows me to distinguish between fake news and true news, I first present two assumptions with micro foundations as the cornerstones of identification.

Definition 1. Denote the response to fake news realized at time τ as $U^F = \left\{y_0 = \overline{R}_1, y_i\right\}_{i=1}^{i=\infty}$ and the response to true news realized prior to time τ as $U^T = \left\{y_0 = \overline{R}_2, y_i\right\}_{i=1}^{i=\infty}$. The response to a news shock we empirically identified through 3 is $U = \{y_0 = R^*, y_i\}_{i=1}^{i=\infty}$.

Assumption 1. *The response to a news shock, either a fake news or a true news, under imperfect information, will be the same before the shock realized. In other words $\overline{R}_1 = \overline{R}_2 = R^*$ and $y_i^F = y_i^T = y_i, \forall y^F \in U^F, y^T \in U^T, y \in U, i \in [0, \tau]$ will hold.*

This assumption is justified given that under imperfect information, agents cannot discern the veracity of news; they simply respond identically to observations triggered by either true or fake news. Thus, it is only under conditions of complete information where the news is fully informative that agents exhibit differing responses before the news' realization at time τ. It is widely recognized that the principle of certainty equivalence applies in the context of first-order linearized state space models, within which Assumption 1 is unequivocally upheld. Further support for this assumption is provided in the appendix D.2.2, where I offer several numerical examples to demonstrate that the aforementioned assumption holds in a state space model under rational expectation.

Assumption 2. *The empirically identified news shock U lies in the medial of response to fake news U^F and response to true news U^T. In other words, $y_i \in \left[y_i^F, y_i^T\right], \forall y^F \in U^F, y^T \in U^T, y \in U, i \in [\tau+1, \infty]$ will hold. Furthermore, the news shock U is a linear combination of U^F and U^T, and $y_i = \alpha y_i^F + \beta y_i^T$ holds.*

Assumption 2 is also reasonable as the identification process 3 is based on expectation error and it cannot differentiate between U^F and U^T, given that both of them impact the expectation error of housing price. Nonetheless, as long as the Data Generating Process (DGP) 2 is a linear equation, the path subsequent to realization of a shock is entirely described by the coefficient Φ, which represents a projection from y_{t-1} to y_t. Therefore, the identified path U is essentially a linear combination of the fake news path U^F and the true news path U^T , which are both intertwined within the posterior observation. In the appendix D.5, I apply the news shock identification strategy 3 to mock data generated by a state space model to demonstrate that assumption 2 is valid.

I now define the identification of fake news as

$$\widehat{y}_i^F = \begin{cases} y_i & i \leq \tau \\ y_i - \frac{e_j' y_{\tau+1}}{e_j' y_0^\tau} y_{i-\tau-1}^\tau & i > \tau \end{cases} \tag{8}$$

where $y_i \in U$, and y_i^τ represents the response path to a contemporaneous shock directly impacting the fundamental variable j, as depicted in equation 5. The fundamental concept here is that the influence of true news realized at time τ can be counteracted by a contemporaneous negative shock, leaving behind only the response to fake news, which has no bearing on variable j or the real economy (subject to a scalar α, which remains unidentified here). This is a logical supposition, given that the true news shock has been influencing the economy since its realization at time τ, and, as long as the shock is independent and identically distributed (iid) and the entire system is linear, it operates (producing real effects) as a contemporaneous shock after τ when it impacts the fundamental. In the appendix D.3 and D.4, I provide two examples that lend micro foundation to this offset effect.

The identification method I am using here aligns with the logical premise first advanced by Wolf and McKay (2022), who propose that we can "replace" the underlying state determinant equation (i.e., policy function) with a counterfactual one by solving a system of linear equations. A set of rescaled fundamental shocks can emulate the old, identified policy function and transform it into a new one by censoring the old impulse response with an additional series of $\{\Theta_{i,\tau}\}_{\tau=0}^{\tau=\infty}$, generated by a fundamental shock. The paths of other endogenous variables, such as GDP, investment, and labor supply, are then determined by the censored path y_i^τ and Wolf and McKay (2022) provide a rigorous proof supporting this argument. Similarly, Hebden and Winkler (2021) and Groot et al. (2021) have also used comparable counterfactual experiments in their research in which the goal was to identify an optimal policy, and they achieved this by solving certain nonlinear problems.

Figure 4 exhibits the empirical response to a deceptive housing demand (housing price fake news) shock. A shock to housing demand arrives (or is announced to household) six quarters ahead, at time 0, but realizes (has fundamental effect) in quarter 5, with a possibility that the news lacks any fundamental effect and is merely noise. Before discerning the true nature of

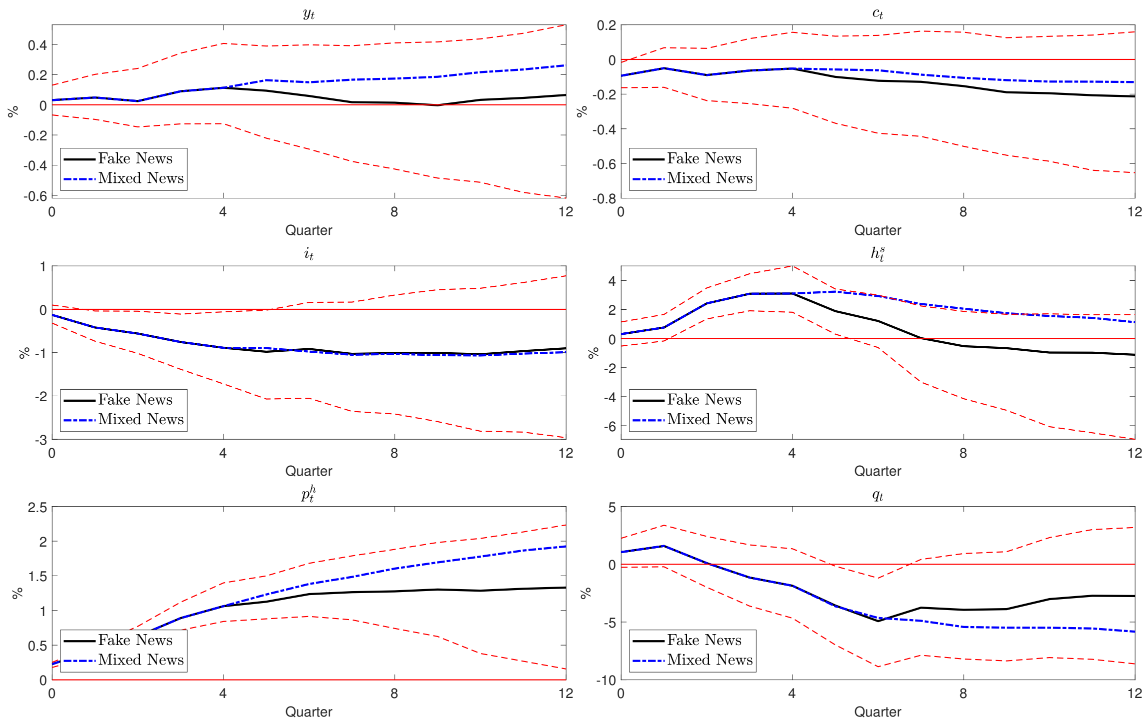


Figure 4: IRF to one unit housing price fake news shock at 90% confidence band

the shock—as either true or fake—agents respond identically to these two shocks, as they are unable to determine the truth. Hence, Figure 4 and 3 share the same responses before period 6, at which point agents commence their attempts to discern whether the news is true or fake[19]. Upon realizing that the news is fake at quarter 5, the housing market boom busts as it lacks further support. Housing price and new construction of residential asset significantly decline, with a 150bps drop in housing price and a 300bps drop in housing supply. Subsequently, the new residential asset construction enters a negative range, indicating a severe and sustained recession triggered by the housing market bubble's bust. Physical investment, initially crowded out due to the housing market boom, only has a mild increase as the subsequent recession yields a lower demand in physical capital. In addition to the stagnation in the housing market, a recession unfolds in the goods market, with output and nondurable consumption dropping immediately after the revelation of the fake news. Due to the scarcity of physical capital during the bust period, the recovery post-recession is muted. This sluggish recovery unveils the drawbacks of housing market boom-bust cycles, where physical capital is crowded out during the boom period, and the resulting scarcity of physical capital leads to a more severe recession during the bust period.

Aside from examining the direction and magnitude of the news shock's effect on housing price, it is vital to consider the news shock's significance. If it does not hold substantial importance in reality, the preceding discussion around the crowd-out effect may lose relevance. Figure 5 presents the historical decomposition of the news shock and fake news shock's influence

[19]They may be informed directly at time 6 or gradually learn that whether the news is true or fake, which depends on the information structure and I provide two examples in appendix to illustrate two different information structures.

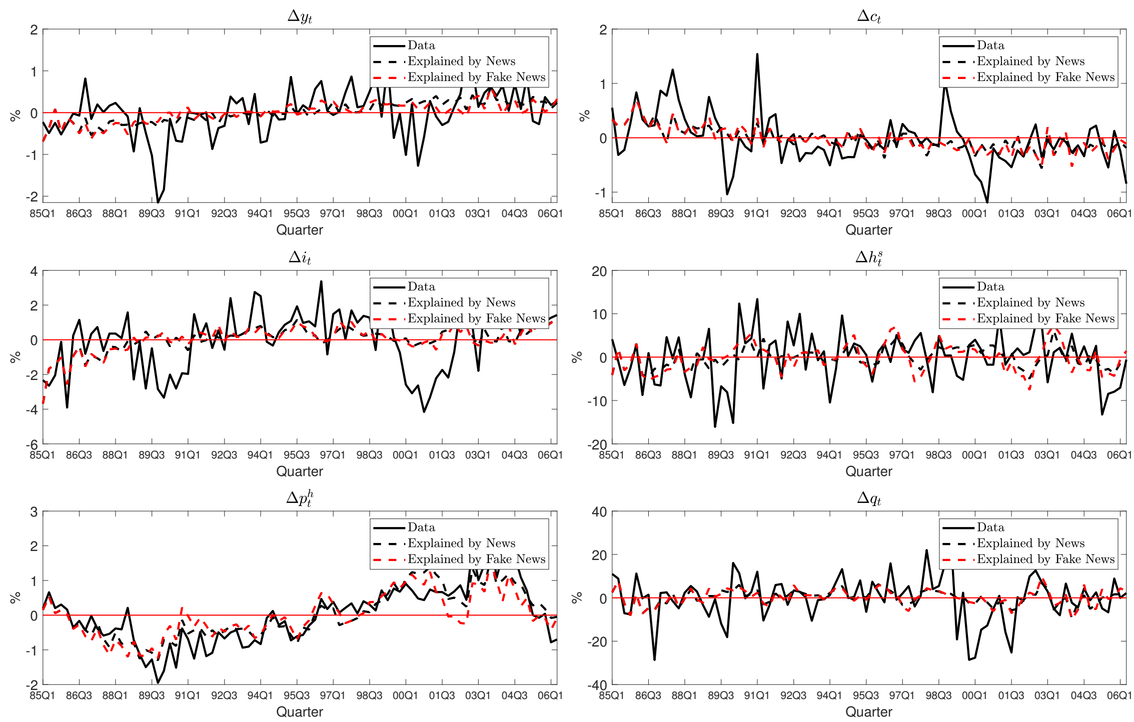
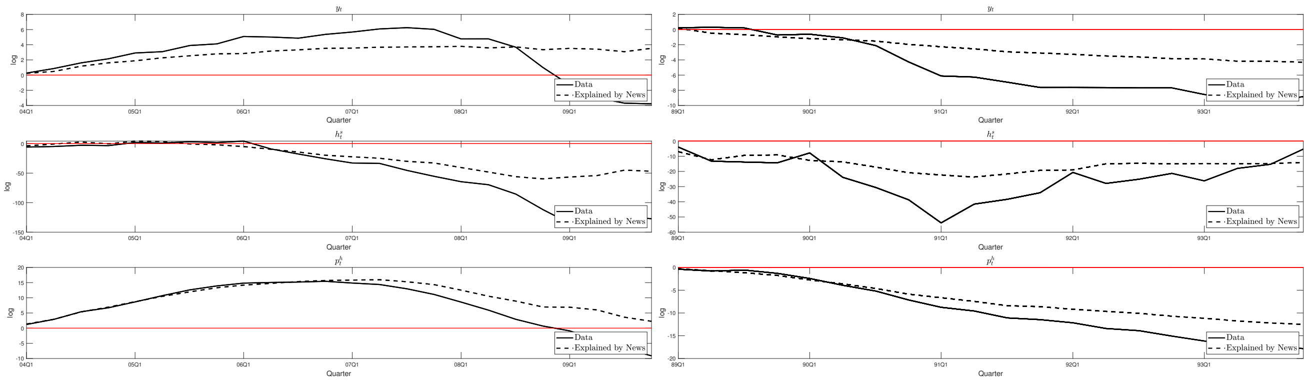


Figure 5: Historical Decomposition of News shock

on various macroeconomic variables. News shock on housing price accounts for a moderate portion of the variance in housing price and new construction, and exerts a modest but not insignificant effect on physical investment and nondurable consumption. To illustrate this significance, I opted for historical variance decomposition that the variables can be explained by the news shock as the measure of the news shock's influence. Roughly 50% of the variance of housing price in data is explained by the fake news and the fake news can also explain 30% of the variation of housing supply. However, only 20% of the variation of physical investment originates from the fake news shock, though the number is not negligible. Conversely, the expiation power attenuates to 14% for stock price, signaling a milder influence than in the housing and capital markets. These suggest that the fake news about housing price explains a significant portion of the boom-bust cycles in the housing market and the capital market due to the crowd-out effect. Nevertheless, based on Sims (2016), the share of variance does not offer a reliable indicator of the relative importance of news shock. Thus, I also use figure 6 to probe the significance of news shock in reality. The figure 6 display how the macro economy grew during the boom-bust period of housing market and what extent can the news shock explain it, by calculating the detrended accumulated growth. The news shock in both figure 6a and 6b draws the housing price at the beginning, yet the housing price drops further than that is explained by news shock in bust period. These divergences implies that the deception of the news may render the bust when household realized the truth and the fake news indeed explains an important and significant share of the housing market bubble preceding the Great Recession.



(a) Historical Decomposition before Great Recession

(b) Historical Decomposition during housing market bust in early 90s

Figure 6: Historical Decomposition of News shock spanning the subprime-debt housing market boom

3 Crowd-out effect of overbuilding: insight from a simple model

Optimistic expectation regarding future housing price engenders a surge in household demand for real estate, inducing a boom in the housing market characterized by inflated housing price and overbuilding. In a context where supply is semi-inelastic, changes on the demand side will not necessarily lead to substantial overbuilding. Conversely, if the supply function possesses sufficient elasticity, a minor demand boom could spur significant overbuilding. The shapes of the supply and demand functions determine the magnitude of overbuilding, and by extension, the degree of crowding in physical capital. This is due to the underlying mechanism through which the crowd-out effect operates: the general equilibrium. Hence, it necessitates the synergy of both supply and demand functions to analyze the crowd-out effect. In this section, I first introduce a simplified Aiyagari-Bewley-Huggett model operating within an incomplete market framework. Subsequently, I utilize this model to demonstrate that overbuilding, influenced by intratemporal substitution, liquidity, precautionary saving, and wealth inequality, leads to the crowd-out effect.

3.1 A simple Aiyagari-Bewley-Huggett model

This framework is grounded in a standard Aiyagari-Bewley-Huggett model wherein households employ wage income and asset returns to meet their consumption and real estate demands. The durable good, in this case housing, is produced by real estate companies in a competitive market utilizing land, capital, and labor. Similarly, non-durable goods are produced in a competitive market with capital and labor as inputs.

It is a standard Aiyagari-Bewley-Huggett model where households use wage income and asset return to fulfill their demand for consumption and real estate. The durable good, house, is produced by real estate companies in complete market with land, capital and labor. Similarly the non-durable good is produced in complete market with capital and labor.

For simplicity I assume that household i provides inelastic labor supply of 1 unit exogenously to solve the problem

$$\max_{c_t^i,h_t^i,a_t^i} \sum_{t=0}^{\infty} \beta^t U^i\left(c_t^i, h_t^i\right) \tag{9}$$

s.t.

$$c_t^i + a_t^i + p_t^H h_t^i = R_t a_{t-1}^i + w_t \varepsilon_t^i + \left(1-\delta^H\right) p_t^H h_{t-1}^i + T_t \tag{10}$$

$$-a_t^i \leq \gamma p_t^H h_t^i \tag{11}$$

where equation 10 is the budget constraint and equation 11 is the collateral constraint. a_t^i could either be positive or negative but in aggregate level is positive as it is the supply of capital which is used to produce durable and non-durable goods. w_t is the real wage and household earns productivity-weighted wage income from which ε_t^i is corresponded idiosyncratic income shock. p_t^H is the real housing price. h_t^i is the unit of houses hold by household i. T_t is the lump-sum transfer to household. For simplicity I further assume the real interest rate is fixed at $\overline{R}$.[20]

The production sector is in a complete market where firms produce non-durable good via $Y_{N,t} = A_{N,t} K_{N,t-1}^{\alpha} L_{N,t}^{1-\alpha}$ and durable good via $Y_{H,t} = A_{H,t} \overline{L}_H^{\theta} K_{H,t-1}^{\nu} L_{H,t}^{1-\nu-\theta}$. The labor market is closed by one unit inelastic labor supply $L_{N,t} + L_{H,t} = 1$ and household provide the capital by $K_{N,t-1} + K_{H,t-1} = K_{t-1} = \int a_{t-1}^i dG_{t-1}$ where G_{t-1} is the cumulative distribution function of household. The non-durable good is used either to consume or to invest in physical capital so the goods market cleaning condition $Y_{N,t} = K_t - (1-\delta)K_{t-1} + C_t$ holds. Meanwhile real estate companies produce all the increment in residential asset by $Y_{H,t} = H_t - (1-\delta^H)H_{t-1}$ where $H_{t-1} = \int h_{t-1}^i dG_{t-1}$.

Proposition 2. *Household will adjust their consumption of non-durable goods based on over-building and precautionary saving. The extent of adjustment is determined by*

$$\widetilde{c}_t = \underbrace{\Phi_H \widetilde{h}_t}_{\text{substitution effect}} - \underbrace{\Phi_\mu \widetilde{\mu}_t}_{\text{credit effect}} + \underbrace{\Phi_{p^H}\left[\frac{1}{1-\left(1-\delta^H\right)\frac{1}{R}} F^H(\widetilde{H}_t) - \frac{\left(1-\delta^H\right)\frac{1}{R}}{1-\left(1-\delta^H\right)\frac{1}{R}} F^H(\widetilde{H}_{t+1})\right]}_{\text{wealth effect}} \tag{12}$$
$$- \underbrace{\Phi_{cov}\widetilde{cov}_t}_{\text{precautionary saving effect}}$$

[20]It is not a too strong assumption since this could happen in many scenarios. For instance the nominal interest rate reaches the ZLB and the price is fixed. Or an open economy where the real interest rate is bounded by the international financial market. In appendix G.1.1 I shows that under a range of parameters the real interest rate will not change at t as long as capital and labor do not change.

where $F^H(\cdot)$ is the inverse supply function,

$$\Phi_H = \frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{13}$$

$$\Phi_\mu = \frac{\mu}{\lambda-\mu}\frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{14}$$

$$\Phi_{p^H} = \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{15}$$

$$\Phi_{cov} = \frac{\eta_c}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}\frac{\beta\left(1-\delta^H\right)\overline{cov}}{h}$$

and $\eta_{c,p^H} = \frac{u_{ch}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}, \eta_{c,p^c} = \frac{u_{hh}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$, $\eta_{h,p^c} = \frac{u_{ch}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$, $\eta_{h,p^h} = \frac{u_{cc}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$, $\eta_{ch} = \frac{u_c u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{ch}$, $\eta_c = \frac{u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$.

Proposition 2 elucidates that any disturbance in the real estate market can propagate to the non-durable goods consumption via four distinct channels: substitution effect, wealth effect, credit effect, and the precautionary savings effect.[21] The directions these four channels take, in terms of how the housing market boom influences the consumption of non-durable goods, is determined by the relative strength of both intertemporal and intratemporal elasticities of substitutions between non-durable and durable goods, and the specific role that housing wealth assumes within budget constraint and credit constraint. When overbuilding transpires in the housing market bubble, positive variations in $\widetilde{h}_t$ and $\widetilde{H}_t$ prompt changes in non-durable consumption via substitution and wealth effects. Meanwhile, it could also endogenously affect consumption through credit and precautionary saving effects. This variation in consumption, set off by a housing market boom, ultimately impacts physical investment, thereby exacerbating the recession in the future, as long as the overall effect is positive.

It merits attention that η_{x,p^y} represents the standard Frisch elasticity of variable x with respect to the relative price of y, serving a crucial role in moderating the impacts of these four effects. If non-durable consumption is more responsive to housing price than to non-durable goods price, a shift in the holdings of housing service will induce a more pronounced effect on non-durable goods consumption, as manifested in Φ_H. Conversely, if households' holdings of housing services respond more substantially to non-durable goods price (compared to housing price), the elasticity of substitution would dampen all four channels. This occurs because the consumption of durable housing becomes more stable, and households do not alter their consumption significantly, suggesting a minor pass-through from housing service consumption to non-durable goods consumption.

[21] Berger et al. (2018) only discussed two of them meticulously but not focused on credit effect and precautionary saving effect. Additionally their goals about decomposition is related to analyze the inequality problem caused by house price inflation.

3.2 Crowd-out effect of overbuilding

The amplification of the crowd-out effect sparked by overbuilding due to intratemporal elasticity of substitution, credit constraints, precautionary saving, and wealth inequality will be discussed herein. Overbuilding intuitively affects the consumption of nondurable goods and crowded-out physical investment, considering the relationship between nondurable consumption and housing as complements at the aggregate level. Similarly, overbuilding tends to ease collateral constraints, facilitating households to borrow more to smooth their consumption demand. Additionally, overbuilding exerts influence on nondurable consumption response via housing price due to the monotonic increasing inverse supply function of residential assets, $F^H(\cdot)$, in a complete market - more new construction leading to higher housing price in equilibrium. As the housing price factors into the budget constraint of household and influences their income, a rise in housing price makes households perceive an increase in wealth, given the dual function of a house as both a utilitarian good and an asset in the budget constraint. This surge in price, arising from a shift in the supply function (a demand shock), implies that overbuilding aligns with house price inflation via the supply side, otherwise an inelastic supply function will not bring any overbuilding from a demand shock.

By aggregating the consumption decision of households from equation 12 and integrating the First Order Conditions (FOC) in supply sectors, a relationship between overbuilding and physical investment can be obtained, as outlined in Proposition 3.

Proposition 3. *The aggregate investment is driven by overbuilding and precautionary saving following*

$$
\begin{aligned}
\widetilde{II}_t = -\Bigg\{ & \left(\Phi_H + \frac{\nu}{\alpha} p^H H\right) \int \widetilde{h}_t^i dG_i - \Phi_\mu \int \widetilde{\mu}_t^i dG_i \\
& + \Phi_{p^H} \left[\frac{1}{1-(1-\delta^H)\frac{1}{R}} F^H(\widetilde{H}_t) - \frac{(1-\delta^H)\frac{1}{R}}{1-(1-\delta^H)\frac{1}{R}} \mathbb{E}_t F^H(\widetilde{H}_{t+1}) \right] \\
& - \Phi_{cov}^i \int \widetilde{cov}_t^i dG_i + \frac{\nu}{\alpha} Y_H p^H F^H(\widetilde{H}_t) \Bigg\}
\end{aligned} \tag{16}
$$

The overbuilding, $\widetilde{H}_t = \int \widetilde{h}_t^i dG_i > 0$, will crowd out physical investment as long as the substitution effect Φ_H and wealth effect Φ_{p^H} are not negative enough and Φ_μ is not positive enough.

Equation 16 reveals that overbuilding results in diminished physical investment and subsequently lower physical capital through distinct mechanisms in the demand and supply sides, at least within a specific parameters range. The term Φ_x pertains to the influence of the pass-through from housing services to the consumption of non-durable goods, whereas the term $\frac{\nu}{\alpha}$ in 16 is connected to the supply-side effect. The following discussion will explore in detail how relative intratemporal elasticity of substitution, credit constraint, precautionary savings, and wealth inequality impact the crowd-out effect instigated by overbuilding.

3.2.1 Intratemporal elasticity of substitution

Intertemporal substitution, extensively explored in relation to the Euler equation and monetary policy, stands in contrast to intratemporal substitution between consumption of durable and non-durable goods, which remains underexplored both theoretically and empirically. In this section, I argue that intratemporal substitution significantly influences household decision-making processes, especially in the context of the crowd-out effect created by overbuilding. Empirical studies in the housing market suggest that intratemporal substitution holds more significance and potency than intertemporal substitution[22], as households, being primarily myopic or financially constrained, often neglect or simply cannot afford to consider future consumption in their present-day decisions. By analyzing the coefficients of the crowd-out effect as delineated in Proposition 12, Corollary 1 concludes that the relative intratemporal substitution can theoretically amplify the crowd-out effect across the demand side of the housing market.

Firstly I define the intertemporal and intratemporal elasticity of substitution below:

Definition 2. The intratemporal elasticity of substitution is

$$\text{IAS} = -\frac{\partial \ln \frac{h}{c}}{\partial \ln \frac{U_h}{U_c}} \tag{17}$$

and the intertemporal elasticity of substitution to consumption bundle is

$$\text{IES} = -\frac{U_{BB}}{U_B}$$

Then based on the definition I obtain following corollary.

Corollary 1. *Ceteris paribus, household with larger intratemporal elasticity of substitution relative to intertemporal elasticity of substitution, as well as their utility function follows the CRRA form, will crowd out less investment through substitution and wealth effect.*

It is easy to understand corollary 1 that non-durable goods and housing services are both normal goods, and if they are substituted more with each other, the crowd-out effect will be further muted, since an increase in consumption of housing service would lead to a corresponding decrease or less increase in non-durable goods consumption. The intratemporal elasticity of substitution gauges the extent to which an increase in housing can be substituted by an increase in consumption at the utility level within a specific period.[23] On the other hand, the intertemporal elasticity of substitution quantifies the inclination to substitute the overall

[22] Khorunzhina (2021) did this vital work empirically.

[23] It is intuitive to focus on U_{ch} which is closely related to the complementarity between house and non-durable good.

consumption bundle over different periods. If $\text{IAS} > \text{IES}$ holds, households are more likely to adjust their consumption between durable and non-durable goods within a given period, rather than across different periods. A relatively larger intratemporal elasticity of substitution implies a lower increase in consumption in response to overbuilding within a given period, as these goods become more substitutable than complementary. The potency of intratemporal substitution is such that it directly influences marginal utility, bypassing the budget constraint, hence any other elements in economy that affects the utilitarian benefit of residential asset will alter the crowd-out effect doubtless.

Proposition 4. *When the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} > \frac{K}{L} > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}$ *holds, substitution effect* Φ_H *and wealth effect* Φ_{p^H} *will decrease as relative intratemporal elasticity of substitution higher. Further, when the aggregate Khun-Tucker multiplier is not too large, credit effect* Φ_μ *will increase as relative intratemporal elasticity of substitution higher.*

Proposition 4 shows that under certain conditions[24] the relative intratemporal elasticity of substitution will manifest a clear impact on substitution, credit and wealth effect. In the subsequent analysis I dispense with these conditions and solve the GE problem quantitatively to provide a profounder analysis on the effect of relative intratemporal elasticity of substitution.

I solve the model 9 with unit intratemporal elasticity such that $\text{IAS} = 1$, and change the intertemporal elasticity from 0.67 to 0.5, effectively increasing the relative intratemporal elasticity. As depicted in Figure 7a, an enlargement in relative intratemporal elasticity results in a contraction of the substitution effect. Theoretically, a preference shock increasing the relative intratemporal elasticity of substitution compared to the intertemporal elasticity will reduce the response of nondurable consumption to a given level of overbuilding, subsequently leading to a smaller extent of crowded-out investment. This outcome can be attributed to the abated complementarity between nondurable goods and housing service due to the enhanced substitution. Additionally, a higher propensity to substitution can alleviate the collateral constraint, given the reduced demand for non-durable goods, thereby causing less households to remain financially constrained in the steady state. For a mathematical elucidation of the above argument, let us consider two economies, a and b. In these two economies, the relative intratemporal elasticity of substitution satisfies $\frac{\text{IAS}_a}{\text{IES}_{c,a}} < \frac{\text{IAS}_b}{\text{IES}_{c,b}}$. Suppose two unexpected tax rebates are given to households in these economies respectively, triggering the same increase in non-durable consumption, $\Delta C_a = \Delta C_b = 0.5$. Given that the intratemporal elasticity in economy a is smaller than that in b, households in a will increase their holdings of durable consumption more, say, $\Delta H_a = 0.5 > \Delta H_b = 0.3$. This increased residential asset holding eases the collateral constraint, with the extent of relief proportionate to the change in residential assets. Therefore, the Karush-Kuhn-Tucker multiplier

[24] The two-state variables Aiyagari-Bewley-Huggett is hard to implement any theory based on Von-Neumann algebra inStokey (1989) as the topology is too complicated. Thus these conditions help to direct the dimension of distribution.

of equation 11 yields $\Delta\mu_a < \Delta\mu_b < 0$, implying $\Phi_\mu^a > \Phi_\mu^b > 0$ in equation 12 as $\Phi_\mu^i = -\frac{\Delta C_i}{\Delta \mu_i}$. This trend is represented in Figure 7b, where the credit constraint progressively expands.

In addition to substitution effect and credit effect, overbuilding will also be passed to the consumption response through the inverse supply function $F^H(\cdot)$ because the residential asset is also a type of asset which enters into the budget constraint, except for acting as consumables in utility function. An inflation(of housing price) in housing market, inspired by overbuilding, also provide liquidity to household as long as they previously hold some amount of house because of the asset's pecuniary character. This wealth effect is amplified as the value, that one unit of housing service provides, now can be transferred to utilitarian value more with a smaller intratemporal elasticity of substitution. The intratemporal consumption decision between durable and nondurable goods, which comes from wealth effect, follows the relative marginal utility equation $\frac{U_{h,t}}{U_{c,t}} = f\left(p_t^+, p_{t+1}^-\right)$. This equation is intuitive and easy to understand. Household can use money to marginally increase one unit of housing servicing at time t and get $U_{h,t}$ unit of extra utility. Alternatively the household can also use the money that affords the housing servicing to buy nondurable consumption and get $U_{c,t} f\left(p_t^+, p_{t+1}^-\right)$ unit of extra utility. The extra unit of nondurable goods is rescaled by the price of housing servicing as the money that affords one unit of housing servicing does not afford the same unit of nondurable goods. If I given a same jump in housing price $\Delta p_{a,t}^H = \Delta p_{b,t}^H > 0$ on RHS and held the housing servicing, there would be an jump in nondurable goods consumption which results in a positive Φ_{p^H} in equation 12. A jump in nondurable goods consumption $\Delta C_t > 0$ will fulfill household's demand for nondurable goods with smaller marginal utility of nondurable goods $\Delta U_{c,t} < 0$ but a higher demand for durable goods(because of complementarity) with larger marginal utility of durable goods $\Delta U_{h,t} > 0$. A larger relative intratemporal elasticity of substitution allows larger variation between marginal utility of housing service and nondurable goods consumption, so a smaller nondurable goods consumption jump can support a given variation($\Delta f\left(p_t, p_{t+1}\right) > 0$) in relative marginal utility. The crowded-out effect is amplified further through the wealth effect and the pass-through from durable goods to nondurable goods. Figure 7c exhibits the decreased strength of wealth channel to crowd-out effect as the relative intratemporal elasticity rises and the one unit housing service becomes less important (can be replaced by nondurable consumption easier). Although in this section I do not quantitatively introduce the aggregate shock into the model and investigate the magnitude change of the precautionary saving effect, it is easy to comprehend that a higher relative intratemporal elasticity of substitution will conduce a smaller precautionary saving effect because the household cherishes the balanced consumption portfolio within a period more than that over periods. To sum up, overbuilding affects crowd-out effect via four channels while three of them are influenced by relative intratemporal elasticity of substitution.

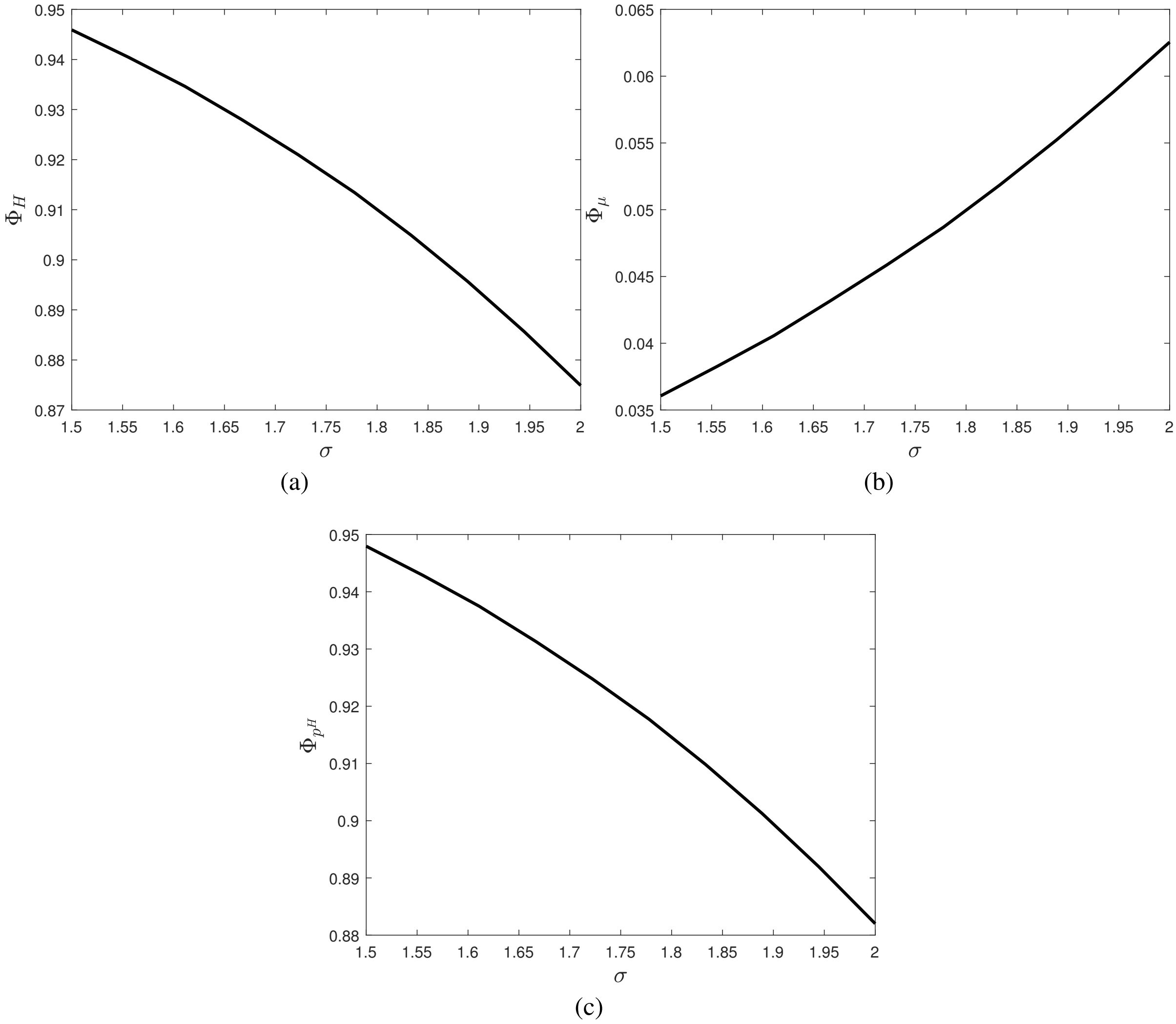


Figure 7: Elasticity of Substitution

3.2.2 Credit constraint and Liquidity

Overbuilding and house market boom push household to spend more money on nondurable goods through substitution effect as now their holding of real estate jumped. Additionally if the economy is incomplete and household cannot fully insure their idiosyncratic shock through financial market, the consumption of household may be constrained by an incomplete market where they cannot borrow as much as they want to confront the bad shock. This credit constraint generates the liquidity problem and some households may be constrained from time to time and not fulfill their consumption demand even though they could repay the amount they borrow in the future. Overbuilding offers more asset that household could use to borrow as collateral and hence it relaxes the previous credit constraint. In figure 8a the extend of financial friction is decreased by increasing the proposition of housing services whose value can be used to borrow money from 0.5 to 0.8. It verifies the argument that a tighter collateral constraint induces a higher substitution effect since one-unit-increased housing servicing becomes more valuable in utility in steady state.

Proposition 5. *When the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} > \frac{K}{L} > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}$ *holds, substitution effect* Φ_H *and wealth effect* Φ_{p^H} *will decrease as collateral constraint is slacker. Further, when the aggregate Khun-Tucker multiplier is not too large, credit effect* Φ_μ *will increase as collateral constraint is slacker.*

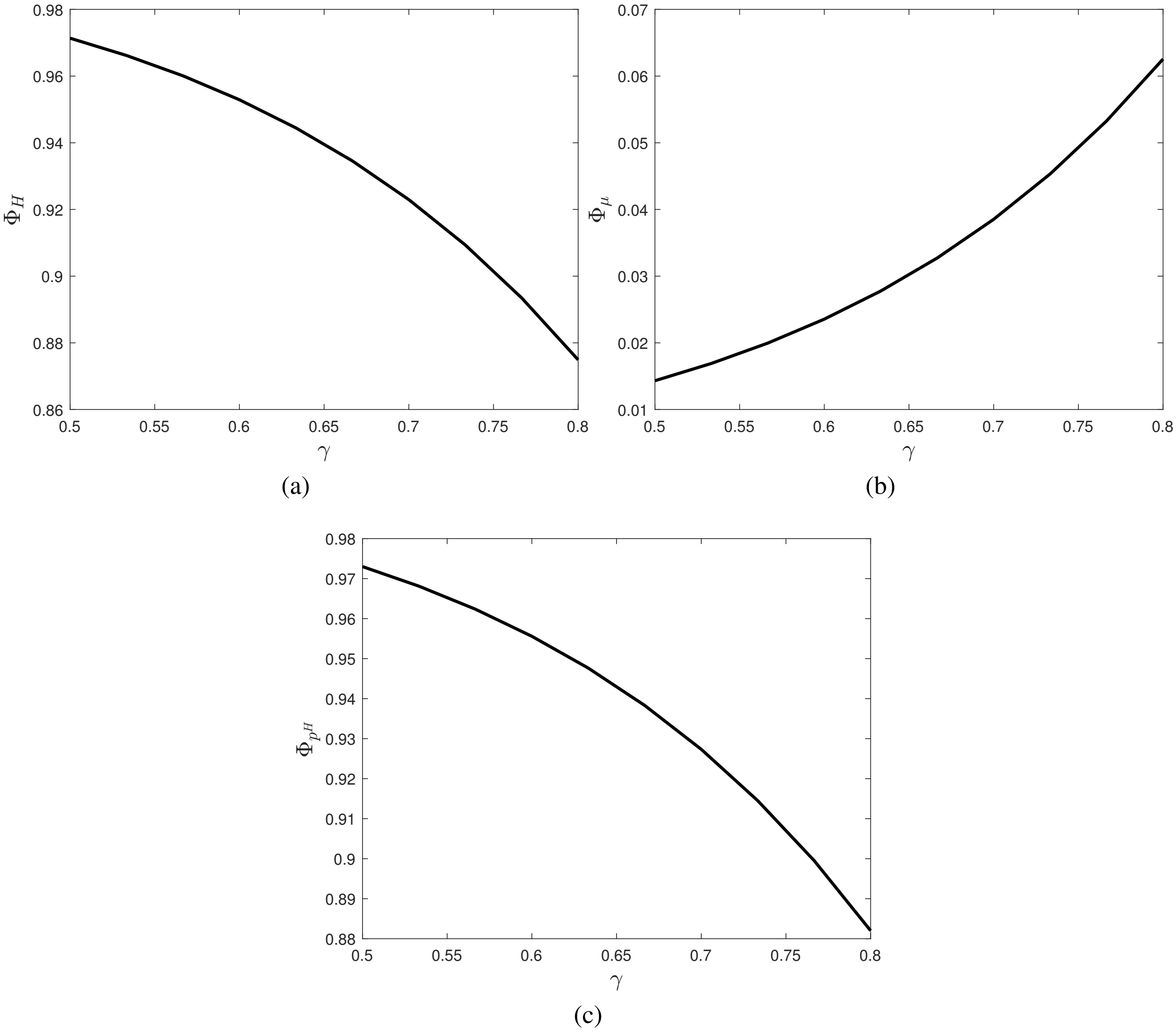
Additionally, a marginal relaxation on the binded collateral constraint represents a smaller K-T multiplier, $\Delta\mu < 0$ in equation 12, and a tighter constraint connects with a smaller nondurable consumption response Φ_μ which ensues a smaller crowded out effect. To understand the credit effect I assume that the unexpected tax rebate spawns the same increased nondurable goods consumption $\Delta C_{a,t} = \Delta C_{b,t}$ and the collateral constraint γ in economy a is tighter than that in economy b and accordingly $\gamma_a < \gamma_b$ will hold in equation 11 as well as in figure 8. A tighter financial constraint reveals a larger K-T multiplier response ergo the absolute change of multiplier in economy a is larger than that in economy $b(\Delta\mu_a < \Delta\mu_b < 0)$. This delineates that a unit change in marginal value of housing servicing in financial constraint performs feebly when the constraint is tight because the unit change in marginal value is now "cheaper" than that in steady state. Figure 7b tells us explicitly the credit crunch (a positive $\widetilde{\mu_t}$) inspired by overbuilding decreases less consumption (or crowd out more investment) when the financial friction is larger.

Comparing to the credit effect, the financial friction works in the opposite direction in wealth effect(but the same in substitution effect). Mathematically a larger financial friction results in a larger K-T multiplier and a larger μ in 15 therefore a larger wealth effect as shown in figure 8c. The mechanism in backdrop is the same as substitution aforementioned since the housing services itself and its price play the same role in collateral constraint 11 and their effect to the pass through should be the same. All these results hold theoretically with several strict constraints which I express in proposition 5.

3.2.3 Precautionary saving and Wealth inequality

Household usually will not consume as much as they would do under the scenario without any idiosyncratic shock or they can perfectly insure the idiosyncratic shock. Household have the propensity to put more income into pocket to save for insuring idiosyncratic shock which we call precautionary saving motive. The last term of equation 16 shows that precautionary saving decreases the consumption adjustment as household save extra $\Phi_{cov}\widetilde{cov}_t$ amount instead of spending out when facing the uncertainty in income.

In addition to the four effects discussed above, substitution effect, credit effect, wealth effect and precautionary saving effect, overbuilding can amplify the crowd-out effect through the lens of business cycle. It is well known that idiosyncratic shock is countercyclical while overbuilding is mostly procyclical. Therefore when the overbuilding happens household are less precautionary since aggregate economic conditions are better and there are less large idiosyncratic shock.

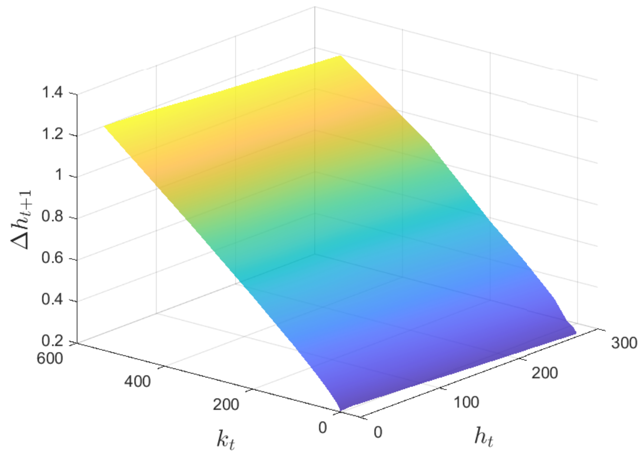


(a) (b) (c)

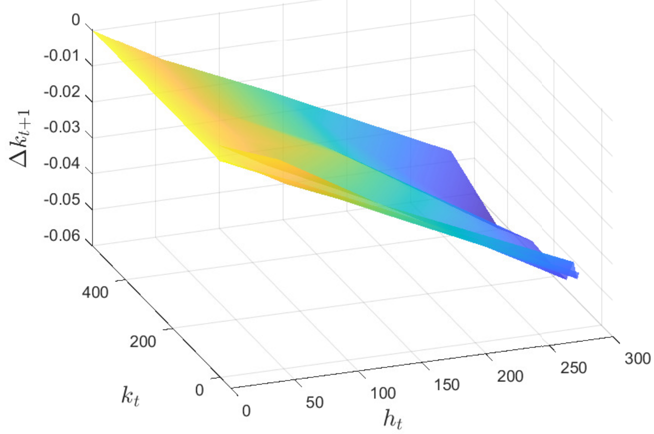
Figure 8: Financial friction

Boom and lower variation of idiosyncratic shock persuades household that economy is going to be better and they become optimistic to consume more and save less. $\widetilde{cov}_t$ in equation 16 will drop which indicates that household save less and consume more when overbuilding and boom arrival. However this amplification is not covered in my numerical experiment which is left for future study.

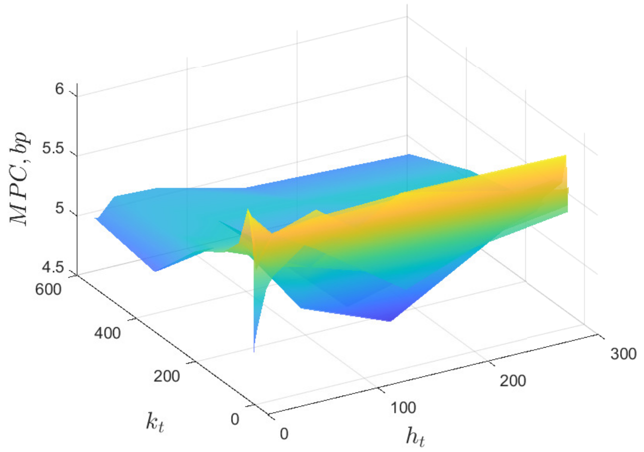
Additionally, the wealth distribution may also manipulate the crowd-out effect triggered by overbuilding in aggregate level. Since the increased holding of housing service is funded by liquid asset and wage income, the absolute amount of large jump per capita in holding of housing service comes from those household who hold a lot of liquid asset and earned high wage income at steady state. After aggregating the consumption decision over household which is shown in equation 16, I can conclude that the distribution of wealth is important as it affects the distribution of coefficient and in turn affect the aggregate crowd-out effect. Figure 9a plots the distribution of changing in holding of housing servicing facing a decrease in house price. The wealthier household who hold a lot of liquid asset is the household who buys more unit



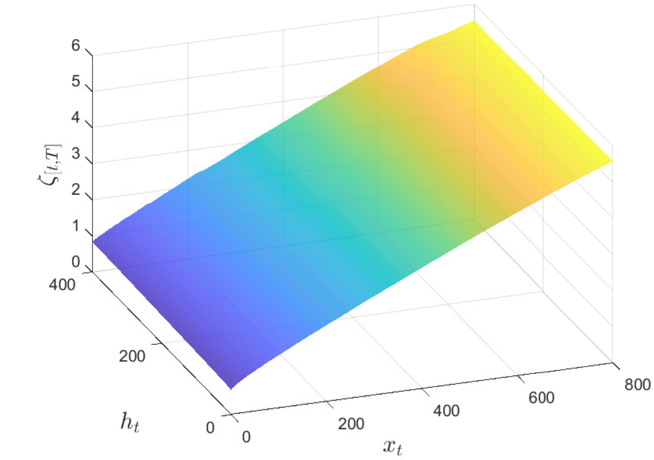
(a)



(b)



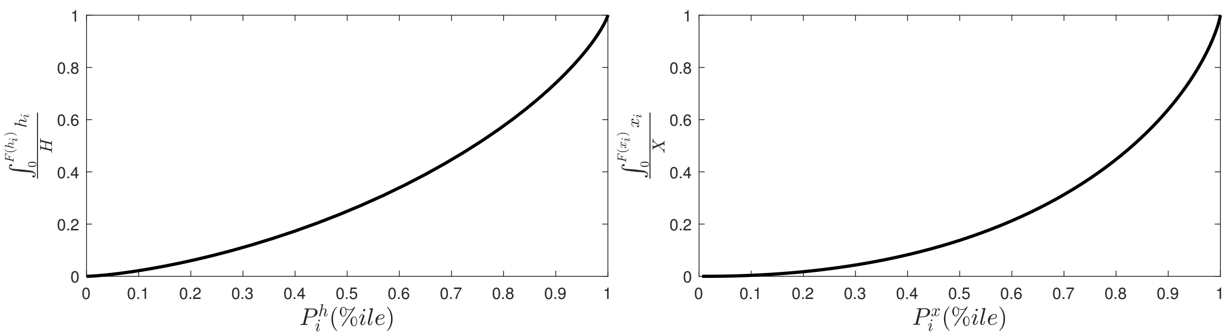
(c)



(d)

Figure 9: Wealth Distribution

of housing service and then who decreases the physical investment as shown in figure 9b. The the cohort mass of the wealth people is small whereas the wealth distribution is left-skewed and the skewness is shown in figure 10a for residential asset and figure 10b for effective liquid asset. The most wealth in economy is held by the least people in the top and this left-skewed wealth distribution amplifies the crowd-out effect of overbuilding through the term $\int \widetilde{h}_t^{\,i} dG_i$ in equation 16. Furthermore, the distribution of MPC is right-skewed(figure 9c) and the standard general equilibrium effect of hand-to-mouth household will also be effective as it works in the pass through of monetary policy. This right-skewed MPC likewise amplifies the crowd-out effect of overbuilding but through the term $\int \widetilde{\mu}_t^i dG_i$ in equation 16. Figure 9d exhibits the wealth distribution effect of a demand-driven boom, which is triggered by an expectation of housing price inflation as I argued in corollary 2, instead of a supply-driven which I use in figure 9a and 9b. The result does not change the attenuation direction formed by wealth distribution which demonstrates that it is independent with type of housing market boom and direction of the change of house price.



(a) Lorenz curve of $F(h_i)$

(b) Lorenz curve of $F(x_i)$

Figure 10: Lorenz curve

3.2.4 Optimistic expectation and overbuilding

Previous arguments are focus on the crowd-out effect generated by overbuilding and we discussed different mechanisms through which this effect works depending on the assumption that overbuilding is already happened. Here I demonstrate that the existence of overbuilding is not a strong assumption and it can easily be created by an optimistic expectation about housing market in the future. When household have a positive expectation about the change of housing price in the future, they will increase their holding of real estate in this period which is similar to the change in consumption induced by intertemporal new keynesian cross. Corollary 2 shows that an increase in the expectation of the housing price in time $T+1$ will marginally provoke $-\left[\beta\left(1-\delta^H\right)\right]^T \Pi_{s=1}^T \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}} \lambda_{t+T+1}/u''_{h^i}$ unit of increase in demand of housing service. If the expectation is driven by optimism or fake news about future, the increased new buildings will become "over"-building as it is not support by the fundamental change in economy but support by a mirage. After this mirage vanishes, the crowd-out effect engenders a recession because of the lack of physical capital produced by the illusion in housing market boom.

Corollary 2. *Ceteris paribus, an positive expectation about the housing price change in time* $T+1$ *will induce a jump in demand of housing service in time t. The response extend follows*

$$\left.\widetilde{h}_t^i\right|_{h_{t+i},\mu_{t+i},\lambda_{t+i},,i\in[1,T]} = \zeta_t^i dp_{t+T+1}^H \tag{18}$$

where $\zeta_t^i = -\frac{1}{u''_{h^i}}\mathbb{E}_t\left[\beta\left(1-\delta^H\right)\right]^T \Pi_{s=1}^T \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}}\lambda_{t+T+1}$

4 Crowd-out effect of overbuilding: Full fledged model

In last section I use a simple model analytically show that an expectation in future housing market boom will inspire household to increase their holding of durable goods' consumption which in turn crowd out the physical investment. This crowd-out effect is influenced by relative intratemporal elasticity of substitution, credit constraint and wealth distribution. In this section I use a full fledged model to analyze the crowd-out effect quantitatively. By linking the model to data I show that news about future can generate a boom-bust cycle in housing market. When the news is fake and the fraud is not realized by household until several periods later, the boom which is supported by a fake news instead of fundamental creates the overbuilding, that induces a large loos in output and consumption during the bust period. I will first introduce the model I used to quantify the drawback of crowd-out effect. Then I use calibration and SMM connect the model with data. In the end I show the large break in economy caused by overbuilding in mirage via some impose response functions.

4.1 Model Setting

4.1.1 Household

Continue household[25] holds housing servicing h and liquid asset b at time t which he takes from last period. He chooses the non-durable consumption c, labor supply l, housing service h' and liquid asset holding b' at time t to solve the optimization problem

$$V(h_{t-1}, b_{t-1}, \varepsilon_{t-1}) = \max_{c,l,b',h'} U(c_t, h_t, l_t) + \beta(1-\theta^d)EV(h_t, b_t, \varepsilon_t)$$

$$\begin{aligned} \text{s.t.} c_t + Q_t b_t + p_t^h \left[h_t - (1-\delta^h)h_{t-1}\right] = R_t Q_{t-1} b_{t-1} + (1-\tau) w_t l_t \varepsilon_t + \Pi_t^h \\ - p_t^h C\left(h_t, h_{t-1}\right) + T_t \end{aligned} \tag{19}$$

$$-Q_t b_t \leq \gamma p_t^h h_t \tag{20}$$

where p_t^h is the relative price of housing unit at time t. R_t is the gross real return of liquid asset which follows $R_t = \frac{Q_t(1-\delta)+r_t}{Q_{t-1}}$. $C\left(h_t, h_{t-1}\right)$ is the adjustment cost function when household want to adjust their holding of housing servicing. γ is the parameter governing the slackness of collateral constraint. δ^h is the depreciation rate. τ is the wage income. Π_t^h is the restitution from construction companies. T is the lump-sum tax transfer payed by government. θ^d is the death rate. ε is the idiosyncratic income shock which follows logarithmic AR1 process with coefficient ρ_ε and standard derivation σ_ε.

[25] Here for simplicity I omit the index for specific household i.

The adjustment function follows the canonical form

$$C\left(h_t, h_{t-1}\right) = \frac{\kappa_1}{\kappa_2}\left(h_{t-1} + \kappa_0\right)\left|\frac{h_t - h_{t-1}}{h_{t-1} + \kappa_0}\right|^{\kappa_2}$$

The utility function follows the CRRA form[26]

$$U(c_t, h_t, l_t) = \frac{\left(c_t^{\phi} h_t^{1-\phi}\right)^{1-\sigma}}{1-\sigma} + \kappa\frac{l_t^{1+\psi}}{1+\psi}$$

4.1.2 Firm

There are two types of firms, construction firms who produce the housing servicing and the non-durable goods producers. All of these two types of producers are staying in complete market but because the construction firms also use exogenous land supply as an input to construct house, they earn non-zero profit which in the end refunded back to their holder, household.

Non-durable goods producer use

$$Y_{N,t} = A_{n,t} K_{n,t}^{\alpha} L_{n,t}^{1-\alpha} \tag{21}$$

to maximizes profit with the cost from real rental rate of capital used by non-durable goods producer K_n and related wage payment to labor L_n .

Similarly, durable goods (housing services) producer use

$$Y_{H,t} = A_{h,t} \overline{LD}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota} \tag{22}$$

to maximizes profit with the cost from real rental rate of capital used by durable goods producer $K_{h,t}$ and related wage payment to labor $L_{h,t}$. The$\overline{LD}_t$ in production function is the exogenous land supply follows $\overline{LD}_t = \overline{LD} A_{L,t}$ and the new construction is homogeneous to each production factor therefore the share of input satisfies $\theta + \nu + \iota = 1$.

4.1.3 Capital Producer

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Lambda_{t,t+\tau} \left\{Q_\tau I_\tau \eta_{I,t} - f\left(I_\tau, I_{\tau-1}\right) I_\tau \eta_{I,t} - I_\tau\right\}$$
$$\text{s.t. } f\left(I_\tau, I_{\tau-1}\right) = \frac{\psi_I}{2}\left(\frac{I_\tau}{I_{\tau-1}} - 1\right)^2$$

[26] Piazzesi et al. (2007) use CEX data suggest that intratemporal elasticity of substitution is close to 1. In other words the utility function form of durable and nondurable goods is close to standard Cobb-Douglas case.

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t \delta_{I,t} = 1 + \frac{\psi_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 \eta_{I,t} + \psi_I\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\eta_{I,t} -$$
$$E_t \beta \Lambda_{t,t+1} \psi_I \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \tag{23}$$

where $\eta_{I,t}$ is the marginal efficiency of investment shock which I follow Justiniano et al. (2011).

4.1.4 Market cleaning

Capital is supplied by household with their liquid asset and labor is supplied in effective form

$$K = \left(1 - \theta^d\right) \int b dG = K_n + K_h$$

$$L = L_h + L_n = \int \varepsilon l dG$$

$$H = \left(1 - \theta^d\right) \int h dG$$

The goods market cleaning condition is

$$C_t + I_t + f\left(I_t, I_{t-1}\right) I_t \eta_{I,t} + p^h C(h', h) = Y_{N,t}$$

where $K' = (1-\delta)K + \delta_I I$

Similarly, the housing market cleaning condition is

$$\left[H' - (1-\delta^h)H\right] = Y_{H,t}$$

The return of gross liquid asset b comes from two component: capital return from firms r and capital gain $\frac{Q'(1-\delta)}{Q}$.

In the end the government close the economy by $T = \tau w L + \theta^d\left(K + p^h H\right)$ and $\Pi^h = p^h Y_H - wL_h - (r - 1 + \delta)K_h$ as all the new born household hold zero liquid asset and housing servicing.

The model contains three types of shock: *contemporaneous unexpected shock, news shock and noise shock* which I introduce detailedly in appendix H.7.1. I introduce the news and noise shock following Chahrour and Jurado (2018) who introduced the news and noise representation to overcome the observational equivalence problem in previous literature such as Schmitt-Grohé and Uribe (2012), Barsky and Sims (2012) and Blanchard et al. (2013).

4.1.5 Shocks

There are two fundamental shocks on the TFP of the two production function 21 and 22 respectively. These two shocks a_t^i follows the standard logarithm AR(1) process $\log(a_t^i) = \rho_a^i \log(a_{t-1}^i) + \varepsilon_t^{a^i}$ where $i \in \{h, n\}$. Thus the TFP of these two production functions follow $A_{n,t} = a_t^n \overline{A}_n$ and $A_{h,t} = a_t^h \overline{A}_h$.

Meanwhile I introduce a preference shock Φ_t^{ϕ} and the shock to its growth rate $\Phi_{g,t}^{\phi}$ to the preference ϕ in utility function in the demand side, cooperating with a land supply shock Φ_t^L and the shock to its growth rate $\Phi_{g,t}^L$ in the supply side to determinate the housing market.

Meanwhile to incorporate the noise and news into the model I assume that the household can get a news related to the shocks up to 8 periods before they realize and I defined them in companion form in equation 93. However the agents cannot perfectly observe these shocks but mixed with noisy observation shock to $\widetilde{\Phi}_t^i$ and $\widetilde{\Phi}_{g,t}^i$ in equation 95.[27]

4.2 Calibration

4.2.1 Parameter

Most of the parameters I used in production side comes from literature which is standard and robust. I relegate them into appendix H.1 which is summarized in table 10. I use the discount factor, disutility to labor supply, and three parameters in production side to match the gross real interest rate at 1.015 quarterly, labor supply at 1, physical investment over GDP at 0.13 and new construction over GDP at 0.05. The physical investment over GDP is estimated from Private Non-Residential Fixed Investment over Gross Domestic Product and the new construction over GDP is estimated from Private Residential Fixed Investment over Gross Domestic Product. The parameters in adjustment cost function is in line with Kaplan et al. (2018) and Auclert et al. (2021). The intertemporal elasticity of substitution and preference between durable and nondurable goods are borrowed from Kaplan et al. (2020). The AR1 coefficient and standard derivation of idiosyncratic shock follow the estimation by McKay et al. (2016). The death rate is estimated from the Underlying Cause of Death provided by Centers for Disease Control and Prevention from 1999 to 2020. All the value of corresponding parameters I used are summarized in table 2.

Table 2: Key Parameter Values

Parameter	Value	Description
β	0.9749	Discount factor
τ	0.20	Labor income tax

[27] I define the news and noise shocks following the suggestion made by Chahrour and Jurado (2018) because this form does not suffer from the observational equivalence problem.

Table 2 – Continued		
Parameter	**Value**	**Description**
κ	-1.28	Disutility to supply labor
θ^d	0.21%	Death rate
γ	0.8	Slackness of collateral constraint
κ_0	0.25	Adjustment cost silent set
κ_1	1.3	Adjustment cost slope
κ_2	2	Adjustment cost curvature
σ	2	Inverse of intertemporal elasticity of substitution
ϕ	0.88	Preference between durable and nondurable
ρ_ε	0.966	AR1 coefficient of income shock
σ_ε	0.25	SD of income shock

4.2.2 Data to Model: Moment Matching

Even though I do not specifically match the moments in distribution, my model generates a lot of merits to replicate the moments extracted from data. Table 3 shows that my model has some nature ability to unveil the reality which I compare the data estimated by Kaplan et al. (2014) and Kaplan et al. (2018) and the moments calculated from model.

Table 3: Distribution Moments

Description	**Data**	**Model**
Poor Hand-to-Mouth Household	0.121	0.1102
Wealthy Hand-to-Mouth Household	0.192	0.2059
Top 10 percent share of Liquid asset	0.8	0.5
Top 10 percent share of Iliquid asset	0.7	0.3

To build the bridge between the model and data, I use full information Bayesian method to estimate the parameters that pertain to the dynamic and business cycle. Particularly I resolve 38 parameters, such as the persistence of shocks, observation matrix and standard derivations of noise shock, from a time series with 7 variables. For similarity I assume the covariance matrix of shocks is a diagonal matrix hence all the shocks are independent and there is no parameters related to covariance terms in estimation. All the details about the estimation are relegated to the appendix H.2.2.2.

Table 4: Real Business Cycle Moments

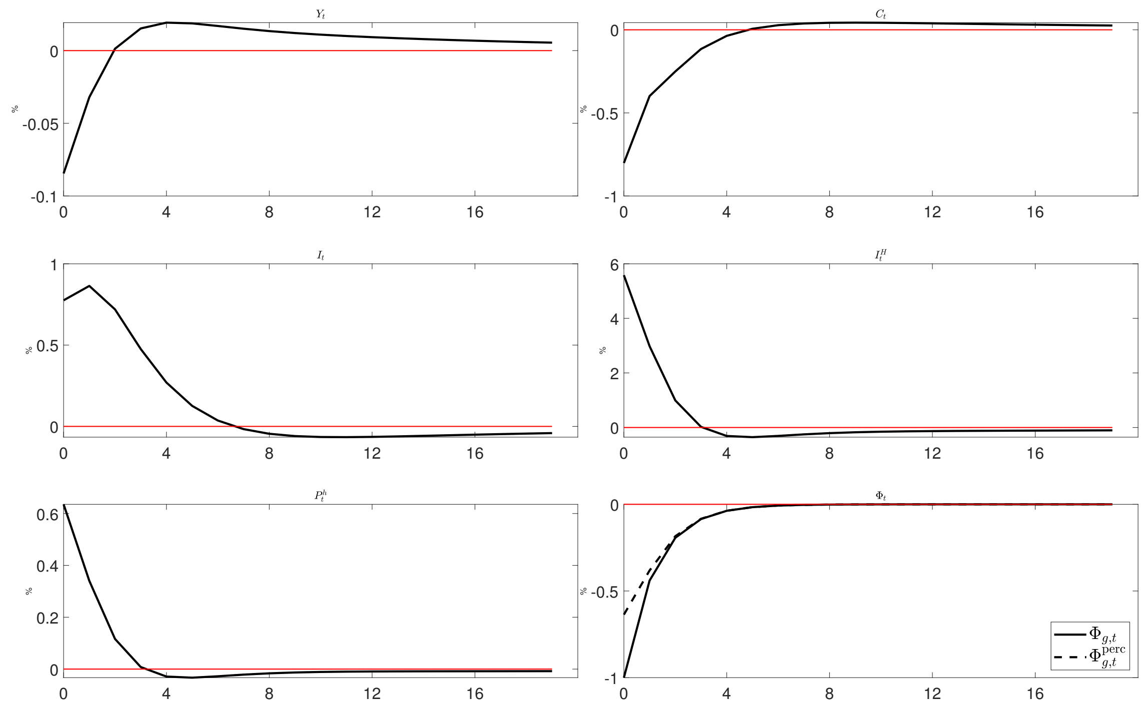
Moments	Description	Model	Data
σ_Y	Standard Derivation of output	0.04	0.02
$\frac{\sigma_{p^H}}{\sigma_Y}$	Relative Standard Derivation between housing price and output	1.57	1.46
$\frac{\sigma_I}{\sigma_Y}$	Relative Standard Derivation between physical investment and output	3.92	3.19
$\frac{\sigma_{I^H}}{\sigma_Y}$	Relative Standard Derivation between new construction and output	12.42	8.88
$\mathrm{corr}(p^H, I^H)$	Correlation between real estate price and new construction	0.42	0.23
$\mathrm{corr}(I, Y)$	Correlation between physical investment and output	0.06	0.19
$\mathrm{corr}(I, Q)$	Correlation between physical investment and capital price	0.40	0.32

The moments in data is calculated by detrending the trend from quarterly time series via hp-filter and for the purpose of compatible compare akin to the filtered data, I also follow the method proposed by Uhlig et al. (1995) and Ravn and Uhlig (2002) to calculate the moments of model in frequency space with some algebraic modifications that are discussed in Appendix H.2.1. Table 4 summarizes the primary moments related to the housing market and physical capital investment on which I focus in this paper. The result shows that the model is in line with the reality and can be used to estimate the economic destruction caused by the real estate over-construction.

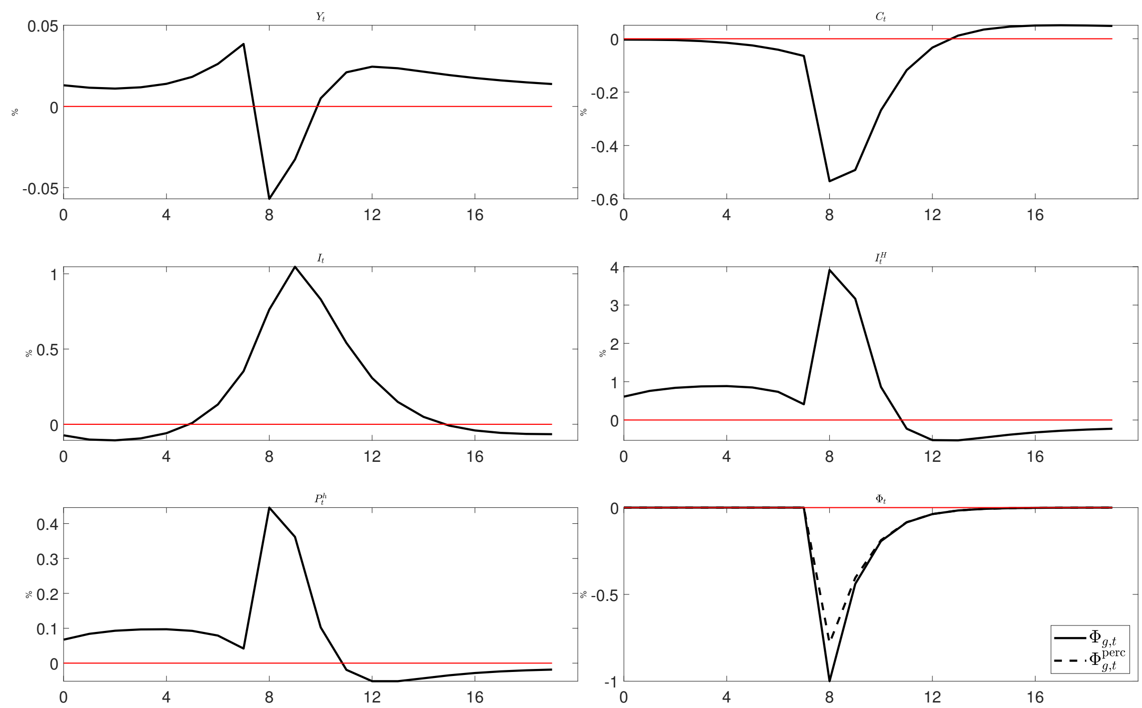
4.3 Quantitative Analysis

4.3.1 Overbuilding and Boom-bust Cycle: News to the Future and Inefficiency of imperfect information

When a contemporaneous preference shock realized, household will decrease their nondurable goods consumption to exchange for more durable goods consumption, housing servicing because they prefer the real estate to the nondurable goods now. This altered preference draws the housing price up because of a demand curve shift to the right, which in the end generates a housing market boom which is shown in figure 11a. A one unit growth shock to preference is only perceived with 0.5 at the peak by the household. The household increase their consumption to housing servicing and this jump in demand increases the construction to the peak of 3 and house price to the peak of 0.6.



(a)



(b)

Figure 11: Contemporaneous and News shock

However if the shock is known by the household ahead of time when it realized, the household will response to this future shock when he known the realization news. They will increase the holding of house right away which pushes up the house price immediately. This will crowd out the physical investment through general equilibrium cycle if the nondurable consumption does not change. Further the household will also increase their consumption either because they are wealthier now fueled by the real estate appreciation or because they can borrow more fund from bank. This will magnify the crowd-out effect as nondurable consumption also entries into the goods market cleaning condition. Figure 11b shows this crowd-out effect triggered by a new shock. After observing a news about preference shock 8 periods later, household increase their holding of housing and there is a boom in housing market. They also increase the nondurable consumption but crowd out the physical investment.

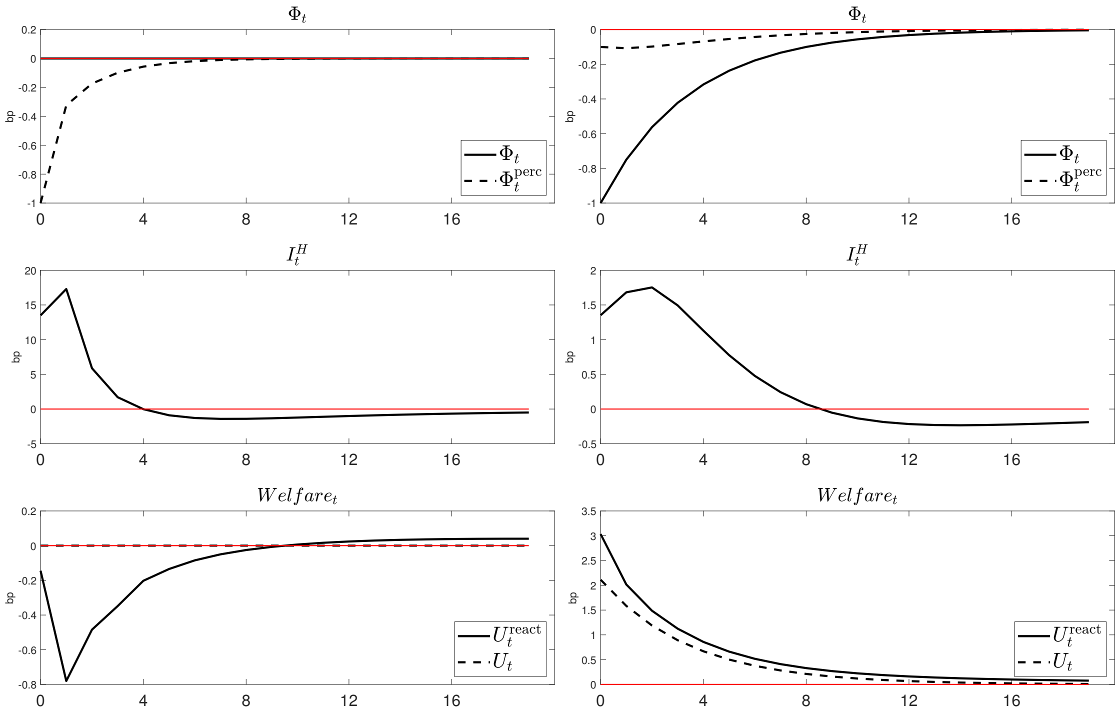


Figure 12: Welfare Loss in Imperfect Information

When the housing market boom is a bubble that is blown by a phantasm, the inefficiency of imperfect information could incur a welfare loss. Figure 12 illustrates the welfare loss caused by imperfect information. RHS is the response of investment and aggregate utility(with unit weight) to a preference shock on nondurable goods. By observing the decrease in contribution of nondurable goods to utility, the household has a perception of this preference shock as which I denotes the dash line in the first row. Because the housing service provides more utility now the household will increase their consumption to housing service and the aggregate welfare jump to 3bp which is shown in the solid line below. If the household does not response to the shock with zero derivation all the time, they will have a relative loss in welfare comparing to the situation that they react because the shock really happened and it is optimal to response to it. Although the household has an absolute increase in welfare because of the distribution effect and existence of hand-to-mouth household. Opposite to the realized preference shock, LHS of figure 12 shows the response of investment and aggregate welfare to a noise shock or observation shock. The household still increase their consumption to housing service because they thought that a preference shock has happened and they loss in welfare from this inappropriate reaction which I denote the solid line in the last row. If they did not react to the noise shock their welfare would have no change at all because nothing had happened which is shown by dash line. The experiment above corroborates the inefficiency of imperfect information as people misleadingly proceed housing market boom and I show that the noise in news, or fake news, can induce a further loss in output and consumption because of crowded-out physical capital.

4.3.2 Overbuilding and Boom-bust Cycle: Fake News

When a pure noise(observation) shock instead of fundamental shock is informed to household, they would response to this shock as what they did to the fundamental shock because of the existence of information friction. Household cannot know the exact magnitude of the shock but a signal contaminated with noise. They response to what they perceived, or in other words their belief, instead of the fundamental shock. Therefore as long as the household believe there is an housing market boom in the future, they will increase their holding of housing service and crowd out the physical investment, which is a chronic poison to them as long as their belief is incorrect and the housing market boom is built on the Babylon tower. When the household across the manifest they need invest more physical capital because they temerariously exchange the physical capital to real estate just before. This large demand to physical capital results in a huge drop in nondurable consumption which follows a heavy loss in welfare. Additionally because the real estate is also a type of wealth which the household used to borrow money from bank, a housing market bust and a deep deflation in house price break the consumption pattern of low-income household and leave them at financial constrained edge, which leads to a further welfare loss.

Figure 13 compares the impulse response to the fake-news preference shock with and without pre-crowded physical capital which demonstrates the large output and welfare loss engendered by crowd-out effect. The blue solid lines are the responses to a contemporaneous noise shock $\widetilde{\Phi}_t^{\phi}$ of non-durable goods' production, non-durable goods consumption, physical investment, new construction and real housing price. The black solid lines represent the responses of them to noisy news $\widetilde{\Phi}_{t+8}^{\phi}$ which is informed to household 8 period ago. When the household knows that there will be an economic boom in the future, they increase the investment in real estate and induce a housing market boom immediately. Because all the household already hold some amount of real estate, this housing market boom spurs higher non-durable goods consumption because of the wealth effect, which is in line with Mian et al. (2013). This further crowds out the physical investment which is shown by the negative response in 13. After the shock "should" realized two years later, at period 9, household are gradually aware the true and increase the physical capital investment a lot to compensate the scarcity of capital caused by crowd-out effect from negative 2 percentage to positive 6 percentage. The bust in housing market leads to a 2.5 percentage drop in housing price and 2 percentage drop in non-durable goods consumption. On the other hand, if the physical capital is not pre-crowded, the economy response is mild and moderate with smaller output loss, consumption privation and housing market bust. There are

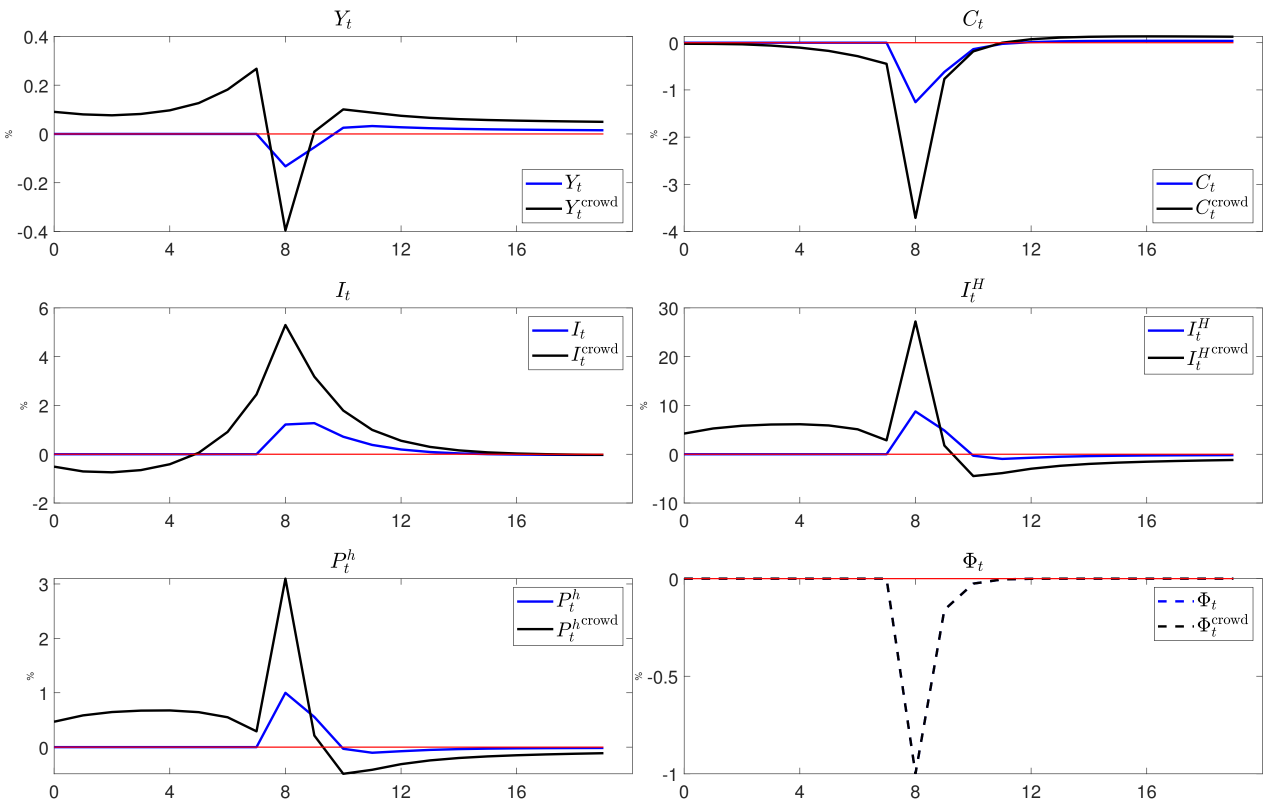


Figure 13: Fake news shock

only half of the loss in non-crowded scenario relative to crowded scenario. Similarly there are only two third of the boom-bust cycle in housing price and new constructions in the non-crowded situation. The difference in impulse response demonstrates the non-negligible drawback of the crowded-out effect in the housing market boom-and-bust cycle.

4.3.3 Idiosyncratic Income shock, Financial Friction, Relative intratemporal elasticity of substitution

To investigate how the crowded-out effect is influenced by the idiosyncratic income shock, financial friction and relative intratemporal elasticity of substitution, I fix the expected jump in house price and change the relative parameters in model in this section. By decreasing the relative intratempral elasticity of substitution with the same amount in section 3.2.1, the blue dash line in figure 14 illustrates the attenuation caused by the relative intratemporal elasticity of substitution. A smaller relative intratemporal elasticity of substitution, from $\mathrm{ES} - \mathrm{EIS} = \frac{1}{2}$ to $\frac{1}{3}$, results in a huge physical investment drop, roughly 3 times larger than that in baseline model. Given this smaller intratemporal elasticity of substitution, household will care less about the substitution in utility between non-durable goods and housing service (in other words more complementarity) which result in a larger increase non-durable goods consumption in figure 14. These lead to a lower investment in physical capital via general equilibrium.

The red dash line in figure 14 depicts the response under a tight credit constraint, which

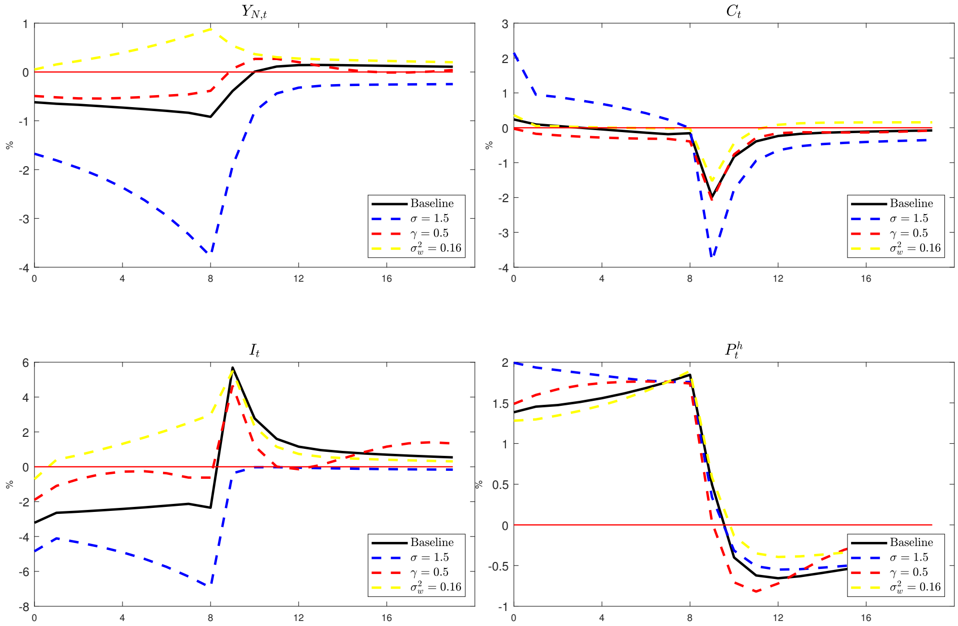


Figure 14: Crowded-out effect comparison

implies an important role of wealth inequality. As shown in section 3.2.2, if we do not consider the wealth distribution (i.e. $\int \widetilde{h}_t^i dG_i$ and $\int \widetilde{\mu}_t^i dG_i$ in equation 16) a tighter financial constraint will result in a severer crowded-out problem because the real estate is more valuable now. However, as shown in section 3.2.3, household cannot increase their non-durable consumption and housing service as much as they want because of financial constraint and wealth inequality. The larger $\widetilde{h}_t^i$ can only be realized in a smaller dG_i and figure 14 shows that this inequality channel dominates other channels. The physical capital is crowded out less than that in baseline model as there are more overwhelmed household who cannot increase their consumption as much as they want.

Additionally I increase the variance of idiosyncratic income shock from $\sigma_w^2 = 0.06$ in baseline model to 0.16 which I characterize with yellow dash line in figure 14. Facing a massive income shock, household will have more precautionary saving motive to hold the asset (to fulfill their consumption demand against potential low income and cash flow state) instead of borrowing money to buy housing services. Even though the household expect a housing market boom they only slightly decrease the physical capital at the first period and then increase until the shock realized. The reason why the physical capital further jumps is that household want to hold more housing services under the effect of expected shock. However they do not want to borrow money and decrease their holding of asset to buy real estate. They can only increase their labor supply to earn more wage income to buy housing services. The complement between labor and physical capital tempts the household to increase their asset instead of decreasing them with a higher asset return, which triggers a positive feedback loop on the boom in physical capital.

4.3.4 Policy Analysis

The quantitative result in last section demonstrates the considerable welfare loss caused by crowd-out effect of overbuilding in housing market after the housing market bubble bust. Accordingly, as long as the policy maker could restrict the extent of housing market bubble, the welfare loss, stemmed from crowd-out effect, would be confined during the bust period because the policy curtails the capital misallocation at the same time. In this section I introduce a macroprudential policy which reduces the ability of equity extraction during boom period endogenously and attenuates the crowd-out effect. Following Galati and Moessner (2013), Angelini et al. (2014) and Suh (2014), I introduce the macroprodential policy rule as a procyclical collateral constraint on capital-output ratio

$$\frac{\gamma_t}{\overline{\gamma}} = \left(\frac{\gamma_{t-1}}{\overline{\gamma}}\right)^{\rho_\gamma} \left(\frac{\upsilon_t}{\overline{\upsilon}}\right)^{\eta_\gamma(1-\rho_\gamma)} \tag{24}$$

where γ_t is the collateral constraint in equation 20 and υ_t is the capital-output ratio. $\overline{\gamma}$ and $\overline{\upsilon}$ are their corresponding value in steady state.

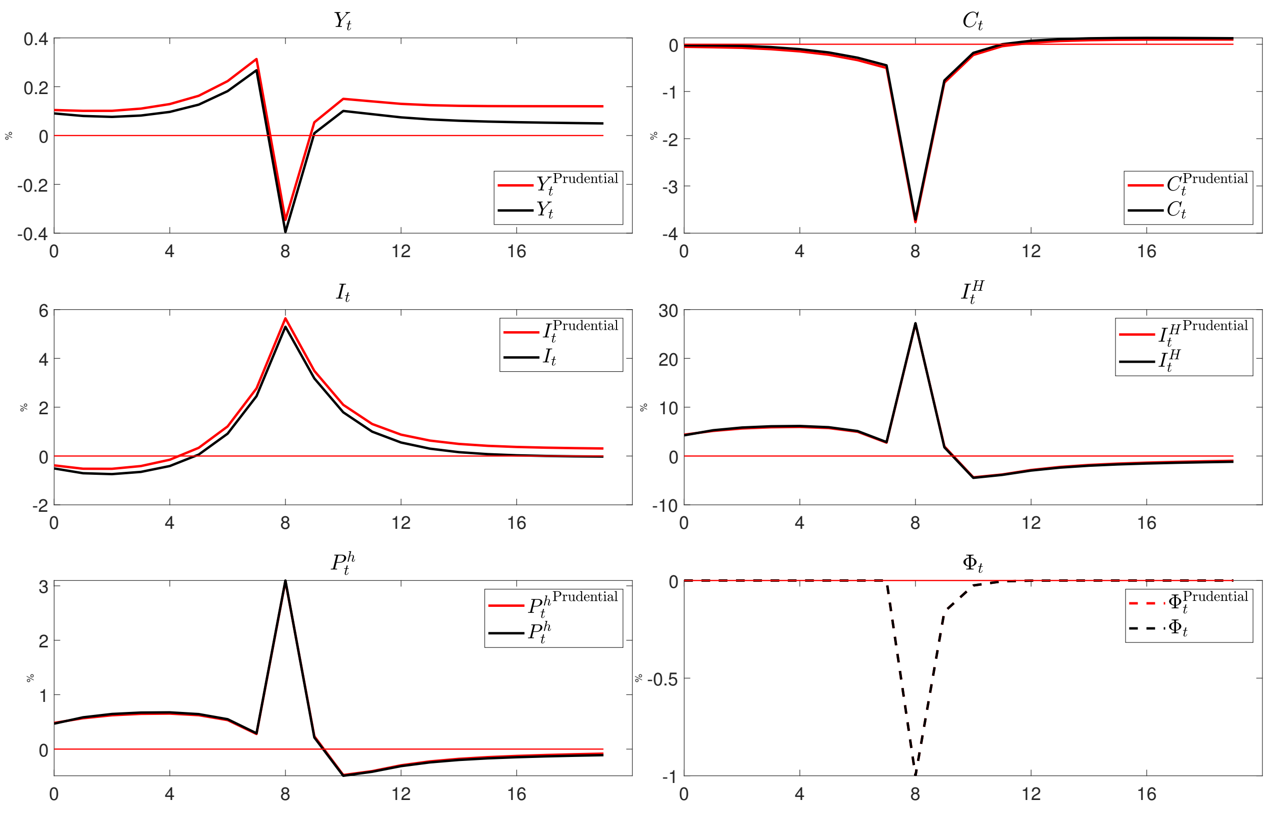


Figure 15: Fake news shock with macroprudential policy

Figure 15 exhibits the physical investment in the model with macroprudential policy stays above that in baseline model and this manifests the ability that macroprudential policy abates the crowd-out effect significantly. Because of the countercyclical restriction on housing market boom, the equity-extraction and asset reallocation (from physical capital to real estate) is modest, and so is the nondurable consumption drop during the bust period. However, because of the policy persistent ρ_γ, the household, especially the poor household, cannot exercise enough capacity of residential asset to smooth their consumption and the red line in figure 15 of nondurable consumption lies below the black line of that in baseline model. Overall the macroprudential policy reduces the welfare loss from 13% in baseline model to 6%, through the same comparison between fake news shock and contemporaneous noisy shock. This substantial decline in welfare loss manifests the main merit of the macroprudential policy that limits the emergence of overheated economy and hence limits the crowd-out effect.

5 Conclusion

This paper documents a new mechanism through which the housing market boom magnifies the recession. An unnecessary jump in residential construction arouse by fake news and imperfect information will blow up a bubble in housing market which is a boom without solid inner filler and not supported by economic foundation. This overbuilding in housing market crowds out physical capital which is used to produce both durable and nondurable goods. The crowd-out effect in physical capital market aggravates the decline in output when the housing market bubble busts because of the deficiency of physical capital. Firms do not have as much as capital they can use to support the optimal production under a specific level of TFP so they will decrease production and labor demand when facing a higher real interest rate and marginal production cost. I use a simple model to argue theoretically that the crowd-out effect of overbuilding is affected by relative intratemporal elasticity of substitution, financial friction, idiosyncratic income shock and wealth distribution. Later the quantitative result from a full-fledged model verifies the argument and demonstrates that the output loss caused by overbuilding is large.

However there are still some problems left for future studies. Even though the imperfect information does not exist the overbuilding and crowd-out effect may still be a significant drawback in the perspective of business cycle as it increases the economic volatility and household leave their first-best equilibrium further. Additionally how can the government introduce an optimal fiscal, monetary, or macroprudential policy to alleviate the crowd-out effect of overbuilding? Is there any complementarity between overbuilding and nominal rigidity in New Keynesian model which will further exacerbate the defect of overbuilding and crowd-out effect?

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A Data Description

Real GDP Y_t is directly downloaded listing "Real Gross Domestic Product" with seasonally adjusted. Real consumption C_t is directly downloaded listing "Real personal consumption expenditures: Nondurable goods" with seasonally adjusted. GDP deflator gdp_{def} is downloaded listing "GDP Implicit Price Deflator in United States" with seasonally adjusted. Nominal nondurable investment I_t^{nom} is downloaded listing "Private Nonresidential Fixed Investment" with seasonally adjusted. I get the real nondurable investment I_t by the formula $I_t = I_t^{\text{nom}}/\text{gdp}_{def} * 100$. The CPI which we take is "Consumer Price Index for All Urban Consumers: All Items Less Shelter in U.S. City Average" since we should consider the correlation between house price and normal CPI. Thus we downloaded the CPI without shelter term. I take the nominal interest rate R_t^{nom} as "Effective Federal Funds Rate". The inflation rate is calculated from the GDP defltor in the form that $\pi_t = \frac{def_t - def_{t-1}}{def_{t-1}}$ (Since we solve the inflation from deflator in quarterly data, the inflation is measured within one quarter instead of annually). Combining the inflation π_t and nominal interest rate R_t^{nom} we can construct the real interest rate $R_t = (\frac{R_t^{\text{nom}}}{100} + 1)/(1 + \pi_t) - 1$ (I divided 100 because the original data is in percentage unit). The house supply H_t is measured by "New Privately-Owned Housing Units Started: Total Units". The nominal mortgage debt MD_t^{nom} comes from "Mortgage Debt Outstanding, All holders (DISCONTINUED)". Since the nominal mortgage debt is in money unit, I can directly get the real mortgage debt value from $MD_t = MD_t^{\text{nom}}/\text{gdp}_{def} * 100$ which is same as we did to get real investment. The real stock price P_t^a is calculated from "NASDAQ Composite Index" and normalized by GDP deflator as I did in constructing real investment and real mortgage debt. The real house price P_t^h is calculated from "All-Transactions Indexes" collected by Federal Housing Finance Agency.

B Identification Step and Robustness Test to VAR Identification

B.1 Identification with sign and zero restricution

Based on the observation and argument, I use a simple SVAR model to decompose the effect of raised house price to investment. Given the model which is similar to Sims et al. (1986)

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + ... + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{e}_t \tag{25}$$

where

$$\boldsymbol{y_t} = \begin{bmatrix} r_t \\ m_t \\ y_t \\ p_t \\ i_t \\ p_t^h \\ c_t \end{bmatrix} \tag{26}$$

r_t is the nominal interest rate; m_t is the money supply; y_t is the real output; p_t is the price level; i_t is the nominal investment; p_t^h is the nominal price of house; c_t is the real consumption of non-durable goods. Most the data comes from FRED, Federal Reserve Bank of St. Louis. I use treasury bill rate represents the nominal interest and GDP deflator for the price level. The price of house comes from FHFA house price index. The detail about it will be discussed at appendix. Meanwhile I use the short-run restriction as well as corresponding sign restriction to decompose the shock term $\boldsymbol{e_t}$ from $\boldsymbol{v}_t$ that

$$\boldsymbol{P}\boldsymbol{e_t} = \boldsymbol{v_t} \tag{27}$$

or detailedly

$$Pe_t \equiv \begin{vmatrix} 1 & b_{11} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & 1 & b_{23} & b_{24} & 0 & 0 & 0 \\ b_{31} & 0 & 1 & 0 & b_{35} & 0 & b_{37} \\ b_{41} & b_{42} & b_{43} & 1 & b_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{56} & 0 \\ b_{61} & 0 & b_{63} & b_{64} & 0 & 1 & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & b_{76} & 0 \end{vmatrix} \begin{bmatrix} e_{rt} \\ e_{mt} \\ e_{yt} \\ e_{pt} \\ e_{it} \\ e_{p^h t} \\ e_{ct} \end{bmatrix} = \begin{bmatrix} v_{rt} \\ v_{mt} \\ v_{yt} \\ v_{pt} \\ v_{it} \\ v_{p^h t} \\ v_{ct} \end{bmatrix}$$

Figure 16 shows the IRF of one unite positive house price shock to output, investment, house price and non-durable goods consumption. The black line is the path of related variable up to 20 period. The read dash line is their related confidence band under 90% calculating by monte-carlo method. We can inspect from IRF that, house price inflation could stimulate the consumption of durable goods as it is long-lasting goods and household could derive out utility by just holding it. The household could feel satisfy and pleased either via living in this house or via owning the house which is valuable every period. Meanwhile the household can obtain utility not only from just holding and enjoying it each period, but also from financial market. The house is a goods that could be consumed. While at the same time it is also a asset that could be collateral and offers more liquidity to household. Household would use this liquidity to smooth their non-durable consumption leisurely, which provide extra benefit to household.

Therefore after observing one unit positive shock in house price, household snap up the house as house it not only a goods but also an asset which we discuss before. This increased demand

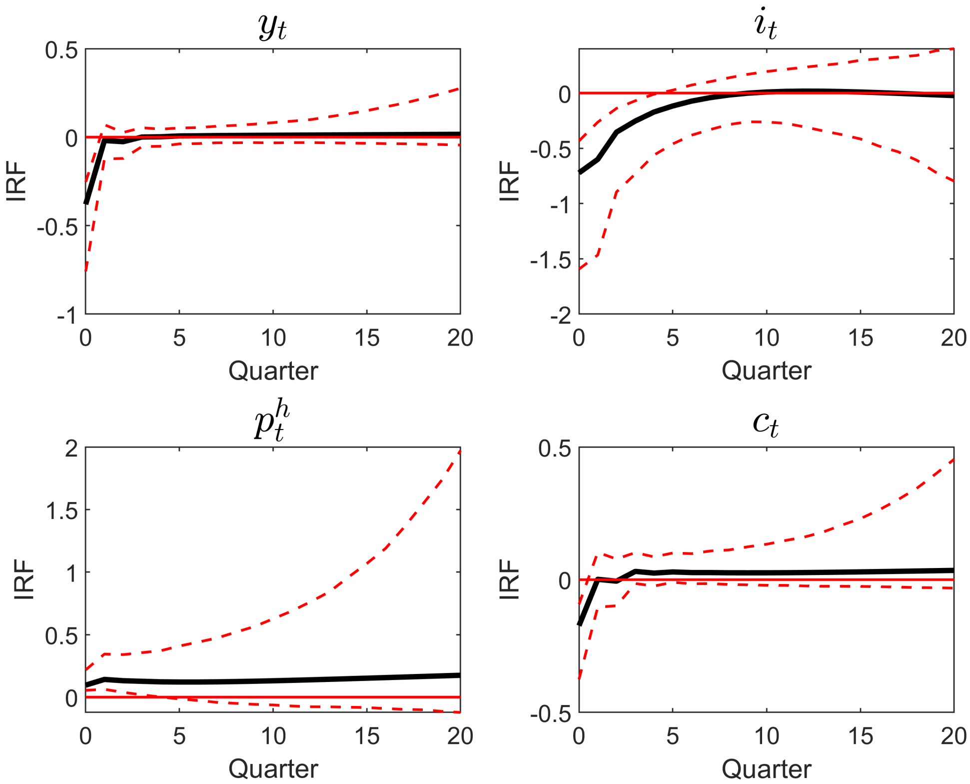


Figure 16: IRF of house price inflation

draw up the house price even more which we can see the house price is raising not only lat the beginning but also later. The house price in the end permanently increased because of increased household demand. This increased house price stimulates household who would borrow more from bank to buy house (the house supply discontinuity will aggravate this channel) or borrow more to help them share the risk as collateral is more expensive. Firms will be more difficult to borrow money to invest and the decreased demand in non-durable goods will also weaken firms' propensity to invest or R&D. Investment is crowded out by this two effects and this is what we can observe from the IRF. Investment drops the most and also spends longest time to recover. Output and non-durable consumption stands behind it. However both of them go back to steady state quickly which indicates that only the first jump in house price affects them. Later household use their more valuable collateral to smooth the consumption as well as output. Thus these two variable converge back quickly while because of strong and amplified effect both in demand and supply side, investment converges much slower than other two variables. This portends that there would be much larger drop in output if recession occurs because the accumulated decreased investment will pass its influence through the capital, a long-lasting things, later.

B.2 Contemporaneous real price shock

B.2.1 Process of estimation and identification

I detrend the main variable by taking logarithm first and first-order difference later. Then I get the detrended real GDP, real consumption, real investment, cpi, house supply, real mortgage debt, stock price and house price in lower-case letter. Then I ordered them in the vector

$$Y_t = [y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h]'$$

I use the data period between 1987Q2 and 2006Q4. Then I add lagged term into the model up to 4 quarter and estimate the model

$$Y = [Y_5, Y_6 ...]$$

$$X_{t-1} = [y_{t-1}, c_{t-1}, i_{t-1}, cpi_{t-1}, r_{t-1}, p_{t-1}^a, hs_{t-1}, md_{t-1}, p_{t-1}^h, y_{t-2}, c_{t-2}, ..., p_{t-4}^h]'$$

$$X = [\mathbf{1}, X_4, X_5, ...]$$

Then use the projection matrix we can solve the factor that

$$\hat{\Phi} = YX'(XX')^{-1}$$

The residue is

$$\hat{e} = Y - \Phi X$$

and the variance of estimation error would be

$$\hat{\Omega} = cov(\hat{e}')$$

To simulate the model we can rewrite the variables into companion form such that

$$\boldsymbol{Y_t} = [y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h, y_{t-1}, c_{t-1}, ..., p_{t-3}^h]'$$

Denote $\hat{P} = \text{chol}(\hat{\Omega})$ and

$$\hat{\boldsymbol{\Phi}} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_p \\ I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_n & 0 \end{bmatrix}$$

where $\Phi(:,2:\text{end}) = [\Phi_1 \Phi_2 \Phi_3 \dots \Phi_p]$ since I have intercept coefficient term with $\mathbf{1}$ in X.

Meanwhile we define

$$\hat{\boldsymbol{P}} = \begin{bmatrix} \hat{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The shock term is

$$\nu_{n\times 1} = [0, 0, ..., 1]'$$

which means there is only one unit shock happened at house price row.

Similarly I should write it in companion form such that

$$\boldsymbol{\nu} = [\nu, \mathbf{0}]$$

Then we can get the IRF that

$$\text{IRF}_t = \hat{\boldsymbol{\Phi}}^t \hat{\boldsymbol{P}} \boldsymbol{\nu}$$

where $t = 0, 1, 2, ..., 20$.

Finally we only take first 1 to n items in IRF_t. Since I take first-order difference to most of the data, at this stage I also calculate the cumsum of IRF to return the accumulated response.

B.2.2 Contemporaneous shock under larger confidence band

B.2.3 News shock under larger confidence band

B.2.4 Alternative detrend Method

Alternatively I also use another method to deal with the data which we call Vector Error Correction Method (VECM) in literature. I add the year number into the model to try to detrend the data. I marked the year with its "number" and add 0.1 to 0.4 on it as the label of quarter. Then I divided these "number" by 1000 to get a comfortable scalar. Specifically we take

$$Y_t = [t, t^2, t^3, y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h]'$$

B.2.5 Confidence Band-MC Method

Here I explain the detailed steps that I used to calculate the confidence band of t he estimation using Monte Carlo method. Since there is no difference in steps between I estimate the confidence band in method I and method II, I only show the first part for simplicity.

I can calculate the estimated variance of the coefficient by

$$\hat{\sigma}^2_{\hat{\Phi}} = \frac{\hat{\Omega} \bigotimes \left(\frac{XX'}{T}\right)^{-1}}{T}$$

Then I draw the coefficient simple $\tilde{\Phi}^{(b)}$ from the distribution

$$vec(\hat{\Phi}) \sim N\left(\text{vec}\left(\hat{\Phi}'\right), \hat{\sigma}^2_{\hat{\Phi}}\right)$$

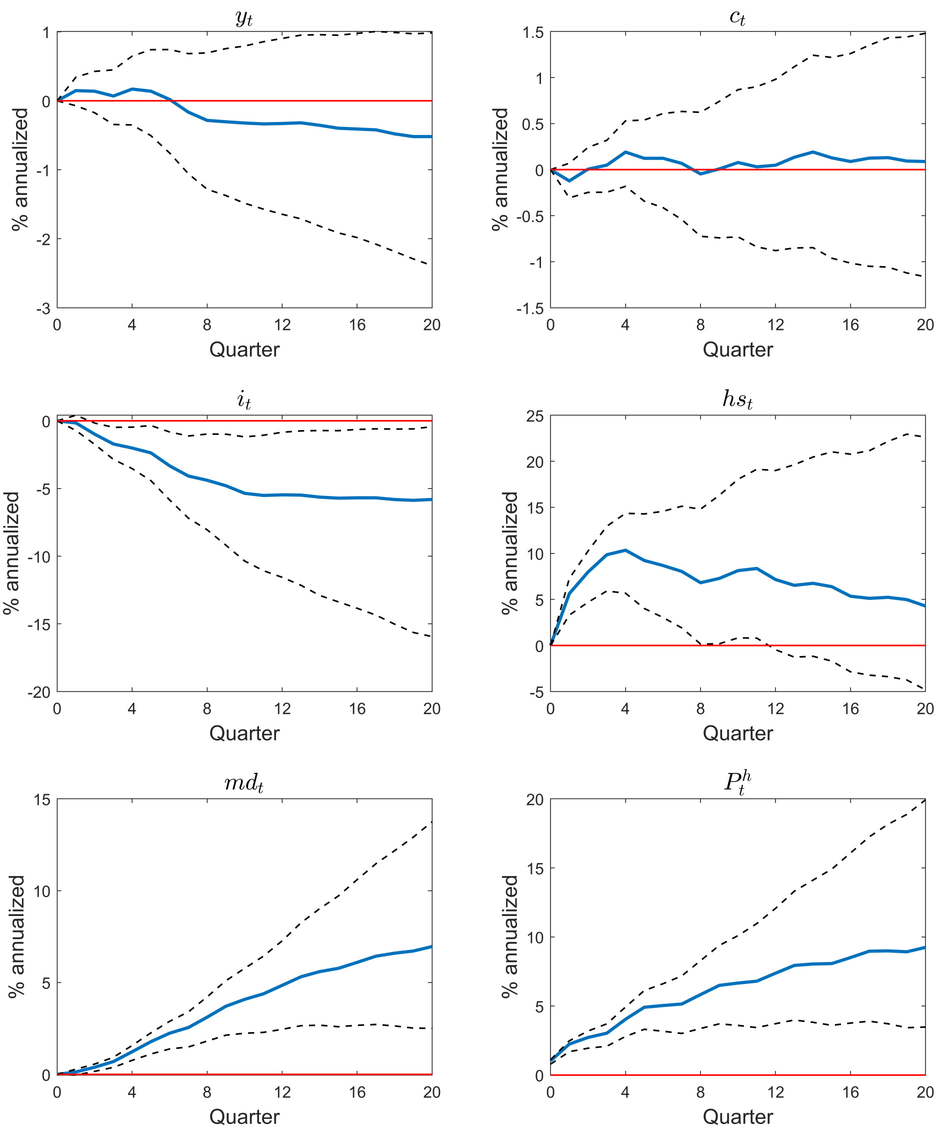


Figure 17: IRF with 90% confidence band

At the same time the estimated variance of the coefficient variance would be

$$\hat{\sigma}^2_{\hat{\Omega}} = \frac{2D_n^+ \left(\hat{\Omega} \bigotimes \hat{\Omega}\right) D_n^{+\prime}}{T}$$

where $D_n^+ = \left(D_n' D_n\right)^{-1} D_n$ is the Moore-Penrose generalized inverse of duplication matrix D_n

I generate the variance simple $\tilde{\Omega}^{(b)}$ from the distribution

$$\text{vech}(\hat{\Omega}) \sim N\left(\text{vech}(\hat{\Omega}), \hat{\sigma}^2_{\hat{\Omega}}\right)$$

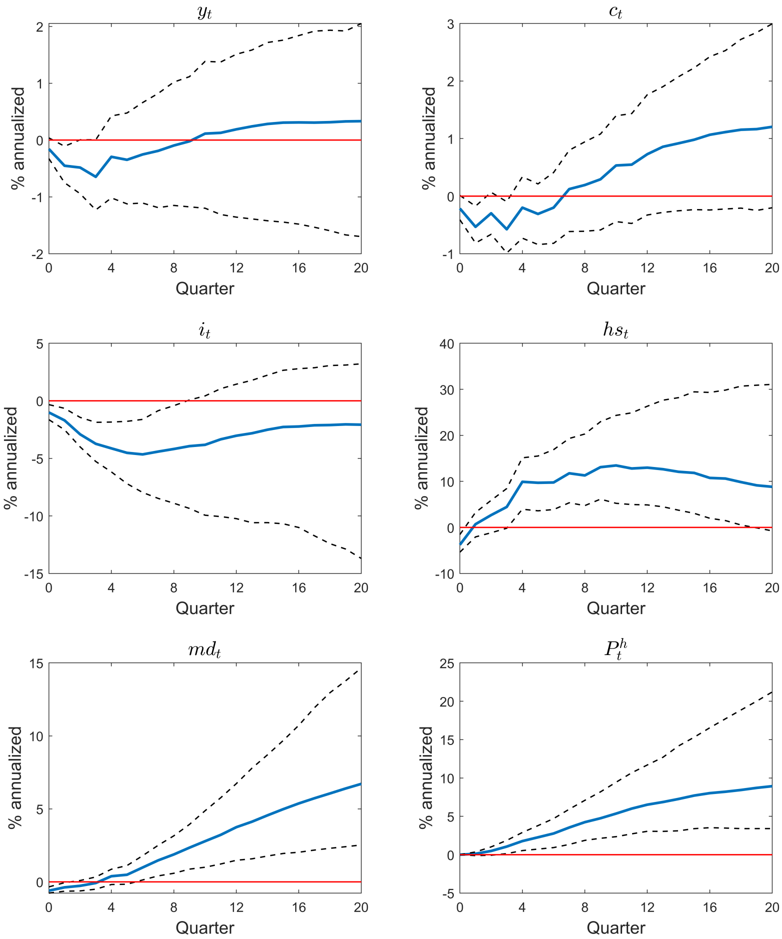


Figure 18: IRF with 90% confidence band

Then use the duplication matrix to transfer back to

$$\text{vec}(\tilde{\Omega}^{(b)}) = D_n \text{vech}(\tilde{\Omega}^{(b)})$$

C Purification Process

In this section I first show that there is another implicit necessary condition of identification. After that I show that given different state space model we cannot arbitrarily add lag and lead term of g_t and $E_t g_{t+6}$ because of the violation of necessary condition. In the end I discuss the detailed purification method I used and the how I pin down the informative span τ through the purification.

C.1 Orthogonal Demand

Now let me consider the news shock under perfect information cases. For simplicity I assume the news is announced one period ahead of the time when it realizes ($\tau = 1$). Given the structure form

$$\begin{bmatrix} 1 & -\alpha_3 & 0 \\ -\alpha_1 & 1 & -\alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ g_t \\ w_t \end{bmatrix} = \begin{bmatrix} \rho_y & 0 & 0 \\ 0 & \rho_g & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ g_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}$$

Where α_1 denotes the effect of monetary policy shock can affect perception via macro-variable y_t. α_2 denotes the endogenous effect of news shock.

Setting $\alpha_1 = 0$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, $\rho_y = 0.6$, $\rho_g = 0.9$ we can get

$$\begin{bmatrix} y_t \\ g_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.9 & 1 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ g_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}$$

We can see w_t has two effects on y_t: contemporaneous effect 0.5 and realization effect 1 one period later. I further denote $\Phi = \begin{bmatrix} 0.6 & 0.9 & 1 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $R_w = [0.5, 0.5, 1]'$, $R_u = [1, 1, 0]'$.

The identification method I used is based on the forecast error and since all the shock is normalized to 1, we can get

$$\begin{aligned} y_{t+3} - E_t y_{t+3} = & \underbrace{R_w}_{w_{t+3}} + \underbrace{\Phi R_w}_{w_{t+2}} + \underbrace{\Phi^2 R_w}_{w_{t+1}} \\ & + \underbrace{R_u}_{u_{t+3}} + \underbrace{\Phi R_u}_{u_{t+2}} + \underbrace{\Phi^2 R_u}_{u_{t+1}} \end{aligned}$$

How can we say that the shock w plays the largest row in explaining $y_{t+3} - E_t y_{t+3}$? No we cannot and the identified news shock might become $R^* = \beta_1 R_w + \beta_2 R_u$. Therefore we need the contemporaneous orthogonal constraint. In other words we use a purified g_t, $\widehat{g}_t$ to rule out the possibility that R_u comes into R^*. Now let us consider the reduced-form VAR again

$$\begin{bmatrix} y_t \\ \widehat{g}_t \\ w_t \end{bmatrix} = \widehat{\Phi} \begin{bmatrix} y_{t-1} \\ \widehat{g}_{t-1} \\ w_{t-1} \end{bmatrix} + \left[0, \widehat{R}_{\widehat{u}}, \widehat{R}_w\right] \begin{bmatrix} \widehat{u}_t \\ w_t \end{bmatrix}$$

As long as $\text{cov}\left(w_t, \widehat{u}_t\right) = 0$, we will have $\widehat{R}'_{\widehat{u}}\widehat{R}_w = 0$. Then even though we still have

$$\begin{aligned} y_{t+3} - E_t y_{t+3} = & \underbrace{\widehat{R}_w}_{w_{t+3}} + \underbrace{\Phi\widehat{R}_w}_{w_{t+2}} + \underbrace{\Phi^2\widehat{R}_w}_{w_{t+1}} \\ & + \underbrace{\widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+3}} + \underbrace{\Phi\widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+2}} + \underbrace{\Phi^2\widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+1}} \end{aligned}$$

we can get $R^* = \widehat{R}_w$ because any combination $R = \beta_1\widehat{R}_w + \beta_2\widehat{R}_{\widehat{u}}$ will be ruled out as $\widehat{R}'_{\widehat{u}}R = \beta_2 \neq 0$

C.2 Another necessary condition of news shock identification: $\text{cov}\left(\widehat{g}_t, w_{t-1}\right) \neq 0$

Given the AR process of g_t follows

$$g_t = \rho_g g_{t-1} + w_{t-1} + u_t + \alpha_2 w_t$$

what I want is to extract the effect of w_t out of g_t. Given

$$E_t g_{t+6} = \rho_g^6 g_t + \rho_g^5 w_t$$

A regression of $E_t g_{t+6}$ on g_t will get the residual $u_t^w = \rho_g^5 w_t$. Then let us run the regression of g_t on u_t^w and clean out the $\alpha_2 w_t$ term in g_t. In the end what we get is the u^{HIM} that

$$\begin{aligned} u_t^{\text{HIM}} &= \rho_g g_{t-1} + w_{t-1} + u_t = \widehat{g}_t \\ &= g_t - \alpha_2 w_t \end{aligned}$$

Pay attention that now $\text{cov}\left(\widehat{g}_t, w_t\right) = 0$ but $\text{cov}\left(\widehat{g}_t, w_{t-1}\right) \neq 0$. I will discuss this inequality later.

Furthermore, it is worth to notice that we cannot observe w_t or w_{t-1}, therefore the DGP would be

$$\begin{bmatrix} y_t \\ \widehat{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widehat{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right] \begin{bmatrix} w_t \\ u_t + \gamma w_{t-1} \end{bmatrix}$$

where $\gamma = 1 + \rho\alpha$ and $Q = \left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right]\left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right]'$.

Therefore as long as $\text{cov}\left(u_t + \gamma w_{t-1}, w_t\right) = 0$, we can get $R^* = \widetilde{R}_w$.

What if we also cleaned out w_{t-1} out of g_t and got $\widetilde{u}_t^{\text{HIM}} = \rho_g g_{t-1} + u_t = \widetilde{g}_t = g_t - \alpha_2 w_t - w_{t-1}$**?** This time both $\text{cov}\left(\widetilde{g}_t, w_t\right) = 0$ and $\text{cov}\left(\widetilde{g}_t, w_{t-1}\right) = 0$ hold. The **can we separate these two models below**

$$\begin{bmatrix} y_t \\ \widetilde{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widetilde{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_{w_t}, \widetilde{R}_{\widetilde{u}}\right] \begin{bmatrix} w_t \\ u_t + \rho_g\alpha w_{t-1} + \rho_g w_{t-2} \end{bmatrix}$$

and

$$\begin{bmatrix} y_t \\ \widetilde{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widetilde{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_{w_{t-1}}, \widetilde{R}_{\widetilde{u}}\right] \begin{bmatrix} w_{t-1} \\ u_t + \rho_g \alpha w_{t-1} + \rho_g w_{t-2} \end{bmatrix}$$

when $\rho_g \alpha \approx 0$? Basically we cannot. Therefore the condition $\operatorname{cov}\left(\widehat{g}_t, w_{t-1}\right) \neq 0$ is necessary.

C.3 Exogenous g_t w.r.t w_t

C.3.1 Perfect Information

C.3.1.1 uniquely identification

Given the fundamental process follows

$$\begin{aligned} g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \\ &= \left(1 - \rho_g L\right)^{-1} w_{t-\tau} + \left(1 - \rho_g L\right)^{-1} w_t^\tau \end{aligned} \tag{28}$$

Then

$$g_{t+\tau|t} = \rho_g^\tau g_t + \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t \tag{29}$$

Therefore lagged expectation up to τ follows

$$\begin{aligned} g_{t|t-\tau} &= \rho_g^\tau g_{t-\tau} + \rho_g^{\tau-1} w_{t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2} + \cdots + w_{t-\tau} \\ &= \rho_g^\tau \left(1 - \rho_g L\right)^{-1} w_{t-\tau}^\tau + \rho_g^\tau \left(1 - \rho_g L\right)^{-1} w_{t-2\tau} + \rho_g^{\tau-1} w_{t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2} + \cdots + w_{t-\tau} \\ &= \rho_g^\tau \left(1 - \rho_g L\right)^{-1} w_{t-\tau}^\tau + \left(1 - \rho_g L\right)^{-1} w_{t-\tau} \end{aligned} \tag{30}$$

Then the projection of g_t on $g_{t|t-\tau}$ yields

$$u_t = g_t - g_{t|t-\tau} \left(g'_{t|t-\tau} g_{t|t-\tau}\right)^{-1} g'_{t|t-\tau} g_t$$

will be almost independent with news shock $w_{t-\tau}$ and exactly independent with w_t as the news term $\left(1 - \rho_g L\right)^{-1} w_{t-\tau}$ can be perfectly purified out. Specifically, for unique τ, the difference $g_t - g_{t|t-\tau} = \left(1 - \rho_g L\right)^{-1} w_t^\tau - \rho_g^\tau \left(1 - \rho_g L\right)^{-1} w_{t-\tau}^\tau$ in which $w_{t-\tau}$ or w_t never emerge.

Figure 19a shows the related numerical exercise.

C.3.1.2 loose identification

Most of time we do not know the number of unique τ or this uniqueness may not even exist. There are several different type of news shock with different information power, i.e. $\tau_1 > \tau_2 > \tau_3 > \cdots > \tau_n$. Therefore I relax the identification method discussed in previous subsection by adding lag and lead terms (relative to $g_{t|t-\tau}$) in to projection. Write the lag of equation 30

$$g_{t-1|t-\tau-1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau-1}^\tau + (1-\rho_g L)^{-1} w_{t-\tau-1}$$
$$\vdots$$
$$g_{t-n|t-\tau-n} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau-n}^\tau + (1-\rho_g L)^{-1} w_{t-\tau-n}$$

and lead

$$g_{t+1|t-\tau+1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau+1}^\tau + (1-\rho_g L)^{-1} w_{t-\tau+1}$$
$$\vdots$$
$$g_{t+\tau-1|t-1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-1}^\tau + (1-\rho_g L)^{-1} w_{t-1} \tag{31}$$

It is easy to comprehend the harmless of this loose identification to $\mathrm{corr}(w_t, u_t)$ as w_t does not emerge either. Meanwhile it is also harmless to $\mathrm{corr}(w_{t-\tau}, u_t)$ as $w_{t-\tau}$ enters into equation 31 with smaller impact coefficient than that in $g_{t|t-\tau}$ and it can still purify the effect of $w_{t-\tau}$ from g_t. However the lead term $w_{t-\tau+1}$, $w_{t-\tau+2}$, ... , w_{t-1} cannot be cleaned out from g_t.

Figure 19b shows the related numerical exercise.

C.3.1.3 arbitrary information power τ

Now we further relax the assumption of information power τ which is arbitrary to the expectation data that we observed, which I denote as k. Basically the previous augment about expectation 30 or 31 but now what we observe and can be used to identify is $g_{t+k|t}$ where $k < \tau$ or $k > \tau$.

When $k > \tau$, W.O.L.G, I assume $k = \tau + 1$, then the observation becomes

$$g_{t|t-k} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau-1}^\tau + (1-\rho_g L)^{-1} w_{t-\tau-1}$$

Furthermore, the lag terms of observation are

$$g_{t-1|t-k-1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau-2}^\tau + (1-\rho_g L)^{-1} w_{t-\tau-2}$$
$$\vdots$$
$$g_{t-n|t-k-n} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau-(n+1)}^\tau + (1-\rho_g L)^{-1} w_{t-\tau-(n+1)} \tag{32}$$

The lead terms of observation are

$$g_{t+1|t-k+1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-\tau}^\tau + (1-\rho_g L)^{-1} w_{t-\tau}$$
$$\vdots$$
$$g_{t+k-1|t-1} = \rho_g^\tau (1-\rho_g L)^{-1} w_{t-1}^\tau + (1-\rho_g L)^{-1} w_{t-1} \tag{33}$$

These two equation 32 and 33 demonstrate that we can still fully purify w_t and almost purify $w_{t-\tau}$.

Figure 19c shows the related numerical exercise.

When $k < \tau$, W.O.L.G, I assume $k = \tau - 1$, then the observation becomes

$$g_{t|t-k} = \rho_g^{\tau} (1 - \rho_g L)^{-1} w_{t-\tau+1}^{\tau} + (1 - \rho_g L)^{-1} w_{t-\tau+1}$$

Furthermore, the lag terms of observation are

$$\begin{aligned} g_{t-1|t-k-1} &= \rho_g^{\tau} (1 - \rho_g L)^{-1} w_{t-\tau}^{\tau} + (1 - \rho_g L)^{-1} w_{t-\tau} \\ &\vdots\vdots \\ g_{t-n|t-k-n} &= \rho_g^{\tau} (1 - \rho_g L)^{-1} w_{t-\tau-n+1}^{\tau} + (1 - \rho_g L)^{-1} w_{t-\tau-n+1} \end{aligned} \tag{34}$$

The lead terms of observation are

$$\begin{aligned} g_{t+1|t-k+1} &= \rho_g^{\tau} (1 - \rho_g L)^{-1} w_{t-\tau+2}^{\tau} + (1 - \rho_g L)^{-1} w_{t-\tau+2} \\ &\vdots\vdots \\ g_{t+k-1|t-1} &= \rho_g^{\tau} (1 - \rho_g L)^{-1} w_{t-1}^{\tau} + (1 - \rho_g L)^{-1} w_{t-1} \end{aligned} \tag{35}$$

These two equation 34 and 35 demonstrate that we can still fully purify w_t and almost purify $w_{t-\tau}$.

Figure 19d shows the related numerical exercise.

C.3.2 Imperfect Information: fundamental impact g_t is observable.

C.3.2.1 uniquely identification

Given the fundamental process follows

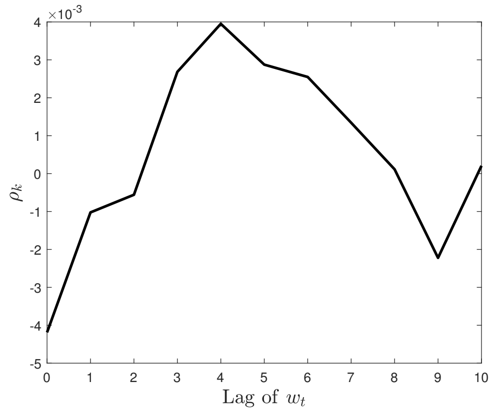
$$\begin{aligned} g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} \\ &= (1 - \rho_g L)^{-1} w_{t-\tau} + (1 - \rho_g L)^{-1} w_t^{\tau} \end{aligned} \tag{36}$$

Then

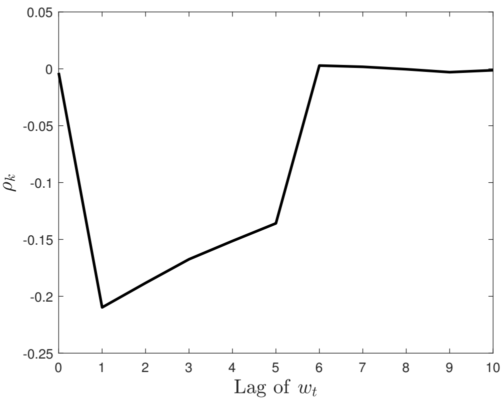
$$g_{t+\tau|t} = \rho_g^{\tau} g_t + \rho_g^{\tau-1} w_{t-\tau+1|t} + \rho_g^{\tau-2} w_{t-\tau+2|t} + \cdots + w_{t|t} \tag{37}$$

where

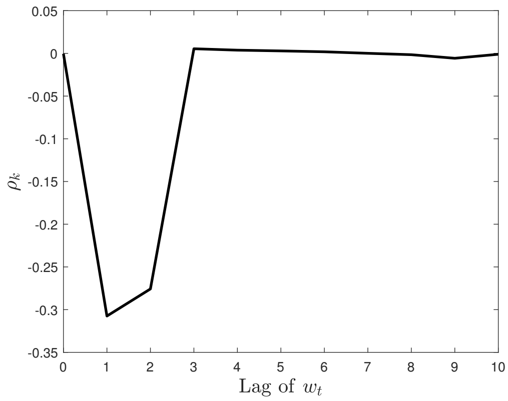
$$\begin{aligned} w_{t-\tau+1|t} &= \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \widetilde{w}_{t-\tau+1} \\ &= \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} (w_{t-\tau+1} + \nu_{t-\tau+1}) \end{aligned}$$



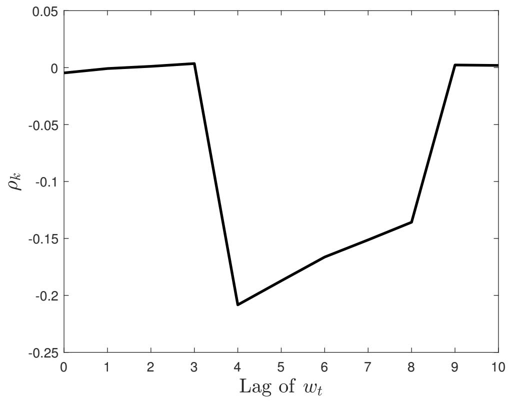
(a) Cross-Correlation between u_t and w_t



(b) Cross-Correlation between u_t and w_t with $n = \tau + 4$



(c) Cross-Correlation between u_t and w_t with $n = \tau + 4$ and $k = \tau + 3$



(d) Cross-Correlation between u_t and w_t with $n = \tau + 4$ and $k = \tau - 3$

Figure 19: Cross-Correlation under Perfect Information (Exogenous g_t)

Therefore lagged expectation up to τ follows

$$\begin{aligned} g_{t|t-\tau} &= \rho_g^{\tau} g_{t-\tau} + \rho_g^{\tau-1} w_{t-2\tau+1|t-\tau} + \rho_g^{\tau-2} w_{t-2\tau+2|t-\tau} + \cdots + w_{t-\tau|t-\tau} \\ &= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau} + \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-2\tau} + \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau|t-\tau} \end{aligned} \tag{38}$$

It is worth to notice that the news shock realized at time t, w_t or $\widetilde{w}_t$, is exactly independent with the residual as it does not emerge neither on LHS or RHS.

In the simple regression case we can get that

$$\begin{aligned} \widehat{\alpha}_{g_{t|t-\tau}} &= \frac{\operatorname{cov}(g_{t|t-\tau}, g_t)}{\operatorname{var}(g_{t|t-\tau})} \\ &\approx \frac{\frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\frac{1-\rho_g^{2\tau}}{1-\rho_g^2}\sigma_w^2}{\frac{1-\rho_g^{2\tau}}{1-\rho_g^2}\operatorname{var}(w_{t-\tau|t-\tau})} = 1 \end{aligned}$$

which follows $\mathrm{cov}(w_t, \nu_t) = 0$. Therefore the residual u_t contains the elements

$$u_t \approx \frac{\sigma_\nu^2}{\sigma_w^2 + \sigma_\nu^2} \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau} - \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sum_{j=0}^{j=\tau-1} \rho_g^j L^j \nu_{t-\tau}$$

Therefore the observation term $\widetilde{w}_{t-\tau}$ is cleaned out as $\mathrm{cov}\left(u_t, \widetilde{w}_{t-\tau}\right) = \frac{\sigma_\nu^2}{\sigma_w^2+\sigma_\nu^2}\sigma_w^2 - \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\sigma_\nu^2 = 0$.

Figure 20a shows the related numerical exercise.

C.3.2.2 loose identification

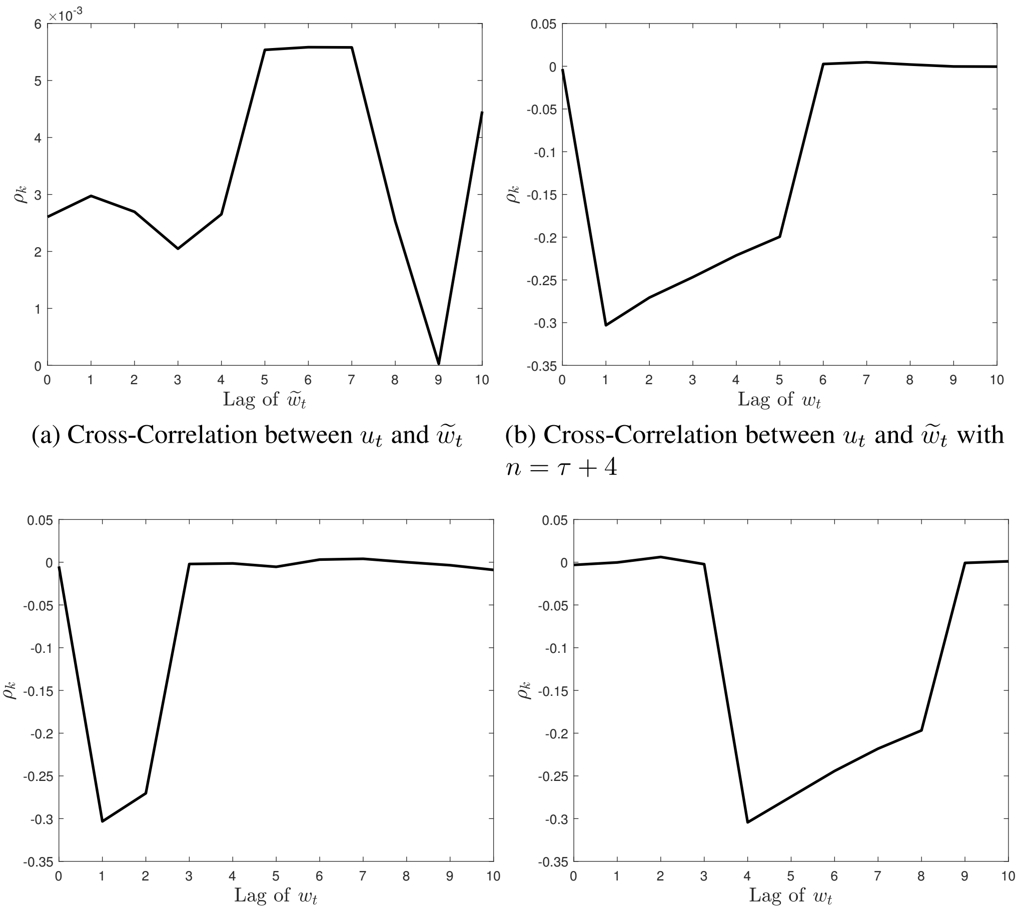
Similar to the cases in perfect information.

Figure 20b shows the related numerical exercise.

C.3.2.3 arbitrary information power τ

Similar to the cases in perfect information.

Figure 20c and 20d shows the related numerical exercise.



(a) Cross-Correlation between u_t and $\widetilde{w}_t$

(b) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$

(c) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau + 3$

(d) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau - 3$

Figure 20: Cross-Correlation under Imperfect Information (Exogenous g_t but observable)

C.3.3 Imperfect Information: fundamental impact g_t is unobservable.

C.3.3.1 uniquely identification

Given the fundamental process follows

$$\begin{aligned} g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \\ &= (1-\rho_g L)^{-1} w_{t-\tau} + (1-\rho_g L)^{-1} w_t^\tau \end{aligned} \tag{39}$$

We can only observe the perception of g_t at t

$$\begin{aligned} g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 \widetilde{g}_t \\ &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 g_t + \gamma_7 \nu_t^\tau \\ &= \rho_g \gamma_2 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 (1-\rho_g L)^{-1} w_{t-\tau} + \gamma_7 (1-\rho_g L)^{-1} w_t^\tau + \gamma_7 \nu_t^\tau \\ &= \gamma_2 (1-\gamma_2\rho_g L)^{-1} w_{t-\tau|t-\tau} + \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-\tau} \\ &+ \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_t^\tau + \gamma_7 (1-\gamma_2\rho_g L)^{-1} \nu_t^\tau \end{aligned} \tag{40}$$

Since now the household cannot observe g_t neither, they have no more other information source to verify the news shock $w_{t-\tau}$. Therefore their perception about it shock will not change as time goes forward, which implies $w_{t-\tau|t-\tau} = w_{t-\tau|t}$.

Then the expectation term follows

$$g_{t+\tau|t} = \rho_g^\tau g_{t|t} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t} \tag{41}$$

Therefore lagged expectation up to τ follows

$$\begin{aligned} g_{t|t-\tau} &= \rho_g^\tau g_{t-\tau|t-\tau} + \rho_g^{\tau-1} w_{t-2\tau+1|t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2|t-2\tau+2} + \cdots + w_{t-\tau|t-\tau} \\ &= \rho_g^\tau \left[\gamma_2 (1-\gamma_2\rho_g L)^{-1} w_{t-2\tau|t-2\tau} + \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-2\tau}\right] \\ &+ \rho_g^\tau \left[\gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-2\tau}^\tau + \gamma_7 (1-\gamma_2\rho_g L)^{-1} \nu_{t-2\tau}^\tau\right] \\ &+ \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau|t-\tau} \end{aligned} \tag{42}$$

To further simplify 40 as

$$\begin{aligned} g_{t|t} &= \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_t^\tau + \gamma_7 (1-\gamma_2\rho_g L)^{-1} \nu_t^\tau \\ &+ \gamma_2 (1-\gamma_2\rho_g L)^{-1} w_{t-\tau|t-\tau} + \gamma_7 \frac{\gamma_2}{\gamma_2 - 1} (1-\gamma_2\rho_g L)^{-1} w_{t-\tau} \\ &+ \gamma_7 \frac{1}{1-\gamma_2} (1-\rho_g L)^{-1} w_{t-\tau} \end{aligned}$$

Since $\gamma_2 + \gamma_7 = 1$, we can get

$$\begin{aligned} g_{t|t} &= \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \left(1 - \rho_g L\right)^{-1} w_t^{\tau} + \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \nu_t^{\tau} \\ &+ \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} - \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau} + \left(1 - \rho_g L\right)^{-1} w_{t-\tau} \end{aligned}$$

Similarly in the simple regression case we can get that

$$\begin{aligned} \widehat{\alpha}_{g_{t|t-\tau}} &= \frac{\mathrm{cov}(g_{t|t-\tau}, g_{t|t})}{\mathrm{var}(g_{t|t-\tau})} \\ &\approx \frac{\Phi}{\frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \mathrm{var}(w_{t-\tau|t-\tau})} = 1 \end{aligned}$$

where $\Phi = \gamma_2 \frac{1-\left(\gamma_2 \rho_g^2\right)^{2\tau}}{1-\gamma_2^2 \rho_g^4} \sigma_{\widetilde{w}}^2 - \gamma_2 \frac{1-\left(\gamma_2 \rho_g^2\right)^{2\tau}}{1-\gamma_2^2 \rho_g^4} \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_w^2 + \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \sigma_w^2 = \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \sigma_w^2$ as $\sigma_{\widetilde{w}}^2 = \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_w^2$.

Therefore the residual u_t contains the elements

$$\begin{aligned} u_t &\approx \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} - \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau} \\ &+ \left(1 - \rho_g L\right)^{-1} w_{t-\tau} - \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau|t-\tau} \end{aligned}$$

However under this scenario the observation term $\widetilde{w}_{t-\tau}$ cannot be cleaned out because

$$\begin{aligned} \mathrm{cov}\left(\widetilde{w}_{t-\tau}, u_t\right) &\approx \gamma_2 \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_{\widetilde{w}}^2 - \gamma_2 \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_w^2 + \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_w^2 - \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_{\widetilde{w}}^2 \\ &= \left(1 - \gamma_2\right) \frac{\sigma_w^2 \sigma_\nu^2}{\sigma_w^2 + \sigma_\nu^2} \neq 0 \end{aligned}$$

Figure 21a shows the related numerical exercise.

C.3.3.2 loose identification

Write the lag of equation 42

$$\begin{aligned} g_{t-1|t-\tau-1} &= \Theta_{t-\tau-1} + \left(1 - \rho_g L\right)^{-1} w_{t-\tau-1|t-\tau-1} \\ &\vdots \\ g_{t-n|t-\tau-n} &= \Theta_{t-\tau-n} + \left(1 - \rho_g L\right)^{-1} w_{t-\tau-n|t-\tau-n} \end{aligned}$$

and lead

$$
\begin{aligned}
g_{t+1|t-\tau+1} &= \Theta_{t-\tau+1} + (1-\rho_g L)^{-1} w_{t-\tau+1|t-\tau+1} \\
&\vdots \\
g_{t+\tau-1|t-1} &= \Theta_{t-1} + (1-\rho_g L)^{-1} w_{t-1|t-1}
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_t = {} & \rho_g^\tau \left[\gamma_2 (1-\gamma_2\rho_g L)^{-1} w_{t-\tau|t-\tau} + \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-\tau}\right] \\
& + \rho_g^\tau \left[\gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-\tau}^\tau + \gamma_7 (1-\gamma_2\rho_g L)^{-1} \nu_{t-\tau}^\tau\right]
\end{aligned}
$$

Similar to the cases in perfect information.

Figure 21b shows the related numerical exercise.

C.3.3.3 arbitrary information power τ

Similar to the cases in perfect information.

Figure 21c and 21d shows the related numerical exercise.

C.4 Endogenous g_t w.r.t w_t

C.4.1 Perfect Information

C.4.1.1 uniquely identification

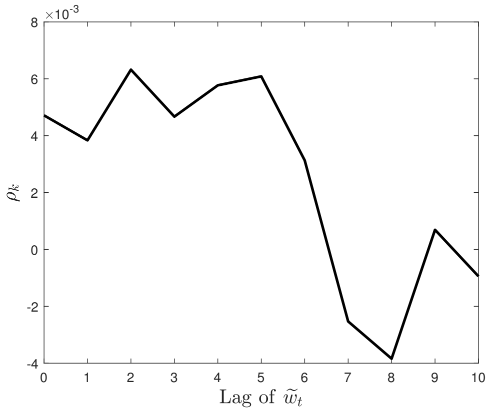
Now let us introduce the endogeneity of w_t on g_t as

$$
g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau + \alpha w_t
$$

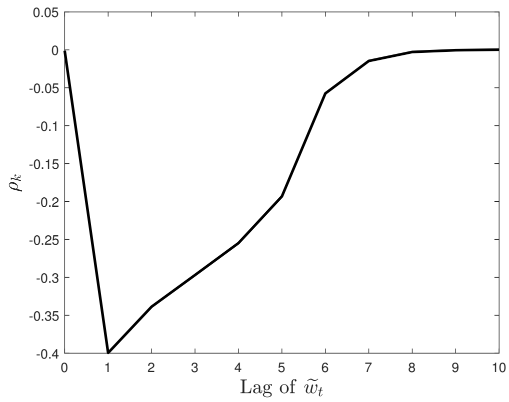
Then the expectation of $g_{t+\tau}$ at time t follows

$$
\begin{aligned}
g_{t+\tau|t} &= \rho_g^\tau g_t + \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t \\
&= \rho_g^{\tau+1} g_{t-1} + \rho_g^\tau (w_{t-\tau} + w_t^\tau + \alpha w_t) \\
&+ \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t
\end{aligned}
$$

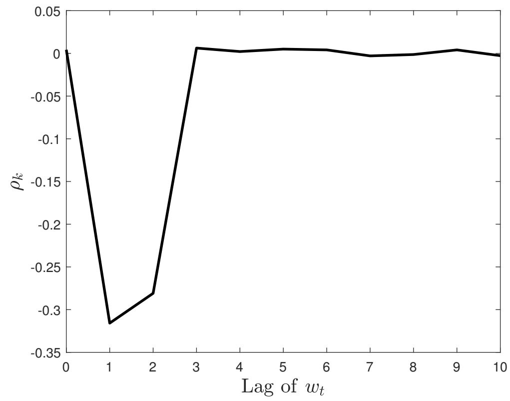
What we need is to clean out w_t from g_t and retain w_t^τ. Therefore we first run the regression of g_t on g_{t-1} (or $g_{t+\tau|t}$ on $g_{t+\tau-1|t-1}$) to get the estimated $\widehat{\rho}_g$. Because the expectation of $g_{t+\tau}$ at t is based on the observation g_t, we can purify the contemporaneous expectation term out of g_t and remain the news part $\sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_t$ through $\widehat{g}_{t+\tau|t} = g_{t+\tau|t} - \rho^\tau g_t$. Then we can clean the news shock w_t out of g_t by simple regression. In the numerical exercise below $\mathrm{corr}\,(u_t, w_t) < 3e^{-3}$ holds.



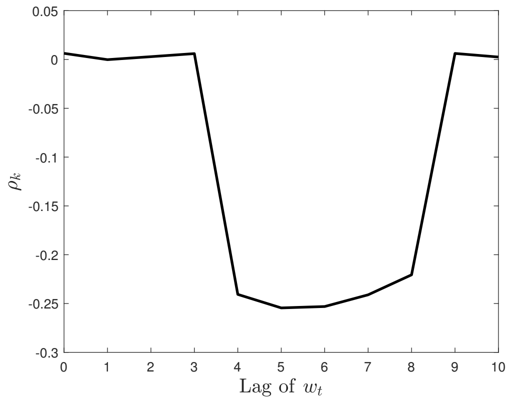
(a) Cross-Correlation between u_t and $\widetilde{w}_t$



(b) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$



(c) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau + 3$



(d) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau - 3$

Figure 21: Cross-Correlation under Imperfect Information (Exogenous g_t but unobservable)

Figure 22a shows the related numerical exercise.

If you further want to clean out $w_{t-\tau}$ (though we in fact do not want and to the contrary we should make sure that $w_{t-\tau}$ exists in g_t), you could use the method mentioned in section C.3.1.2 to yield

Figure 22b shows the related numerical exercise.

C.4.1.2 loose identification

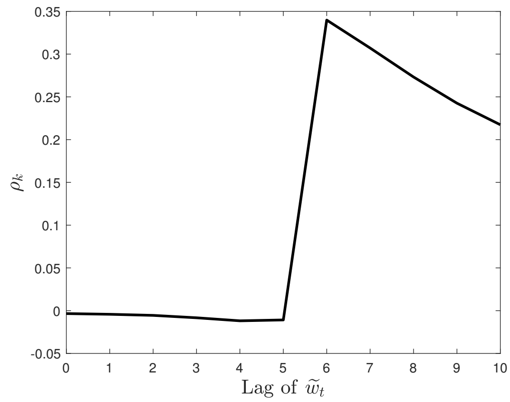
Similar to the arguments in section C.3.1.2, I relax the identification method discussed in previous subsection by adding lead terms (relative to $\widehat{g}_{t+\tau|t}$) in to projection. Write the lead of equation 30

$$\widehat{g}_{t+\tau|t} = \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t = \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_t$$

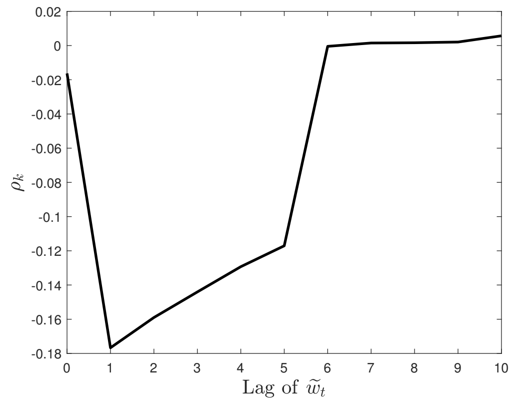
$$\vdots$$

$$\widehat{g}_{t+\tau+n|t+n} = \rho_g^{\tau-1} w_{t-\tau+1+n} + \rho_g^{\tau-2} w_{t-\tau+2+n} + \cdots + w_{t+n} = \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t+n}$$

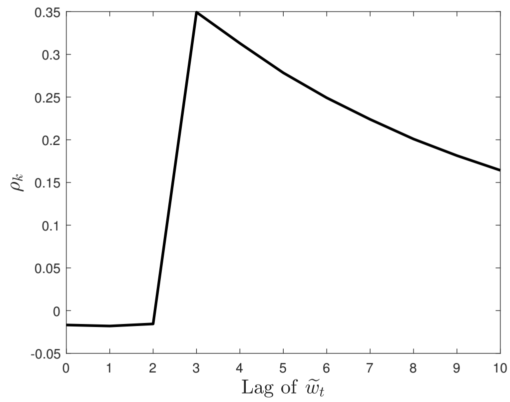
Figure 22c shows the related numerical exercise. It is worth to notice that the approximately zero of $\text{cov}\,(u_t, w_t)$ results from the estimation error of $\widehat{\rho}_g$. If we use the true ρ_g to conduct the purification process, $\text{cov}\,(u_t, w_t)$ will be exactly zero as figure 22d shows.



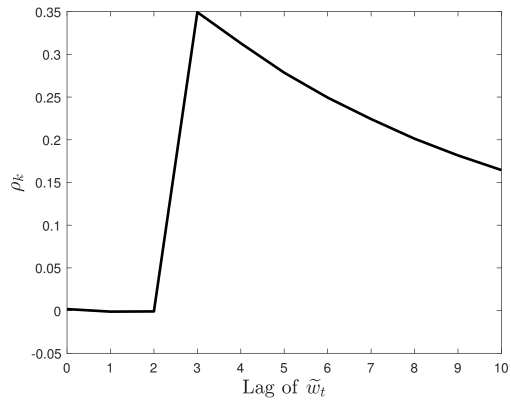
(a) Cross-Correlation between u_t and w_t



(b) Cross-Correlation betweenu_t and w_t



(c) Cross-Correlation between u_t and $w_t$with $n = \tau + 4$



(d) Cross-Correlation between u_t and w_t with true ρ_g and $n = \tau + 4$

Figure 22: Cross-Correlation under Perfect Information-1 (Endogenous g_t)

Moreover, write the lag terms

$$
\begin{aligned}
g_{t+\tau-1|t-1} &= \rho_g^{\tau-1} w_{t-\tau} + \rho_g^{\tau-2} w_{t-\tau+1} + \cdots + w_{t-1} \\
&\vdots \\
g_{t-m|t-\tau-m} &= \rho_g^{\tau-1} w_{t-2\tau-m+1} + \rho_g^{\tau-2} w_{t-2\tau-m+2} + \cdots + w_{t-\tau-m}
\end{aligned}
$$

It seems harmless to add the lagged term into purification regression and figure 23a verifies this argument.

C.4.1.3 arbitrary information power τ

Similar to section C.3.1.3, now we observe and can be used to identify is $g_{t+k|t}$ where $k < \tau$ or $k > \tau$ instead of $g_{t+\tau|\tau}$.

When $k > \tau$, W.O.L.G, I assume $k = \tau + 1$, then the observation becomes

$$
\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau+1} + \rho_g^{k-2} w_{t-\tau+2} + \cdots + \rho_g w_t = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_t
$$

Furthermore, the lead terms of observation are

$$
\begin{aligned}
\widehat{g}_{t+k|t} &= \rho_g^{k-1} w_{t-\tau+1} + \rho_g^{k-2} w_{t-\tau+2} + \cdots + \rho_g w_t = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_t \\
&\vdots \\
\widehat{g}_{t+k+n|t+n} &= \rho_g^{k-1} w_{t-\tau+1+n} + \rho_g^{k-2} w_{t-\tau+2+n} + \cdots + \rho_g w_{t+n} = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_{t+n}
\end{aligned}
$$

The lag terms of observation are

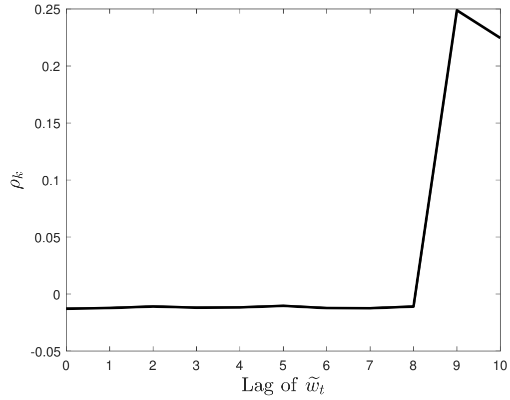
$$
\begin{aligned}
g_{t+\tau-1|t-1} &= \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + \rho_g w_{t-1} \\
&\vdots \\
g_{t-m|t-\tau-m} &= \rho_g^{k-1} w_{t-2\tau-m+1} + \rho_g^{k-2} w_{t-2\tau-m+2} + \cdots + \rho_g w_{t-\tau-m}
\end{aligned}
$$

Therefore we can add both lead and lag terms into purification regression safely and figure 23b verifies this argument.

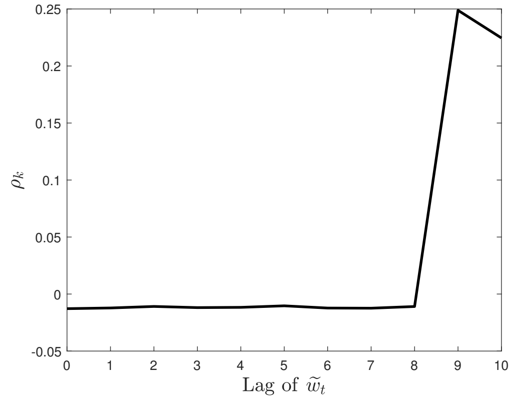
When $k < \tau$, W.O.L.G, I assume $k = \tau - 1$, then the observation becomes

$$
\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + w_{t-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t-1}
$$

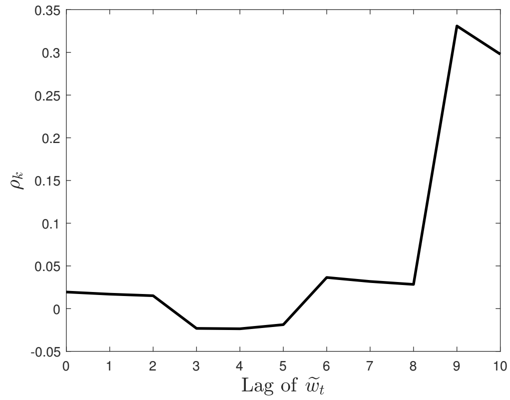
Thus we cannot uniquely clean out w_t from g_t with $\widehat{g}_{t+k|t}$ in which w_t does not emerge. When



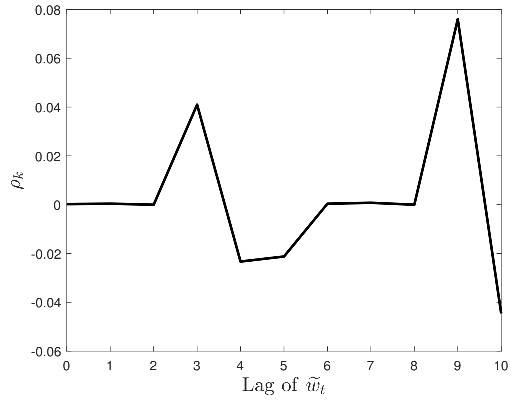
(a) Cross-Correlation between u_t and $w_t$with $m = \tau$ and $n = \tau + 4$



(b) Cross-Correlation between u_t and $w_t$with $m = \tau$ and $n = \tau + 4$



(c) Cross-Correlation between u_t and $w_t$with $k = \tau - 3$ and $n = \tau + 4$



(d) Cross-Correlation between u_t and $w_t$with $k = \tau - 3$, $m = \tau + 1$ and $n = \tau + 4$

Figure 23: Cross-Correlation under Perfect Information-2 (Endogenous g_t)

we impose loose identification and add the lead term

$$\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + w_{t-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t-1}$$

$$\vdots$$

$$\widehat{g}_{t+k+n|t+n} = \rho_g^{k-1} w_{t-\tau+n} + \rho_g^{k-2} w_{t-\tau+1+n} + \cdots + w_{t+n-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t+n-1}$$

the news term w_t is embedded into $\widehat{g}_{t+k+1|t+1}$ and we can clean out w_t via the loose identification.

Figure 23c shows this identification result.

Similar to the argument in loose identification, since the lagged terms does not contains any information about contemporaneous news shock, it is harmless to add the lag part into identification and figure 23d shows the numerical result.

C.4.2 Imperfect Information: fundamental impact g_t is unobservable.

C.4.2.1 uniquely identification

Given the fundamental process follows[28]

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau + \alpha w_t$$

$$g_{t|t} = \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_7 \widetilde{g}_t \tag{43}$$

Then the expectation follows

$$g_{t+\tau|t} = \rho_g^\tau g_{t|t} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

$$g_{t+\tau+1|t+1} = \rho_g^\tau g_{t+1|t+1} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

Therefore the estimation step of AR coefficient cannot be the autoregression on perception $g_{t|t}$ but on the expectation $g_{t+\tau|t}$. Given the forward-looking news estimation

$$\widehat{g}_{t+\tau|t} = \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

$$\widehat{g}_{t+\tau-1|t-1} = \rho_g^{\tau-1} w_{t-\tau|t-\tau} + \rho_g^{\tau-2} w_{t-\tau+1|t-\tau+1} + \cdots + w_{t-1|t-1}$$

Everything goes back to the perfect information cases and all the arguments in perfect information case will also be true under imperfect information case. Figure 24 shows the experiment result of this identification result.

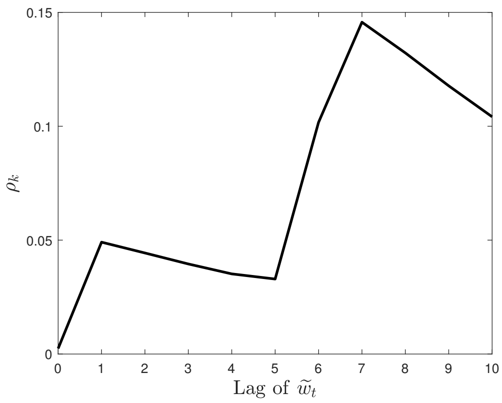


Figure 24: Cross-Correlation under Imperfect Information (Endogenous unobservable g_t)

[28]In section D.4 I provide rigorous proof of equation 43.

C.5 Endogenality, Heteroscedasticity and Biased-estimation Problem during Purification

C.5.1 get w_t out of 6

Because g_t contains w_t, if we run the regression of $E_t g_{t+6}$ on g_t there will be an endogenality problem (residual is correlated with independent variable) and the estimated ρ_g^6 is biased. Therefore I use the model

$$E_t g_{t+6} = \rho_g^7 g_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t \tag{44}$$

If we run the regression of $E_t g_{t+6}$ on g_{t-1}, we can get $u_t^E = \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$

The problem is that g_{t-1} contains w_{t-1}, w_{t-2}, w_{t-3} too, as $g_{t-1} = \rho_g g_{t-2} + w_{t-4} + u_{t-1} + \alpha_2 w_{t-1}$, and the endogenality problem still hold.

Adding the lag span may identify up to scale

$$E_t g_{t+6} = \rho_g^8 g_{t-2} + \rho_g^7 w_{t-4} + \rho_g^7 u_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \left(\rho_g^4 + \rho_g^7 \alpha_2\right) w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$$

because $\mathrm{cov}(g_{t-2}, \rho_g^5 w_{t-2}) < \mathrm{cov}(g_{t-1}, \rho_g^4 w_{t-1})$ holds and in the end the endogenality in first step will be solved. However, we should also care about the trade-off problem here, because when we add the lag span we actually introduce more term into residual, especially u_t and u_{t-1}. This will introduce the endogenality problem into our second regression step: run regression of g_t on $u_t^E = \rho_g^7 w_{t-4} + \rho_g^7 u_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \left(\rho_g^4 + \rho_g^7 \alpha_2\right) w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$.

C.5.2 run regression of 6 on w_t

Assume I use the regression of equation 44 and get

$$u_t^E = \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t \tag{45}$$

we cannot directly run regression of g_t on u_t^E because there are three elements (w_t, u_t, and w_{t-3}) in g_t that are correlated with u_t^E. Given the regression $g_t = \gamma_1 u_t^E + \varepsilon_t$ we cannot make sure that $\mathrm{cov}(\varepsilon_t, w_t) = 0$ (Through simulated data, it is indeed not zero or close to zero up to scale) because a lot of elements in u_t^E correlate with the non-w_t elements in g_t such as u_t and w_{t-3} which will change the projection and cause $\gamma_1 \neq \left(\rho_g^6 \alpha_2 + \rho_g^3\right)$, the coefficient in front of w_t in 45. To solve the problem I further add the lead term of u_t^E into the second step of purification. For instance, if I use the regression $g_t = \gamma_1 u_{t+3}^E + \varepsilon_t$ instead of $g_t = \gamma_1 u_t^E + \varepsilon_t$, the problem can be solved, as in $u_{t+3}^E = \rho_g^6 w_t + \rho_g^6 u_{t+3} + \rho_g^5 w_{t+1} + \rho_g^4 w_{t+2} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_{t+3}$ the only element that correlates with g_t is w_t.

Therefore the only problem left is that how to determine the informative power of news shock? If 6 becomes

$$g_t = \rho_g g_{t-1} + \alpha_1 y_t + w_{t-1} + u_t + \alpha_2 w_t$$

the equation

$$u_t^E = \rho_g^6 w_{t-1} + \rho_g^6 u_t + \left(\rho_g^6 \alpha_2 + \rho_g^5\right) w_t \tag{46}$$

will hold and we may use u_{t+1}^E to clean the g_t yet not u_{t+3}^E. Meanwhile, when 6 becomes

$$g_t = \rho_g g_{t-1} + \alpha_1 y_t + w_{t-9} + u_t + \alpha_2 w_t$$

the equation

$$u_t^E = \rho_g^6 w_{t-9} + \rho_g^6 u_t + \rho_g^5 w_{t-8} + \rho_g^4 w_{t-7} + \rho_g^3 w_{t-6} + \rho_g^2 w_{t-5} + \rho_g w_{t-4} + w_{t-3} + \rho_g^6 \alpha_2 w_t \tag{47}$$

will hold and we may use u_{t+9}^E.

By observing the equation 46 and 47, we find that it is possible to use ACF of u_t^E to pin down the informative power of news τ because difference news with different informative power will imply different "MA" process with different shape of ACF. For instance, if $\tau = 1$ holds, equation 46 will imply that the ACF will converge to zero quickly at second lag. To the contrary, if $\tau = 9$ holds, equation 47 will imply that the ACF will converge to zero sluggishly at ninth lag. Hence the speed of convergence of ACF will help us to find the informative power of housing price news shock even we only have the expectation data up to six month later.

C.6 Purified perception on the status of housing market

The first task to purify w_t out of g_t in equation 6 is to find appropriate macro variables x_t which affects the perception of the status of housing market. Taking an overall consideration on the data constraint and efficiency, I use real interest rate, inflation, M2 supply, unemployment rate and nondurable consumption as the independent macro variables that affect the perception g_t. Because of lack of monthly investment data, I use the real interest rate to reveal the effect of physical capital and investment. The inflation rate and M2 supply reveal the effect of normal friction in New-Keynesian and monetary theory. The unemployment rate and consumption reflect the effect in labor and goods market. By adding the contemporaneous and lagged term of these macro variables in table 5 I show that people's perception are more based on previous macro variables as they may not have the data related to the contemporaneous macro status.

It is harder to determine the optimal lag interval of each macro variables as these macro variables are persistent in themselves. The last three columns in table 6 show that it is inappropriate to add lagged term in third order of M2 supply and unemployment and second order of nondurable consumption. Column 2 and column 3 imply that there is no marginal benefit in adding more lagged terms of real interest and inflation. Because of the inertia of real interest rate (inflation), third (fourth) order in lag r_{t-3} (π_{t-4}) is significant yet this significance comes from

Table 5: Contemporaneous Macro Variables' effect

	Dependent variable:								
	HIM								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
EFF_rate_1		3.177							
		(23.554)							
CPI_1			47.354						
			(109.226)						
M2_1				−44.437					
				(64.726)					
GDP_1					−3.086				
					(56.138)				
Consumption_1						79.407			
						(71.478)			
Unemployment_1							0.005		
							(0.041)		
HIM_1	0.987**	0.916*	0.655	0.762**	0.977*	1.205**	1.041***	0.803	0.840
	(0.502)	(0.483)	(0.410)	(0.352)	(0.504)	(0.596)	(0.350)	(1.311)	(0.810)
HIM_2								−0.218	−1.366
								(1.219)	(1.204)
HIM_3									0.832
									(0.879)
Observations	273	273	273	273	273	273	273	271	269
R^2	-0.904	-0.775	-0.380	-0.523	-0.885	-1.359	-1.010	-0.601	-2.774
Adjusted R^2	-0.911	-0.788	-0.390	-0.534	-0.899	-1.376	-1.025	-0.613	-2.817
Residual Std. Error	4.840 (df = 272)	4.682 (df = 271)	4.129 (df = 271)	4.337 (df = 271)	4.825 (df = 271)	5.397 (df = 271)	4.983 (df = 271)	4.443 (df = 269)	6.835 (df = 266)

Note: *p<0.1; **p<0.05; ***p<0.01

the smaller lagged term r_{t-2} (π_{t-3}). Furthermore, because of monetary policy, the real interest rate moves endogenously with inflation and the lagged third order term of interest rate r_{t-3} also hurts the significance of inflation rate.

As I argued in main body, equation 6 is only for illustration purpose and the true formula of status perception may contains more lagged term or even the expectation term $E_t g_{t+6}$. By adding more lagged term of perception g_t in model 6 I show that the maximized lag number of g_t is 1 and further lagged terms are insignificant via the column 2 to column 6 in table 7. Column 8 and column 9 in table 7 shows that more lagged term of expectation will not provide extra explanation power on the dependent variable. While, it is more complicate to decide whether add previous expectation in equation 6 as column 7 shows that it is significant to add it. However, since the expectation $E_t g_{t+6}$ itself is based on the perception g_t, its significant property is not surprising and the key point is the marginal benefit of adding the expectation term. Column 7 shows that the coefficient of g_{t-1} decreases from 0.84 to 0.16 and the coefficient of expectation term is close to that of g_{t-1}. This means the expectation term does not introduce new explanation power but shares with g_{t-1} as $E_{t-1}g_{t+5}$ is a function of g_{t-1}. Additionally, the inflation rate, M2 supply, unemployment rate and nondurable consumption, those macro variables, become insignificant after adding the over-interpolation term $E_{t-1}g_{t+5}$. Therefore in baseline model 6 I do not add the expectation term because it is not an efficient and profitable explanatory variable.

In baseline model I only use 5 macro variables to indicate the effect of macroeconomics on household's perception on the status of housing market because other macro variables are not significant in explaining the perception. Table 8 provides the robustness check on adding more macro variables into purification. Moreover, since in the last step after purification I embed the purified g_t into VAR identification, any macroeconomic effect that is missed here will be covered later.

In addition to get the near "MA" process of news shock u_t^E, I also need to find out the informative power of news since until now I do not know whether the form of u_t^E follows equation 46 or 47 (or some other forms). As discussed in C.5.2, the ACF of residual in first step of purification, u_t^E, implies the informative power of news and the speed of its convergence to zero refers how many period ahead that the news is announced to household. Figure 25 shows that the news is informed to household 16-17 months before it realizes, roughly 5 quarters to 6 quarters. Table 9 provides more evidence to the informative power of news by using different lead term of u_t^E in second regression and the column 6 to column 9 demonstrate and verify the result in ACF of u_t^E.

D Micro Foundation to Identification and Tests

In this section I provide some micro foundation related to fake-news identification in section 2.3 and some tests to my identification as proof to the reliability. I first provide several different

Table 6: Lagged Macro Variables' effect

	Dependent variable:								
	HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
EFF_rate_1	529.794***								
	(141.326)								
CPI_1			1,732.599***						
			(206.905)						
M2_1				305.905*					
				(169.201)					
GDP_1					−139.443				
					(149.262)				
Consumption_1						410.650***			
						(121.748)			
Unemployment_1							0.717***		
							(0.070)		
HIM	5.898***	5.330**	1.825	3.526*	4.964*	1.409	1.722**	7.096	17.048
	(2.212)	(2.559)	(1.262)	(2.038)	(2.654)	(1.240)	(0.781)	(8.978)	(29.615)
HIM_1								−2.621	7.141
								(11.052)	(19.888)
HIM_2								2.335	4.185
								(7.265)	(18.814)
HIM_3									−12.493
									(30.945)
Observations	381	381	381	381	381	381	381	379	377
R^2	-10.048	-8.689	-0.567	-3.723	-7.507	-0.510	-0.363	-17.973	-152.309
Adjusted R^2	-10.107	-8.715	-0.576	-3.748	-7.552	-0.518	-0.371	-18.125	-153.953
Residual Std. Error	19.987 (df = 379)	18.693 (df = 380)	7.528 (df = 379)	13.068 (df = 379)	17.539 (df = 379)	7.388 (df = 379)	7.021 (df = 379)	26.296 (df = 376)	75.037 (df = 373)
Note:	*p<0.1; **p<0.05; ***p<0.01								

Table 7: Lagged Effect of HIM_t and $E_t\mathrm{HIM}_{t+6}$

	Dependent variable:								
	HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
HIM_1	0.824***	0.919***	0.934***	0.933***	0.933***	0.930***	0.120**	0.119**	0.134**
	(0.020)	(0.074)	(0.073)	(0.074)	(0.075)	(0.076)	(0.051)	(0.054)	(0.056)
HIM_2		−0.101	−0.091	−0.087	−0.086	−0.077			
		(0.075)	(0.081)	(0.083)	(0.083)	(0.083)			
HIM_3			−0.026	0.026	0.024	0.024			
			(0.068)	(0.084)	(0.083)	(0.083)			
HIM_4				−0.058	−0.040	−0.056			
				(0.065)	(0.068)	(0.067)			
HIM_5					−0.018	−0.028			
					(0.060)	(0.080)			
HIM_6						0.022			
						(0.064)			
HIME6_1							0.850***	0.849***	0.843***
							(0.058)	(0.068)	(0.069)
HIME6_2								0.002	0.035
								(0.058)	(0.078)
HIME6_3									−0.048
									(0.058)
Unemployment_1	0.212***	0.211***	0.209***	0.211***	0.212***	0.213***	0.109**	0.109**	0.108**
	(0.074)	(0.072)	(0.074)	(0.076)	(0.078)	(0.078)	(0.053)	(0.054)	(0.052)
Unemployment_2	−0.201***	−0.203***	−0.202***	−0.205***	−0.207***	−0.208***	−0.101*	−0.101*	−0.100**
	(0.073)	(0.072)	(0.074)	(0.076)	(0.077)	(0.078)	(0.052)	(0.053)	(0.051)
Consumption_1	6.957**	7.069**	7.355**	6.955**	7.009**	6.657**	4.285	4.290	4.821*
	(3.375)	(3.347)	(3.364)	(3.349)	(3.361)	(3.378)	(2.666)	(2.710)	(2.775)
Constant	0.154**	0.174***	0.177**	0.188**	0.191***	0.190**	−0.019	−0.020	−0.011
	(0.067)	(0.067)	(0.071)	(0.074)	(0.073)	(0.076)	(0.046)	(0.047)	(0.047)
Observations	418	418	417	416	415	414	418	418	417
R^2	0.930	0.930	0.931	0.931	0.931	0.931	0.955	0.955	0.955
Adjusted R^2	0.929	0.929	0.930	0.930	0.930	0.930	0.954	0.954	0.955
Residual Std. Error	0.201 (df = 413)	0.200 (df = 412)	0.200 (df = 410)	0.200 (df = 408)	0.200 (df = 406)	0.200 (df = 404)	0.161 (df = 412)	0.161 (df = 411)	0.161 (df = 409)
F Statistic	1,367.493*** (df = 4; 413)	1,098.471*** (df = 5; 412)	919.660*** (df = 6; 410)	787.041*** (df = 7; 408)	685.423*** (df = 8; 406)	609.366*** (df = 9; 404)	1,742.976*** (df = 5; 412)	1,448.960*** (df = 6; 411)	1,250.463*** (df = 7; 409)

Note: *p<0.1; **p<0.05; ***p<0.01

Table 8: More Macro Variables

	Dependent variable: HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
HIM_1	0.824^{***} (0.020)	0.825^{***} (0.020)	0.823^{***} (0.020)	0.824^{***} (0.020)	0.822^{***} (0.020)	0.828^{***} (0.019)	0.836^{***} (0.021)	0.834^{***} (0.021)	0.826^{***} (0.020)
Unemployment_1	0.212^{***} (0.074)	0.202^{***} (0.071)	0.212^{***} (0.074)	0.210^{***} (0.074)	0.218^{***} (0.074)	0.191^{***} (0.069)	0.126^{*} (0.067)	0.207^{***} (0.073)	0.193^{***} (0.069)
Unemployment_2	-0.201^{***} (0.073)	-0.189^{***} (0.070)	-0.202^{***} (0.072)	-0.199^{***} (0.073)	-0.208^{***} (0.073)	-0.179^{***} (0.069)	-0.114^{*} (0.066)	-0.062 (0.072)	-0.182^{***} (0.068)
Unemployment_3								-0.132^{*} (0.074)	
Consumption_1	6.957^{**} (3.375)	7.880^{**} (3.552)	7.116^{**} (3.337)	6.450^{*} (3.645)	7.428^{**} (3.377)	6.643^{*} (3.596)	7.530^{**} (3.558)	7.113^{**} (3.535)	7.051^{*} (3.959)
Consumption_2									1.508 (4.504)
GDP_1		-1.279 (1.809)				-1.474 (2.013)	-0.587 (1.704)	-0.840 (1.994)	-1.564 (2.013)
GDP_2						-2.641 (1.965)	-1.336 (1.880)	-1.796 (1.923)	-2.769 (1.964)
GDP_3							-3.791^{*} (2.067)		
EFF_rate_1			-0.764 (1.925)			-2.232 (5.334)	-0.571 (5.328)	-1.842 (5.187)	-2.255 (5.344)
EFF_rate_2						2.529 (5.316)	-95.932^{***} (27.871)	2.094 (5.252)	2.415 (5.372)
EFF_rate_3							97.156^{***} (26.995)		
CPI_Inflation_1				-1.756 (4.598)		-6.036 (7.271)	-101.540^{***} (27.253)	-4.691 (7.268)	-6.419 (7.341)
CPI_Inflation_2						-14.951^{**} (6.562)	84.560^{***} (28.514)	-13.799^{**} (6.705)	-14.214^{**} (6.822)
CPI_Inflation_3							-15.348^{**} (6.036)		
M2_1					-2.433 (2.663)	-6.478^{*} (3.395)	-8.174^{***} (3.169)	-6.116^{*} (3.434)	-6.568^{*} (3.371)
M2_2						-4.471 (3.229)	-6.534^{*} (3.399)	-4.216 (3.166)	-4.384 (3.192)
M2_3							-3.477 (3.203)		
Constant	0.154^{**} (0.067)	0.144^{**} (0.069)	0.167^{**} (0.066)	0.160^{**} (0.069)	0.162^{**} (0.068)	0.218^{***} (0.079)	0.248^{***} (0.081)	0.202^{**} (0.081)	0.216^{***} (0.079)
Observations	418	418	418	418	418	418	417	417	418
R^2	0.930	0.930	0.930	0.930	0.930	0.932	0.937	0.933	0.932
Adjusted R^2	0.929	0.929	0.929	0.929	0.929	0.930	0.934	0.931	0.930
Residual Std. Error	0.201 (df = 413)	0.201 (df = 412)	0.201 (df = 412)	0.201 (df = 412)	0.201 (df = 412)	0.200 (df = 405)	0.193 (df = 400)	0.199 (df = 403)	0.200 (df = 404)
F Statistic	$1{,}367.493^{***}$ (df = 4; 413)	$1{,}092.883^{***}$ (df = 5; 412)	$1{,}092.046^{***}$ (df = 5; 412)	$1{,}091.858^{***}$ (df = 5; 412)	$1{,}094.656^{***}$ (df = 5; 412)	461.162^{***} (df = 12; 405)	370.635^{***} (df = 16; 400)	429.907^{***} (df = 13; 403)	424.794^{***} (df = 13; 404)

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table 9: Lead of u_t^E and the Informative power of news shock

	Dependent variable:							
	HIM							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
HIM_1	0.982^{***}	0.982^{***}	0.982^{***}	0.984^{***}	0.982^{***}	0.982^{***}	0.982^{***}	0.983^{***}
	(0.011)	(0.011)	(0.011)	(0.012)	(0.011)	(0.011)	(0.012)	(0.012)
Consumption_1	8.385^{***}	8.524^{***}	8.421^{***}	7.573^{**}	8.322^{***}	8.334^{***}	8.265^{***}	8.307^{***}
	(2.520)	(2.483)	(2.677)	(2.968)	(2.533)	(2.541)	(2.521)	(2.518)
EFF_rate_1	-2.278^{**}	-2.583^{*}	-2.271^{**}	-2.701^{**}	-2.262^{**}	-2.266^{**}	-2.236^{**}	-2.251^{**}
	(1.028)	(1.353)	(0.990)	(1.221)	(1.030)	(1.028)	(1.036)	(1.036)
CPI_Inflation_1		1.595						
		(3.526)						
M2_1			−0.085					
			(2.416)					
Unemployment_1				0.001				
				(0.002)				
uHIME6L14					−0.004		−0.006	−0.003
					(0.042)		(0.043)	(0.043)
uHIME6L15					−0.036	−0.037	−0.035	−0.036
					(0.045)	(0.042)	(0.045)	(0.045)
uHIME6L16	-0.094^{**}	-0.096^{**}	-0.094^{**}	-0.094^{**}	-0.079^{*}	-0.079^{*}	-0.076^{*}	-0.076^{*}
	(0.038)	(0.038)	(0.038)	(0.038)	(0.041)	(0.041)	(0.040)	(0.041)
uHIME6L17	0.093^{**}	0.093^{**}	0.093^{**}	0.095^{**}	0.097^{**}	0.097^{**}	0.108^{**}	0.104^{**}
	(0.039)	(0.039)	(0.039)	(0.039)	(0.040)	(0.040)	(0.046)	(0.046)
uHIME6L18							−0.026	−0.037
							(0.047)	(0.049)
uHIME6L19								0.029
								(0.046)
Observations	401	401	401	401	401	401	400	399
R^2	0.971	0.971	0.971	0.971	0.971	0.971	0.971	0.971
Adjusted R^2	0.971	0.971	0.971	0.971	0.971	0.971	0.971	0.971
Residual Std. Error	0.155 (df = 396)	0.155 (df = 395)	0.155 (df = 395)	0.155 (df = 395)	0.155 (df = 394)	0.155 (df = 395)	0.156 (df = 392)	0.156 (df = 390)
F Statistic	$2,677.787^{***}$ (df = 5; 396)	$2,227.826^{***}$ (df = 6; 395)	$2,225.869^{***}$ (df = 6; 395)	$2,228.874^{***}$ (df = 6; 395)	$1,906.734^{***}$ (df = 7; 394)	$2,230.111^{***}$ (df = 6; 395)	$1,655.839^{***}$ (df = 8; 392)	$1,462.320^{***}$ (df = 9; 390)
Note:								$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

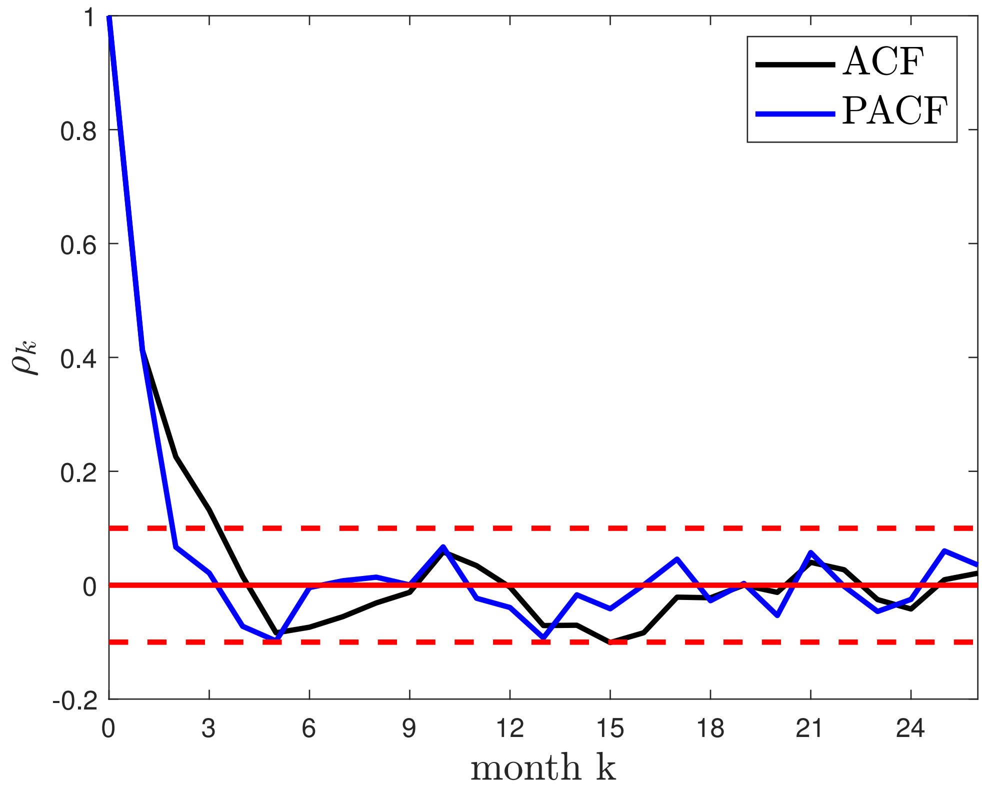


Figure 25: ACF and PACF of u_t^{EHIM6}

setting about news and fake news in the literature. Then I describe the standard rbc model that I used to provide some numerical examples and micro foundation to the identification in main page.

D.1 Literature in modeling the news and fake news

D.1.1 Perfect News

This type of "fake news" is the setting following Christiano et al. (2008), Schmitt-Grohé and Uribe (2012), Barsky et al. (2015) and Sims (2016) in which household gets a news about a shock ν_τ realized at time τ which is true for sure. However after the household reaches at time τ there is an identical negative unexpected shock $-\nu$ just offsetting the effect of positive shock ν_τ. Comparing to the setting in equation 49, in which household gets a news about ν_τ via ϵ (and totally believe it) but is misled because the observation ϵ is generated by noise w, Anderson and Moore (2012) and Chahrour and Jurado (2018) shows that this type of "fake news" shock is *observational equivalent.*[29] To theoretically formulate this type of fake news shock, we can consider the shock series

$$\phi_t = \nu_{0,t} + \nu_{1,t-\tau} \tag{48}$$

[29]They call this representation to fundamental and belief as *news representation* and the representation in equation 49 as a *noise representation.*

where $\nu_{0,t}$ and $\nu_{1,t-\tau}$ are iid over time and follow

$$\begin{bmatrix} \nu_{0,t} \\ \nu_{1,t} \end{bmatrix} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \begin{bmatrix} \sigma^2_{\nu,0} & 0 \\ 0 & \sigma^2_{\nu,1} \end{bmatrix}\right)$$

D.1.2 Noisy News

This type of news is used by Lorenzoni (2009), Baxter et al. (2011), Barsky and Sims (2012), Blanchard et al. (2013), et al. The most intuitive one.

$$\epsilon_t = \nu_{t+\tau} + w_t \tag{49}$$

where ν is the true news shock observed by agents τ periods ahead and w is the noise or fake news shock. These two shocks are independent with each other and follow

$$\begin{bmatrix} \nu_t \\ w_t \end{bmatrix} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \begin{bmatrix} \sigma^2_{\nu} & 0 \\ 0 & \sigma^2_{w} \end{bmatrix}\right)$$

D.1.3 Fake News

It is worth to notice that when we consider the dynamic cases of equation 48 and 49, everything and every realization of $\nu_{0,t}$, $\nu_{1,t-\tau}$, $\nu_{t+\tau}$ and w_t could happen. Given $\phi_t = 1$, different combination such as $(\nu_{0,t} = 0.5, \nu_{1,t-\tau} = 0.5)$ or $(\nu_{0,t} = 1.5, \nu_{1,t-\tau} = -0.5)$ may all hold. Similarly given $\epsilon_t = 1$, $(\nu_{t+\tau} = 0.5, w_t = 0.5)$ or $(\nu_{t+\tau} = -0.5, w_t = 1.5)$ may all hold.

In this section what I am considering is the "pure shock" scenario or the impulse response to a single shock. In other words, for instance, one unit realization of noisy news $\epsilon_t = 1$ can only come from $\nu_{t+\tau} = 1$ or $w_t = 1$. It does not mean I have an implicit restriction on the shock $\nu_{t+\tau}$ and w_t that $\nu_{t+\tau}w_t = 0$. They are iid shocks. Similarly, given one unit realization of perfect news $\nu_{1,t-\tau} = 1$, it can be true news $\nu_{0,t} = 0$ or fake news $\nu_{0,t} = -1$. It does not mean I have an implicit restriction on the shock $\nu_{0,t}$ and $\nu_{1,t-\tau}$ that $\mathrm{corr}(\nu_{0,t}, \nu_{1,t-\tau}) = -1$. They are iid shocks.

D.1.4 Fake News in Perfect News

To model a fake news in perfect news model, there is a realization of perfect news $\nu_{1,t-\tau} = 1$ at time $t - \tau$ and known by household, though this shock would have fundamental effect later, at time t. Then at time t there is an unexpected contemporaneous shock $\nu_{0,t} = -1$ to "neutralize" or "offset" the perfect news effect to make the fundamental stay at the beginning. The VAR identification to this type of fake news is easy. Because all the news in this model is true or perfectly foreseen by household, we just need to find a news shock first. Then at time τ there is a same shock but an opposite direction. We only need to identify the response to shock once.

Sims (2016) did this identification.

D.1.5 Fake News in Noisy News

To model a fake news in noisy news model, there is a realization of observation $\epsilon_t = 1$ at time t which can either be a signal to a fundamental shock in the future, time $t+\tau$, $\nu_{t+\tau} = 1$, or be a noisy $w_t = 1$, which does not have any fundamental effect to the economy. In noisy news model given an observation $\epsilon_t = 1$ household will response to their perception to the true news $\nu_{t+\tau|t}$ which is smaller than ϵ_t under rational expectation and we can write it as $\nu_{t+\tau|t} = \alpha\epsilon_t$ where $\alpha < 1$. There exist learning and belief updating in this type of modeling and theoretically their is no point when household "realizes" that the news is fake. For fake news their perception converge to zero faster than that in true news. In other words, $\lim_{i\to\infty} \nu_{t+\tau|t+\tau+i} = 0$ will be faster for fake news than true news.

To model the "awareness" of fake news, we now consider a scenario in which no more information about shock $\nu_{t+\tau}$ is delivered to household throughout time $t+1$ and time $t+\tau-1$. Therefore the belief to $\nu_{t+\tau}$ of household will not be updated and $\nu_{t+\tau|t} = \nu_{t+\tau|t+1} = \cdots = \nu_{t+\tau|t+\tau-1}$. However when the news realize at time $t+\tau$, household gets a further signal, or information to it. In other words household can also observe $\epsilon^{\tau}_{t+\tau} = \nu_{t+\tau} + w^{\tau}_{t+\tau}$ and this new observation $\epsilon^{\tau}_{t+\tau}$ will update or twist the household's belief to shock $\nu_{t+\tau}$. Therefore their exists a value of $w^{\tau}_{t+\tau}$ which can "correct" the belief of household. Thus, $\nu_{t+\tau|t+\tau} = 0$ and household at time $t+\tau$ realize that the news $\nu_{t+\tau}$ which they known at time t is a fake news.

D.2 Numerical test to identification: A simple RBC model

D.2.1 Equations used to solve the state space model

In this subsection I describe a simple 8 variables RBC model to test my identification strategy and show that it can successfully recover the impulse response to news and fake news shocks. I will first introduce the DSGE model briefly and then show that my identification process works well by comparing the identified empirical impulse response with the theoretical one.

The 8 variables RBC model is a standard one in which household provides labor and earns labor income. Given the labor income and capital return, which is paid by firms with real rental rate as they rent capital to produce goods, the household decides their investment and consumption level. In additional to these endogenous variation there is an exogenous government spending shock following equation 50 and other 4 standard shocks such as TFP shock and preference shock.

Household

$$c_t^{-\sigma} = \beta R_{t+1} c_{t+1}^{-\sigma}$$

$$h_t^{\varphi} = w c_t^{-\sigma}$$

Firm

$$R_t = \alpha \frac{y_t}{k_{t-1}} + \delta - 1$$

$$w_t = (1-\alpha)\frac{y_t}{h_t}$$

$$y_t = A_t k_{t-t}^{\alpha} h_t^{1-\alpha}$$

Market Cleaning

$$y_t = c_t + I_t + \log(G_t)$$

$$I_t = k_t - (1-\delta)k_{t-1}$$

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} \tag{50}$$

The household cannot know the value of G_t and w_t but a signal to then

$$\widetilde{g}_t = g_t + \nu_t^{\tau}$$

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

Household at time $t-\tau$ will have a perception of $w_{t-\tau}$ given the observation $\widetilde{w}_{t-\tau}$ and I denote it as $w_{t-\tau|t-\tau} = \theta\widetilde{w}_{t-\tau}$

Denote $\widetilde{w}_t^i$ as an observation to shock w_{t-i}. For example, a news shock w_t will have effect on G at $t+\tau$. At time $t+1$ household gets a new observation related to w_t, $\widetilde{w}_{t+1}^1$, in addition to the old observation of w_t at time t $\widetilde{w}_t$. I further assume

$$\widetilde{w}_{t-\tau+1}^1 = \widetilde{w}_{t-\tau+2}^2 = \cdots = \widetilde{w}_{t-1}^{\tau-1} = 0$$

holds. Therefore

$$w_{t-\tau|t-\tau} = w_{t-\tau|t-\tau+1} = w_{t-\tau|t-\tau+2} = \cdots = w_{t-\tau|t-1}$$

D.2.2 Quantitative Exercise

D.2.2.1 Same perception: $g_{t|t}^{\nu} = g_{t|t}^{w} = g_{t|t}^{\nu+\nu^{\tau}}$

Notation: Throughout exercise 1 to 3, imperfect information holds.

1) Only noisy shock $\nu_{t-\tau}$;

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν_t^{τ};

3) A news shock $w_{t-\tau}$.

D.2.2.2 Same observation at time $t-\tau$: $\widetilde{w}_{t-\tau}$

Notation: Throughout exercise 1 to 2, imperfect information holds. In exercise 3, it is the type of perfect news.

1) Only noisy shock $\nu_{t-\tau}$;

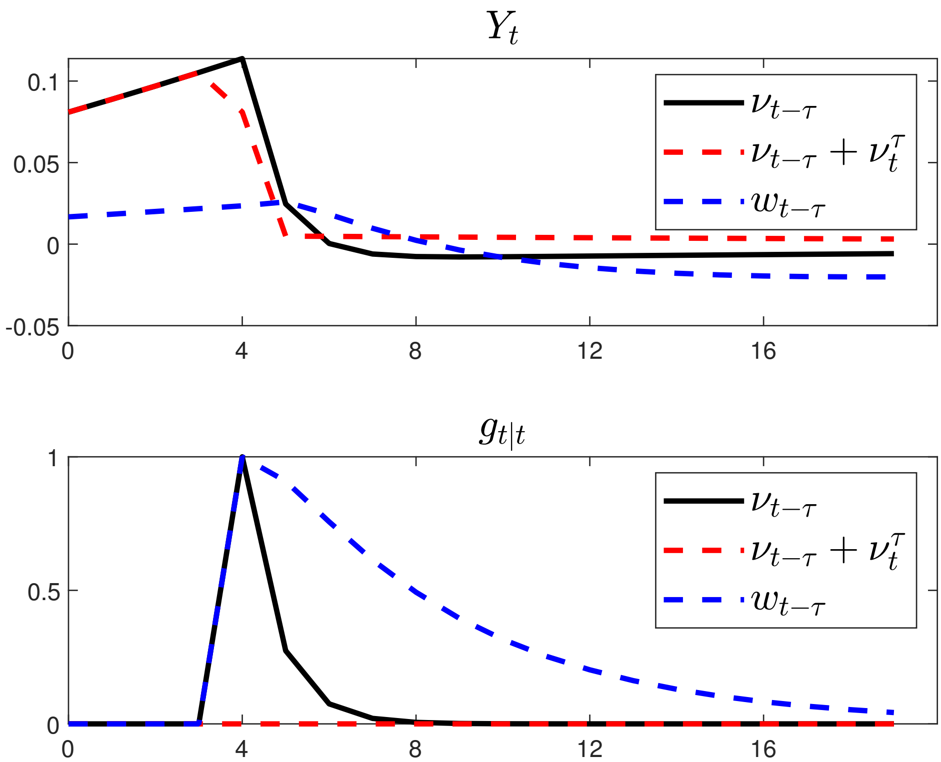


Figure 26: Same Perception $g^w_{t|t} = g^{w^\tau}_{t|t} = g^{w+\nu^\tau}_{t|t}$

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν_t^τ;

3) A perfect news shock $w_{t-\tau}$.

D.2.2.3 Same observation at time $t-\tau$: $\widetilde{w}_{t-\tau}$

Notation: Throughout exercise 1 to 3, imperfect information holds.

1) Only noisy shock $\nu_{t-\tau}$;

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν_t^τ;

3) A news shock $w_{t-\tau}$.

D.3 Two examples of "offset" identification (g_t is exogenous w.r.t w_t)

Denote the fundamental impact (i.e. housing demand variation, TFP) g_t follows an AR1 process

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \tag{51}$$

where $w_{t-\tau}$ is the news shock known by household at time $t-\tau$ yet has real effect at time t, w_t^τ is the contemporaneous shock. Because of the imperfect information, household cannot know

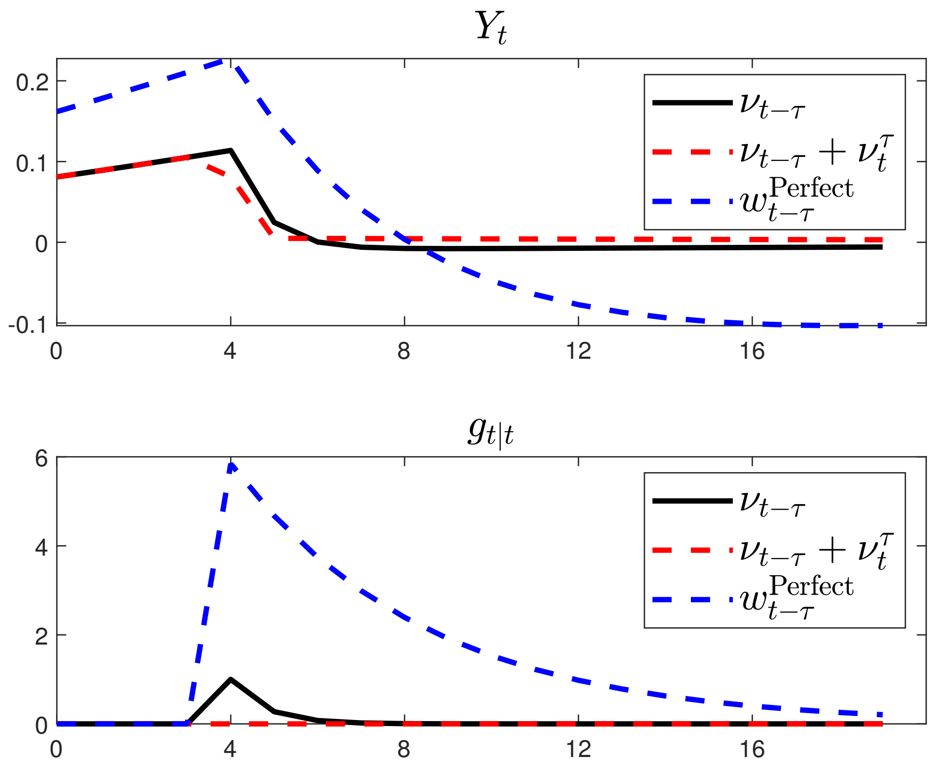


Figure 27: Same observation $\widetilde{w}_{t-\tau}$

the exact value of news shock $w_{t-\tau}$ but an observation to it with noisy shock

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

where $\widetilde{w}_{t-\tau}$ is the observation to $w_{t-\tau}$ but may be contaminated by a noisy $\nu_{t-\tau}$ which does not have any real effect to economy. There are two scenarios that household comprehend whether the jump in observation $\widetilde{w}_{t-\tau}$ comes from news $w_{t-\tau}$ or noisy $\nu_{t-\tau}$ which I call 1). suddenly realization and 2). realization by learning.

D.3.1 The fundamental impact g_t is observable.

When the fundamental impact g_t is observable, whether the news $\widetilde{w}_{t-\tau}$ is true or fake is informed to household via g_t at time t without any delay. Since it is the impact g_t that affects the economy through which the shock $w_{t-\tau}$ and w_t^τ affect the economy, the household only care about the impact value g_t is $w_{t-\tau}$ (true news) or 0 (fake news). Therefore $y_{i-\tau-1}^\tau$ in equation 8 works as a contemporaneous shock w_t^τ offsets the true shock realized at t, $w_{t-\tau}$ and generates $g_t = 0$ which is what the fake news $\nu_{t-\tau}$ would do. This scenario is a standard one in literature and Christiano et al. (2008), Schmitt-Grohé and Uribe (2012), Barsky et al. (2015) and Sims (2016) did the similar process to generate fake news.

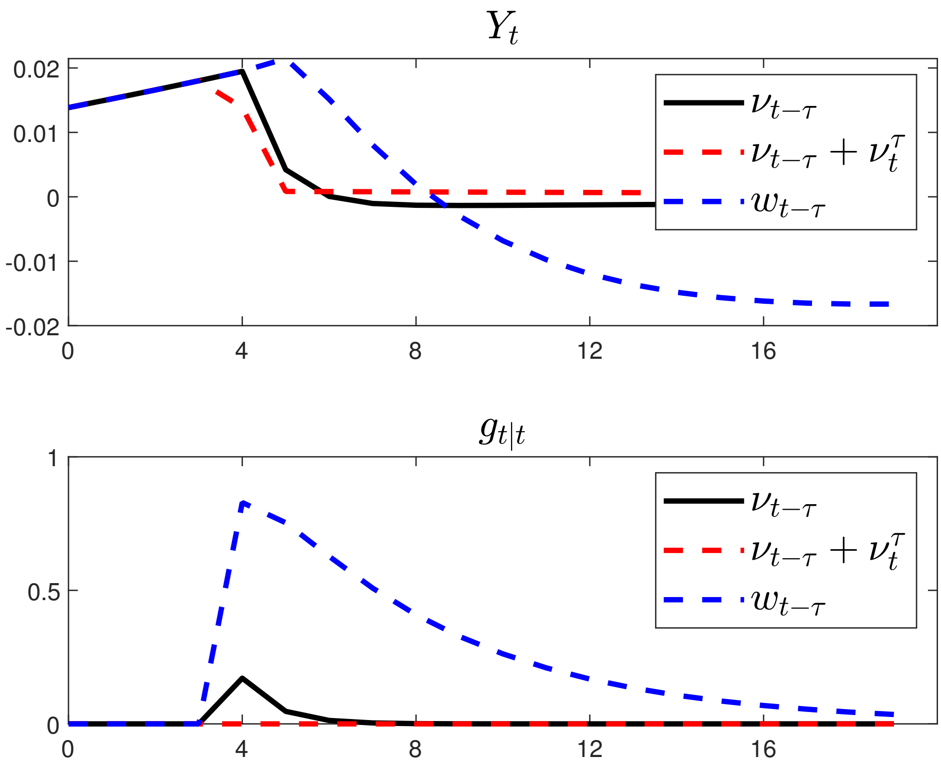


Figure 28: Same observation $\widetilde{w}_{t-\tau}$

D.3.2 The fundamental impact g_t is unobservable.

When the fundamental impact g_t is unobservable, there is no other signal that household can use to infer whether $\widetilde{w}_{t-\tau}$ comes from $w_{t-\tau}$ or $\nu_{t-\tau}$ but learn through observation gradually. In this scenario household cannot know g_t but an observation to it $\widetilde{g}_t$ following

$$\widetilde{g}_t = g_t + \nu_t^\tau$$

I can show that the perception to the fundamental impact at time t, $g_{t|t}$ follows

$$\begin{aligned} g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 \widetilde{g}_t \\ &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_3 g_{t-1} + \gamma_4 w_{t-\tau} + \gamma_5 \nu_t^\tau + \gamma_6 w_t^\tau \end{aligned} \tag{52}$$

where $\gamma_1 = \rho \left[1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}\right]$, $\gamma_2 = 1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$, $\gamma_3 = \gamma_7 \rho$ and $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$ which is the Kalman gain. z_{11} can be solved from the positive root of quadratic equation

$$z_{11}^2 + \left(\sigma^2_{\nu^\tau} - \rho^2\sigma^2_{\nu^\tau} - \sigma^2_w - \sigma^2_{w^\tau} + \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right) z_{11} - \sigma^2_{\nu^\tau}\left(\sigma^2_w + \sigma^2_{w^\tau} - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right) = 0$$

Therefore the only difference between fake news and true news at time t is the term $\gamma_4 w_{t-\tau}$ which comes from the observation $\widetilde{g}_t$ as it truly spur a jump in g_t, though the household cannot distinguish whether this jump is caused by realized news $w_{t-\tau}$ or contemporaneous shock w_t^τ and ν_t^τ. That is the reason why these three terms share the same coefficient $\gamma_4 = \gamma_5 = \gamma_6$, and similarly $y^{\tau}_{i-\tau-1}$ in equation 8 works as a contemporaneous shock w^{τ}_t which offsets the effect of true shock $w_{t-\tau}$ at time t.

D.3.3 Proof of equation 52

Firstly I assume the law of motion of the shock g_t follows

$$g_t = \rho g_{t-1} + w_{t-\tau} + w^{\tau}_t$$

where $w_{t-\tau}$ is a shock realized at $t-\tau$ yet has effect on t. w^{τ}_t is a contemporaneous unexpected shock realized at time t.

The household cannot know the value of the value of shock underneath g_t and w_t but a signal to then

$$\widetilde{g}_t = g_t + \nu^{\tau}_t$$

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

Household at time $t-\tau$ will have a perception of $w_{t-\tau}$ given the observation $\widetilde{w}_{t-\tau}$ and I denote it as $w_{t-\tau|t-\tau} = \theta\widetilde{w}_{t-\tau}$

Denote $\widetilde{w}^{i}_t$ as an observation to shock w_{t-i}. For example, a news shock w_t will have effect on G at $t+\tau$. At time $t+1$ household gets a new observation related to w_t, $\widetilde{w}^{1}_{t+1}$, in addition to the old observation of w_t at time t $\widetilde{w}_t$. I further assume

$$\widetilde{w}^{1}_{t-\tau+1} = \widetilde{w}^{2}_{t-\tau+2} = \cdots = \widetilde{w}^{\tau-1}_{t-1} = 0$$

holds. Therefore

$$w_{t-\tau|t-\tau} = w_{t-\tau|t-\tau+1} = w_{t-\tau|t-\tau+2} = \cdots = w_{t-\tau|t-1}$$

Above system of equation can be written as a state equation

$$\begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix} = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ w_{t-\tau} \end{bmatrix} + \begin{bmatrix} w^{\tau}_t \\ w_{t-\tau+1} \end{bmatrix}$$

and observation(moment) equation

$$\begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix} + \begin{bmatrix} \nu^{\tau}_t \\ \nu_{t-\tau+1} \end{bmatrix}$$

For simplicity I denote $y_t = \begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix}$, $\widetilde{y}_t = \begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \end{bmatrix}$, $B = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$\omega_t = \begin{bmatrix} w_t^{\tau} \\ w_{t-\tau+1} \end{bmatrix}$ and $v_t = \begin{bmatrix} \nu_t^{\tau} \\ \nu_{t-\tau+1} \end{bmatrix}$.

Following Hamilton (2020) we can solve the conditional expectation of the variance of $Z = \Sigma_y(t|t)$ follows

$$B\left[Z - Z\left(Z + \Sigma_\nu\right)^{-1} Z\right] B' + \Sigma_\omega = Z \tag{53}$$

where I omit the observation matrix H as it is an identity matrix.

Since the second row of B is zero, the matrix $D = BXB'$ must follow $D = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}$. Plugging the matrix D back to equation 53 yidelds $D + \Sigma_\omega = Z$. Therefore we must have

$$Z = \begin{bmatrix} d + \sigma_{w^\tau}^2 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} = \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix}$$

By solving the equation

$$\begin{bmatrix} z_{11} - \sigma_{w^\tau}^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \begin{bmatrix} (z_{11} + \sigma_{\nu^\tau}^2)^{-1} & 0 \\ 0 & (\sigma_w^2 + \sigma_\nu^2)^{-1} \end{bmatrix} \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \right\} \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix}$$

we can solve out z_{11} as the positive root of quadratic equation

$$z_{11}^2 + \left(\sigma_{\nu^\tau}^2 - \rho^2\sigma_{\nu^\tau}^2 - \sigma_w^2 - \sigma_{w^\tau}^2 + \frac{\sigma_w^4}{\sigma_w^2 + \sigma_\nu^2}\right) z_{11} - \sigma_{\nu^\tau}^2 \left(\sigma_w^2 + \sigma_{w^\tau}^2 - \frac{\sigma_w^4}{\sigma_w^2 + \sigma_\nu^2}\right) = 0$$

Then we can solve the law of motion of perception(conditional expectation) of y_t as $y_{t|t} = (I - PH)\, By_{t-1|t-1} + P\widetilde{y}_t$ where P is the Kalman gain following $P = ZH'\left(HZH' + \Sigma_v\right)^{-1}$.

D.4 Two examples of "offset" identification (g_t is endogenous w.r.t w_t)

Denote the fundamental impact (i.e. housing demand variation, TFP) g_t follows an AR1 process

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} + \alpha w_t \tag{54}$$

where $w_{t-\tau}$ is the news shock known by household at time $t-\tau$ yet has real effect at time t, w_t^{τ} is the contemporaneous shock. Because of the imperfect information, household cannot know the exact value of news shock $w_{t-\tau}$ but an observation to it with noisy shock

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

where $\widetilde{w}_{t-\tau}$ is the observation to $w_{t-\tau}$ but may be contaminated by a noisy $\nu_{t-\tau}$ which does not have any real effect to economy.

This is similar to the equation 51 and I will also discuss two scenarios that household comprehend whether the jump in observation $\widetilde{w}_{t-\tau}$ comes from news $w_{t-\tau}$ or noisy $\nu_{t-\tau}$ which I call 1). suddenly realization and 2). realization by learning.

D.4.1 The fundamental impact g_t is observable.

When the fundamental impact g_t is observable, whether the news $\widetilde{w}_{t-\tau}$ is true or fake is informed to household via g_t at time t without any delay. Similar to the exogenous case, it is the g_t that affects the economy instead of w_t or $\widetilde{w}_t$ in the end. Therefore as long as g_t can be fully observed, the endogenous effect of w_t will not play any role based on imperfect information here as household at time t will not care about this endogeneity but only g_t. Therefore even we change the assumption of endogenous effect and assume that g_t response to the observation $\widetilde{w}_t$ or perception $w_{t|t}$ the result will not change as long as household perfectly knows g_t.

D.4.2 The fundamental impact g_t is unobservable.

I can show that the perception to the fundamental impact at time t, $g_{t|t}$ follows

$$
\begin{aligned}
g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_7 \widetilde{g}_t \\
&= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_3 g_{t-1} + \gamma_4 w_{t-\tau} + \gamma_5 \nu_t^{\tau} + \gamma_6 w_t^{\tau}
\end{aligned}
\tag{55}
$$

where $\gamma_1 = \rho\left[1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}\right]$, $\gamma_2 = 1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$, $\gamma_3 = \gamma_7\rho$ and $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$ which is the Kalman gain. z_{11} can be solved from the positive root of quadratic equation

$$
z_{11}^2 + \left(\sigma^2_{\nu^\tau} - \rho^2\sigma^2_{\nu^\tau} - \sigma^2_{w^\tau} - (1+\alpha^2)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w+\sigma^2_\nu}\right]\right) z_{11} - \sigma^2_{\nu^\tau}\left(\sigma^2_{w^\tau} + (1+\alpha^2)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w+\sigma^2_\nu}\right]\right) = 0
$$

D.4.3 Proof of equation 55

Similar to the proof of equation 52, above system of equation can be written as a state equation

$$
\begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ w_{t-\tau} \\ w_t \end{bmatrix} + \begin{bmatrix} w_t^\tau \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}
$$

and observation(moment) equation

$$
\begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \\ \widetilde{w}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t^\tau \\ \nu_{t-\tau+1} \\ \nu_{t+1} \end{bmatrix}
$$

For simplicity I denote $y_t = \begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}$, $\widetilde{y}_t = \begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \\ \widetilde{w}_{t+1} \end{bmatrix}$, $B = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\omega_t = \begin{bmatrix} w_t^\tau \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}$ and $v_t = \begin{bmatrix} \nu_t^\tau \\ \nu_{t-\tau+1} \\ \nu_{t+1} \end{bmatrix}$.

Following Hamilton (2020) we can solve the conditional expectation of the variance of $Z = \Sigma_y(t|t)$ follows

$$B\left[Z - Z\left(Z + \Sigma_\nu\right)^{-1} Z\right] B' + \Sigma_\omega = Z \tag{56}$$

where I omit the observation matrix H as it is an identity matrix.

Since the second row of B is zero, the matrix $D = BXB'$ must follow $D = \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Plugging the matrix D back to equation 56 yidelds $D + \Sigma_\omega = Z$. Therefore we must have

$$Z = \begin{bmatrix} d + \sigma^2_{w^\tau} & 0 & 0 \\ 0 & \sigma^2_w & 0 \\ 0 & 0 & \sigma^2_w \end{bmatrix} = \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma^2_w & 0 \\ 0 & 0 & \sigma^2_w \end{bmatrix}$$

By solving the equation

$$\begin{bmatrix} z_{11} - \sigma^2_{w^\tau} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma^2_w & 0 \\ 0 & 0 & \sigma^2_w \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma^2_w & 0 \\ 0 & 0 & \sigma^2_w \end{bmatrix} \begin{bmatrix} (z_{11} + \sigma^2_{\nu^\tau})^{-1} & 0 & 0 \\ 0 & (\sigma^2_w + \sigma^2_\nu)^{-1} & 0 \\ 0 & 0 & (\sigma^2_w + \sigma^2_\nu)^{-1} \end{bmatrix} \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma^2_w & 0 \\ 0 & 0 & \sigma^2_w \end{bmatrix} \right\} \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can solve out z_{11} as the positive root of quadratic equation

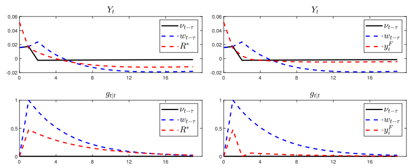
$$z_{11}^2 + \left(\sigma^2_{\nu^\tau} - \rho^2\sigma^2_{\nu^\tau} - \sigma^2_{w^\tau} - \left(1+\alpha^2\right)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right]\right) z_{11} - \sigma^2_{\nu^\tau}\left(\sigma^2_{w^\tau} + \left(1+\alpha^2\right)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right]\right) = 0$$

D.5 Identification Test

D.5.1 The fundamental impact g_t is observable.

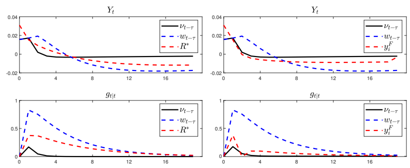
D.5.2 The fundamental impact g_t is unobservable.

Figure 30 shows the result of the identification test.



(a) Identification to news shock R^* (b) Identification to fake news shock R_F^*

Figure 29: Identification Test to observable fundamental impact



(a) Identification to news shock R^* (b) Identification to fake news shock R_F^*

Figure 30: Identification Test to unobservable fundamental impact

E Perturbation result around the Simple Model

E.1 Proof of Proposition 2

The Lagrangian of the problem 9 could be written as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U^i \left(c_t^i, h_t^i\right) + \sum_{t=0}^{\infty} \lambda_t^i \left[R_t a_{t-1}^i + w_t \varepsilon_t^i + \left(1-\delta^H\right) p_t^H h_{t-1}^i + \pi_t^i + \pi_t^{H,i} - c_t^i - a_t^i - p_t^H h_t^i\right]$$
$$+ \sum_{t=0}^{\infty} \mu_t^i \left(p_t^H h_t^i + a_t^i\right)$$

I omit the superscript i henceforth for convenience. Then the first order condition would be

$$U_{c_t} = \lambda_t \tag{57}$$

$$-\lambda_t + \mu_t + \beta E_t R_{t+1} \lambda_{t+1} = 0 \tag{58}$$

$$U_{h_t} - \lambda_t p_t^H + \mu_t p_t^H + \beta \left(1-\delta^H\right) E_t \lambda_{t+1} p_{t+1}^H = 0 \tag{59}$$

To break the expectation I can rearrange the equation 59 as

$$U_{h_t} = \left(\lambda_t - \mu_t\right) p_t^H - \left(1-\delta^H\right)\left(\lambda_t - \mu_t\right) \frac{1}{E_t R_{t+1}} E_t p_{t+1}^H + \beta\left(1-\delta^H\right) \frac{cov\left(\lambda_{t+1}, R_{t+1}\right)}{E_t R_{t+1}} E_t p_{t+1}^H \tag{60}$$
$$- \beta\left(1-\delta^H\right) cov\left(\lambda_{t+1}, p_{t+1}^H\right)$$

Since the interest rate here is not related to the issue we want to solve, I further assume the exogenous TFP of non-durable goods production function is constant. Together with some assumption on the production function of durable and non-durable goods[30], $R_{t+1} = R_t = \overline{R}$ and $cov\left(\lambda_{t+1}, R_{t+1}\right) = 0$ will hold. Combining this assumption I log linearize equation 60 to get

$$\widetilde{U}_{h_t} = \frac{(\lambda-\mu)\left[p^H - \left(1-\delta^H\right)p^H\frac{1}{R}\right]}{U_h}\left\{\frac{\lambda}{\lambda-\mu}\widetilde{\lambda}_t - \frac{\mu}{\lambda-\mu}\widetilde{\mu}_t + \frac{p^H}{p^H - \left(1-\delta^H\right)p^H\frac{1}{R}}\widetilde{p}_t^H - \right. \tag{61}$$

$$\left.\frac{\left(1-\delta^H\right)p^H\frac{1}{R}}{p^H - \left(1-\delta^H\right)p^H\frac{1}{R}}\widetilde{p}_{t+1}^H\right\} - \frac{\beta\left(1-\delta^H\right)\overline{cov}}{U_h}\widetilde{cov}_t$$

where $\widetilde{cov}_t$ is the percentage derivation from steady state of $cov\left(\lambda_t, p_t^H\right)$

Then following Etheridge (2019) I expand U_{c_t} around its steady-state value U_c to get

$$U_{c_t} \approx U_c + U_{cc}c\widetilde{c}_t + U_{ch}h\widetilde{h}_t$$

I rearrange above equation to get

$$\frac{U_{c_t} - U_c}{U_c} = d\ln u_{c_t} = \widetilde{U}_{c_t} = \frac{U_{cc}c}{U_c}\widetilde{c}_t + \frac{U_{ch}h}{U_c}\widetilde{h}_t \tag{62}$$

Similarly expanding U_{h_t}around its steady-state value U_h gives

$$\frac{U_{h_t} - U_h}{U_h} = d\ln u_{h_t} = \widetilde{U}_{h_t} = \frac{U_{hc}c}{U_h}\widetilde{c}_t + \frac{U_{hh}h}{U_h}\widetilde{h}_t \tag{63}$$

Perturbing around its steady state for equation 57 returns

$$\widetilde{U}_{c_t} = \widetilde{\lambda}_t \tag{64}$$

Combining equation 61, 62, 63 and 64 I can solve out

$$\widetilde{c}_t = \left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}\right)\widetilde{\lambda}_t - \frac{\mu}{\lambda-\mu}\eta_{c,p^H}\widetilde{\mu}_t + \eta_{c,p^H}\left[\frac{1}{1-\left(1-\delta^H\right)\frac{1}{R}}\widetilde{p}_t^H - \right.$$
$$\left.\frac{\left(1-\delta^H\right)\frac{1}{R}}{1-\left(1-\delta^H\right)\frac{1}{R}}\widetilde{p}_{t+1}^H\right] - \frac{U_{ch}}{U_{ch}^2 - U_{cc}U_{hh}}\frac{\beta\left(1-\delta^H\right)\overline{cov}}{c}\widetilde{cov}_t$$

[30]The related assumptions are described at appendix G.1.1.

Then plugging back equation 57 gives

$$\widetilde{c}_t = \frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}\widetilde{h}_t - \frac{\mu}{\lambda-\mu}\frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}\widetilde{\mu}_t + \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}\left[\frac{1}{1-\left(1-\delta^H\right)\frac{1}{R}}\widetilde{p}_t^H - \frac{\left(1-\delta^H\right)\frac{1}{R}}{1-\left(1-\delta^H\right)\frac{1}{R}}\widetilde{p}_{t+1}^H\right] - \frac{\eta_c}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}\frac{\beta\left(1-\delta^H\right)\overline{cov}}{h}\widetilde{cov}_t$$

where η_{h,p^c}, η_{h,p^h}, η_{c,p^H}, η_{c,p^c}, η_{ch} and η_c are

$$\eta_{c,p^H} = \frac{u_{ch}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$$

$$\eta_{c,p^c} = \frac{u_{hh}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$$

$$\eta_{h,p^c} = \frac{u_{ch}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$$

$$\eta_{h,p^h} = \frac{u_{cc}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$$

$$\eta_{ch} = \frac{u_c u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{ch}$$

$$\eta_c = \frac{u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$$

E.2 Derivation of the Definition of Intratemporal Elasticity of substitution 17

Firstly, following the standard procedure I first define the optimization problem

$$\max_{c,h} u(c,h)$$

$$\text{s.t. } c + p^h h = y$$

where c is the consumption,p^h is the relative price of housing services and y is the exogenous income. The interior solution implies

$$p^h = \frac{u_h}{u_c}$$

which is used to define the intratemporal elasticity of substitution

$$\begin{aligned} \text{ES} &= -\frac{d\ln\left(\frac{c}{h}\right)}{d\ln\left(p^h\right)} \\ &= -\frac{d\ln\left(\frac{c}{h}\right)}{d\ln\left(\frac{U_c}{U_h}\right)} \end{aligned}$$

E.3 Proof of Proposition 3

I first use the same production function 21 and 22 which I defined at section 4. Since the sample model in section 3 is frictionless in adjusting housing and physical capital, the goods market clearing condition should be

$$\begin{aligned} Y &= Y_H + Y_N \\ &= C + I_N + I_H \end{aligned}$$

where $Y_H = I_H$ and $Y_N = C + I_N$

Combining equation 74 and the market clearing condition of capital I can get

$$\alpha Y_{N,t} + \nu P_t^H Y_{H,t} = (r_t + \delta) K_{t-1}$$

Taking differential on both side of above equation around their steady state will yield

$$\alpha dY_{N,t} + \nu Y_H dP_t^H + \nu P^H dY_{H,t} = 0$$

because the total capital K_{t-1} is predetermined and r_t is fixed by assumption. Further because the amount of total housing service at time $t-1$, H_{t-1} is predetermined, above equation can be rewritten to

$$\alpha dY_{N,t} + \nu Y_H dP_t^H + \nu P^H dH_t = 0$$

Plugging this back to goods market clearing condition will return the general equilibrium condition of crowd-out effect

$$-I_N \widetilde{I}_{N,t} = C\widetilde{C}_t + \frac{\nu}{\alpha} Y_H P^H \widetilde{P}_t^H + \frac{\nu}{\alpha} P^H H \widetilde{H}_t$$

Finally the equation 16 can be obtained by plugging equation 12 into above equation.

E.4 Proof of Corollary 1

If the household utility function follows the standard CRRA form

$$u_t = \frac{\left(\phi c_t^\gamma + (1-\phi) s_t^{1-\gamma}\right)^{\frac{1-\sigma}{1-\gamma}}}{1-\sigma}$$

Therefore the intratempral elasticity of substitution will be $\mathrm{ES} = \frac{1}{\gamma}$ and the intertemporal elasticity of substitution will be $\mathrm{EIS} = \frac{1}{\sigma}$ and $u_{ch} = \phi(1-\phi)(\gamma-\sigma)c^{\gamma-\sigma-1}h^{-\gamma}\left[\phi + (1-\phi)(\frac{h}{c})^{1-\gamma}\right]^{\frac{\gamma-\sigma}{1-\gamma}}$. Then based on the definition of relative force of substitution effect Φ_H and wealth effect Φ_{p^H} the prove process is straightforward.

E.5 Proof of Corollary 2

Iterating equation 59 forward with expectation at t on both side, I can eliminate the intra-price term until time $T+1$ with the chain rule of expectation

$$\begin{aligned} U_{h_t} + (\mu_t - \lambda_t)\, p_t^H + \beta\left(1-\delta^H\right) E_t \lambda_{t+1} p_{t+1}^H &= 0 \\ U_{h_{t+1}} + (\mu_{t+1} - \lambda_{t+1})\, p_{t+1}^H + \beta\left(1-\delta^H\right) E_{t+1} \lambda_{t+2} p_{t+2}^H &= 0 \\ U_{h_{t+2}} + (\mu_{t+2} - \lambda_{t+2})\, p_{t+2}^H + \beta\left(1-\delta^H\right) E_{t+2} \lambda_{t+3} p_{t+3}^H &= 0 \\ &\vdots \\ U_{h_{t+T}} + (\mu_{t+T} - \lambda_{t+T})\, p_{t+T}^H + \beta\left(1-\delta^H\right) E_{t+T} \lambda_{t+T+1} p_{t+T+1}^H &= 0 \end{aligned} \tag{65}$$

Multiple $\frac{\beta\left(1-\delta^H\right)\lambda_{t+i}}{\lambda_{t+i}-\mu_{t+i}}$ on both side of above equation will yield (here I only take equation 65 as an example)

$$\frac{\beta\left(1-\delta^H\right)\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}} U_{h_{t+1}} - \beta\left(1-\delta^H\right)\lambda_{t+1} p_{t+1}^H + \beta\left(1-\delta^H\right)\frac{\beta\left(1-\delta^H\right)\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}} E_{t+1}\lambda_{t+2} p_{t+2}^H = 0$$

The last term can be rearranged to $\left[\beta\left(1-\delta^H\right)\right]^2 E_{t+1}\frac{\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}}\lambda_{t+2}p_{t+2}^H$ because the term $\frac{\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}}$ only contains the term at time $t+1$ which is known at time $t+1$. Then take expectation with the information at time t on both side of this equation to aggregate as

$$U_{h_t} + \mathbb{E}_t \sum_{i=1}^{T} \left[\beta\left(1-\delta^H\right)\right]^i \left[\prod_{s=1}^{i} \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}}\right] U_{h_{t+i}} + \mathbb{E}_t \left[\beta\left(1-\delta^H\right)\right]^T \prod_{s=1}^{T} \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}} \lambda_{t+T+1} p_{t+T+1}^H = 0$$

Equation 18 can be derived by take total differential on both side to above equation.

E.6 Proof of Proposition 4 and 5

The proposition 4 is a straight result of Lemma 6, 9 and 10. Similarly proposition 5 is a straight result of Lemma 14, 16 and 17.

Lemma 1. *When the utility function follows Cobb-Douglas formula 91, the monotonicity of parameter Φ_H, Φ_μ and Φ_{p^H} is equivalent to $\widetilde{\Phi}_H = \frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H} - \widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$, $\widetilde{\Phi}_\mu = \frac{\mu}{\lambda-\mu}\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ and $\Phi_{p^H} = \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ where $\widetilde{\eta}_{c,p^H} = \phi(1-\phi)^2(1-\sigma)$, $\widetilde{\eta}_{c,p^c} = \phi(1-\phi)\left[(1-\phi)(1-\sigma)-1\right]$, $\widetilde{\eta}_{h,p^c} = \phi^2(1-\phi)(1-\sigma)$, $\widetilde{\eta}_{h,p^H} = \phi(1-\phi)\left[\phi(1-\sigma)-1\right]$ and $\widetilde{\eta}_{ch} = \phi(1-\phi)$.*

Proof. Because the proposition 2 and equation 12 is derived around aggregate consumption and residential asset, by plugging the marginal utility function into equation 13, 14 and 15 and rearranging the algebraic structure, we can solve above equations. □

Lemma 2. *If $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \sigma} > 0$ hold.*

Proof. Simplify the formula of Φ_H to $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$. If $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)}{\partial\sigma}\left(\eta_{h,p^c}-\eta_{h,p^h}\right)<\frac{\partial\left(\eta_{h,p^c}-\eta_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)$ holds. Further it is easy to check that as long as $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)}{\partial\sigma}\left(\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}\right)<\frac{\partial\left(\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)$ holds, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ. Because of Lemma 1 we only need to check $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)<\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)$. Meanwhile $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}+\frac{\lambda}{\lambda-\mu}\frac{\partial\widetilde{\eta}_{c,p^H}}{\partial\sigma}-\frac{\partial\widetilde{\eta}_{c,p^c}}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}-\frac{\lambda}{\lambda-\mu}\phi(1-\phi)^2+\phi(1-\phi)^2$ holds. Therefore as long as $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$, $\frac{\lambda}{\lambda-\mu}>1$, $\widetilde{\eta}_{c,p^H}<0$ and $\widetilde{\eta}_{h,p^h}<0$, we will have $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}<0$ and the inequality $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)<\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)$ will hold.

Additionally, it is easy to yield $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}=\frac{\partial\widetilde{\eta}_{h,p^c}}{\partial\sigma}-\frac{\partial\widetilde{\eta}_{h,p^h}}{\partial\sigma}\frac{\lambda}{\lambda-\mu}-\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h}=-\phi^2(1-\phi)+\frac{\lambda}{\lambda-\mu}\phi^2(1-\phi)-\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h}>\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}=-\phi^2(1-\phi)+\phi^2(1-\phi)$ as $\frac{\lambda}{\lambda-\mu}>1$, $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$ and $\widetilde{\eta}_{h,p^h}<0$. Therefore by rescaling the inequality $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)>\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)$ will hold and $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ. $\square$

Lemma 3. *If $\frac{\lambda}{\lambda-\mu}\geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$ hold, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ will be monotonic decreasing in σ.*

Proof. Based on Lemma 1, it is equivalent to show that $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)<\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)$. Because $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}=\frac{\partial(\phi(1-\phi))}{\partial\sigma}=0$, we only need to prove $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}<0$. It is easy to verify that $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}+\frac{\lambda}{\lambda-\mu}\frac{\partial\widetilde{\eta}_{c,p^H}}{\partial\sigma}-\frac{\partial\widetilde{\eta}_{c,p^c}}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}-\frac{\lambda}{\lambda-\mu}\phi(1-\phi)^2+\phi(1-\phi)^2<0$ when $\widetilde{\eta}_{c,p^H}<0$, $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$ and $\frac{\lambda}{\lambda-\mu}>0$. $\square$

Lemma 4. *The stationary capital over effective labor ratio will increase as σ increases in Aiyagari-Bewley-Huggett model 9 when the housing supply is fixed and initial housing distribution over dynamic path is exogenous.*

Proof. The problem 9 can be write as the instantaneous payoff function

$$\max\sum_{t=0}^{\infty}\beta^t\nu\left(c_t,a_t\right) \tag{66}$$

where $\nu=\frac{\left(c_t^{\phi}h_t^{*1-\phi}\right)^{1-\sigma}}{1-\sigma}$ and $h^*=\max\left(\frac{1-\phi}{\phi}\left[p^H-(1-\delta^H)\frac{p^H}{R}\right]^{-1}c_t,\frac{-a_t}{\gamma p^H}\right)$ and the constraint correspondence

$$\Gamma\left(a_{t-1}, c_t, i_{s,t}, \varepsilon_t\right) = \left\{ \left(a_t, c_{t+1}, i_{s,t}, \varepsilon_t\right) \in \left[-\frac{(1-\phi)\gamma p^H}{\phi\left(p^H - (1-\delta^H)\frac{p^H}{R}\right)} c_t, \bar{a} \right] \times [0, \bar{c}] \times \left[-\underline{i}_s, \bar{i}_s \right] : \right.$$
$$\left. a_t \leq R(Q)a_{t-1} + w(Q)\varepsilon_t - p^H i_{s,t} + T - c_t \right\} \tag{67}$$

Because the aggregate housing supply is fixed, the problem is partial on remain sectors and take the housing price as an exogenous parameter (and the general equilibrium will in the end be pinned down through find the price that match the fixed housing supply with the housing demand $\int h^* g(h^*)di$). Then the real rental rate $R(Q)$ and real wage $w(Q)$ will be a function of real effective capital over labor ratio $Q = \frac{K}{AL}$.

Then following the theorem 5 and proposition 1 in Acemoglu and Jensen (2015), σ is a positive shock and Q is monotonic increasing in σ. □

Lemma 5. *$\frac{\lambda}{\lambda-\mu} \geq 1$, $\frac{\partial \frac{\mu}{\lambda}}{\partial \sigma} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \sigma} > 0$ holds in Aiyagari-Bewley-Huggett model 9 when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. $\frac{\lambda}{\lambda-\mu} = \frac{1}{1-\frac{\mu}{\lambda}} > 1$ is obvious as λ is the marginal utility which is a positive number in 57 and μ is the Khun-Tucker multiplier which is also positive. Following Lemma 4 we know that when σ increases, Q will also increase. Because of the market clearing condition $AK^\alpha L^{1-\alpha} = C + \delta K$ we can solve $\frac{\partial C}{\partial \sigma} = \frac{\partial\left(AK^\alpha L^{1-\alpha} - \delta K\right)}{\partial K}\frac{\partial K}{\partial \sigma} = \left(\alpha A(\frac{K}{L})^{\alpha-1} - \delta\right)\frac{\partial K}{\partial \sigma} > 0$. Therefore the marginal utility λ is a monotonic decreasing function of σ.

Additionally, by integrating and combining equation 78 and 81 across household we can get the relationship between aggregate Khun-Tucker multiplier and marginal utility $\mu = (\beta R - 1)\lambda$. Therefore as long as $\beta R < 1$ holds, the Khun-Tucker multiplier will have the opposite monotonicity as λ and it is guaranteed by $K < \left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$. Hence, we can yield $\frac{\partial \frac{\mu}{\lambda}}{\partial \sigma} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \sigma} > 0$. □

Lemma 6. *The substitution effect Φ_H will decrease as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. Lemma 6 is a direct inference from Lemma 2, 3, 4 and 5. □

Lemma 7. *If $\frac{\eta_{ch}}{\eta_{h,p^c} - \eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \sigma} > 0$ hold.*

Proof. Similar to Lemma 2, because of Lemma 1, given $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma}\left(\widetilde{\eta}_{h,p^c} - \widetilde{\eta}_{h,p^h}\right) < \frac{\partial\left(\widetilde{\eta}_{h,p^c} - \widetilde{\eta}_{h,p^h}\right)}{\partial \sigma}\widetilde{\eta}_{ch}$, we need to check $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) < \frac{\partial\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial \sigma}\widetilde{\eta}_{ch}$. Since $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma} = 0$, we only need to check $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch} > \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$ which is true because $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} > \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}$ (shown in Lemma 2) and $\widetilde{\eta}_{ch} > 0$. □

Lemma 8. $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ *will be also monotonic decreasing in* σ*, as long as* $\frac{\lambda}{\lambda-\mu} \geq 1$ *and* $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ *hold.*

Proof. Following Lemma 1, we can get the monotonicity of $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ by checking

$$\frac{\partial\widetilde{\eta}_{ch}}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)=0<\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$$

Because $\widetilde{\eta}_{ch} > 0$, we need $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} > 0$ to let $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ monotonic decreasing in σ. It is straightforward as $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} = -\phi^2(1-\phi) - \frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h} + \frac{\lambda}{\lambda-\mu}\phi^2(1-\phi) > 0$ because of $\widetilde{\eta}_{h,p^h} < 0$, $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$. □

Lemma 9. *The wealth effect* Φ_{p^H} *will decrease as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds.*

Proof. Lemma 9 is a direct inference from Lemma 5 and 8. □

Lemma 10. *The credit effect* Φ_μ *will increase as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous ;* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds and the aggregate Khun-Tucker multiplier is not too large.*

Proof. Based on lemma 1 we can show that $\frac{\partial\Phi_{p^H}}{\partial\sigma} \cong \frac{\partial\left(\frac{\mu}{\lambda-\mu}\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma} = \frac{\mu}{\lambda-\mu}\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma} + \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma}$. Further because $\frac{\lambda}{\lambda-\mu} > 1$, which comes from Lemma 5 and 8, the inequality $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}} > \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}} = 1$ holds. Meanwhile since $\frac{\mu}{\lambda-\mu} = \frac{1}{\frac{\lambda}{\mu}-1}$ and $\frac{\partial\frac{\mu}{\lambda}}{\partial\sigma} > 0$ hold, $\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma} > 0$ is obvious.

As $\frac{\lambda}{\lambda-\mu} > 1$ and $\lambda > 0$, we must have $\frac{\mu}{\lambda-\mu} > 0$. Combining Lemma 8, we can yield the conclusion that $\frac{\partial\Phi_{p^H}}{\partial\sigma} > 0$ as long as μ is not too large to induce $\left|\frac{\mu}{\lambda-\mu}\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma}\right| > \left|\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma}\right|$. □

Lemma 11. *The stationary capital over effective labor ratio will increase as collateral constraint* γ *increases in Aiyagari-Bewley-Huggett model 9 when the housing supply is fixed and initial housing distribution over dynamic path is exogenous.*

Proof. Similar to the proof process of Lemma 4, we can reconstruct the how problem to payoff function 66 and constraint 67. Then because the collateral constraint is endogenous, we first need to explore the direction of $\frac{\partial \frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} c_t}{\partial \gamma}$ which I will show by induction below.

If $\frac{\partial c_t}{\partial \gamma} \geq -\frac{c_t}{\gamma}$, then $\frac{\partial \frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} c_t}{\partial \gamma} = \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} c_t + \frac{(1-\phi)p^H\gamma}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} \frac{\partial c_t}{\partial \gamma} \geq 0$ will hold with a slacker constraint. Further we can show that $\frac{\partial h^*}{\partial \gamma} \geq \frac{(1-\phi)}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} \frac{\partial c_t}{\partial \gamma}$. By taking derivative with respect to γ on both side of the budge constraint in 67 we know that $\frac{\partial a_t}{\partial \gamma} \leq -\left[1 + \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\right] \frac{\partial c_t}{\partial \gamma} \leq \left[1 + \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\right] \frac{c_t}{\gamma} < \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} c_t + \frac{(1-\phi)p^H\gamma}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} \frac{\partial c_t}{\partial \gamma}$. However this means the decreasing speed of a_t is larger than the decreasing speed of collateral constraint, which violates the meaning of collateral constraint. Therefore $\frac{\partial c_t}{\partial \gamma} < -\frac{c_t}{\gamma}$ will hold and we can yield $\frac{\partial \frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)} c_t}{\partial \gamma} < 0$ for sure.

Then based on the Lemma 1, Theorem 5 and Proposition 1 in Acemoglu and Jensen (2015), γ is a positive shock and the stationary capital over effective labor ratio Q is monotonic increasing in γ. □

Lemma 12. *$\frac{\lambda}{\lambda-\mu} \geq 1$, $\frac{\partial \frac{\mu}{\lambda}}{\partial \gamma} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ holds in Aiyagari-Bewley-Huggett model 9 when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. The demonstration process is similar to Lemma 5 as γ is also a positive price following Lemma 11 and it shares the same monotonicity as σ on $\frac{\mu}{\lambda}$ and $\frac{\lambda}{\lambda-\mu}$ when the stationary consumption C increases. □

Lemma 13. *$\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ is monotonic decreasing in γ, as long as $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ hold.*

Proof. Because of Lemma 1, we only need to check whether $\frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ is monotonic decreasing in γ. It is easy to calculate

$$\begin{aligned}
&\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\gamma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)-\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\gamma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)\\
&=\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\gamma}\widetilde{\eta}_{c,p^H}\left[\left(1-\frac{\lambda}{\lambda-\mu}\right)\widetilde{\eta}_{h,p^c}+\phi(1-\phi)\right]+\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\gamma}\widetilde{\eta}_{h,p^h}\left[\left(\frac{\lambda}{\lambda-\mu}-1\right)\widetilde{\eta}_{h,p^c}+\phi(1-\phi)\right]\\
&=\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\gamma}\phi(1-\phi)\left(\widetilde{\eta}_{c,p^H}+\widetilde{\eta}_{h,p^h}\right)
\end{aligned}$$

Hence $\frac{\partial\left(\frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\gamma} < 0$ holds as $\widetilde{\eta}_{c,p^H}+\widetilde{\eta}_{h,p^h} < 0$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\gamma} > 0$. □

Lemma 14. *The substitution effect Φ_H will decrease as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds.*

Proof. It is a straightforward conclusion from Lemma 12 and 13. □

Lemma 15. $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ *will be monotonic decreasing in γ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ hold.*

Proof. Because of Lemma 1, we only need to check whether $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ is monotonic decreasing in γ. It is easy to calculate

$$\frac{\partial \widetilde{\eta}_{ch}}{\partial \gamma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) - \frac{\partial\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial \gamma}\widetilde{\eta}_{ch} = \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\widetilde{\eta}_{h,p^h}\widetilde{\eta}_{ch}$$

Hence $\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial \gamma} < 0$ holds as $\widetilde{\eta}_{h,p^h} < 0$, $\widetilde{\eta}_{ch} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$. □

Lemma 16. *The wealth effect Φ_{p^H} will decrease as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds.*

Proof. Lemma 16 is a direct inference from Lemma 12 and 15. □

Lemma 17. *The credit effect Φ_μ will increase as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous ;* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds and the aggregate Khun-Tucker multiplier is not too large.*

Proof. Similar to Lemma 10, we can yield $\frac{\partial \frac{\mu}{\lambda-\mu}}{\partial \gamma} > 0$ because $\frac{\mu}{\lambda-\mu} = \frac{1}{\frac{\lambda}{\mu}-1}$ and $\frac{\partial \frac{\mu}{\lambda}}{\partial \gamma} > 0$ from Lemma 12. Therefore as long as the aggregate Khun-Tucker multiplier is not too large to violate $\frac{\partial \frac{\mu}{\lambda-\mu}}{\partial \gamma}\left|\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right| > \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\frac{\mu}{\lambda-\mu}\phi(1-\phi)\left|\widetilde{\eta}_{c,p^H} + \widetilde{\eta}_{h,p^h}\right|$, the credit effect is monotonic increasing in γ because $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}} > 0$ which we can induce from $\frac{\lambda}{\lambda-\mu} \geq 1$ in Lemma 12 and 1. □

F Toy model with global solution

Given the budget constraint of household

$$c_0 + a_1 + p_0\left[s_1 - (1-\delta^h)s_0\right] = (1+R_0)a_0 + w_0 + \pi_0^h + \pi_0$$

$$c_1 + a_2 + p_1\left[s_2 - (1-\delta^h)s_1\right] = (1+R_1)a_1 + w_1 + \pi_1^h + \pi_1$$

$$c_2 = (1+R_2)a_2 + p_2(1-\delta^h)s_2 + w_2 + \pi_2^h + \pi_2$$

From utility function and FOC of household we can get the key equation

$$u_{c_0}\left[p_0 - \frac{1}{1+R_1}(1-\delta^h)p_1\right] = u_{s_1} \tag{68}$$

Then if we assume the utility function is non-separable such that

$$u_t = \frac{\left(c_t^{\nu} s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}$$

By using the Euler equation of consumption as well as housing we can simplify equation 68 to

$$\left[p_0 - \frac{1}{1+R_1}(1-\delta^h)p_1\right] = \frac{c_1}{s_1^{\Phi} s_0^{\Psi}}$$

F.1 General equilibrium is important

A perturb happened at p_1 will decrease c_1 which in tern decrease c_2, If p_0, s_1 and R_1 not change.(This is the total effect of substitution and income as we derive from max utility which means from Marshallian demand function. This is pseudo-effect as we assume s_1 fixed)

However this analysis is based on the assumption that p_0, s_1 and R_1 will not change. Now we assume s_1 is not changed. Meanwhile the production is $Y_t = Aa_t$ so that $R_t = MPK = A$ which means R_1 will also be fixed. Which direction of p_0 changed?

The answer is that any small perturb increased happened in p_1 which returns $\tilde{p_1} = p_1 + \varepsilon$, p_0 will increase relative amount to make sure $p_0 - (1-\delta^h)p_1$ is fixed. This tells us that c_1 will in fact not change at all.[31]

Later we can also proof that given the decreasing return to scale production function such as $Y_t = Aa_t^{\alpha}$ will not change the result.

Intuition: Given p_1 increased, the household want to buy more s_1 at period 0. The fixed s_1 will caused p_0 increases a lot to even offset the wealth effect. If we assume s_1 increases and p_0 not change ($s_1$supply increased to the level that just fulfill the demand and p_0 does not change) the direction of c_1 will depends on the extent of increased s_1 and intratemperal substitution and intertemperal substitution). Another condition, p_0 increases more than related to $\frac{1}{1+R_1}(1-\delta^h)p_1$ is somehow less likely as an expectation causes a much higher inflation this period.

[31]The proof process is simple using induction. Given p_0 increases little but not enough to offset total decreased c_1. Then c_1 and c_0 will decreases little. Then using budget constraint, a_1 and $a_2$will relatively changed. Then to the final period we can get a contradiction. Inversely given p_0 increases a lot to result in c_1 increassing, we can get similar contradiction.

F.2 House supply is the key to determine non-durable consumption

Now we losse the assumption that s_1 does not change. From last section we know that under general equilibrium as long as the house supply does not increase, then no matter how large changed in p_1, c_1 will not change anymore because p_0 will adjusted one-to-one with it.

This give us the argument that the house supply or elasticity of house supply is much more important than scholar's focusing, as most of time we just take it as an IV in empirical research.

A right-hand shift in period 0 house demand(caused by a perturb in p_1) happened, the elasticity of house supply then determine the equilibrium changed in s. We have prove at previous section that when $e_1 = 0$, the increased p_0 will caused c_0 not change. In other words, under the most increased p_0, c_0 not changed. Then assume $e_1 > 0$, Δp_0 will decrease. LHS of equation 68 decrease. But because the intratemporal effect is larger than intertemporal effect, c_1 and c_0 will increase. In other words, the degree of elasticity of house supply determinate the non-durable consumption.

F.3 Unseparable utility function

F.3.1 partial effect

If the utility function is

$$u_t = \frac{\left(c_t^{\nu} s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}$$

then we will have

$$s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} = \beta R_1 s_1^{(1-\sigma)(1-\nu)} c_1^{\nu(1-\sigma)-1}$$

$$s_1^{(1-\nu)(1-\sigma)} c_1^{\nu(1-\sigma)-1} = \beta R_2 s_2^{(1-\sigma)(1-\nu)} c_2^{\nu(1-\sigma)-1}$$

$$\nu s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} p_0 = \beta \nu s_1^{(1-\sigma)(1-\nu)} c_1^{\nu(1-\sigma)-1} p_1 (1-\delta^h) + \beta(1-\nu) c_1^{\nu(1-\sigma)} s_1^{\nu(\sigma-1)-\sigma}$$

$$\nu s_1^{(1-\nu)(1-\sigma)} c_1^{\nu(1-\sigma)-1} p_1 = \beta \nu s_2^{(1-\sigma)(1-\nu)} c_2^{\nu(1-\sigma)-1} p_2 (1-\delta^h) + \beta(1-\nu) c_2^{\nu(1-\sigma)} s_2^{\nu(\sigma-1)-\sigma}$$

Then we will solve out c_1, c_2, s_1, s_2 by these four equations

$$c_1 = \left[\frac{1}{\beta R_1}\right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}} \left\{\frac{\nu}{1-\nu}\frac{1}{\beta}\left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right)\right]\right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}\right]^{-\frac{1}{\sigma}}$$

$$s_1 = \left\{ \frac{\nu}{1-\nu} \frac{1}{\beta} \frac{s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}}{c_1^{\nu(1-\sigma)}} \left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right) \right] \right\}^{\frac{1}{(1-\nu)(1-\sigma)-1}}$$
$$= \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}$$
$$\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta} \left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right) \right] \right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1} \frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[\frac{1}{\beta R_1} \right]^{\frac{\nu(1-\sigma)}{\sigma}}$$

$$c_2 = \left[\frac{1}{\beta^2 R_1 R_2} \right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}}$$
$$\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^2 R_1} \left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right) \right] \right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}$$

$$s_2 = \left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^2 R_1} \frac{s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}}{c_2^{\nu(1-\sigma)}} \left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right) \right] \right\}^{\frac{1}{(1-\nu)(1-\sigma)-1}}$$
$$= \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}$$
$$\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^2 R_1} \left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right) \right] \right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1} \frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[\frac{1}{\beta^2 R_1 R_2} \right]^{\frac{\nu(1-\sigma)}{\sigma}}$$

Under infinite horizon we will have

$$c_t = \left[\frac{1}{\beta^t \prod_{i=1}^t R_i} \right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}}$$
$$\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^t \prod_{i=1}^{t-1} R_i} \left[p_{t-1} - \frac{1}{R_t} p_t \left(1-\delta^h\right) \right] \right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}$$

$$s_t = \left[\frac{1}{\beta^t \prod_{i=1}^t R_i} \right]^{\frac{\nu(1-\sigma)}{\sigma}}$$
$$\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^t \prod_{i=1}^{t-1} R_i} \left[p_{t-1} - \frac{1}{R_t} p_t \left(1-\delta^h\right) \right] \right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1} \frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}$$

F.3.2 Other utility function

If the utility function is

$$u_t = log\left(c_t^{\nu} s_t^{1-\nu}\right)$$

then no GE effect

If the utility function is

$$u_t = log\left(c_t^{\nu} + s_t^{1-\nu}\right)$$

still unsolvable.

F.4 Standard utility function

F.4.1 general effect

No we assume that the utility function is no longer logarithmic such that

$$u_t = \frac{\left(c_t^{\nu} s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}$$

Then we have two key market cleaning condition that

$$a_2 = A_1 a_1^{\alpha} - c_1 + (1-\delta)a_1 = A_1 a_1^{\alpha} - c_0 \left(\beta R_1\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + (1-\delta)a_1$$

$$(1-\delta)a_2 + A_2 a_2^{\alpha} = c_2 = c_0 \left(\beta^2 R_1 R_2\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}$$

Based on these two equations we can rewrite equation as

$$\begin{aligned}
&(1-\delta)\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + \right.\\
&\qquad\qquad \left.(1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)\right] +\\
&A_2\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}\ \alpha} + \right.
\end{aligned} \tag{69}$$

$$\begin{aligned}
&\left.(1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)\right] =\\
&c_0\left\{\beta^2\alpha^2 A_1 A_2\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha-1}\right.\\
&\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right.\\
&\left.\left.+(1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta)a_0 - c_0\right)\right]^{\alpha-1}\right\}^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}
\end{aligned}$$

Similarly we set $\alpha = 1$, equation 69 becomes

$$(1-\delta)\left[A_1\left(A_0a_0+(1-\delta)a_0-c_0\right)-c_0\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}+\right.$$
$$\left.(1-\delta)\left(A_0a_0+(1-\delta)a_0-c_0\right)\right]+$$
$$A_2\left[A_1\left(A_0a_0+(1-\delta)a_0-c_0\right)-c_0\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}\alpha}+\right.$$
$$\left.(1-\delta)\left(A_0a_0+(1-\delta)a_0-c_0\right)\right]=$$
$$c_0\left(\beta^2\alpha^2A_1A_2\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}$$

Now we can solve the c_0 as

$$c_0=\frac{\left(A_2+1-\delta\right)\left(A_1+1-\delta\right)\left(A_0a_0+(1-\delta)a_0\right)}{\left(A_2+1-\delta\right)\left[A_1+1-\delta+\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right]+\left(\beta^2\alpha^2A_1A_2\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}}$$
$$=\frac{\left(A_2+1-\delta\right)\left(A_1+1-\delta\right)\left(A_0a_0+(1-\delta)a_0\right)}{\left(A_2+1-\delta\right)\left[A_1+1-\delta+\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{(1-\delta^h)s_0+\bar{s}_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right]+\left(\beta^2\alpha^2A_1A_2\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{\bar{s}_2+(1-\delta^h)\bar{s}_1+(1-\delta^h)^2s_0}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}}$$

Under the GE and determined economy, c_0 can only be decided by the equalized house stock. It is intuitive as in the end because all excess profit are payback by construction companies and consumption is mainly determined by IES & market cleaning condition. If we assume that good market clean does not involve construction industry, the house market can only affect the consumption via the Euler equation of asset. Here $\bar{s}_2$ decreases will lead p_2 increase, but it increase c_0 at the same time.

F.4.2 Infinite horizon condition

The market cleaning condition will be

$$a_1 = A_0a_0^\alpha + (1-\delta)a_0 - c_0$$

$$a_2 = A_1a_1^\alpha - c_1 + (1-\delta)a_1$$

$$a_3 = A_2a_2^\alpha - c_2 + (1-\delta)a_2$$

$$(1-\delta)a_\infty + A_\infty a_\infty^\alpha = c_\infty = c_0\left(\beta^3R_1R_2R_3\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_3}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}$$

$$
\begin{aligned}
c_0 &= \frac{\left(A_0 a_0 + (1-\delta) a_0\right) \prod_{t=1}^{\infty} \left(A_t + 1 - \delta\right)}{\sum_{t=1}^{T} \left[\prod_{i=t}^{T} \left(A_i + 1 - \delta\right)\right] \left(\beta^{t-1} \alpha^{t-1} \prod_{i=0}^{t-1} A_i\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_{t-1}}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + \left(\beta^T \alpha^T \prod_{t=0}^{T} A_t\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_T}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}} \\
&= \frac{\left(A_0 a_0 + (1-\delta) a_0\right) \prod_{t=1}^{T} \left(A_t + 1 - \delta\right)}{\sum_{t=1}^{T} \left[\prod_{i=t}^{T} \left(A_i + 1 - \delta\right)\right] \left(\beta^{t-1} \alpha^{t-1} \prod_{i=0}^{t-1} A_i\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{\sum_{i=0}^{t-1} (1-\delta^h)^i \bar{s}_i}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + \left(\beta^T \alpha^T \prod_{t=0}^{T} A_t\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{\sum_{i=0}^{T} (1-\delta^h)^i \bar{s}_i}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}}
\end{aligned}
$$

when normalizes $A_0 = 1$

F.5 Separable utility function

F.5.1 partial effect

$$
\begin{aligned}
c_1 &= c_0 \left(\beta R_1\right)^{\frac{1}{\sigma}} \\
c_2 &= c_0 \left(\beta^2 R_1 R_2\right)^{\frac{1}{\sigma}} \\
s_1 &= \left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1}{\nu}} \\
s_2 &= \left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1}{\nu}}
\end{aligned}
$$

$$
\begin{aligned}
c_0 \left(\beta^2 R_1 R_2\right)^{\frac{1}{\sigma}} + R_2 c_0 \left(\beta R_1\right)^{\frac{1}{\sigma}} + R_1 R_2 c_0 + \\
R_2 p_1 \left\{\left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1}{\nu}} - (1-\delta^h) \left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1}{\nu}}\right\} + \\
R_1 R_2 p_0 \left\{\left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1}{\nu}} - p_0 (1-\delta^h)\right\} = \\
R_0 R_1 R_2 a_0 + R_1 R_2 \left(w_0 + \pi_0\right) + R_2 \left(w_1 + \pi_1\right) + w_2 + \pi_2 \\
+ p_2 (1-\delta^h) \left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1}{\nu}}
\end{aligned}
$$

$$
\begin{aligned}
F_{p_1} &= R_2 \left\{\left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1}{\nu}} - (1-\delta^h) \left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1}{\nu}}\right\} \\
&+ R_2 p_1 \left\{-\frac{1}{\nu} R_2 \left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1+\nu}{\nu}} - \frac{1}{\nu} (1-\delta^h)^2 \left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1+\nu}{\nu}}\right\} \\
&+ \frac{(1-\delta^h)}{\nu} R_1 R_2 p_0 \left[p_0 R_1 - p_1 (1-\delta^h)\right]^{-\frac{1+\nu}{\nu}} + \frac{1}{\nu} p_2 R_2 (1-\delta^h) \left[p_1 R_2 - p_2 (1-\delta^h)\right]^{-\frac{1+\nu}{\nu}}
\end{aligned}
$$

$$
F_{c_0} = \left(\beta^2 R_1 R_2\right)^{\frac{1}{\sigma}} + R_2 \left(\beta R_1\right)^{\frac{1}{\sigma}} + R_1 R_2
$$

F.5.2 general effect

$$
a_1 = A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0
$$

$$
\begin{aligned}
a_2 = & A_1 \left[A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right]^{\alpha} - c_0 \left[\beta \alpha A_1 \left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha-1}\right]^{\frac{1}{\sigma}} \\
& + (1-\delta) \left[A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right]
\end{aligned}
$$

we can solve c_0 by

$$(1-\delta)a_2 + A_2 a_2^{\alpha} = c_0 \left(\beta^2\alpha^2 A_1 A_2 \left(a_1 a_2\right)^{\alpha-1}\right)^{\frac{1}{\sigma}}$$

which means it is predetermined.

G Equilibrium condition of the full fledged model

G.1 Focs

G.1.1 Focs in production sector

In this section I show that there exists an knife-edge equilibrium in which along the dynamic transition path real rental rate and wage is fixed, as long as the TFP does not change.

The non-durable goods producer solve the problem

$$\max_{K_n, L_n} A_n K_{n,t}^{\alpha} L_{n,t}^{1-\alpha} - (r_t+\delta)K_{n,t} - wL_{n,t}$$

to yield the Foc

$$(1-\alpha)\, A_n K_{n,t}^{\alpha} L_{n,t}^{-\alpha} = w_t \tag{70}$$

and

$$\alpha A_n K_{n,t-1}^{\alpha-1} L_{n,t}^{1-\alpha} = r_t + \delta \tag{71}$$

Similarly the durable goods producer solve the problem

$$\max_{K_h, L_h} \Pi^h = p_t^h A_h \overline{L}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota} - (r_t+\delta)K_{h,t} - wL_h$$

to yield the Foc

$$\iota A_h p_t^h \overline{L}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota-1} = w_t \tag{72}$$

and

$$\nu A_h p_t^h \overline{L}_t^{\theta} K_{h,t}^{\nu-1} L_{h,t}^{\iota} = r_t + \delta \tag{73}$$

Combine equation 71 and 73 will yield

$$\frac{\nu p_t^h Y_{H,t}}{K_{h,t}} = r_t + \delta = \frac{\alpha Y_{N,t}}{K_{n,t}} \tag{74}$$

It is easy to check that when $\frac{\iota}{\nu} = \frac{1-\alpha}{\alpha}$ the real rental rate and wage at time t is fixed, as long as the total capital used at time t, K_{t-1} and labor L_t is fixed. I attach the proof process below.

By dividing equation 70, 71, 72 and 73 with each other I can get the relative input sharing condition

$$\frac{\iota\alpha}{\nu\left(1-\alpha\right)}\frac{K_{h,t}}{K_{n,t}}\frac{L_{n,t}}{L_{h,t}}=1$$

when $\frac{\iota}{\nu}=\frac{1-\alpha}{\alpha}$ holds, above equation will change to $\frac{K_{h,t}}{K_{n,t}}=\frac{L_{n,t}}{L_{h,t}}$.

Furthermore, the relative value of $K_{n,t}$ and $L_{n,t}$ can be pinned down with the market clearing condition $K_{H,t-1}=K_{h,t}+K_{n,t}$ and $L_t=L_{h,t}+L_{n,t}$. In section 3 I assume that the labor supply is exogenous which will help to demonstrate that the relative value of $K_{n,t}$ and $L_{n,t}$ follows

$$\frac{K_{n,t}}{L_{n,t}}=\frac{K_{H,t-1}}{L}\frac{1+\frac{K_{n,t}}{L_{n,t}}}{1+\frac{K_{h,t}}{L_{h,t}}}$$

Because $K_{H,t-1}$ is predetermined and $\frac{K_{h,t}}{K_{n,t}}=\frac{L_{n,t}}{L_{h,t}}$, the $\frac{K_{n,t}}{L_{n,t}}$ is fixed. Therefore r_t is fixed from equation 74.

G.1.2 Focs in consumer sector

The household solve the problem

$$V(h_{t-1},x_{t-1},\varepsilon_{t-1})=\max_{h_t,x_t,l_t,c_t}U\left(c_t,h_t,l_t\right)+\beta EV(h_t,x_t,\varepsilon_t)$$

$$\begin{aligned}\text{s.t.}c_t+x_t+\left(1-\gamma\right)p_t^h h_t=&\left[\left(1-\delta^h\right)p_t^h-\gamma R_t p_{t-1}^h\right]h_{t-1}+R_t x_{t-1}\\&+(1-\tau)w_t l_t\varepsilon_{t-1}-p_t^h C\left(h_t,h_{t-1}\right)+T_t\end{aligned}\tag{75}$$

and

$$x_t\geq 0$$

The related Lagrange is

$$\begin{aligned}\mathcal{L}=&U\left(c_t,h_t,l_t\right)+\beta E_t V(h_t,x_t,\varepsilon_t)\\&+\lambda_t\left[c_t+x_t+\left(1-\gamma\right)p_t^h h_t-\left[\left(1-\delta^h\right)p_t^h-\gamma R_t p_{t-1}^h\right]h_{t-1}\right.\\&\left.-R_t x_{t-1}-(1-\tau)w_t l_t\varepsilon_{t-1}+p_t^h C\left(h_t,h_{t-1}\right)-T_t\right]\\&+\mu_t x_t\end{aligned}$$

Then the FOCs related to consumer's problem will be

$$U_{c,t}+\lambda_t=0\tag{76}$$

$$U_{h,t}+\beta E_t V_{h,t}+\lambda_t\left(1-\gamma+C_{h,t}\right)p_t^h=0\tag{77}$$

$$\beta E_t V_{x,t}+\lambda_t+\mu_t=0\tag{78}$$

$$U_{l,t} - \lambda_t(1-\tau)w_t\varepsilon_{t-1} = 0 \tag{79}$$

The envelop conditions are

$$V_h(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = -\lambda_t \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h - C_{h_{t-1}}\left(h_t, h_{t-1}\right) p_t^h\right] \tag{80}$$

$$V_x(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = -\lambda_t R_t \tag{81}$$

G.1.3 Steady State condition in production sector

G.2 Alternative Setting to Capital Producer

G.2.1 Capital Producer(Setting I)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned} &\max\left(Q_t - 1\right) I_t - f\left(I_t, K_{t-1}\right) K_{t-1} \\ \text{s.t. } f\left(I_t, K_{t-1}\right) = &\frac{\psi_{I,1}}{\psi_{I,2}}\left(\frac{I_t}{K_{t-1}} - \overline{\delta}\right)^{\psi_{I,2}} \end{aligned}$$

where $\overline{\delta}$ is the steady-state investment rate.

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = 1 + \psi_{I,1}\left(\frac{I_t}{K_{t-1}} - \overline{\delta}\right)^{\psi_{I,2}-1}$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + f\left(I_t, K_{t-1}\right) K_{t-1} + p^h C(h', h)$$

G.2.2 Capital Producer(Setting II)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned} &\max Q_t I_t - f\left(I_t, K_{t-1}\right) K_{t-1} \\ \text{s.t. } f\left(I_t, K_{t-1}\right) = &\frac{\overline{\delta}^{-1/\phi}}{1+1/\phi}\left(\frac{I_t}{K_{t-1}}\right)^{1+1/\phi} + \frac{\overline{\delta}}{\phi+1} \end{aligned}$$

where $\overline{\delta}$ is the steady-state investment rate following $\overline{\delta} = \frac{\overline{I}}{\overline{K}}$

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = \left(\frac{I_t}{K_{t-1}\overline{\delta}} \right)^{1+1/\phi}$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + f\left(I_t, K_{t-1}\right) K_{t-1} + p^h C\left(h', h\right)$$

G.2.3 Capital Producer(Setting III)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned} \max Q_t f\left(I_t, K_{t-1}\right) K_{t-1} - I_t \\ \text{s.t. } f\left(I_t, K_{t-1}\right) = & \frac{\overline{\delta}^{1/\phi}}{1-1/\phi} \left(\frac{I_t}{K_{t-1}} \right)^{1-1/\phi} - \frac{\overline{\delta}}{\phi+1} \end{aligned}$$

where $\overline{\delta}$ is the steady-state investment rate.

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = \left(\frac{I_t}{K_{t-1}\overline{\delta}} \right)^{1-1/\phi}$$

and the law of motion of capital will become

$$K_t = (1-\delta)K_{t-1} + f\left(I_t, K_{t-1}\right) K_{t-1}$$

The goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + p^h C\left(h', h\right)$$

G.2.4 Capital Producer(Setting IV)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Lambda_{t,t+\tau} \left\{ (Q_\tau - 1) I_\tau - f(I_\tau, I_{\tau-1}) I_\tau \right\}$$

$$\text{s.t. } f(I_\tau, I_{\tau-1}) = \frac{\psi_{I,1}}{\psi_{I,2}} \left(\frac{I_\tau}{I_{\tau-1}} - 1 \right)^{\psi_{I,2}}$$

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = 1 + \frac{\psi_{I,1}}{\psi_{I,2}} \left(\frac{I_t}{I_{t-1}} - 1 \right)^{\psi_{I,2}} + \psi_{I,1} \left(\frac{I_t}{I_{t-1}} - 1 \right)^{\psi_{I,2}-1} \frac{I_t}{I_{t-1}} -$$
$$E_t \beta \Lambda_{t,t+1} \psi_{I,1} \left(\frac{I_{t+1}}{I_t} - 1 \right)^{\psi_{I,2}-1} \left(\frac{I_{t+1}}{I_t} \right)^2$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + f(I_t, I_{t-1}) I_{t-1} + p^h C(h', h)$$

H Numerical solution

H.1 Calibration to full fledged model

All the parameters related to production sector are selected from literature. The depreciation rate of physical capital is 0.03 which implies 12% annually. The depreciation rate of housing service is estimated from data which is constructed by Rognlie et al. (2018) as my model in supply side is too simple to use the gross GDP in NIPA. Therefore I use the GDP constructed by Rognlie et al. (2018) which is more suitable to this simple supply side. The depreciation rate of housing service is roughly 1.9% quarterly which is in line with Kaplan et al. (2020). The relative share of production factors in construction function ν, θ and ι comes from Favilukis et al. (2017). The last three parameters, exogenous land supply, TFP in production function and TFP in construction function, together with other parameters in household problem, are selected to match the real gross rate, labor demand, liquid asset over GDP and iliquid asset over GDP.

Table 10: Parameter Values from Calibration

Parameter	Value	Description
δ	0.03	Depreciation rate of physical capital

Table 10 – Continued		
Parameter	**Value**	**Description**
δ^h	0.01873	Depreciation rate of housing service
α	0.36	Capital share in production function
ν	0.27	Capital share in construction function
ι	0.36	Labor share in construction function
θ	0.1	Land share in construction function
$\overline{LD}$	4.95	Land supply
A_n	0.67	TFP in production function
A_h	2.75	TFP in construction function

Table 11: Presetted Parameter Values

Parameter	**Value**	**Description**
σ_L	0	Depreciation rate of physical capital
$\sigma_{m_4^L}$	∞	Depreciation rate of housing service
$\sigma_{m_4^\phi}$	∞	Capital share in production function
m_1^L	1	Capital share in construction function
m_2^L	1	Labor share in construction function
m_3^L	1	Land share in construction function
m_4^L	0	Land supply
m_1^ϕ	1	TFP in production function
m_2^ϕ	1	
m_3^ϕ	1	
m_4^ϕ	0	TFP in construction function

H.2 Bayesian estimation to full fledged model

I use Bayesian method to estimate the parameters that control the impulse response and transition path such as the AR1 coefficients ρ_a^i, the observation matrix H and related covariance matrix $\eta\eta'$ and $\epsilon\epsilon'$. Since the data process itself is not stationary it is not appropriate to use the full-information Bayesian and if we used the statistic method to detrend such as first-order difference and hp filter, the Bayesian update rule would not be further used and the posterior $p\left(\theta|Y^T\right) \propto p\left(Y^T|\theta\right)p\left(\theta\right)$ would be unsolvable as $p\left(Y^T|\theta\right)$ was unknown. Therefore I use GMM to match the moments in data and model to proceed the estimation. In this subsection I first introduce the moments I used to match the data and then explain the Bayesian estimation strategy in detail.

H.2.1 Moments Selection and Theoretical moments after filter

I impose hp filter on the data and calculate moments from the cyclical elements such as the autocovariance of output, standard derivation of output, physical investment, new constructed residential estate, relative housing price and their related covariance. The covariance between output and physical investment $\text{cov}(y_t, I_t)$ captures the general equilibrium $Y = C + I$. Similarly the covariance between residential investment and physical investment $\text{cov}(I_t^H, I_t)$ captures the crowded-out effect. The covariance between new constructed residential estate and relative housing price capture the demand and supply equilibrium in the housing market. All these eight moments are summarized in vector $g(\cdot) = \Psi$ following

$$\Psi = \begin{bmatrix} \varrho_m' & \sigma_{m,m}' & \sigma_{m,n}' \end{bmatrix}'$$

where ϱ_m is the vector that contains the autocovariance moments (ρ_m^i represents the $\text{AR}(i)$'s coefficient of variable m)

$$\varrho_m = \begin{bmatrix} \rho_y^1 & \rho_c^1 & \rho_I^1 & \rho_{I_H}^1 & \rho_{p_H}^1 & \rho_Q^1 \end{bmatrix}'$$

$\sigma_{m,m}$ is the vector that contains the standard derivation moments

$$\sigma_{m,m} = \begin{bmatrix} \sigma_y & \sigma_c & \sigma_I & \sigma_{p_H} & \sigma_Q \end{bmatrix}'$$

$\sigma_{m,n}$ is the vector that contains the covariance moments of variables $\phi_v = \begin{bmatrix} y & c & I & I_H & p_H & Q & R \end{bmatrix}'$

$$\sigma_{m,n} = \begin{bmatrix} \sigma_{y,c} & \sigma_{y,I} & \sigma_{y,I_H} & \sigma_{y,p_H} & \sigma_{y,Q} & \sigma_{y,R} & \sigma_{c,I_H} & \cdots & \sigma_{Q,R} \end{bmatrix}'$$

Moreover I solve the theoretical moments from model after hp filter by switching to frequency domain and the spectrum. After some algebra I can solve the covariance matrix

$$\mathbb{E}\left[\widetilde{Y}_t \widetilde{Y}_{t-1}\right] = \int_{-\pi}^{\pi} g^{\text{HP}}(\omega) e^{i\omega k} d\omega$$

where $\widetilde{Y}_t = \begin{bmatrix} s_t' & s_{t|t}' & Ec_{t+1}' \end{bmatrix}'$ in equation 102. The spectral density of HP filter $g^{\text{HP}}(\omega)$ follows $g^{\text{HP}}(\omega) = h^2(\omega) g(\omega)$. $h(\omega) = \frac{4\lambda(1-\cos(\omega))^2}{1+4\lambda(1-\cos(\omega))^2}$ is the transfer function of HP derived from King and Rebelo (1993). The spectral density of state and control variables Y_t is solved by

$$g(\omega) = \begin{bmatrix} I_{ns} & 0_{ns,nq} \\ M_{21} e^{-i\omega} & D_2 \\ 0_{nq,ns} & I_{nq} \end{bmatrix} f(\omega) \begin{bmatrix} I_{ns} & M_{21}' e^{i\omega} & 0_{ns,nq} \\ 0_{nq,ns} & D_2' & I_{nq} \end{bmatrix} = W f(\omega) W' \tag{82}$$

$$f(\omega) = \frac{1}{2\pi} \begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ I_{nq} \end{bmatrix} \Sigma \left[D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1}, I_{nq}\right] \tag{83}$$

where ns is the number of state variables and nq is the number of shocks. M and D come from the policy function 108 and Σ is the covariance matrix of shocks. Because I assume the shock term Ξ_t in system 102 follows standard normal distribution and all the covariance terms are absorbed in η and ϵ, Σ in equation 83 is an identity matrix.

W.L.O.G, I assume the shock Ξ_t in equation 108 is independent with each other and all the covariance term is stored in response D. Therefore the covariance term Σ in equation 83 is an identity matrix and the equation can be further simplified as

$$f(\omega) = \frac{1}{2\pi} \begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & I_{nq} \end{bmatrix}$$

Then equation 82 becomes

$$\begin{aligned} g(\omega) &= \frac{1}{2\pi} \begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & I_{nq} \end{bmatrix} W' \\ &+ \frac{1}{2\pi} \begin{bmatrix} 0 & 0 \\ D_2 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & D_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{ns} & M_{21}' e^{i\omega} & 0_{ns,nq} \\ 0_{nq,ns} & D_2' & I_{nq} \end{bmatrix} W' \\ &= \frac{1}{2\pi} \left(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4\right) \end{aligned}$$

where

$$\Upsilon_1 = \begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & M_{21} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' & M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & I_{nq} \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} 0 & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_2' & 0 \\ 0 & M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_2' & 0 \\ 0 & D_2' & 0 \end{bmatrix}$$

$$\Upsilon_3 = \begin{bmatrix} 0 & 0 & 0 \\ D_2 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & D_2 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & D_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Upsilon_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_2 D_2' & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To further decrease the computation burden it is easy to show that $M_{21}\left(I_{ns}-M_{11}\mathrm{e}^{-i\omega}\right)^{-1}=\mathrm{e}^{i\omega}M_{21}U_M\left(\mathrm{e}^{i\omega}I_{ns}-T_M\right)^{-1}U_M'$ where $M_{11}=U_MT_MU_M'$ is prederived from Schur decomposition.

H.2.2 Bayesian GMM

H.2.2.1 Moment Matching: Imperfect Information Bayesian Estimation

Following Rotemberg and Woodford (1997), Christiano et al. (2005) and Barsky and Sims (2012), to construct the asymptotic properties of the moments which I select to conduct the Bayesian GMM, I first construct the auxiliary variable ψ_t

$$\psi_t=\begin{bmatrix} y_t & c_t & I_t & I_{t,H} & p_{t,H} & Q_t & R_t & y_ty_{t-1} & c_tc_{t-1} & \cdots & y_t^2 & c_t^2 & \cdots & p_{t,H}^2 & y_tc_t & y_tI_t & \cdots & Q_tR_t \end{bmatrix}'$$

Additionally I define the moment function as $g(\cdot)$ which yields the moments

$$g(\psi_t)=\Psi$$

If the sample estimation of ψ_t is $\widehat{\psi}$ the moment function is well defined as

$$g(\widehat{\psi})=\begin{bmatrix} \hat{\psi}_{y_ty_{t-1}}-\hat{\psi}_y^2 \\ \hat{\psi}_{c_tc_{t-1}}-\hat{\psi}_c^2 \\ \vdots \\ \sqrt{\hat{\psi}_{y^2}-\hat{\psi}_y^2} \\ \sqrt{\hat{\psi}_{c^2}-\hat{\psi}_c^2} \\ \vdots \\ \hat{\psi}_{yc}-\hat{\psi}_y\hat{\psi}_c \\ \hat{\psi}_{yI}-\hat{\psi}_y\hat{\psi}_I \\ \vdots \\ \hat{\psi}_{QR}-\hat{\psi}_Q\hat{\psi}_R \end{bmatrix}$$

Therefore the Jacobian of moment function $\Gamma_g(\cdot)$ should be

$$\Gamma_g(\widehat{\psi}) = \frac{\partial g}{\partial \psi} = \begin{bmatrix} -2\mu_y & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\mu_c & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\mu_y}{\sigma_y} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_y} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu_c}{\sigma_c} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_c} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\frac{\mu_Q}{\sigma_Q} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_Q} & 0 & \cdots & 0 & 0 & 0 \\ -\mu_c & -\mu_y & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ -\mu_I & 0 & -\mu_y & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\mu_R & -\mu_Q & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

By applying the Delta Method the sample estimation of moments $\widehat{\Psi}$ has the following asymptotic properities

$$\sqrt{T}\left(\widehat{\Psi} - \Psi\right) \xrightarrow{d} N\left(0, \Gamma_g \Sigma \Gamma_g'\right)$$

where Σ is the LRV of ψ_t.

H.2.2.2 Full Information Bayesian Estimation

There are 38 parameters to be estimated via bayesian method and most of them govern the dynamic transition path of the economy. Firstly, six out of thirty-eight parameters are the AR1 coefficient of the shocks' process: ρ_{A_n} and ρ_{A_h} relate to the TFP of output and construction sector; ρ_L and ρ_{L_g} relate to the supply side of the housing market, land supply, and they are similar to the form defined in equation 93; ρ_ϕ and ρ_{ϕ_g} relate to the demand side of the housing market, preference on residential asset, and they are just the form defined in equation 93. Then eight parameters correspond to the standard derivation of above six shock series with two news shock on supply and demand side of the housing market. Additionally eight parameters, in supply and demand side of the housing market, associate with the observation or imperfect information process (H in equation 96) and another eight parameters pertain to the standard derivation of these observation noisy (ϵ in equation 96). Then one parameter affects the capital price, which is in the capital production function (ψ_I in equation 23). In the end the left seven parameters are the standard derivation of measure error of the seven data series that I used to estimate: output, nondurable consumption, physical investment, new construction, housing price, stock price and real interest rate. The whole estimation process is overestimated as there are 38 parameters in model but 77 moments (49 in coefficient matrix and 28 in the covariance matrix of residual).

Following Smets and Wouters (2007) and Rudebusch and Swanson (2012), I use the standard random walk metropolis-hastings (RWMH) algorithm to conduct the bayesian estimation and the data I used are per capita series to get a stationary time series. However most of the data does not pass the unit-root test and thus I further use the first order difference method to detrend the data, because I do not introduce the trend (growth) elements in the model. Moreover, to ensure the compatible between the model and data, I rearrange the state equation 108 of the model to

$$\begin{bmatrix} \widetilde{Y}_t \\ \widetilde{Y}_{t-1} \end{bmatrix} = \begin{bmatrix} M & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} \widetilde{Y}_{t-1} \\ \widetilde{Y}_{t-2} \end{bmatrix} + D \begin{bmatrix} \Xi_t \\ 0 \end{bmatrix} \tag{84}$$

Therefore the measurement equation should change to

$$Y_t = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} \widetilde{Y}_t \\ \widetilde{Y}_{t-1} \end{bmatrix} + \Xi_t \tag{85}$$

The likelihood function can be solved from the Kalman Filter from the state equation 84 and measurement equation 85. Based on the recommendation of Herbst and Schorfheide (2016), I use gradient based MLE method to proceed the estimation to get the asymptotic variance of the parameters (the inverse Hessian of the likelihood function) and the prior mean of the parameters. Following Schmitt-Grohé and Uribe (2012), Blanchard et al. (2013) and Christiano et al. (2014) the prior standard derivations that pertain to AR1 coefficient are 0.1 and others that associate with variance are 1.

Table 12: Bayesian Estimation

Parameter	**Distribution**	**Prior**		**Posterior**	
		mean	s.d.	mean	s.d.
ρ_{A_n}	Beta	0.5	0.2		
ρ_{A_h}	Beta	0.5	0.2		
ρ_L	Beta	0.5	0.2		
ρ_{L_g}	Beta	0.5	0.2		
ρ_ϕ	Beta	0.5	0.2		
ρ_{ϕ_g}	Beta	0.5	0.2		
σ_{A_n}	InvGamma	0.1	1		
σ_{A_h}	InvGamma	0.1	1		
σ_{L_g}	InvGamma	0.1	1		
σ_ϕ	InvGamma	0.1	1		
σ_{ϕ_g}	InvGamma	0.1	1		
$\sigma_{L_{g,\varepsilon}}$	InvGamma	0.1	1		
$\sigma_{\phi_{g,\varepsilon}}$	InvGamma	0.1	1		
$\sigma_{m_1^L}$	InvGamma	0.1	1		

Table 12 – Continued					
Parameter	**Distribution**	**Prior**		**Posterior**	
		mean	s.d.	mean	s.d.
$\sigma_{m_2^L}$	InvGamma	0.1	1		
$\sigma_{m_3^L}$	InvGamma	0.1	1		
$\sigma_{m_1^\phi}$	InvGamma	0.1	1		
$\sigma_{m_2^\phi}$	InvGamma	0.1	1		
$\sigma_{m_3^\phi}$	InvGamma	0.1	1		
ϕ_I	Gamma	1.728	1		

H.3 Solution method to simple model

H.3.1 Reconstruction

Similar to the section H.7.1, I replace the saving a_t by the effective asset holding x_t which follows $x_t = \gamma p_t^H h_t + a_t$. Then the problem 3 change to

$$\max_{c_t, h_t, x_t} \sum_{t=0}^{\infty} \beta^t U\left(c_t, h_t\right) \tag{86}$$

s.t.

$$c_t + x_t + (1-\gamma)\, p_t^H h_t = R_t x_{t-1} + w_t \varepsilon_t + \left[(1-\delta^H) p_t^H - \gamma R_t p_{t-1}^H\right] h_{t-1} + T_t \tag{87}$$

$$x_t \geq 0$$

The related FOCs 57, 58 and 59 will become

$$U_{c_t} = \lambda_t \tag{88}$$

$$-\lambda_t + \mu_t + \beta E_t R_{t+1} \lambda_{t+1} = 0 \tag{89}$$

$$U_{h_t} - (1-\gamma)\, \lambda_t p_t^H + \beta E_t \lambda_{t+1} \left[(1-\delta^H) p_{t+1}^H - \gamma R_{t+1} p_t^H\right] = 0 \tag{90}$$

Similar to the full fledged model, I assume the utility function $U\left(c_t, h_t\right)$ follows the Cobb-Douglas formula

$$U\left(c_t, h_t\right) = \frac{\left(c_t^{\phi} h_t^{1-\phi}\right)^{1-\sigma}}{1-\sigma} \tag{91}$$

Since I assume there is no aggregate shock existing in the simple model, R_{t+1}, p^H_{t+1} and p^H_t can be perfectly expected. Therefore for non-constrained household there exists a static relationship between c_t and h_t from the combining of equation 88, 89 and 90

$$c_t = \frac{\phi}{1-\phi} h_t \left[p^H_t - (1-\delta^H) \frac{p^H_{t+1}}{R_{t+1}} \right] \tag{92}$$

When the collateral constraint is binding, it is worth to notice that the two FOC 58 and 89 have the same form. Therefore the Khun-Tucker multiplier is the same between the two model, the original one and the reconstructed one. To sum up, the problem 86 degenerates to a one state x_t problem which can be solved easily by value function iteration.

H.3.2 Solution Steps

Since in this simple problem I use Cobb-Douglas utility function where intratemporal elasticity of substitution between housing service and non-durable consumption is constant at 1, the consumption and housing servicing is homogeneous in degree 1 (linear) in the frictionless scenario. Therefore it is solvable to use value function iteration method.

1. Take an initial guess about value function $V(h_{-1}, x_{-1}, \varepsilon_{-1}) = \sum_{t=0}^{\infty} \beta^t U(c_t, h_t)$. If h_0, x_0 is still on grid I can remove the expectation with $\widetilde{V}(h_0, x_0, \varepsilon_{-1}) = E_0 V(h_0, x_0, \varepsilon_0) = \Pi V(h_0, x_0, \varepsilon_0)$ as h_0, x_0 is determined at time 0.

2. If the budget constraint is not binding, equation 92 will always hold. Therefore given an initial guess of $h_0(h_{-1}, x_{-1}, \varepsilon_{-1})$, I can get the unique mapping $x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ and $c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ through budget constraint 87 and equation 92. Then it is easy to find

$$h_0^{uc}(h_{-1}, x_{-1}, \varepsilon_{-1}) = \underset{h_0}{\text{argmax}} U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta \widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$$

where $\widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$ can be solved from linear interpolation on the on-grid value $\widetilde{V}(h_0, x_0, \varepsilon_{-1})$ in last step. I also define and save the value

$$\text{RHS}^{UC} = \max U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta \widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$$

.

3. If the budget constraint is binding, the Euler equation does not hold anymore. Therefore the mapping between h_0 and c_0 is no longer useful. However the effective wealth is known as now the household is constrained so $x_0(h_{-1}, x_{-1}, \varepsilon_{-1}) = 0$. Given any guess of $h_0(h_{-1}, x_{-1}, \varepsilon_{-1})$ the consumption $c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ can be solved from budget

constraint 87. Then it is easy to find

$$h_0^c(h_{-1},x_{-1},\varepsilon_{-1}) = \underset{h_0}{\text{argmax}} U\left[c_0(h_0,h_{-1},x_{-1},\varepsilon_{-1}),h_0\right] + \beta\widetilde{V}\left[h_0,0,\varepsilon_{-1}\right]$$

where $\widetilde{V}\left[h_0,0,\varepsilon_{-1}\right]$ can be solved from linear interpolation on the on-grid value $\widetilde{V}(h_0,0,\varepsilon_{-1})$ in step 1. I also define and save the value

$$\text{RHS}^C = \max U\left[c_0(h_0,h_{-1},x_{-1},\varepsilon_{-1}),h_0\right] + \beta\widetilde{V}\left[h_0,0,\varepsilon_{-1}\right]$$

.

4. Because the result of constrained optimization in convex function optimization problem is always inferior than that of unconstrained optimization, the updated value function $V(h_{-1},x_{-1},\varepsilon_{-1})$ will follows

$$V(h_{-1},x_{-1},\varepsilon_{-1}) = \begin{cases} \text{RHS}^{UC} & x_0^{uc} \geq 0 \\ \text{RHS}^{C} & x_0^{c} < 0 \end{cases}$$

Update the value function and go back to step 1.

H.4 Solution method to simple model with separable utility function

H.4.1 Reconstruction and new FOCs

Change the utility function from 91 to the separable utility function

$$U(c_t,h_t) = \frac{\phi c_t^{1-\sigma} + (1-\phi)h_t^{1-\sigma}}{1-\sigma}$$

Then the mapping from c_t to h_t under the frictionless scenario changes to

$$c_t = \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{\sigma}} \left[p_t^H - (1-\delta^H)\frac{p_{t+1}^H}{R_{t+1}}\right]^{\frac{1}{\sigma}} h_t$$

H.5 Expected news shock

Then denote the "fundamental" variable X_t as

$$X_t = \begin{bmatrix} \log\Phi_t^i & \log\Phi_{g,t}^i & \varepsilon_t^8 & \varepsilon_{t-1}^8 & \varepsilon_{t-2}^8 & \varepsilon_{t-3}^8 & \varepsilon_{t-4}^8 & \varepsilon_{t-5}^8 & \varepsilon_{t-6}^8 & \varepsilon_{t-7}^8 \end{bmatrix}' \tag{93}$$

Then X_t follows

$$X_t = B^s X_{t-1} + \eta w_t \tag{94}$$

where

$$B^s = \begin{bmatrix} \rho_a & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \rho_g & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{10\times 10}$$

$$\eta = \begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & \sigma_g^8 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}_{10\times 3}$$

$$\boldsymbol{w}_t = \begin{bmatrix} w_t^a \\ w_t^g \\ w_t^8 \end{bmatrix}$$

However household can only observe the variable $\widetilde{X}_t$ such that

$$\widetilde{X}_t = \begin{bmatrix} \log \widetilde{\Phi}_t & \log \widetilde{\Phi}_{g,t} & \widehat{\varepsilon}_t^8 & \widehat{\varepsilon}_{t-1}^8 & \widehat{\varepsilon}_{t-2}^8 & \widehat{\varepsilon}_{t-3}^8 & \widehat{\varepsilon}_{t-4}^8 & \widehat{\varepsilon}_{t-5}^8 & \widehat{\varepsilon}_{t-6}^8 & \widehat{\varepsilon}_{t-7}^8 \end{bmatrix}' \tag{95}$$

which follows

$$\widetilde{X}_t = HX_t + \epsilon v \tag{96}$$

where

$$H = \begin{bmatrix} H_{3\times 3}^{11} & 0_{3\times 5} \\ 0_{5\times 3} & m_4 I_{5\times 5} \end{bmatrix}$$

$$H^{11} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$m \in \mathbb{R}^+$$

$$\epsilon = \begin{bmatrix} \sigma_a^s & 0 & 0 & \cdots & 0 \\ 0 & \sigma_g^s & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{g1}^s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{g8}^s \end{bmatrix}_{10\times 10}$$

$$v_t = \begin{bmatrix} v_t^a \\ v_t^g \\ v_t^{g1} \\ \vdots \\ v_t^{g8} \end{bmatrix}$$

H.6 Kalman Filter

Even though the household can successfully observe A_t at time t, he cannot observe g_t at time t. This make the household harder to estimate the A_{t+1} as $E_t \log(A_{t+1}) = \rho_a \log A_t + E_t \log g_t$. Thus we need get $g_{t|t}$ to get the expectation of A_{t+1}. Based on the Kalman filter and equation 94 and 96, we can solve out the perception of g_t by household as[32]

$$X_{t+1|t+1} = A^s X_{t|t} + P^s \widetilde{X}_{t+1} \tag{97}$$

where P^s is the Kalman gain and $A^s = (I - P^s H)B^s$

H.7 Model Reconstruction and Solution Process

The computation process follows the augmented endogenous gird method which is proposed by Auclert et al. (2021).

H.7.1 Preliminaries

I define the risk-adjusted expected value function as

$$\widetilde{V}(h_t, b_t, \varepsilon_{t-1}) = \beta EV(h_t, b_t, \varepsilon_t)$$

Therefore the marginal risk-adjusted expected value should be

$$\widetilde{V}_h(h_t, b_t, \varepsilon_{t-1}) = \beta EV_h(h_t, b_t, \varepsilon_t)$$

and

$$\widetilde{V}_b(h_t, b_t, \varepsilon_{t-1}) = \beta EV_b(h_t, b_t, \varepsilon_t)$$

To simplify the computation process, I further define the auxiliary variable x_t as t he effective asset holding which follows $x_t = \gamma p_t^h h_t + b_t$. Therefore the budget constraint 10 becomes

$$\begin{aligned} c_t + x_t + (1-\gamma)\, p_t^h h_t = & \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} \\ & + (1-\tau) w_t l_t \varepsilon_{t-1} - p_t^h C\left(h_t, h_{t-1}\right) + T_t \end{aligned} \tag{98}$$

[32]For the reference Hamilton (2020) provides rigorous proof to this equation.

Correspondingly collateral constraint becomes

$$x_t \geq 0$$

H.7.2 Decision Problems

The household solve the problem

$$V(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = \max_{h_t, x_t, l_t, c_t} U(c_t, h_t, l_t) + \beta E V(h_t, x_t, \varepsilon_t)$$

$$\begin{aligned}\text{s.t.} c_t + x_t + (1-\gamma)\, p_t^h h_t = & \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} \\ & + (1-\tau) w_t l_t \varepsilon_{t-1} - p_t^h C(h_t, h_{t-1}) + T_t\end{aligned}$$

and

$$x_t \geq 0$$

H.7.3 Solve step

1. Take the initial guess to marginal value function at time $t+1$ as $V_h(h_t, x_t, \varepsilon_t)$ and $V_x(h_t, x_t, \varepsilon_t)$

2. Solve the expectation problem on marginal value function to get risk-adjusted expected value function

   $$\widetilde{V}_h(h_t, x_t, \varepsilon_{t-1}) = \beta \Pi V_h(h_t, x_t, \varepsilon_t)$$

   and

   $$\widetilde{V}_x(h_t, x_t, \varepsilon_{t-1}) = \beta \Pi V_x(h_t, x_t, \varepsilon_t)$$

3. Assuming the collateral constraint is unconstrained, I can combine equation 76, 77 and 78 to get

   $$F(h_t, x_t, \varepsilon_{t-1}, h_{t-1}) = \frac{U_{h,t} + \widetilde{V}_h}{p_t^h \widetilde{V}_x} - (1 - \gamma + C_{h,t}) = 0$$

   Further because the unseparable utility function $U(c_t, h_t, l_t)$ is homogeneous between c_t and h_t, $U_{h,t}$ can be written as a function of $\widetilde{V}_x$

   $$U_{h,t} = (1-\phi)\left(\frac{\widetilde{V}_x}{\phi}\right)^{\frac{\phi(1-\sigma)}{\phi(1-\sigma)-1}} h_t^{\frac{\phi(1-\phi)(1-\sigma)^2}{1-\phi(1-\sigma)} + (1-\phi)(1-\sigma) - 1} \tag{99}$$

   This can be used to solve $h_t(h_{t-1}, x_t, \varepsilon_{t-1})$. The related mapping weight can also be used to map $\widetilde{V}_x(h_t, x_t, \varepsilon_{t-1})$ into $\widetilde{V}_x(h_{t-1}, x_t, \varepsilon_{t-1})$. Then $c(h_{t-1}, x_t, \varepsilon_{t-1})$ and $l(h_{t-1}, x_t, \varepsilon_{t-1})$

can be solved straightforward from

$$c\left(h_{t-1},x_t,\varepsilon_{t-1}\right)=\left(\frac{\widetilde{V}_x(h_{t-1},x_t,\varepsilon_{t-1})}{\phi}\right)^{\frac{1}{\phi(1-\sigma)-1}}h_t\left(h_{t-1},x_t,\varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{1-\phi(1-\sigma)}}\tag{100}$$

and

$$l\left(h_{t-1},x_t,\varepsilon_{t-1}\right)=\left(-\phi\frac{(1-\tau)w_t\varepsilon_{t-1}}{\kappa}\right)^{\frac{1}{\psi}}c\left(h_{t-1},x_t,\varepsilon_{t-1}\right)^{\frac{\phi(1-\sigma)-1}{\psi}}h_t\left(h_{t-1},x_t,\varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{\psi}}\tag{101}$$

4. Then the effective asset holding can be solved from budget constraint

$$\begin{aligned}x_{t-1}\left(h_{t-1},x_t,\varepsilon_{t-1}\right)=&\frac{c\left(h_{t-1},x_t,\varepsilon_{t-1}\right)+x_t+(1-\gamma)\,p_t^h h_t\left(h_{t-1},x_t,\varepsilon_{t-1}\right)}{R_t}\\&-\frac{\left[\left(1-\delta^h\right)p_t^h-\gamma R_t p_{t-1}^h\right]h_{t-1}+(1-\tau)\varepsilon_{t-1}w_t l\left(h_{t-1},x_t,\varepsilon_{t-1}\right)+T_t}{R_t}\\&+\frac{p_t^h C\left(h_t\left(h_{t-1},x_t,\varepsilon_{t-1}\right),h_{t-1}\right)}{R_t}\end{aligned}$$

Now invert above function $x_{t-1}\left(h_{t-1},x_t,\varepsilon_{t-1}\right)$ to $x_t\left(h_{t-1},x_{t-1},\varepsilon_{t-1}\right)$. After this invert process the function $h_t\left(h_{t-1},x_t,\varepsilon_{t-1}\right)$ can be mapped to $h_t\left(h_{t-1},x_{t-1},\varepsilon_{t-1}\right)$ by the function $x_t\left(h_{t-1},x_{t-1},\varepsilon_{t-1}\right)$.

5. Assuming the collateral constraint is constrained, I further define the relative Khun-Tucker multiplier as $\widetilde{\mu}_t(h_t,0,\varepsilon_{t-1})=\frac{\mu_t}{\widetilde{V}_x(h_t,0,\varepsilon_{t-1})}$ so that equation 78 becomes

$$U_{c,t}=\left(1+\widetilde{\mu}_t\right)\widetilde{V}_x$$

Therefore the equation 99 changes to

$$U_{h,t}=(1-\phi)\left(\frac{\left(1+\widetilde{\mu}_t\right)\widetilde{V}_x}{\phi}\right)^{\frac{\phi(1-\sigma)}{\phi(1-\sigma)-1}}h_t^{\frac{\phi(1-\phi)(1-\sigma)^2}{1-\phi(1-\sigma)}+(1-\phi)(1-\sigma)-1}$$

Similar to the process in step 3 this can be used to solve $h_t\left(h_{t-1},\widetilde{\mu}_t,\varepsilon_{t-1}\right)$ from

$$F\left(h_t,\widetilde{\mu}_t,\varepsilon_{t-1},h_{t-1}\right)=\frac{1}{1+\widetilde{\mu}_t}\frac{U_{h,t}+\widetilde{V}_h}{p_t^h\widetilde{V}_x}-\left(1-\gamma+C_{h,t}\right)=0$$

and equation 100 changes to

$$c\left(h_{t-1},\widetilde{\mu}_t,\varepsilon_{t-1}\right)=\left(\frac{\left(1+\widetilde{\mu}_t\right)\widetilde{V}_x(h_t,0,\varepsilon_{t-1})}{\phi h_t\left(h_{t-1},\widetilde{\mu}_t,\varepsilon_{t-1}\right)^{(1-\phi)(1-\sigma)}}\right)^{\frac{1}{\phi(1-\sigma)-1}}$$

and corresponded optimal labor supply $l\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ from equation 101.

6. The effective asset holding under the constraint scenario can be solved from budget constraint

$$\begin{aligned}
x_{t-1}\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) = & \frac{c\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) + (1-\gamma)\, p_t^h h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)}{R_t} \\
& - \frac{\left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + (1-\tau)\varepsilon_{t-1} w_t l\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) + T_t}{R_t} \\
& + \frac{p_t^h C\left(h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right), h_{t-1}\right)}{R_t}
\end{aligned}$$

Now invert above function $x_{t-1}\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ to $\widetilde{\mu}_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$. After this invert process the function $h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ can be mapped to $h_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$.[33] It is worth to notice that $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ is already known such that $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) = 0$.

7. Compare $x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ and $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ to select the largest elemental value. Then replace the unconstrained optimal housing service choice $h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ with $h_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$. Then for each grid point solve the nonlinear equation

$$\begin{aligned}
c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) = & \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} \\
& + (1-\tau) w_t \varepsilon_{t-1} \left(-\phi \frac{(1-\tau) w_t \varepsilon_{t-1}}{\kappa}\right)^{\frac{1}{\psi}} \\
& c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)^{\frac{\phi(1-\sigma)-1}{\psi}} h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{\psi}} \\
& - p_t^h C\left(h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right), h_{t-1}\right) + T_t \\
& - x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) - (1-\gamma)\, p_t^h h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)
\end{aligned}$$

Then update the marginal value function through the envelop condition 80 and 81

$$V_h(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = U_{c,t}\left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h - C_{h_{t-1}}\left(h_t, h_{t-1}\right) p_t^h\right]$$

$$V_x(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = U_{c,t} R_t$$

H.8 Solve Rational Expectation model with imperfect information

Following Baxter et al. (2011) and Hürtgen (2014), I first solve perfect information model

$$AY_t = BY_{t-1} + C^{\text{pseo}} \Xi_t \tag{102}$$

where $Y_t = \begin{bmatrix} s_t' & Ec_{t+1}' \end{bmatrix}'$ where s_t is the vector of state variable and c_t is the vector of control variable. Ξ_t is the vector of pseudo-shock and composed with fundamental shock w_t

[33] Here I use c in superscript as the notation to "constrained".

and noisy shock v_t such that $\Xi_t = \begin{bmatrix} w_t' & v_t' \end{bmatrix}'$. The effect of shock C^{pseo} naturally becomes $C^{\text{pseo}} = \begin{bmatrix} P^s H\eta \\ P^s\epsilon \end{bmatrix}$ where P^s is the Kalman gain from equation 97. This linear model can be easily solved by Klein (2000) to yield $Y_t = PY_{t-1} + Q\Xi_t$. Take partition on P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

It is widely known that to solve the linear rational expectation model we pre-impose the restriction that $P_{12} = 0$ and $P_{22} = 0$. Further because of the holding of CEQ under first-order perturbation method, the policy function of control variables c_t will follow

$$c_t = P_{21}s_{t-1|t-1} + Q_2^w w_t + Q_2^v v_t \tag{103}$$

where Q_2^w and Q_2^v are subset of Q^w and Q^v which comes from Q such that $Q = \begin{bmatrix} Q^w & Q^s \end{bmatrix}$. Plug equation 103 into partition of equation 102 but replace $C^{\text{pseo}}\Xi_t$ with true fundamental shock process ηw_t such that

$$A_{11}s_t + A_{12}Ec_{t+1} = B_{11}s_{t-1} + B_{12}c_t + \eta w_t$$

$$A_{11}s_t + A_{12}P_{21}s_{t|t} = B_{11}s_{t-1} + B_{12}\left(P_{21}s_{t-1|t-1} + Q_2^w w_t + Q_2^v v_t\right) + \eta w_t \tag{104}$$

It is worth to notice that here I use the first ns linear equations of equation 102 which is not free of choice yet a simplification in notation. The basic purpose now is to solve the law of motion of perceived state variable $s_{t|t}$ therefore we need ns "core" linear equations related to state variables to pin down ns state variable $s_{t|t}$. The word "core" refers to those equations that affect state variables directly, or more specifically, the law of motion of state variables. For instance, if we want to select one out of two linear equations in 102, 1) Euler equation $-\sigma\widetilde{c}_t = \widetilde{R}_t - \sigma\widetilde{c}_{t+1}$ and 2) Law of Motion of Capital $K\widetilde{k}_t = I\widetilde{I}_t + K\widetilde{k}_{t-1}$, which is used in equation 104, we should select the equation 2 because the equation 1 is implicitly comprised in the mapping from $s_{t-1|t-1}$ to c_t in equation 103. Otherwise we redundantly use the linear constraints and the matrix $A_{11} + A_{12}P_{21}G$ in equation 107 will not be well-defined.

Furthermore, the law of motion of perception of unobservable variables could be derived through plugging equation 96 into equation 97 to yield

$$X_{t|t} = A^s X_{t-1|t-1} + P^s H X_t + P^s \epsilon v_t \tag{105}$$

However, It is not all the state variables s_t that is unobservable, so I rewrite the law of motion of perceived state variable $s_{t|t}$ below. Without loss of generality, I assume the unobservable state variables lay on the last nx row (in this paper $nx = 10$ as equation 93 shows).

$$s_{t|t} = Fs_{t-1|t-1} + Gs_t + G_{P^s}\epsilon v_t \tag{106}$$

where $F = \begin{bmatrix} 0 & 0 \\ 0 & A^s \end{bmatrix}$, $G = \begin{bmatrix} I & 0 \\ 0 & P^s H \end{bmatrix}$ and $G_{P^s} = \begin{bmatrix} 0 \\ P^s \end{bmatrix}$.

And then plug equation 106 back to above equation 104

$$A_{11}s_t + A_{12}P_{21}\left(Fs_{t-1|t-1} + Gs_t + G_{P^s}\epsilon v_t\right) = B_{11}s_{t-1} + B_{12}\left(P_{21}s_{t-1|t-1} + Q_2^w w_t + Q_2^v v_t\right) + \eta w_t$$

$$\begin{aligned}(A_{11} + A_{12}P_{21}G)\, s_t = B_{11}s_{t-1} + (B_{12}P_{21} - A_{12}P_{21}F)\, s_{t-1|t-1} + (B_{12}Q_2^w + \eta)\, w_t \\ + (B_{12}Q_2^v - A_{12}P_{21}G_{P^s}\epsilon)\, v_t\end{aligned} \tag{107}$$

Simplify above equation to

$$\widetilde{Y}_t = M\widetilde{Y}_{t-1} + D\Xi_t \tag{108}$$

where

$$\widetilde{Y}_t = \begin{bmatrix} s_t \\ s_{t|t} \\ c_t \end{bmatrix}$$

$$A_L = \begin{bmatrix} I & 0 & 0 \\ -G & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$B_L = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} & 0 \\ 0 & F & 0 \\ 0 & P_{21} & 0 \end{bmatrix}$$

$$C_L = \begin{bmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ 0 & P^s\epsilon \\ Q_2^w & Q_2^v \end{bmatrix}$$

$M = A_L^{-1}B_L$, $D = A_L^{-1}C_L$, $\widetilde{P}_{11} = (A_{11} + A_{12}P_{21}G)^{-1}B_{11}$,

$\widetilde{P}_{12} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}P_{21} - A_{12}P_{21}F)$, $\widetilde{Q}_{11} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}Q_2^w + \eta)$

and

$\widetilde{Q}_{12} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}Q_2^v - A_{12}P_{21}G_{P^s}\epsilon)$.

H.9 Solve Rational Expectation model with imperfect information in second order

H.9.1 Necessity

Given the utility function $U_t(c_t, h_t)$ where c_t is the nondurable consumption and h_t is the residential asset, we can take taylor expansion around the steady states to yield

$$U_t(c_t, h_t) \approx \overline{U} + U_c\widetilde{c}_t + U_h\widetilde{h}_t + \frac{1}{2}U_{cc}\widetilde{c}_t^2 + \frac{1}{2}U_{hh}\widetilde{h}_t^2 + U_{hc}\widetilde{c}_t\widetilde{h}_t + \circ_t$$

where $\circ_t$ is the higher order term. However I cannot use $\widetilde{c}_t$ as the result in first order because of two reason:

1) the precautionary saving motive will disappear as now $\frac{\partial \widetilde{c}_t}{\partial \sigma^2} = 0$. Then the quadratic term will be misspecified in dynamic path and the calculated welfare will be incorrect.

2) In the heterogeneous agent model, there is no steady state for each household and above taylor expansion will not exist.

Therefore I propose the method below to conduct the second-order perturbation under imperfect information.

The main trick I used is that the certainty equivalence will still hold, only in the information updated process in second order perturbation. Now consider the policy function as

$$\begin{aligned} y_t &= p_1 y_{t-1} + p_2 y_{t-1}^2 + \sigma p_3 y_{t-1}\varepsilon_t + k_1 x_{t-1|t-1} + k_2 x_{t-1|t-1}^2 + k_3 y_{t-1} x_{t-1|t-1} + k_4 x_{t-1|t-1}\varepsilon_t \\ &+ q_1\sigma\varepsilon_t + q_2\sigma^2 \end{aligned}$$

where y_t is the standard variables that we know it perfectly but x_t is the variable that we cannot perfectly observe. σ^2 represents the change in the variance of shock term and q_2 is just the precautionary saving effect.

The only difference between imperfect information model and perfect information model is that all the policy related to perception, k_1, k_2, k_3... are affected by σ^2 as people form their expectation through the variance of the shock. However, because it is affect by the quadratic form of variance, σ^2, instead of standard derivation σ, its final effect is third order and in second order case. For instance, $\frac{\partial k_1}{\partial \sigma}\Big|_{\sigma=0} = 0$ holds, therefore $\frac{\partial^2 k_1 x_{t-1|t-1}}{\partial\sigma\partial\sigma} = 0$ at steady states.

H.9.2 Steps

Write the system of equations as

$$G(x_{t-1}, y_t, x_t, y_{t+1}, \sigma) = F(x_{t-1}, u_t, u_{t+1}, \sigma) = 0$$

However since the η can be calculated from the covariance matrix of the shock ϵ_t (a shock on the variance of the model. It is a nk vector yet if we consider it is the shock on the variance u_t, we can set some elements in ϵ_t as zero), we can leave it into Σ_ϵ.

Take second-order approximation

$$\begin{aligned}F(x_{t-1},u_t,u_{t+1},\sigma) &= F^1(x_{t-1},u_t,u_{t+1},\sigma)\\ &+\frac{1}{2}\left[F_{xx}\left(x_{t-1}\otimes x_{t-1}\right)+F_{uu}\left(u\otimes u\right)+F_{u'u'}\left(u'\otimes u'\right)+F_{\sigma\sigma}\sigma^2\right]\\ &+F_{xu}\left(x\otimes u\right)+F_{xu'}\left(x\otimes u'\right)+F_{y\sigma}\sigma x+F_{uu'}\left(u\otimes u'\right)+F_{u\sigma}u_t\sigma+F_{u'\sigma}u'\sigma\end{aligned}$$

Because u and u' are the linear innovation to the state variable x and x', F_u is just a constant matrix such that $F_u = G_{x'}\frac{\partial x'}{\partial u}+G_y\frac{\partial y}{\partial u}+G_{y'}\frac{\partial y'}{\partial x}\frac{\partial x}{\partial u}$. This can be verified through the second-order policy functions

$$x_t = \frac{1}{2}h_{\sigma\sigma}\sigma^2+h_x x_{t-1}+h_u u_t+\frac{1}{2}h_{xx}\left(x_{t-1}\otimes x_{t-1}\right)+\frac{1}{2}h_{uu}\left(u_t\otimes u_t\right)+h_{xu}\left(x_{t-1}\otimes u_t\right)$$

and

$$y_t = \frac{1}{2}g_{\sigma\sigma}\sigma^2+g_x x_{t-1}+g_u u_t+\frac{1}{2}g_{xx}\left(x_{t-1}\otimes x_{t-1}\right)+\frac{1}{2}g_{uu}\left(u_t\otimes u_t\right)+g_{xu}\left(x_{t-1}\otimes u_t\right)$$

$$y_{t+1} = \frac{1}{2}g_{\sigma\sigma}\sigma^2+g_x x_t+g_u u_{t+1}+\frac{1}{2}g_{xx}\left(x_t\otimes x_t\right)+\frac{1}{2}g_{uu}\left(u_{t+1}\otimes u_{t+1}\right)+g_{xu}\left(x_t\otimes u_{t+1}\right)$$

Therefore $F_{yu}=F_{yu'}=F_{uu'}=F_{u\sigma}u_t=F_{u'\sigma}=0$. Simplify to

$$\begin{aligned}\mathbb{E}_t\left\{F(x_{t-1},u_t,u_{t+1},\sigma)\right\} &= \mathbb{E}_t\left\{F^1(x_{t-1},u_t,u_{t+1},\sigma)\right\}\\ &+\frac{1}{2}\left[F_{xx}\left(x_{t-1}\otimes x_{t-1}\right)+F_{uu}\left(u\otimes u\right)+F_{u'u'}\sigma^2\overrightarrow{\Sigma}_\epsilon+F_{\sigma\sigma}\sigma^2\right]\\ &+F_{xu}\left(x\otimes u\right)+F_{u\sigma}u_t\sigma+F_{y\sigma}\sigma x\end{aligned}$$

To understand the $\overrightarrow{\Sigma}_\epsilon$ and $\overline{\sigma}=0$, let use write u_t as $u_t=\varepsilon_t+\sigma\epsilon_t$ where $\Sigma_\varepsilon = I$ and $\overrightarrow{\Sigma}_\epsilon = \operatorname{vec}(\Sigma_\epsilon)$. The shock ε_t represents the first order shock that household does not take into account its variance into policy function (yet it indeed has the variance). ϵ_t is the second order shock that household takes into account its variance and has precautionary saving motive. Therefore the existence of $\overrightarrow{\Sigma}_\epsilon$ matches that meaning that we only care about the add-on variance of u_t that has second order effect. Therefore the first order effect of u_t or u_{t+1} is zero (or even not zero is already considered in $F^1(x_{t-1},u_t,u_{t+1},\sigma)$).

Further, the chain rule in partial derivative can only work when the "differential point" is fixed. For instance, the condition

$$x_{t-1} = \frac{1}{2}h_{\sigma\sigma}\sigma^2+h_x x_{t-2}+h_u u_{t-1}+\frac{1}{2}h_{xx}\left(x_{t-2}\otimes x_{t-2}\right)+\frac{1}{2}h_{uu}\left(u_{t-1}\otimes u_{t-1}\right)+h_{xu}\left(x_{t-1}\otimes u_{t-1}\right)$$

also hold. Does $\frac{\partial G}{\partial\sigma^2} = ... + \frac{\partial G}{\partial x_{t-1}}\frac{\partial x_{t-1}}{\partial x_{t-2}}\frac{\partial x_{t-2}}{\partial\sigma^2}$ hold? NO! Because $\frac{\partial x_{t-1}}{\partial x_{t-2}}$ and $\frac{\partial x_{t-2}}{\partial\sigma^2}$ exist is conditional on the the condition that we know x_{t-2}, which we do not know.

Now let me solve them one by one. Firstly, write the function of x_t, y_t and y_{t+1}[34]

$$
\begin{aligned}
F_{xx} = {} & G_y g_{xx} + G_{x'} h_{xx} + G_{y'} \left[g_x h_{xx} + g_{xx} \left(h_x \otimes h_x \right) \right] \\
& + G_{xx} \left(I_{nk} \otimes I_{nk} \right) + G_{xy} \left(I_{nk} \otimes g_x \right) + G_{xx'} \left(I_{nk} \otimes h_x \right) + G_{xy'} \left(I_{nk} \otimes g_x h_x \right) \\
& + G_{yx} \left(g_x \otimes I_{nk} \right) + G_{yy} \left(g_x \otimes g_x \right) + G_{yx'} \left(g_x \otimes h_x \right) + G_{yy'} \left(g_x \otimes g_x h_x \right) \\
& + G_{x'x} \left(h_x \otimes I_{nk} \right) + G_{x'y} \left(h_x \otimes g_x \right) + G_{x'x'} \left(h_x \otimes h_x \right) + G_{x'y'} \left(h_x \otimes g_x h_x \right) \\
& + G_{y'x} \left(g_x h_x \otimes I_{nk} \right) + G_{y'y} \left(g_x h_x \otimes g_x \right) + G_{y'x'} \left(g_x h_x \otimes h_x \right) + G_{y'y'} \left(g_x h_x \otimes g_x h_x \right) \\
& = 0
\end{aligned}
$$

Rewrite it as

$$
\begin{bmatrix} G_{x'} + G_{y'} g_x & G_y \end{bmatrix} \begin{bmatrix} h_{xx} \\ g_{xx} \end{bmatrix} + \begin{bmatrix} 0 & G_{y'} \end{bmatrix} \begin{bmatrix} h_{xx} \\ g_{xx} \end{bmatrix} \left(h_x \otimes h_x \right) + B_x = 0
$$

Secondly

$$
\begin{aligned}
F_{uu} = {} & G_y g_{uu} + G_{x'} h_{uu} + G_{y'} \left[g_x h_{uu} + g_{xx} \left(h_u \otimes h_u \right) \right] \\
& + G_{yy} \left(g_u \otimes g_u \right) + G_{yx'} \left(g_u \otimes h_u \right) + G_{yy'} \left(g_u \otimes g_x h_u \right) \\
& + G_{x'y} \left(h_u \otimes g_u \right) + G_{x'x'} \left(h_u \otimes h_u \right) + G_{x'y'} \left(h_u \otimes g_x h_u \right) \\
& + G_{y'x'} \left(g_x h_u \otimes h_u \right) + G_{y'y} \left(g_x h_u \otimes g_u \right) + G_{y'y'} \left(g_x h_u \otimes g_x h_u \right) \\
& = 0
\end{aligned}
$$

Rewrite it as

$$
\begin{bmatrix} G_{x'} + G_{y'} g_x & G_y \end{bmatrix} \begin{bmatrix} h_{uu} \\ g_{uu} \end{bmatrix} + B_{u1} = 0
$$

Thirdly

$$
\begin{aligned}
F_{xu} = {} & G_y g_{xu} + G_{x'} h_{xu} + G_{y'} \left[g_x h_{xu} + g_{xx} \left(h_x \otimes h_u \right) \right] \\
& + G_{xy} \left(I_{nk} \otimes g_u \right) + G_{xx'} \left(I_{nk} \otimes h_u \right) + G_{xy'} \left(I_{nk} \otimes g_x h_u \right) \\
& + G_{yy} \left(g_x \otimes g_x \right) + G_{yx'} \left(g_x \otimes h_u \right) + G_{yy'} \left(g_x \otimes g_x h_u \right) \\
& + G_{x'y} \left(h_x \otimes g_u \right) + G_{x'x'} \left(h_x \otimes h_u \right) + G_{x'y'} \left(h_x \otimes g_x h_u \right) \\
& + G_{y'y} \left(g_x h_x \otimes g_u \right) + G_{y'x'} \left(g_x h_x \otimes h_u \right) + G_{y'y'} \left(g_x h_x \otimes g_x h_u \right) \\
& = 0
\end{aligned}
$$

Rewrite it as

$$
\begin{bmatrix} G_{x'} + G_{y'} g_x & G_y \end{bmatrix} \begin{bmatrix} h_{xu} \\ g_{xu} \end{bmatrix} + B_{u2} = 0
$$

[34] $\frac{1}{2} \frac{\partial^2 h_{xx} (x_{t-1} \otimes x_{t-1})}{\partial x_{t-1} \partial x_{t-1}} = \frac{1}{2} 2 h_{xx} = h_{xx}$

Forthly

$$F_{\sigma\sigma} = G_x h_{\sigma\sigma} + G_y \left[g_{\sigma\sigma} + g_x h_{\sigma\sigma}\right] + G_{x'} \left[h_{\sigma\sigma} + h_x h_{\sigma\sigma}\right] + G_{y'} \left[g_{\sigma\sigma} + g_x h_{\sigma\sigma} + g_x h_x h_{\sigma\sigma}\right]$$

where h_{u,σ^2} and g_{u,σ^2} is solved from the perturbation around the first order policy function. Even though $u_t = \varepsilon_t + \sigma\epsilon_t$, because at time t u_t is already realized, there is no expectation in front ϵ_t, $G_{yy}\left(g_u \otimes g_u\right)\left(\epsilon_t \otimes \epsilon_t\right) = G_y g_{uu}\left(I_{nu} \otimes I_{nu}\right)\left(\epsilon_t \otimes \epsilon_t\right) = ... = 0$ will hold around the steady state $\epsilon = 0$. Throughout the calculation of F_{xx}, F_{uu}, F_{xu} and $F_{\sigma\sigma}$, we do not need to care about the shock coefficient η because $G_{uu} = 0$. All of its effect is already implied in h_u and g_u.

Furthermore, there is no higher order expectation effect here (up to second order) such as $G_y g_{u,\sigma^2}\overline{u} + G_{x'} h_{u,\sigma^2}\overline{u} + G_{y'}\left[g_{u,\sigma^2} + g_x h_{u,\sigma^2}\right]\overline{u}$ as $\overline{u} = 0$. Yet higher order approximation will have this problem. Meanwhile remember that in first order even we have $\overline{u} > 0$, because $\overline{\sigma} = 0$, the first order effect $G_y g_{u,\sigma}\overline{u\sigma} = h_{u,\sigma}\overline{u\sigma} = 0$. The reason is that the policy will not derivative until second order or higher because of σ^2, the variance is second order. Then the effect of this derivation, derivation in dynamic with x_t or $x_t \otimes x_t$, is at least third-order which will be zero under second-order approximation.

and

$$F_{u'u'} = G_{y'} g_{uu} + G_{y'y'}\left(g_u \otimes g_u\right)$$

Therefore

$$F_{u'u'}\overrightarrow{\Sigma}_\epsilon\sigma^2 + F_{\sigma\sigma}\sigma^2 = \left(F_{u'u'}\overrightarrow{\Sigma}_\epsilon + F_{\sigma\sigma}\right)\sigma^2 = 0$$

holds, which is equivalent to

$$F_{u'u'}\overrightarrow{\Sigma}_\epsilon + F_{\sigma\sigma} = 0$$

Rearrange to

$$\begin{bmatrix} G_x + G_{x'} + G_{y'} g_x & G_y + G_{y'} \end{bmatrix} \begin{bmatrix} h_{\sigma\sigma} \\ g_{\sigma\sigma} \end{bmatrix} + \left\{G_{y'} g_{uu} + G_{y'y'}\left(g_u \otimes g_u\right)\right\}\overrightarrow{\Sigma}_\epsilon = 0$$

Taylor expansion around

$$K(z_{t-1}, u_t, u_{t+1}, \sigma) = L(z_{t-1}, y_t, z_t, y_{t+1}, \sigma) = 0$$

Guess policy function

$$z_t = \frac{1}{2}p_{\sigma\sigma}\sigma^2 + p_z z_{t-1} + p_u u_t + \frac{1}{2}p_{zz}\left(z_{t-1} \otimes z_{t-1}\right) + \frac{1}{2}p_{uu}\left(u_t \otimes u_t\right) + p_{zu}\left(z_{t-1} \otimes u_t\right)$$

where $z_t = \begin{bmatrix} x_t \\ x_{t|t} \end{bmatrix}$ with the known function

$$\begin{aligned} y_t &= \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_x x_{t-1|x-1} + g_u u_t + \frac{1}{2} g_{xx}\left(x_{t-1|t-1} \otimes x_{t-1|t-1}\right) + \frac{1}{2} g_{uu}\left(u_t \otimes u_t\right) + g_{xu}\left(x_{t-1|t-1} \otimes u_t\right) \\ &= \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_x m_2 z_{t-1} + g_u u_t + \frac{1}{2} g_{xx}\left(m_2 \otimes m_2\right)\left(z_{t-1} \otimes z_{t-1}\right) + \frac{1}{2} g_{uu}\left(u_t \otimes u_t\right) + g_{xu}\left(m_2 \otimes I_{nu}\right)\left(z_{t-1} \otimes \right. \end{aligned}$$

and

$$\begin{aligned} y_{t+1} &= \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_x x_{t|t} + g_u u_{t+1} + \frac{1}{2} g_{xx}\left(x_{t|t} \otimes x_{t|t}\right) + \frac{1}{2} g_{uu}\left(u_{t+1} \otimes u_{t+1}\right) + g_{xu}\left(x_{t|t} \otimes u_{t+1}\right) \\ &= \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_x m_2 z_t + g_u u_{t+1} + \frac{1}{2} g_{xx}\left(m_2 \otimes m_2\right)\left(z_t \otimes z_t\right) + \frac{1}{2} g_{uu}\left(u_{t+1} \otimes u_{t+1}\right) + g_{xu}\left(m_2 \otimes I_{nu}\right)\left(z_t \otimes \right. \end{aligned}$$

where $m_2 = \begin{bmatrix} 0_{nk} & I_{nk} \end{bmatrix}$

Take second-order approximation

$$\begin{aligned} K(z_{t-1}, u_t, u_{t+1}, \sigma) = {} & K^1(z_{t-1}, u_t, u_{t+1}, \sigma) \\ & + \frac{1}{2}\left[K_{zz}\left(z \otimes z\right) + K_{uu}\left(u \otimes u\right) + K_{u'u'}\left(u' \otimes u'\right) + K_{\sigma\sigma}\sigma^2\right] \\ & + K_{zu}\left(z \otimes u\right) + K_{zu'}\left(z \otimes u'\right) + K_{z\sigma}\sigma z + K_{uu'}\left(u \otimes u'\right) + K_{u\sigma} u_t \sigma + K_{u'\sigma} u' \sigma \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}_t\left\{K(z_{t-1}, u_t, u_{t+1}, \sigma)\right\} = {} & \mathbb{E}_t\left\{K^1(z_{t-1}, u_t, u_{t+1}, \sigma)\right\} \\ & + \frac{1}{2}\left[K_{zz}\left(z \otimes z\right) + F_{uu}\left(u \otimes u\right) + F_{u'u'}\overrightarrow{\Sigma}_{\epsilon}\sigma^2 + F_{\sigma\sigma}\sigma^2\right] \\ & + K_{zu}\left(z \otimes u\right) + K_{z\sigma}\sigma z + K_{u\sigma} u_t \sigma \end{aligned}$$

Now let me solve them one by one. Firstly, write the function of x_t, y_t and y_{t+1}

$$\begin{aligned} K_{zz} &= L_y g_{xx}\left(m_2 \otimes m_2\right) + L_{z'} p_{zz} + L_{y'}\left[g_x m_2 p_{zz} + g_{xx}\left(h_x \otimes h_x\right)\left(m_2 \otimes m_2\right)\left(p_z \otimes p_z\right)\right] \\ &\quad + L_{zz}\left(I_{2nk} \otimes I_{2nk}\right) + L_{zy}\left(I_{2nk} \otimes g_x m_2\right) + L_{zz'}\left(I_{2nk} \otimes p_z\right) + L_{zy'}\left(I_{2nk} \otimes g_x m_2 p_z\right) \\ &\quad + L_{yz}\left(g_x m_2 \otimes I_{2nk}\right) + L_{yy}\left(g_x m_2 \otimes g_x m_2\right) + L_{yz'}\left(g_x m_2 \otimes p_z\right) + L_{yy'}\left(g_x m_2 \otimes g_x m_2 p_z\right) \\ &\quad + L_{z'z}\left(p_z \otimes I_{2nk}\right) + L_{z'y}\left(p_z \otimes g_x m_2\right) + L_{z'z'}\left(p_z \otimes p_z\right) + L_{z'y'}\left(p_z \otimes g_x m_2 p_z\right) \\ &\quad + L_{y'z}\left(g_x m_2 p_z \otimes I_{2nk}\right) + L_{y'y}\left(g_x m_2 p_z \otimes g_x m_2\right) + L_{y'z'}\left(g_x m_2 p_z \otimes p_z\right) + L_{y'y'}\left(g_x m_2 p_z \otimes g_x m_2 p_z\right) \\ &= 0 \end{aligned}$$

Then p_{zz} is solved by

$$\left(L_{z'} + L_{y'} g_x m_2\right) p_{zz} + C_x = 0$$

Secondly,

$$
\begin{aligned}
K_{uu} &= L_y g_{uu} + L_{z'} p_{uu} + L_{y'} \left[g_x m_2 p_{uu} + g_{xx} \left(m_2 \otimes m_2 \right) \left(p_u \otimes p_u \right) \right] \\
&+ L_{yy} \left(g_u \otimes g_u \right) + L_{yz'} \left(g_u \otimes p_u \right) + L_{yy'} \left(g_u \otimes g_x m_2 p_u \right) \\
&+ L_{z'y} \left(p_u \otimes g_u \right) + L_{z'z'} \left(p_u \otimes p_u \right) + L_{z'y'} \left(p_u \otimes g_x m_2 p_u \right) \\
&+ L_{y'y} \left(g_x m_{2v} p_u \otimes g_u \right) + L_{y'z'} \left(g_x m_2 p_u \otimes p_u \right) + L_{y'y'} \left(g_x m_2 p_u \otimes g_x m_2 p_u \right) \\
&= 0
\end{aligned}
$$

Then p_{uu} is solved by

$$
\left(L_{z'} + L_{y'} g_x m_2 \right) p_{uu} + C_{u1} = 0
$$

Thirdly,

$$
\begin{aligned}
K_{zu} &= L_y g_{xu} \left(m_2 \otimes I_{nu} \right) + L_{z'} p_{zu} + L_{y'} \left[g_x m_2 p_{zu} + g_{xx} \left(m_2 \otimes m_2 \right) \left(p_z \otimes p_u \right) \right] \\
&+ L_{zy} \left(I_{2nk} \otimes g_u \right) + L_{zz'} \left(I_{2nk} \otimes p_u \right) + L_{zy'} \left(I_{2nk} \otimes g_x m_2 p_u \right) \\
&+ L_{yy} \left(g_x m_2 \otimes g_u \right) + L_{yz'} \left(g_x m_2 \otimes p_u \right) + L_{yy'} \left(g_x m_2 \otimes g_x m_2 p_u \right) \\
&+ L_{z'y} \left(p_z \otimes g_u \right) + L_{z'z'} \left(p_z \otimes p_u \right) + L_{z'y'} \left(p_z \otimes g_x m_2 p_u \right) \\
&+ L_{y'y} \left(g_x m_2 p_z \otimes g_u \right) + L_{y'z'} \left(g_x m_2 p_z \otimes p_u \right) + L_{y'y'} \left(g_x m_2 p_z \otimes g_x m_2 p_u \right) \\
&= 0
\end{aligned}
$$

Then p_{zu} is solved by

$$
\left(L_{z'} + L_{y'} g_x m_2 \right) p_{zu} + C_{u2} = 0
$$

Finally we have two approximations

$$
K_{\sigma\sigma} = L_z p_{\sigma\sigma} + L_y \left[g_{\sigma\sigma} + g_x m_2 p_{\sigma\sigma} \right] + L_{z'} \left[p_{\sigma\sigma} + p_z p_{\sigma\sigma} \right] + L_{y'} \left[g_{\sigma\sigma} + g_x m_2 p_{\sigma\sigma} + g_x m_2 p_z p_{\sigma\sigma} \right]
$$

and

$$
K_{u'u'} = L_{y'} g_{uu} + L_{y'y'} \left(g_u \otimes g_u \right)
$$

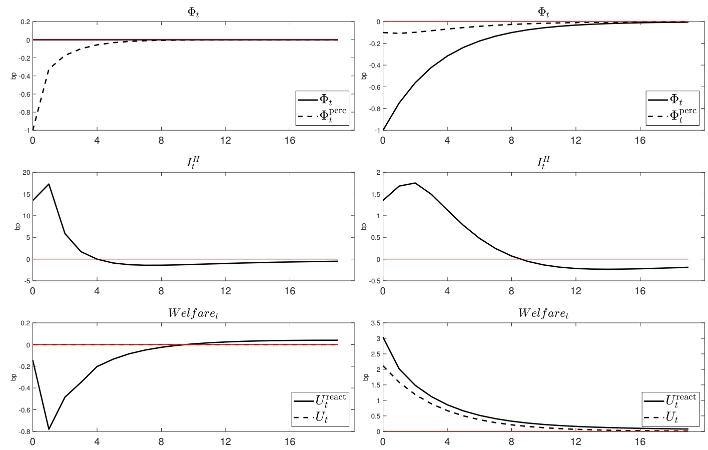
Because of

$$
K_{\sigma\sigma} + K_{u'u'} \overrightarrow{\Sigma}_{\epsilon} = 0
$$

The $p_{\sigma\sigma}$ is solved by

$$
\left[L_z + L_y g_x m_2 + L_{z'} \left(1 + p_z \right) + L_{y'} g_x m_2 \left(1 + p_z \right) \right] p_{\sigma\sigma} + C_{\sigma} = 0
$$

H.10 Arguments to fake news and inefficiency



Φ_t
Φ_t
Φ_t^{perc}
I_t^H
$Welfare_t$
U_t^{react}
U_t
bp