Principled priors for Bayesian inference of circular models

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SUPPLEMENTARY MATERIAL

A Appendix: Derivation of PC Priors

A.1 The PC Prior for the Concentration Parameter of Von Mises Distribution

A.1.1 With Circular Uniform Base Model

With the base model is at $\kappa_0 = 0$, then the KLD in Equation 3.3 becomes

$$\text{KLD}\left(\mathcal{VM} \middle\| \mathcal{VM}_0\right)|_{\kappa_0=0} = \frac{\kappa \mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)} - \log\left(\mathcal{I}_0\left(\kappa\right)\right). \tag{1}$$

Then, the distance is

$$d\left(\kappa\right) = \sqrt{\frac{\kappa \mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)} - \log\left(\mathcal{I}_0\left(\kappa\right)\right)}, \tag{2}$$

and its derivative is

$$\frac{\partial d(\kappa)}{\partial \kappa} = \frac{\frac{\kappa(\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa))}{2\mathcal{I}_0(\kappa)} - \frac{\kappa \mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2}}{2d(\kappa)}, \tag{3}$$

which is positive for $\kappa \in [0, \infty)$. Thus,

$$
\begin{aligned}
\pi(\kappa) &= \lambda \exp\{-\lambda d(\kappa)\} \left|\frac{\partial d(\kappa)}{\partial \kappa}\right| = \lambda \exp\{-\lambda d(\kappa)\} \frac{\frac{\kappa(\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa))}{2\mathcal{I}_0(\kappa)} - \frac{\kappa \mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2}}{2d(\kappa)} \\
&= \lambda \exp\left\{-\lambda \sqrt{\frac{\kappa \mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)} - \log\left(\mathcal{I}_0(\kappa)\right)}\right\} \frac{\frac{\kappa(\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa))}{2\mathcal{I}_0(\kappa)} - \frac{\kappa \mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2}}{2\sqrt{\frac{\kappa \mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)} - \log\left(\mathcal{I}_0(\kappa)\right)}},
\end{aligned} \tag{4}
$$

which is given in Proposition 3.1.

A.1.2 With Point Mass Base Model

If $\kappa_0 \gg \kappa$ is chosen as the base model, we derive the corresponding PC prior following the approach given in Appendix A.1 of Simpson et al. (2017). The derivation start with approximating the KLD using an asymptotic expansion for $\log\left(\mathcal{I}_0\left(\kappa_0\right)\right)$ as $\kappa_0 \to \infty$:

$$
\log\left(\mathcal{I}_0\left(\kappa_0\right)\right) \overset{\kappa_0 \to \infty}{\sim} \kappa_0 - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\kappa_0\right) + \frac{1}{8\kappa_0} + \mathcal{O}\left(\frac{1}{\kappa_0^2}\right). \tag{5}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{KLD}\left(\mathcal{VM} \| \mathcal{VM}_0\right)}{\kappa_0} &\overset{\kappa_0 \to \infty}{\sim} -\frac{\log\left(\mathcal{I}_0\left(\kappa\right)\right)}{\kappa_0} + \frac{\log\left(\mathcal{I}_0\left(\kappa_0\right)\right)}{\kappa_0} + \left(\kappa - \kappa_0\right)\frac{\mathcal{I}_1(\kappa)}{\kappa_0 \mathcal{I}_0(\kappa)} \\
&= 1 - \frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)} + \mathcal{O}\left(\frac{1}{\kappa_0^2}\right).
\end{aligned} \tag{6}
$$

Notice that for any PC prior with the KLD in the form of $\frac{\mathrm{KLD}}{g(\xi_0)} = h(\xi)$ where $g(\xi_0)$ is a function of ξ_0 and $h(\xi)$ is a function of ξ, by setting $d(\xi) = \sqrt{h(\xi)}$, we can re-define the PC prior density as

$$
d(\xi) = \tilde{\lambda} \exp\left\{-\tilde{\lambda}\frac{d(\xi)}{\sqrt{g(\xi_0)}}\right\} \left|\frac{1}{\sqrt{g(\xi_0)}}\frac{\partial d(\xi)}{\partial \xi}\right|. \tag{7}
$$

Then, with $\lambda = \frac{\tilde{\lambda}}{\sqrt{g(\xi_0)}}$, this formulation is equivalent to Equation 3.2.

For the KLD given in Equation 6, $g(\kappa_0) = \kappa_0$, and $h(\kappa) = 1 - \frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)}$. Thus, the distance is

$$
d(\kappa) = \sqrt{1 - \frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)}}, \tag{8}
$$

and its derivative is

$$
\frac{\partial d(\kappa)}{\partial \kappa} = \frac{\left(\frac{\mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2} - \frac{\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa)}{2\mathcal{I}_0(\kappa)}\right)}{2d(\kappa)}, \tag{9}
$$

which is negative for $\kappa \in [0, \infty)$. The derivation of the pdf of this PC prior (Proposition 3.2) is then given below:

$$
\begin{aligned}
\pi(\kappa) &= \tilde{\lambda}\exp\left\{-\tilde{\lambda}\frac{d(\kappa)}{\sqrt{\kappa_0}}\right\}\left|\frac{1}{\sqrt{\kappa_0}}\frac{\partial d(\kappa)}{\partial \kappa}\right| = \lambda\exp\{-\lambda d(\kappa)\}\frac{\left(\frac{\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa)}{2\mathcal{I}_0(\kappa)}-\frac{\mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2}\right)}{2d(\kappa)} \\
&= \lambda\exp\left\{\lambda\sqrt{1-\frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)}}\right\}\frac{\left(\frac{\mathcal{I}_0(\kappa)+\mathcal{I}_2(\kappa)}{2\mathcal{I}_0(\kappa)}-\frac{\mathcal{I}_1(\kappa)^2}{\mathcal{I}_0(\kappa)^2}\right)}{2\sqrt{1-\frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)}}},
\end{aligned} \tag{10}
$$

where $\lambda = \frac{\tilde{\lambda}}{\sqrt{\kappa_0}}$.

A.2 The PC Prior for the Concentration Parameter of Cardioid Distribution

A.2.1 With Circular Uniform Base Model

With the circular uniform base model ($\ell_0 = 0$), approximating KLD through asymptotic expansion with respect to ℓ_0 yields

$$
-\frac{\ell}{\ell_0}\left(1-\sqrt{1-4\ell_0^2}\right) \overset{\ell_0\to 0_+}{\sim} -\ell\left(2\ell_0+2\ell_0^3\right)+\mathcal{O}\left(\ell_0^4\right)=0, \tag{11}
$$

and

$$
\frac{1}{2}\log\left(1-\sqrt{1-4\ell_0^2}\right) \overset{\ell_0\to 0_+}{\sim} \frac{\log(2)}{2}+\log(\ell_0)+\frac{1}{2}\ell_0^2+\mathcal{O}\left(\ell_0^4\right). \tag{12}
$$

Substituting them into Equation 3.10, we obtain

$$
\text{KLD}\left(\mathcal{C}\middle\|\mathcal{C}_0\right) \overset{\ell_0\to 0_+}{\sim} 1-\sqrt{1-4\ell^2}+\log(\ell)+\frac{1}{2}\log\left(\frac{1+\sqrt{1-4\ell^2}}{1-\sqrt{1-4\ell^2}}\right)+\mathcal{O}\left(\ell_0^4\right), \tag{13}
$$

and the distance is therefore

$$
d(\ell) = \sqrt{\left(1-\sqrt{1-4\ell^2}+\log(\ell)+\frac{1}{2}\log\left(\frac{1+\sqrt{1-4\ell^2}}{1-\sqrt{1-4\ell^2}}\right)\right)}. \tag{14}
$$

Thus,

$$
\frac{\partial d(\ell)}{\partial \ell} = \frac{2\ell}{\left(1+\sqrt{1-4\ell^2}\right)d(\ell)}, \tag{15}
$$

which is positive for $\ell \in \left[0, \frac{1}{2}\right)$. Therefore,

$$
\pi(\ell) = \lambda\exp\{-\lambda d(\ell)\}\left|\frac{\partial d(\ell)}{\partial \ell}\right| = \lambda\exp\{-\lambda d(\ell)\}\frac{2\ell}{\left(1+\sqrt{1-4\ell^2}\right)d(\ell)}. \tag{16}
$$

The support for ℓ is $\left[0, \frac{1}{2}\right)$, whilst the support for the variable of an exponential distribution is $[0, \infty)$. Thus, an additional normalizing constant is required and is given by:

$$\int_0^{\frac{1}{2}} \pi(\ell)\, d\ell = 1 - \exp\left\{-\lambda\sqrt{1-\log(2)}\right\}. \tag{17}$$

The CDF for this PC prior is

$$F(\ell) = \frac{\int \pi(\ell)\, d\ell}{\int_0^{\frac{1}{2}} \pi(\ell)\, d\ell} = \frac{1 - \exp\left\{-\lambda\sqrt{1-\sqrt{1-4\ell^2}+\log(\ell)+\frac{1}{2}\log\left(\frac{1+\sqrt{1-4\ell^2}}{1-\sqrt{1-4\ell^2}}\right)}\right\}}{1-\exp\left\{-\lambda\sqrt{1-\log(2)}\right\}}. \tag{18}$$

Therefore, the exact expression for the pdf of this PC prior is

$$\pi(\ell) = \frac{\partial F(\ell)}{\partial \ell} = \lambda \exp\{-\lambda d(\ell)\} \frac{2\ell \exp\left\{\lambda\sqrt{1-\log(2)}\right\}}{\left(\exp\left\{\lambda\sqrt{1-\log(2)}\right\}-1\right)\left(1+\sqrt{1-4\ell^2}\right) d(\ell)}. \tag{19}$$

A.2.2 With Cardioid Curve Base Model

When the base model is at $\ell_0 \to \frac{1}{2}$,

$$\lim_{\ell_0 \to \frac{1}{2}} \mathrm{KLD}\left(\mathcal{C} \middle\| \mathcal{C}_0\right) = 1 + \log(2) - 2\ell - \sqrt{1-4\ell^2} + \log(\ell) + \frac{1}{2}\log\left(\frac{1+\sqrt{1-4\ell^2}}{1-\sqrt{1-4\ell^2}}\right). \tag{20}$$

Therefore,

$$d(\ell) = \sqrt{1+\log(2)-2\ell-\sqrt{1-4\ell^2}+\log(\ell)+\frac{1}{2}\log\left(\frac{1+\sqrt{1-4\ell^2}}{1-\sqrt{1-4\ell^2}}\right)}, \tag{21}$$

and the derivative is

$$\frac{\partial d(\ell)}{\partial \ell} = -\frac{2\ell+\sqrt{1-4\ell^2}-1}{2\ell d(\ell)}, \tag{22}$$

which is negative for $\ell \in \left[0, \frac{1}{2}\right)$. Thus,

$$\pi(\ell) = \lambda \exp\{-\lambda d(\ell)\} \left|\frac{\partial d(\ell)}{\partial \ell}\right| = \lambda \exp\{-\lambda d(\ell)\} \frac{2\ell+\sqrt{1-4\ell^2}-1}{2\ell d(\ell)}. \tag{23}$$

For this PC prior, the normalizing constant is

$$\int_0^{\frac{1}{2}} \pi(\ell)\, d\ell = 1. \tag{24}$$

Therefore, the CDF is

$$F(\ell) = \exp\{-\lambda d(\ell)\}, \tag{25}$$

and the density remains to be

$$\pi\left(\ell\right) = \lambda \exp\left\{-\lambda d\left(\ell\right)\right\} \frac{2\ell + \sqrt{1-4\ell^2} - 1}{2\ell d\left(\ell\right)}. \tag{26}$$

A.3 The PC Prior for the Concentration Parameter of Wrapped Cauchy Distribution

From Equation 3.19, the distance is

$$d\left(\rho\right) = \sqrt{-\log\left(1-\rho^2\right)}, \tag{27}$$

and its derivative is

$$\frac{\partial d(\rho)}{\partial \rho} = \frac{\rho}{\left(1-\rho^2\right)\sqrt{-\log\left(1-\rho^2\right)}} = \frac{\rho}{\left(1-\rho^2\right) d(\rho)}, \tag{28}$$

which is positive for $\rho \in [0, 1)$. Therefore, the density is

$$\begin{aligned} \pi\left(\rho\right) &= \lambda \exp\left\{-\lambda d(\rho)\right\} \left|\frac{\partial d(\rho)}{\partial \rho}\right| = \lambda \exp\left\{-\lambda d(\rho)\right\} \frac{\rho}{\left(1-\rho^2\right) d(\rho)} \\ &= \lambda \exp\left\{-\lambda \sqrt{-\log\left(1-\rho^2\right)}\right\} \frac{\rho}{\left(1-\rho^2\right)\sqrt{-\log\left(1-\rho^2\right)}}. \end{aligned} \tag{29}$$

Since the normalizing constant is $\int_0^1 \pi\left(\rho\right) d\rho = 1$, the density remain the same.

In addition, to derive the expression of the beta prior in terms of distance scale, we rewrite Equation 27 as Equation 30.

$$\rho\left(d\right) = \sqrt{1 - \exp\left\{-d^2\right\}}. \tag{30}$$

Therefore,

$$\begin{aligned} \pi\left(d\right) &= \frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \rho\left(d\right)^{a-1}\left(1-\rho\left(d\right)\right)^{b-1} \left|\frac{\partial \rho\left(d\right)}{\partial d}\right| \\ &= \frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \left(\sqrt{1-\exp\left\{-d^2\right\}}\right)^{a-1} \left(1-\sqrt{1-\exp\left\{-d^2\right\}}\right)^{b-1} \frac{d \exp\left\{-d^2\right\}}{\sqrt{1-\exp\left\{-d^2\right\}}}. \end{aligned} \tag{31}$$