Overbuilding and Underinvestment over Housing Boom-Bust Cycles

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Abstract

In this paper, I unveil a novel mechanism through which a housing market boom can lead to a deep recession by decreasing the physical investment and rendering capital scarce. This inefficiency arises from a crowding-out effect: the available liquidity, which could otherwise be channeled into firms' capital investments (e.g., factories, equipment, R&D), is redirected toward the residential sector. The crowded-out physical investment subsequently amplifies the losses of the bust and prolongs the duration of the recession. Employing a new identification method of a shock that generates housing boom-bust cycles via a structural vector regression model, this paper empirically verifies the crowding-out effect and finds that a 2% jump in housing prices can crowd out 1% physical investment at the peak. Then, I develop a heterogeneous household model to quantify this welfare effects of this novel mechanism, It documents that the crowding-out effect can account for up to 13% of the welfare losses during the recession period. Finally, I show that a macroprudential policy targeting the overheated housing market can significantly alleviate the crowding-out effect and welfare losses.

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1 Introduction

The economic downturn that followed the Great Recession in 2007 generated a significant upswing in unemployment rates and declines in output, consumption, and investment.[1] Numerous researchers have endeavored to comprehend the source of this recession and explore the mechanisms through which the source spread. The consensus among many scholars posits that the boom and subsequent bust in the housing market exacerbated the collapse of financial markets, leading to a recession, yet people have not reached agreement on how this boom-bust cycle led to the recession. The Great Recession lingered for an extended period, a phenomenon some attribute to behavioral inefficiency such as self-fulfilling equilibrium and “animal spirits”[2], liquidity traps[3], and the zero lower bound (ZLB).[4] These channels typically suggest that the fallout from the housing market bust had tangible economic impacts, mainly through financial friction on the supply side by influencing production. Moreover, on the demand side, real estate served as collateral, enabling households to borrow money and smooth consumption patterns[5], but after the recession, the fall in price of real estate significantly eroded household wealth, adversely affecting the real economy. In this paper, I propose a new mechanism, through which a housing market boom preceding a recession can precipitate the economic downturn and contribute to economic malaise.

First, we turn our attention to the statistical characteristics of the data, which offer insights into the mechanisms discussed in this paper. Figure 1 displays the quantity of both nonresidential and residential investment as a share of GDP, spanning from 1975 to 2012. The shaded areas represent the boom periods as per NBER business cycle data, with the numbers above signifying the average shares during these booms. Notably, the boom that began in 2001, subsequently interrupted by the Great Recession at the close of 2007, features a peak in residential investment and a trough in nonresidential investment.

The focus of this study is on the mechanism through which an increase in investment in residential assets, which lacks support from underlying economic fundamentals, generates capital scarcity. For simplicity, I define this investment without fundamental support as overbuilding. Limited theoretical frameworks have been employed to elucidate how a housing market boom might absorb substantial liquidity. When this boom is inefficient and becomes a bubble,[6] the

[1] as examined by Mian and Sufi (2010) and Grusky et al. (2011).

[2] Islam and Verick (2011) and Cochrane (2011) discuss this problem.

[3] Brunnermeier (2009), Ivashina and Scharfstein (2010) and Jermann and Quadrini (2012)argue that the lack of liquidity of financial institution, mostly referring to the commercial bank, helps the crisis diffuse around and induce large recession.

[4] Christiano et al. (2015), Fisher (2015), Guerrieri and Lorenzoni (2017) and Bayer et al. (2019) did these works.

[5] Eggertsson and Krugman (2012), Mian and Sufi (2010), Mian and Sufi (2014) and Qian (2023)discuss this problem. Households extracted their equity via collateral during the boom period, which substantially increased consumption. This constructed a mirage through general equilibrium. When the bust came, people struggled against rapid constraint tightening, which led to the Great Recession.

[6] Throughout this paper, I define a bubble as an inefficient boom, i.e., a boom that is not supported by fundamentals.

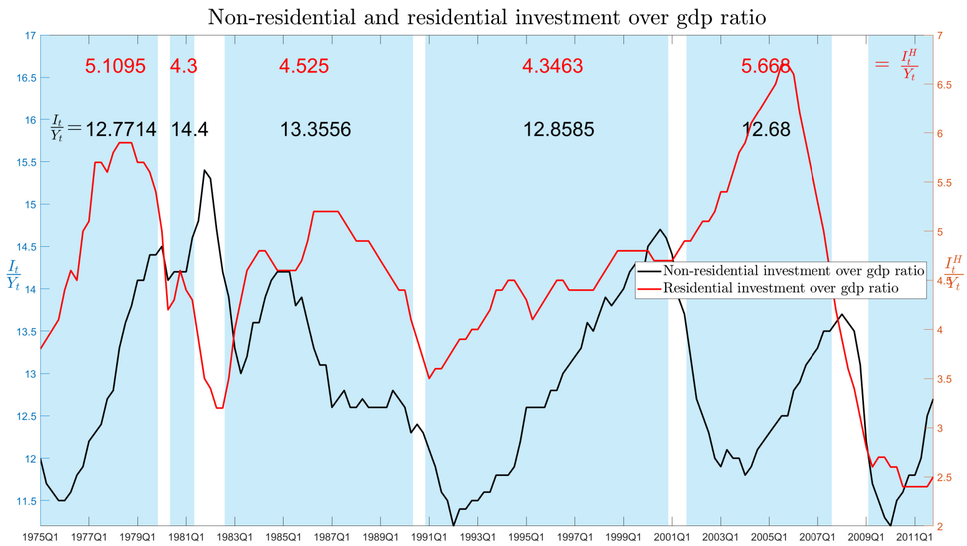


Figure 1: Nonresidential and residential investment share of GDP, over NBER Boom

available liquidity, which could otherwise be channeled into firms' capital investments (e.g., factories, equipment, R&D), is redirected toward the residential sector. This can be revealed by a significant negative correlation $\mathrm{corr}(\frac{I_t}{y_t}, \frac{I_t^H}{y_t}) = -0.3355$ between residential and nonresidential investment over GDP ratio.

Notably, in this paper, I introduce imperfect information (misguided household beliefs) to inflate the bubble because it is the most convenient and suitable way to generate the crowding-out effect. However, the crowding-out effect is not unique to imperfect information and other factors, such as real frictions (e.g., financial accelerators, shadow banking, search and matching, moral hazard) and behavioral frictions (sentiment shocks, irrational expectations) can also produce inefficiency. This suboptimal reallocation of liquidity within financial institutions results in inefficiencies compared to a first-best allocation scenario. During a housing market boom, financial institutions exhibit a proclivity for directing loans toward the household sector, at the expense of other sector. Owing to the increased influx of liquidity into the residential real estate market, there is a concomitant reduction in the allocation of liquidity to physical capital, which is used by the supply side of the economy. This effect is especially pronounced when the liquidity supply is inelastic and resistant to expansion, as the amount that flows into real estate market will not be compensated by increased total liquidity but decreased physical investment. Furthermore, due to general equilibrium effects, the increase in consumption during a housing boom amplifies the reduction in physical investment. This phenomenon can be comprehended through the goods market clearing condition, which allocates output into three categories: housing, consumption, and physical investment. Assuming a fixed labor supply and predetermined capital, a jump in investment in residential assets and a rise in consumption yield decreased physical investment---a situation I call the *crowding-out effect.*

This paper first empirically demonstrates the existence of the crowding-out effect and its significance in explaining the shortfall in physical investment subsequent to a housing market boom. To probe the intricacies of the housing market's boom-and-bust cycle, I introduce a novel identification strategy aimed at exploring the effect of a news (to housing price inflation) shock in the context of imperfect information. Upon receiving news that housing prices are expected to rise in the future, households will react immediately, as this alters their expectations. Specifically, the crowding-out effect is not limited to news shocks and imperfect information, although these provide the most straightforward illustration of the effect. A single news shock can generate a prolonged housing market boom, but to produce the same boom with other shocks, I require strong assumptions regarding their persistence.

Within this context, households cannot verify whether news about future inflation in housing price is true or fake before the news is realized, because of imperfect information. I call the fake news shock as a news shock that does not realize when it should. Because the fake news shock ultimately does not change the fundamentals, people's reactions to it are suboptimal and inefficient. The subsequent market bust ensues when households eventually find that the news is fake, thereby prompting the adjustment of market dynamics. The empirical results reveal that a 2% increase in housing prices can yield a 1% decrease in physical investment at the peak. After the market busts, a 1% decline in housing prices correlates with a 0.1% decrease in consumption, which implies moderate welfare loss.

This paper then employs an Bewley-Huggett-Aiyagari model, which characterizes housing and consumption, financial friction, and heterogeneous household, to offer rigorous analytical results explicating the formation of physical capital scarcity. Each of these three elements plays a pivotal role in analyzing and determining the crowding-out effect. As asserted in literature, housing and consumption are closely connected with each other, and together, they (C and H) pin down the crowding-out effect (ΔI), given the goods market cleaning condition $Y = C + I + \Delta H$. The presence of financial friction and household heterogeneity will quantitatively impact the crowding-out effect, exhibiting a potent amplification effect. The analysis to the crowding-out effect, based on these characters, identifies three pivotal factors that modulate the crowding-out effect, each corresponding to a distinct functional role that residential assets play in the economy.

The first characteristic of the residential asset, utilitarian feature (residential asset offers utility to households directly), corresponds to the relative intratemporal elasticity of substitution (IAS) to intertemporal elasticity of substitution (IES) between housing and consumption, which has been extensively scrutinized within the housing literature. However, most analytical frameworks are partial equilibrium and confined to the housing sector for a considerable period of time.[7]The interplay between the intratemporal and intertemporal elasticity of substi-

[7]For instance, when the utility function is separable in housing and consumption, the relative intratemporal elasticity is always one, i.e., $\frac{\text{IAS}}{\text{IES}} = 1$. Iacoviello (2005), Liu et al. (2019) and Greenwald (2018)used the separable utility function to analyze their problems. However because their models lack an intratemporal channel, they can only weight other elements such as bubbles, self-fulling expectations and multiple credit constraints to generate a sufficient consumption response to housing prices. In contrast, Berger et al. (2018) and Kaplan et al. (2020)used the nonseparable utility function to discuss the housing problem and focused more on the consumption response, which

tution has been overlooked within a general equilibrium framework and the importance of the pass-through mechanism between housing and consumption in determining the crowding-out effect. In such a standard Ramsey equilibrium setting, variations in housing prices are solely driven by the substitution effect. However, there are two substitution effects, intratemporal substitution and intertemporal substitution, that determine the total substitution effect and the relative elasticity between them governs the immediate response of consumption to changes in housing prices. When the relative intratemporal elasticity of substitution exceeds one[8] and continues to grow, the demand for intratemporal consumption smoothing supersedes that for intertemporal consumption smoothing (the increased holding of residential asset decreases the marginal utility of consumption at this period, and households must determine consumption by balancing consumption smoothing between this period, $\frac{U_H}{U_C} = f(p_{H,}, p'_H)$, and the next period $U_C = \beta R U_{C'}$). Consequently, there is either a modest increase or even a decline in consumption, as the complementarity between consumption and housing services weakens---in other words, the substitution effect becomes increasingly pronounced. Therefore, the crowding-out effect is attenuated, as, in accordance with the market-clearing condition, an increase in residential investment is associated with a more modest rise in consumption.

In addition to the relative intratemporal elasticity of substitution, the financial friction also exerts the crowding-out effect, a concept well-embedded within the literature.[9] A housing market bubble driven by demand shocks elevates housing prices and triggers overbuilding, in addition to a shift in demand. As a result, the boom in the residential property market alleviates credit constraints of households. Households who previously constrained by liquidity expend their spending---a phenomenon termed "equity extraction" that was first proposed by Bhutta and Keys (2016). Therefore, an rise in financial frictions enhances the distributional marginal propensity to consume (MPC) effect, thereby magnifying the crowding-out effect as the increase in consumption indicates a decrease in physical investment.

Moreover, household heterogeneity further amplifies the crowding-out effect through idiosyncratic income shocks and wealth distribution. Unlike representative agent models, households with uninsured income---subject to idiosyncratic shocks---exhibit a precautionary saving motive and consequently maintain a higher saving rate. During periods when income risk is countercyclical[10], overbuilding tends to coincide with economic upswings. Lower risk encourages households to reduce capital accumulation, thereby intensifying the crowding out of investment. Beyond the uncertainty channel, households with greater disposable income are also the primary driver of overbuilding. Conversely, households facing tighter budget constraints tend to have a higher

requires the intratemporal effect.

[8] Khorunzhina (2021) provides empirical evidence of $\frac{\text{IAS}}{\text{IES}} > 1$ in the housing market.

[9] Garriga and Hedlund (2020),Hurst et al. (2016),Bailey et al. (2019), Garriga et al. (2017), Gorea and Midrigin (2017) and Chen et al. (2020) contribute to this literature and investigate how financial frictions influence the cross effect between housing and consumption.

[10] Debortoli and Galí (2017), Acharya and Dogra (2020) and Bilbiie and Ragot (2021) analyzed this problem linked with monetary policy theoretically. Storesletten et al. (2004), Schulhofer-Wohl (2011) and Guvenen et al. (2014) analyzed the countercyclical idiosyncratic shock empirically.

MPC and therefore exhibit greater increases in consumption, facilitated by equity extraction. Consequently, the more the wealth distribution skews to the left and the MPC distribution to the right[11], the more pronounced the crowding-out effect becomes.

Finally, using a full-fledged heterogeneous agent model with financial frictions, I integrate the theory with the real world via full-information Bayesian estimation, demonstrating that the model is in line with key moments related to the market of residential and nonresidential assets in the business cycle. By comparing the scenarios with and without crowding-out effect, I demonstrate that the crowding-out effect can explain up to 13% of the welfare losses during the recession period. Furthermore, by implementing a countercyclical macroprudential policy to control credit expansion capacity and overheated housing market, a policymaker could calm the boom-bust cycle and approximately halve the welfare loss generated by the crowding-out effect.

This paper offers several noteworthy contributions to the existing literature. First, it establishes a novel link between the housing market boom (overbuilding) preceding the recession and the recession itself. A lot of researches suggest that the housing market boom and recession are driven by expectations and speculation as opposed to sustainable growth. This is evidenced by the work of Landvoigt (2017), McQuinn et al. (2021) and Kaplan et al. (2020), among others. Other studies have posited that the credit supply also played a significant role, a perspective supported by Campbell and Cocco (2007), Favara and Imbs (2015), Favilukis et al. (2017), Justiniano et al. (2019), Mian and Sufi (2022) and Martínez (2023). However, real estate only functioned as an asset in the context of collateral constraints among these studies, and the inherent recession comes from the demand side that is initiated by the collapse in housing market. They omit the supply-side effect of the recession, while many studies contend that capital misallocation contributed significantly to the Great Recession, such as Justiniano et al. (2010) and Justiniano et al. (2011), with supply-side effects accounting for nearly 40% of the economic downturn. When the major companies could undertake extensive margin investments through self-financing, as outlined in Bachmann et al. (2013) and Winberry (2016),, the housing market boom not only impacted investment in the construction sector (Boldrin et al. (2013)) but also diverted physical investment from other sectors. Using this perspective, I investigate, the crowding-out effect, for which Chakraborty et al. (2018) provides evidence via micro data.

My paper is related to several literature. First, Dong et al. (2022) and Dong et al. (2023). They also employs the term "crowding-out" to describe the investment tradeoff and capital misallocation between housing and non-housing sectors. However, their conceptualization of "crowding-out" aligns more closely with firms' balance sheet portfolio adjustments in partial equilibrium, and my paper focuses on another aspect, the household sector with general equilibrium, whose view maps more to reality[12], given that enterprises do not hold the majority of

[11] "In 2019, the top 10% of U.S. households controlled more than 70 percent of total household wealth" as argued by Batty et al. (2020) and related data can be found in Distributional Financial Accounts on the Federal Reserve website. Orchard et al. (2022) demonstrates that the MPC distribution is heavily right-skewed.

[12] Kaplan et al. (2014) shows that "Housing equity forms the majority of illiquid wealth for households in every country with the exception of Germany".

residential assets, nor do these assets play a pivotal role in production activity. Meanwhile, my paper is also closely related to Rognlie et al. (2018). They proposed that an exogenous investment hangover at the outset precipitated a demand-driven recession due to high real interest rates, nominal rigidity, and the ZLB on monetary policy. Conversely, my paper argues that the financial friction and household heterogeneity play important roles in crowding out physical investment. Furthermore, my paper illustrates that even in the absence of nominal rigidity, overbuilding can also catalyze a supply-driven recession with significant welfare loss.

Second, this paper not only provides a new explanation for the severity of the Great Recession but also sheds light on elements of policy failure as discussed by Mitman (2016) and Antunes et al. (2020). Accordingly, it also makes a valuable contribution to the literature on macroprudential policy. Since the recession is propelled by both supply and demand dynamics, singular stimulus efforts in the demand sector fail to effectively counteract the economic decline. Neither of the aforementioned studies considers the supply of housing services, although Khan and Thomas (2008)demonstrated that a general equilibrium framework could yield entirely distinct results. My research extends the findings of Chodorow-Reich et al. (2021), Chahrour and Gaballo (2021) and Beaudry et al. (2018) and emphasizes investment in the production (of consumption) sector, arguing that overbuilding exacerbated the crowding-out effect and incited a more profound recession, which could be dramatically attenuated by macroprudential policy on the supply side.

Furthermore, this research contributes to the literature with a methodological advancement: a new implementation of the SVAR identification strategy for distinctly identifying news and fake news shocks with endogenous contemporaneous effect, predicated on the approach of Wolf and McKay (2022). Almost all the previous identification methods to news shock, such as Barsky and Sims (2012), Blanchard et al. (2013), Barsky et al. (2015) and Sims (2016), identified a TFP shock that is observable and exogenous, and its news does not have any contemporaneous effect on itself. However, there are a many shocks that cannot be directly observed, such as news about inflation or monetary policy news shocks.[13] My paper extends the identification strategy to be applicable for the news shock with the contemporaneous endogeneity, as well as fake news shock.

Numerous studies emphasize the importance of household heterogeneity in explaining the housing boom-and-bust cycle, either empirically, such as Etheridge (2019), Mian et al. (2013), Li et al. (2016) and Díaz and Luengo-Prado (2010), or theoretically, such as Kaplan et al. (2020), Favilukis et al. (2017) and Garriga and Hedlund (2020). This paper builds a model that demonstrates that the distribution of wealth and income is pivotal in determining the strength of overbuilding and supplements the literature on how expectations and animal spirits can fuel a boom. To solve the model with imperfect information, earlier research either employed a guess-and-verify approach, as in Lorenzoni (2009) and Barsky and Sims (2012), or a reconstruction

[13] For instance, news that indicating a decrease in the federal funds rate in the future will persuade households to increase their consumption in the current period, but this contemporaneous economic boom will increase the federal funds rate in the present.

methodology, as demonstrated by Baxter et al. (2011), Blanchard et al. (2013) and Hürtgen (2014), to solve imperfect information DSGE models. However, these methods necessitate specific analytical equations to regulate the unobserved state variable with other state variables, which is unfeasible to derive from a heterogeneous agent model due to its extensive number of state variables. To achieve this, I propose an enhancement in the numerical solution approach for handling an intricate heterogeneous agent model with imperfect information at both the first and second order. Following Uhlig (2001), I reconstruct the linearized model and solve the policy function via a new system of equations.

In Section 2, I use an identification strategy that I proposed to analyze the crowding-out effect generated by news and fake news shocks to housing prices. Later, in Section 3, I investigate how the crowding-out effect is influenced by four elements in the economy. In Section 4, I quantitatively investigate the welfare loss due to crowding-out effect through the lens of a full-fledged heterogeneous agent model. In the last section, I conclude the paper.

2 Empirical evidence

In this section, I first introduce the new identification algorithm to the news shock with contemporaneous endogeneity. Using the new method, I then demonstrate that the crowding-out effect is empirically significant. After investigating the news shock, I further extend the algorithm to the fake news shock and show that the crowding-out effect is significant in explaining the recession after the housing market bust.

2.1 Real price news shock

In the appendix B.2.2, I illustrate that a contemporaneous real price shock can also empirically measure the crowding-out effect from a housing market boom. While the mechanisms I have proposed in this paper may be theoretically valid, they might not accurately portray the realities leading up to the Great Recession. The source for the prerecession housing market boom extends beyond merely an exogenous contemporaneous real housing price shock. Other variables such as optimistic expectations, excess credit supply, and a secular decline in interest rates also contributed to this boom. To delve deeper into this issue, this section employs a SVAR model to identify the effect of a news shock on housing demand. My objective is to answer the following question: given future expectations of housing price inflation, how would other economic components respond to this anticipatory shock? I adopt, with minor modifications, the method proposed by Barsky and Sims (2011) (henceforth referred to as 'BS'). Through this approach, the news shock is identified as the component that can account for the largest forecast-error variance of the housing price while maintaining some orthogonal restrictions to exclude the effect of other contemporaneous shocks. This orthogonal restriction procedure is designed to mitigate any risk that an unexpected contemporaneous shock realized in the future could influence the forecast error. Furthermore, rather than adopting the level specification used by BS, I process the data using a hybrid specification or detrending method as mentioned in the previous section. This alternative approach was necessary because the data I used failed the unit-root test in the level specification.

First, I propose the reduced-form VAR system as

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Phi_3 y_{t-3} + ... + u_t \tag{1}$$

where the residual follows $u_t = Q\varepsilon_t$, $\varepsilon \sim N(0, I)$ and $\Omega = \mathrm{var}(u_t) = QQ'$. Moreover I assume that P is the Cholesky decomposition of the covariance matrix of residual u_t, so $P = \mathrm{chol}(\Omega)$ will hold. I further define the "news" vector $R = [r_1, r_2, ..., r_{N-1}, r_N]'$ where r_i represents the unknown parameters of the vector R that need to be estimated. It measures the effect of housing-price-change news. The response to the news will be PR, and by introducing this "vector shock" R, I can directly solve for the response to the news shock and avoid drawing difference alternative orthogonal matrix.

Note that solving the response vector R, instead of solving the response matrix Q, is more convenient and can provide an analytical solution as argued by Uhlig et al. (2004). As long as the orthogonal assumption 3 holds, we can find an orthogonal matrix $\mathcal{Q}$ that satisfies $\mathcal{Q}'R = e_i$ where $i \in [1, N] \cap \mathbb{N}$. Multiply $\mathcal{Q}$ by the LHS to yield $R = \mathcal{Q}e_i$, and hence R is just the ith column of $\mathcal{Q}$. Throughout this paper, I will combine these two definitions, namely, 1). response vector R and 2) a shock R, because they represent the same thing in the identification problem.

After proposing the VAR formula, I define the forecast error decomposition along the horizontal up to time h as

$$\mathrm{fevd}^i_{n,h} = \frac{e'_n \mathrm{var}(y^i_{t+h} - E_{t-1}y^i_{t+h})e_n}{e'_n \mathrm{var}(y_{t+h} - E_{t-1}y_{t+h})e_n}$$

the economic meaning of which is that the proposition of variance of variable n's expectation error that can be explained by shock i from time 0 to time h. The total forecast error from 0 to period H with unit weight should be $\mathrm{fevd}_n = \sum_{h=0}^{H} \mathrm{fevd}_{n,h}$ where $H = 12$.[14] The superscript i in vector y_t denotes the impulse response spurred by shock i, and the subscript n in vector y_t (equivalent to $e'_n y_t$) denotes the nth variable in vector y_t. Therefore, $y^i_{n,t}$ denotes the response of variable n at time t to shock i, and I will use this notation throughout the discussion in this section.

To identify the news shock, I solve problem 7 below that identifies a shock R^* that can best

[14] Uhlig et al. (2004) and Barsky and Sims (2011) discussed the weight-selection problem and arbitrary maximized horizontal problem. Based on their argument, I choose the unit weight and 3-year forecasting as the baseline cases, which is reasonable and robust in the range from 5 quarters to 40 quarters.

explain the variance in the expectation error of housing prices.

$$R^* = \text{argmaxfevd}_n = \text{argmax} \sum_{h=0}^{H} \frac{e_n' \left(\sum_{s=0}^{h} \Phi^s P R R' P' \Phi'^s \right) e_n}{e_n' \left(\sum_{s=0}^{h} \Phi^s P P' \Phi'^s \right) e_n} \tag{2}$$

s.t

$$R'R = 1 \tag{3}$$

$$e_j' P R = 0 \tag{4}$$

The first constraint 3 guarantees the orthogonality of response R^* and ensures the unit realization of the news shock that pertains to the corresponding column of orthogonal matrix Q; otherwise, there always exists an infinitely large shock $e_n' R = \infty$ that renders the identification meaningless. Additionally, it indicates that the existence of maximization problem 2 as the Hessian of the objective function is semi-positive definite where the maximized point is not on the saddle point. The second constraint 4 rules out any contemporaneous shock in the future that influences the expectation error. In essence, there are two type of shocks that can affect the expectation error $y_{t+h} - E_{t-1} y_{t+h}$: one is the news shock that arrives at time t but realizes at a future time from $t+1$ to $t+h$ (based on the type of news and how informative it is); the second is the contemporaneous shock that arrives at any time from t to $t+h$ $\varepsilon_{t+i}, \forall i \in [0, h]$. It would be inappropriate to posit that the news shock accounts for more variation in the expectation error than the contemporaneous shock. Sims (2016) asserts that, typically, this proposition does not hold in reality. As such, I necessitate this secondary constraint 4 to segregate the effects of the contemporaneous shock from the identified R^*. The objective of the above problem 2 is to pinpoint a shock, apart from any contemporaneous shock that influences variable j, which can best explain the expectation error. Appendix C.1 discusses the requirement of the orthogonal restriction in detail.

While the method of identification employed here is not exclusive to news about housing prices—news about endogenous variables such as commodity prices, marginal costs or inflation could also fit—I limit the focus to the housing market in this paper. Here, i denotes the housing price news shock, and $y_{n,t}$ represents the housing price. Given that the identified news shock R^* is subject to sign, I further impose a sign restriction on the impulse response $y_{n,t}$ to generate a positive demand shock on the housing price. The final issue in the identification of 2 involves finding a variable j in constraint 4 that aids in eliminating the possibility of a contemporaneous shock during identification.

Before elaborating on the construction of variable j that has zero contemporaneous effect of the news shock i, it is worth discussing proposition 1. This proposition highlights that canonical identification techniques, such as zero restriction, sign restriction, and long-run restriction, are ineffective for identifying the news shock in this context without constructing or finding variable

j.

Proposition 1. *The identification of a news shock R^* through Equation 2 is unique to covariance of the residual $\Omega = PP'$ from VAR's DGP 1.*

Proof. Give the covariance matrix of the residual from the DGP 1, the Cholesky P is unique to covariance matrix Ω. Following Rubio-Ramirez et al. (2010), we know that any identification of the DGP is unique to PQ where Q is an orthogonal matrix. To identify the news shock, I solve the maximization problem 2 to obtain the news shock R^* that maximizes fevd$_n$ subjecting to two constraints, 3 and 4, and the rotation Q is identity $Q = I$. However, when the rotation Q is not identified, i.e., for any different response matrix $P\widetilde{Q}$, the optimization problem that helps to find $\widetilde{R}^*$ from $g\left(\widetilde{R}\right) = 0$ is equivalent to that employed to find R^* from $g\left(f\left(R\right)\right) = 0$ as long as $f\left(R\right) = \widetilde{R}$ holds. If the mapping $f(\cdot)$ and its inverse $f^{-1}(\cdot)$ are all bijections, for any $\widetilde{R} \in \mathbb{R}^N$ there will exist a unique $R \in \mathbb{R}^N$ that satisfies $f\left(R\right) = \widetilde{R}$. It is easy to set $f^{-1}(\widetilde{R}) = \widetilde{Q}\widetilde{R}$ and $f(R) = \widetilde{Q}'R$. Therefore, the corresponding identified news shock $\widetilde{R}$ must satisfy $\widetilde{R}^* = \widetilde{Q}'R^*$ because of Equation 2, and the impulse response of the news shock is identical to the Cholesky identification $P\widetilde{Q}\widetilde{Q}'R^* = PR^*$. □

Proposition 1 intuitively suggests that news or information is neutral to the fundamentals, and individuals respond to it based on their perception or belief about the reliability of the news. Whether the news is genuine or false can only be discerned after the fundamental shock is realized and observed by economic agents several periods later. Therefore, the initial response to the news at time zero is unique to the covariance matrix, and the authenticity of the news, along with the corresponding response, cannot be determined by any rotation method on Cholesky P.

Proposition 1 above raises the following question: Why should we construct variable j rather than seek one that is observable in reality? This deviation from the standard news literature, where scholars typically focus on TFP shocks and the underlying exogenous TFP is observable or calculable from data, arises due to the unobservable nature of the demand shock and the exogenous fundamental variation path. As such, our task is to identify a variable j that is correlated with the contemporaneous variation of housing demand within the demand function, which I denote as the direct fundamental impact. The term "fundamental impact" refers to an index of the core elements that drive the housing demand function, i.e., the preference ϕ_t in the Cobb-Douglas utility function $U(c_t, h_t, l_t) = \frac{\left(c_t^{\phi_t} h_t^{1-\phi_t}\right)^{1-\sigma}}{1-\sigma} + \kappa \frac{l_t^{1+\psi}}{1+\psi}$, following $\phi_t = (1-\rho_\phi)\overline{\phi} + \rho_\phi \phi_{t-1} + w_{t-\tau} + w_t^\tau$ where $w_{t-\tau}$ is the news shock to housing demand. The modifier "direct" indicates that variable y_t^j reflects the contemporaneous impact ϕ_t, rather than ϕ_{t+i}. Moreover, when imperfect information exists and households cannot precisely observe the fundamentals, as discussed in Section 2.2, the fundamental impact y_t^j should serve as an indicator for the perceived fundamentals $\phi_{t|t}$, rather than the true fundamentals. Consequently, survey data appear to be the most suitable source of information for excluding contemporaneous shocks via constraint 4. However, neither the true fundamentals nor the perceived fundamentals are observable, and all observations in the survey relating to the impact of fundamentals are endogenous, tainted by macro variables and the endogenous response to news shocks. Therefore, this paper proposes a method to clean the endogenous perception data and eliminate the contemporaneous endogenous news effect.

Before discussing the purification process, I first describe the data that I can use to remove endogenous perceptions of the status of the housing market. In this paper, I use the NAHB/Wells Fargo Housing Market Index (HMI), which is a monthly survey of NAHB members regarding their perception of the current status of the housing market Υ_t(in equation 5), and their expectation over the next six months $E_t\Upsilon_{t+6}$(in Equation 6).

To intuitively elucidate the purification process, let us consider a model with perfect information. Assume that Υ_t represents survey data about people's perception of the housing market and follows the relationship

$$\Upsilon_t = \rho\Upsilon_{t-1} + \alpha_1 x_t + w_{t-\tau} + u_t + \alpha_2 w_t \tag{5}$$

where x_t stands for any macroeconomic variable such as the interest rate, GDP, unemployment rate, etc. The coefficient α_1 quantifies the cross-linkages between macroeconomics and perceptions of fundamentals. For instance, a monetary policy shock may initially affect the interest rate and output, leading to a commensurate change in Υ_t. In this context, $w_{t-\tau}$ represents a news shock announced τ periods ahead. Moreover, u_t denotes a contemporaneous shock, and α_2 captures the endogenous contemporaneous effect induced by news shock w_t. If households anticipate the realization of the shock three periods ahead, they would react in the present time. Because of this contemporaneous response α_2, news shock w_t will exert an endogenous effect at the time of its arrival, in addition to the direct effect occurring three periods later when the shock materializes. Under rational expectations, the expectation about housing market status τ periods ahead will follow

$$E_t\Upsilon_{t+6} = \begin{cases} \rho^6\Upsilon_t + \alpha_3 x_t + \sum_{n=1}^{n=\tau} \rho^{6-n} w_{t-\tau+n} & \tau \leq 6 \\ \rho^6\Upsilon_t + \alpha_3 x_t + \sum_{n=1}^{n=6} \rho^{6-n} w_{t-\tau+n} & \tau > 6 \end{cases} \tag{6}$$

To simplify our discussion, I have deliberately omitted terms with additional lags, such as Υ_{t-2}, x_{t-1}, in Equations 5 and 6. These terms may indeed manifest in these models, and as such, I performed a range of robustness tests to investigate these independent variables in Appendix C.6 and better understand the underlying models. However, note that these equations make an implicit assumption: any other macroeconomic shocks, such as a monetary policy shock, TFP shock, or marginal cost shock, will influence the status of the housing market solely through macro variables x_t, without any direct effects. This assumption parallels the notion that Υ_t occupies the first row of y_t in Equation 1, corresponding to the first column of the Cholesky P.

The basic idea of this purification process is to identify the parameters ρ, α_1, α_2 and w_{-1},

w_{-2}... that yield the purified housing market status, denoted as $\widehat{\Upsilon}_t = \Upsilon_t - \alpha_2 w_t$. However, canonical regression-based methods cannot be used here because of endogeneity and the imperfect identification problem. For instance, even for the simplest model of 5 (or 6) without aggregate effects, the regression of Υ_t on Υ_{t-1} ($E_t\Upsilon_{t+6}$ on Υ_t) will yield biased results because $w_{t-\tau}$ is already embedded into $\Upsilon_{t-1}(\Upsilon_t)$. Furthermore, the residual of this regression represents a "near" moving average process that contains several components instead of $w_{t-\tau}$ itself. Thus, the second regression, a regression of Υ_t on the residual or its lagged and lead terms will not be exactly α_2, and $\widehat{\Upsilon}_t$ will still encompass some amount of contemporaneous effect, w_t. In addition to addressing the standard issues of endogeneity and heteroskedasticity that are common in OLS regression, another crucial challenge must be overcome: understanding the informative power of news w_t, specifically how far in advance households become aware of it. This challenge will directly impact the structure of the expectation 6 and, subsequently, the structure of the residual, which I use to extract the w_t term from Υ_t. Given that the only observable expectation linked to a six-period lead, the form of the expectation would take a different form when the news arrives at different periods prior to realization. Therefore, I use the maximum likelihood estimation method to estimate and purify the contemporaneous endogenous effect α_2 and the likelihood of different informative power of the news can be used to determine how many periods ahead that the news is announced to households. In Appendix C.3 and C.3, I provide a range of numerical and empirical tests demonstrating that this purification method can effectively eliminate the endogenous news effect w_t from the perception of housing market status Υ_t, albeit to a certain scale. Additionally, I also conduct a series of robustness checks by using an instrumental variable to purify Υ_t through 2SLS regression analysis.

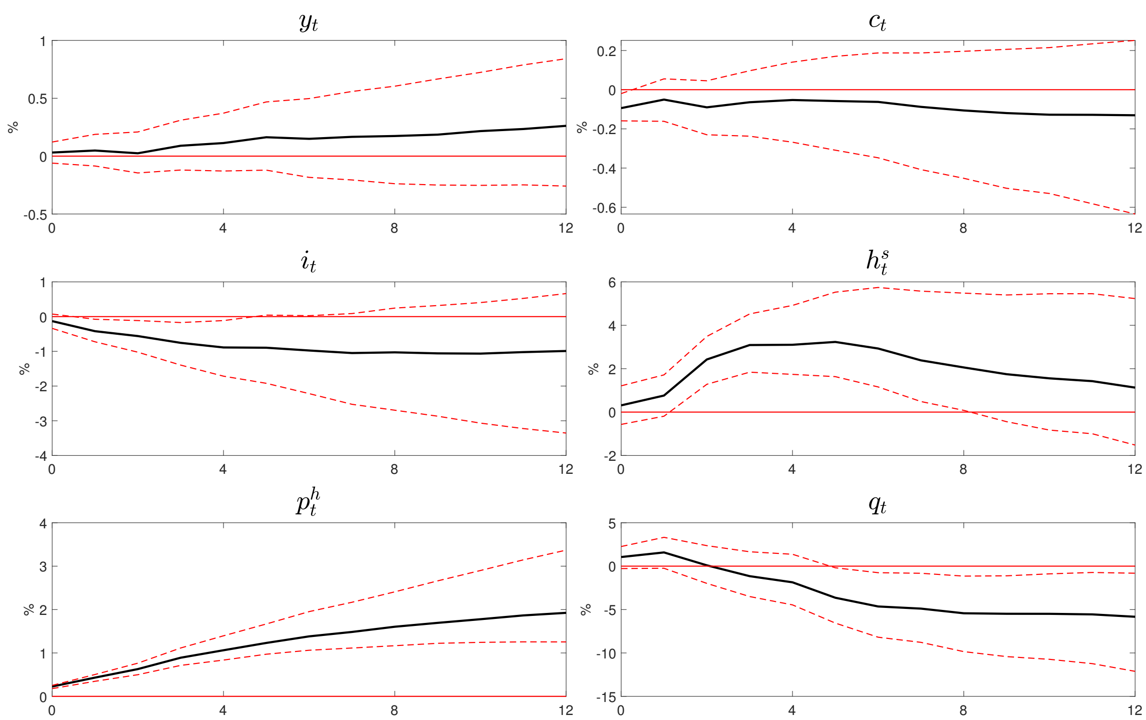


Figure 2: IRF to one unit housing price news shock at 90% confidence band

Figure 2 presents the IRFs of a one-unit news shock that delivers information about future housing price to agents, with the red dashed lines indicating the 90% confidence band, which visibly confirms the significant crowding-out effect. The pattern of the housing price response closely mirrors that observed in the contemporaneous shock, although the boom in the housing market is nearly twice as significant. Housing prices progressively rise from 30bps to a peak of 200bps, approximately five times larger than that under the contemporaneous shock. This marked expansion in the housing market, driven by expectations and news shocks, triggers a decrease in capital prices five times larger than the surge in housing prices. Households considerably decrease their capital holdings, even entering into negative positions (in debt), thereby depressing capital prices due to reduced demand. This reveals the crowding-out effect as a manifestation of capital misallocation at the micro level. These observations underscore the effective identification of news shocks and demonstrate the reliability and transparency of the results. The study aligns with existing literature that attributes housing market booms primarily to expectations and slackness in the credit market. Additionally, it highlights the sensitivity and fragility of the housing market during the prerecession period, as the market could be triggered into a boom merely by initial expectations, culminating in a considerable peak without any observed hesitations or declines. This housing market boom also coincides with significant overbuilding, which is five times greater than that observed under contemporaneous shocks, culminating at 300bps. Concurrently, the output experiences a slight yet insignificant increase due to general equilibrium effects, with the economy overheating. The contemporaneous response of consumption exhibits a small but insignificant decrease (10bps), potentially resulting from a stronger substitution effect than the wealth effect, which is revealed in new evidence from survey data (Kuang et al. (2023)). As previously observed, substantial physical investment is crowded out during periods of housing market booms and overbuilding. In comparison to the response of physical investment to a contemporaneous housing price shock, investment is crowded out to a greater but milder extent (compared to the difference in housing price and stock price), reaching up to 100bps. This is reasonable, as the crowding-out effect arises from the general equilibrium among investment, output, and consumption, which continues to experience a mild change.

2.2 Real price fake news shock

Due to Proposition 1, canonical identification methodologies such as sign restriction (Uhlig (2005)) and short-run restriction ((Sims (1980) and Basu et al. (2006)) are insufficient to differentiate true news from fake news within the previously identified news shock. As an alternative, I introduce a novel identification strategy through which the effect of true news about future housing prices is refined by a contemporaneous shock during the realization of the news shock, thus isolating the effect of fake news.

In Section 2.1, I previously presented the concept of the news shock R^*, representing the shock that best accounts for the expectation error over the next H periods from period 0. However, this shock is agnostic to its own status and does not yield any insights regarding whether it is a true or fake news. This is because it is identified based on expectation error, devoid of any proxy for the underlying "fundamental situation", and both fake and true news can elicit identical responses before the news type is realized. Despite the neutrality of the news shock R^* and our inability to directly identify fake news prior to its realization, I design the strategy to differentiate between fake and true news by adjusting the combined news with a contemporaneous shock and refining the preceding impulse response. Before introducing this identification strategy, which allows me to distinguish between fake news and true news, I first present two assumptions with microfoundations as the basis of identification.

Definition 1. Denote the response to fake news realized at time τ as $U^F = \left\{y_0 = \overline{R}_1, y_i\right\}_{i=1}^{i=\infty}$ and the response to true news realized prior to time τ as $U^T = \left\{y_0 = \overline{R}_2, y_i\right\}_{i=1}^{i=\infty}$. The response to a news shock we empirically identified through 2 is $U = \{y_0 = R^*, y_i\}_{i=1}^{i=\infty}$.

Assumption 1. *The response to a news shock, either fake or true news, under imperfect information, will be the same before the shock realized. In other words, $\overline{R}_1 = \overline{R}_2 = R^*$ and $y_i^F = y_i^T = y_i, \forall y^F \in U^F, y^T \in U^T, y \in U, i \in [0, \tau]$ will hold.*

This assumption is justified given that under imperfect information, agents cannot discern the veracity of news; they simply respond identically to observations triggered by either true or fake news. Thus, it is only under conditions of complete information where the news is fully informative that agents exhibit differing responses before news realization at time τ. It is widely recognized that the principle of certainty equivalence applies in the context of first-order linearized state space models, within which Assumption 1 is unequivocally upheld. Further support for this assumption is provided in Appendix D.2.2, where I offer several numerical examples to demonstrate that the aforementioned assumption holds in a state space model under rational expectations.

Assumption 2. *The empirically identified news shock U lies in the medial of the response to fake news U^F and response to true news U^T. In other words, $y_i \in \left[y_i^F, y_i^T\right], \forall y^F \in U^F, y^T \in U^T, y \in U, i \in [\tau+1, \infty]$ will hold. Furthermore, the news shock U is a linear combination of U^F and U^T, and $y_i = \alpha y_i^F + \beta y_i^T$ holds.*

Assumption 2 is also reasonable because the identification process 2 is based on expectation error and cannot differentiate between U^F and U^T, as both of them impact the expectation error of housing prices. Nonetheless, as long as the data generating process (DGP) 1 is a linear equation, the path subsequent to realization of a shock is entirely described by the coefficient Φ, which represents a projection from y_{t-1} to y_t. Therefore, the identified path U is essentially a linear combination of the fake news path U^F and the true news path U^T , which are both intertwined within the posterior observation. In Appendix D.5, I apply the news shock identification strategy 2 to mock data generated by a state space model to demonstrate that Assumption 2 is valid.

I now define the identification of fake news as

$$\widehat{y}_i^F = \begin{cases} y_i & i \leq \tau \\ y_i - \frac{e_j' y_{\tau+1}}{e_j' y_0^\tau} y_{i-\tau-1}^\tau & i > \tau \end{cases} \tag{7}$$

where $y_i \in U$ and y_i^τ represents the response path to a contemporaneous shock directly impacting the fundamental variable j, as depicted in Equation 4. The fundamental concept here is that the influence of true news realized at time τ can be counteracted by a contemporaneous negative shock, leaving behind only the response to fake news, which has no bearing on variable j or the real economy (subject to a scalar α, which remains unidentified here). This is a logical supposition, given that the true news shock has been influencing the economy since its realization at time τ, and as long as the shock is independent and identically distributed (iid) and the entire system is linear, it operates (producing real effects) as a contemporaneous shock after τ when it impacts the fundamentals. In Appendices D.3 and D.4, I provide two examples that lend microfoundation to this offset effect.

The identification method I employ aligns with the logical premise first advanced by Wolf and McKay (2022), who propose that we can "replace" the underlying state determinant equation (i.e., policy function) with a counterfactual version by solving a system of linear equations. A set of rescaled fundamental shocks can emulate the old, identified policy function and transform it into a new function by censoring the old impulse response with an additional series of $\{\Theta_{i,\tau}\}_{\tau=0}^{\tau=\infty}$, generated by a fundamental shock. The paths of other endogenous variables, such as GDP, investment, and labor supply, are then determined by the censored path y_i^τ, and Wolf and McKay (2022) provide a rigorous proof supporting this argument. Similarly, Hebden and Winkler (2021) and Groot et al. (2021) have also used comparable counterfactual experiments in their research in which the goal was to identify an optimal policy, and they achieved this by solving certain nonlinear problems.

Figure 3 exhibits the empirical response to a deceptive housing demand (housing price fake news) shock. A shock to housing demand arrives (or is announced to households) six quarters ahead, at time 0, but realizes (has fundamental effect) in quarter 5, with a possibility that the news lacks any fundamental effect and is merely noise. Before discerning the true nature of the shock—as either true or fake—agents respond identically to these two shocks, as they are unable to determine the truth. Hence, Figures 3 and 2 share the same responses before period 6, at which point agents commence their attempts to discern whether the news is true or fake[15]. Upon realizing that the news is fake at quarter 5, the housing market boom busts because it lacks further support. Housing prices and new construction of residential assets decline significantly, with a 150bps drop in housing prices and a 300bps drop in housing supply. Subsequently, new

[15] They may be informed directly at time 6 or gradually learn that whether the news is true or fake, which depends on the information structure, and I provide two examples in the appendix to illustrate two different information structures.

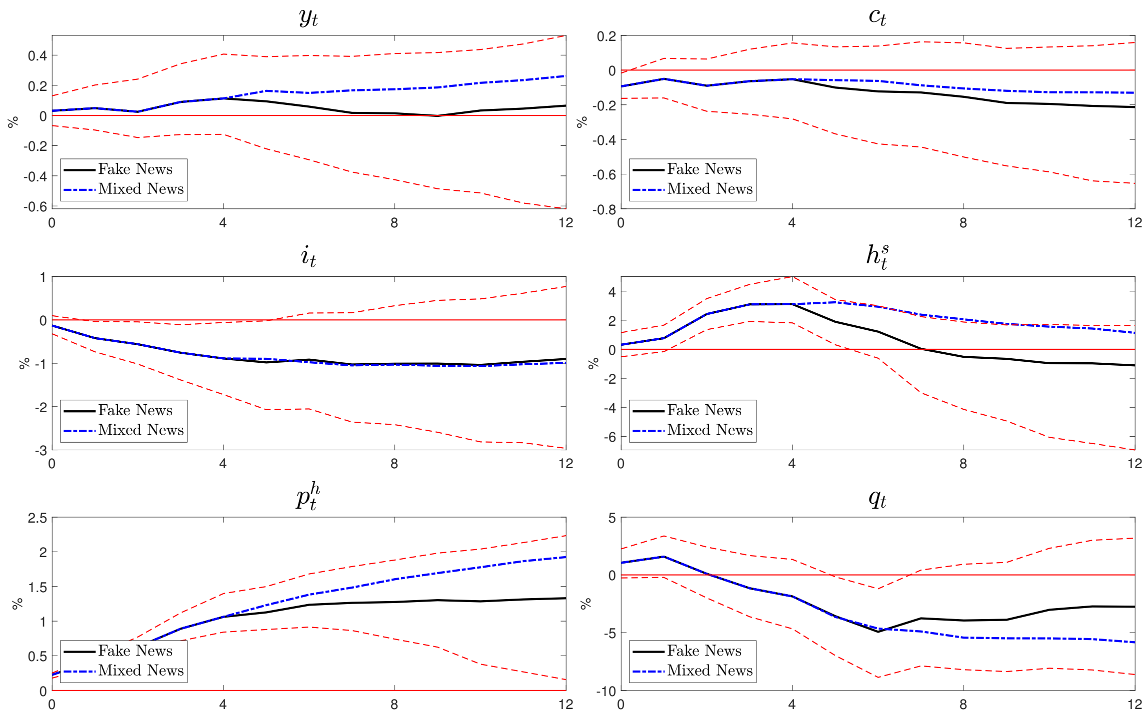


Figure 3: IRF to one unit housing price fake news shock at 90% confidence band

residential asset construction enters a negative range, indicating a severe and sustained recession triggered by the housing market bubble's bust. Physical investment, initially crowded out due to the housing market boom, only has a mild increase because the subsequent recession yields a lower demand for physical capital. In addition to the stagnation in the housing market, a recession unfolds in the goods market, with output and consumption dropping immediately after the revelation of the fake news. Due to the scarcity of physical capital during the bust period, the postrecession recovery is muted. This sluggish recovery unveils the drawbacks of housing market boom-bust cycles, where physical capital is crowded out during the boom period, and the resulting scarcity of physical capital leads to a more severe recession during the bust period.

In addition to examining the direction and magnitude of the news shock's effect on housing prices, it is vital to consider a news shock's significance. If it does not hold substantial importance in reality, the preceding discussion around the crowding-out effect may lose relevance. Figure 4 presents the historical decomposition of the news shock and fake news shock's influence on various macroeconomic variables. A news shock to housing prices accounts for a moderate portion of the variance in housing prices and new construction and exerts a modest but not insignificant effect on physical investment and consumption. To illustrate this significance, I opted for historical variance decomposition whereby the variables can be explained by the news shock as a measure of the news shock's influence. Approximately 50% of the variance in housing prices in the data is explained by fake news, and fake news can also explain 30% of the variation of housing supply. However, only 20% of the variation in physical investment originates from the fake news shock, although the number is not negligible. Conversely, the explanatory power attenuates to 14% for stock prices, signaling a milder influence than in the housing and capital

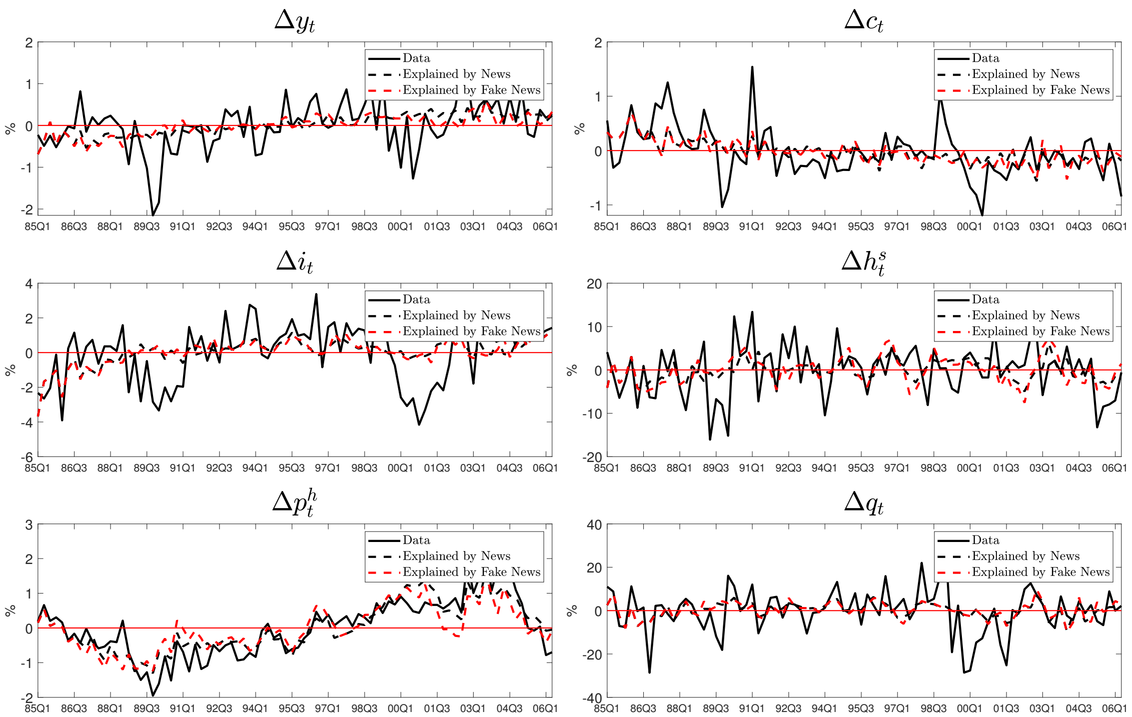
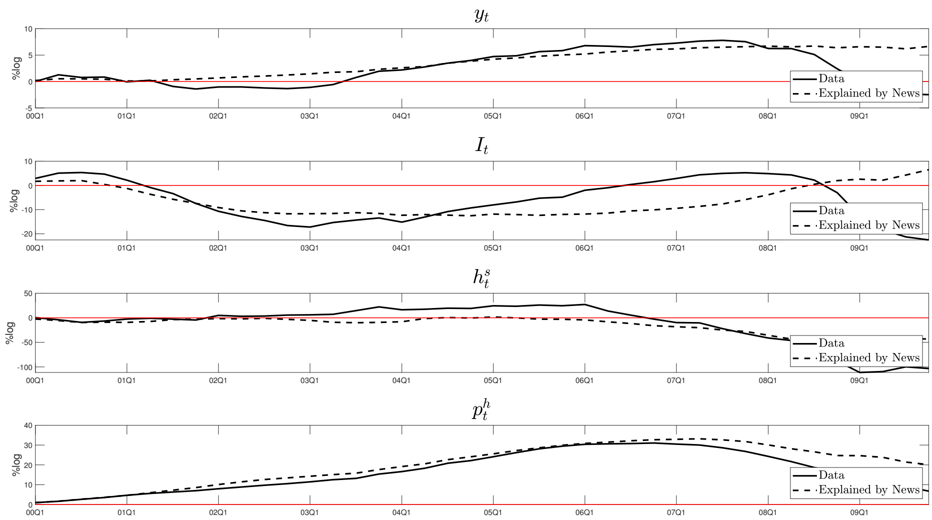


Figure 4: Historical Decomposition of News shock

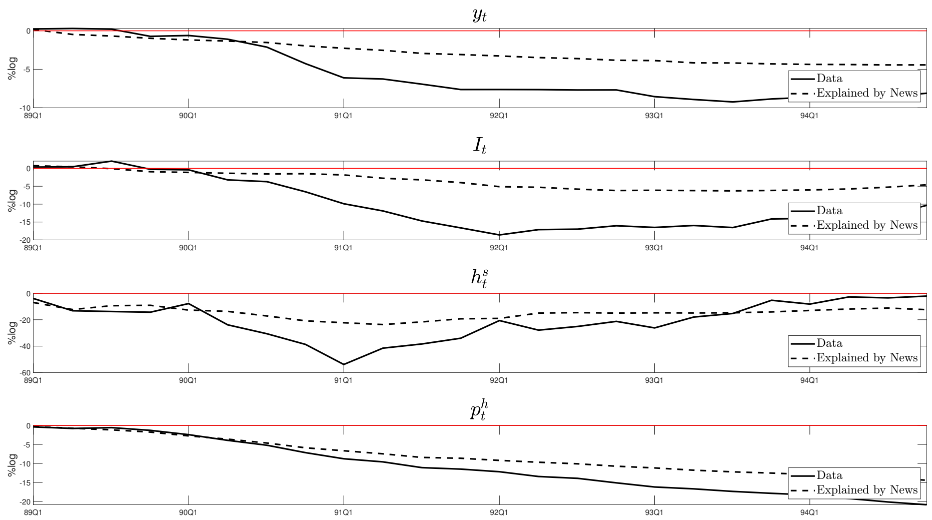
markets. These suggest that fake news about housing prices explains a significant portion of the boom-bust cycles in the housing market and the capital market due to the crowding-out effect. Nevertheless, based on Sims (2016), the share of variance does not offer a reliable indicator of the relative importance of news shocks. Thus, I also use Figure 5 to probe the significance of news shocks in reality. Figure 5 displays how the macroeconomy grew during the boom-bust period of the housing market and the extent to which a news shock can explain it by calculating the detrended accumulated growth. The news shock in Figures 5a and 5b initially increases housing prices, but the housing price drops further than is explained by the news shock in the bust period. These divergences imply that the deception generated the news may yield a bust when households realized the truth and that fake news indeed explains an important and significant share of the housing market bubble preceding the Great Recession. Furthermore, in both 5a and 5b, we observe that the crowding-out effect is significant, as physical investment substantially declines among the housing market boom.

3 Crowding-out effect of overbuilding: Insights from a simple model

Optimistic expectations regarding future housing prices engenders a surge in household demand for real estate, inducing a boom in the housing market characterized by inflated housing prices and overbuilding. In a context where supply is semi-inelastic, changes on the demand side will not necessarily lead to substantial overbuilding. Conversely, if the supply function possesses



(a) Historical Decomposition before Great Recession



(b) Historical Decomposition during housing market bust in early 90s

Figure 5: Historical Decomposition of News shock spanning two housing market booms

sufficient elasticity, a minor demand boom could spur significant overbuilding. The shapes of the supply and demand functions determine the magnitude of overbuilding and, by extension, the degree of crowding in physical capital. This is due to the underlying mechanism through which the crowding-out effect operates: the general equilibrium. Hence, it necessitates the synergy of both supply and demand functions to analyze the crowding-out effect. In this section, I first introduce a simplified Bewley-Huggett-Aiyagari model operating within an incomplete market framework. Subsequently, I utilize this model to demonstrate that overbuilding, influenced by intratemporal substitution, liquidity, precautionary saving, and wealth inequality, leads to the crowding-out effect.

3.1 A simple Bewley-Huggett-Aiyagari model

This framework is grounded in a standard Aiyagari-Bewley-Huggett model wherein households employ wage income and asset returns to meet their consumption and real estate demands. The durable good, in this case housing, is produced by real estate companies in a competitive market utilizing land, capital, and labor. Similarly, consumption are produced in a competitive market with capital and labor as inputs.

It is a standard Aiyagari-Bewley-Huggett model where households use wage income and asset returns to fulfill their demand for consumption and real estate. The durable good, housing, is produced by real estate companies in a complete market with land, capital and labor. Similarly, the consumption is produced in a complete market with capital and labor.

For simplicity, I assume that household i exogenously provides an inelastic labor supply of 1 unit to solve the problem

$$\max_{c_t^i, h_t^i, a_t^i} \sum_{t=0}^{\infty} \beta^t U^i \left(c_t^i, h_t^i\right) \tag{8}$$

s.t.

$$c_t^i + a_t^i + p_t^H h_t^i = R_t a_{t-1}^i + w_t \varepsilon_t^i + \left(1 - \delta^H\right) p_t^H h_{t-1}^i + T_t \tag{9}$$

$$-a_t^i \leq \gamma p_t^H h_t^i \tag{10}$$

where Equation 9 is the budget constraint and Equation 10 is the collateral constraint. a_t^i could either be positive or negative, but in aggregate it is positive because the supply of capital is used to produce housing, consumption and physical capital. w_t is the real wage, and the household earns productivity-weighted wage income from which ε_t^i corresponds to the idiosyncratic income shock. p_t^H is the real housing price. h_t^i is the unit of houses hold by household i. T_t is the lump-sum transfer to the household. For simplicity, I further assume the real interest rate is fixed at $\overline{R}$.[16]

The production sector is in a complete market where firms produce the nondurable good via $Y_{N,t} = A_{N,t} K_{N,t-1}^{\alpha} L_{N,t}^{1-\alpha}$ and the durable good via $Y_{H,t} = A_{H,t} \overline{L}_H^{\theta} K_{H,t-1}^{\nu} L_{H,t}^{1-\nu-\theta}$. The labor market is closed by one unit of inelastic labor supply $L_{N,t} + L_{H,t} = 1$ and households provide capital by $K_{N,t-1} + K_{H,t-1} = K_{t-1} = \int a_{t-1}^i dG_{t-1}$ where G_{t-1} is the cumulative distribution function of households. The nondurable good is used either for consumption or investment in physical capital, so the goods market cleaning condition $Y_{N,t} = K_t - (1-\delta)K_{t-1} + C_t$ holds. Furthermore, real estate companies produce all the increase in residential assets by $Y_{H,t} = H_t - (1-\delta^H)H_{t-1}$ where $H_{t-1} = \int h_{t-1}^i dG_{t-1}$.

Proposition 2. *Households will adjust their consumption of nondurable goods based on overbuilding and precautionary saving. The extent of adjustment is determined by*

[16]This is not an overly strong assumption since this could happen in many scenarios. For instance, the nominal interest rate reaches the ZLB, and the price is fixed. Alternatively, we could have an open economy where the real interest rate is bounded by the international financial market. In Appendix G.1.1, I show that under a range of parameters, the real interest rate will not change at t as long as capital and labor do not change.

$$\widetilde{c}_t = \underbrace{\Phi_H \widetilde{h}_t}_{\text{substitution effect}} - \underbrace{\Phi_\mu \widetilde{\mu}_t}_{\text{credit effect}} + \underbrace{\Phi_{p^H}\left[\frac{1}{1-(1-\delta^H)\frac{1}{R}} F^H(\widetilde{H}_t) - \frac{(1-\delta^H)\frac{1}{R}}{1-(1-\delta^H)\frac{1}{R}} F^H(\widetilde{H}_{t+1})\right]}_{\text{wealth effect}} \tag{11}$$

$$- \underbrace{\Phi_{cov} \widetilde{cov}_t}_{\text{precautionary saving effect}}$$

where $F^H(\cdot)$ is the inverse supply function,

$$\Phi_H = \frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{12}$$

$$\Phi_\mu = \frac{\mu}{\lambda-\mu} \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{13}$$

$$\Phi_{p^H} = \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \tag{14}$$

$$\Phi_{cov} = \frac{\eta_c}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}} \frac{\beta\left(1-\delta^H\right)\overline{cov}}{h}$$

and $\eta_{c,p^H} = \frac{u_{ch}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}, \eta_{c,p^c} = \frac{u_{hh}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$, $\eta_{h,p^c} = \frac{u_{ch}u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$, $\eta_{h,p^h} = \frac{u_{cc}u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{h}$, $\eta_{ch} = \frac{u_c u_h}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{ch}$, $\eta_c = \frac{u_c}{u_{ch}^2 - u_{cc}u_{hh}}\frac{1}{c}$.

Proposition 2 elucidates that any disturbance in the real estate market can propagate to consumption via four distinct channels: the substitution effect, wealth effect, credit effect, and precautionary savings effect.[17] The directions these four channels take, in terms of how the housing market boom influences the consumption, is determined by the relative strength of both intertemporal and intratemporal elasticities of substitution between nondurable and durable goods and the specific role that housing wealth assumes within the budget constraint and credit constraint. When overbuilding transpires in a housing market bubble, positive variations in $\widetilde{h}_t$ and $\widetilde{H}_t$ prompt changes in consumption via substitution and wealth effects. Furthermore, it could also endogenously affect consumption through credit and precautionary saving effects. This variation in consumption, engendered by a housing market boom, ultimately impacts physical investment, thereby exacerbating the recession in the future, as long as the overall effect is positive.

It merits attention that η_{x,p^y} represents the standard Frisch elasticity of variable x with respect to the relative price of y, serving a crucial role in moderating the impacts of these four effects. If consumption is more responsive to housing prices than to consumption goods' prices,

[17] Berger et al. (2018) only discussed two of them meticulously and did not focus on the credit effect and precautionary saving effect. Additionally, their goal of decomposition is related to analyzing the inequality problem caused by house price inflation.

a shift in the holdings of housing will induce a more pronounced effect on consumption, as manifested in Φ_H. Conversely, if households' holdings of housing respond more substantially to consumption goods prices (than to housing prices), the elasticity of substitution would dampen all four channels. This occurs because the consumption of durable housing becomes more stable, and households do not significantly alter their consumption, suggesting a minor pass-through from housing to consumption.

3.2 Crowding-out effect of overbuilding

The amplification of the crowding-out effect sparked by overbuilding due to the intratemporal elasticity of substitution, credit constraints, precautionary saving, and wealth inequality will be discussed herein. Overbuilding intuitively affects the consumption and crowded-out physical investment, considering the relationship between consumption and housing as complements at the aggregate level. Similarly, overbuilding tends to ease collateral constraints, facilitating households to borrow more to smooth their consumption demand. Additionally, overbuilding exerts influence on the consumption response via housing prices due to the monotonic increasing inverse supply function of residential assets, $F^H(\cdot)$, in a complete market – more new construction leading to higher housing prices in equilibrium. As the housing price factors into the budget constraint of the household and influences their income, a rise in housing prices makes households perceive an increase in wealth, given the dual function of a house as both a utilitarian good and an asset in the budget constraint. This surge in price, arising from a shift in the supply function (a demand shock), implies that overbuilding aligns with house price inflation via the supply side; otherwise, an inelastic supply function will not generate any overbuilding from a demand shock.

By aggregating the consumption decision of households from Equation 11 and integrating the first-order conditions (FOCs) in supply sectors, a relationship between overbuilding and physical investment can be obtained, as outlined in Proposition 3.

Proposition 3. *The aggregate investment is driven by overbuilding and precautionary saving following*

$$\begin{aligned} \widetilde{II}_t = -\Bigg\{ &\left(\Phi_H + \frac{\nu}{\alpha} p^H H\right) \int \widetilde{h}_t^i dG_i - \Phi_\mu \int \widetilde{\mu}_t^i dG_i \\ &+ \Phi_{p^H} \left[\frac{1}{1-(1-\delta^H)\frac{1}{R}} F^H(\widetilde{H}_t) - \frac{(1-\delta^H)\frac{1}{R}}{1-(1-\delta^H)\frac{1}{R}} \mathbb{E}_t F^H(\widetilde{H}_{t+1}) \right] \\ &- \Phi_{cov}^i \int \widetilde{cov}_t^i dG_i + \frac{\nu}{\alpha} Y_H p^H F^H(\widetilde{H}_t) \Bigg\} \end{aligned} \tag{15}$$

The overbuilding, $\widetilde{H}_t = \int \widetilde{h}_t^i dG_i > 0$, will crowd out physical investment as long as the substitution effect Φ_H and wealth effect Φ_{p^H} are not negative enough and Φ_μ is not positive enough.

Equation 15 reveals that overbuilding results in diminished physical investment and subsequently lower physical capital through distinct mechanisms on the demand and supply sides, at least within a specific parameter range. The term Φ_x pertains to the influence of the pass-through from housing to the consumption, whereas the term $\frac{\nu}{\alpha}$ in 15 is connected to the supply-side effect. The following discussion will explore in detail how the relative intratemporal elasticity of substitution, credit constraint, precautionary savings, and wealth inequality impact the crowding-out effect instigated by overbuilding.

3.2.1 Intratemporal elasticity of substitution

Intertemporal substitution, extensively explored in relation to the Euler equation and monetary policy, stands in contrast to intratemporal substitution between housing and consumption, which remains underexplored both theoretically and empirically. In this section, I argue that intratemporal substitution significantly influences household decision-making processes, especially in the context of the crowding-out effect created by overbuilding. Empirical studies in the housing market suggest that intratemporal substitution holds more significance and potency than intertemporal substitution[18], as households, being primarily myopic or financially constrained, often neglect or simply cannot afford to consider future consumption in their present-day decisions. By analyzing the coefficients of the crowding-out effect as delineated in Proposition 11, Corollary 1 concludes that the relative intratemporal substitution can theoretically amplify the crowding-out effect across the demand side of the housing market.

First, I define the intertemporal and intratemporal elasticity of substitution below:

Definition 2. The intratemporal elasticity of substitution is

$$\mathrm{IAS} = -\frac{\partial \ln \frac{h}{c}}{\partial \ln \frac{U_h}{U_c}} \tag{16}$$

and the intertemporal elasticity of substitution to consumption bundle is

$$\mathrm{IES} = -\frac{U_{BB}}{U_B}$$

Then, based on the definition, I obtain the following corollary.

Corollary 1. *Ceteris paribus, households with a higher intratemporal elasticity of substitution relative to their intertemporal elasticity of substitution and with a CRRA utility function, will crowd out less investment through the substitution and wealth effects.*

It is easy to understand Corollary 1 that consumption and housing services are both normal goods, and if their degree of substitution is high , the crowding-out effect will be further muted since an increase in housing consumption would lead to a corresponding decrease or smaller

[18] Khorunzhina (2021) conducted this vital empirical work.

increase in consumption. The intratemporal elasticity of substitution gauges the extent to which an increase in housing can be substituted by an increase in consumption for a given utility level within a specific period.[19] On the other hand, the intertemporal elasticity of substitution quantifies the inclination to substitute the overall consumption bundle over different periods. If $\mathrm{IAS} > \mathrm{IES}$ holds, households are more likely to adjust their holding of housing and consumption within a given period, rather than across different periods. A relatively larger intratemporal elasticity of substitution implies a lower increase in consumption in response to overbuilding within a given period, as these goods become more substitutable than complementary. The potency of intratemporal substitution is such that it directly influences marginal utility, bypassing the budget constraint; hence, any other elements in the economy that affects the utility of residential assets will doubtless alter the crowding-out effect.

Proposition 4. *When the housing supply is fixed, the initial housing distribution over the dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} > \frac{K}{L} > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}$ *holds, the substitution effect* Φ_H *and wealth effect* Φ_{p^H} *will decrease as the relative intratemporal elasticity of substitution increases. Furthermore, when the aggregate Khun-Tucker multiplier is not too large, the credit effect* Φ_μ *will increase in the relative intratemporal elasticity of substitution.*

Proposition 4 shows that under certain conditions,[20] the relative intratemporal elasticity of substitution will have a clear impact on the substitution, credit and wealth effects. In the subsequent analysis, I dispense with these conditions and quantitatively solve the GE problem to provide a more detailed analysis of the effect of the relative intratemporal elasticity of substitution.

I solve model 8 with unit intratemporal elasticity such that $\mathrm{IAS} = 1$ and change the intertemporal elasticity from 0.67 to 0.5, effectively increasing the relative intratemporal elasticity. As depicted in Figure 6a, increasing the relative intratemporal elasticity results in a contraction of the substitution effect. Theoretically, a preference shock increasing the relative intratemporal elasticity of substitution compared to the intertemporal elasticity will reduce the response of consumption to a given level of overbuilding, subsequently leading to a smaller extent of crowded-out investment. This outcome can be attributed to the abated complementarity between consumption and housing due to the enhanced substitution. Additionally, a higher propensity for substitution can alleviate the collateral constraint, given the reduced demand for consumption, thereby causing fewer households to remain financially constrained in steady state. For a mathematical elucidation of the above argument, let us consider two economies, a and b. In these two economies, the relative intratemporal elasticity of substitution satisfies $\frac{\mathrm{IAS}_a}{\mathrm{IES}_{c,a}} < \frac{\mathrm{IAS}_b}{\mathrm{IES}_{c,b}}$. Suppose that an unexpected tax rebate is given to households in each of these economies, triggering the

[19] It is intuitive to focus on U_{ch}, which is closely related to the complementarity between housing and consumption.

[20] It is difficult to implement two state variable Bewley-Huggett-Aiyagari model under theory based on Von-Neumann algebra inStokey (1989) because the topology is too complicated. Thus, these conditions help to direct the dimension of distribution.

same increase in consumption, $\Delta C_a = \Delta C_b = 0.5$. Given that the intratemporal elasticity in economy a is smaller than that in b, households in a will increase their durable consumption by more, say, $\Delta H_a = 0.5 > \Delta H_b = 0.3$. This increased residential asset holding eases the collateral constraint, with the extent of relief being proportionate to the change in residential assets. Therefore, the Karush-Kuhn-Tucker multiplier in Equation 10 yields $\Delta\mu_a < \Delta\mu_b < 0$, implying $\Phi_\mu^a > \Phi_\mu^b > 0$ in Equation 11 as $\Phi_\mu^i = -\frac{\Delta C_i}{\Delta \mu_i}$. This trend is represented in Figure 6b, where the credit constraint progressively expands.

In addition to substitution and credit effects, overbuilding also influences pass-through consumption responses through the inverse supply function $F^H(\cdot)$. Note that residential assets not only act as consumable goods within a utility function but also act as a type of asset within the budget constraint. A surge in housing prices, often stimulated by overbuilding, augments household liquidity as long as households previously hold some amount of housing. The resulting wealth effect is amplified when a unit of housing, in terms of value, translates to a higher utility under a diminished intratemporal elasticity of substitution. Intratemporal decisions between housing and consumption, driven by this wealth effect, adhere to equation $\frac{U_{h,t}}{U_{c,t}} = f\left(p_t^+, p_{t+1}^-\right)$, which is rather intuitive. Consider a scenario where households buy one additional unit of housing at time t and obtain $U_{h,t}$ units of utility. Alternatively, these households could expend equivalent money on consumption, obtaining $U_{c,t} f\left(p_t^+, p_{t+1}^-\right)$ units of additional utility. Here, the unit of consumption is scaled against the relative housing price. Introducing an equivalent jump in housing prices $\Delta p_{a,t}^H = \Delta p_{b,t}^H > 0$ in both economies a and b while holding housing fixed triggers a spike in consumption, leading to a positive Φ_{p^H} in Equation 11. Such a jump in consumption $\Delta C_t > 0$ aligns with the reduced marginal utility of consumption $\Delta U_{c,t} < 0$ and an increased demand for durable goods (owing to their complementarity), resulting in an increased marginal utility of durable goods $\Delta U_{h,t} > 0$. A larger relative intratemporal elasticity of substitution permits greater disparities between the marginal utilities of housing and consumption. Consequently, an uptick in consumption can be sufficient to sustain a given variation ($\Delta f\left(p_t, p_{t+1}\right) > 0$) in relative marginal utility, which means that a one-unit increase in housing prices induces a smaller increase in consumption in the current period. This further amplifies the crowding-out effect via wealth effects and the pass-through from durable to consumption. Figure 6c exhibits the decreased influence of the wealth channel on the crowding-out effect as relative intratemporal elasticity increases, marking one unit of housing less important (it can more easily be replaced by consumption). While this section eschews a quantitative introduction of aggregate shocks into our model and does not address the varying magnitude of the precautionary saving effect, it remains evident that a higher relative intratemporal elasticity of substitution encourages a diminished precautionary saving effect because the household prefers balanced consumption portfolios within a period to portfolios over multiple periods. In conclusion, overbuilding impacts the crowding-out effect through four channels, with three being significantly influenced by the relative intratemporal elasticity of substitution.

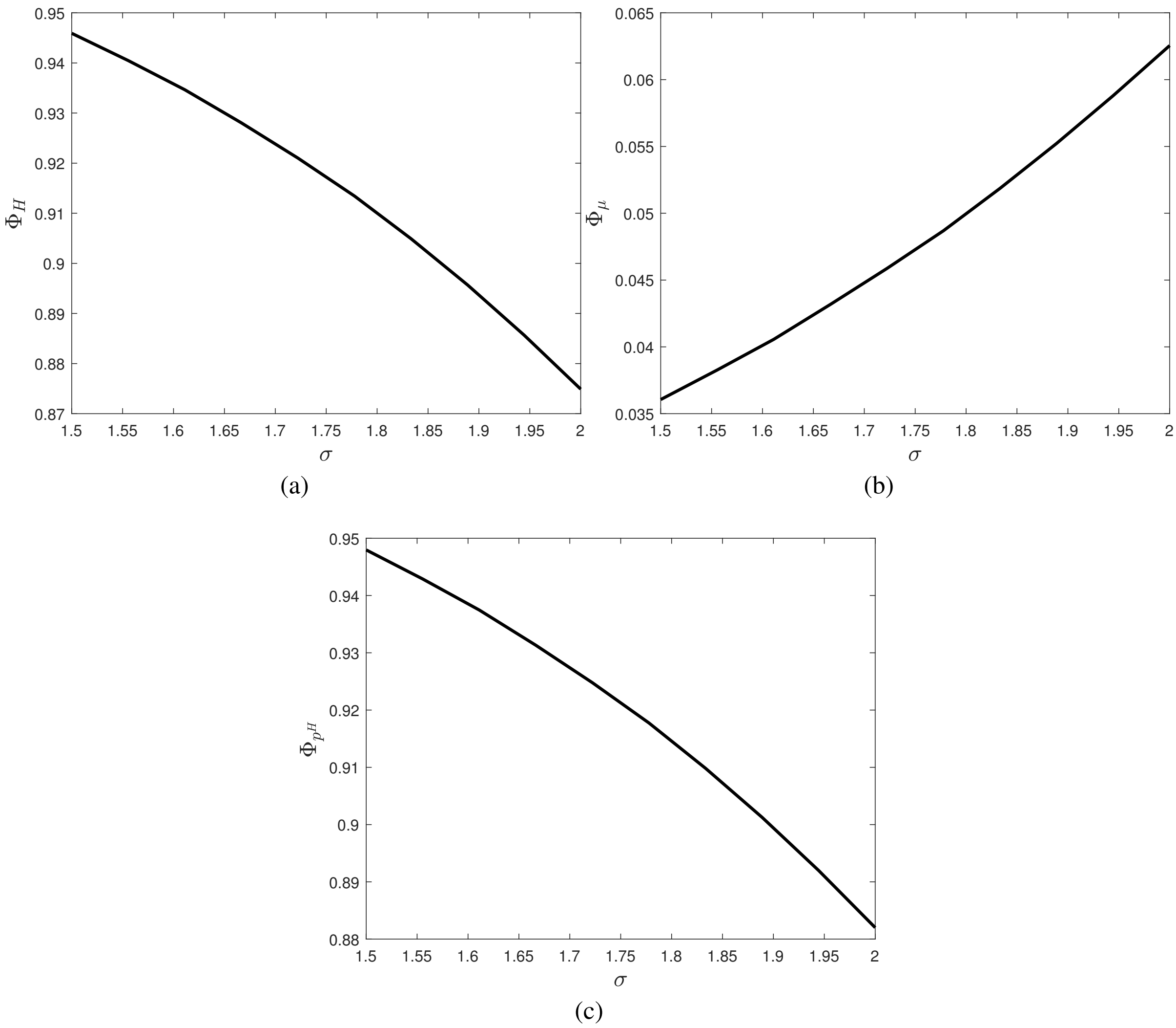


Figure 6: Elasticity of Substitution

3.2.2 Credit constraint and liquidity

Overbuilding and housing market booms influence household consumption, a shift that is primarily attributable to the substitution effect. Additionally, in an incomplete market, where households cannot fully insure themselves against idiosyncratic shocks via financial markets, households' consumption patterns may be bounded by market constraints, limiting their borrowing capabilities to address adverse shocks. These credit constraints give rise to liquidity challenges. Consequently, certain households occasionally face constraints, impeding them from satisfying their consumption demands, even if they are able to repay their future borrowing. Overbuilding introduces a higher volume of assets that households can employ as collateral, ameliorating the loss introduced by the credit constraints. In Figure 7a, the extent of financial friction decreases, attributed to an increase in the proportion of housing value that can be leveraged for borrowing—from 0.5 to 0.8. This verifies the assertion that stricter collateral constraints augment the substitution effect, as the marginal utility of housing is higher in a steady-state scenario.

Proposition 5. *When the housing supply is fixed, the initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} > \frac{K}{L} > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}$ *holds, the substitution effect* Φ_H *and wealth effect* Φ_{p^H} *will decrease as the collateral constraint slackens. Furthermore, when the aggregate Khun-Tucker multiplier is not too large, the credit effect* Φ_μ *will increase as collateral constraint slackens.*

Moreover, a marginal relaxation of the binding collateral constraint is associated with a reduced K-T multiplier, as indicated by $\Delta\mu < 0$ in Equation 11. In contrast, a tighter constraint corresponds to a diminished consumption response, Φ_μ, which in turn leads to a smaller crowding-out effect. To clarify the credit effect, consider an assumption where an unanticipated tax rebate leads to equivalent increases in consumption in economies a and b, denoted as $\Delta C_{a,t} = \Delta C_{b,t}$. If the collateral constraint, γ, in economy a is more stringent than in economy b, then $\gamma_a < \gamma_b$ will hold both in Equation 10 and in Figure 7. A stricter financial constraint reveals a more pronounced K-T multiplier response. Thus, the absolute change in the multiplier in economy a surpasses that in economy b ($\Delta\mu_a < \Delta\mu_b < 0$). This suggests that under a tight financial constraint, a unit change in the marginal value of housing is less effective. The reason is that under such circumstances, a unit change in marginal value is comparatively "cheaper" than its steady-state counterpart. Figure 6b explicitly demonstrates that a credit crunch (a positive $\widetilde{\mu}_t$), triggered by overbuilding, leads to a less pronounced reduction in consumption (or a greater crowding out of investment) when financial friction is more substantial.

In contrast to the credit effect, financial friction operates inversely concerning the wealth effect and substitution effect. Mathematically, a larger financial friction leads to an increased K-T multiplier and a larger μ, resulting in a more pronounced wealth effect, as depicted in Figure 7c. The underlying mechanism mirrors that of substitution, given that both housing services and their pricing play the same role within the collateral constraint 10. Their influence on the pass-through is consistent. All these results hold theoretically and are derived under certain stringent conditions, as expressed in Proposition 5.

3.2.3 Precautionary saving and wealth inequality

Households tend to exhibit less consumption than in the absence of idiosyncratic shocks or if they possessed perfect insurance against such shocks. This propensity toward saving as a safeguard against unforeseen idiosyncratic shocks is referred to as the precautionary saving motive. The final term in Equation 15 elucidates that precautionary saving decreases consumption, as households allocate an additional amount $\Phi_{cov}\widetilde{cov}_t$ to savings rather than expenditure in the face of income uncertainties.

Beyond the four previously discussed effects – the substitution, credit, wealth, and precautionary saving effects – overbuilding can magnify the crowding-out effect over the business cycle. It is a recognized fact that idiosyncratic shocks are countercyclical, whereas overbuilding tends

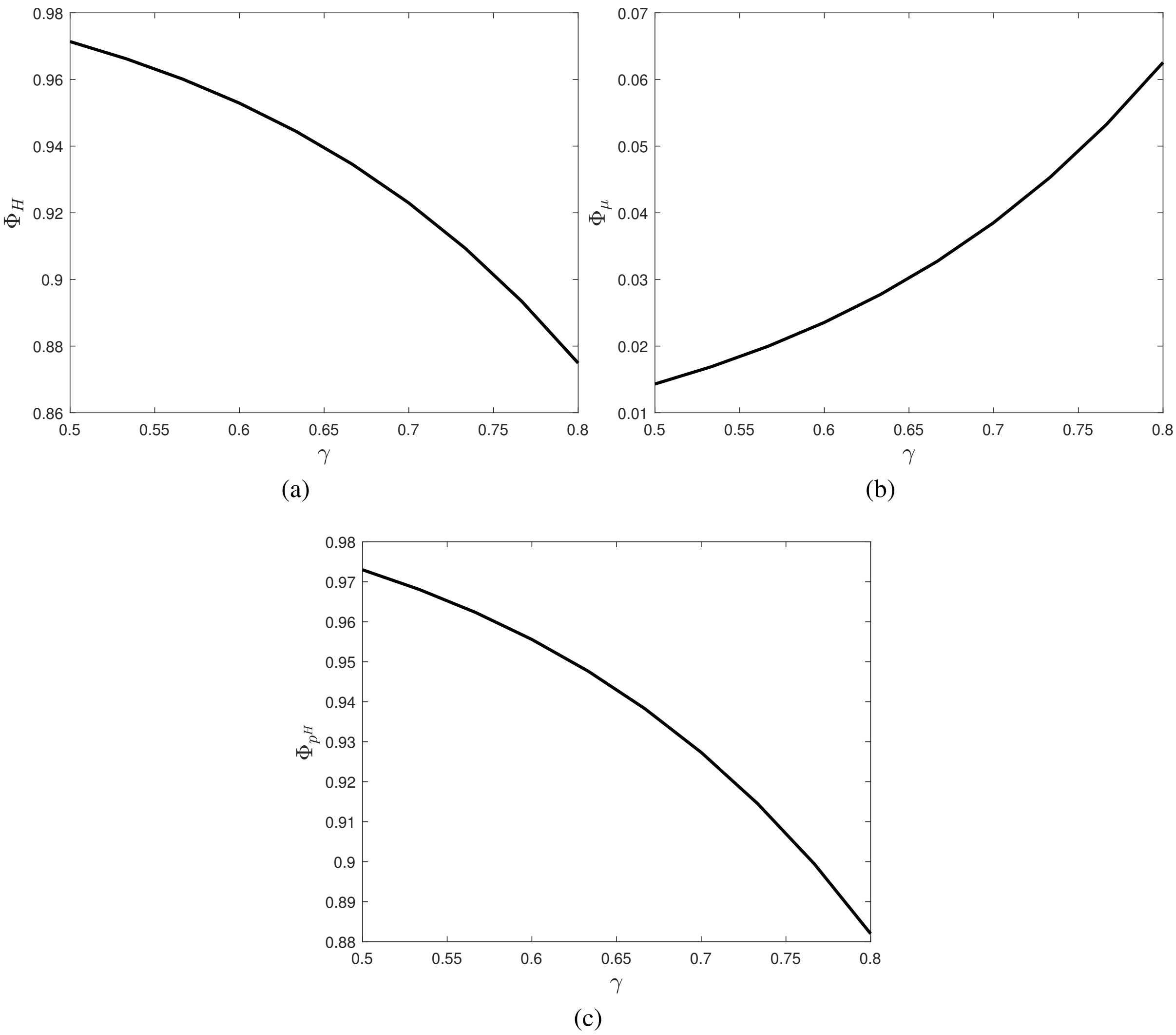


Figure 7: Financial friction

to be procyclical. Consequently, during periods of overbuilding, households exhibit reduced precautionary behavior due to improved aggregate economic conditions and diminished severe idiosyncratic shocks. A booming economy combined with lower idiosyncratic shock variability emboldens households, leading them to increase consumption and reduce savings. The term $\widetilde{cov}_t$ in Equation 15 will decrease, indicating more consumption and less savings during overbuilding and economic upturns. Nevertheless, this amplification effect is beyond the scope of my current numerical experimentation and remains a subject for future research.

Furthermore, the wealth distribution can potentially influence the crowding-out effect initiated by overbuilding via the aggregation process. Given that increased holdings of housing are financed through liquid assets and wage income, the most significant per capita jump in housing asset holdings typically comes from households possessing abundant liquid assets and earning high incomes. Aggregating the consumption decisions across households, as presented in Equation 15, reveals the significance of the wealth distribution, particularly with respect to the distribution of coefficients, subsequently affecting the aggregate crowding-out effect. Figure 8a

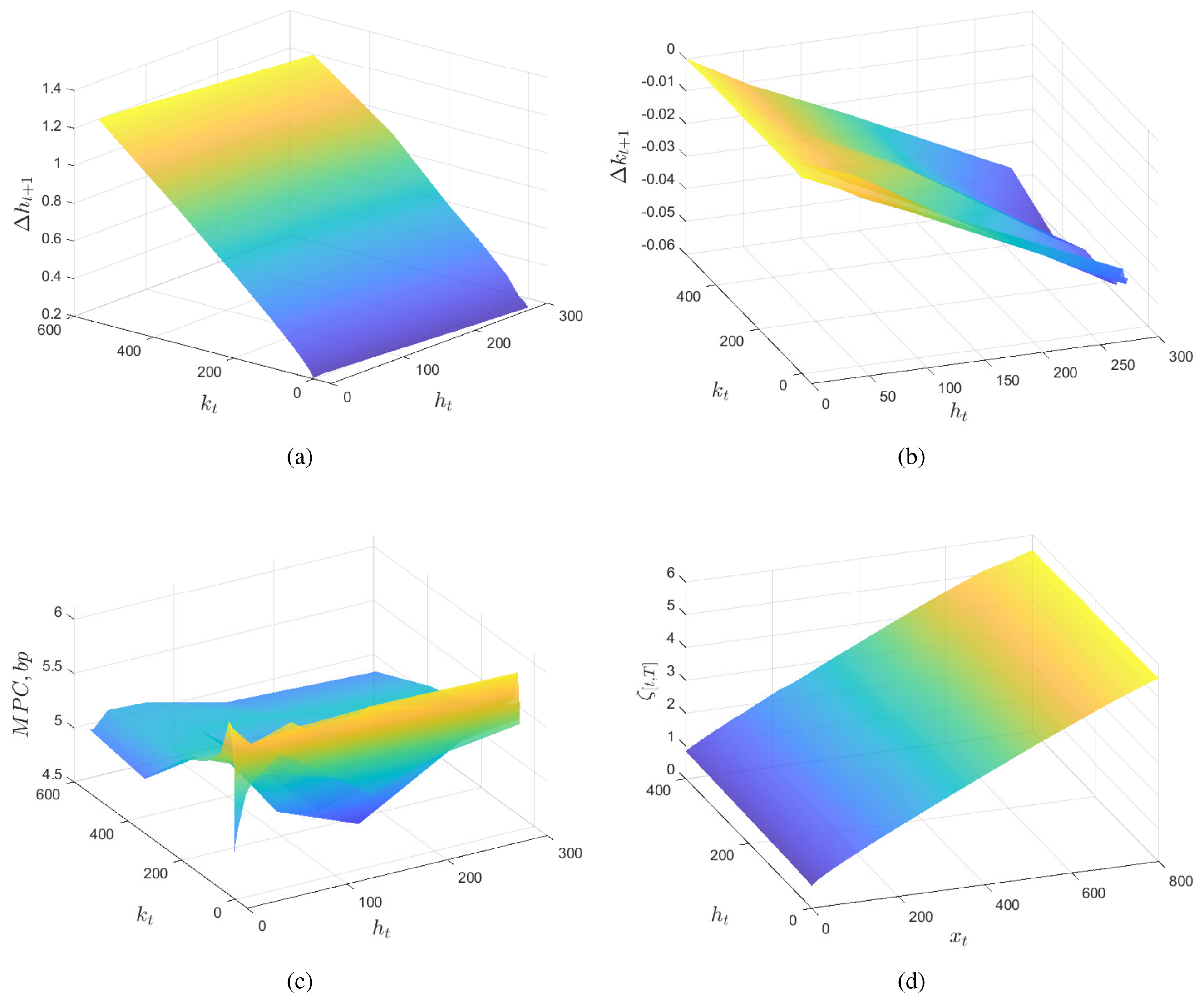
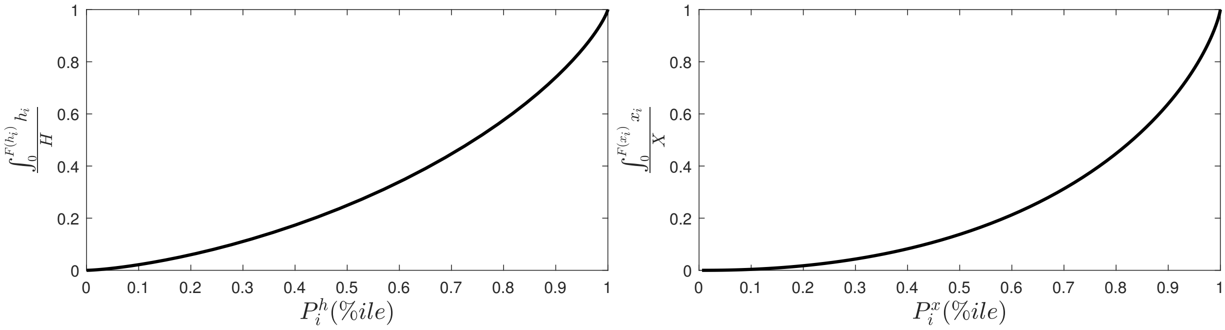


Figure 8: Wealth Distribution

delineates the distribution changes in housing holdings when housing prices decrease. Wealthy households with significant liquid assets are the primary purchasers of additional housing units, subsequently decreasing the physical investment, as illustrated in Figure 8b. Although the cohort mass is numerically small, the distribution of wealth is significantly left-skewed, with the skewness being evident in Figures 9a (for residential assets) and 9b (for effective liquid assets). The most wealth is concentrated among a minority at the top tier, and this skewed wealth distribution accentuates the overbuilding-induced crowding-out effect, as represented by the term $\int \widetilde{h_t}^i dG_i$ in Equation 15. Additionally, with the distribution of the MPC being right-skewed (Figure 8c), the standard general equilibrium effect for hand-to-mouth households remains valid, especially in the monetary policy pass-through context. This right-skewed MPC also intensifies the crowding-out effect, albeit through the term $\int \widetilde{\mu}_t^i dG_i$ in Equation 15. Figure 8d illustrates the wealth distribution effect of a demand-driven boom, which I argued in Corollary 2 arises from anticipated housing price inflation, in contrast to the supply-driven booms represented in Figures 8a and 8b.



(a) Lorenz curve of $F(h_i)$ (b) Lorenz curve of $F(x_i)$

Figure 9: Lorenz curve

3.2.4 Optimistic expectations and overbuilding

The previous discussions have primarily centered on the crowding-out effect generated by overbuilding, examining the various mechanisms through which this effect manifests, contingent on the assumption of the occurrence of overbuilding. In this section, I contend that the presumption of overbuilding is not strong; indeed, an optimistic expectation regarding the future housing market can create overbuilding. When households have positive expectations regarding future housing prices, they tend to augment their current real estate holdings. This behavior parallels the consumption adjustments driven by the intertemporal New Keynesian framework. Corollary 2 shows that an upswing in the anticipated housing price at time $T+1$ induces a marginal surge of $-\left[\beta\left(1-\delta^H\right)\right]^T \Pi_{s=1}^T \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}} \lambda_{t+T+1}/u''_{h^i}$ units in housing demand. If such expectations are fueled by misplaced optimism or unfounded news, the ensuing rise in construction may well translate to overbuilding. This is because such expansion is not rooted in foundational shifts but is instead supported by an illusion. Once this illusion dissipates, the crowding-out effect could catalyze a recession, given the lack of physical capital that was misdirected during the housing market boom.

Corollary 2. *Ceteris paribus, a positive expectation about the housing price change at time $T+1$ will induce a jump in demand for housing at time t. The response extends as follows:*

$$\widetilde{h}_t^i\Big|_{h_{t+i},\mu_{t+i},\lambda_{t+i},,i\in[1,T]} = \zeta_t^i dp_{t+T+1}^H \tag{17}$$

where $\zeta_t^i = -\frac{1}{u''_{h^i}}\mathbb{E}_t\left[\beta\left(1-\delta^H\right)\right]^T \Pi_{s=1}^T \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}}\lambda_{t+T+1}$

4 Crowding-out effect of overbuilding: Full-fledged model

In the preceding section, I utilized a simple model to demonstrate that expectations of a future housing market boom can motivate households to augment their consumption of durable goods. This, in turn, can crowd out physical investment. This crowding-out effect is influenced by several factors, namely the relative intratemporal elasticity of substitution, credit constraints, and the distribution of wealth. In this section, I employ a full-fledged model to provide a quantitative analysis of the crowding-out effect. By aligning this model with empirical data, I intend to elucidate how news regarding the future can generate a boom-bust cycle in the housing market. Particularly, if such news proves to be inaccurate and households only realize this after a certain period, the ensuing boom—supported by misinformation rather than economic fundamentals—will induce overbuilding. This misallocation can subsequently lead to significant declines in both output and consumption during the bust phase. To proceed, I first describe the model adopted for this quantitative analysis. Next, calibration and the full-information Bayesian method will be used to integrate the model with empirical data. Finally, I highlight the severe recession resulting from overbuilding, as evidenced through certain IRFs.

4.1 Model Setting

4.1.1 Household

Assume that the household[21] holds houses h_{t-1} and a liquid asset b_{t-1} at time t, which he takes from the previous period. He chooses consumption c_t, labor supply l_t, houses h_t and liquid asset holding b_t at time t to solve the optimization problem

$$V(h_{t-1}, b_{t-1}, \varepsilon_{t-1}) = \max_{c,l,b',h'} U(c_t, h_t, l_t) + \beta EV(h_t, b_t, \varepsilon_t)$$

$$\begin{aligned} \text{s.t.} c_t + Q_t b_t + p_t^h \left[h_t - (1-\delta^h)h_{t-1}\right] = & R_t Q_{t-1} b_{t-1} + (1-\tau) w_t l_t \varepsilon_t + \Pi_t^h \\ & - p_t^h C\left(h_t, h_{t-1}\right) + T_t \end{aligned} \tag{18}$$

$$-Q_t b_t \leq \gamma p_t^h h_t \tag{19}$$

where p_t^h is the relative price of a housing unit at time t. R_t is the gross real return of the liquid asset, which follows $R_t = \frac{Q_t(1-\delta)+r_t}{Q_{t-1}}$. $C\left(h_t, h_{t-1}\right)$ is the adjustment cost function when the household wants to adjust its holdings of housing. γ is the parameter governing the slackness of the collateral constraint. δ^h is the depreciation rate. τ is wage income. Π_t^h is the profit rebated from construction companies. T is the lump-sum tax transfer payed by the government. ε_t is

[21] Here, for simplicity, I omit the index for a specific household i.

the idiosyncratic income shock that follows a logarithmic AR1 process with coefficient ρ_{ε} and standard derivation σ_{ε}.

The adjustment function follows the canonical form

$$C\left(h_t, h_{t-1}\right) = \frac{\kappa_1}{\kappa_2}\left(h_{t-1} + \kappa_0\right)\left|\frac{h_t - h_{t-1}}{h_{t-1} + \kappa_0}\right|^{\kappa_2}$$

The utility function follows the CRRA form[22]

$$U(c_t, h_t, l_t) = \frac{\left(c_t^{\phi} h_t^{1-\phi}\right)^{1-\sigma}}{1-\sigma} + \kappa \frac{l_t^{1+\psi}}{1+\psi}$$

4.1.2 Firm

There are two types of firms, construction firms that produce housing and the nondurable goods producers. Both of these two types of producers operate in a complete market, but because the construction firms also use exogenous land supply as an input to construct housing, they earn a nonzero profit, which is ultimately repaid to their shareholder, the household.

Nondurable goods producers use

$$Y_{N,t} = A_{n,t} K_{n,t}^{\alpha} L_{n,t}^{1-\alpha} \tag{20}$$

to maximize profit with the costs coming from the real rental rate of capital K_n and the related wage payment to labor L_n .

Similarly, durable goods (housing) producers use

$$Y_{H,t} = A_{h,t} \mathscr{L}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota} \tag{21}$$

to maximize profit with the cost coming from the real rental rate of capital $K_{h,t}$ and the related wage payment to labor $L_{h,t}$. The $\mathscr{L}_t^{\theta}$ in the production function is the exogenous land supply and follows $\mathscr{L}_t^{\theta} = \overline{\mathscr{L}} A_{L,t}$, and the new constructions are homogeneous to each production factor; hence, the share of input satisfies $\theta + \nu + \iota = 1$.

[22]Piazzesi et al. (2007) use CEX data suggest that intratemporal elasticity of substitution is close to 1. In other words the utility function form of durable and consumption is close to standard Cobb-Douglas case.

4.1.3 Capital Producer

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned}
&\max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Lambda_{t,t+\tau} \left\{ Q_\tau I_\tau \eta_{I,t} - f\left(I_\tau, I_{\tau-1}\right) I_\tau \eta_{I,t} - I_\tau \right\} \\
&\text{s.t. } f\left(I_\tau, I_{\tau-1}\right) = \frac{\psi_I}{2} \left(\frac{I_\tau}{I_{\tau-1}} - 1 \right)^2
\end{aligned}$$

By solving above optimization problem, I obtain the capital price as a convex function of investment, which is shown below

$$\begin{aligned}
Q_t \delta_{I,t} = 1 + \frac{\psi_I}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \eta_{I,t} + \psi_I \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \eta_{I,t} - \\
E_t \beta \Lambda_{t,t+1} \psi_I \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2
\end{aligned} \tag{22}$$

where $\eta_{I,t}$ is the marginal efficiency of the investment shock, following Justiniano et al. (2011).

4.1.4 Market cleaning

Capital is supplied by households with their gross net liquid assets, and labor is supplied in effective form

$$\begin{aligned}
K_t &= \int b_t dG_t = K_{n,t} + K_{h,t} \\
L_t &= L_{h,t} + L_{n,t} = \int \varepsilon_t l dG_t \\
H_t &= \int h_t dG_t
\end{aligned}$$

The goods market cleaning condition is

$$C_t + I_t + f\left(I_t, I_{t-1}\right) I_t \eta_{I,t} + p_t^h \int C(h_t, h_{t-1}) dG_t = Y_{N,t}$$

where $K_t = (1-\delta)K_{t-1} + \eta_{I,t} I_t$ and G_t is the cumulative distribution function.

Similarly, the housing market cleaning condition is

$$\left[H_t - (1-\delta^h) H_t\right] = Y_{H,t}$$

The return on gross liquid assets b_t comes from two components: capital return from firms r_t and capital gain $\frac{Q_t(1-\delta)}{Q_{t-1}}$.

Finally, the government closes the economy by $T = \tau w L$ and $\Pi_t^h = p_t^h Y_{H,t} - w_t L_{h,t} - (r_t -$

$1 + \delta)K_{h,t}$.

4.1.5 Shocks

The model contains three types of shock: a *contemporaneous unexpected shock, news shock and noise shock*. There are two fundamental shocks to the TFP of the two production functions 20 and 21. These two shocks a_t^i follow the standard logarithmic AR(1) process $\log(a_t^i) = \rho_a^i \log(a_{t-1}^i) + \varepsilon_t^{a^i}$ where $i \in \{h, n\}$. Thus, the TFPs of these two production functions follow $A_{n,t} = a_t^n \overline{A}_n$ and $A_{h,t} = a_t^h \overline{A}_h$. I introduce a preference shock Φ_t^ϕ to the preference ϕ in the utility function on the demand side, cooperating with a land supply shock Φ_t^L and to determine the housing market.

Moreover, to incorporate noise and news into the model, I assume that the household can obtain news related to the shocks up to 8 periods before the shocks realize, and I define them in companion form in Equation 93. However, the agents cannot perfectly observe these shocks but do so in conjunction with a noisy observation shock to $\widetilde{\Phi}_t^i$ in Equation 95.[23] I relegate details about the news and noise shocks to Appendix H.7.1, in which I introduce the news and noise shock following Chahrour and Jurado (2018), who introduced the news and noise representation to overcome the observational equivalence problem in previous literature such as Schmitt-Grohé and Uribe (2012), Barsky and Sims (2012) and Blanchard et al. (2013).

4.2 Calibration

4.2.1 Parameters

Most of the parameters on the production side come from the literature and are standard and robust. These parameters have been relegated to Appendix H.1 and summarized in Table 9. I use the discount factor, disutility to labor supply, and three parameters on the production side to match the gross quarterly real interest rate of 1.015 , labor supply of 1, physical investment-to-GDP ratio of 0.13 and new construction-to-GDP ratio of 0.05. The proportion of physical investment to GDP is estimated from private nonresidential fixed investment relative to GDP. Similarly, the ratio of new construction to GDP is computed based on private residential fixed investment over GDP. The parameters in the adjustment cost function are in line with Kaplan et al. (2018) and Auclert et al. (2021). The intertemporal elasticity of substitution and preference between housing and consumption are borrowed from Kaplan et al. (2020). The AR1 coefficient and standard derivation of the idiosyncratic shock follow the estimation by McKay et al. (2016). All the values of corresponding parameters are summarized in Table 1.

[23] I define the news and noise shocks following the suggestion made by Chahrour and Jurado (2018) because this form does not suffer from the observational equivalence problem.

Table 1: Key Parameter Values

Parameter	Value	Description
β	0.9749	Discount factor
τ	0.20	Labor income tax
κ	-1.28	Disutility to supply labor
γ	0.8	Slackness of collateral constraint
κ_0	0.25	Adjustment cost silent set
κ_1	1.3	Adjustment cost slope
κ_2	2	Adjustment cost curvature
σ	2	Inverse of intertemporal elasticity of substitution
ϕ	0.88	Preference between housing and consumption
ρ_ε	0.966	AR1 coefficient of income shock
σ_ε	0.25	SD of income shock

4.2.2 Data to Model: Moment Matching

Although I do not specifically match the moments in the distribution, my model has considerable merit in replicating the moments extracted from data. Table 2 shows that my model has some natural ability to reflect reality when I compare the data estimated by Kaplan et al. (2014) and Kaplan et al. (2018) and the moments calculated from my model.

Table 2: Distribution Moments

Description	Data	Model
Poor Hand-to-Mouth Household	0.121	0.1102
Wealthy Hand-to-Mouth Household	0.192	0.2059
Top 10 percent share of Liquid asset	0.8	0.5
Top 10 percent share of Illiquid asset	0.7	0.3

To build a bridge between the model and data, I use full information Bayesian method to estimate the parameters that pertain to the dynamic and business cycle. Particularly, I resolve parameters in 7 shock series from 7 variables. For similarity, I assume that the covariance matrix of shocks is a diagonal matrix hence that all the shocks are independent and there are no parameters related to covariance terms in the estimation. All details about the estimation are relegated to Appendix H.2.2.2.

Table 3: Real Business Cycle Moments

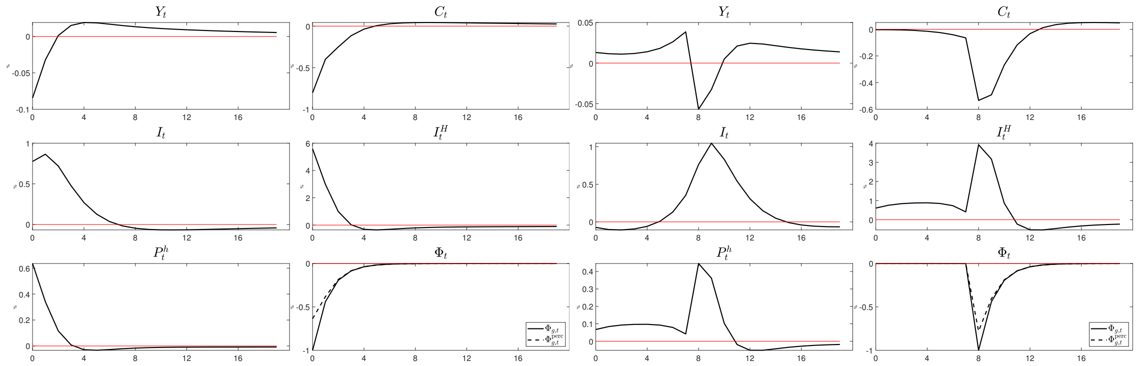
Moments	Description	Model	Data
σ_Y	Standard deviation of output	0.04	0.02
$\frac{\sigma_{p^H}}{\sigma_Y}$	Relative standard deviation between housing price and output	1.57	1.46
$\frac{\sigma_I}{\sigma_Y}$	Relative standard deviation between physical investment and output	3.92	3.19
$\frac{\sigma_{I^H}}{\sigma_Y}$	Relative standard deviation between new construction and output	12.42	8.88
$\text{corr}(p^H, I^H)$	Correlation between real estate price and new construction	0.42	0.23
$\text{corr}(I, I^H)$	Correlation between physical investment and new construction	-0.15	-0.28
$\text{corr}(I, Y)$	Correlation between physical investment and output	0.06	0.19
$\text{corr}(I, Q)$	Correlation between physical investment and capital price	0.40	0.32

The moments in the data in Table 3 are calculated by detrending the trend from quarterly time series via the HP filter, and for the purpose of comparability to the filtered data, I also follow the method proposed by Uhlig et al. (1995) and Ravn and Uhlig (2002) to calculate the model moments in the frequency domain with some algebraic modifications that are discussed in Appendix H.2.1. Table 3 summarizes the primary moments related to the housing market and physical capital investment on which I focus in this paper. The results show that the model is in line with reality.

4.3 Quantitative Analysis

4.3.1 Overbuilding and boom-bust cycle: News in the future and inefficiency of imperfect information

Upon the realization of a contemporaneous preference shock, households tend to reduce their consumption in favor of increased durable consumption, particularly housing services. Such a preference shift generates an increase in housing prices, owing to a rightward shift of the demand curve and a housing market boom, as depicted in Figure 10a. Interestingly, a one-unit preference shock translates to a 0.6 perception of the shock, because of the imperfect information. Consequently, they increase their consumption of houses, leading to a jump in construction and housing prices.



(a) (b)

Figure 10: Contemporaneous and News shock

However, if households know this shock in advance, they will also respond to this shock in advance. They increase their holdings of houses instantaneously, leading to a jump in housing prices. This overbuilding in real estate can displace physical investments through general equilibrium effects. Moreover, households might also increase their overall consumption, either due to higher wage income or the ability to secure more loans from financial institutions, if the wealth effect is strong enough. This has the potential to exacerbate the crowding-out effect, especially as consumption becomes a part of the goods market equilibrium condition. However, in Figure 10b, the estimation result indicates that an impending preference shock in advance corresponds to a small wealth effect. Concurrently, while their consumption does not increase significantly, the other demand shocks, such as the credit shock or depreciation shock, may lead to a significant jump in consumption.

In scenarios where by illusions rather than fundamental adjustments, the inefficiencies stemming from imperfect information can incur welfare losses. Figure 11 illustrates the welfare loss from such imperfect information. The right column delineates the investment responses and welfare variation following a shock to preferences. Observing the diminishing contributions of consumption to overall utility, households perceive this shift as the dashed line in the top row. Given the higher utility derived from housing services, households increase their consumption in this sector, resulting in a rise in aggregate welfare. In the absence of any response to the shock, they will lose some welfare with respect to the situation when they react, as the reaction is derived from the optimization problem. However, an increase in welfare emerges due to distributional effects and the presence of hand-to-mouth households.

Opposite to the realized preference shock, the left column of Figure 11 illustrates the responses to a noise shock. Misinterpreting this as a preference shock, households increase their investment in residential assets. This misguided response inflicts a welfare loss on households, as represented by the solid line at the bottom. In the absence of reactions to this noise shock, welfare would remain unchanged, as nothing fundamentally happens. These experiments corroborate the inefficiencies associated with imperfect information, whereby individuals can be misled into proceeding housing market booms. The experiments elucidate how fake news can potentially

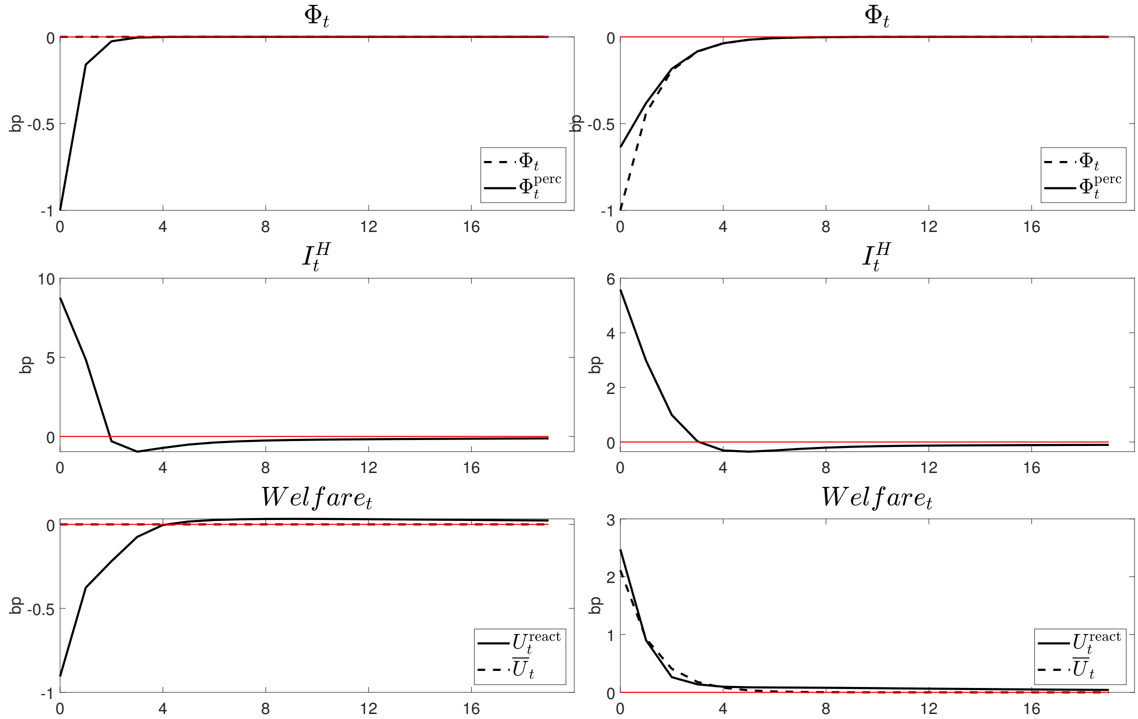


Figure 11: Welfare Loss in Imperfect Information

trigger further losses in output and consumption due to crowded-out physical capital.

4.3.2 Overbuilding and Boom-bust Cycle: Fake News

Upon receiving a noisy shock, households react based on the same dynamics they would attribute to a fundamental change. This behavior stems from the existence of information frictions. Households, in essence, do not possess the capability to discern the precise magnitude of the shock. Instead, they response based on a signal that might be contaminated by noise. Consequently, their actions are anchored to their perceptions or beliefs, rather than the underlying factual shock. In scenarios where households anticipate a future housing market boom, they increase their housing holdings and decrease their savings. This can be detrimental in the long run, especially if their beliefs are misguided and the perceived housing boom is a fantasy. Upon this realization, households recognize that they must urgently invest more resources in physical capital, having previously shifted their focus to real estate. This increase in demand for physical capital results in a decline in consumption, culminating in a significant welfare loss. Additionally, the role of real estate, as a form of wealth (which households typically leverage to secure loans), contributes to the welfare loss during a housing market downturn. As house prices drop, consumption, especially of lower-income households, decline significantly and the financial market disruption exacerbates welfare losses.

Figure 12 compares the impulse responses to a fake-news preference shock, in scenarios with and without preexisting crowded-out physical capital, and demonstrates the large output and welfare loss resulting from the crowding-out effect. The blue solid lines depict responses

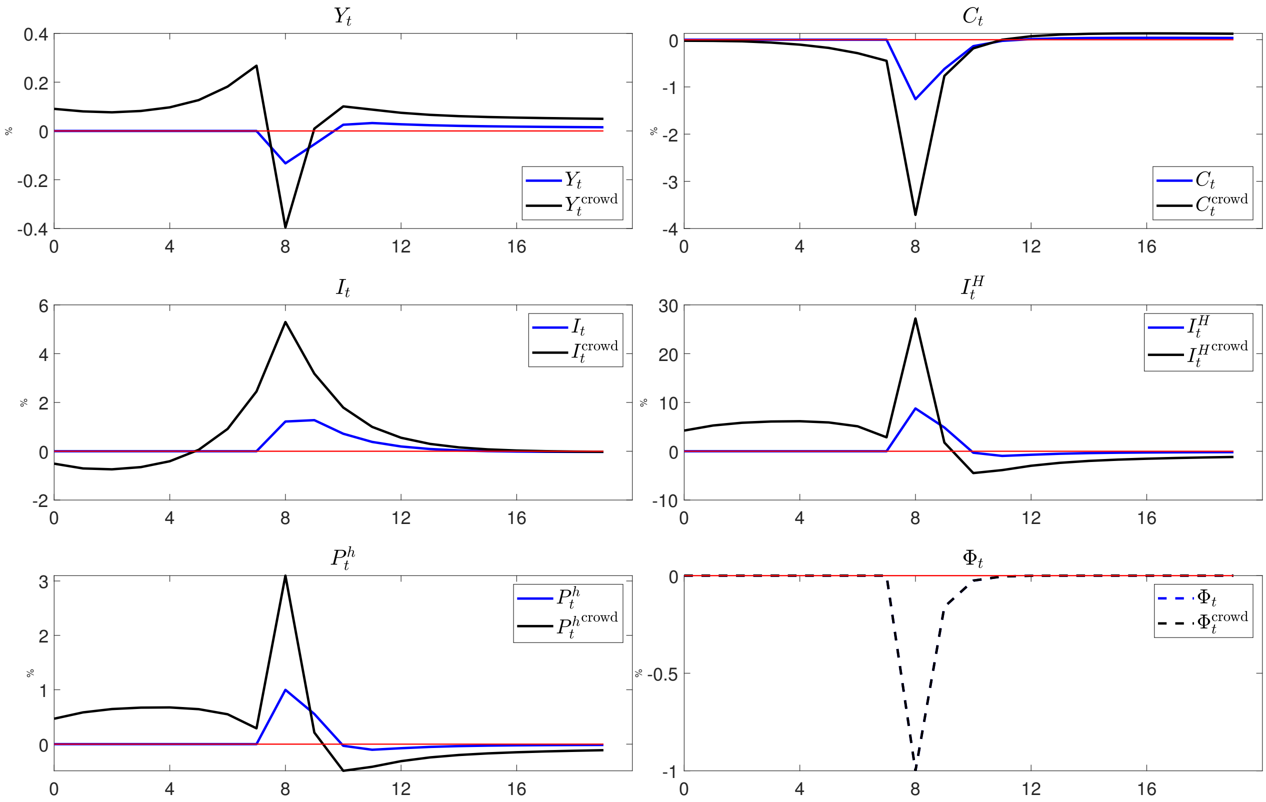


Figure 12: Fake news shock

to a contemporaneous noise shock, $\widetilde{\Phi}_t^{\phi}$, with respect to production, consumption, physical investment, new construction, and real housing prices. The black solid lines represent the responses to the noisy news shock, $\widetilde{\Phi}_{t+8}^{\phi}$, that is disclosed to households eight periods ahead. At the announcement of a potential economic boom in the future, households increase their real estate investments, inducing an immediate housing market boom. This housing market boom spurs a mild response in consumption, because of a smaller wealth effect of the preference shock, rather than credit shock that is argued in Mian et al. (2013). As the forecasted boom materializes two years later (in the ninth period), households gradually become aware of the truth, thereby increasing their savings because of the high real interest rate originating from the deficit created by earlier crowding out. This is accompanied by a housing market downturn, with a 3.5% drop in housing prices. Conversely, in scenarios without prior crowded-out physical capital, economic responses are considerably more tempered, characterized by lesser output losses and milder market fluctuations. The drop in housing prices and consumption are approximately one-third of their counterparts in the crowded-out case. This difference in impulse response demonstrates the crowding-out effect within housing market boom-bust cycles.

In period 8, households alter their perceptions of the fundamental economic framework, as this is the point when the shock takes effect, adjusting their understanding of foundational economic shifts. Households reduce their nondurable consumption at this time, driven by the dominant substitution effect overtaking the wealth effect. Notably, this dominant substitution effect might not be as significant under a non-preference shock, such as a credit crunch shock.

Under a preference shock, households derive greater utility from substituting housing with non-durable consumption. However, given the illiquidity of real estate, they choose to invest more in residential assets, consequently diminishing their marginal utility for consumption. This decrease in marginal utility amplifies households' propensity to defer consumption to the future, leading to an increase in the stochastic discount factor. The increased stochastic discount factor, in turn, boosts the price of capital, rendering savings in physical capital more appealing to households. This dynamic explains the observed surge in physical investment.

4.3.3 Idiosyncratic income shock, financial friction, relative intratemporal elasticity of substitution

In this section, the focus is on elucidating how the crowding-out effect is influenced by factors in the economy, such as idiosyncratic income shocks, financial frictions, and the relative intratemporal elasticity of substitution. To undertake this investigation, I maintain a constant expected jump in housing prices while varying relevant parameters. A modification of the relative intratemporal elasticity of substitution is illustrated by the blue dashed line in Figure 13. Specifically, a reduction in this relative elasticity (from $\frac{\text{IAS}}{\text{IES}} = 2$ to 1.5) results in a large drop in physical investment. This diminished elasticity implies that households exhibit lesser utility substitution between consumption and housing (suggesting greater complementarity), yielding a smaller decline in consumption. Consequently, through general equilibrium effects, investment in physical capital decreases further.

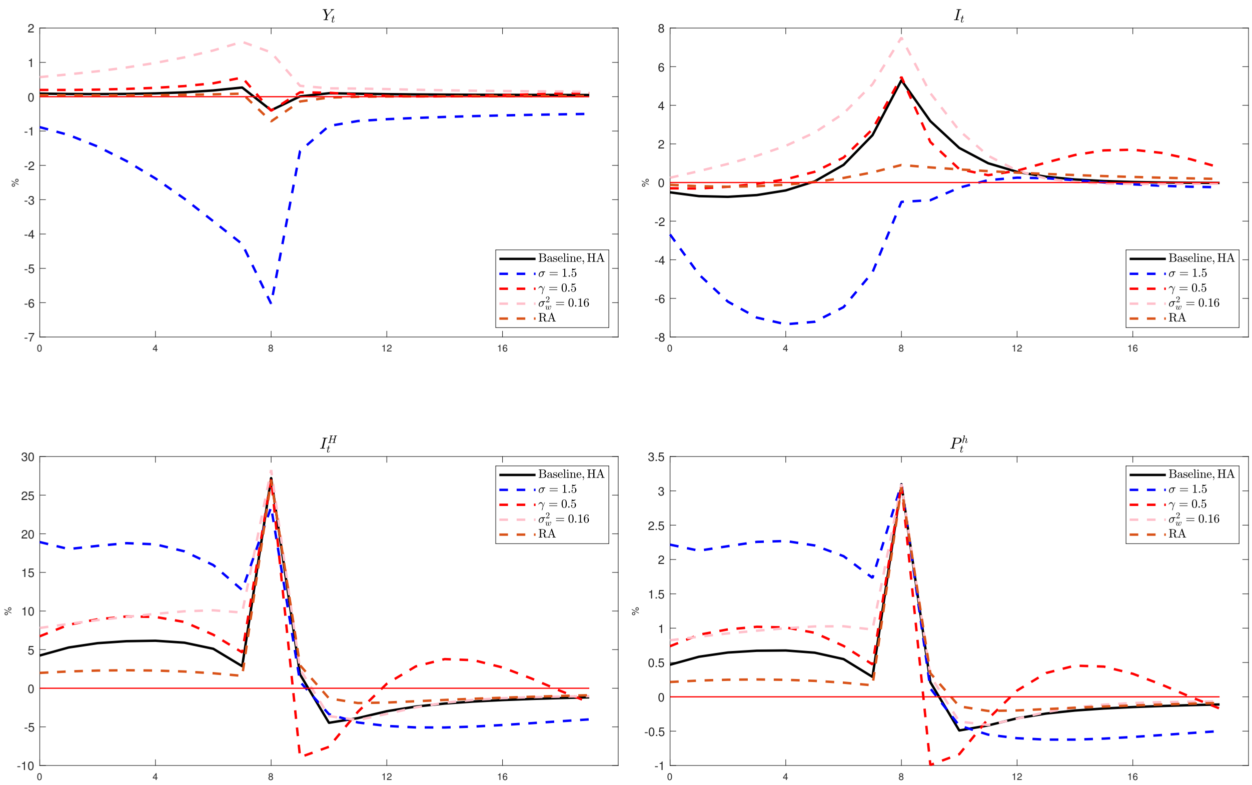
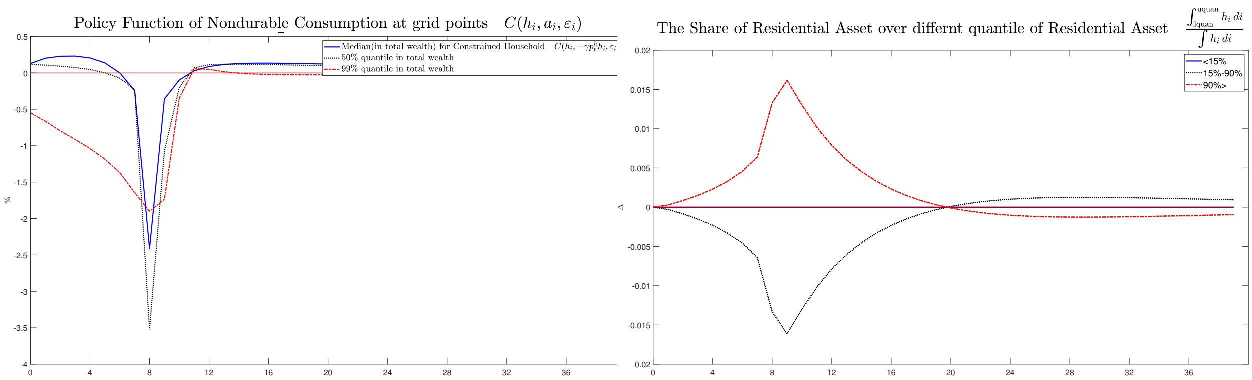


Figure 13: Crowded-out effect comparison

The red dashed line in Figure 13 depicts the response under a tight credit constraint, which implies an important role of wealth inequality. As shown in Section 3.2.2, if we do not consider the wealth distribution (i.e., $\int \widetilde{h}_t^i dG_i$ and $\int \widetilde{\mu}_t^i dG_i$ in Equation 15), a tighter financial constraint will result in a more severe crowding-out problem because real estate is more valuable now. However, as shown in Section 3.2.3, households cannot increase their consumption and housing service as much as they want to because of financial constraints and wealth inequality. The larger $\widetilde{h}_t^i$ can only be realized in a smaller dG_i, and Figure 13 shows that this inequality channel dominates other channels. Physical capital is crowded out less than in baseline model because there are more overwhelmed households that cannot increase their consumption as much as they want.

Additionally, I increase the variance in the idiosyncratic income shock from $\sigma_w^2 = 0.06$ in the baseline model to 0.16, which I characterize by the orange dashed line in Figure 13. Facing a massive income shock, households will have a larger precautionary saving motive to hold the asset (to fulfill their consumption demand against potential low income and cash flow) instead of borrowing money to buy housing. Although the households expect a housing market boom, they only slightly decrease their physical capital in the first period and then increase it until the shock us realized. The reason that physical capital jumps further is that households want to hold more housing services under the effect of an expected shock. However, they do not want to borrow money and decrease their asset holdings to buy real estate. They can only increase their labor supply to earn more wage income to buy housing. The complementarity between labor and physical capital tempts households to increase their assets instead of decreasing them with a higher asset return, which triggers a positive feedback loop on the boom in physical capital.



(a) (b)

Figure 14: Distributional Effect

Figure 14 provides a deeper examination of household heterogeneity and the effects on distribution. Figure 14a illustrates the percentage deviation of the policy function related to consumption across various levels of wealth distribution, based on fixed grid points. Among the poor and middle-income groups, limited liquidity obstructs substantial investment in residential assets. Consequently, as housing prices rise, the wealth effect surpasses the substitution effect, prompting an increase in their consumption. Conversely, affluent individuals, endowed with ample liquidity, can invest more substantially in residential assets. For this demographic, the substitution between non-durable consumption and housing assets leads to a reduction in non-durable consumption. Therefore, the presence of hand-to-mouth households, characterized by a high MPC, amplifies the crowding-out effect, as evidenced by the jump in consumption represented by the blue solid line in Figure 14a. Simultaneously, Figure 14b presents shifts in the proportion of residential assets across different quantiles within the housing distribution, with a notable increase in the residential asset share held by wealthy individuals. This trend verifies the prior assertion that wealth inequality exacerbates the crowding-out effect (those who desire residential assets most are those who have the greatest investment capability). Furthermore, Figure 15 displays changes in the mass of hand-to-mouth households stimulated by fake news shocks. At period 8, when households perceive a preference for housing over consumption, a significant number rush into the real estate market, becoming wealthy hand-to-mouth households. However, upon quickly discovering that the shock is fake, they dispose of the overinvested real estate, deflating the housing market prices. The rapid downturn in the housing market induces economic loss and forces some households into a hand-to-mouth status, generating another spike at period 11. The increased proportion of hand-to-mouth households further exacerbates the economic downturn from the demand side.

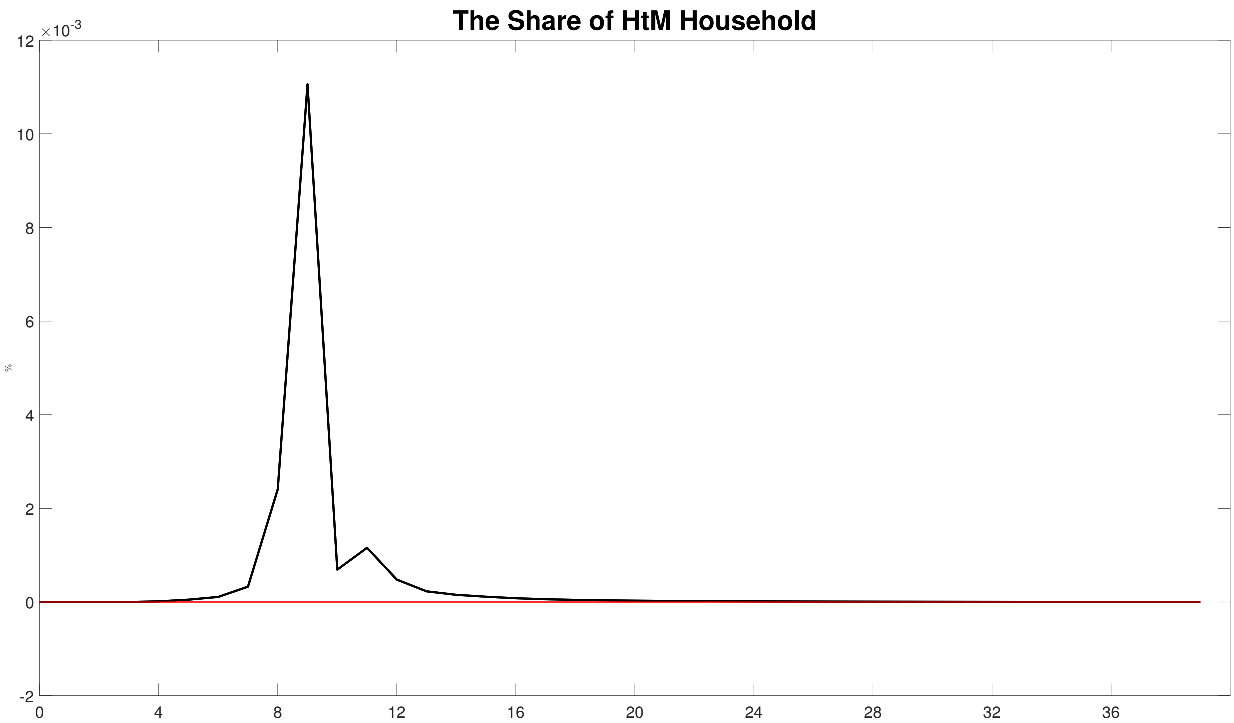


Figure 15: The Share of HtM Households

4.3.4 Policy Analysis

The quantitative results from the preceding section highlights the significant welfare losses stemming from the crowding-out effect of overbuilding in the housing market after a bust. It stands to reason that if policymakers can effectively restrict the amplitude of housing market bubbles, they could likewise diminish the welfare losses arising from these crowding-out effects during bust periods, primarily by minimizing capital misallocation. In this section, I introduce a macroprudential policy designed to dampen equity extraction during boom periods, consequently mitigating the crowding-out effect. Drawing inspiration from the works of Galati and Moessner (2013), Angelini et al. (2014) and Suh (2014), I incorporate a macroprudential policy rule as a countercyclical collateral constraint on the capital-output ratio.

$$\frac{\gamma_t}{\overline{\gamma}} = \left(\frac{\gamma_{t-1}}{\overline{\gamma}}\right)^{\rho_\gamma} \left(\frac{\upsilon_t}{\overline{\upsilon}}\right)^{\eta_\gamma(1-\rho_\gamma)} \tag{23}$$

where γ_t is the collateral constraint in Equation 19 and υ_t is the capital-output ratio. $\overline{\gamma}$ and $\overline{\upsilon}$ are their corresponding values in steady state, and $\eta_\gamma = 1.5$.

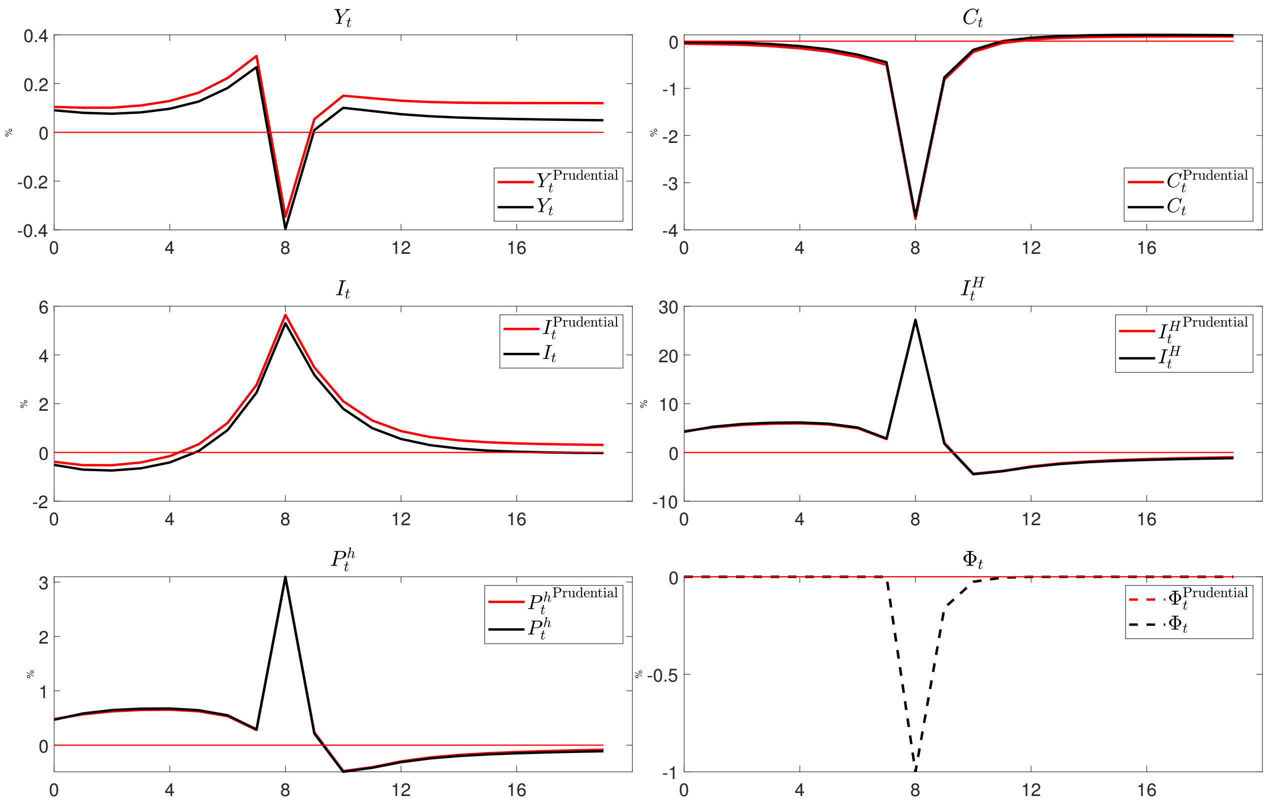


Figure 16: Fake news shock with macroprudential policy

Figure 16 demonstrates that in a model integrating this macroprudential policy, physical investment consistently remains above its counterpart in the baseline model. This manifests the potency of macroprudential policies in significantly moderating the crowding-out effect. Given the countercyclical limitations imposed during housing market surges, both equity extraction and asset reallocation are tempered, leading to a more moderate decline in consumption during downturns. However, due to the persistence of the policy effect denoted by ρ_γ, households, particularly low-income households, face constraints in leveraging their residential assets to stabilize their consumption. Overall, the macroprudential policy reduces the welfare loss from an initial 13% in the baseline model to a revised 6%. Such a substantial reduction in welfare losses manifests the main merit of macroprudential policies: their capacity to limit the overheated economy and, hence, limit the crowding-out effect.

5 Conclusion

This paper documents a new mechanism through which a housing market boom magnifies a recession. An unnecessary jump in residential construction spurred by fake news and imperfect information will inflate a bubble in the housing market, which is a boom without a solid basis and not supported by economic fundamentals. This overbuilding in the housing market crowds out physical capital that is used to produce both durable and nondurable goods. The crowding-out effect in the physical capital market aggravates the decline in output when the housing market bubble busts because of the deficiency of physical capital. Firms do not have as much as capital to support the optimal production under a specific level of TFP, so they will decrease production and labor demand when facing a higher real interest rate and marginal production cost. I use a simple model to argue theoretically that the crowding-out effect of overbuilding is affected by relative intratemporal elasticity of substitution, financial friction, an idiosyncratic income shock and wealth distribution. Later, the quantitative result from a full-fledged model verifies the argument and demonstrates that the output loss caused by overbuilding is large.

However there are still some problems left for future studies. Even if imperfect information did not exist, overbuilding and the crowding-out effect may still be a significant problem from a business cycle perspective because they increase the economic volatility and households diverge further from their first-best equilibrium. Additionally, how can the government introduce an optimal fiscal, monetary, or macroprudential policy to alleviate the crowding-out effect of overbuilding? Is there any complementarity between overbuilding and nominal rigidity in New Keynesian models that would further exacerbate overbuilding and the crowding-out effect?

References

Acemoglu, Daron, and Martin Kaae Jensen. 2015. "Robust comparative statics in large dynamic economies." *Journal of Political Economy* 123 (3): 587–640.

Acharya, Sushant, and Keshav Dogra. 2020. "Understanding HANK: Insights from a PRANK." *Econometrica* 88 (3): 1113–1158.

Anderson, Brian DO, and John B Moore. 2012. *Optimal filtering*. Courier Corporation.

Angelini, Paolo, Stefano Neri, and Fabio Panetta. 2014. "The interaction between capital requirements and monetary policy." *Journal of money, credit and Banking* 46 (6): 1073–1112.

Antunes, António, Tiago Cavalcanti, Caterina Mendicino, Marcel Peruffo, and Anne Villamil. 2020. "Tighter Credit and Consumer Bankruptcy Insurance."

Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. 2021. "Using the sequence-space Jacobian to solve and estimate heterogeneous-agent models." *Econometrica* 89 (5): 2375–2408.

Bachmann, Rüdiger, Ricardo J Caballero, and Eduardo MRA Engel. 2013. "Aggregate implications of lumpy investment: new evidence and a DSGE model." *American Economic Journal: Macroeconomics* 5 (4): 29–67.

Bailey, Michael, Eduardo Dávila, Theresa Kuchler, and Johannes Stroebel. 2019. "House price beliefs and mortgage leverage choice." *The Review of Economic Studies* 86 (6): 2403–2452.

Barsky, Robert B, Susanto Basu, and Keyoung Lee. 2015. "Whither news shocks?" *NBER macroeconomics annual* 29 (1): 225–264.

Barsky, Robert B, and Eric R Sims. 2011. "News shocks and business cycles." *Journal of monetary Economics* 58 (3): 273–289.

Barsky, Robert B, and Eric R Sims. 2012. "Information, animal spirits, and the meaning of innovations in consumer confidence." *American Economic Review* 102 (4): 1343–1377.

Basu, Susanto, John G Fernald, and Miles S Kimball. 2006. "Are technology improvements contractionary?" *American Economic Review* 96 (5): 1418–1448.

Batty, Michael, Jesse Bricker, Joseph Briggs et al. 2020. "The Distributional Financial Accounts of the United States." *Measuring and Understanding the Distribution and Intra/Inter-Generational Mobility of Income and Wea*.

Baxter, Brad, Liam Graham, and Stephen Wright. 2011. "Invertible and non-invertible information sets in linear rational expectations models." *Journal of Economic Dynamics and control* 35 (3): 295–311.

Bayer, Christian, Ralph Lütticke, Lien Pham-Dao, and Volker Tjaden. 2019. "Precautionary savings, illiquid assets, and the aggregate consequences of shocks to household income risk." *Econometrica* 87 (1): 255–290.

Beaudry, Paul, Dana Galizia, and Franck Portier. 2018. "Reconciling Hayek's and Keynes' views of recessions." *The Review of Economic Studies* 85 (1): 119–156.

Berger, David, Veronica Guerrieri, Guido Lorenzoni, and Joseph Vavra. 2018. "House prices and consumer spending." *The Review of Economic Studies* 85 (3): 1502–1542.

Bernanke, Ben S, and Ilian Mihov. 1998. "Measuring monetary policy." *The quarterly journal of economics* 113 (3): 869–902.

Bhutta, Neil, and Benjamin J Keys. 2016. "Interest rates and equity extraction during the housing boom." *American Economic Review* 106 (7): 1742–1774.

Bilbiie, Florin O, and Xavier Ragot. 2021. "Optimal monetary policy and liquidity with heterogeneous households." *Review of Economic Dynamics* 41 71–95.

Blanchard, Olivier J, Jean-Paul L'Huillier, and Guido Lorenzoni. 2013. "News, noise, and fluctuations: An empirical exploration." *American Economic Review* 103 (7): 3045–3070.

Boldrin, Michele, Carlos Garriga, Adrian Peralta-Alva, and Juan M Sánchez. 2013. "Reconstructing the great recession." *FRB of St. Louis Working Paper No*.

Brunnermeier, Markus K. 2009. "Deciphering the liquidity and credit crunch 2007-2008." *Journal of Economic perspectives* 23 (1): 77–100.

Campbell, John Y, and Joao F Cocco. 2007. "How do house prices affect consumption? Evidence from micro data." *Journal of monetary Economics* 54 (3): 591–621.

Chahrour, Ryan, and Gaetano Gaballo. 2021. "Learning from house prices: Amplification and business fluctuations." *The Review of Economic Studies* 88 (4): 1720–1759.

Chahrour, Ryan, and Kyle Jurado. 2018. "News or noise? The missing link." *American Economic Review* 108 (7): 1702–36.

Chakraborty, Indraneel, Itay Goldstein, and Andrew MacKinlay. 2018. "Housing price booms and crowding-out effects in bank lending." *The Review of Financial Studies* 31 (7): 2806–2853.

Chen, Hui, Michael Michaux, and Nikolai Roussanov. 2020. "Houses as ATMs: mortgage refinancing and macroeconomic uncertainty." *The Journal of Finance* 75 (1): 323–375.

Chodorow-Reich, Gabriel, Adam M Guren, and Timothy J McQuade. 2021. "The 2000s Housing Cycle With 2020 Hindsight: A Neo-Kindlebergerian View."Technical report, National Bureau of Economic Research.

Christiano, Lawrence J, Martin Eichenbaum, and Charles L Evans. 1999. "Monetary policy shocks: What have we learned and to what end?" *Handbook of macroeconomics* 1 65–148.

Christiano, Lawrence J, Martin Eichenbaum, and Charles L Evans. 2005. "Nominal rigidities and the dynamic effects of a shock to monetary policy." *Journal of political Economy* 113 (1): 1–45.

Christiano, Lawrence J, Martin S Eichenbaum, and Mathias Trabandt. 2015. "Understanding the great recession." *American Economic Journal: Macroeconomics* 7 (1): 110–67.

Christiano, Lawrence J, Cosmin L Ilut, Roberto Motto, and Massimo Rostagno. 2008. "Monetary policy and stock market boom-bust cycles."

Christiano, Lawrence J, Roberto Motto, and Massimo Rostagno. 2014. "Risk shocks." *American Economic Review* 104 (1): 27–65.

Cochrane, John H. 2011. "Understanding policy in the great recession: Some unpleasant fiscal arithmetic." *European Economic Review* 55 (1): 2–30.

Debortoli, Davide, and Jordi Galí. 2017. "Monetary policy with heterogeneous agents: Insights from TANK models."

Díaz, Antonia, and María José Luengo-Prado. 2010. "The wealth distribution with durable goods." *International Economic Review* 51 (1): 143–170.

Dong, Feng, Yumei Guo, Yuchao Peng, and Zhiwei Xu. 2022. "Economic slowdown and housing dynamics in China: A tale of two investments by firms." *Journal of Money, Credit and Banking* 54 (6): 1839–1874.

Dong, Feng, Yang Jiao, and Haoning Sun. 2023. "Inefficient Bubbly Booms."Technical report, Working Paper.

Eggertsson, Gauti B, and Paul Krugman. 2012. "Debt, deleveraging, and the liquidity trap: A Fisher-Minsky-Koo approach." *The Quarterly Journal of Economics* 127 (3): 1469–1513.

Etheridge, Ben. 2019. "House prices and consumption inequality." *International Economic Review* 60 (4): 1781–1822.

Favara, Giovanni, and Jean Imbs. 2015. "Credit supply and the price of housing." *American Economic Review* 105 (3): 958–92.

Favilukis, Jack, Sydney C Ludvigson, and Stijn Van Nieuwerburgh. 2017. "The macroeconomic effects of housing wealth, housing finance, and limited risk sharing in general equilibrium." *Journal of Political Economy* 125 (1): 140–223.

Fisher, Jonas DM. 2015. "On the structural interpretation of the Smets–Wouters "risk premium" shock." *Journal of Money, Credit and Banking* 47 (2-3): 511–516.

Galati, Gabriele, and Richhild Moessner. 2013. "Macroprudential policy–a literature review." *Journal of Economic Surveys* 27 (5): 846–878.

Garriga, Carlos, and Aaron Hedlund. 2020. "Mortgage debt, consumption, and illiquid housing markets in the great recession." *American Economic Review* 110 (6): 1603–34.

Garriga, Carlos, Finn E Kydland, and Roman Šustek. 2017. "Mortgages and monetary policy." *The Review of Financial Studies* 30 (10): 3337–3375.

Gertler, Mark, and Peter Karadi. 2015. "Monetary policy surprises, credit costs, and economic activity." *American Economic Journal: Macroeconomics* 7 (1): 44–76.

Gorea, Denis, and Virgiliu Midrigin. 2017. "Liquidity constraints in the us housing market." *NBER Working Paper* (w23345): .

Greenwald, Daniel. 2018. "The mortgage credit channel of macroeconomic transmission."

Groot, Oliver, Falk Mazelis, Roberto Motto, and Annukka Ristiniemi. 2021. "A toolkit for computing constrained optimal policy projections (copps)."

Grusky, David B, Bruce Western, and Christopher Wimer. 2011. "The consequences of the great recession." *The great recession* 3–20.

Guerrieri, Veronica, and Guido Lorenzoni. 2017. "Credit crises, precautionary savings, and the liquidity trap." *The Quarterly Journal of Economics* 132 (3): 1427–1467.

Guvenen, Fatih, Serdar Ozkan, and Jae Song. 2014. "The nature of countercyclical income risk." *Journal of Political Economy* 122 (3): 621–660.

Hamilton, James Douglas. 2020. *Time series analysis*. Princeton university press.

Hebden, James, and Fabian Winkler. 2021. "Impulse-Based Computation of Policy Counterfactuals."

Herbst, Edward P, and Frank Schorfheide. 2016. *Bayesian estimation of DSGE models*. Princeton University Press.

Hurst, Erik, Benjamin J Keys, Amit Seru, and Joseph Vavra. 2016. "Regional redistribution through the US mortgage market." *American Economic Review* 106 (10): 2982–3028.

Hürtgen, Patrick. 2014. "Consumer misperceptions, uncertain fundamentals, and the business cycle." *Journal of Economic Dynamics and Control* 40 279–292.

Iacoviello, Matteo. 2005. "House prices, borrowing constraints, and monetary policy in the business cycle." *American economic review* 95 (3): 739–764.

Islam, I, and S Verick. 2011. "The great recession of 2008–09: Causes, consequences and policy responses." In *From the great recession to labour market recovery*, 19–52, Springer.

Ivashina, Victoria, and David Scharfstein. 2010. "Bank lending during the financial crisis of 2008." *Journal of Financial economics* 97 (3): 319–338.

Jermann, Urban, and Vincenzo Quadrini. 2012. "Macroeconomic effects of financial shocks." *American Economic Review* 102 (1): 238–71.

Justiniano, Alejandro, Giorgio E Primiceri, and Andrea Tambalotti. 2010. "Investment shocks and business cycles." *Journal of Monetary Economics* 57 (2): 132–145.

Justiniano, Alejandro, Giorgio E Primiceri, and Andrea Tambalotti. 2011. "Investment shocks and the relative price of investment." *Review of Economic Dynamics* 14 (1): 102–121.

Justiniano, Alejandro, Giorgio E Primiceri, and Andrea Tambalotti. 2019. "Credit supply and the housing boom." *Journal of Political Economy* 127 (3): 1317–1350.

Kaplan, Greg, Kurt Mitman, and Giovanni L Violante. 2020. "The housing boom and bust: Model meets evidence."Technical Report 9.

Kaplan, Greg, Benjamin Moll, and Giovanni L Violante. 2018. "Monetary policy according to HANK." *American Economic Review* 108 (3): 697–743.

Kaplan, Greg, Giovanni L Violante, and Justin Weidner. 2014. "The wealthy hand-to-mouth."Technical report, National Bureau of Economic Research.

Khan, Aubhik, and Julia K Thomas. 2008. "Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics." *Econometrica* 76 (2): 395–436.

Khorunzhina, Natalia. 2021. "Intratemporal nonseparability between housing and nondurable consumption: evidence from reinvestment in housing stock." *Journal of Monetary Economics* 117 658–670.

King, Robert G, and Sergio T Rebelo. 1993. "Low frequency filtering and real business cycles." *Journal of Economic dynamics and Control* 17 (1-2): 207–231.

Klein, Paul. 2000. "Using the generalized Schur form to solve a multivariate linear rational expectations model." *Journal of economic dynamics and control* 24 (10): 1405–1423.

Kuang, Pei, Kaushik Mitra, Li Tang, and Shihan Xie. 2023. "Macroprudential Policy and Housing Market Expectations." *Available at SSRN 4447472*.

Landvoigt, Tim. 2017. "Housing demand during the boom: The role of expectations and credit constraints." *The Review of Financial Studies* 30 (6): 1865–1902.

Li, Wenli, Haiyong Liu, Fang Yang, and Rui Yao. 2016. "Housing over time and over the life cycle: a structural estimation." *International Economic Review* 57 (4): 1237–1260.

Liu, Zheng, Pengfei Wang, and Tao Zha. 2019. "A theory of housing demand shocks."Technical report, National Bureau of Economic Research.

Lorenzoni, Guido. 2009. "A theory of demand shocks." *American Economic Review* 99 (5): 2050–2084.

Martínez, Carlos Cañizares. 2023. "Leaning against housing booms fueled by credit."Technical report, Working Paper.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson. 2016. "The power of forward guidance revisited." *American Economic Review* 106 (10): 3133–3158.

McQuinn, Kieran, Teresa Monteiro, and Conor O'Toole. 2021. "House price expectations, labour market developments and the house price to rent ratio: A user cost of capital approach." *The Journal of Real Estate Finance and Economics* 62 (1): 25–47.

Mian, Atif, Kamalesh Rao, and Amir Sufi. 2013. "Household balance sheets, consumption, and the economic slump." *The Quarterly Journal of Economics* 128 (4): 1687–1726.

Mian, Atif, and Amir Sufi. 2010. "The great recession: Lessons from microeconomic data." *American Economic Review* 100 (2): 51–56.

Mian, Atif, and Amir Sufi. 2014. "What explains the 2007–2009 drop in employment?" *Econometrica* 82 (6): 2197–2223.

Mian, Atif, and Amir Sufi. 2022. "Credit supply and housing speculation." *The Review of Financial Studies* 35 (2): 680–719.

Mitman, Kurt. 2016. "Macroeconomic effects of bankruptcy and foreclosure policies." *American Economic Review* 106 (8): 2219–55.

Orchard, Jacob, Valerie A Ramey, and Johannes Wieland. 2022. "Micro MPCs and macro counterfactuals: the case of the 2008 rebates."Technical report, Working Paper.

Piazzesi, Monika, Martin Schneider, and Selale Tuzel. 2007. "Housing, consumption and asset pricing." *Journal of Financial Economics* 83 (3): 531–569.

Qian, Wei. 2023. "House price expectations and household consumption." *Journal of Economic Dynamics and Control* 151 104652.

Ravn, Morten O, and Harald Uhlig. 2002. "On adjusting the Hodrick-Prescott filter for the frequency of observations." *Review of economics and statistics* 84 (2): 371–376.

Rognlie, Matthew, Andrei Shleifer, and Alp Simsek. 2018. "Investment hangover and the great recession." *American Economic Journal: Macroeconomics* 10 (2): 113–53.

Romer, Christina D, and David H Romer. 2004. "A new measure of monetary shocks: Derivation and implications." *American economic review* 94 (4): 1055–1084.

Rotemberg, Julio J, and Michael Woodford. 1997. "An optimization-based econometric framework for the evaluation of monetary policy." *NBER macroeconomics annual* 12 297–346.

Rubio-Ramirez, Juan F, Daniel F Waggoner, and Tao Zha. 2010. "Structural vector autoregressions: Theory of identification and algorithms for inference." *The Review of Economic Studies* 77 (2): 665–696.

Rudebusch, Glenn D, and Eric T Swanson. 2012. "The bond premium in a DSGE model with long-run real and nominal risks." *American Economic Journal: Macroeconomics* 4 (1): 105–143.

Schmitt-Grohé, Stephanie, and Martin Uribe. 2012. "What's news in business cycles." *Econometrica* 80 (6): 2733–2764.

Schulhofer-Wohl, Sam. 2011. "Heterogeneity and tests of risk sharing." *Journal of Political Economy* 119 (5): 925–958.

Sims, Christopher A et al. 1986. "Are forecasting models usable for policy analysis?" *Quarterly Review* (Win): 2–16.

Sims, Christopher A. 1980. "Macroeconomics and reality." *Econometrica: journal of the Econometric Society* 1–48.

Sims, Christopher A. 1998. "Comment on Glenn Rudebusch's" Do measures of monetary policy in a VAR make sense?"." *International Economic Review* 39 (4): 933–941.

Sims, Eric. 2016. "What's news in News? A cautionary note on using a variance decomposition to assess the quantitative importance of news shocks." *Journal of Economic Dynamics and Control* 73 41–60.

Smets, Frank, and Rafael Wouters. 2007. "Shocks and frictions in US business cycles: A Bayesian DSGE approach." *American economic review* 97 (3): 586–606.

Stokey, Nancy L. 1989. *Recursive methods in economic dynamics*. Harvard University Press.

Storesletten, Kjetil, Chris I Telmer, and Amir Yaron. 2004. "Cyclical dynamics in idiosyncratic labor market risk." *Journal of political Economy* 112 (3): 695–717.

Suh, Hyunduk. 2014. "Dichotomy between macroprudential policy and monetary policy on credit and inflation." *Economics Letters* 122 (2): 144–149.

Uhlig, Harald et al. 2004. "What moves GNP?" In *Econometric Society 2004 North American Winter Meetings*, (636): , Econometric Society.

Uhlig, Harald. 2001. "Chapter 3 A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily." In *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, . 10.1093/0199248273.003.0003.

Uhlig, Harald. 2005. "What are the effects of monetary policy on output? Results from an agnostic identification procedure." *Journal of Monetary Economics* 52 (2): 381–419.

Uhlig, Harald F et al. 1995. "A toolkit for analyzing nonlinear dynamic stochastic models easily."

Winberry, Thomas. 2016. "Lumpy investment, business cycles, and stimulus policy." *Job market paper, Princeton University/University of Chicago Booth School of Business*.

Wolf, Christian K, and Alisdair McKay. 2022. "What Can Time-Series Regressions Tell Us About Policy Counterfactuals?"Technical report, National Bureau of Economic Research.

A Data Description

Real GDP Y_t is directly downloaded listing "Real Gross Domestic Product" with seasonally adjusted. Real consumption C_t is directly downloaded listing "Real personal consumption expenditures: Nondurable goods" with seasonally adjusted. GDP deflator gdp_{def} is downloaded listing "GDP Implicit Price Deflator in United States" with seasonally adjusted. Nominal nondurable investment I_t^{nom} is downloaded listing "Private Nonresidential Fixed Investment" with seasonally adjusted. I get the real nondurable investment I_t by the formula $I_t = I_t^{\text{nom}}/\text{gdp}_{def} * 100$. The CPI which we take is "Consumer Price Index for All Urban Consumers: All Items Less Shelter in U.S. City Average" since we should consider the correlation between house price and normal CPI. Thus we downloaded the CPI without shelter term. I take the nominal interest rate R_t^{nom} as "Effective Federal Funds Rate". The inflation rate is calculated from the GDP defltor in the form that $\pi_t = \frac{def_t - def_{t-1}}{def_{t-1}}$ (Since we solve the inflation from deflator in quarterly data, the inflation is measured within one quarter instead of annually). Combining the inflation π_t and nominal interest rate R_t^{nom} we can construct the real interest rate $R_t = (\frac{R_t^{\text{nom}}}{100} + 1)/(1 + \pi_t) - 1$ (I divided 100 because the original data is in percentage unit). The house supply H_t is measured by "New Privately-Owned Housing Units Started: Total Units". The nominal mortgage debt MD_t^{nom} comes from "Mortgage Debt Outstanding, All holders (DISCONTINUED)". Since the nominal mortgage debt is in money unit, I can directly get the real mortgage debt value from $MD_t = MD_t^{\text{nom}}/\text{gdp}_{def} * 100$ which is same as we did to get real investment. The real stock price P_t^a is calculated from "NASDAQ Composite Index" and normalized by GDP deflator as I did in constructing real investment and real mortgage debt. The real house price P_t^h is calculated from "All-Transactions Indexes" collected by Federal Housing Finance Agency.

B Identification Step and Robustness Test to VAR Identification

B.1 Identification with sign and zero restricution

Based on the observation and argument, I use a simple SVAR model to decompose the effect of raised house price to investment. Given the model which is similar to Sims et al. (1986)

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + ... + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{e}_t \tag{24}$$

where

$$\boldsymbol{y_t} = \begin{bmatrix} r_t \\ m_t \\ y_t \\ p_t \\ i_t \\ p_t^h \\ c_t \end{bmatrix} \tag{25}$$

r_t is the nominal interest rate; m_t is the money supply; y_t is the real output; p_t is the price level; i_t is the nominal investment; p_t^h is the nominal price of house; c_t is the real consumption of non-durable goods. Most the data comes from FRED, Federal Reserve Bank of St. Louis. I use treasury bill rate represents the nominal interest and GDP deflator for the price level. The price of house comes from FHFA house price index. The detail about it will be discussed at appendix. Meanwhile I use the short-run restriction as well as corresponding sign restriction to decompose the shock term $\boldsymbol{e_t}$ from $\boldsymbol{v}_t$ that

$$\boldsymbol{Pe_t} = \boldsymbol{v_t} \tag{26}$$

or detailedly

$$Pe_t \equiv \begin{vmatrix} 1 & b_{11} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & 1 & b_{23} & b_{24} & 0 & 0 & 0 \\ b_{31} & 0 & 1 & 0 & b_{35} & 0 & b_{37} \\ b_{41} & b_{42} & b_{43} & 1 & b_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{56} & 0 \\ b_{61} & 0 & b_{63} & b_{64} & 0 & 1 & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & b_{76} & 0 \end{vmatrix} \begin{bmatrix} e_{rt} \\ e_{mt} \\ e_{yt} \\ e_{pt} \\ e_{it} \\ e_{p^h t} \\ e_{ct} \end{bmatrix} = \begin{bmatrix} v_{rt} \\ v_{mt} \\ v_{yt} \\ v_{pt} \\ v_{it} \\ v_{p^h t} \\ v_{ct} \end{bmatrix}$$

Figure 17 shows the IRF of one unite positive house price shock to output, investment, house price and non-durable goods consumption. The black line is the path of related variable up to 20 period. The read dash line is their related confidence band under 90% calculating by monte-carlo method. We can inspect from IRF that, house price inflation could stimulate the consumption of durable goods as it is long-lasting goods and household could derive out utility by just holding it. The household could feel satisfy and pleased either via living in this house or via owning the house which is valuable every period. Meanwhile the household can obtain utility not only from just holding and enjoying it each period, but also from financial market. The house is a goods that could be consumed. While at the same time it is also a asset that could be collateral and offers more liquidity to household. Household would use this liquidity to smooth their non-durable consumption leisurely, which provide extra benefit to household.

Therefore after observing one unit positive shock in house price, household snap up the house as house it not only a goods but also an asset which we discuss before. This increased demand

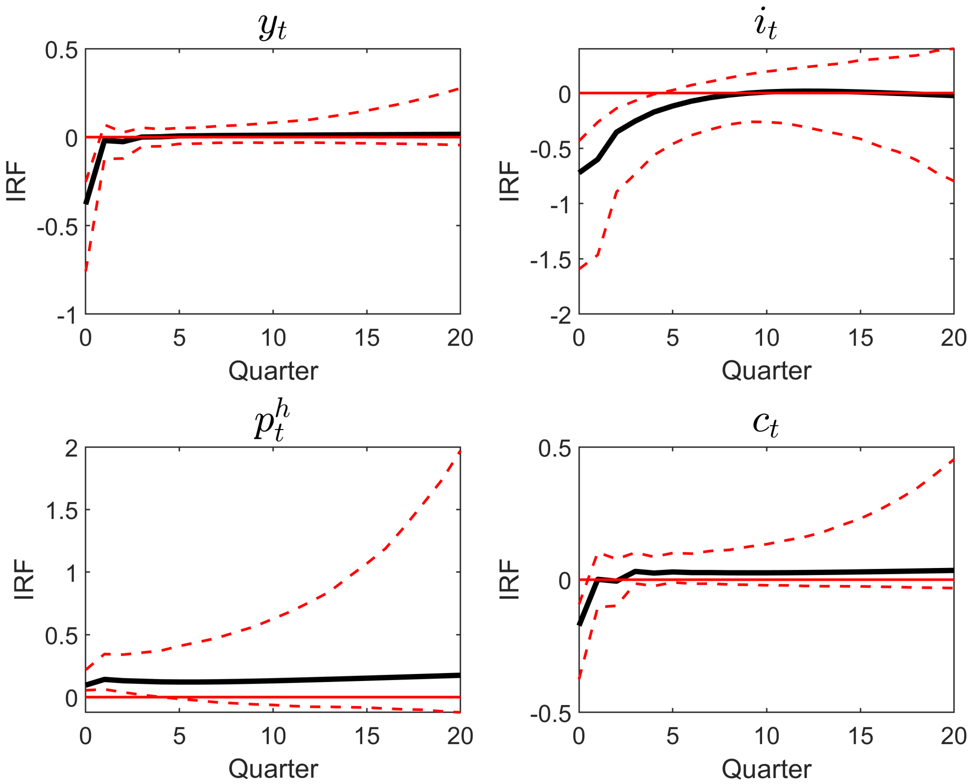


Figure 17: IRF of house price inflation

draw up the house price even more which we can see the house price is raising not only lat the beginning but also later. The house price in the end permanently increased because of increased household demand. This increased house price stimulates household who would borrow more from bank to buy house (the house supply discontinuity will aggravate this channel) or borrow more to help them share the risk as collateral is more expensive. Firms will be more difficult to borrow money to invest and the decreased demand in non-durable goods will also weaken firms' propensity to invest or R&D. Investment is crowded out by this two effects and this is what we can observe from the IRF. Investment drops the most and also spends longest time to recover. Output and non-durable consumption stands behind it. However both of them go back to steady state quickly which indicates that only the first jump in house price affects them. Later household use their more valuable collateral to smooth the consumption as well as output. Thus these two variable converge back quickly while because of strong and amplified effect both in demand and supply side, investment converges much slower than other two variables. This portends that there would be much larger drop in output if recession occurs because the accumulated decreased investment will pass its influence through the capital, a long-lasting things, later.

B.2 Contemporaneous real price shock

B.2.1 Process of estimation and identification

I detrend the main variable by taking logarithm first and first-order difference later. Then I get the detrended real GDP, real consumption, real investment, cpi, house supply, real mortgage debt, stock price and house price in lower-case letter. Then I ordered them in the vector

$$Y_t = [y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h]'$$

I use the data period between 1987Q2 and 2006Q4. Then I add lagged term into the model up to 4 quarter and estimate the model

$$Y = [Y_5, Y_6 ...]$$

$$X_{t-1} = [y_{t-1}, c_{t-1}, i_{t-1}, cpi_{t-1}, r_{t-1}, p_{t-1}^a, hs_{t-1}, md_{t-1}, p_{t-1}^h, y_{t-2}, c_{t-2}, ..., p_{t-4}^h]'$$

$$X = [\mathbf{1}, X_4, X_5, ...]$$

Then use the projection matrix we can solve the factor that

$$\hat{\Phi} = YX'(XX')^{-1}$$

The residue is

$$\hat{e} = Y - \Phi X$$

and the variance of estimation error would be

$$\hat{\Omega} = cov(\hat{e}')$$

To simulate the model we can rewrite the variables into companion form such that

$$\boldsymbol{Y_t} = [y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h, y_{t-1}, c_{t-1}, ..., p_{t-3}^h]'$$

Denote $\hat{P} = \text{chol}(\hat{\Omega})$ and

$$\hat{\boldsymbol{\Phi}} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_p \\ I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_n & 0 \end{bmatrix}$$

where $\Phi(:,2:\text{end}) = [\Phi_1 \Phi_2 \Phi_3 \dots \Phi_p]$ since I have intercept coefficient term with $\mathbf{1}$ in X.

Meanwhile we define

$$\hat{\boldsymbol{P}} = \begin{bmatrix} \hat{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The shock term is

$$\nu_{n\times 1} = [0, 0, ..., 1]'$$

which means there is only one unit shock happened at house price row.

Similarly I should write it in companion form such that

$$\boldsymbol{\nu} = [\nu, \mathbf{0}]$$

Then we can get the IRF that

$$\text{IRF}_t = \hat{\boldsymbol{\Phi}}^t \hat{\boldsymbol{P}} \boldsymbol{\nu}$$

where $t = 0, 1, 2, ..., 20$.

Finally we only take first 1 to n items in IRF_t. Since I take first-order difference to most of the data, at this stage I also calculate the cumsum of IRF to return the accumulated response.

B.2.2 Contemporaneous real price shock

Figure 17 in the appendix sheds light on the crowding-out effect engendered by a housing market boom. However, given the speed at which the impulse response function (IRF) reverts to the steady-state, it may not generate a significant scarcity in physical capital, thereby rendering the crowding-out effect less consequential in this rudimentary identification test. Moreover, the identification method I employed, namely Sims et al. (1986), has been critiqued for its potential overemphasis on identifying the underlying shocks, occasionally leading to artificial unreliability. To surmount these limitations, I utilize an alternative canonical workhorse identification method, the Cholesky decomposition, to identify the effect of contemporaneous housing price shocks. Following the method of Bernanke and Mihov (1998), Cholesky decomposition ensures that the shock can initially only impact the last variable, while the variables that precede it will not be contemporaneously influenced by the shock. Throughout this section, I argue the implications of the crowding-out effect incited by a housing market boom devoid of fundamental support. Therefore, I place the housing price at the end to simulate a nonfundamental housing price boom, where only the housing price is stimulated initially. As a result, a single unit housing price shock triggers the movement of other variables, following the inherent relationship and mechanism (Φ in Equation 1).Inspired by existing literature, I order the economic variables in the data vector Y_t as

$$Y_t = [\Upsilon_t, y_t, c_t, i_t, r_t, r_t^d, q_t, h_t^s, p_t^h]' \tag{27}$$

where Υ_t is the NAHB/Wells Fargo Housing Market Index; y_t is real GDP; c_t is real consumption plus services; i_t is real investment in the nonresidential sector; r_t is the real interest rate; r_t^d is the real mortgage debt rate; q_t is the real stock price index; h_t^s is the real housing supply; and p_t^h is the real housing price. I select the time interval between 1985Q1 and 2007Q2 when the housing market boom reached its peak before the Great Recession.[24] I add housing market index Υ_t to the estimation for comparative purposes because it is further used in Sections 2.1 and 2.2. All the variables are in logarithmic form and detrended by hybrid specification, a method through which I use all nonstationary variables as the growth rate, and all the variables in Y_t pass the unit-root test.

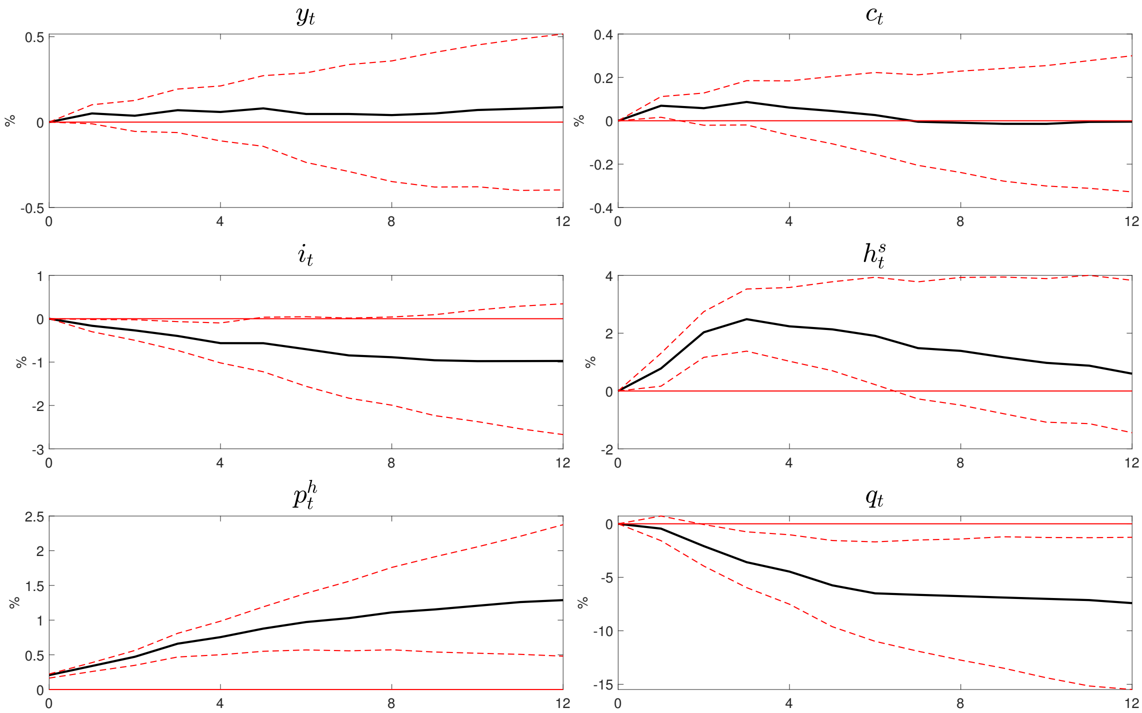


Figure 18: IRF to one unit house price jump

Figure 18presents the impulse responses to a one-unit housing price shock, encased within a 90% confidence band. It reveals that a 10 basis point (bp) initial increase in housing price p_t^h instigates a housing market boom, escalating the housing price to a peak of 60bp four quarters later. This is approximately six times larger than the original increase. Individuals lacking sufficient residential asset holdings display optimism and a strong desire to acquire more houses. This, in turn, shifts the demand curve of residential assets upward as both price and quantity increase simultaneously. However, these individuals make only a partial down payment for the asset value, borrowing the remainder from commercial banks as mortgage debt. In parallel, those who already possess housing leverage the increased housing price to extract equity and generate liquidity, particularly if they are financially constrained and require more liquidity to meet their consumption needs. Nevertheless, the initial impacts on the consumption and output are insignificant or negligible, potentially due to identification problems or data issues as argued by Sims (1998), Christiano et al. (1999) and Romer and Romer (2004). Investment in the production (of consumption) sector declines throughout the entire period, stabilizing after two years at

[24]In the appendix, I perform robustness tests to this span selection by extending the data to 2019Q4 with a shadow rate or 1-year treasury bond rate that is proposed by Gertler and Karadi (2015). The crowding-out effect exists in all these robustness tests.

approximately 1% annualized. This clearly uncovers the crowding-out effect. It demonstrates that the crowding-out effect is potent and sensitive to housing prices -- a 10bp increase in housing price engenders a 100bp decrease in investment. This overreaction suggests an underlying conduit that transmits and amplifies the flow from the housing price to physical investment, and the decline in capital demand decreases the capital price by up to 7%. Observations reveal that an increase in the housing price corresponds to an increase in housing supply in the same direction, affirming the two key arguments discussed previously: overbuilding and a crowding-out effect spurred by a nonfundamental housing price demand shock. Furthermore, the nonexponential expansion in housing supply sheds light on the shape of the supply function in the housing market, which is not fully inelastic, contradicting the assumption made in the literature.

B.2.3 Alternative detrend Method

Alternatively I also use another method to deal with the data which we call Vector Error Correction Method (VECM) in literature. I add the year number into the model to try to detrend the data. I marked the year with its "number" and add 0.1 to 0.4 on it as the label of quarter. Then I divided these "number" by 1000 to get a comfortable scalar. Specifically we take

$$Y_t = [t, t^2, t^3, y_t, c_t, i_t, cpi_t, r_t, p_t^a, hs_t, md_t, p_t^h]'$$

B.2.4 Confidence Band-MC Method

Here I explain the detailed steps that I used to calculate the confidence band of t he estimation using Monte Carlo method. Since there is no difference in steps between I estimate the confidence band in method I and method II, I only show the first part for simplicity.

I can calculate the estimated variance of the coefficient by

$$\hat{\sigma}^2_{\hat{\Phi}} = \frac{\hat{\Omega} \bigotimes \left(\frac{XX'}{T}\right)^{-1}}{T}$$

Then I draw the coefficient simple $\tilde{\Phi}^{(b)}$ from the distribution

$$vec(\hat{\Phi}) \sim N\left(\text{vec}\left(\hat{\Phi}'\right), \hat{\sigma}^2_{\hat{\Phi}}\right)$$

At the same time the estimated variance of the coefficient variance would be

$$\hat{\sigma}^2_{\hat{\Omega}} = \frac{2D_n^+ \left(\hat{\Omega} \bigotimes \hat{\Omega}\right) D_n^{+\prime}}{T}$$

where $D_n^+ = (D_n' D_n)^{-1} D_n$ is the Moore-Penrose generalized inverse of duplication matrix D_n

I generate the variance simple $\tilde{\Omega}^{(b)}$ from the distribution

$$\text{vech}(\hat{\Omega}) \sim N\left(\text{vech}(\hat{\Omega}), \hat{\sigma}^2_{\hat{\Omega}}\right)$$

Then use the duplication matrix to transfer back to

$$\text{vec}(\tilde{\Omega}^{(b)}) = D_n \text{vech}(\tilde{\Omega}^{(b)})$$

C Purification Process

In this section I first show that there is another implicit necessary condition of identification. After that I show that given different state space model we cannot arbitrarily add lag and lead term of g_t and $E_t g_{t+6}$ because of the violation of necessary condition. In the end I discuss the detailed purification method I used and the how I pin down the informative span τ through the purification.

C.1 Orthogonal Demand

Now let me consider the news shock under perfect information cases. For simplicity I assume the news is announced one period ahead of the time when it realizes ($\tau = 1$). Given the structure form

$$\begin{bmatrix} 1 & -\alpha_3 & 0 \\ -\alpha_1 & 1 & -\alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ g_t \\ w_t \end{bmatrix} = \begin{bmatrix} \rho_y & 0 & 0 \\ 0 & \rho_g & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ g_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}$$

Where α_1 denotes the effect of monetary policy shock can affect perception via macro-variable y_t. α_2 denotes the endogenous effect of news shock.

Setting $\alpha_1 = 0$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, $\rho_y = 0.6$, $\rho_g = 0.9$ we can get

$$\begin{bmatrix} y_t \\ g_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.9 & 1 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ g_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}$$

We can see w_t has two effects on y_t: contemporaneous effect 0.5 and realization effect 1 one period later. I further denote $\Phi = \begin{bmatrix} 0.6 & 0.9 & 1 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $R_w = [0.5, 0.5, 1]'$, $R_u = [1, 1, 0]'$.

The identification method I used is based on the forecast error and since all the shock is normalized to 1, we can get

$$\begin{aligned} y_{t+3} - E_t y_{t+3} = & \underbrace{R_w}_{w_{t+3}} + \underbrace{\Phi R_w}_{w_{t+2}} + \underbrace{\Phi^2 R_w}_{w_{t+1}} \\ & + \underbrace{R_u}_{u_{t+3}} + \underbrace{\Phi R_u}_{u_{t+2}} + \underbrace{\Phi^2 R_u}_{u_{t+1}} \end{aligned}$$

How can we say that the shock w plays the largest row in explaining $y_{t+3} - E_t y_{t+3}$? No we cannot and the identified news shock might become $R^* = \beta_1 R_w + \beta_2 R_u$. Therefore we need the contemporaneous orthogonal constraint. In other words we use a purified g_t, $\widehat{g}_t$ to rule out the possibility that R_u comes into R^*. Now let us consider the reduced-form VAR again

$$\begin{bmatrix} y_t \\ \widehat{g}_t \\ w_t \end{bmatrix} = \widehat{\Phi} \begin{bmatrix} y_{t-1} \\ \widehat{g}_{t-1} \\ w_{t-1} \end{bmatrix} + \left[0, \widehat{R}_{\widehat{u}}, \widehat{R}_w\right] \begin{bmatrix} \widehat{u}_t \\ w_t \end{bmatrix}$$

As long as $\operatorname{cov}(w_t, \widehat{u}_t) = 0$, we will have $\widehat{R}'_{\widehat{u}} \widehat{R}_w = 0$. Then even though we still have

$$\begin{aligned} y_{t+3} - E_t y_{t+3} = & \underbrace{\widehat{R}_w}_{w_{t+3}} + \underbrace{\Phi \widehat{R}_w}_{w_{t+2}} + \underbrace{\Phi^2 \widehat{R}_w}_{w_{t+1}} \\ & + \underbrace{\widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+3}} + \underbrace{\Phi \widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+2}} + \underbrace{\Phi^2 \widehat{R}_{\widehat{u}}}_{\widehat{u}_{t+1}} \end{aligned}$$

we can get $R^* = \widehat{R}_w$ because any combination $R = \beta_1 \widehat{R}_w + \beta_2 \widehat{R}_{\widehat{u}}$ will be ruled out as $\widehat{R}'_{\widehat{u}} R = \beta_2 \neq 0$

C.2 Another necessary condition of news shock identification: $\operatorname{cov}(\widehat{g}_t, w_{t-1}) \neq 0$

Given the AR process of g_t follows

$$g_t = \rho_g g_{t-1} + w_{t-1} + u_t + \alpha_2 w_t$$

what I want is to extract the effect of w_t out of g_t. Given

$$E_t g_{t+6} = \rho_g^6 g_t + \rho_g^5 w_t$$

A regression of $E_t g_{t+6}$ on g_t will get the residual $u_t^w = \rho_g^5 w_t$. Then let us run the regression of g_t on u_t^w and clean out the $\alpha_2 w_t$ term in g_t. In the end what we get is the u^{HIM} that

$$\begin{aligned} u_t^{\text{HIM}} &= \rho_g g_{t-1} + w_{t-1} + u_t = \widehat{g}_t \\ &= g_t - \alpha_2 w_t \end{aligned}$$

Pay attention that now $\operatorname{cov}(\widehat{g}_t, w_t) = 0$ but $\operatorname{cov}(\widehat{g}_t, w_{t-1}) \neq 0$. I will discuss this inequality later.

Furthermore, it is worth to notice that we cannot observe w_t or w_{t-1}, therefore the DGP would be

$$\begin{bmatrix} y_t \\ \widehat{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widehat{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right] \begin{bmatrix} w_t \\ u_t + \gamma w_{t-1} \end{bmatrix}$$

where $\gamma = 1 + \rho\alpha$ and $Q = \left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right] \left[\widetilde{R}_w, \widetilde{R}_{\widehat{u}}\right]'$.

Therefore as long as $\operatorname{cov}(u_t + \gamma w_{t-1}, w_t) = 0$, we can get $R^* = \widetilde{R}_w$.

What if we also cleaned out w_{t-1} out of g_t and got $\widetilde{u}_t^{\text{HIM}} = \rho_g g_{t-1} + u_t = \widetilde{g}_t = g_t - \alpha_2 w_t - w_{t-1}$**?** This time both $\operatorname{cov}(\widetilde{g}_t, w_t) = 0$ and $\operatorname{cov}(\widetilde{g}_t, w_{t-1}) = 0$ hold. The **can we separate these two models below**

$$\begin{bmatrix} y_t \\ \widetilde{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widetilde{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_{w_t}, \widetilde{R}_{\widetilde{u}}\right] \begin{bmatrix} w_t \\ u_t + \rho_g \alpha w_{t-1} + \rho_g w_{t-2} \end{bmatrix}$$

and

$$\begin{bmatrix} y_t \\ \widetilde{g}_t \end{bmatrix} = \widetilde{\Phi} \begin{bmatrix} y_{t-1} \\ \widetilde{g}_{t-1} \end{bmatrix} + \left[\widetilde{R}_{w_{t-1}}, \widetilde{R}_{\widetilde{u}}\right] \begin{bmatrix} w_{t-1} \\ u_t + \rho_g \alpha w_{t-1} + \rho_g w_{t-2} \end{bmatrix}$$

when $\rho_g \alpha \approx 0$? Basically we cannot. Therefore the condition $\operatorname{cov}(\widehat{g}_t, w_{t-1}) \neq 0$ is necessary.

C.3 Exogenous g_t w.r.t w_t

C.3.1 Perfect Information

C.3.1.1 uniquely identification

Given the fundamental process follows

$$\begin{aligned} g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} \\ &= (1 - \rho_g L)^{-1} w_{t-\tau} + (1 - \rho_g L)^{-1} w_t^{\tau} \end{aligned} \tag{28}$$

Then

$$g_{t+\tau|t} = \rho_g^{\tau} g_t + \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t \tag{29}$$

Therefore lagged expectation up to τ follows

$$
\begin{aligned}
g_{t|t-\tau} &= \rho_g^{\tau} g_{t-\tau} + \rho_g^{\tau-1} w_{t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2} + \cdots + w_{t-\tau} \\
&= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau} + \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-2\tau} + \rho_g^{\tau-1} w_{t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2} + \cdots + w_{t-\tau} \\
&= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau} + \left(1-\rho_g L\right)^{-1} w_{t-\tau} \qquad (30)
\end{aligned}
$$

Then the projection of g_t on $g_{t|t-\tau}$ yields

$$u_t = g_t - g_{t|t-\tau} \left(g'_{t|t-\tau} g_{t|t-\tau}\right)^{-1} g'_{t|t-\tau} g_t$$

will be almost independent with news shock $w_{t-\tau}$ and exactly independent with w_t as the news term $\left(1-\rho_g L\right)^{-1} w_{t-\tau}$ can be perfectly purified out. Specifically, for unique τ, the difference $g_t - g_{t|t-\tau} = \left(1-\rho_g L\right)^{-1} w_t^{\tau} - \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau}$ in which $w_{t-\tau}$ or w_t never emerge.

Figure 19a shows the related numerical exercise.

C.3.1.2 loose identification

Most of time we do not know the number of unique τ or this uniqueness may not even exist. There are several different type of news shock with different information power, i.e. $\tau_1 > \tau_2 > \tau_3 > \cdots > \tau_n$. Therefore I relax the identification method discussed in previous subsection by adding lag and lead terms (relative to $g_{t|t-\tau}$) in to projection. Write the lag of equation 30

$$
\begin{aligned}
g_{t-1|t-\tau-1} &= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau-1}^{\tau} + \left(1-\rho_g L\right)^{-1} w_{t-\tau-1} \\
&\vdots \\
g_{t-n|t-\tau-n} &= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau-n}^{\tau} + \left(1-\rho_g L\right)^{-1} w_{t-\tau-n}
\end{aligned}
$$

and lead

$$
\begin{aligned}
g_{t+1|t-\tau+1} &= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-\tau+1}^{\tau} + \left(1-\rho_g L\right)^{-1} w_{t-\tau+1} \\
&\vdots \\
g_{t+\tau-1|t-1} &= \rho_g^{\tau} \left(1-\rho_g L\right)^{-1} w_{t-1}^{\tau} + \left(1-\rho_g L\right)^{-1} w_{t-1} \qquad (31)
\end{aligned}
$$

It is easy to comprehend the harmless of this loose identification to $\mathrm{corr}(w_t, u_t)$ as w_t does not emerge either. Meanwhile it is also harmless to $\mathrm{corr}(w_{t-\tau}, u_t)$ as $w_{t-\tau}$ enters into equation 31 with smaller impact coefficient than that in $g_{t|t-\tau}$ and it can still purify the effect of $w_{t-\tau}$ from g_t. However the lead term $w_{t-\tau+1}$, $w_{t-\tau+2}$, ... , w_{t-1} cannot be cleaned out from g_t.

Figure 19b shows the related numerical exercise.

C.3.1.3 arbitrary information power τ

Now we further relax the assumption of information power τ which is arbitrary to the expectation data that we observed, which I denote as k. Basically the previous augment about expectation 30 or 31 but now what we observe and can be used to identify is $g_{t+k|t}$ where $k<\tau$ or $k>\tau$.

When $k>\tau$, W.O.L.G, I assume $k=\tau+1$, then the observation becomes

$$g_{t|t-k}=\rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau-1}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau-1}$$

Furthermore, the lag terms of observation are

$$\begin{aligned} g_{t-1|t-k-1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau-2}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau-2} \\ &\vdots \\ g_{t-n|t-k-n} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau-(n+1)}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau-(n+1)} \end{aligned} \tag{32}$$

The lead terms of observation are

$$\begin{aligned} g_{t+1|t-k+1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau} \\ &\vdots \\ g_{t+k-1|t-1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-1}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-1} \end{aligned} \tag{33}$$

These two equation 32 and 33 demonstrate that we can still fully purify w_t and almost purify $w_{t-\tau}$.

Figure 19c shows the related numerical exercise.

When $k<\tau$, W.O.L.G, I assume $k=\tau-1$, then the observation becomes

$$g_{t|t-k}=\rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau+1}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau+1}$$

Furthermore, the lag terms of observation are

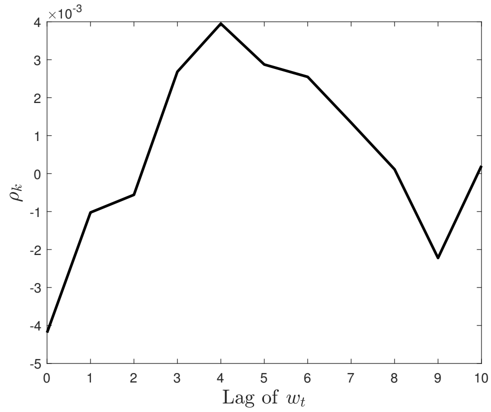
$$\begin{aligned} g_{t-1|t-k-1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau} \\ &\vdots \\ g_{t-n|t-k-n} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau-n+1}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau-n+1} \end{aligned} \tag{34}$$

The lead terms of observation are

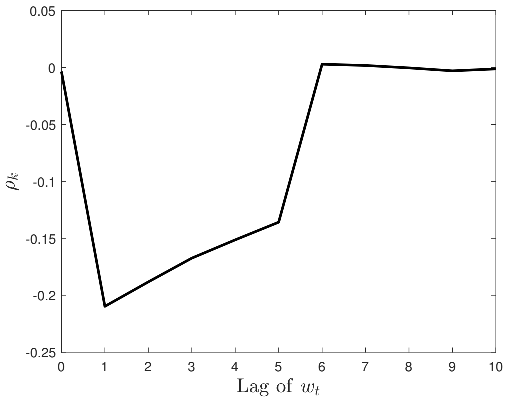
$$\begin{aligned} g_{t+1|t-k+1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-\tau+2}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-\tau+2} \\ &\vdots \\ g_{t+k-1|t-1} &= \rho_g^{\tau}\left(1-\rho_g L\right)^{-1} w_{t-1}^{\tau}+\left(1-\rho_g L\right)^{-1} w_{t-1} \end{aligned} \tag{35}$$

These two equation 34 and 35 demonstrate that we can still fully purify w_t and almost purify $w_{t-\tau}$.

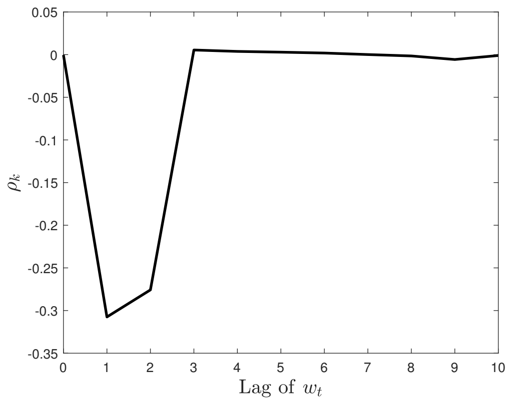
Figure 19d shows the related numerical exercise.



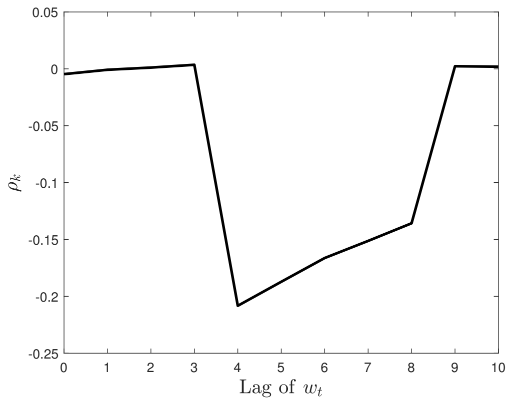
(a) Cross-Correlation between u_t and w_t



(b) Cross-Correlation between u_t and w_t with $n = \tau + 4$



(c) Cross-Correlation between u_t and w_t with $n = \tau + 4$ and $k = \tau + 3$



(d) Cross-Correlation between u_t and w_t with $n = \tau + 4$ and $k = \tau - 3$

Figure 19: Cross-Correlation under Perfect Information (Exogenous g_t)

C.3.2 Imperfect Information: fundamental impact g_t is observable.

C.3.2.1 uniquely identification

Given the fundamental process follows

$$
\begin{aligned}
g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \\
&= (1 - \rho_g L)^{-1} w_{t-\tau} + (1 - \rho_g L)^{-1} w_t^\tau
\end{aligned}
\tag{36}
$$

Then

$$
g_{t+\tau|t} = \rho_g^\tau g_t + \rho_g^{\tau-1} w_{t-\tau+1|t} + \rho_g^{\tau-2} w_{t-\tau+2|t} + \cdots + w_{t|t} \tag{37}
$$

where

$$\begin{aligned}
w_{t-\tau+1|t} &= \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\widetilde{w}_{t-\tau+1} \\
&= \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\left(w_{t-\tau+1}+\nu_{t-\tau+1}\right)
\end{aligned}$$

Therefore lagged expectation up to τ follows

$$\begin{aligned}
g_{t|t-\tau} &= \rho_g^\tau g_{t-\tau}+\rho_g^{\tau-1}w_{t-2\tau+1|t-\tau}+\rho_g^{\tau-2}w_{t-2\tau+2|t-\tau}+\cdots+w_{t-\tau|t-\tau} \\
&= \rho_g^\tau\left(1-\rho_g L\right)^{-1}w_{t-\tau}^\tau+\rho_g^\tau\left(1-\rho_g L\right)^{-1}w_{t-2\tau}+\sum_{j=0}^{j=\tau-1}\rho_g^j L^j w_{t-\tau|t-\tau}
\end{aligned} \tag{38}$$

It is worth to notice that the news shock realized at time t, w_t or $\widetilde{w}_t$, is exactly independent with the residual as it does not emerge neither on LHS or RHS.

In the simple regression case we can get that

$$\begin{aligned}
\widehat{\alpha}_{g_{t|t-\tau}} &= \frac{\operatorname{cov}(g_{t|t-\tau},g_t)}{\operatorname{var}(g_{t|t-\tau})} \\
&\approx \frac{\frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\frac{1-\rho_g^{2\tau}}{1-\rho_g^2}\sigma_w^2}{\frac{1-\rho_g^{2\tau}}{1-\rho_g^2}\operatorname{var}(w_{t-\tau|t-\tau})}=1
\end{aligned}$$

which follows $\operatorname{cov}(w_t,\nu_t)=0$. Therefore the residual u_t contains the elements

$$u_t \approx \frac{\sigma_\nu^2}{\sigma_w^2+\sigma_\nu^2}\sum_{j=0}^{j=\tau-1}\rho_g^j L^j w_{t-\tau}-\frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\sum_{j=0}^{j=\tau-1}\rho_g^j L^j \nu_{t-\tau}$$

Therefore the observation term $\widetilde{w}_{t-\tau}$ is cleaned out as $\operatorname{cov}\left(u_t,\widetilde{w}_{t-\tau}\right)=\frac{\sigma_\nu^2}{\sigma_w^2+\sigma_\nu^2}\sigma_w^2-\frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2}\sigma_\nu^2=0.$

Figure 20a shows the related numerical exercise.

C.3.2.2 loose identification

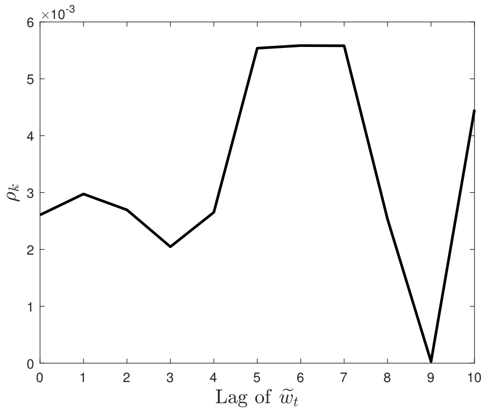
Similar to the cases in perfect information.

Figure 20b shows the related numerical exercise.

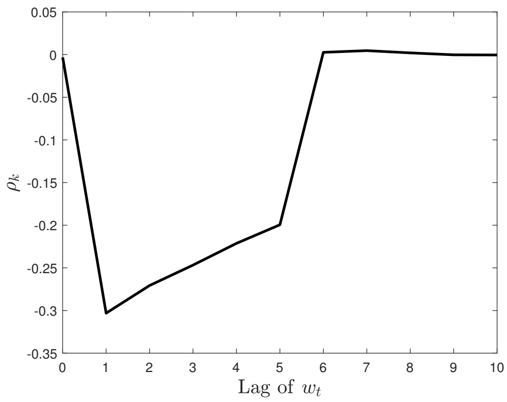
C.3.2.3 arbitrary information power τ

Similar to the cases in perfect information.

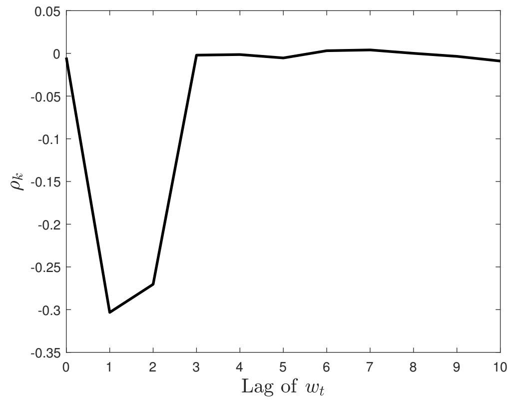
Figure 20c and 20d shows the related numerical exercise.



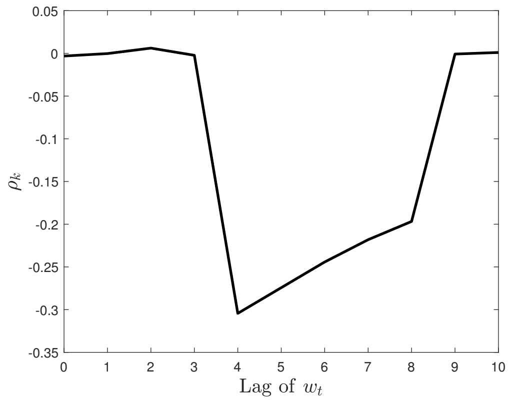
(a) Cross-Correlation between u_t and $\widetilde{w}_t$



(b) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$



(c) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau + 3$



(d) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau - 3$

Figure 20: Cross-Correlation under Imperfect Information (Exogenous g_t but observable)

C.3.3 Imperfect Information: fundamental impact g_t is unobservable.

C.3.3.1 uniquely identification

Given the fundamental process follows

$$\begin{aligned} g_t &= \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \\ &= (1-\rho_g L)^{-1} w_{t-\tau} + (1-\rho_g L)^{-1} w_t^\tau \end{aligned} \tag{39}$$

We can only observe the perception of g_t at t

$$\begin{aligned} g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 \widetilde{g}_t \\ &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 g_t + \gamma_7 \nu_t^\tau \\ &= \rho_g \gamma_2 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 (1-\rho_g L)^{-1} w_{t-\tau} + \gamma_7 (1-\rho_g L)^{-1} w_t^\tau + \gamma_7 \nu_t^\tau \\ &= \gamma_2 (1-\gamma_2\rho_g L)^{-1} w_{t-\tau|t-\tau} + \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_{t-\tau} \\ &+ \gamma_7 (1-\gamma_2\rho_g L)^{-1} (1-\rho_g L)^{-1} w_t^\tau + \gamma_7 (1-\gamma_2\rho_g L)^{-1} \nu_t^\tau \end{aligned} \tag{40}$$

Since now the household cannot observe g_t neither, they have no more other information source to verify the news shock $w_{t-\tau}$. Therefore their perception about it shock will not change as time goes forward, which implies $w_{t-\tau|t-\tau} = w_{t-\tau|t}$.

Then the expectation term follows

$$g_{t+\tau|t} = \rho_g^\tau g_{t|t} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t} \tag{41}$$

Therefore lagged expectation up to τ follows

$$\begin{aligned} g_{t|t-\tau} &= \rho_g^\tau g_{t-\tau|t-\tau} + \rho_g^{\tau-1} w_{t-2\tau+1|t-2\tau+1} + \rho_g^{\tau-2} w_{t-2\tau+2|t-2\tau+2} + \cdots + w_{t-\tau|t-\tau} \\ &= \rho_g^\tau \left[\gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-2\tau|t-2\tau} + \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \left(1 - \rho_g L\right)^{-1} w_{t-2\tau}\right] \\ &+ \rho_g^\tau \left[\gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \left(1 - \rho_g L\right)^{-1} w_{t-2\tau}^\tau + \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \nu_{t-2\tau}^\tau\right] \\ &+ \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau|t-\tau} \end{aligned} \tag{42}$$

To further simplify 40 as

$$\begin{aligned} g_{t|t} &= \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \left(1 - \rho_g L\right)^{-1} w_t^\tau + \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \nu_t^\tau \\ &+ \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} + \gamma_7 \frac{\gamma_2}{\gamma_2 - 1} \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau} \\ &+ \gamma_7 \frac{1}{1 - \gamma_2} \left(1 - \rho_g L\right)^{-1} w_{t-\tau} \end{aligned}$$

Since $\gamma_2 + \gamma_7 = 1$, we can get

$$\begin{aligned} g_{t|t} &= \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \left(1 - \rho_g L\right)^{-1} w_t^\tau + \gamma_7 \left(1 - \gamma_2 \rho_g L\right)^{-1} \nu_t^\tau \\ &+ \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} - \gamma_2 \left(1 - \gamma_2 \rho_g L\right)^{-1} w_{t-\tau} + \left(1 - \rho_g L\right)^{-1} w_{t-\tau} \end{aligned}$$

Similarly in the simple regression case we can get that

$$\begin{aligned} \widehat{\alpha}_{g_{t|t-\tau}} &= \frac{\operatorname{cov}(g_{t|t-\tau}, g_{t|t})}{\operatorname{var}(g_{t|t-\tau})} \\ &\approx \frac{\Phi}{\frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \operatorname{var}(w_{t-\tau|t-\tau})} = 1 \end{aligned}$$

where $\Phi = \gamma_2 \frac{1-\left(\gamma_2 \rho_g^2\right)^{2\tau}}{1-\gamma_2^2 \rho_g^4} \sigma_{\widetilde{w}}^2 - \gamma_2 \frac{1-\left(\gamma_2 \rho_g^2\right)^{2\tau}}{1-\gamma_2^2 \rho_g^4} \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_w^2 + \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \sigma_w^2 = \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \frac{1-\rho_g^{2\tau}}{1-\rho_g^2} \sigma_w^2$ as $\sigma_{\widetilde{w}}^2 = \frac{\sigma_w^2}{\sigma_w^2 + \sigma_\nu^2} \sigma_w^2$.

Therefore the residual u_t contains the elements

$$\begin{aligned} u_t \approx & \gamma_2 \left(1-\gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} - \gamma_2 \left(1-\gamma_2 \rho_g L\right)^{-1} w_{t-\tau} \\ & + \left(1-\rho_g L\right)^{-1} w_{t-\tau} - \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t-\tau|t-\tau} \end{aligned}$$

However under this scenario the observation term $\widetilde{w}_{t-\tau}$ cannot be cleaned out because

$$\begin{aligned} \operatorname{cov}\left(\widetilde{w}_{t-\tau}, u_t\right) & \approx \gamma_2 \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_{\widetilde{w}}^2 - \gamma_2 \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_w^2 + \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_w^2 - \frac{\sigma_w^2}{\sigma_w^2+\sigma_\nu^2} \sigma_{\widetilde{w}}^2 \\ & = \left(1-\gamma_2\right) \frac{\sigma_w^2 \sigma_\nu^2}{\sigma_w^2+\sigma_\nu^2} \neq 0 \end{aligned}$$

Figure 21a shows the related numerical exercise.

C.3.3.2 loose identification

Write the lag of equation 42

$$\begin{aligned} g_{t-1|t-\tau-1} & = \Theta_{t-\tau-1} + \left(1-\rho_g L\right)^{-1} w_{t-\tau-1|t-\tau-1} \\ & \vdots \\ g_{t-n|t-\tau-n} & = \Theta_{t-\tau-n} + \left(1-\rho_g L\right)^{-1} w_{t-\tau-n|t-\tau-n} \end{aligned}$$

and lead

$$\begin{aligned} g_{t+1|t-\tau+1} & = \Theta_{t-\tau+1} + \left(1-\rho_g L\right)^{-1} w_{t-\tau+1|t-\tau+1} \\ & \vdots \\ g_{t+\tau-1|t-1} & = \Theta_{t-1} + \left(1-\rho_g L\right)^{-1} w_{t-1|t-1} \end{aligned}$$

where

$$\begin{aligned} \Theta_t = & \rho_g^\tau \left[\gamma_2 \left(1-\gamma_2 \rho_g L\right)^{-1} w_{t-\tau|t-\tau} + \gamma_7 \left(1-\gamma_2 \rho_g L\right)^{-1} \left(1-\rho_g L\right)^{-1} w_{t-\tau}\right] \\ & + \rho_g^\tau \left[\gamma_7 \left(1-\gamma_2 \rho_g L\right)^{-1} \left(1-\rho_g L\right)^{-1} w_{t-\tau}^\tau + \gamma_7 \left(1-\gamma_2 \rho_g L\right)^{-1} \nu_{t-\tau}^\tau\right] \end{aligned}$$

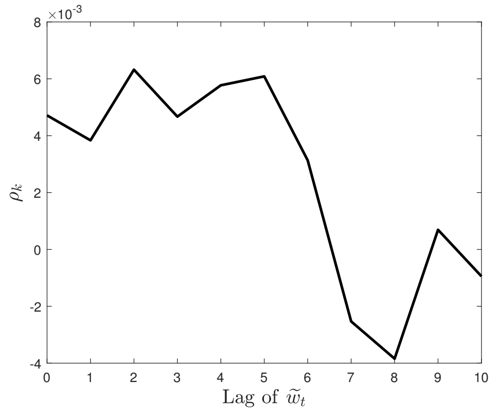
Similar to the cases in perfect information.

Figure 21b shows the related numerical exercise.

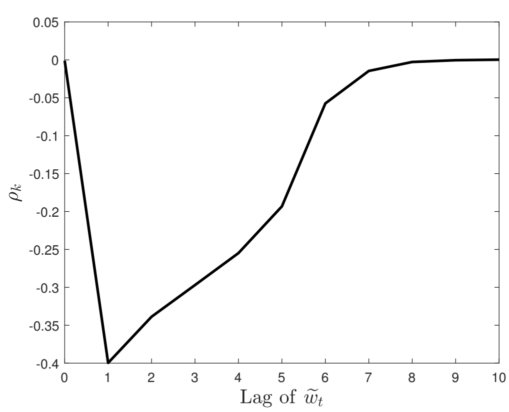
C.3.3.3 arbitrary information power τ

Similar to the cases in perfect information.

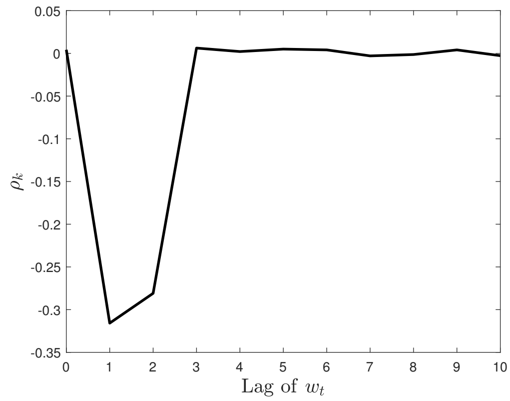
Figure 21c and 21d shows the related numerical exercise.



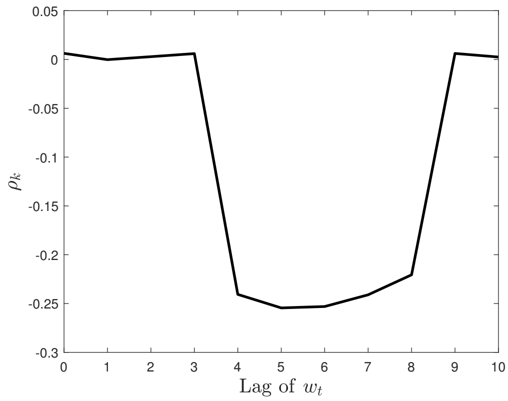
(a) Cross-Correlation between u_t and $\widetilde{w}_t$



(b) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$



(c) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau + 3$



(d) Cross-Correlation between u_t and $\widetilde{w}_t$ with $n = \tau + 4$ and $k = \tau - 3$

Figure 21: Cross-Correlation under Imperfect Information (Exogenous g_t but unobservable)

C.4 Endogenous g_t w.r.t w_t

C.4.1 Perfect Information

C.4.1.1 uniquely identification

Now let us introduce the endogeneity of w_t on g_t as

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} + \alpha w_t$$

Then the expectation of $g_{t+\tau}$ at time t follows

$$\begin{aligned}
g_{t+\tau|t} &= \rho_g^{\tau} g_t + \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t \\
&= \rho_g^{\tau+1} g_{t-1} + \rho_g^{\tau} \left(w_{t-\tau} + w_t^{\tau} + \alpha w_t \right) \\
&\quad + \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t
\end{aligned}$$

What we need is to clean out w_t from g_t and retain w_t^{τ}. Therefore we first run the regression of g_t on g_{t-1} (or $g_{t+\tau|t}$ on $g_{t+\tau-1|t-1}$) to get the estimated $\widehat{\rho}_g$. Because the expectation of $g_{t+\tau}$ at t is based on the observation g_t, we can purify the contemporaneous expectation term out of g_t and remain the news part $\sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_t$ through $\widehat{g}_{t+\tau|t} = g_{t+\tau|t} - \rho^{\tau} g_t$. Then we can clean the news shock w_t out of g_t by simple regression. In the numerical exercise below $\operatorname{corr}(u_t, w_t) < 3e^{-3}$ holds.

Figure 22a shows the related numerical exercise.

If you further want to clean out $w_{t-\tau}$ (though we in fact do not want and to the contrary we should make sure that $w_{t-\tau}$ exists in g_t), you could use the method mentioned in section C.3.1.2 to yield

Figure 22b shows the related numerical exercise.

C.4.1.2 loose identification

Similar to the arguments in section C.3.1.2, I relax the identification method discussed in previous subsection by adding lead terms (relative to $\widehat{g}_{t+\tau|t}$) in to projection. Write the lead of equation 30

$$
\begin{aligned}
\widehat{g}_{t+\tau|t} &= \rho_g^{\tau-1} w_{t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2} + \cdots + w_t = \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_t \\
&\vdots \\
\widehat{g}_{t+\tau+n|t+n} &= \rho_g^{\tau-1} w_{t-\tau+1+n} + \rho_g^{\tau-2} w_{t-\tau+2+n} + \cdots + w_{t+n} = \sum_{j=0}^{j=\tau-1} \rho_g^j L^j w_{t+n}
\end{aligned}
$$

Figure 22c shows the related numerical exercise. It is worth to notice that the approximately zero of $\operatorname{cov}(u_t, w_t)$ results from the estimation error of $\widehat{\rho}_g$. If we use the true ρ_g to conduct the purification process, $\operatorname{cov}(u_t, w_t)$ will be exactly zero as figure 22d shows.

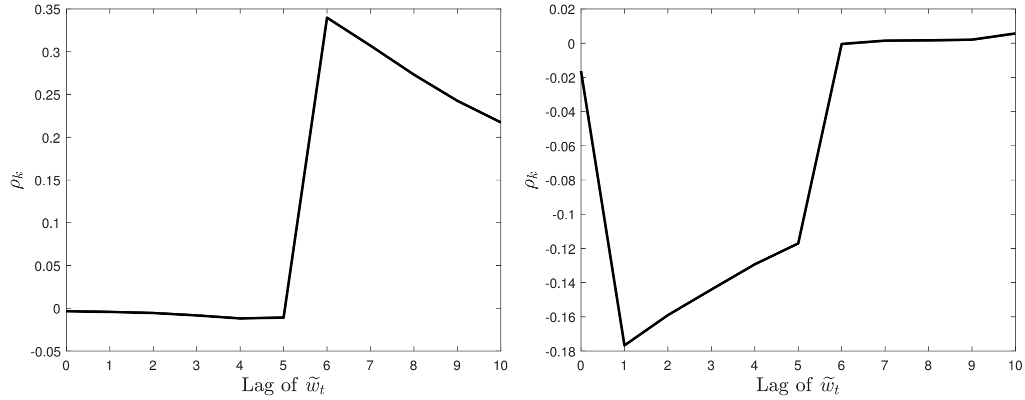
Moreover, write the lag terms

$$
\begin{aligned}
g_{t+\tau-1|t-1} &= \rho_g^{\tau-1} w_{t-\tau} + \rho_g^{\tau-2} w_{t-\tau+1} + \cdots + w_{t-1} \\
&\vdots \\
g_{t-m|t-\tau-m} &= \rho_g^{\tau-1} w_{t-2\tau-m+1} + \rho_g^{\tau-2} w_{t-2\tau-m+2} + \cdots + w_{t-\tau-m}
\end{aligned}
$$

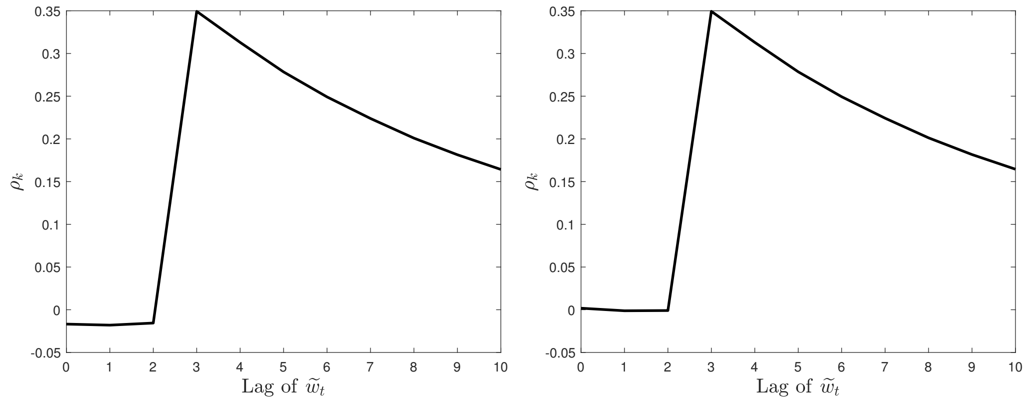
It seems harmless to add the lagged term into purification regression and figure 23a verifies this argument.

C.4.1.3 arbitrary information power τ

Similar to section C.3.1.3, now we observe and can be used to identify is $g_{t+k|t}$ where $k < \tau$ or $k > \tau$ instead of $g_{t+\tau|\tau}$.



(a) Cross-Correlation between u_t and w_t (b) Cross-Correlation betweenu_t and w_t



(c) Cross-Correlation between u_t and $w_t$with $n = \tau + 4$ (d) Cross-Correlation between u_t and w_t with true ρ_g and $n = \tau + 4$

Figure 22: Cross-Correlation under Perfect Information-1 (Endogenous g_t)

When $k > \tau$, W.O.L.G, I assume $k = \tau + 1$, then the observation becomes

$$\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau+1} + \rho_g^{k-2} w_{t-\tau+2} + \cdots + \rho_g w_t = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_t$$

Furthermore, the lead terms of observation are

$$\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau+1} + \rho_g^{k-2} w_{t-\tau+2} + \cdots + \rho_g w_t = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_t$$

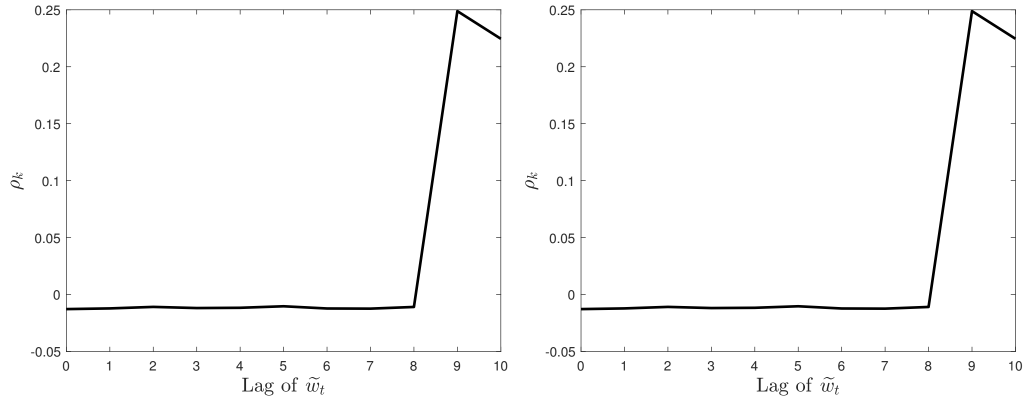
$$\vdots\vdots$$

$$\widehat{g}_{t+k+n|t+n} = \rho_g^{k-1} w_{t-\tau+1+n} + \rho_g^{k-2} w_{t-\tau+2+n} + \cdots + \rho_g w_{t+n} = \sum_{j=1}^{j=k-1} \rho_g^j L^j w_{t+n}$$

The lag terms of observation are

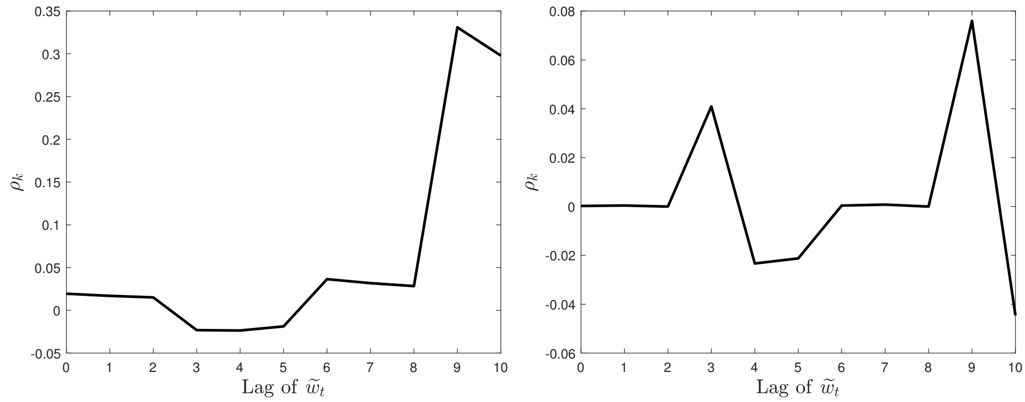
$$g_{t+\tau-1|t-1} = \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + \rho_g w_{t-1}$$
$$\vdots$$
$$g_{t-m|t-\tau-m} = \rho_g^{k-1} w_{t-2\tau-m+1} + \rho_g^{k-2} w_{t-2\tau-m+2} + \cdots + \rho_g w_{t-\tau-m}$$

Therefore we can add both lead and lag terms into purification regression safely and figure 23b verifies this argument.



(a) Cross-Correlation between u_t and $w_t$with $m = \tau$ and $n = \tau + 4$

(b) Cross-Correlation between u_t and $w_t$with $m = \tau$ and $n = \tau + 4$



(c) Cross-Correlation between u_t and $w_t$with $k = \tau - 3$ and $n = \tau + 4$

(d) Cross-Correlation between u_t and $w_t$with $k = \tau - 3$, $m = \tau + 1$ and $n = \tau + 4$

Figure 23: Cross-Correlation under Perfect Information-2 (Endogenous g_t)

When $k < \tau$, W.O.L.G, I assume $k = \tau - 1$, then the observation becomes

$$\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + w_{t-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t-1}$$

Thus we cannot uniquely clean out w_t from g_t with $\widehat{g}_{t+k|t}$ in which w_t does not emerge. When we impose loose identification and add the lead term

$$\widehat{g}_{t+k|t} = \rho_g^{k-1} w_{t-\tau} + \rho_g^{k-2} w_{t-\tau+1} + \cdots + w_{t-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t-1}$$

$$\vdots$$

$$\widehat{g}_{t+k+n|t+n} = \rho_g^{k-1} w_{t-\tau+n} + \rho_g^{k-2} w_{t-\tau+1+n} + \cdots + w_{t+n-1} = \sum_{j=0}^{j=k-1} \rho_g^j L^j w_{t+n-1}$$

the news term w_t is embedded into $\widehat{g}_{t+k+1|t+1}$ and we can clean out w_t via the loose identification.

Figure 23c shows this identification result.

Similar to the argument in loose identification, since the lagged terms does not contains any information about contemporaneous news shock, it is harmless to add the lag part into identification and figure 23d shows the numerical result.

C.4.2 Imperfect Information: fundamental impact g_t is unobservable.

C.4.2.1 uniquely identification

Given the fundamental process follows[25]

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} + \alpha w_t$$

$$g_{t|t} = \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_7 \widetilde{g}_t \tag{43}$$

Then the expectation follows

$$g_{t+\tau|t} = \rho_g^{\tau} g_{t|t} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

$$g_{t+\tau+1|t+1} = \rho_g^{\tau} g_{t+1|t+1} + \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

Therefore the estimation step of AR coefficient cannot be the autoregression on perception $g_{t|t}$ but on the expectation $g_{t+\tau|t}$. Given the forward-looking news estimation

$$\widehat{g}_{t+\tau|t} = \rho_g^{\tau-1} w_{t-\tau+1|t-\tau+1} + \rho_g^{\tau-2} w_{t-\tau+2|t-\tau+2} + \cdots + w_{t|t}$$

$$\widehat{g}_{t+\tau-1|t-1} = \rho_g^{\tau-1} w_{t-\tau|t-\tau} + \rho_g^{\tau-2} w_{t-\tau+1|t-\tau+1} + \cdots + w_{t-1|t-1}$$

Everything goes back to the perfect information cases and all the arguments in perfect information case will also be true under imperfect information case. Figure 24 shows the experiment result of this identification result.

[25] In section D.4 I provide rigorous proof of equation 43.

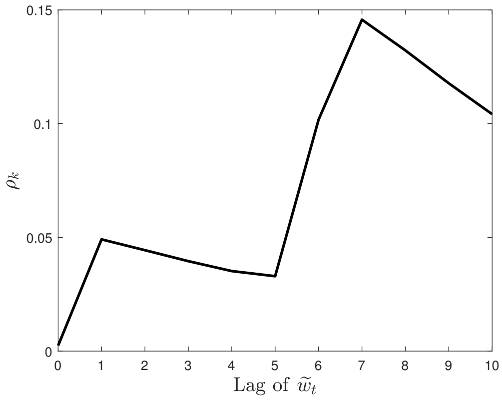


Figure 24: Cross-Correlation under Imperfect Information (Endogenous unobservable g_t)

C.5 Endogenality, Heteroscedasticity and Biased-estimation Problem during Purification

C.5.1 get w_t out of 5

Because g_t contains w_t, if we run the regression of $E_t g_{t+6}$ on g_t there will be an endogenality problem (residual is correlated with independent variable) and the estimated ρ_g^6 is biased. Therefore I use the model

$$E_t g_{t+6} = \rho_g^7 g_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t \tag{44}$$

If we run the regression of $E_t g_{t+6}$ on g_{t-1}, we can get $u_t^E = \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$

The problem is that g_{t-1} contains w_{t-1}, w_{t-2}, w_{t-3} too, as $g_{t-1} = \rho_g g_{t-2} + w_{t-4} + u_{t-1} + \alpha_2 w_{t-1}$, and the endogenality problem still hold.

Adding the lag span may identify up to scale

$$E_t g_{t+6} = \rho_g^8 g_{t-2} + \rho_g^7 w_{t-4} + \rho_g^7 u_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \left(\rho_g^4 + \rho_g^7 \alpha_2\right) w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$$

because $\operatorname{cov}(g_{t-2}, \rho_g^5 w_{t-2}) < \operatorname{cov}(g_{t-1}, \rho_g^4 w_{t-1})$ holds and in the end the endogenality in first step will be solved. However, we should also care about the trade-off problem here, because when we add the lag span we actually introduce more term into residual, especially u_t and u_{t-1}. This will introduce the endogenality problem into our second regression step: run regression of g_t on $u_t^E = \rho_g^7 w_{t-4} + \rho_g^7 u_{t-1} + \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \left(\rho_g^4 + \rho_g^7 \alpha_2\right) w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t$.

C.5.2 run regression of 5 on w_t

Assume I use the regression of equation 44 and get

$$u_t^E = \rho_g^6 w_{t-3} + \rho_g^6 u_t + \rho_g^5 w_{t-2} + \rho_g^4 w_{t-1} + \left(\rho_g^6 \alpha_2 + \rho_g^3\right) w_t \tag{45}$$

we cannot directly run regression of g_t on u_t^E because there are three elements (w_t, u_t, and w_{t-3}) in g_t that are correlated with u_t^E. Given the regression $g_t = \gamma_1 u_t^E + \varepsilon_t$ we cannot make sure that $\operatorname{cov}(\varepsilon_t, w_t) = 0$ (Through simulated data, it is indeed not zero or close to zero up to scale) because a lot of elements in u_t^E correlate with the non-w_t elements in g_t such as u_t and w_{t-3} which will change the projection and cause $\gamma_1 \neq \left(\rho_g^6\alpha_2 + \rho_g^3\right)$, the coefficient in front of w_t in 45. To solve the problem I further add the lead term of u_t^E into the second step of purification. For instance, if I use the regression $g_t = \gamma_1 u_{t+3}^E + \varepsilon_t$ instead of $g_t = \gamma_1 u_t^E + \varepsilon_t$, the problem can be solved, as in $u_{t+3}^E = \rho_g^6 w_t + \rho_g^6 u_{t+3} + \rho_g^5 w_{t+1} + \rho_g^4 w_{t+2} + \left(\rho_g^6\alpha_2 + \rho_g^3\right) w_{t+3}$ the only element that correlates with g_t is w_t.

Therefore the only problem left is that how to determine the informative power of news shock? If 5 becomes

$$g_t = \rho_g g_{t-1} + \alpha_1 y_t + w_{t-1} + u_t + \alpha_2 w_t$$

the equation

$$u_t^E = \rho_g^6 w_{t-1} + \rho_g^6 u_t + \left(\rho_g^6\alpha_2 + \rho_g^5\right) w_t \tag{46}$$

will hold and we may use u_{t+1}^E to clean the g_t yet not u_{t+3}^E. Meanwhile, when 5 becomes

$$g_t = \rho_g g_{t-1} + \alpha_1 y_t + w_{t-9} + u_t + \alpha_2 w_t$$

the equation

$$u_t^E = \rho_g^6 w_{t-9} + \rho_g^6 u_t + \rho_g^5 w_{t-8} + \rho_g^4 w_{t-7} + \rho_g^3 w_{t-6} + \rho_g^2 w_{t-5} + \rho_g w_{t-4} + w_{t-3} + \rho_g^6\alpha_2 w_t \tag{47}$$

will hold and we may use u_{t+9}^E.

By observing the equation 46 and 47, we find that it is possible to use ACF of u_t^E to pin down the informative power of news τ because difference news with different informative power will imply different “MA” process with different shape of ACF. For instance, if $\tau = 1$ holds, equation 46 will imply that the ACF will converge to zero quickly at second lag. To the contrary, if $\tau = 9$ holds, equation 47 will imply that the ACF will converge to zero sluggishly at ninth lag. Hence the speed of convergence of ACF will help us to find the informative power of housing price news shock even we only have the expectation data up to six month later.

C.6 Purified perception on the status of housing market

The first task to purify w_t out of g_t in equation 5 is to find appropriate macro variables x_t which affects the perception of the status of housing market. Taking an overall consideration on the data constraint and efficiency, I use real interest rate, inflation, M2 supply, unemployment rate and nondurable consumption as the independent macro variables that affect the perception g_t. Because of lack of monthly investment data, I use the real interest rate to reveal the effect of physical capital and investment. The inflation rate and M2 supply reveal the effect of normal friction in New-Keynesian and monetary theory. The unemployment rate and consumption reflect the effect in labor and goods market. By adding the contemporaneous and lagged term of these macro variables in table 4 I show that people's perception are more based on previous macro variables as they may not have the data related to the contemporaneous macro status.

It is harder to determine the optimal lag interval of each macro variables as these macro variables are persistent in themselves. The last three columns in table 5 show that it is inappropriate to add lagged term in third order of M2 supply and unemployment and second order of nondurable consumption. Column 2 and column 3 imply that there is no marginal benefit in adding more lagged terms of real interest and inflation. Because of the inertia of real interest rate (inflation), third (fourth) order in lag r_{t-3} (π_{t-4}) is significant yet this significance comes from the smaller lagged term r_{t-2} (π_{t-3}). Furthermore, because of monetary policy, the real interest rate moves endogenously with inflation and the lagged third order term of interest rate r_{t-3} also hurts the significance of inflation rate.

As I argued in main body, equation 5 is only for illustration purpose and the true formula of status perception may contains more lagged term or even the expectation term $E_t g_{t+6}$. By adding more lagged term of perception g_t in model 5 I show that the maximized lag number of g_t is 1 and further lagged terms are insignificant via the column 2 to column 6 in table 6. Column 8 and column 9 in table 6 shows that more lagged term of expectation will not provide extra explanation power on the dependent variable. While, it is more complicate to decide whether add previous expectation in equation 5 as column 7 shows that it is significant to add it. However, since the expectation $E_t g_{t+6}$ itself is based on the perception g_t, its significant property is not surprising and the key point is the marginal benefit of adding the expectation term. Column 7 shows that the coefficient of g_{t-1} decreases from 0.84 to 0.16 and the coefficient of expectation term is close to that of g_{t-1}. This means the expectation term does not introduce new explanation power but shares with g_{t-1} as $E_{t-1} g_{t+5}$ is a function of g_{t-1}. Additionally, the inflation rate, M2 supply, unemployment rate and nondurable consumption, those macro variables, become insignificant after adding the over-interpolation term $E_{t-1} g_{t+5}$. Therefore in baseline model 5 I do not add the expectation term because it is not an efficient and profitable explanatory variable.

In baseline model I only use 5 macro variables to indicate the effect of macroeconomics on household's perception on the status of housing market because other macro variables are not significant in explaining the perception. Table 7 provides the robustness check on adding more macro variables into purification. Moreover, since in the last step after purification I embed the purified g_t into VAR identification, any macroeconomic effect that is missed here will be covered later.

In addition to get the near "MA" process of news shock u_t^E, I also need to find out the informative power of news since until now I do not know whether the form of u_t^E follows equation 46 or 47 (or some other forms). As discussed in C.5.2, the ACF of residual in first step of purification, u_t^E, implies the informative power of news and the speed of its convergence to zero refers how many period ahead that the news is announced to household. Figure 25 shows

Table 4: Contemporaneous Macro Variables' effect

	Dependent variable:								
	HIM								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
EFF_rate_1		3.177							
		(23.554)							
CPI_1			47.354						
			(109.226)						
M2_1				−44.437					
				(64.726)					
GDP_1					−3.086				
					(56.138)				
Consumption_1						79.407			
						(71.478)			
Unemployment_1							0.005		
							(0.041)		
HIM_1	0.987^{**}	0.916^{*}	0.655	0.762^{**}	0.977^{*}	1.205^{**}	1.041^{***}	0.803	0.840
	(0.502)	(0.483)	(0.410)	(0.352)	(0.504)	(0.596)	(0.350)	(1.311)	(0.810)
HIM_2								−0.218	−1.366
								(1.219)	(1.204)
HIM_3									0.832
									(0.879)
Observations	273	273	273	273	273	273	273	271	269
R^2	-0.904	-0.775	-0.380	-0.523	-0.885	-1.359	-1.010	-0.601	-2.774
Adjusted R^2	-0.911	-0.788	-0.390	-0.534	-0.899	-1.376	-1.025	-0.613	-2.817
Residual Std. Error	4.840 (df = 272)	4.682 (df = 271)	4.129 (df = 271)	4.337 (df = 271)	4.825 (df = 271)	5.397 (df = 271)	4.983 (df = 271)	4.443 (df = 269)	6.835 (df = 266)
Note:									*p<0.1; **p<0.05; ***p<0.01

Table 5: Lagged Macro Variables' effect

	Dependent variable:								
	HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
EFF_rate_1	529.794***								
	(141.326)								
CPI_1			1,732.599***						
			(206.905)						
M2_1				305.905*					
				(169.201)					
GDP_1					−139.443				
					(149.262)				
Consumption_1						410.650***			
						(121.748)			
Unemployment_1							0.717***		
							(0.070)		
HIM	5.898***	5.330**	1.825	3.526*	4.964*	1.409	1.722**	7.096	17.048
	(2.212)	(2.559)	(1.262)	(2.038)	(2.654)	(1.240)	(0.781)	(8.978)	(29.615)
HIM_1								−2.621	7.141
								(11.052)	(19.888)
HIM_2								2.335	4.185
								(7.265)	(18.814)
HIM_3									−12.493
									(30.945)
Observations	381	381	381	381	381	381	381	379	377
R^2	-10.048	-8.689	-0.567	-3.723	-7.507	-0.510	-0.363	-17.973	-152.309
Adjusted R^2	-10.107	-8.715	-0.576	-3.748	-7.552	-0.518	-0.371	-18.125	-153.953
Residual Std. Error	19.987 (df = 379)	18.693 (df = 380)	7.528 (df = 379)	13.068 (df = 379)	17.539 (df = 379)	7.388 (df = 379)	7.021 (df = 379)	26.296 (df = 376)	75.037 (df = 373)
Note:									*p<0.1; **p<0.05; ***p<0.01

Table 6: Lagged Effect of HIM_t and $E_t\mathrm{HIM}_{t+6}$

	Dependent variable:								
	HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
HIM_1	0.824^{***}	0.919^{***}	0.934^{***}	0.933^{***}	0.933^{***}	0.930^{***}	0.120^{**}	0.119^{**}	0.134^{**}
	(0.020)	(0.074)	(0.073)	(0.074)	(0.075)	(0.076)	(0.051)	(0.054)	(0.056)
HIM_2		−0.101	−0.091	−0.087	−0.086	−0.077			
		(0.075)	(0.081)	(0.083)	(0.083)	(0.083)			
HIM_3			−0.026	0.026	0.024	0.024			
			(0.068)	(0.084)	(0.083)	(0.083)			
HIM_4				−0.058	−0.040	−0.056			
				(0.065)	(0.068)	(0.067)			
HIM_5					−0.018	−0.028			
					(0.060)	(0.080)			
HIM_6						0.022			
						(0.064)			
HIME6_1							0.850^{***}	0.849^{***}	0.843^{***}
							(0.058)	(0.068)	(0.069)
HIME6_2								0.002	0.035
								(0.058)	(0.078)
HIME6_3									−0.048
									(0.058)
Unemployment_1	0.212^{***}	0.211^{***}	0.209^{***}	0.211^{***}	0.212^{***}	0.213^{***}	0.109^{**}	0.109^{**}	0.108^{**}
	(0.074)	(0.072)	(0.074)	(0.076)	(0.078)	(0.078)	(0.053)	(0.054)	(0.052)
Unemployment_2	-0.201^{***}	-0.203^{***}	-0.202^{***}	-0.205^{***}	-0.207^{***}	-0.208^{***}	-0.101^{*}	-0.101^{*}	-0.100^{**}
	(0.073)	(0.072)	(0.074)	(0.076)	(0.077)	(0.078)	(0.052)	(0.053)	(0.051)
Consumption_1	6.957^{**}	7.069^{**}	7.355^{**}	6.955^{**}	7.009^{**}	6.657^{**}	4.285	4.290	4.821^{*}
	(3.375)	(3.347)	(3.364)	(3.349)	(3.361)	(3.378)	(2.666)	(2.710)	(2.775)
Constant	0.154^{**}	0.174^{***}	0.177^{**}	0.188^{**}	0.191^{***}	0.190^{**}	−0.019	−0.020	−0.011
	(0.067)	(0.067)	(0.071)	(0.074)	(0.073)	(0.076)	(0.046)	(0.047)	(0.047)
Observations	418	418	417	416	415	414	418	418	417
R^2	0.930	0.930	0.931	0.931	0.931	0.931	0.955	0.955	0.955
Adjusted R^2	0.929	0.929	0.930	0.930	0.930	0.930	0.954	0.954	0.955
Residual Std. Error	0.201 (df = 413)	0.200 (df = 412)	0.200 (df = 410)	0.200 (df = 408)	0.200 (df = 406)	0.200 (df = 404)	0.161 (df = 412)	0.161 (df = 411)	0.161 (df = 409)
F Statistic	$1,367.493^{***}$ (df = 4; 413)	$1,098.471^{***}$ (df = 5; 412)	919.660^{***} (df = 6; 410)	787.041^{***} (df = 7; 408)	685.423^{***} (df = 8; 406)	609.366^{***} (df = 9; 404)	$1,742.976^{***}$ (df = 5; 412)	$1,448.960^{***}$ (df = 6; 411)	$1,250.463^{***}$ (df = 7; 409)

Note: *p<0.1; **p<0.05; ***p<0.01

Table 7: More Macro Variables

	Dependent variable:								
	HIME6								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
HIM_1	0.824***	0.825***	0.823***	0.824***	0.822***	0.828***	0.836***	0.834***	0.826***
	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.019)	(0.021)	(0.021)	(0.020)
Unemployment_1	0.212***	0.202***	0.212***	0.210***	0.218***	0.191***	0.126*	0.207***	0.193***
	(0.074)	(0.071)	(0.074)	(0.074)	(0.074)	(0.069)	(0.067)	(0.073)	(0.069)
Unemployment_2	−0.201***	−0.189***	−0.202***	−0.199***	−0.208***	−0.179***	−0.114*	−0.062	−0.182***
	(0.073)	(0.070)	(0.072)	(0.073)	(0.073)	(0.069)	(0.066)	(0.072)	(0.068)
Unemployment_3								−0.132*	
								(0.074)	
Consumption_1	6.957**	7.880**	7.116**	6.450*	7.428**	6.643*	7.530**	7.113**	7.051*
	(3.375)	(3.552)	(3.337)	(3.645)	(3.377)	(3.596)	(3.558)	(3.535)	(3.959)
Consumption_2									1.508
									(4.504)
GDP_1		−1.279				−1.474	−0.587	−0.840	−1.564
		(1.809)				(2.013)	(1.704)	(1.994)	(2.013)
GDP_2						−2.641	−1.336	−1.796	−2.769
						(1.965)	(1.880)	(1.923)	(1.964)
GDP_3							−3.791*		
							(2.067)		
EFF_rate_1			−0.764			−2.232	−0.571	−1.842	−2.255
			(1.925)			(5.334)	(5.328)	(5.187)	(5.344)
EFF_rate_2						2.529	−95.932***	2.094	2.415
						(5.316)	(27.871)	(5.252)	(5.372)
EFF_rate_3							97.156***		
							(26.995)		
CPI_Inflation_1				−1.756		−6.036	−101.540***	−4.691	−6.419
				(4.598)		(7.271)	(27.253)	(7.268)	(7.341)
CPI_Inflation_2						−14.951**	84.560***	−13.799**	−14.214**
						(6.562)	(28.514)	(6.705)	(6.822)
CPI_Inflation_3							−15.348**		
							(6.036)		
M2_1					−2.433	−6.478*	−8.174***	−6.116*	−6.568*
					(2.663)	(3.395)	(3.169)	(3.434)	(3.371)
M2_2						−4.471	−6.534*	−4.216	−4.384
						(3.229)	(3.399)	(3.166)	(3.192)
M2_3							−3.477		
							(3.203)		
Constant	0.154**	0.144**	0.167**	0.160**	0.162**	0.218***	0.248***	0.202**	0.216***
	(0.067)	(0.069)	(0.066)	(0.069)	(0.068)	(0.079)	(0.081)	(0.081)	(0.079)
Observations	418	418	418	418	418	418	417	417	418
R^2	0.930	0.930	0.930	0.930	0.930	0.932	0.937	0.933	0.932
Adjusted R^2	0.929	0.929	0.929	0.929	0.929	0.930	0.934	0.931	0.930
Residual Std. Error	0.201 (df = 413)	0.201 (df = 412)	0.201 (df = 412)	0.201 (df = 412)	0.201 (df = 412)	0.200 (df = 405)	0.193 (df = 400)	0.199 (df = 403)	0.200 (df = 404)
F Statistic	1,367.493*** (df = 4; 413)	1,092.883*** (df = 5; 412)	1,092.046*** (df = 5; 412)	1,091.858*** (df = 5; 412)	1,094.656*** (df = 5; 412)	461.162*** (df = 12; 405)	370.635*** (df = 16; 400)	429.907*** (df = 13; 403)	424.794*** (df = 13; 404)

Note: *p<0.1; **p<0.05; ***p<0.01

that the news is informed to household 16-17 months before it realizes, roughly 5 quarters to 6 quarters. Table 8 provides more evidence to the informative power of news by using different lead term of u_t^E in second regression and the column 6 to column 9 demonstrate and verify the result in ACF of u_t^E.

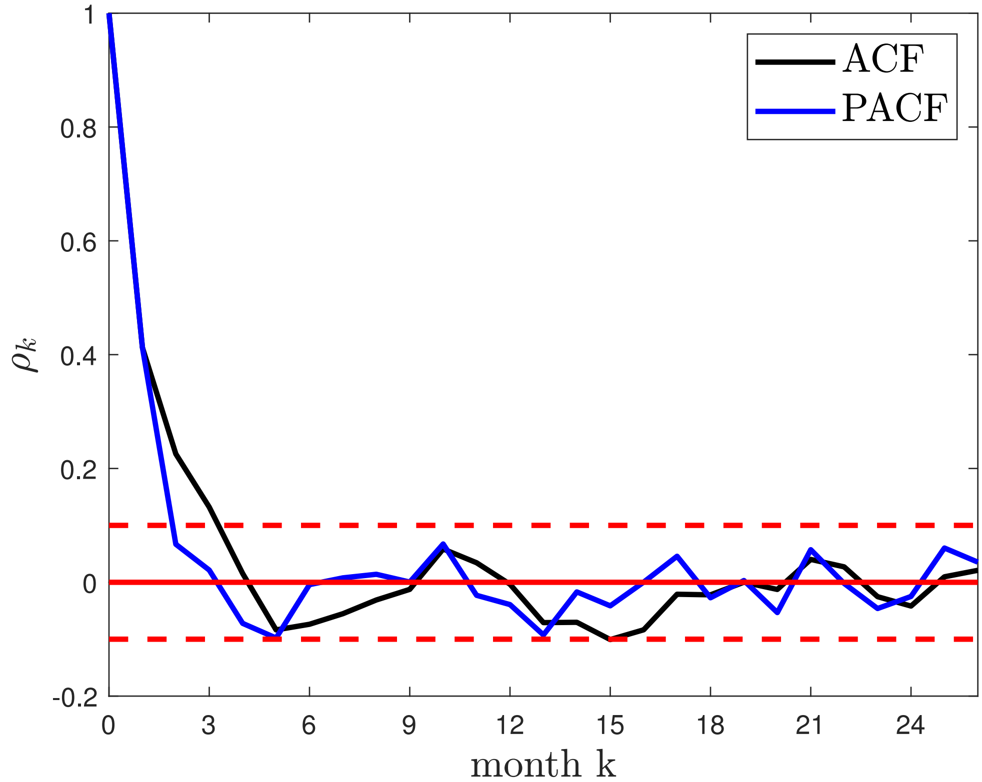


Figure 25: ACF and PACF of u_t^{EHIM6}

D Micro Foundation to Identification and Tests

In this section I provide some micro foundation related to fake-news identification in section 2.2 and some tests to my identification as proof to the reliability. I first provide several different setting about news and fake news in the literature. Then I describe the standard rbc model that I used to provide some numerical examples and micro foundation to the identification in main page.

D.1 Literature in modeling the news and fake news

D.1.1 Perfect News

This type of "fake news" is the setting following Christiano et al. (2008), Schmitt-Grohé and Uribe (2012), Barsky et al. (2015) and Sims (2016) in which household gets a news about a shock ν_τ realized at time τ which is true for sure. However after the household reaches at time τ

Table 8: Lead of u_t^E and the Informative power of news shock

	Dependent variable:							
	HIM							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
HIM_1	0.982***	0.982***	0.982***	0.984***	0.982***	0.982***	0.982***	0.983***
	(0.011)	(0.011)	(0.011)	(0.012)	(0.011)	(0.011)	(0.012)	(0.012)
Consumption_1	8.385***	8.524***	8.421***	7.573**	8.322***	8.334***	8.265***	8.307***
	(2.520)	(2.483)	(2.677)	(2.968)	(2.533)	(2.541)	(2.521)	(2.518)
EFF_rate_1	−2.278**	−2.583*	−2.271**	−2.701**	−2.262**	−2.266**	−2.236**	−2.251**
	(1.028)	(1.353)	(0.990)	(1.221)	(1.030)	(1.028)	(1.036)	(1.036)
CPI_Inflation_1		1.595						
		(3.526)						
M2_1			−0.085					
			(2.416)					
Unemployment_1				0.001				
				(0.002)				
uHIME6L14					−0.004		−0.006	−0.003
					(0.042)		(0.043)	(0.043)
uHIME6L15					−0.036	−0.037	−0.035	−0.036
					(0.045)	(0.042)	(0.045)	(0.045)
uHIME6L16	−0.094**	−0.096**	−0.094**	−0.094**	−0.079*	−0.079*	−0.076*	−0.076*
	(0.038)	(0.038)	(0.038)	(0.038)	(0.041)	(0.041)	(0.040)	(0.041)
uHIME6L17	0.093**	0.093**	0.093**	0.095**	0.097**	0.097**	0.108**	0.104**
	(0.039)	(0.039)	(0.039)	(0.039)	(0.040)	(0.040)	(0.046)	(0.046)
uHIME6L18							−0.026	−0.037
							(0.047)	(0.049)
uHIME6L19								0.029
								(0.046)
Observations	401	401	401	401	401	401	400	399
R^2	0.971	0.971	0.971	0.971	0.971	0.971	0.971	0.971
Adjusted R^2	0.971	0.971	0.971	0.971	0.971	0.971	0.971	0.971
Residual Std. Error	0.155 (df = 396)	0.155 (df = 395)	0.155 (df = 395)	0.155 (df = 395)	0.155 (df = 394)	0.155 (df = 395)	0.156 (df = 392)	0.156 (df = 390)
F Statistic	2,677.787*** (df = 5; 396)	2,227.826*** (df = 6; 395)	2,225.869*** (df = 6; 395)	2,228.874*** (df = 6; 395)	1,906.734*** (df = 7; 394)	2,230.111*** (df = 6; 395)	1,655.839*** (df = 8; 392)	1,462.320*** (df = 9; 390)

Note: *p<0.1; **p<0.05; ***p<0.01

there is an identical negative unexpected shock $-\nu$ just offsetting the effect of positive shock ν_τ. Comparing to the setting in equation 49, in which household gets a news about ν_τ via ϵ (and totally believe it) but is misled because the observation ϵ is generated by noise w, Anderson and Moore (2012) and Chahrour and Jurado (2018) shows that this type of "fake news" shock is *observational equivalent.*[26] To theoretically formulate this type of fake news shock, we can consider the shock series

$$\phi_t = \nu_{0,t} + \nu_{1,t-\tau} \tag{48}$$

where $\nu_{0,t}$ and $\nu_{1,t-\tau}$ are iid over time and follow

$$\begin{bmatrix} \nu_{0,t} \\ \nu_{1,t} \end{bmatrix} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \begin{bmatrix} \sigma^2_{\nu,0} & 0 \\ 0 & \sigma^2_{\nu,1} \end{bmatrix}\right)$$

D.1.2 Noisy News

This type of news is used by Lorenzoni (2009), Baxter et al. (2011), Barsky and Sims (2012), Blanchard et al. (2013), et al. The most intuitive one.

$$\epsilon_t = \nu_{t+\tau} + w_t \tag{49}$$

where ν is the true news shock observed by agents τ periods ahead and w is the noise or fake news shock. These two shocks are independent with each other and follow

$$\begin{bmatrix} \nu_t \\ w_t \end{bmatrix} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \begin{bmatrix} \sigma^2_{\nu} & 0 \\ 0 & \sigma^2_{w} \end{bmatrix}\right)$$

D.1.3 Fake News

It is worth to notice that when we consider the dynamic cases of equation 48 and 49, everything and every realization of $\nu_{0,t}$, $\nu_{1,t-\tau}$, $\nu_{t+\tau}$ and w_t could happen. Given $\phi_t = 1$, different combination such as $(\nu_{0,t} = 0.5, \nu_{1,t-\tau} = 0.5)$ or $(\nu_{0,t} = 1.5, \nu_{1,t-\tau} = -0.5)$ may all hold. Similarly given $\epsilon_t = 1$, $(\nu_{t+\tau} = 0.5, w_t = 0.5)$ or $(\nu_{t+\tau} = -0.5, w_t = 1.5)$ may all hold.

In this section what I am considering is the "pure shock" scenario or the impulse response to a single shock. In other words, for instance, one unit realization of noisy news $\epsilon_t = 1$ can only come from $\nu_{t+\tau} = 1$ or $w_t = 1$. It does not mean I have an implicit restriction on the shock $\nu_{t+\tau}$ and w_t that $\nu_{t+\tau} w_t = 0$. They are iid shocks. Similarly, given one unit realization of perfect news $\nu_{1,t-\tau} = 1$, it can be true news $\nu_{0,t} = 0$ or fake news $\nu_{0,t} = -1$. It does not mean I have an implicit restriction on the shock $\nu_{0,t}$ and $\nu_{1,t-\tau}$ that $\text{corr}(\nu_{0,t}, \nu_{1,t-\tau}) = -1$. They are iid shocks.

[26] They call this representation to fundamental and belief as *news representation* and the representation in equation 49 as a *noise representation.*

D.1.4 Fake News in Perfect News

To model a fake news in perfect news model, there is a realization of perfect news $\nu_{1,t-\tau} = 1$ at time $t - \tau$ and known by household, though this shock would have fundamental effect later, at time t. Then at time t there is an unexpected contemporaneous shock $\nu_{0,t} = -1$ to "neutralize" or "offset" the perfect news effect to make the fundamental stay at the beginning. The VAR identification to this type of fake news is easy. Because all the news in this model is true or perfectly foreseen by household, we just need to find a news shock first. Then at time τ there is a same shock but an opposite direction. We only need to identify the response to shock once.

Sims (2016) did this identification.

D.1.5 Fake News in Noisy News

To model a fake news in noisy news model, there is a realization of observation $\epsilon_t = 1$ at time t which can either be a signal to a fundamental shock in the future, time $t + \tau$, $\nu_{t+\tau} = 1$, or be a noisy $w_t = 1$, which does not have any fundamental effect to the economy. In noisy news model given an observation $\epsilon_t = 1$ household will response to their perception to the true news $\nu_{t+\tau|t}$ which is smaller than ϵ_t under rational expectation and we can write it as $\nu_{t+\tau|t} = \alpha\epsilon_t$ where $\alpha < 1$. There exist learning and belief updating in this type of modeling and theoretically their is no point when household "realizes" that the news is fake. For fake news their perception converge to zero faster than that in true news. In other words, $\lim_{i\to\infty} \nu_{t+\tau|t+\tau+i} = 0$ will be faster for fake news than true news.

To model the "awareness" of fake news, we now consider a scenario in which no more information about shock $\nu_{t+\tau}$ is delivered to household throughout time $t + 1$ and time $t + \tau - 1$. Therefore the belief to $\nu_{t+\tau}$ of household will not be updated and $\nu_{t+\tau|t} = \nu_{t+\tau|t+1} = \cdots = \nu_{t+\tau|t+\tau-1}$. However when the news realize at time $t + \tau$, household gets a further signal, or information to it. In other words household can also observe $\epsilon^{\tau}_{t+\tau} = \nu_{t+\tau} + w^{\tau}_{t+\tau}$ and this new observation $\epsilon^{\tau}_{t+\tau}$ will update or twist the household's belief to shock $\nu_{t+\tau}$. Therefore their exists a value of $w^{\tau}_{t+\tau}$ which can "correct" the belief of household. Thus, $\nu_{t+\tau|t+\tau} = 0$ and household at time $t + \tau$ realize that the news $\nu_{t+\tau}$ which they known at time t is a fake news.

D.2 Numerical test to identification: A simple RBC model

D.2.1 Equations used to solve the state space model

In this subsection I describe a simple 8 variables RBC model to test my identification strategy and show that it can successfully recover the impulse response to news and fake news shocks. I will first introduce the DSGE model briefly and then show that my identification process works well by comparing the identified empirical impulse response with the theoretical one.

The 8 variables RBC model is a standard one in which household provides labor and earns labor income. Given the labor income and capital return, which is paid by firms with real rental rate as they rent capital to produce goods, the household decides their investment and consumption level. In additional to these endogenous variation there is an exogenous government spending shock following equation 50 and other 4 standard shocks such as TFP shock and preference shock.

Household

$$c_t^{-\sigma} = \beta R_{t+1} c_{t+1}^{-\sigma}$$

$$h_t^{\varphi} = w c_t^{-\sigma}$$

Firm

$$R_t = \alpha \frac{y_t}{k_{t-1}} + \delta - 1$$

$$w_t = (1-\alpha)\frac{y_t}{h_t}$$

$$y_t = A_t k_{t-t}^{\alpha} h_t^{1-\alpha}$$

Market Cleaning

$$y_t = c_t + I_t + \log(G_t)$$

$$I_t = k_t - (1-\delta)k_{t-1}$$

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^{\tau} \tag{50}$$

The household cannot know the value of G_t and w_t but a signal to then

$$\widetilde{g}_t = g_t + \nu_t^{\tau}$$

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

Household at time $t-\tau$ will have a perception of $w_{t-\tau}$ given the observation $\widetilde{w}_{t-\tau}$ and I denote it as $w_{t-\tau|t-\tau} = \theta \widetilde{w}_{t-\tau}$

Denote $\widetilde{w}_t^i$ as an observation to shock w_{t-i}. For example, a news shock w_t will have effect on G at $t+\tau$. At time $t+1$ household gets a new observation related to w_t, $\widetilde{w}_{t+1}^1$, in addition to the old observation of w_t at time t $\widetilde{w}_t$. I further assume

$$\widetilde{w}_{t-\tau+1}^1 = \widetilde{w}_{t-\tau+2}^2 = \cdots = \widetilde{w}_{t-1}^{\tau-1} = 0$$

holds. Therefore

$$w_{t-\tau|t-\tau} = w_{t-\tau|t-\tau+1} = w_{t-\tau|t-\tau+2} = \cdots = w_{t-\tau|t-1}$$

D.2.2 Quantitative Exercise

D.2.2.1 Same perception: $g^{\nu}_{t|t} = g^{w}_{t|t} = g^{\nu+\nu^{\tau}}_{t|t}$

Notation: Throughout exercise 1 to 3, imperfect information holds.

1) Only noisy shock $\nu_{t-\tau}$;

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν^{τ}_t;

3) A news shock $w_{t-\tau}$.

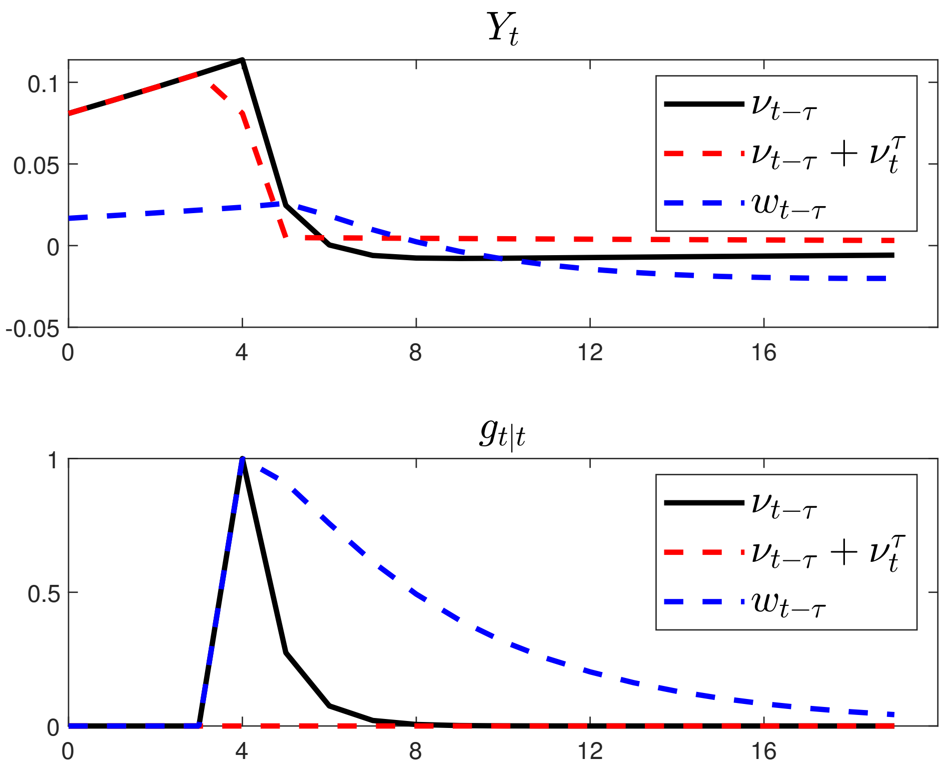


Figure 26: Same Perception $g^{w}_{t|t} = g^{w^{\tau}}_{t|t} = g^{w+\nu^{\tau}}_{t|t}$

D.2.2.2 Same observation at time $t-\tau$: $\widetilde{w}_{t-\tau}$

Notation: Throughout exercise 1 to 2, imperfect information holds. In exercise 3, it is the type of perfect news.

1) Only noisy shock $\nu_{t-\tau}$;

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν^{τ}_t;

3) A perfect news shock $w_{t-\tau}$.

D.2.2.3 Same observation at time $t-\tau$: $\widetilde{w}_{t-\tau}$

Notation: Throughout exercise 1 to 3, imperfect information holds.

1) Only noisy shock $\nu_{t-\tau}$;

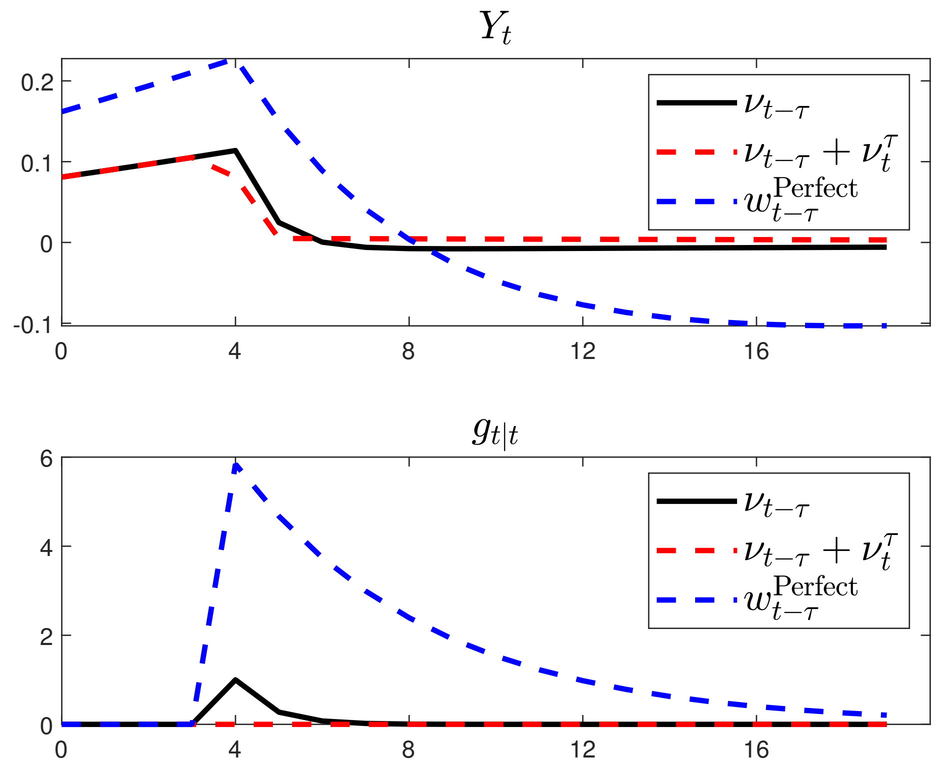


Figure 27: Same observation $\widetilde{w}_{t-\tau}$

2) Fake news shock. A noisy shock on $w_{t-\tau}$ at time $t-\tau$, $\nu_{t-\tau}$, as well as a negative noisy shock on g_t at time t, ν_t^τ;

3) A news shock $w_{t-\tau}$.

D.3 Two examples of "offset" identification (g_t is exogenous w.r.t w_t)

Denote the fundamental impact (i.e. housing demand variation, TFP) g_t follows an AR1 process

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau \tag{51}$$

where $w_{t-\tau}$ is the news shock known by household at time $t-\tau$ yet has real effect at time t, w_t^τ is the contemporaneous shock. Because of the imperfect information, household cannot know the exact value of news shock $w_{t-\tau}$ but an observation to it with noisy shock

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

where $\widetilde{w}_{t-\tau}$ is the observation to $w_{t-\tau}$ but may be contaminated by a noisy $\nu_{t-\tau}$ which does not have any real effect to economy. There are two scenarios that household comprehend whether the jump in observation $\widetilde{w}_{t-\tau}$ comes from news $w_{t-\tau}$ or noisy $\nu_{t-\tau}$ which I call 1). suddenly realization and 2). realization by learning.

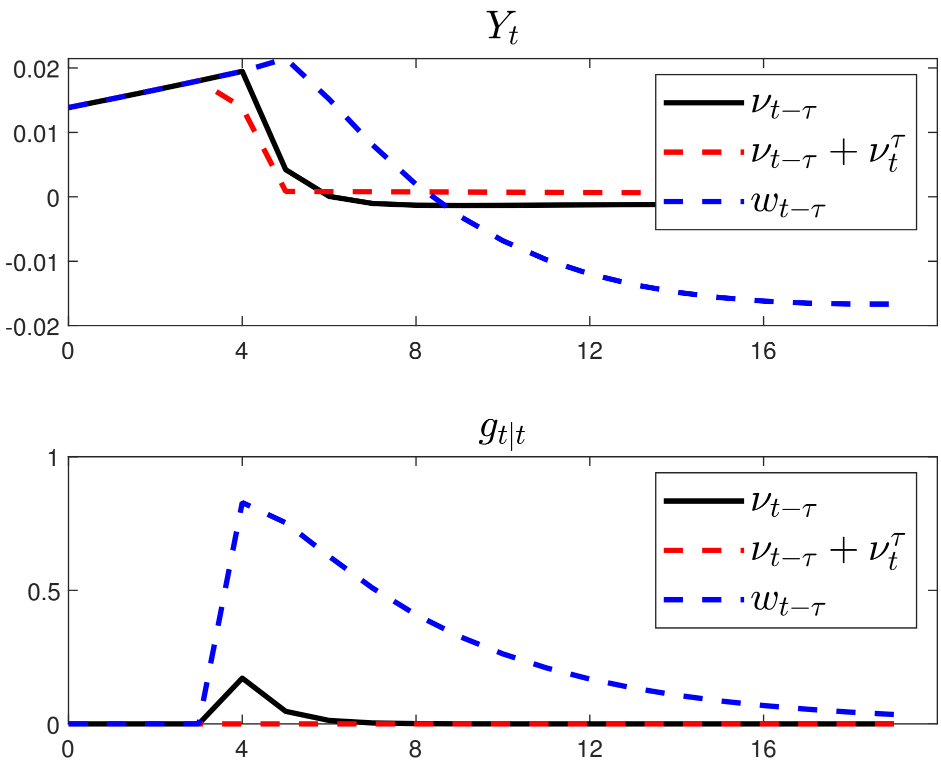


Figure 28: Same observation $\widetilde{w}_{t-\tau}$

D.3.1 The fundamental impact g_t is observable.

When the fundamental impact g_t is observable, whether the news $\widetilde{w}_{t-\tau}$ is true or fake is informed to household via g_t at time t without any delay. Since it is the impact g_t that affects the economy through which the shock $w_{t-\tau}$ and w_t^τ affect the economy, the household only care about the impact value g_t is $w_{t-\tau}$ (true news) or 0 (fake news). Therefore $y_{i-\tau-1}^\tau$ in equation 7 works as a contemporaneous shock w_t^τ offsets the true shock realized at t, $w_{t-\tau}$ and generates $g_t = 0$ which is what the fake news $\nu_{t-\tau}$ would do. This scenario is a standard one in literature and Christiano et al. (2008), Schmitt-Grohé and Uribe (2012), Barsky et al. (2015) and Sims (2016) did the similar process to generate fake news.

D.3.2 The fundamental impact g_t is unobservable.

When the fundamental impact g_t is unobservable, there is no other signal that household can use to infer whether $\widetilde{w}_{t-\tau}$ comes from $w_{t-\tau}$ or $\nu_{t-\tau}$ but learn through observation gradually. In this scenario household cannot know g_t but an observation to it $\widetilde{g}_t$ following

$$\widetilde{g}_t = g_t + \nu_t^\tau$$

I can show that the perception to the fundamental impact at time t, $g_{t|t}$ follows

$$\begin{aligned} g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_7 \widetilde{g}_t \\ &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_3 g_{t-1} + \gamma_4 w_{t-\tau} + \gamma_5 \nu_t^\tau + \gamma_6 w_t^\tau \end{aligned} \tag{52}$$

where $\gamma_1 = \rho\left[1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}\right]$, $\gamma_2 = 1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$, $\gamma_3 = \gamma_7\rho$ and $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$ which is the Kalman gain. z_{11} can be solved from the positive root of quadratic equation

$$z_{11}^2 + \left(\sigma^2_{\nu^\tau} - \rho^2\sigma^2_{\nu^\tau} - \sigma^2_w - \sigma^2_{w^\tau} + \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right) z_{11} - \sigma^2_{\nu^\tau}\left(\sigma^2_w + \sigma^2_{w^\tau} - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right) = 0$$

Therefore the only difference between fake news and true news at time t is the term $\gamma_4 w_{t-\tau}$ which comes from the observation $\widetilde{g}_t$ as it truly spur a jump in g_t, though the household cannot distinguish whether this jump is caused by realized news $w_{t-\tau}$ or contemporaneous shock w_t^τ and ν_t^τ. That is the reason why these three terms share the same coefficient $\gamma_4 = \gamma_5 = \gamma_6$, and similarly $y^\tau_{i-\tau-1}$ in equation 7 works as a contemporaneous shock w_t^τ which offsets the effect of true shock $w_{t-\tau}$ at time t.

D.3.3 Proof of equation 52

Firstly I assume the law of motion of the shock g_t follows

$$g_t = \rho g_{t-1} + w_{t-\tau} + w_t^\tau$$

where $w_{t-\tau}$ is a shock realized at $t-\tau$ yet has effect on t. w_t^τ is a contemporaneous unexpected shock realized at time t.

The household cannot know the value of the value of shock underneath g_t and w_t but a signal to then

$$\widetilde{g}_t = g_t + \nu_t^\tau$$

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

Household at time $t-\tau$ will have a perception of $w_{t-\tau}$ given the observation $\widetilde{w}_{t-\tau}$ and I denote it as $w_{t-\tau|t-\tau} = \theta\widetilde{w}_{t-\tau}$

Denote $\widetilde{w}_t^i$ as an observation to shock w_{t-i}. For example, a news shock w_t will have effect on G at $t+\tau$. At time $t+1$ household gets a new observation related to w_t, $\widetilde{w}^1_{t+1}$, in addition to the old observation of w_t at time t $\widetilde{w}_t$. I further assume

$$\widetilde{w}^1_{t-\tau+1} = \widetilde{w}^2_{t-\tau+2} = \cdots = \widetilde{w}^{\tau-1}_{t-1} = 0$$

holds. Therefore

$$w_{t-\tau|t-\tau} = w_{t-\tau|t-\tau+1} = w_{t-\tau|t-\tau+2} = \cdots = w_{t-\tau|t-1}$$

Above system of equation can be written as a state equation

$$\begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix} = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ w_{t-\tau} \end{bmatrix} + \begin{bmatrix} w_t^\tau \\ w_{t-\tau+1} \end{bmatrix}$$

and observation(moment) equation

$$\begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix} + \begin{bmatrix} \nu_t^\tau \\ \nu_{t-\tau+1} \end{bmatrix}$$

For simplicity I denote $y_t = \begin{bmatrix} g_t \\ w_{t-\tau+1} \end{bmatrix}$, $\widetilde{y}_t = \begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \end{bmatrix}$, $B = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\omega_t = \begin{bmatrix} w_t^\tau \\ w_{t-\tau+1} \end{bmatrix}$ and $v_t = \begin{bmatrix} \nu_t^\tau \\ \nu_{t-\tau+1} \end{bmatrix}$.

Following Hamilton (2020) we can solve the conditional expectation of the variance of $Z = \Sigma_y(t|t)$ follows

$$B \left[Z - Z \left(Z + \Sigma_\nu \right)^{-1} Z \right] B' + \Sigma_\omega = Z \tag{53}$$

where I omit the observation matrix H as it is an identity matrix.

Since the second row of B is zero, the matrix $D = BXB'$ must follow $D = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}$. Plugging the matrix D back to equation 53 yidelds $D + \Sigma_\omega = Z$. Therefore we must have

$$Z = \begin{bmatrix} d + \sigma_{w^\tau}^2 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} = \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix}$$

By solving the equation

$$\begin{bmatrix} z_{11} - \sigma_{w^\tau}^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \begin{bmatrix} (z_{11} + \sigma_{\nu^\tau}^2)^{-1} & 0 \\ 0 & (\sigma_w^2 + \sigma_\nu^2)^{-1} \end{bmatrix} \begin{bmatrix} z_{11} & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \right\} \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix}$$

we can solve out z_{11} as the positive root of quadratic equation

$$z_{11}^2 + \left(\sigma_{\nu^\tau}^2 - \rho^2 \sigma_{\nu^\tau}^2 - \sigma_w^2 - \sigma_{w^\tau}^2 + \frac{\sigma_w^4}{\sigma_w^2 + \sigma_\nu^2} \right) z_{11} - \sigma_{\nu^\tau}^2 \left(\sigma_w^2 + \sigma_{w^\tau}^2 - \frac{\sigma_w^4}{\sigma_w^2 + \sigma_\nu^2} \right) = 0$$

Then we can solve the law of motion of perception(conditional expectation) of y_t as $y_{t|t} =$

$(I - PH) By_{t-1|t-1} + P\widetilde{y}_t$ where P is the Kalman gain following $P = ZH' (HZH' + \Sigma_v)^{-1}$.

D.4 Two examples of "offset" identification (g_t is endogenous w.r.t w_t)

Denote the fundamental impact (i.e. housing demand variation, TFP) g_t follows an AR1 process

$$g_t = \rho_g g_{t-1} + w_{t-\tau} + w_t^\tau + \alpha w_t \tag{54}$$

where $w_{t-\tau}$ is the news shock known by household at time $t - \tau$ yet has real effect at time t, w_t^τ is the contemporaneous shock. Because of the imperfect information, household cannot know the exact value of news shock $w_{t-\tau}$ but an observation to it with noisy shock

$$\widetilde{w}_{t-\tau} = w_{t-\tau} + \nu_{t-\tau}$$

where $\widetilde{w}_{t-\tau}$ is the observation to $w_{t-\tau}$ but may be contaminated by a noisy $\nu_{t-\tau}$ which does not have any real effect to economy.

This is similar to the equation 51 and I will also discuss two scenarios that household comprehend whether the jump in observation $\widetilde{w}_{t-\tau}$ comes from news $w_{t-\tau}$ or noisy $\nu_{t-\tau}$ which I call 1). suddenly realization and 2). realization by learning.

D.4.1 The fundamental impact g_t is observable.

When the fundamental impact g_t is observable, whether the news $\widetilde{w}_{t-\tau}$ is true or fake is informed to household via g_t at time t without any delay. Similar to the exogenous case, it is the g_t that affects the economy instead of w_t or $\widetilde{w}_t$ in the end. Therefore as long as g_t can be fully observed, the endogenous effect of w_t will not play any role based on imperfect information here as household at time t will not care about this endogeneity but only g_t. Therefore even we change the assumption of endogenous effect and assume that g_t response to the observation $\widetilde{w}_t$ or perception $w_{t|t}$ the result will not change as long as household perfectly knows g_t.

D.4.2 The fundamental impact g_t is unobservable.

I can show that the perception to the fundamental impact at time t, $g_{t|t}$ follows

$$\begin{aligned} g_{t|t} &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_7 \widetilde{g}_t \\ &= \gamma_1 g_{t-1|t-1} + \gamma_2 w_{t-\tau|t-\tau} + \gamma_2 \alpha w_{t|t} + \gamma_3 g_{t-1} + \gamma_4 w_{t-\tau} + \gamma_5 \nu_t^\tau + \gamma_6 w_t^\tau \end{aligned} \tag{55}$$

where $\gamma_1 = \rho \left[1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}\right]$, $\gamma_2 = 1 - \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$, $\gamma_3 = \gamma_7 \rho$ and $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \frac{z_{11}}{z_{11}+\sigma^2_{\nu^\tau}}$ which is the Kalman gain. z_{11} can be solved from the positive root of quadratic equation

$$z_{11}^2 + \left(\sigma^2_{\nu^\tau} - \rho^2\sigma^2_{\nu^\tau} - \sigma^2_{w^\tau} - (1+\alpha^2)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right]\right) z_{11} - \sigma^2_{\nu^\tau}\left(\sigma^2_{w^\tau} + (1+\alpha^2)\left[\sigma^2_w - \frac{\sigma^4_w}{\sigma^2_w + \sigma^2_\nu}\right]\right) = 0$$

D.4.3 Proof of equation 55

Similar to the proof of equation 52, above system of equation can be written as a state equation

$$\begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ w_{t-\tau} \\ w_t \end{bmatrix} + \begin{bmatrix} w_t^{\tau} \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}$$

and observation(moment) equation

$$\begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \\ \widetilde{w}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t^{\tau} \\ \nu_{t-\tau+1} \\ \nu_{t+1} \end{bmatrix}$$

For simplicity I denote $y_t = \begin{bmatrix} g_t \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}$, $\widetilde{y}_t = \begin{bmatrix} \widetilde{g}_t \\ \widetilde{w}_{t-\tau+1} \\ \widetilde{w}_{t+1} \end{bmatrix}$, $B = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\omega_t = \begin{bmatrix} w_t^{\tau} \\ w_{t-\tau+1} \\ w_{t+1} \end{bmatrix}$ and $v_t = \begin{bmatrix} \nu_t^{\tau} \\ \nu_{t-\tau+1} \\ \nu_{t+1} \end{bmatrix}$.

Following Hamilton (2020) we can solve the conditional expectation of the variance of $Z = \Sigma_y(t|t)$ follows

$$B\left[Z - Z\left(Z + \Sigma_\nu\right)^{-1} Z\right] B' + \Sigma_\omega = Z \tag{56}$$

where I omit the observation matrix H as it is an identity matrix.

Since the second row of B is zero, the matrix $D = BXB'$ must follow $D = \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Plugging the matrix D back to equation 56 yidelds $D + \Sigma_\omega = Z$. Therefore we must have

$$Z = \begin{bmatrix} d + \sigma^2_{w^\tau} & 0 & 0 \\ 0 & \sigma_w^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} = \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma_w^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix}$$

By solving the equation

$$\begin{bmatrix} z_{11} - \sigma^2_{w^\tau} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma_w^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} \right.$$

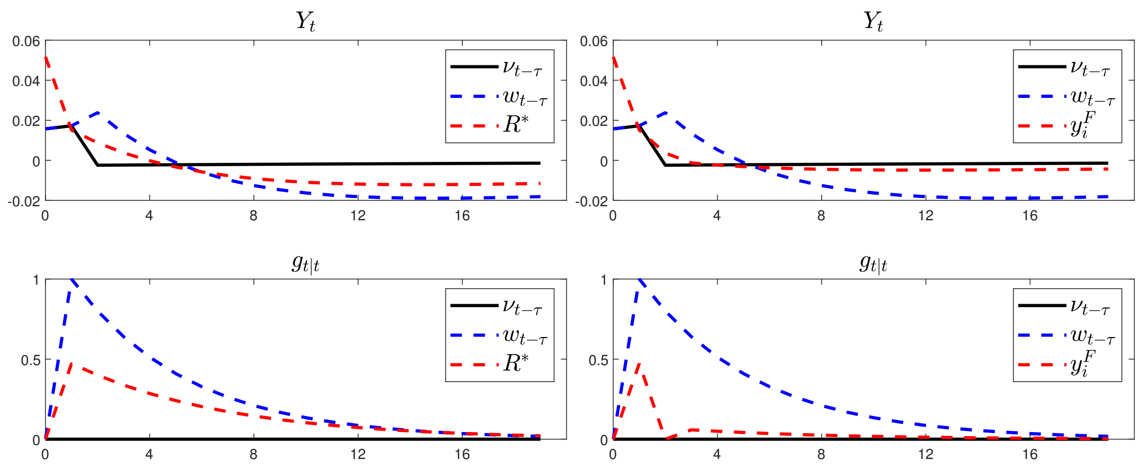
$$\left. - \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma_w^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} \begin{bmatrix} \left(z_{11} + \sigma^2_{\nu^\tau}\right)^{-1} & 0 & 0 \\ 0 & \left(\sigma_w^2 + \sigma_\nu^2\right)^{-1} & 0 \\ 0 & 0 & \left(\sigma_w^2 + \sigma_\nu^2\right)^{-1} \end{bmatrix} \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & \sigma_w^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} \right\} \begin{bmatrix} \rho & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can solve out z_{11} as the positive root of quadratic equation

$$z_{11}^2+\left(\sigma_{\nu^\tau}^2-\rho^2\sigma_{\nu^\tau}^2-\sigma_{w^\tau}^2-(1+\alpha^2)\left[\sigma_w^2-\frac{\sigma_w^4}{\sigma_w^2+\sigma_\nu^2}\right]\right)z_{11}-\sigma_{\nu^\tau}^2\left(\sigma_{w^\tau}^2+(1+\alpha^2)\left[\sigma_w^2-\frac{\sigma_w^4}{\sigma_w^2+\sigma_\nu^2}\right]\right)=0$$

D.5 Identification Test

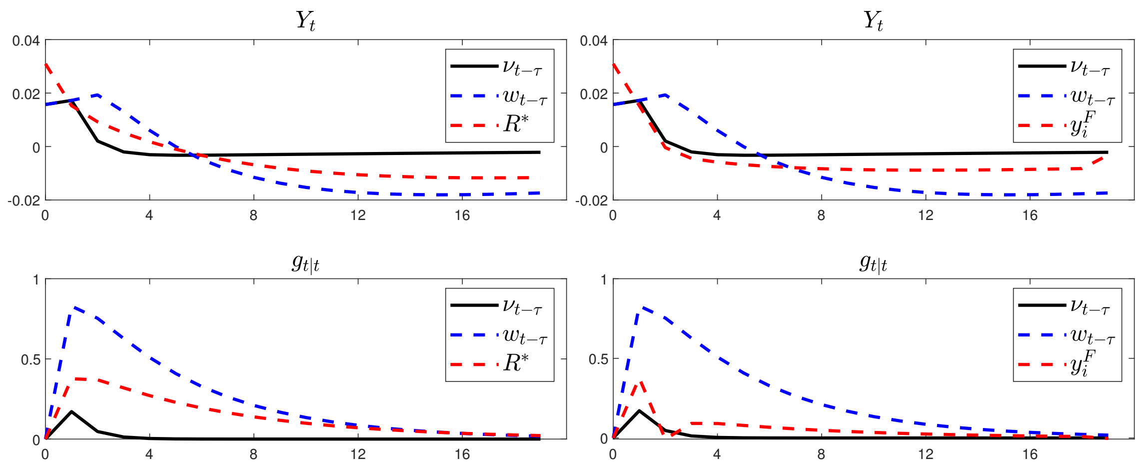
D.5.1 The fundamental impact g_t is observable.



(a) Identification to news shock R^* (b) Identification to fake news shock R_F^*

Figure 29: Identification Test to observable fundamental impact

D.5.2 The fundamental impact g_t is unobservable.



(a) Identification to news shock R^* (b) Identification to fake news shock R_F^*

Figure 30: Identification Test to unobservable fundamental impact

Figure 30 shows the result of the identification test.

E Perturbation result around the Simple Model

E.1 Proof of Proposition 2

The Lagrangian of the problem 8 could be written as

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^{t}U^{i}\left(c_{t}^{i},h_{t}^{i}\right)+\sum_{t=0}^{\infty}\lambda_{t}^{i}\left[R_{t}a_{t-1}^{i}+w_{t}\varepsilon_{t}^{i}+\left(1-\delta^{H}\right)p_{t}^{H}h_{t-1}^{i}+\pi_{t}^{i}+\pi_{t}^{H,i}-c_{t}^{i}-a_{t}^{i}-p_{t}^{H}h_{t}^{i}\right]$$
$$+\sum_{t=0}^{\infty}\mu_{t}^{i}\left(p_{t}^{H}h_{t}^{i}+a_{t}^{i}\right)$$

I omit the superscript i henceforth for convenience. Then the first order condition would be

$$U_{c_t}=\lambda_t \tag{57}$$

$$-\lambda_t+\mu_t+\beta E_t R_{t+1}\lambda_{t+1}=0 \tag{58}$$

$$U_{h_t}-\lambda_t p_t^H+\mu_t p_t^H+\beta\left(1-\delta^H\right)E_t\lambda_{t+1}p_{t+1}^H=0 \tag{59}$$

To break the expectation I can rearrange the equation 59 as

$$U_{h_t}=\left(\lambda_t-\mu_t\right)p_t^H-\left(1-\delta^H\right)\left(\lambda_t-\mu_t\right)\frac{1}{E_tR_{t+1}}E_tp_{t+1}^H+\beta\left(1-\delta^H\right)\frac{cov\left(\lambda_{t+1},R_{t+1}\right)}{E_tR_{t+1}}E_tp_{t+1}^H \tag{60}$$
$$-\beta\left(1-\delta^H\right)cov\left(\lambda_{t+1},p_{t+1}^H\right)$$

Since the interest rate here is not related to the issue we want to solve, I further assume the exogenous TFP of non-durable goods production function is constant. Together with some assumption on the production function of durable and non-durable goods[27], $R_{t+1}=R_t=\overline{R}$ and $cov\left(\lambda_{t+1},R_{t+1}\right)=0$ will hold. Combining this assumption I log linearize equation 60 to get

$$\widetilde{U}_{h_t}=\frac{\left(\lambda-\mu\right)\left[p^H-\left(1-\delta^H\right)p^H\frac{1}{R}\right]}{U_h}\left\{\frac{\lambda}{\lambda-\mu}\widetilde{\lambda}_t-\frac{\mu}{\lambda-\mu}\widetilde{\mu}_t+\frac{p^H}{p^H-\left(1-\delta^H\right)p^H\frac{1}{R}}\widetilde{p}_t^H-\right. \tag{61}$$
$$\left.\frac{\left(1-\delta^H\right)p^H\frac{1}{R}}{p^H-\left(1-\delta^H\right)p^H\frac{1}{R}}\widetilde{p}_{t+1}^H\right\}-\frac{\beta\left(1-\delta^H\right)\overline{cov}}{U_h}\widetilde{cov}_t$$

where $\widetilde{cov}_t$ is the percentage derivation from steady state of $cov\left(\lambda_t,p_t^H\right)$

[27] The related assumptions are described at appendix G.1.1.

Then following Etheridge (2019) I expand U_{c_t} around its steady-state value U_c to get

$$U_{c_t} \approx U_c + U_{cc} c \widetilde{c}_t + U_{ch} h \widetilde{h}_t$$

I rearrange above equation to get

$$\frac{U_{c_t} - U_c}{U_c} = d \ln u_{c_t} = \widetilde{U}_{c_t} = \frac{U_{cc} c}{U_c} \widetilde{c}_t + \frac{U_{ch} h}{U_c} \widetilde{h}_t \tag{62}$$

Similarly expanding U_{h_t}around its steady-state value U_h gives

$$\frac{U_{h_t} - U_h}{U_h} = d \ln u_{h_t} = \widetilde{U}_{h_t} = \frac{U_{hc} c}{U_h} \widetilde{c}_t + \frac{U_{hh} h}{U_h} \widetilde{h}_t \tag{63}$$

Perturbing around its steady state for equation 57 returns

$$\widetilde{U}_{c_t} = \widetilde{\lambda}_t \tag{64}$$

Combining equation 61, 62, 63 and 64 I can solve out

$$\widetilde{c}_t = \left(\frac{\lambda}{\lambda - \mu} \eta_{c,p^H} - \eta_{c,p^c} \right) \widetilde{\lambda}_t - \frac{\mu}{\lambda - \mu} \eta_{c,p^H} \widetilde{\mu}_t + \eta_{c,p^H} \left[\frac{1}{1 - (1 - \delta^H) \frac{1}{R}} \widetilde{p}_t^H - \frac{(1 - \delta^H) \frac{1}{R}}{1 - (1 - \delta^H) \frac{1}{R}} \widetilde{p}_{t+1}^H \right] - \frac{U_{ch}}{U_{ch}^2 - U_{cc} U_{hh}} \frac{\beta (1 - \delta^H) \overline{cov}}{c} \widetilde{cov}_t$$

Then plugging back equation 57 gives

$$\widetilde{c}_t = \frac{\frac{\lambda}{\lambda - \mu} \eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \frac{\lambda}{\lambda - \mu} \eta_{h,p^h}} \widetilde{h}_t - \frac{\mu}{\lambda - \mu} \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda - \mu} \eta_{h,p^h}} \widetilde{\mu}_t + \frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda - \mu} \eta_{h,p^h}} \left[\frac{1}{1 - (1 - \delta^H) \frac{1}{R}} \widetilde{p}_t^H - \frac{(1 - \delta^H) \frac{1}{R}}{1 - (1 - \delta^H) \frac{1}{R}} \widetilde{p}_{t+1}^H \right] - \frac{\eta_c}{\eta_{h,p^c} - \frac{\lambda}{\lambda - \mu} \eta_{h,p^h}} \frac{\beta (1 - \delta^H) \overline{cov}}{h} \widetilde{cov}_t$$

where η_{h,p^c}, η_{h,p^h}, η_{c,p^H}, η_{c,p^c}, η_{ch} and η_c are

$$\eta_{c,p^H} = \frac{u_{ch} u_h}{u_{ch}^2 - u_{cc} u_{hh}} \frac{1}{c}$$

$$\eta_{c,p^c} = \frac{u_{hh} u_c}{u_{ch}^2 - u_{cc} u_{hh}} \frac{1}{c}$$

$$\eta_{h,p^c} = \frac{u_{ch} u_c}{u_{ch}^2 - u_{cc} u_{hh}} \frac{1}{h}$$

$$\eta_{h,p^h} = \frac{u_{cc} u_h}{u_{ch}^2 - u_{cc} u_{hh}} \frac{1}{h}$$

$$\eta_{ch} = \frac{u_c u_h}{u_{ch}^2 - u_{cc}u_{hh}} \frac{1}{ch}$$

$$\eta_c = \frac{u_c}{u_{ch}^2 - u_{cc}u_{hh}} \frac{1}{c}$$

E.2 Derivation of the Definition of Intratemporal Elasticity of substitution 16

Firstly, following the standard procedure I first define the optimization problem

$$\max_{c,h} u(c,h)$$

$$\text{s.t. } c + p^h h = y$$

where c is the consumption,p^h is the relative price of housing services and y is the exogenous income. The interior solution implies

$$p^h = \frac{u_h}{u_c}$$

which is used to define the intratemporal elasticity of substitution

$$\begin{aligned} \text{ES} &= -\frac{d\ln\left(\frac{c}{h}\right)}{d\ln\left(p^h\right)} \\ &= -\frac{d\ln\left(\frac{c}{h}\right)}{d\ln\left(\frac{U_c}{U_h}\right)} \end{aligned}$$

E.3 Proof of Proposition 3

I first use the same production function 20 and 21 which I defined at section 4. Since the sample model in section 3 is frictionless in adjusting housing and physical capital, the goods market clearing condition should be

$$\begin{aligned} Y &= Y_H + Y_N \\ &= C + I_N + I_H \end{aligned}$$

where $Y_H = I_H$ and $Y_N = C + I_N$

Combining equation 74 and the market clearing condition of capital I can get

$$\alpha Y_{N,t} + \nu P_t^H Y_{H,t} = (r_t + \delta) K_{t-1}$$

Taking differential on both side of above equation around their steady state will yield

$$\alpha dY_{N,t} + \nu Y_H dP_t^H + \nu P^H dY_{H,t} = 0$$

because the total capital K_{t-1} is predetermined and r_t is fixed by assumption. Further because the amount of total housing service at time $t-1$, H_{t-1} is predetermined, above equation can be rewritten to

$$\alpha dY_{N,t} + \nu Y_H dP_t^H + \nu P^H dH_t = 0$$

Plugging this back to goods market clearing condition will return the general equilibrium condition of crowding-out effect

$$-I_N \widetilde{I}_{N,t} = C\widetilde{C}_t + \frac{\nu}{\alpha} Y_H P^H \widetilde{P}_t^H + \frac{\nu}{\alpha} P^H H \widetilde{H}_t$$

Finally the equation 15 can be obtained by plugging equation 11 into above equation.

E.4 Proof of Corollary 1

If the household utility function follows the standard CRRA form

$$u_t = \frac{\left(\phi c_t^{\gamma} + (1-\phi) s_t^{1-\gamma}\right)^{\frac{1-\sigma}{1-\gamma}}}{1-\sigma}$$

Therefore the intratempral elasticity of substitution will be $\text{ES} = \frac{1}{\gamma}$ and the intertemporal elasticity of substitution will be $\text{EIS} = \frac{1}{\sigma}$ and $u_{ch} = \phi(1-\phi)(\gamma-\sigma)c^{\gamma-\sigma-1}h^{-\gamma}\left[\phi + (1-\phi)(\frac{h}{c})^{1-\gamma}\right]^{\frac{\gamma-\sigma}{1-\gamma}}$. Then based on the definition of relative force of substitution effect Φ_H and wealth effect Φ_{p^H} the prove process is straightforward.

E.5 Proof of Corollary 2

Iterating equation 59 forward with expectation at t on both side, I can eliminate the intra-price term until time $T+1$ with the chain rule of expectation

$$\begin{aligned} U_{h_t} + (\mu_t - \lambda_t) p_t^H + \beta\left(1-\delta^H\right) E_t \lambda_{t+1} p_{t+1}^H &= 0 \\ U_{h_{t+1}} + (\mu_{t+1} - \lambda_{t+1}) p_{t+1}^H + \beta\left(1-\delta^H\right) E_{t+1} \lambda_{t+2} p_{t+2}^H &= 0 \\ U_{h_{t+2}} + (\mu_{t+2} - \lambda_{t+2}) p_{t+2}^H + \beta\left(1-\delta^H\right) E_{t+2} \lambda_{t+3} p_{t+3}^H &= 0 \\ &\vdots \\ U_{h_{t+T}} + (\mu_{t+T} - \lambda_{t+T}) p_{t+T}^H + \beta\left(1-\delta^H\right) E_{t+T} \lambda_{t+T+1} p_{t+T+1}^H &= 0 \end{aligned} \tag{65}$$

Multiple $\frac{\beta(1-\delta^H)\lambda_{t+i}}{\lambda_{t+i}-\mu_{t+i}}$ on both side of above equation will yield (here I only take equation 65 as an example)

$$\frac{\beta\left(1-\delta^H\right)\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}} U_{h_{t+1}} - \beta\left(1-\delta^H\right)\lambda_{t+1} p_{t+1}^H + \beta\left(1-\delta^H\right)\frac{\beta\left(1-\delta^H\right)\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}} E_{t+1}\lambda_{t+2} p_{t+2}^H = 0$$

The last term can be rearranged to $\left[\beta\left(1-\delta^{H}\right)\right]^{2} E_{t+1} \frac{\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}} \lambda_{t+2} p_{t+2}^{H}$ because the term $\frac{\lambda_{t+1}}{\lambda_{t+1}-\mu_{t+1}}$ only contains the term at time $t+1$ which is known at time $t+1$. Then take expectation with the information at time t on both side of this equation to aggregate as

$$U_{h_t}+\mathbb{E}_t \sum_{i=1}^{T}\left[\beta\left(1-\delta^{H}\right)\right]^{i}\left[\prod_{s=1}^{i} \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}}\right] U_{h_{t+i}}+\mathbb{E}_t\left[\beta\left(1-\delta^{H}\right)\right]^{T} \prod_{s=1}^{T} \frac{\lambda_{t+s}}{\lambda_{t+s}-\mu_{t+s}} \lambda_{t+T+1} p_{t+T+1}^{H}=0$$

Equation 17 can be derived by take total differential on both side to above equation.

E.6 Proof of Proposition 4 and 5

The proposition 4 is a straight result of Lemma 6, 9 and 10. Similarly proposition 5 is a straight result of Lemma 14, 16 and 17.

Lemma 1. *When the utility function follows Cobb-Douglas formula 91, the monotonicity of parameter Φ_H, Φ_μ and Φ_{p^H} is equivalent to $\widetilde{\Phi}_H=\frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$, $\widetilde{\Phi}_\mu=\frac{\mu}{\lambda-\mu}\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ and $\Phi_{p^H}=\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ where $\widetilde{\eta}_{c,p^H}=\phi(1-\phi)^2(1-\sigma)$, $\widetilde{\eta}_{c,p^c}=\phi(1-\phi)\left[(1-\phi)(1-\sigma)-1\right]$, $\widetilde{\eta}_{h,p^c}=\phi^2(1-\phi)(1-\sigma)$, $\widetilde{\eta}_{h,p^H}=\phi(1-\phi)\left[\phi(1-\sigma)-1\right]$ and $\widetilde{\eta}_{ch}=\phi(1-\phi)$.*

Proof. Because the proposition 2 and equation 11 is derived around aggregate consumption and residential asset, by plugging the marginal utility function into equation 12, 13 and 14 and rearranging the algebraic structure, we can solve above equations. □

Lemma 2. *If $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ, as long as $\frac{\lambda}{\lambda-\mu}\geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$ hold.*

Proof. Simplify the formula of Φ_H to $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$. If $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)}{\partial\sigma}\left(\eta_{h,p^c}-\eta_{h,p^h}\right)<\frac{\partial\left(\eta_{h,p^c}-\eta_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)$ holds. Further it is easy to check that as long as $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)}{\partial\sigma}\left(\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}\right)<\frac{\partial\left(\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}\right)$ holds, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ. Because of Lemma 1 we only need to check $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)<\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)$. Meanwhile $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}+\frac{\lambda}{\lambda-\mu}\frac{\partial\widetilde{\eta}_{c,p^H}}{\partial\sigma}-\frac{\partial\widetilde{\eta}_{c,p^c}}{\partial\sigma}=\widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}-\frac{\lambda}{\lambda-\mu}\phi(1-\phi)^2+\phi(1-\phi)^2$ holds. Therefore as long as $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$, $\frac{\lambda}{\lambda-\mu}>1$, $\widetilde{\eta}_{c,p^H}<0$ and $\widetilde{\eta}_{h,p^h}<0$, we will have $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}<0$ and the inequality $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)<\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)$ will hold.

Additionally, it is easy to yield $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}=\frac{\partial\widetilde{\eta}_{h,p^c}}{\partial\sigma}-\frac{\partial\widetilde{\eta}_{h,p^h}}{\partial\sigma}\frac{\lambda}{\lambda-\mu}-\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h}=-\phi^2(1-\phi)+\frac{\lambda}{\lambda-\mu}\phi^2(1-\phi)-\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h}>\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}=-\phi^2(1-\phi)+\phi^2(1-\phi)$ as $\frac{\lambda}{\lambda-\mu}>1$, $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}>0$

and $\widetilde{\eta}_{h,p^h} < 0$. Therefore by rescaling the inequality $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right) > \frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)$ will hold and $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ. □

Lemma 3. *If $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ hold, $\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H}-\eta_{c,p^c}}{\eta_{h,p^c}-\eta_{h,p^h}}$ will be monotonic decreasing in σ.*

Proof. Based on Lemma 1, it is equivalent to show that $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma}\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right) < \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)$. Because $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} = \frac{\partial(\phi(1-\phi))}{\partial\sigma} = 0$, we only need to prove $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma} < 0$. It is easy to verify that $\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H}-\widetilde{\eta}_{c,p^c}\right)}{\partial\sigma} = \widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} + \frac{\lambda}{\lambda-\mu}\frac{\partial\widetilde{\eta}_{c,p^H}}{\partial\sigma} - \frac{\partial\widetilde{\eta}_{c,p^c}}{\partial\sigma} = \widetilde{\eta}_{c,p^H}\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} - \frac{\lambda}{\lambda-\mu}\phi(1-\phi)^2 + \phi(1-\phi)^2 < 0$ when $\widetilde{\eta}_{c,p^H} < 0$, $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ and $\frac{\lambda}{\lambda-\mu} > 0$. □

Lemma 4. *The stationary capital over effective labor ratio will increase as σ increases in Aiyagari-Bewley-Huggett model 8 when the housing supply is fixed and initial housing distribution over dynamic path is exogenous.*

Proof. The problem 8 can be write as the instantaneous payoff function

$$\max\sum_{t=0}^{\infty}\beta^t\nu\left(c_t,a_t\right) \tag{66}$$

where $\nu = \frac{\left(c_t^{\phi}h_t^{*1-\phi}\right)^{1-\sigma}}{1-\sigma}$ and $h^* = \max\left(\frac{1-\phi}{\phi}\left[p^H-(1-\delta^H)\frac{p^H}{R}\right]^{-1}c_t, \frac{-a_t}{\gamma p^H}\right)$ and the constraint correspondence

$$\Gamma\left(a_{t-1},c_t,i_{s,t},\varepsilon_t\right) = \left\{\left(a_t,c_{t+1},i_{s,t},\varepsilon_t\right) \in \left[-\frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t,\overline{a}\right]\times[0,\overline{c}]\times\left[-\underline{i}_s,\overline{i}_s\right]:\right.$$
$$\left. a_t \leq R(Q)a_{t-1}+w(Q)\varepsilon_t-p^Hi_{s,t}+T-c_t\right\} \tag{67}$$

Because the aggregate housing supply is fixed, the problem is partial on remain sectors and take the housing price as an exogenous parameter (and the general equilibrium will in the end be pinned down through find the price that match the fixed housing supply with the housing demand $\int h^*g(h^*)di$). Then the real rental rate $R(Q)$ and real wage $w(Q)$ will be a function of real effective capital over labor ratio $Q = \frac{K}{AL}$.

Then following the theorem 5 and proposition 1 in Acemoglu and Jensen (2015), σ is a positive shock and Q is monotonic increasing in σ. □

Lemma 5. *$\frac{\lambda}{\lambda-\mu} \geq 1$, $\frac{\partial\frac{\mu}{\lambda}}{\partial\sigma} > 0$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ holds in Aiyagari-Bewley-Huggett model 8 when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}}L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}L$ holds.*

Proof. $\frac{\lambda}{\lambda-\mu} = \frac{1}{1-\frac{\mu}{\lambda}} > 1$ is obvious as λ is the marginal utility which is a positive number in 57 and μ is the Khun-Tucker multiplier which is also positive. Following Lemma 4 we know that when σ increases, Q will also increase. Because of the market clearing condition $AK^{\alpha}L^{1-\alpha} = C + \delta K$ we can solve $\frac{\partial C}{\partial \sigma} = \frac{\partial\left(AK^{\alpha}L^{1-\alpha}-\delta K\right)}{\partial K}\frac{\partial K}{\partial \sigma} = \left(\alpha A(\frac{K}{L})^{\alpha-1}-\delta\right)\frac{\partial K}{\partial \sigma} > 0$. Therefore the marginal utility λ is a monotonic decreasing function of σ.

Additionally, by integrating and combining equation 78 and 81 across household we can get the relationship between aggregate Khun-Tucker multiplier and marginal utility $\mu = (\beta R - 1)\lambda$. Therefore as long as $\beta R < 1$ holds, the Khun-Tucker multiplier will have the opposite monotonicity as λ and it is guaranteed by $K < \left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$. Hence, we can yield $\frac{\partial \frac{\mu}{\lambda}}{\partial \sigma} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \sigma} > 0$. ∎

Lemma 6. *The substitution effect Φ_H will decrease as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. Lemma 6 is a direct inference from Lemma 2, 3, 4 and 5. ∎

Lemma 7. *If $\frac{\eta_{ch}}{\eta_{h,p^c}-\eta_{h,p^h}}$ is monotonic decreasing in σ, $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be also monotonic decreasing in σ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ hold.*

Proof. Similar to Lemma 2, because of Lemma 1, given $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma}\left(\widetilde{\eta}_{h,p^c} - \widetilde{\eta}_{h,p^h}\right) < \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$, we need to check $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) < \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$. Since $\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma} = 0$, we only need to check $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch} > \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$ which is true because $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} > \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}$ (shown in Lemma 2) and $\widetilde{\eta}_{ch} > 0$. ∎

Lemma 8. $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ *will be also monotonic decreasing in σ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$ hold.*

Proof. Following Lemma 1, we can get the monotonicity of $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ by checking

$$\frac{\partial \widetilde{\eta}_{ch}}{\partial \sigma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) = 0 < \frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma}\widetilde{\eta}_{ch}$$

Because $\widetilde{\eta}_{ch} > 0$, we need $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} > 0$ to let $\frac{\eta_{ch}}{\eta_{h,p^c}-\frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ monotonic decreasing in σ. It is straightforward as $\frac{\partial\left(\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial\sigma} = -\phi^2(1-\phi) - \frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma}\widetilde{\eta}_{h,p^h} + \frac{\lambda}{\lambda-\mu}\phi^2(1-\phi) > 0$ because of $\widetilde{\eta}_{h,p^h} < 0$, $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial\frac{\lambda}{\lambda-\mu}}{\partial\sigma} > 0$. ∎

Lemma 9. *The wealth effect Φ_{p^H} will decrease as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds.*

Proof. Lemma 9 is a direct inference from Lemma 5 and 8. □

Lemma 10. *The credit effect Φ_μ will increase as relative intratemporal elasticity of substitution higher, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous ;* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds and the aggregate Khun-Tucker multiplier is not too large.*

Proof. Based on lemma 1 we can show that $\frac{\partial \Phi_{p^H}}{\partial \sigma} \cong \frac{\partial\left(\frac{\mu}{\lambda-\mu}\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma} = \frac{\mu}{\lambda-\mu}\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma} + \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma}$. Further because $\frac{\lambda}{\lambda-\mu} > 1$, which comes from Lemma 5 and 8, the inequality $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}} > \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\widetilde{\eta}_{h,p^h}} = 1$ holds. Meanwhile since $\frac{\mu}{\lambda-\mu} = \frac{1}{\frac{\lambda}{\mu}-1}$ and $\frac{\partial\frac{\mu}{\lambda}}{\partial\sigma} > 0$ hold, $\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma} > 0$ is obvious.

As $\frac{\lambda}{\lambda-\mu} > 1$ and $\lambda > 0$, we must have $\frac{\mu}{\lambda-\mu} > 0$. Combining Lemma 8, we can yield the conclusion that $\frac{\partial\Phi_{p^H}}{\partial\sigma} > 0$ as long as μ is not too large to induce $\left|\frac{\mu}{\lambda-\mu}\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial\sigma}\right| > \left|\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c}-\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\frac{\partial\frac{\mu}{\lambda-\mu}}{\partial\sigma}\right|$. □

Lemma 11. *The stationary capital over effective labor ratio will increase as collateral constraint γ increases in Aiyagari-Bewley-Huggett model 8 when the housing supply is fixed and initial housing distribution over dynamic path is exogenous.*

Proof. Similar to the proof process of Lemma 4, we can reconstruct the how problem to payoff function 66 and constraint 67. Then because the collateral constraint is endogenous, we first need to explore the direction of $\frac{\partial\frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t}{\partial\gamma}$ which I will show by induction below.

If $\frac{\partial c_t}{\partial\gamma} \geq -\frac{c_t}{\gamma}$, then $\frac{\partial\frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t}{\partial\gamma} = \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t + \frac{(1-\phi)p^H\gamma}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\frac{\partial c_t}{\partial\gamma} \geq 0$ will hold with a slacker constraint. Further we can show that $\frac{\partial h^*}{\partial\gamma} \geq \frac{(1-\phi)}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\frac{\partial c_t}{\partial\gamma}$. By taking derivative with respect to γ on both side of the budge constraint in 67 we know that $\frac{\partial a_t}{\partial\gamma} \leq -\left[1+\frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\right]\frac{\partial c_t}{\partial\gamma} \leq \left[1+\frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\right]\frac{c_t}{\gamma} < \frac{(1-\phi)p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t + \frac{(1-\phi)p^H\gamma}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}\frac{\partial c_t}{\partial\gamma}$. However this means the decreasing speed of a_t is larger than the decreasing speed of collateral constraint, which violates the meaning of collateral constraint. Therefore $\frac{\partial c_t}{\partial\gamma} < -\frac{c_t}{\gamma}$ will hold and we can yield $\frac{\partial\frac{(1-\phi)\gamma p^H}{\phi\left(p^H-(1-\delta^H)\frac{p^H}{R}\right)}c_t}{\partial\gamma} < 0$ for sure.

Then based on the Lemma 1, Theorem 5 and Proposition 1 in Acemoglu and Jensen (2015), γ is a positive shock and the stationary capital over effective labor ratio Q is monotonic increasing in γ. $\square$

Lemma 12. *$\frac{\lambda}{\lambda-\mu} \geq 1$, $\frac{\partial \frac{\mu}{\lambda}}{\partial \gamma} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ holds in Aiyagari-Bewley-Huggett model 8 when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. The demonstration process is similar to Lemma 5 as γ is also a positive price following Lemma 11 and it shares the same monotonicity as σ on $\frac{\mu}{\lambda}$ and $\frac{\lambda}{\lambda-\mu}$ when the stationary consumption C increases. $\square$

Lemma 13. *$\frac{\frac{\lambda}{\lambda-\mu}\eta_{c,p^H} - \eta_{c,p^c}}{\eta_{h,p^c} - \eta_{h,p^h}}$ is monotonic decreasing in γ, as long as $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ hold.*

Proof. Because of Lemma 1, we only need to check whether $\frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H} - \widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ is monotonic decreasing in γ. It is easy to calculate

$$\frac{\partial\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H} - \widetilde{\eta}_{c,p^c}\right)}{\partial \gamma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) - \frac{\partial\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial \gamma}\left(\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H} - \widetilde{\eta}_{c,p^c}\right)$$
$$= \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\widetilde{\eta}_{c,p^H}\left[\left(1 - \frac{\lambda}{\lambda-\mu}\right)\widetilde{\eta}_{h,p^c} + \phi(1-\phi)\right] + \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\widetilde{\eta}_{h,p^h}\left[\left(\frac{\lambda}{\lambda-\mu} - 1\right)\widetilde{\eta}_{h,p^c} + \phi(1-\phi)\right]$$
$$= \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\phi(1-\phi)\left(\widetilde{\eta}_{c,p^H} + \widetilde{\eta}_{h,p^h}\right)$$

Hence $\frac{\partial\left(\frac{\frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{c,p^H} - \widetilde{\eta}_{c,p^c}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial \gamma} < 0$ holds as $\widetilde{\eta}_{c,p^H} + \widetilde{\eta}_{h,p^h} < 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$. $\square$

Lemma 14. *The substitution effect Φ_H will decrease as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ holds.*

Proof. It is a straightforward conclusion from Lemma 12 and 13. $\square$

Lemma 15. *$\frac{\eta_{ch}}{\eta_{h,p^c} - \frac{\lambda}{\lambda-\mu}\eta_{h,p^h}}$ will be monotonic decreasing in γ, as long as $\frac{\lambda}{\lambda-\mu} \geq 1$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$ hold.*

Proof. Because of Lemma 1, we only need to check whether $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}$ is monotonic decreasing in γ. It is easy to calculate

$$\frac{\partial \widetilde{\eta}_{ch}}{\partial \gamma}\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right) - \frac{\partial\left(\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}\right)}{\partial \gamma}\widetilde{\eta}_{ch} = \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma}\widetilde{\eta}_{h,p^h}\widetilde{\eta}_{ch}$$

Hence $\frac{\partial\left(\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}}\right)}{\partial \gamma} < 0$ holds as $\widetilde{\eta}_{h,p^h} < 0$, $\widetilde{\eta}_{ch} > 0$ and $\frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} > 0$. $\square$

Lemma 16. *The wealth effect Φ_{p^H} will decrease as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous and* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds.*

Proof. Lemma 16 is a direct inference from Lemma 12 and 15. $\square$

Lemma 17. *The credit effect Φ_μ will increase as collateral constraint is slacker, when the housing supply is fixed; initial housing distribution over dynamic path is exogenous ;* $\left(\frac{\frac{1-\beta}{\beta}}{\alpha A}\right)^{\frac{1}{\alpha-1}} L > K > \left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$ *holds and the aggregate Khun-Tucker multiplier is not too large.*

Proof. Similar to Lemma 10, we can yield $\frac{\partial \frac{\mu}{\lambda-\mu}}{\partial \gamma} > 0$ because $\frac{\mu}{\lambda-\mu} = \frac{1}{\frac{\lambda}{\mu}-1}$ and $\frac{\partial \frac{\mu}{\lambda}}{\partial \gamma} > 0$ from Lemma 12. Therefore as long as the aggregate Khun-Tucker multiplier is not too large to violate $\frac{\partial \frac{\mu}{\lambda-\mu}}{\partial \gamma} \left| \frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}} \right| > \frac{\partial \frac{\lambda}{\lambda-\mu}}{\partial \gamma} \frac{\mu}{\lambda-\mu}\phi(1-\phi)\left|\widetilde{\eta}_{c,p^H} + \widetilde{\eta}_{h,p^h}\right|$, the credit effect is monotonic increasing in γ because $\frac{\widetilde{\eta}_{ch}}{\widetilde{\eta}_{h,p^c} - \frac{\lambda}{\lambda-\mu}\widetilde{\eta}_{h,p^h}} > 0$ which we can induce from $\frac{\lambda}{\lambda-\mu} \geq 1$ in Lemma 12 and 1. $\square$

F Toy model with global solution

Given the budget constraint of household

$$c_0 + a_1 + p_0\left[s_1 - (1-\delta^h)s_0\right] = (1+R_0)a_0 + w_0 + \pi_0^h + \pi_0$$

$$c_1 + a_2 + p_1\left[s_2 - (1-\delta^h)s_1\right] = (1+R_1)a_1 + w_1 + \pi_1^h + \pi_1$$

$$c_2 = (1+R_2)a_2 + p_2(1-\delta^h)s_2 + w_2 + \pi_2^h + \pi_2$$

From utility function and FOC of household we can get the key equation

$$u_{c_0}\left[p_0 - \frac{1}{1+R_1}(1-\delta^h)p_1\right] = u_{s_1} \tag{68}$$

Then if we assume the utility function is non-separable such that

$$u_t = \frac{\left(c_t^\nu s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}$$

By using the Euler equation of consumption as well as housing we can simplify equation 68 to

$$\left[p_0 - \frac{1}{1+R_1}(1-\delta^h)p_1\right] = \frac{c_1}{s_1^\Phi s_0^\Psi}$$

F.1 General equilibrium is important

A perturb happened at p_1 will decrease c_1 which in tern decrease c_2, If p_0, s_1 and R_1 not change.(This is the total effect of substitution and income as we derive from max utility which means from Marshallian demand function. This is pseudo-effect as we assume s_1 fixed)

However this analysis is based on the assumption that p_0, s_1 and R_1 will not change. Now we assume s_1 is not changed. Meanwhile the production is $Y_t = Aa_t$ so that $R_t = MPK = A$ which means R_1 will also be fixed. Which direction of p_0 changed?

The answer is that any small perturb increased happened in p_1 which returns $\tilde{p_1} = p_1 + \varepsilon$, p_0 will increase relative amount to make sure $p_0 - (1 - \delta^h)p_1$ is fixed. This tells us that c_1 will in fact not change at all.[28]

Later we can also proof that given the decreasing return to scale production function such as $Y_t = Aa_t^\alpha$ will not change the result.

Intuition: Given p_1 increased, the household want to buy more s_1 at period 0. The fixed s_1 will caused p_0 increases a lot to even offset the wealth effect. If we assume s_1 increases and p_0 not change ($s_1$supply increased to the level that just fulfill the demand and p_0 does not change) the direction of c_1 will depends on the extent of increased s_1 and intratemperal substitution and intertemperal substitution). Another condition, p_0 increases more than related to $\frac{1}{1+R_1}(1 - \delta^h)p_1$ is somehow less likely as an expectation causes a much higher inflation this period.

F.2 House supply is the key to determine non-durable consumption

Now we losse the assumption that s_1 does not change. From last section we know that under general equilibrium as long as the house supply does not increase, then no matter how large changed in p_1, c_1 will not change anymore because p_0 will adjusted one-to-one with it.

This give us the argument that the house supply or elasticity of house supply is much more important than scholar's focusing, as most of time we just take it as an IV in empirical research.

A right-hand shift in period 0 house demand(caused by a perturb in p_1) happened, the elasticity of house supply then determine the equilibrium changed in s. We have prove at previous section that when $e_1 = 0$, the increased p_0 will caused c_0 not change. In other words, under the most increased p_0, c_0 not changed. Then assume $e_1 > 0$, Δp_0 will decrease. LHS of equation 68 decrease. But because the intratemporal effect is larger than intertemporal effect, c_1 and c_0 will increase. In other words, the degree of elasticity of house supply determinate the non-durable consumption.

[28]The proof process is simple using induction. Given p_0 increases little but not enough to offset total decreased c_1. Then c_1 and c_0 will decreases little. Then using budget constraint, a_1 and $a_2$will relatively changed. Then to the final period we can get a contradiction. Inversely given p_0 increases a lot to result in c_1 increassing, we can get similar contradiction.

F.3 Unseparable utility function

F.3.1 partial effect

If the utility function is

$$u_t = \frac{\left(c_t^{\nu} s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}$$

then we will have

$$\begin{aligned}
s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} &= \beta R_1 s_1^{(1-\sigma)(1-\nu)} c_1^{\nu(1-\sigma)-1} \\
s_1^{(1-\nu)(1-\sigma)} c_1^{\nu(1-\sigma)-1} &= \beta R_2 s_2^{(1-\sigma)(1-\nu)} c_2^{\nu(1-\sigma)-1} \\
\nu s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} p_0 &= \beta \nu s_1^{(1-\sigma)(1-\nu)} c_1^{\nu(1-\sigma)-1} p_1(1-\delta^h) + \beta(1-\nu) c_1^{\nu(1-\sigma)} s_1^{\nu(\sigma-1)-\sigma}
\end{aligned}$$

$$\nu s_1^{(1-\nu)(1-\sigma)} c_1^{\nu(1-\sigma)-1} p_1 = \beta \nu s_2^{(1-\sigma)(1-\nu)} c_2^{\nu(1-\sigma)-1} p_2(1-\delta^h) + \beta(1-\nu) c_2^{\nu(1-\sigma)} s_2^{\nu(\sigma-1)-\sigma}$$

Then we will solve out c_1, c_2, s_1, s_2 by these four equations

$$c_1 = \left[\frac{1}{\beta R_1}\right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}} \left\{\frac{\nu}{1-\nu}\frac{1}{\beta}\left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right)\right]\right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}\right]^{-\frac{1}{\sigma}}$$

$$\begin{aligned}
s_1 &= \left\{\frac{\nu}{1-\nu}\frac{1}{\beta}\frac{s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}}{c_1^{\nu(1-\sigma)}}\left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right)\right]\right\}^{\frac{1}{(1-\nu)(1-\sigma)-1}} \\
&= \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}\right]^{-\frac{1}{\sigma}} \\
&\quad \left\{\frac{\nu}{1-\nu}\frac{1}{\beta}\left[p_0 - \frac{1}{R_1} p_1 \left(1-\delta^h\right)\right]\right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1}\frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[\frac{1}{\beta R_1}\right]^{\frac{\nu(1-\sigma)}{\sigma}}
\end{aligned}$$

$$\begin{aligned}
c_2 = &\left[\frac{1}{\beta^2 R_1 R_2}\right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}} \\
&\left\{\frac{\nu}{1-\nu}\frac{1}{\beta^2 R_1}\left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right)\right]\right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}\right]^{-\frac{1}{\sigma}}
\end{aligned}$$

$$
\begin{aligned}
s_2 &= \left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^2 R_1} \frac{s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1}}{c_2^{\nu(1-\sigma)}} \left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right) \right] \right\}^{\frac{1}{(1-\nu)(1-\sigma)-1}} \\
&= \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}} \\
&\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^2 R_1} \left[p_1 - \frac{1}{R_2} p_2 \left(1-\delta^h\right) \right] \right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1} \frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[\frac{1}{\beta^2 R_1 R_2} \right]^{\frac{\nu(1-\sigma)}{\sigma}}
\end{aligned}
$$

Under infinite horizon we will have

$$
\begin{aligned}
c_t &= \left[\frac{1}{\beta^t \prod_{i=1}^t R_i} \right]^{\frac{(1-\nu)(1-\sigma)-1}{\sigma}} \\
&\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^t \prod_{i=1}^{t-1} R_i} \left[p_{t-1} - \frac{1}{R_t} p_t \left(1-\delta^h\right) \right] \right\}^{-\frac{(1-\nu)(1-\sigma)}{\sigma}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
s_t &= \left[\frac{1}{\beta^t \prod_{i=1}^t R_i} \right]^{\frac{\nu(1-\sigma)}{\sigma}} \\
&\left\{ \frac{\nu}{1-\nu} \frac{1}{\beta^t \prod_{i=1}^{t-1} R_i} \left[p_{t-1} - \frac{1}{R_t} p_t \left(1-\delta^h\right) \right] \right\}^{\frac{(1-\nu)(1-\sigma)}{(1-\nu)(1-\sigma)-1} \frac{\nu(1-\sigma)}{\sigma} + \frac{1}{(1-\nu)(1-\sigma)-1}} \left[s_0^{(1-\nu)(1-\sigma)} c_0^{\nu(1-\sigma)-1} \right]^{-\frac{1}{\sigma}}
\end{aligned}
$$

F.3.2 Other utility function

If the utility function is

$$
u_t = log\left(c_t^{\nu} s_t^{1-\nu}\right)
$$

then no GE effect

If the utility function is

$$
u_t = log\left(c_t^{\nu} + s_t^{1-\nu}\right)
$$

still unsolvable.

F.4 Standard utility function

F.4.1 general effect

No we assume that the utility function is no longer logarithmic such that

$$
u_t = \frac{\left(c_t^{\nu} s_t^{1-\nu}\right)^{1-\sigma}}{1-\sigma}
$$

Then we have two key market cleaning condition that

$$a_2 = A_1 a_1^{\alpha} - c_1 + (1-\delta) a_1 = A_1 a_1^{\alpha} - c_0 \left(\beta R_1\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + (1-\delta) a_1$$

$$(1-\delta) a_2 + A_2 a_2^{\alpha} = c_2 = c_0 \left(\beta^2 R_1 R_2\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}$$

Based on these two equations we can rewrite equation as

$$\begin{aligned}
&(1-\delta)\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + \right. \\
&\qquad \left. (1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)\right] + \\
&A_2\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}\alpha} + \right. \\
&\qquad \left. (1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)\right] = \\
&c_0\left\{\beta^2\alpha^2 A_1 A_2\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha-1}\right. \\
&\left[A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha} - c_0\left(\beta\alpha A_1\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)^{\alpha-1}\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right. \\
&\qquad \left. \left. +(1-\delta)\left(A_0 a_0^{\alpha} + (1-\delta) a_0 - c_0\right)\right]^{\alpha-1}\right\}^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}
\end{aligned} \tag{69}$$

Similarly we set $\alpha = 1$, equation 69 becomes

$$\begin{aligned}
&(1-\delta)\left[A_1\left(A_0 a_0 + (1-\delta) a_0 - c_0\right) - c_0\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}} + \right. \\
&\qquad \left. (1-\delta)\left(A_0 a_0 + (1-\delta) a_0 - c_0\right)\right] + \\
&A_2\left[A_1\left(A_0 a_0 + (1-\delta) a_0 - c_0\right) - c_0\left(\beta\alpha A_1\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}\alpha} + \right. \\
&\qquad \left. (1-\delta)\left(A_0 a_0 + (1-\delta) a_0 - c_0\right)\right] = \\
&c_0\left(\beta^2\alpha^2 A_1 A_2\right)^{\frac{1}{1-\nu(1-\sigma)}} \left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}
\end{aligned}$$

Now we can solve the c_0 as

$$
\begin{aligned}
c_0 &= \frac{(A_2+1-\delta)\,(A_1+1-\delta)\,(A_0a_0+(1-\delta)a_0)}{(A_2+1-\delta)\left[A_1+1-\delta+(\beta\alpha A_1)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right]+(\beta^2\alpha^2A_1A_2)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_2}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}} \\
&= \frac{(A_2+1-\delta)\,(A_1+1-\delta)\,(A_0a_0+(1-\delta)a_0)}{(A_2+1-\delta)\left[A_1+1-\delta+(\beta\alpha A_1)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{(1-\delta^h)s_0+\bar{s}_1}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}\right]+(\beta^2\alpha^2A_1A_2)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{\bar{s}_2+(1-\delta^h)\bar{s}_1+(1-\delta^h)^2s_0}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}}
\end{aligned}
$$

Under the GE and determined economy, c_0 can only be decided by the equalized house stock. It is intuitive as in the end because all excess profit are payback by construction companies and consumption is mainly determined by IES & market cleaning condition. If we assume that good market clean does not involve construction industry, the house market can only affect the consumption via the Euler equation of asset. Here $\bar{s}_2$ decreases will lead p_2 increase, but it increase c_0 at the same time.

F.4.2 Infinite horizon condition

The market cleaning condition will be

$$a_1 = A_0a_0^\alpha + (1-\delta)a_0 - c_0$$

$$a_2 = A_1a_1^\alpha - c_1 + (1-\delta)a_1$$

$$a_3 = A_2a_2^\alpha - c_2 + (1-\delta)a_2$$

$$(1-\delta)a_\infty + A_\infty a_\infty^\alpha = c_\infty = c_0\left(\beta^3R_1R_2R_3\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_3}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}$$

$$
\begin{aligned}
c_0 &= \frac{(A_0a_0+(1-\delta)a_0)\prod_{t=1}^{\infty}(A_t+1-\delta)}{\sum_{t=1}^{T}\left[\prod_{i=t}^{T}(A_i+1-\delta)\right]\left(\beta^{t-1}\alpha^{t-1}\prod_{i=0}^{t-1}A_i\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_{t-1}}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}+\left(\beta^T\alpha^T\prod_{t=0}^{T}A_t\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{s_T}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}} \\
&= \frac{(A_0a_0+(1-\delta)a_0)\prod_{t=1}^{T}(A_t+1-\delta)}{\sum_{t=1}^{T}\left[\prod_{i=t}^{T}(A_i+1-\delta)\right]\left(\beta^{t-1}\alpha^{t-1}\prod_{i=0}^{t-1}A_i\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{\sum_{i=0}^{t-1}(1-\delta^h)^i\bar{s}_i}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}+\left(\beta^T\alpha^T\prod_{t=0}^{T}A_t\right)^{\frac{1}{1-\nu(1-\sigma)}}\left(\frac{s_0}{\sum_{i=0}^{T}(1-\delta^h)^i\bar{s}_i}\right)^{\frac{(1-\nu)(1-\sigma)}{\nu(1-\sigma)-1}}}
\end{aligned}
$$

when normalizes $A_0 = 1$

F.5 Separable utility function

F.5.1 partial effect

$$c_1 = c_0\left(\beta R_1\right)^{\frac{1}{\sigma}}$$

$$c_2 = c_0\left(\beta^2R_1R_2\right)^{\frac{1}{\sigma}}$$

$$s_1 = \left[p_0R_1 - p_1(1-\delta^h)\right]^{-\frac{1}{\nu}}$$

$$s_2 = \left[p_1R_2 - p_2(1-\delta^h)\right]^{-\frac{1}{\nu}}$$

$$
\begin{aligned}
c_0\left(\beta^2 R_1 R_2\right)^{\frac{1}{\sigma}} + R_2 c_0 \left(\beta R_1\right)^{\frac{1}{\sigma}} + R_1 R_2 c_0 + \\
R_2 p_1 \left\{\left[p_1 R_2 - p_2(1-\delta^h)\right]^{-\frac{1}{\nu}} - (1-\delta^h)\left[p_0 R_1 - p_1(1-\delta^h)\right]^{-\frac{1}{\nu}}\right\} + \\
R_1 R_2 p_0 \left\{\left[p_0 R_1 - p_1(1-\delta^h)\right]^{-\frac{1}{\nu}} - p_0(1-\delta^h)\right\} = \\
R_0 R_1 R_2 a_0 + R_1 R_2 \left(w_0 + \pi_0\right) + R_2 \left(w_1 + \pi_1\right) + w_2 + \pi_2 \\
+ p_2(1-\delta^h)\left[p_1 R_2 - p_2(1-\delta^h)\right]^{-\frac{1}{\nu}}
\end{aligned}
$$

$$
\begin{aligned}
F_{p_1} &= R_2 \left\{\left[p_1 R_2 - p_2(1-\delta^h)\right]^{-\frac{1}{\nu}} - (1-\delta^h)\left[p_0 R_1 - p_1(1-\delta^h)\right]^{-\frac{1}{\nu}}\right\} \\
&+ R_2 p_1 \left\{-\frac{1}{\nu} R_2 \left[p_1 R_2 - p_2(1-\delta^h)\right]^{-\frac{1+\nu}{\nu}} - \frac{1}{\nu}(1-\delta^h)^2 \left[p_0 R_1 - p_1(1-\delta^h)\right]^{-\frac{1+\nu}{\nu}}\right\} \\
&+ \frac{(1-\delta^h)}{\nu} R_1 R_2 p_0 \left[p_0 R_1 - p_1(1-\delta^h)\right]^{-\frac{1+\nu}{\nu}} + \frac{1}{\nu} p_2 R_2 (1-\delta^h)\left[p_1 R_2 - p_2(1-\delta^h)\right]^{-\frac{1+\nu}{\nu}}
\end{aligned}
$$

$$
F_{c_0} = \left(\beta^2 R_1 R_2\right)^{\frac{1}{\sigma}} + R_2 \left(\beta R_1\right)^{\frac{1}{\sigma}} + R_1 R_2
$$

F.5.2 general effect

$$
a_1 = A_0 a_0^\alpha + (1-\delta) a_0 - c_0
$$

$$
\begin{aligned}
a_2 = & A_1 \left[A_0 a_0^\alpha + (1-\delta)a_0 - c_0\right]^\alpha - c_0 \left[\beta\alpha A_1 \left(A_0 a_0^\alpha + (1-\delta)a_0 - c_0\right)^{\alpha-1}\right]^{\frac{1}{\sigma}} \\
& + (1-\delta)\left[A_0 a_0^\alpha + (1-\delta)a_0 - c_0\right]
\end{aligned}
$$

we can solve c_0 by

$$
(1-\delta)a_2 + A_2 a_2^\alpha = c_0 \left(\beta^2 \alpha^2 A_1 A_2 \left(a_1 a_2\right)^{\alpha-1}\right)^{\frac{1}{\sigma}}
$$

which means it is predetermined.

G Equilibrium condition of the full fledged model

G.1 Focs

G.1.1 Focs in production sector

In this section I show that there exists an knife-edge equilibrium in which along the dynamic transition path real rental rate and wage is fixed, as long as the TFP does not change.

The non-durable goods producer solve the problem

$$
\max_{K_n, L_n} A_n K_{n,t}^\alpha L_{n,t}^{1-\alpha} - (r_t + \delta) K_{n,t} - w L_{n,t}
$$

to yield the Foc

$$(1-\alpha) A_n K_{n,t}^{\alpha} L_{n,t}^{-\alpha} = w_t \tag{70}$$

and

$$\alpha A_n K_{n,t-1}^{\alpha-1} L_{n,t}^{1-\alpha} = r_t + \delta \tag{71}$$

Similarly the durable goods producer solve the problem

$$\max_{K_h, L_h} \Pi^h = p_t^h A_h \overline{L}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota} - (r_t + \delta) K_{h,t} - w L_h$$

to yield the Foc

$$\iota A_h p_t^h \overline{L}_t^{\theta} K_{h,t}^{\nu} L_{h,t}^{\iota-1} = w_t \tag{72}$$

and

$$\nu A_h p_t^h \overline{L}_t^{\theta} K_{h,t}^{\nu-1} L_{h,t}^{\iota} = r_t + \delta \tag{73}$$

Combine equation 71 and 73 will yield

$$\frac{\nu p_t^h Y_{H,t}}{K_{h,t}} = r_t + \delta = \frac{\alpha Y_{N,t}}{K_{n,t}} \tag{74}$$

It is easy to check that when $\frac{\iota}{\nu} = \frac{1-\alpha}{\alpha}$ the real rental rate and wage at time t is fixed, as long as the total capital used at time t, K_{t-1} and labor L_t is fixed. I attach the proof process below.

By dividing equation 70, 71, 72 and 73 with each other I can get the relative input sharing condition

$$\frac{\iota \alpha}{\nu (1-\alpha)} \frac{K_{h,t}}{K_{n,t}} \frac{L_{n,t}}{L_{h,t}} = 1$$

when $\frac{\iota}{\nu} = \frac{1-\alpha}{\alpha}$ holds, above equation will change to $\frac{K_{h,t}}{K_{n,t}} = \frac{L_{n,t}}{L_{h,t}}$.

Furthermore, the relative value of $K_{n,t}$ and $L_{n,t}$ can be pinned down with the market clearing condition $K_{H,t-1} = K_{h,t} + K_{n,t}$ and $L_t = L_{h,t} + L_{n,t}$. In section 3 I assume that the labor supply is exogenous which will help to demonstrate that the relative value of $K_{n,t}$ and $L_{n,t}$ follows

$$\frac{K_{n,t}}{L_{n,t}} = \frac{K_{H,t-1}}{L} \frac{1 + \frac{K_{n,t}}{L_{n,t}}}{1 + \frac{K_{h,t}}{L_{h,t}}}$$

Because $K_{H,t-1}$ is predetermined and $\frac{K_{h,t}}{K_{n,t}} = \frac{L_{n,t}}{L_{h,t}}$, the $\frac{K_{n,t}}{L_{n,t}}$ is fixed. Therefore r_t is fixed from equation 74.

G.1.2 Focs in consumer sector

The household solve the problem

$$V(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = \max_{h_t, x_t, l_t, c_t} U(c_t, h_t, l_t) + \beta EV(h_t, x_t, \varepsilon_t)$$

$$
\begin{aligned}
\text{s.t.} c_t + x_t + (1-\gamma)\, p_t^h h_t = & \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} \\
& + (1-\tau) w_t l_t \varepsilon_{t-1} - p_t^h C\left(h_t, h_{t-1}\right) + T_t
\end{aligned} \tag{75}
$$

and

$$
x_t \geq 0
$$

The related Lagrange is

$$
\begin{aligned}
\mathcal{L} = & U\left(c_t, h_t, l_t\right) + \beta E_t V\left(h_t, x_t, \varepsilon_t\right) \\
& + \lambda_t \left[c_t + x_t + (1-\gamma)\, p_t^h h_t - \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1}\right. \\
& \left. - R_t x_{t-1} - (1-\tau) w_t l_t \varepsilon_{t-1} + p_t^h C\left(h_t, h_{t-1}\right) - T_t\right] \\
& + \mu_t x_t
\end{aligned}
$$

Then the FOCs related to consumer's problem will be

$$
U_{c,t} + \lambda_t = 0 \tag{76}
$$

$$
U_{h,t} + \beta E_t V_{h,t} + \lambda_t \left(1 - \gamma + C_{h,t}\right) p_t^h = 0 \tag{77}
$$

$$
\beta E_t V_{x,t} + \lambda_t + \mu_t = 0 \tag{78}
$$

$$
U_{l,t} - \lambda_t (1-\tau) w_t \varepsilon_{t-1} = 0 \tag{79}
$$

The envelop conditions are

$$
V_h(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = -\lambda_t \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h - C_{h_{t-1}}\left(h_t, h_{t-1}\right) p_t^h\right] \tag{80}
$$

$$
V_x(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = -\lambda_t R_t \tag{81}
$$

G.1.3 Steady State condition in production sector

G.2 Alternative Setting to Capital Producer

G.2.1 Capital Producer(Setting I)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$
\begin{aligned}
& \max\left(Q_t - 1\right) I_t - f\left(I_t, K_{t-1}\right) K_{t-1} \\
\text{s.t. } f\left(I_t, K_{t-1}\right) = & \frac{\psi_{I,1}}{\psi_{I,2}} \left(\frac{I_t}{K_{t-1}} - \overline{\delta}\right)^{\psi_{I,2}}
\end{aligned}
$$

where $\overline{\delta}$ is the steady-state investment rate.

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = 1 + \psi_{I,1}\left(\frac{I_t}{K_{t-1}} - \overline{\delta}\right)^{\psi_{I,2}-1}$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + f\left(I_t, K_{t-1}\right)K_{t-1} + p^h C(h', h)$$

G.2.2 Capital Producer(Setting II)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned}
&\max Q_t I_t - f\left(I_t, K_{t-1}\right)K_{t-1} \\
\text{s.t. } f\left(I_t, K_{t-1}\right) =& \frac{\overline{\delta}^{-1/\phi}}{1+1/\phi}\left(\frac{I_t}{K_{t-1}}\right)^{1+1/\phi} + \frac{\overline{\delta}}{\phi+1}
\end{aligned}$$

where $\overline{\delta}$ is the steady-state investment rate following $\overline{\delta} = \frac{\overline{I}}{\overline{K}}$

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = \left(\frac{I_t}{K_{t-1}\overline{\delta}}\right)^{1+1/\phi}$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + f\left(I_t, K_{t-1}\right)K_{t-1} + p^h C\left(h', h\right)$$

G.2.3 Capital Producer(Setting III)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned}
&\max Q_t f\left(I_t, K_{t-1}\right)K_{t-1} - I_t \\
\text{s.t. } f\left(I_t, K_{t-1}\right) =& \frac{\overline{\delta}^{1/\phi}}{1-1/\phi}\left(\frac{I_t}{K_{t-1}}\right)^{1-1/\phi} - \frac{\overline{\delta}}{\phi+1}
\end{aligned}$$

where $\overline{\delta}$ is the steady-state investment rate.

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$Q_t = \left(\frac{I_t}{K_{t-1}\bar{\delta}}\right)^{1-1/\phi}$$

and the law of motion of capital will become

$$K_t = (1-\delta)K_{t-1} + f\left(I_t, K_{t-1}\right)K_{t-1}$$

The goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + p^h C\left(h', h\right)$$

G.2.4 Capital Producer(Setting IV)

The capital producer uses final nondurable goods Y_N to produce capital following the maximization problem

$$\begin{aligned}
&\max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t}\Lambda_{t,t+\tau}\left\{\left(Q_\tau - 1\right)I_\tau - f\left(I_\tau, I_{\tau-1}\right)I_\tau\right\} \\
&\text{s.t. } f\left(I_\tau, I_{\tau-1}\right) = \frac{\psi_{I,1}}{\psi_{I,2}}\left(\frac{I_\tau}{I_{\tau-1}} - 1\right)^{\psi_{I,2}}
\end{aligned}$$

By solving above optimization problem I could get the capital price as a convex function of investment which is shown below

$$\begin{aligned}
Q_t = 1 + \frac{\psi_{I,1}}{\psi_{I,2}}\left(\frac{I_t}{I_{t-1}} - 1\right)^{\psi_{I,2}} + \psi_{I,1}\left(\frac{I_t}{I_{t-1}} - 1\right)^{\psi_{I,2}-1}\frac{I_t}{I_{t-1}} - \\
E_t\beta\Lambda_{t,t+1}\psi_{I,1}\left(\frac{I_{t+1}}{I_t} - 1\right)^{\psi_{I,2}-1}\left(\frac{I_{t+1}}{I_t}\right)^2
\end{aligned}$$

So the goods market clearing condition will become

$$Y_{N,t} = C_t + I_t + f\left(I_t, I_{t-1}\right)I_{t-1} + p^h C(h', h)$$

H Numerical solution

H.1 Calibration to full fledged model

All the parameters related to production sector are selected from literature. The depreciation rate of physical capital is 0.03 which implies 12% annually. The depreciation rate of housing service is estimated from data which is constructed by Rognlie et al. (2018) as my model in supply side is too simple to use the gross GDP in NIPA. Therefore I use the GDP constructed by Rognlie et al. (2018) which is more suitable to this simple supply side. The depreciation rate of housing service is roughly 1.9% quarterly which is in line with Kaplan et al. (2020). The relative share of production factors in construction function ν, θ and ι comes from Favilukis et al. (2017). The last three parameters, exogenous land supply, TFP in production function and TFP in construction function, together with other parameters in household problem, are selected to match the real gross rate, labor demand, liquid asset over GDP and iliquid asset over GDP.

Table 9: Parameter Values from Calibration

Parameter	Value	Description
δ	0.03	Depreciation rate of physical capital
δ^h	0.01873	Depreciation rate of housing service
α	0.36	Capital share in production function
ν	0.27	Capital share in construction function
ι	0.36	Labor share in construction function
θ	0.1	Land share in construction function
$\overline{\mathscr{L}}$	4.95	Land supply
A_n	0.67	TFP in production function
A_h	2.75	TFP in construction function

Table 10: Presetted Parameter Values

Parameter	Value	Description
σ_L	0	Depreciation rate of physical capital
$\sigma_{m_4^L}$	∞	Depreciation rate of housing service
$\sigma_{m_4^\phi}$	∞	Capital share in production function
m_1^L	1	Capital share in construction function
m_2^L	1	Labor share in construction function
m_3^L	1	Land share in construction function
m_4^L	0	Land supply
m_1^ϕ	1	TFP in production function
m_2^ϕ	1	
m_3^ϕ	1	
m_4^ϕ	0	TFP in construction function

H.2 Bayesian estimation to full fledged model

I use Bayesian method to estimate the parameters that control the impulse response and transition path such as the AR1 coefficients ρ_a^i, the observation matrix H and related covariance matrix $\eta\eta'$ and $\epsilon\epsilon'$. Since the data process itself is not stationary it is not appropriate to use the full-information Bayesian and if we used the statistic method to detrend such as first-order difference and hp filter, the Bayesian update rule would not be further used and the posterior $p\left(\theta|Y^T\right) \propto p\left(Y^T|\theta\right) p\left(\theta\right)$ would be unsolvable as $p\left(Y^T|\theta\right)$ was unknown. Therefore I use GMM to match the moments in data and model to proceed the estimation. In this subsection I first introduce the moments I used to match the data and then explain the Bayesian estimation strategy in detail.

H.2.1 Moments Selection and Theoretical moments after filter

I impose hp filter on the data and calculate moments from the cyclical elements such as the autocovariance of output, standard derivation of output, physical investment, new constructed residential estate, relative housing price and their related covariance. The covariance between output and physical investment $\mathrm{cov}(y_t, I_t)$ captures the general equilibrium $Y = C + I$. Similarly the covariance between residential investment and physical investment $\mathrm{cov}(I_t^H, I_t)$ captures the crowded-out effect. The covariance between new constructed residential estate and relative housing price capture the demand and supply equilibrium in the housing market. All these eight moments are summarized in vector $g(\cdot) = \Psi$ following

$$\Psi = \left[\begin{array}{ccc} \varrho_m' & \sigma_{m,m}' & \sigma_{m,n}' \end{array}\right]'$$

where ϱ_m is the vector that contains the autocovariance moments (ρ_m^i represents the $\mathrm{AR}(i)$'s coefficient of variable m)

$$\varrho_m = \left[\begin{array}{cccccc} \rho_y^1 & \rho_c^1 & \rho_I^1 & \rho_{I_H}^1 & \rho_{p_H}^1 & \rho_Q^1 \end{array}\right]'$$

$\sigma_{m,m}$ is the vector that contains the standard derivation moments

$$\sigma_{m,m} = \left[\begin{array}{ccccc} \sigma_y & \sigma_c & \sigma_I & \sigma_{p_H} & \sigma_Q \end{array}\right]'$$

$\sigma_{m,n}$ is the vector that contains the covariance moments of variables $\phi_v = \left[\begin{array}{ccccccc} y & c & I & I_H & p_H & Q & R \end{array}\right]'$

$$\sigma_{m,n} = \left[\begin{array}{ccccccccc} \sigma_{y,c} & \sigma_{y,I} & \sigma_{y,I_H} & \sigma_{y,p_H} & \sigma_{y,Q} & \sigma_{y,R} & \sigma_{c,I_H} & \cdots & \sigma_{Q,R} \end{array}\right]'$$

Moreover I solve the theoretical moments from model after hp filter by switching to frequency domain and the spectrum. After some algebra I can solve the covariance matrix

$$\mathbb{E}\left[\widetilde{Y}_t\widetilde{Y}_{t-1}\right] = \int_{-\pi}^{\pi} g^{\mathrm{HP}}(\omega)e^{i\omega k}d\omega$$

where $\widetilde{Y}_t = \begin{bmatrix} s_t' & s_{t|t}' & Ec_{t+1}' \end{bmatrix}'$ in equation 102. The spectral density of HP filter $g^{\mathrm{HP}}(\omega)$ follows $g^{\mathrm{HP}}(\omega) = h^2(\omega)g(\omega)$. $h(\omega) = \frac{4\lambda(1-\cos(\omega))^2}{1+4\lambda(1-\cos(\omega))^2}$ is the transfer function of HP derived from King and Rebelo (1993). The spectral density of state and control variables Y_t is solved by

$$g(\omega) = \begin{bmatrix} I_{ns} & 0_{ns,nq} \\ M_{21}\mathrm{e}^{-i\omega} & D_2 \\ 0_{nq,ns} & I_{nq} \end{bmatrix} f(\omega) \begin{bmatrix} I_{ns} & M_{21}'e^{i\omega} & 0_{ns,nq} \\ 0_{nq,ns} & D_2' & I_{nq} \end{bmatrix} = Wf(\omega)W' \tag{82}$$

$$f(\omega) = \frac{1}{2\pi}\begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ I_{nq} \end{bmatrix} \Sigma \left[D_1' \left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1}, I_{nq}\right] \tag{83}$$

where ns is the number of state variables and nq is the number of shocks. M and D come from the policy function 108 and Σ is the covariance matrix of shocks. Because I assume the shock term Ξ_t in system 102 follows standard normal distribution and all the covariance terms are absorbed in η and ϵ, Σ in equation 83 is an identity matrix.

W.L.O.G, I assume the shock Ξ_t in equation 108 is independent with each other and all the covariance term is stored in response D. Therefore the covariance term Σ in equation 83 is an identity matrix and the equation can be further simplified as

$$f(\omega) = \frac{1}{2\pi}\begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & I_{nq} \end{bmatrix}$$

Then equation 82 becomes

$$\begin{aligned} g(\omega) &= \frac{1}{2\pi}\begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ M_{21}\mathrm{e}^{-i\omega}\left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & M_{21}\mathrm{e}^{-i\omega}\left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & I_{nq} \end{bmatrix} W' \\ &+ \frac{1}{2\pi}\begin{bmatrix} 0 & 0 \\ D_2D_1'\left(I_{ns} - M_{11}'e^{i\omega}\right)^{-1} & D_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_{ns} & M_{21}'e^{i\omega} & 0_{ns,nq} \\ 0_{nq,ns} & D_2' & I_{nq} \end{bmatrix} W' \\ &= \frac{1}{2\pi}\left(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4\right) \end{aligned}$$

where

$$\Upsilon_1 = \begin{bmatrix} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & M_{21} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' & M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 \\ D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & I_{nq} \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} 0 & \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_2' & 0 \\ 0 & M_{21}\mathrm{e}^{-i\omega} \left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} D_1 D_2' & 0 \\ 0 & D_2' & 0 \end{bmatrix}$$

$$\Upsilon_3 = \begin{bmatrix} 0 & 0 & 0 \\ D_2 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} & D_2 D_1' \left(I_{ns} - M_{11}' e^{i\omega}\right)^{-1} M_{21}' e^{i\omega} & D_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Upsilon_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_2 D_2' & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To further decrease the computation burden it is easy to show that $M_{21}\left(I_{ns} - M_{11}\mathrm{e}^{-i\omega}\right)^{-1} = \mathrm{e}^{i\omega} M_{21} U_M \left(\mathrm{e}^{i\omega} I_{ns} - T_M\right)^{-1} U_M'$ where $M_{11} = U_M T_M U_M'$ is prederived from Schur decomposition.

H.2.2 Bayesian GMM

H.2.2.1 Moment Matching: Imperfect Information Bayesian Estimation

Following Rotemberg and Woodford (1997), Christiano et al. (2005) and Barsky and Sims (2012), to construct the asymptotic properties of the moments which I select to conduct the Bayesian GMM, I first construct the auxiliary variable ψ_t

$$\psi_t = \begin{bmatrix} y_t & c_t & I_t & I_{t,H} & p_{t,H} & Q_t & R_t & y_t y_{t-1} & c_t c_{t-1} & \cdots & y_t^2 & c_t^2 & \cdots & p_{t,H}^2 & y_t c_t & y_t I_t & \cdots & Q_t R_t \end{bmatrix}'$$

Additionally I define the moment function as $g(\cdot)$ which yields the moments

$$g(\psi_t) = \Psi$$

If the sample estimation of ψ_t is $\widehat{\psi}$ the moment function is well defined as

$$
g(\widehat{\psi}) = \begin{bmatrix} \hat{\psi}_{y_t y_{t-1}} - \hat{\psi}_y^2 \\ \hat{\psi}_{c_t c_{t-1}} - \hat{\psi}_c^2 \\ \vdots \\ \sqrt{\hat{\psi}_{y^2} - \hat{\psi}_y^2} \\ \sqrt{\hat{\psi}_{c^2} - \hat{\psi}_c^2} \\ \vdots \\ \hat{\psi}_{yc} - \hat{\psi}_y \hat{\psi}_c \\ \hat{\psi}_{yI} - \hat{\psi}_y \hat{\psi}_I \\ \vdots \\ \hat{\psi}_{QR} - \hat{\psi}_Q \hat{\psi}_R \end{bmatrix}
$$

Therefore the Jacobian of moment function $\Gamma_g(\cdot)$ should be

$$
\Gamma_g(\widehat{\psi}) = \frac{\partial g}{\partial \psi} = \begin{bmatrix}
-2\mu_y & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2\mu_c & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\frac{\mu_y}{\sigma_y} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_y} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\mu_c}{\sigma_c} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_c} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\frac{\mu_Q}{\sigma_Q} & 0 & \cdots & 0 & 0 & \frac{1}{2\sigma_Q} & 0 & \cdots & 0 & 0 & 0 \\
-\mu_c & -\mu_y & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
-\mu_I & 0 & -\mu_y & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\mu_R & -\mu_Q & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

By applying the Delta Method the sample estimation of moments $\widehat{\Psi}$ has the following asymptotic properities

$$
\sqrt{T}\left(\widehat{\Psi} - \Psi\right) \xrightarrow{d} N\left(0, \Gamma_g \Sigma \Gamma_g'\right)
$$

where Σ is the LRV of ψ_t.

H.2.2.2 Full Information Bayesian Estimation

There are 38 parameters to be estimated via bayesian method and most of them govern the dynamic transition path of the economy. Firstly, six out of thirty-eight parameters are the AR1 coefficient of the shocks' process: ρ_{A_n} and ρ_{A_h} relate to the TFP of output and construction sector; ρ_L and ρ_{L_g} relate to the supply side of the housing market, land supply, and they are similar to the form defined in equation 93; ρ_ϕ and ρ_{ϕ_g} relate to the demand side of the housing market, preference on residential asset, and they are just the form defined in equation 93. Then eight parameters correspond to the standard derivation of above six shock series with two news shock on supply and demand side of the housing market. Additionally eight parameters, in supply and demand side of the housing market, associate with the observation or imperfect information process (H in equation 96) and another eight parameters pertain to the standard derivation of these observation noisy (ϵ in equation 96). Then one parameter affects the capital price, which is in the capital production function (ψ_I in equation 22). In the end the left seven parameters are the standard derivation of measure error of the seven data series that I used to estimate: output, nondurable consumption, physical investment, new construction, housing price, stock price and real interest rate. The whole estimation process is overestimated as there are 38 parameters in model but 77 moments (49 in coefficient matrix and 28 in the covariance matrix of residual).

Following Smets and Wouters (2007) and Rudebusch and Swanson (2012), I use the standard random walk metropolis-hastings (RWMH) algorithm to conduct the bayesian estimation and the data I used are per capita series to get a stationary time series. However most of the data does not pass the unit-root test and thus I further use the first order difference method to detrend the data, because I do not introduce the trend (growth) elements in the model. Moreover, to ensure the compatible between the model and data, I rearrange the state equation 108 of the model to

$$\begin{bmatrix} \widetilde{Y}_t \\ \widetilde{Y}_{t-1} \end{bmatrix} = \begin{bmatrix} M & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} \widetilde{Y}_{t-1} \\ \widetilde{Y}_{t-2} \end{bmatrix} + D \begin{bmatrix} \Xi_t \\ 0 \end{bmatrix} \tag{84}$$

Therefore the measurement equation should change to

$$Y_t = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} \widetilde{Y}_t \\ \widetilde{Y}_{t-1} \end{bmatrix} + \Xi_t \tag{85}$$

The likelihood function can be solved from the Kalman Filter from the state equation 84 and measurement equation 85. Based on the recommendation of Herbst and Schorfheide (2016), I use gradient based MLE method to proceed the estimation to get the asymptotic variance of the parameters (the inverse Hessian of the likelihood function) and the prior mean of the parameters. Following Schmitt-Grohé and Uribe (2012), Blanchard et al. (2013) and Christiano et al. (2014) the prior standard derivations that pertain to AR1 coefficient are 0.1 and others that associate with variance are 1.

Table 11: Bayesian Estimation

Parameter	Distribution	Prior		Posterior
		mean	s.d.	mean
ρ_{A_n}	Beta	0.5	0.2	
ρ_{A_h}	Beta	0.5	0.2	
ρ_L	Beta	0.5	0.2	
ρ_ϕ	Beta	0.5	0.2	
σ_{A_n}	InvGamma	0.1	1	
σ_{A_h}	InvGamma	0.1	1	
σ_{L_g}	InvGamma	0.1	1	
σ_ϕ	InvGamma	0.1	1	
$\sigma_{m_1^L}$	InvGamma	0.1	1	
$\sigma_{m_2^L}$	InvGamma	0.1	1	
$\sigma_{m_3^L}$	InvGamma	0.1	1	
$\sigma_{m_1^\phi}$	InvGamma	0.1	1	
$\sigma_{m_2^\phi}$	InvGamma	0.1	1	
$\sigma_{m_3^\phi}$	InvGamma	0.1	1	
ϕ_I	Gamma	1.728	1	

H.3 Solution method to simple model

H.3.1 Reconstruction

Similar to the section H.7.1, I replace the saving a_t by the effective asset holding x_t which follows $x_t = \gamma p_t^H h_t + a_t$. Then the problem 8 change to

$$\max_{c_t, h_t, x_t} \sum_{t=0}^{\infty} \beta^t U\left(c_t, h_t\right) \tag{86}$$

s.t.

$$c_t + x_t + (1-\gamma)\, p_t^H h_t = R_t x_{t-1} + w_t \varepsilon_t + \left[(1-\delta^H) p_t^H - \gamma R_t p_{t-1}^H\right] h_{t-1} + T_t \tag{87}$$

$$x_t \geq 0$$

The related FOCs 57, 58 and 59 will become

$$U_{c_t} = \lambda_t \tag{88}$$

$$-\lambda_t + \mu_t + \beta E_t R_{t+1}\lambda_{t+1} = 0 \tag{89}$$

$$U_{h_t} - (1-\gamma)\,\lambda_t p_t^H + \beta E_t \lambda_{t+1}\left[(1-\delta^H)p_{t+1}^H - \gamma R_{t+1} p_t^H\right] = 0 \tag{90}$$

Similar to the full fledged model, I assume the utility function $U(c_t, h_t)$ follows the Cobb-Douglas formula

$$U(c_t, h_t) = \frac{\left(c_t^{\phi} h_t^{1-\phi}\right)^{1-\sigma}}{1-\sigma} \tag{91}$$

Since I assume there is no aggregate shock existing in the simple model, R_{t+1}, p_{t+1}^H and p_t^H can be perfectly expected. Therefore for non-constrained household there exists a static relationship between c_t and h_t from the combining of equation 88, 89 and 90

$$c_t = \frac{\phi}{1-\phi} h_t \left[p_t^H - (1-\delta^H)\frac{p_{t+1}^H}{R_{t+1}}\right] \tag{92}$$

When the collateral constraint is binding, it is worth to notice that the two FOC 58 and 89 have the same form. Therefore the Khun-Tucker multiplier is the same between the two model, the original one and the reconstructed one. To sum up, the problem 86 degenerates to a one state x_t problem which can be solved easily by value function iteration.

H.3.2 Solution Steps

Since in this simple problem I use Cobb-Douglas utility function where intratemporal elasticity of substitution between housing service and non-durable consumption is constant at 1, the consumption and housing servicing is homogeneous in degree 1 (linear) in the frictionless scenario. Therefore it is solvable to use value function iteration method.

1. Take an initial guess about value function $V(h_{-1}, x_{-1}, \varepsilon_{-1}) = \sum_{t=0}^{\infty} \beta^t U(c_t, h_t)$. If h_0, x_0 is still on grid I can remove the expectation with $\widetilde{V}(h_0, x_0, \varepsilon_{-1}) = E_0 V(h_0, x_0, \varepsilon_0) = \Pi V(h_0, x_0, \varepsilon_0)$ as h_0, x_0 is determined at time 0.

2. If the budget constraint is not binding, equation 92 will always hold. Therefore given an initial guess of $h_0(h_{-1}, x_{-1}, \varepsilon_{-1})$, I can get the unique mapping $x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ and $c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ through budget constraint 87 and equation 92. Then it is easy to find

   $$h_0^{uc}(h_{-1}, x_{-1}, \varepsilon_{-1}) = \underset{h_0}{\text{argmax}} U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta\widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$$

   where $\widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$ can be solved from linear interpolation on the on-grid value $\widetilde{V}(h_0, x_0, \varepsilon_{-1})$ in last step. I also define and save the value

   $$\text{RHS}^{UC} = \max U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta\widetilde{V}\left[h_0, x_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), \varepsilon_{-1}\right]$$

   .

3. If the budget constraint is binding, the Euler equation does not hold anymore. Therefore the mapping between h_0 and c_0 is no longer useful. However the effective wealth is known as now the household is constrained so $x_0(h_{-1}, x_{-1}, \varepsilon_{-1}) = 0$. Given any guess of $h_0(h_{-1}, x_{-1}, \varepsilon_{-1})$ the consumption $c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1})$ can be solved from budget constraint 87. Then it is easy to find

$$h_0^c(h_{-1}, x_{-1}, \varepsilon_{-1}) = \underset{h_0}{\text{argmax}} U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta \widetilde{V}\left[h_0, 0, \varepsilon_{-1}\right]$$

where $\widetilde{V}\left[h_0, 0, \varepsilon_{-1}\right]$ can be solved from linear interpolation on the on-grid value $\widetilde{V}(h_0, 0, \varepsilon_{-1})$ in step 1. I also define and save the value

$$\text{RHS}^C = \max U\left[c_0(h_0, h_{-1}, x_{-1}, \varepsilon_{-1}), h_0\right] + \beta \widetilde{V}\left[h_0, 0, \varepsilon_{-1}\right]$$

.

4. Because the result of constrained optimization in convex function optimization problem is always inferior than that of unconstrained optimization, the updated value function $V(h_{-1}, x_{-1}, \varepsilon_{-1})$ will follows

$$V(h_{-1}, x_{-1}, \varepsilon_{-1}) = \begin{cases} \text{RHS}^{UC} & x_0^{uc} \geq 0 \\ \text{RHS}^{C} & x_0^{c} < 0 \end{cases}$$

Update the value function and go back to step 1.

H.4 Solution method to simple model with separable utility function

H.4.1 Reconstruction and new FOCs

Change the utility function from 91 to the separable utility function

$$U\left(c_t, h_t\right) = \frac{\phi c_t^{1-\sigma} + (1-\phi) h_t^{1-\sigma}}{1-\sigma}$$

Then the mapping from c_t to h_t under the frictionless scenario changes to

$$c_t = \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{\sigma}} \left[p_t^H - (1-\delta^H)\frac{p_{t+1}^H}{R_{t+1}}\right]^{\frac{1}{\sigma}} h_t$$

H.5 Expected news shock

Then denote the "fundamental" variable X_t as

$$X_t = \left[\begin{array}{cccccccccc} \log \Phi_t^i & \log \Phi_{g,t}^i & \varepsilon_t^8 & \varepsilon_{t-1}^8 & \varepsilon_{t-2}^8 & \varepsilon_{t-3}^8 & \varepsilon_{t-4}^8 & \varepsilon_{t-5}^8 & \varepsilon_{t-6}^8 & \varepsilon_{t-7}^8 \end{array}\right]' \tag{93}$$

Then X_t follows

$$X_t = B^s X_{t-1} + \eta w_t \tag{94}$$

where

$$B^s = \begin{bmatrix} \rho_a & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \rho_g & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{10\times 10}$$

$$\eta = \begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & \sigma_g^8 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}_{10\times 3}$$

$$\boldsymbol{w}_t = \begin{bmatrix} w_t^a \\ w_t^g \\ w_t^8 \end{bmatrix}$$

However household can only observe the variable $\widetilde{X}_t$ such that

$$\widetilde{X}_t = \begin{bmatrix} \log \widetilde{\Phi}_t & \log \widetilde{\Phi}_{g,t} & \widetilde{\varepsilon}_t^8 & \widetilde{\varepsilon}_{t-1}^8 & \widetilde{\varepsilon}_{t-2}^8 & \widetilde{\varepsilon}_{t-3}^8 & \widetilde{\varepsilon}_{t-4}^8 & \widetilde{\varepsilon}_{t-5}^8 & \widetilde{\varepsilon}_{t-6}^8 & \widetilde{\varepsilon}_{t-7}^8 \end{bmatrix}' \tag{95}$$

which follows

$$\widetilde{X}_t = H X_t + \epsilon v \tag{96}$$

where

$$H = \begin{bmatrix} H_{3\times 3}^{11} & 0_{3\times 5} \\ 0_{5\times 3} & m_4 I_{5\times 5} \end{bmatrix}$$

$$H^{11} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$m \in \mathbb{R}^+$$

$$\epsilon = \begin{bmatrix} \sigma_a^s & 0 & 0 & \cdots & 0 \\ 0 & \sigma_g^s & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{g1}^s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{g8}^s \end{bmatrix}_{10\times 10}$$

$$v_t = \begin{bmatrix} v_t^a \\ v_t^g \\ v_t^{g1} \\ \vdots \\ v_t^{g8} \end{bmatrix}$$

H.6 Kalman Filter

Even though the household can successfully observe A_t at time t, he cannot observe g_t at time t. This make the household harder to estimate the A_{t+1} as $E_t \log(A_{t+1}) = \rho_a \log A_t + E_t \log g_t$. Thus we need get $g_{t|t}$ to get the expectation of A_{t+1}. Based on the Kalman filter and equation 94 and 96, we can solve out the perception of g_t by household as[29]

$$X_{t+1|t+1} = A^s X_{t|t} + P^s \widetilde{X}_{t+1} \tag{97}$$

where P^s is the Kalman gain and $A^s = (I - P^s H)B^s$

H.7 Model Reconstruction and Solution Process

The computation process follows the augmented endogenous gird method which is proposed by Auclert et al. (2021).

H.7.1 Preliminaries

I define the risk-adjusted expected value function as

$$\widetilde{V}(h_t, b_t, \varepsilon_{t-1}) = \beta EV(h_t, b_t, \varepsilon_t)$$

Therefore the marginal risk-adjusted expected value should be

$$\widetilde{V}_h(h_t, b_t, \varepsilon_{t-1}) = \beta EV_h(h_t, b_t, \varepsilon_t)$$

and

$$\widetilde{V}_b(h_t, b_t, \varepsilon_{t-1}) = \beta EV_b(h_t, b_t, \varepsilon_t)$$

[29]For the reference Hamilton (2020) provides rigorous proof to this equation.

To simplify the computation process, I further define the auxiliary variable x_t as t he effective asset holding which follows $x_t = \gamma p_t^h h_t + b_t$. Therefore the budget constraint 9 becomes

$$c_t + x_t + (1-\gamma) p_t^h h_t = \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} + (1-\tau) w_t l_t \varepsilon_{t-1} - p_t^h C\left(h_t, h_{t-1}\right) + T_t \tag{98}$$

Correspondingly collateral constraint becomes

$$x_t \geq 0$$

H.7.2 Decision Problems

The household solve the problem

$$V(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = \max_{h_t, x_t, l_t, c_t} U\left(c_t, h_t, l_t\right) + \beta E V(h_t, x_t, \varepsilon_t)$$

$$\text{s.t.} c_t + x_t + (1-\gamma) p_t^h h_t = \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} + (1-\tau) w_t l_t \varepsilon_{t-1} - p_t^h C\left(h_t, h_{t-1}\right) + T_t$$

and

$$x_t \geq 0$$

H.7.3 Solve step

1. Take the initial guess to marginal value function at time $t+1$ as $V_h(h_t, x_t, \varepsilon_t)$ and $V_x(h_t, x_t, \varepsilon_t)$
2. Solve the expectation problem on marginal value function to get risk-adjusted expected value function
$$\widetilde{V}_h(h_t, x_t, \varepsilon_{t-1}) = \beta \Pi V_h(h_t, x_t, \varepsilon_t)$$
and
$$\widetilde{V}_x(h_t, x_t, \varepsilon_{t-1}) = \beta \Pi V_x(h_t, x_t, \varepsilon_t)$$
3. Assuming the collateral constraint is unconstrained, I can combine equation 76, 77 and 78 to get
$$F\left(h_t, x_t, \varepsilon_{t-1}, h_{t-1}\right) = \frac{U_{h,t} + \widetilde{V}_h}{p_t^h \widetilde{V}_x} - (1 - \gamma + C_{h,t}) = 0$$
Further because the unseparable utility function $U\left(c_t, h_t, l_t\right)$ is homogeneous between c_t

and h_t, $U_{h,t}$ can be written as a function of $\widetilde{V}_x$

$$U_{h,t} = (1-\phi)\left(\frac{\widetilde{V}_x}{\phi}\right)^{\frac{\phi(1-\sigma)}{\phi(1-\sigma)-1}} h_t^{\frac{\phi(1-\phi)(1-\sigma)^2}{1-\phi(1-\sigma)}+(1-\phi)(1-\sigma)-1} \tag{99}$$

This can be used to solve $h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)$. The related mapping weight can also be used to map $\widetilde{V}_x(h_t, x_t, \varepsilon_{t-1})$ into $\widetilde{V}_x(h_{t-1}, x_t, \varepsilon_{t-1})$. Then $c\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)$ and $l\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)$ can be solved straightforward from

$$c\left(h_{t-1}, x_t, \varepsilon_{t-1}\right) = \left(\frac{\widetilde{V}_x(h_{t-1}, x_t, \varepsilon_{t-1})}{\phi}\right)^{\frac{1}{\phi(1-\sigma)-1}} h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{1-\phi(1-\sigma)}} \tag{100}$$

and

$$l\left(h_{t-1}, x_t, \varepsilon_{t-1}\right) = \left(-\phi\frac{(1-\tau)w_t\varepsilon_{t-1}}{\kappa}\right)^{\frac{1}{\psi}} c\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)^{\frac{\phi(1-\sigma)-1}{\psi}} h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{\psi}} \tag{101}$$

4. Then the effective asset holding can be solved from budget constraint

$$\begin{aligned} x_{t-1}\left(h_{t-1}, x_t, \varepsilon_{t-1}\right) = & \frac{c\left(h_{t-1}, x_t, \varepsilon_{t-1}\right) + x_t + (1-\gamma)\, p_t^h h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)}{R_t} \\ & - \frac{\left[\left(1-\delta^h\right)p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + (1-\tau)\varepsilon_{t-1} w_t l\left(h_{t-1}, x_t, \varepsilon_{t-1}\right) + T_t}{R_t} \\ & + \frac{p_t^h C\left(h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right), h_{t-1}\right)}{R_t} \end{aligned}$$

Now invert above function $x_{t-1}\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)$ to $x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$. After this invert process the function $h_t\left(h_{t-1}, x_t, \varepsilon_{t-1}\right)$ can be mapped to $h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ by the function $x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$.

5. Assuming the collateral constraint is constrained, I further define the relative Khun-Tucker multiplier as $\widetilde{\mu}_t(h_t, 0, \varepsilon_{t-1}) = \frac{\mu_t}{\widetilde{V}_x(h_t, 0, \varepsilon_{t-1})}$ so that equation 78 becomes

$$U_{c,t} = \left(1+\widetilde{\mu}_t\right)\widetilde{V}_x$$

Therefore the equation 99 changes to

$$U_{h,t} = (1-\phi)\left(\frac{\left(1+\widetilde{\mu}_t\right)\widetilde{V}_x}{\phi}\right)^{\frac{\phi(1-\sigma)}{\phi(1-\sigma)-1}} h_t^{\frac{\phi(1-\phi)(1-\sigma)^2}{1-\phi(1-\sigma)}+(1-\phi)(1-\sigma)-1}$$

Similar to the process in step 3 this can be used to solve $h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ from

$$F\left(h_t, \widetilde{\mu}_t, \varepsilon_{t-1}, h_{t-1}\right) = \frac{1}{1+\widetilde{\mu}_t} \frac{U_{h,t} + \widetilde{V}_h}{p_t^h \widetilde{V}_x} - \left(1 - \gamma + C_{h,t}\right) = 0$$

and equation 100 changes to

$$c\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) = \left(\frac{\left(1+\widetilde{\mu}_t\right) \widetilde{V}_x(h_t, 0, \varepsilon_{t-1})}{\phi h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)^{(1-\phi)(1-\sigma)}}\right)^{\frac{1}{\phi(1-\sigma)-1}}$$

and corresponded optimal labor supply $l\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ from equation 101.

6. The effective asset holding under the constraint scenario can be solved from budget constraint

$$\begin{aligned} x_{t-1}\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) = & \frac{c\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) + (1-\gamma) p_t^h h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)}{R_t} \\ & - \frac{\left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + (1-\tau)\varepsilon_{t-1} w_t l\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right) + T_t}{R_t} \\ & + \frac{p_t^h C\left(h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right), h_{t-1}\right)}{R_t} \end{aligned}$$

Now invert above function $x_{t-1}\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ to $\widetilde{\mu}_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$. After this invert process the function $h_t\left(h_{t-1}, \widetilde{\mu}_t, \varepsilon_{t-1}\right)$ can be mapped to $h_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$.[30] It is worth to notice that $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ is already known such that $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) = 0$.

7. Compare $x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ and $x_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ to select the largest elemental value. Then replace the unconstrained optimal housing service choice $h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$ with $h_t^c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)$. Then for each grid point solve the nonlinear equation

$$\begin{aligned} c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) = & \left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h\right] h_{t-1} + R_t x_{t-1} \\ & + (1-\tau) w_t \varepsilon_{t-1} \left(-\phi \frac{(1-\tau) w_t \varepsilon_{t-1}}{\kappa}\right)^{\frac{1}{\psi}} \\ & c\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)^{\frac{\phi(1-\sigma)-1}{\psi}} h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right)^{\frac{(1-\phi)(1-\sigma)}{\psi}} \\ & - p_t^h C\left(h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right), h_{t-1}\right) + T_t \\ & - x_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) - (1-\gamma) p_t^h h_t\left(h_{t-1}, x_{t-1}, \varepsilon_{t-1}\right) \end{aligned}$$

Then update the marginal value function through the envelop condition 80 and 81

$$V_h(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = U_{c,t}\left[\left(1-\delta^h\right) p_t^h - \gamma R_t p_{t-1}^h - C_{h_{t-1}}\left(h_t, h_{t-1}\right) p_t^h\right]$$

[30]Here I use c in superscript as the notation to "constrained".

$$V_x(h_{t-1}, x_{t-1}, \varepsilon_{t-1}) = U_{c,t} R_t$$

H.8 Solve Rational Expectation model with imperfect information

Following Baxter et al. (2011) and Hürtgen (2014), I first solve perfect information model

$$AY_t = BY_{t-1} + C^{\text{pseo}} \Xi_t \tag{102}$$

where $Y_t = \begin{bmatrix} s_t' & Ec_{t+1}' \end{bmatrix}'$ where s_t is the vector of state variable and c_t is the vector of control variable. Ξ_t is the vector of pseudo-shock and composed with fundamental shock w_t and noisy shock v_t such that $\Xi_t = \begin{bmatrix} w_t' & v_t' \end{bmatrix}'$. The effect of shock C^{pseo} naturally becomes $C^{\text{pseo}} = \begin{bmatrix} P^s H \eta \\ P^s \epsilon \end{bmatrix}$ where P^s is the Kalman gain from equation 97. This linear model can be easily solved by Klein (2000) to yield $Y_t = PY_{t-1} + Q\Xi_t$. Take partition on P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

It is widely known that to solve the linear rational expectation model we pre-impose the restriction that $P_{12} = 0$ and $P_{22} = 0$. Further because of the holding of CEQ under first-order perturbation method, the policy function of control variables c_t will follow

$$c_t = P_{21} s_{t-1|t-1} + Q_2^w w_t + Q_2^v v_t \tag{103}$$

where Q_2^w and Q_2^v are subset of Q^w and Q^v which comes from Q such that $Q = \begin{bmatrix} Q^w & Q^s \end{bmatrix}$. Plug equation 103 into partition of equation 102 but replace $C^{\text{pseo}}\Xi_t$ with true fundamental shock process ηw_t such that

$$A_{11} s_t + A_{12} E c_{t+1} = B_{11} s_{t-1} + B_{12} c_t + \eta w_t$$

$$A_{11} s_t + A_{12} P_{21} s_{t|t} = B_{11} s_{t-1} + B_{12} \left(P_{21} s_{t-1|t-1} + Q_2^w w_t + Q_2^v v_t \right) + \eta w_t \tag{104}$$

It is worth to notice that here I use the first ns linear equations of equation 102 which is not free of choice yet a simplification in notation. The basic purpose now is to solve the law of motion of perceived state variable $s_{t|t}$ therefore we need ns "core" linear equations related to state variables to pin down ns state variable $s_{t|t}$. The word "core" refers to those equations that affect state variables directly, or more specifically, the law of motion of state variables. For instance, if we want to select one out of two linear equations in 102, 1) Euler equation $-\sigma \widetilde{c}_t = \widetilde{R}_t - \sigma \widetilde{c}_{t+1}$ and 2) Law of Motion of Capital $K \widetilde{k}_t = I \widetilde{I}_t + K \widetilde{k}_{t-1}$, which is used in equation 104, we should select the equation 2 because the equation 1 is implicitly comprised in the mapping from $s_{t-1|t-1}$ to c_t in equation 103. Otherwise we redundantly use the linear constraints and the matrix $A_{11}+A_{12}P_{21}G$ in equation 107 will not be well-defined.

Furthermore, the law of motion of perception of unobservable variables could be derived through plugging equation 96 into equation 97 to yield

$$X_{t|t}=A^sX_{t-1|t-1}+P^sHX_t+P^s\epsilon v_t \tag{105}$$

However, It is not all the state variables s_t that is unobservable, so I rewrite the law of motion of perceived state variable $s_{t|t}$ below. Without loss of generality, I assume the unobservable state variables lay on the last nx row (in this paper $nx=10$ as equation 93 shows).

$$s_{t|t}=Fs_{t-1|t-1}+Gs_t+G_{P^s}\epsilon v_t \tag{106}$$

where $F=\begin{bmatrix}0&0\\0&A^s\end{bmatrix}$, $G=\begin{bmatrix}I&0\\0&P^sH\end{bmatrix}$ and $G_{P^s}=\begin{bmatrix}0\\P^s\end{bmatrix}$.

And then plug equation 106 back to above equation 104

$$A_{11}s_t+A_{12}P_{21}\left(Fs_{t-1|t-1}+Gs_t+G_{P^s}\epsilon v_t\right)=B_{11}s_{t-1}+B_{12}\left(P_{21}s_{t-1|t-1}+Q_2^w w_t+Q_2^v v_t\right)+\eta w_t$$

$$\begin{aligned}\left(A_{11}+A_{12}P_{21}G\right)s_t=B_{11}s_{t-1}+\left(B_{12}P_{21}-A_{12}P_{21}F\right)s_{t-1|t-1}+\left(B_{12}Q_2^w+\eta\right)w_t\\+\left(B_{12}Q_2^v-A_{12}P_{21}G_{P^s}\epsilon\right)v_t\end{aligned} \tag{107}$$

Simplify above equation to

$$\widetilde{Y}_t=M\widetilde{Y}_{t-1}+D\Xi_t \tag{108}$$

where

$$\widetilde{Y}_t=\begin{bmatrix}s_t\\s_{t|t}\\c_t\end{bmatrix}$$

$$A_L=\begin{bmatrix}I&0&0\\-G&I&0\\0&0&I\end{bmatrix}$$

$$B_L=\begin{bmatrix}\widetilde{P}_{11}&\widetilde{P}_{12}&0\\0&F&0\\0&P_{21}&0\end{bmatrix}$$

$$C_L=\begin{bmatrix}\widetilde{Q}_{11}&\widetilde{Q}_{12}\\0&P^s\epsilon\\Q_2^w&Q_2^v\end{bmatrix}$$

$M=A_L^{-1}B_L$, $D=A_L^{-1}C_L$, $\widetilde{P}_{11}=\left(A_{11}+A_{12}P_{21}G\right)^{-1}B_{11}$,

$\widetilde{P}_{12} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}P_{21} - A_{12}P_{21}F)$, $\widetilde{Q}_{11} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}Q_2^w + \eta)$
and
$\widetilde{Q}_{12} = (A_{11} + A_{12}P_{21}G)^{-1}(B_{12}Q_2^v - A_{12}P_{21}G_{P^s\epsilon})$.

H.9 Solve Rational Expectation model with imperfect information in second order

H.9.1 Necessity

Given the utility function $U_t(c_t, h_t)$ where c_t is the nondurable consumption and h_t is the residential asset, we can take taylor expansion around the steady states to yield

$$U_t(c_t, h_t) \approx \overline{U} + U_c\widetilde{c}_t + U_h\widetilde{h}_t + \frac{1}{2}U_{cc}\widetilde{c}_t^2 + \frac{1}{2}U_{hh}\widetilde{h}_t^2 + U_{hc}\widetilde{c}_t\widetilde{h}_t + \circ_t$$

where $\circ_t$ is the higher order term. However I cannot use $\widetilde{c}_t$ as the result in first order because of two reason:

1) the precautionary saving motive will disappear as now $\frac{\partial\widetilde{c}_t}{\partial\sigma^2} = 0$. Then the quadratic term will be misspecified in dynamic path and the calculated welfare will be incorrect.

2) In the heterogeneous agent model, there is no steady state for each household and above taylor expansion will not exist.

Therefore I propose the method below to conduct the second-order perturbation under imperfect information.

The main trick I used is that the certainty equivalence will still hold, only in the information updated process in second order perturbation. Now consider the policy function as

$$\begin{aligned} y_t = {} & p_1y_{t-1} + p_2y_{t-1}^2 + \sigma p_3y_{t-1}\varepsilon_t + k_1x_{t-1|t-1} + k_2x_{t-1|t-1}^2 + k_3y_{t-1}x_{t-1|t-1} + k_4x_{t-1|t-1}\varepsilon_t \\ & + q_1\sigma\varepsilon_t + q_2\sigma^2 \end{aligned}$$

where y_t is the standard variables that we know it perfectly but x_t is the variable that we cannot perfectly observe. σ^2 represents the change in the variance of shock term and q_2 is just the precautionary saving effect.

The only difference between imperfect information model and perfect information model is that all the policy related to perception, k_1, k_2, k_3... are affected by σ^2 as people form their expectation through the variance of the shock. However, because it is affect by the quadratic form of variance, σ^2, instead of standard derivation σ, its final effect is third order and in second order case. For instance, $\frac{\partial k_1}{\partial\sigma}\Big|_{\sigma=0} = 0$ holds, therefore $\frac{\partial^2 k_1x_{t-1|t-1}}{\partial\sigma\partial\sigma} = 0$ at steady states.

H.9.2 Steps

Write the system of equations as

$$G(x_{t-1}, y_t, x_t, y_{t+1}, \sigma) = F(x_{t-1}, u_t, u_{t+1}, \sigma) = 0$$

However since the η can be calculated from the covariance matrix of the shock ϵ_t (a shock on the variance of the model. It is a nk vector yet if we consider it is the shock on the variance u_t, we can set some elements in ϵ_t as zero), we can leave it into Σ_ϵ.

Take second-order approximation

$$\begin{aligned}
F(x_{t-1}, u_t, u_{t+1}, \sigma) = {} & F^1(x_{t-1}, u_t, u_{t+1}, \sigma) \\
& + \frac{1}{2}\left[F_{xx}\left(x_{t-1} \otimes x_{t-1}\right) + F_{uu}\left(u \otimes u\right) + F_{u'u'}\left(u' \otimes u'\right) + F_{\sigma\sigma}\sigma^2\right] \\
& + F_{xu}\left(x \otimes u\right) + F_{xu'}\left(x \otimes u'\right) + F_{y\sigma}\sigma x + F_{uu'}\left(u \otimes u'\right) + F_{u\sigma}u_t\sigma + F_{u'\sigma}u'\sigma
\end{aligned}$$

Because u and u' are the linear innovation to the state variable x and x', F_u is just a constant matrix such that $F_u = G_{x'}\frac{\partial x'}{\partial u} + G_y\frac{\partial y}{\partial u} + G_{y'}\frac{\partial y'}{\partial x}\frac{\partial x}{\partial u}$. This can be verified through the second-order policy functions

$$x_t = \frac{1}{2}h_{\sigma\sigma}\sigma^2 + h_x x_{t-1} + h_u u_t + \frac{1}{2}h_{xx}\left(x_{t-1} \otimes x_{t-1}\right) + \frac{1}{2}h_{uu}\left(u_t \otimes u_t\right) + h_{xu}\left(x_{t-1} \otimes u_t\right)$$

and

$$y_t = \frac{1}{2}g_{\sigma\sigma}\sigma^2 + g_x x_{t-1} + g_u u_t + \frac{1}{2}g_{xx}\left(x_{t-1} \otimes x_{t-1}\right) + \frac{1}{2}g_{uu}\left(u_t \otimes u_t\right) + g_{xu}\left(x_{t-1} \otimes u_t\right)$$

$$y_{t+1} = \frac{1}{2}g_{\sigma\sigma}\sigma^2 + g_x x_t + g_u u_{t+1} + \frac{1}{2}g_{xx}\left(x_t \otimes x_t\right) + \frac{1}{2}g_{uu}\left(u_{t+1} \otimes u_{t+1}\right) + g_{xu}\left(x_t \otimes u_{t+1}\right)$$

Therefore $F_{yu} = F_{yu'} = F_{uu'} = F_{u\sigma}u_t = F_{u'\sigma} = 0$. Simplify to

$$\begin{aligned}
\mathbb{E}_t\left\{F(x_{t-1}, u_t, u_{t+1}, \sigma)\right\} = {} & \mathbb{E}_t\left\{F^1(x_{t-1}, u_t, u_{t+1}, \sigma)\right\} \\
& + \frac{1}{2}\left[F_{xx}\left(x_{t-1} \otimes x_{t-1}\right) + F_{uu}\left(u \otimes u\right) + F_{u'u'}\sigma^2\overrightarrow{\Sigma}_\epsilon + F_{\sigma\sigma}\sigma^2\right] \\
& + F_{xu}\left(x \otimes u\right) + F_{u\sigma}u_t\sigma + F_{y\sigma}\sigma x
\end{aligned}$$

To understand the $\overrightarrow{\Sigma}_\epsilon$ and $\overline{\sigma} = 0$, let use write u_t as $u_t = \varepsilon_t + \sigma\epsilon_t$ where $\Sigma_\varepsilon = I$ and $\overrightarrow{\Sigma}_\epsilon = \operatorname{vec}\left(\Sigma_\epsilon\right)$. The shock ε_t represents the first order shock that household does not take into account its variance into policy function (yet it indeed has the variance). ϵ_t is the second order shock that household takes into account its variance and has precautionary saving motive. Therefore the existence of $\overrightarrow{\Sigma}_\epsilon$ matches that meaning that we only care about the add-on variance of u_t that has second order effect. Therefore the first order effect of u_t or u_{t+1} is zero (or even not zero is already considered in $F^1(x_{t-1}, u_t, u_{t+1}, \sigma)$).

Further, the chain rule in partial derivative can only work when the "differential point" is fixed. For instance, the condition

$$x_{t-1} = \frac{1}{2}h_{\sigma\sigma}\sigma^2 + h_x x_{t-2} + h_u u_{t-1} + \frac{1}{2}h_{xx}\left(x_{t-2}\otimes x_{t-2}\right) + \frac{1}{2}h_{uu}\left(u_{t-1}\otimes u_{t-1}\right) + h_{xu}\left(x_{t-1}\otimes u_{t-1}\right)$$

also hold. Does $\frac{\partial G}{\partial \sigma^2} = \ldots + \frac{\partial G}{\partial x_{t-1}}\frac{\partial x_{t-1}}{\partial x_{t-2}}\frac{\partial x_{t-2}}{\partial \sigma^2}$ hold? NO! Because $\frac{\partial x_{t-1}}{\partial x_{t-2}}$ and $\frac{\partial x_{t-2}}{\partial \sigma^2}$ exist is conditional on the the condition that we know x_{t-2}, which we do not know.

Now let me solve them one by one. Firstly, write the function of x_t, y_t and y_{t+1}[31]

$$\begin{aligned}
F_{xx} &= G_y g_{xx} + G_{x'} h_{xx} + G_{y'}\left[g_x h_{xx} + g_{xx}\left(h_x \otimes h_x\right)\right] \\
&+ G_{xx}\left(I_{nk}\otimes I_{nk}\right) + G_{xy}\left(I_{nk}\otimes g_x\right) + G_{xx'}\left(I_{nk}\otimes h_x\right) + G_{xy'}\left(I_{nk}\otimes g_x h_x\right) \\
&+ G_{yx}\left(g_x \otimes I_{nk}\right) + G_{yy}\left(g_x\otimes g_x\right) + G_{yx'}\left(g_x\otimes h_x\right) + G_{yy'}\left(g_x\otimes g_x h_x\right) \\
&+ G_{x'x}\left(h_x\otimes I_{nk}\right) + G_{x'y}\left(h_x\otimes g_x\right) + G_{x'x'}\left(h_x\otimes h_x\right) + G_{x'y'}\left(h_x\otimes g_x h_x\right) \\
&+ G_{y'x}\left(g_x h_x\otimes I_{nk}\right) + G_{y'y}\left(g_x h_x\otimes g_x\right) + G_{y'x'}\left(g_x h_x\otimes h_x\right) + G_{y'y'}\left(g_x h_x\otimes g_x h_x\right) \\
&= 0
\end{aligned}$$

Rewrite it as

$$\begin{bmatrix} G_{x'} + G_{y'} g_x & G_y \end{bmatrix}\begin{bmatrix} h_{xx} \\ g_{xx}\end{bmatrix} + \begin{bmatrix} 0 & G_{y'}\end{bmatrix}\begin{bmatrix} h_{xx} \\ g_{xx}\end{bmatrix}\left(h_x\otimes h_x\right) + B_x = 0$$

Secondly

$$\begin{aligned}
F_{uu} &= G_y g_{uu} + G_{x'} h_{uu} + G_{y'}\left[g_x h_{uu} + g_{xx}\left(h_u\otimes h_u\right)\right] \\
&+ G_{yy}\left(g_u\otimes g_u\right) + G_{yx'}\left(g_u\otimes h_u\right) + G_{yy'}\left(g_u\otimes g_x h_u\right) \\
&+ G_{x'y}\left(h_u\otimes g_u\right) + G_{x'x'}\left(h_u\otimes h_u\right) + G_{x'y'}\left(h_u\otimes g_x h_u\right) \\
&+ G_{y'x'}\left(g_x h_u\otimes h_u\right) + G_{y'y}\left(g_x h_u\otimes g_u\right) + G_{y'y'}\left(g_x h_u\otimes g_x h_u\right) \\
&= 0
\end{aligned}$$

Rewrite it as

$$\begin{bmatrix} G_{x'} + G_{y'} g_x & G_y \end{bmatrix}\begin{bmatrix} h_{uu} \\ g_{uu}\end{bmatrix} + B_{u1} = 0$$

31 $\frac{1}{2}\frac{\partial^2 h_{xx}(x_{t-1}\otimes x_{t-1})}{\partial x_{t-1}\partial x_{t-1}} = \frac{1}{2}2h_{xx} = h_{xx}$

Thirdly

$$\begin{aligned}F_{xu} &= G_y g_{xu} + G_{x'} h_{xu} + G_{y'} \left[g_x h_{xu} + g_{xx} \left(h_x \otimes h_u \right) \right] \\ &+ G_{xy} \left(I_{nk} \otimes g_u \right) + G_{xx'} \left(I_{nk} \otimes h_u \right) + G_{xy'} \left(I_{nk} \otimes g_x h_u \right) \\ &+ G_{yy} \left(g_x \otimes g_x \right) + G_{yx'} \left(g_x \otimes h_u \right) + G_{yy'} \left(g_x \otimes g_x h_u \right) \\ &+ G_{x'y} \left(h_x \otimes g_u \right) + G_{x'x'} \left(h_x \otimes h_u \right) + G_{x'y'} \left(h_x \otimes g_x h_u \right) \\ &+ G_{y'y} \left(g_x h_x \otimes g_u \right) + G_{y'x'} \left(g_x h_x \otimes h_u \right) + G_{y'y'} \left(g_x h_x \otimes g_x h_u \right) \\ &= 0\end{aligned}$$

Rewrite it as

$$\left[\begin{array}{cc} G_{x'} + G_{y'} g_x & G_y \end{array} \right] \left[\begin{array}{c} h_{xu} \\ g_{xu} \end{array} \right] + B_{u2} = 0$$

Forthly

$$F_{\sigma\sigma} = G_x h_{\sigma\sigma} + G_y \left[g_{\sigma\sigma} + g_x h_{\sigma\sigma} \right] + G_{x'} \left[h_{\sigma\sigma} + h_x h_{\sigma\sigma} \right] + G_{y'} \left[g_{\sigma\sigma} + g_x h_{\sigma\sigma} + g_x h_x h_{\sigma\sigma} \right]$$

where h_{u,σ^2} and g_{u,σ^2} is solved from the perturbation around the first order policy function. Even though $u_t = \varepsilon_t + \sigma\epsilon_t$, because at time t u_t is already realized, there is no expectation in front ϵ_t, $G_{yy} \left(g_u \otimes g_u \right) \left(\epsilon_t \otimes \epsilon_t \right) = G_y g_{uu} \left(I_{nu} \otimes I_{nu} \right) \left(\epsilon_t \otimes \epsilon_t \right) = ... = 0$ will hold around the steady state $\epsilon = 0$. Throughout the calculation of F_{xx}, F_{uu}, F_{xu} and $F_{\sigma\sigma}$, we do not need to care about the shock coefficient η because $G_{uu} = 0$. All of its effect is already implied in h_u and g_u.

Furthermore, there is no higher order expectation effect here (up to second order) such as $G_y g_{u,\sigma^2} \overline{u} + G_{x'} h_{u,\sigma^2} \overline{u} + G_{y'} \left[g_{u,\sigma^2} + g_x h_{u,\sigma^2} \right] \overline{u}$ as $\overline{u} = 0$. Yet higher order approximation will have this problem. Meanwhile remember that in first order even we have $\overline{u} > 0$, because $\overline{\sigma} = 0$, the first order effect $G_y g_{u,\sigma} \overline{u\sigma} = h_{u,\sigma} \overline{u\sigma} = 0$. The reason is that the policy will not derivative until second order or higher because of σ^2, the variance is second order. Then the effect of this derivation, derivation in dynamic with x_t or $x_t \otimes x_t$, is at least third-order which will be zero under second-order approximation.

and

$$F_{u'u'} = G_{y'} g_{uu} + G_{y'y'} \left(g_u \otimes g_u \right)$$

Therefore

$$F_{u'u'} \overrightarrow{\Sigma}_\epsilon \sigma^2 + F_{\sigma\sigma} \sigma^2 = \left(F_{u'u'} \overrightarrow{\Sigma}_\epsilon + F_{\sigma\sigma} \right) \sigma^2 = 0$$

holds, which is equivalent to

$$F_{u'u'} \overrightarrow{\Sigma}_\epsilon + F_{\sigma\sigma} = 0$$

Rearrange to

$$\left[\begin{array}{cc} G_x + G_{x'} + G_{y'} g_x & G_y + G_{y'} \end{array}\right] \left[\begin{array}{c} h_{\sigma\sigma} \\ g_{\sigma\sigma} \end{array}\right] + \left\{ G_{y'} g_{uu} + G_{y'y'} \left(g_u \otimes g_u \right) \right\} \overrightarrow{\Sigma}_{\epsilon} = 0$$

Taylor expansion around

$$K(z_{t-1}, u_t, u_{t+1}, \sigma) = L(z_{t-1}, y_t, z_t, y_{t+1}, \sigma) = 0$$

Guess policy function

$$z_t = \frac{1}{2} p_{\sigma\sigma} \sigma^2 + p_z z_{t-1} + p_u u_t + \frac{1}{2} p_{zz} \left(z_{t-1} \otimes z_{t-1} \right) + \frac{1}{2} p_{uu} \left(u_t \otimes u_t \right) + p_{zu} \left(z_{t-1} \otimes u_t \right)$$

where $z_t = \left[\begin{array}{c} x_t \\ x_{t|t} \end{array}\right]$ with the known function

$$\begin{aligned} y_t &= \frac{1}{2} g_{\sigma\sigma} \sigma^2 + g_x x_{t-1|x-1} + g_u u_t + \frac{1}{2} g_{xx} \left(x_{t-1|t-1} \otimes x_{t-1|t-1} \right) + \frac{1}{2} g_{uu} \left(u_t \otimes u_t \right) + g_{xu} \left(x_{t-1|t-1} \otimes u_t \right) \\ &= \frac{1}{2} g_{\sigma\sigma} \sigma^2 + g_x m_2 z_{t-1} + g_u u_t + \frac{1}{2} g_{xx} \left(m_2 \otimes m_2 \right) \left(z_{t-1} \otimes z_{t-1} \right) + \frac{1}{2} g_{uu} \left(u_t \otimes u_t \right) + g_{xu} \left(m_2 \otimes I_{nu} \right) \left(z_{t-1} \otimes \right. \end{aligned}$$

and

$$\begin{aligned} y_{t+1} &= \frac{1}{2} g_{\sigma\sigma} \sigma^2 + g_x x_{t|t} + g_u u_{t+1} + \frac{1}{2} g_{xx} \left(x_{t|t} \otimes x_{t|t} \right) + \frac{1}{2} g_{uu} \left(u_{t+1} \otimes u_{t+1} \right) + g_{xu} \left(x_{t|t} \otimes u_{t+1} \right) \\ &= \frac{1}{2} g_{\sigma\sigma} \sigma^2 + g_x m_2 z_t + g_u u_{t+1} + \frac{1}{2} g_{xx} \left(m_2 \otimes m_2 \right) \left(z_t \otimes z_t \right) + \frac{1}{2} g_{uu} \left(u_{t+1} \otimes u_{t+1} \right) + g_{xu} \left(m_2 \otimes I_{nu} \right) \left(z_t \otimes \right. \end{aligned}$$

where $m_2 = \left[\begin{array}{cc} 0_{nk} & I_{nk} \end{array}\right]$

Take second-order approximation

$$\begin{aligned} K(z_{t-1}, u_t, u_{t+1}, \sigma) = {} & K^1(z_{t-1}, u_t, u_{t+1}, \sigma) \\ & + \frac{1}{2} \left[K_{zz} \left(z \otimes z \right) + K_{uu} \left(u \otimes u \right) + K_{u'u'} \left(u' \otimes u' \right) + K_{\sigma\sigma} \sigma^2 \right] \\ & + K_{zu} \left(z \otimes u \right) + K_{zu'} \left(z \otimes u' \right) + K_{z\sigma} \sigma z + K_{uu'} \left(u \otimes u' \right) + K_{u\sigma} u_t \sigma + K_{u'\sigma} u' \sigma \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}_t \left\{ K(z_{t-1}, u_t, u_{t+1}, \sigma) \right\} = {} & \mathbb{E}_t \left\{ K^1(z_{t-1}, u_t, u_{t+1}, \sigma) \right\} \\ & + \frac{1}{2} \left[K_{zz} \left(z \otimes z \right) + F_{uu} \left(u \otimes u \right) + F_{u'u'} \overrightarrow{\Sigma}_{\epsilon} \sigma^2 + F_{\sigma\sigma} \sigma^2 \right] \\ & + K_{zu} \left(z \otimes u \right) + K_{z\sigma} \sigma z + K_{u\sigma} u_t \sigma \end{aligned}$$

Now let me solve them one by one. Firstly, write the function of x_t, y_t and y_{t+1}

$$\begin{aligned}
K_{zz} &= L_y g_{xx} \left(m_2 \otimes m_2\right) + L_{z'} p_{zz} + L_{y'} \left[g_x m_2 p_{zz} + g_{xx} \left(h_x \otimes h_x\right) \left(m_2 \otimes m_2\right) \left(p_z \otimes p_z\right)\right] \\
&+ L_{zz} \left(I_{2nk} \otimes I_{2nk}\right) + L_{zy} \left(I_{2nk} \otimes g_x m_2\right) + L_{zz'} \left(I_{2nk} \otimes p_z\right) + L_{zy'} \left(I_{2nk} \otimes g_x m_2 p_z\right) \\
&+ L_{yz} \left(g_x m_2 \otimes I_{2nk}\right) + L_{yy} \left(g_x m_2 \otimes g_x m_2\right) + L_{yz'} \left(g_x m_2 \otimes p_z\right) + L_{yy'} \left(g_x m_2 \otimes g_x m_2 p_z\right) \\
&+ L_{z'z} \left(p_z \otimes I_{2nk}\right) + L_{z'y} \left(p_z \otimes g_x m_2\right) + L_{z'z'} \left(p_z \otimes p_z\right) + L_{z'y'} \left(p_z \otimes g_x m_2 p_z\right) \\
&+ L_{y'z} \left(g_x m_2 p_z \otimes I_{2nk}\right) + L_{y'y} \left(g_x m_2 p_z \otimes g_x m_2\right) + L_{y'z'} \left(g_x m_2 p_z \otimes p_z\right) + L_{y'y'} \left(g_x m_2 p_z \otimes g_x m_2 p_z\right) \\
&= 0
\end{aligned}$$

Then p_{zz} is solved by

$$\left(L_{z'} + L_{y'} g_x m_2\right) p_{zz} + C_x = 0$$

Secondly,

$$\begin{aligned}
K_{uu} &= L_y g_{uu} + L_{z'} p_{uu} + L_{y'} \left[g_x m_2 p_{uu} + g_{xx} \left(m_2 \otimes m_2\right) \left(p_u \otimes p_u\right)\right] \\
&+ L_{yy} \left(g_u \otimes g_u\right) + L_{yz'} \left(g_u \otimes p_u\right) + L_{yy'} \left(g_u \otimes g_x m_2 p_u\right) \\
&+ L_{z'y} \left(p_u \otimes g_u\right) + L_{z'z'} \left(p_u \otimes p_u\right) + L_{z'y'} \left(p_u \otimes g_x m_2 p_u\right) \\
&+ L_{y'y} \left(g_x m_{2v} p_u \otimes g_u\right) + L_{y'z'} \left(g_x m_2 p_u \otimes p_u\right) + L_{y'y'} \left(g_x m_2 p_u \otimes g_x m_2 p_u\right) \\
&= 0
\end{aligned}$$

Then p_{uu} is solved by

$$\left(L_{z'} + L_{y'} g_x m_2\right) p_{uu} + C_{u1} = 0$$

Thirdly,

$$\begin{aligned}
K_{zu} &= L_y g_{xu} \left(m_2 \otimes I_{nu}\right) + L_{z'} p_{zu} + L_{y'} \left[g_x m_2 p_{zu} + g_{xx} \left(m_2 \otimes m_2\right) \left(p_z \otimes p_u\right)\right] \\
&+ L_{zy} \left(I_{2nk} \otimes g_u\right) + L_{zz'} \left(I_{2nk} \otimes p_u\right) + L_{zy'} \left(I_{2nk} \otimes g_x m_2 p_u\right) \\
&+ L_{yy} \left(g_x m_2 \otimes g_u\right) + L_{yz'} \left(g_x m_2 \otimes p_u\right) + L_{yy'} \left(g_x m_2 \otimes g_x m_2 p_u\right) \\
&+ L_{z'y} \left(p_z \otimes g_u\right) + L_{z'z'} \left(p_z \otimes p_u\right) + L_{z'y'} \left(p_z \otimes g_x m_2 p_u\right) \\
&+ L_{y'y} \left(g_x m_2 p_z \otimes g_u\right) + L_{y'z'} \left(g_x m_2 p_z \otimes p_u\right) + L_{y'y'} \left(g_x m_2 p_z \otimes g_x m_2 p_u\right) \\
&= 0
\end{aligned}$$

Then p_{zu} is solved by

$$\left(L_{z'} + L_{y'} g_x m_2\right) p_{zu} + C_{u2} = 0$$

Finally we have two approximations

$$K_{\sigma\sigma} = L_z p_{\sigma\sigma} + L_y \left[g_{\sigma\sigma} + g_x m_2 p_{\sigma\sigma}\right] + L_{z'} \left[p_{\sigma\sigma} + p_z p_{\sigma\sigma}\right] + L_{y'} \left[g_{\sigma\sigma} + g_x m_2 p_{\sigma\sigma} + g_x m_2 p_z p_{\sigma\sigma}\right]$$

and

$$K_{u'u'} = L_{y'} g_{uu} + L_{y'y'} \left(g_u \otimes g_u \right)$$

Because of

$$K_{\sigma\sigma} + K_{u'u'} \overrightarrow{\Sigma}_{\epsilon} = 0$$

The $p_{\sigma\sigma}$ is solved by

$$\left[L_z + L_y g_x m_2 + L_{z'} \left(1 + p_z \right) + L_{y'} g_x m_2 \left(1 + p_z \right) \right] p_{\sigma\sigma} + C_{\sigma} = 0$$