1. Suppose $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$ for some unknown $\sigma^2 > 0$. Let $\hat{\boldsymbol{y}} = \boldsymbol{P_X y}$. By the results on multivariate normal distributions from slide set 1 ("Preliminaries"), we have

$$\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I}).$$

(a) Notice that we want to find the distribution of a linear transformation of $\boldsymbol{y}$:

$$\begin{bmatrix} \hat{\boldsymbol{y}} \\ \boldsymbol{y} - \hat{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X y} \\ \boldsymbol{y} - \boldsymbol{P_X y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{y}. \tag{1}$$

Together, slides 13–14 of the first slide set ("Preliminaries") imply that a linear transformation, say $\boldsymbol{Ax} + \boldsymbol{b}$, of a normal random variable $\boldsymbol{x}$ is also normal:

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \implies \boldsymbol{Ax} + \boldsymbol{b} \sim \mathcal{N}(\boldsymbol{A\mu} + \boldsymbol{b}, \boldsymbol{A\Sigma A}').$$

The mean of the linear transformation in (1), since $\boldsymbol{P_X X} = \boldsymbol{X}$, is

$$\begin{aligned}
\mathrm{E}\left(\begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{y}\right) &= \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \mathrm{E}(\boldsymbol{y}) \\
&= \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{X\beta} \\
&= \begin{bmatrix} \boldsymbol{P_X X\beta} \\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{X\beta} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{P_X X\beta} \\ \boldsymbol{IX\beta} - \boldsymbol{P_X X\beta} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{X\beta} \\ \boldsymbol{X\beta} - \boldsymbol{X\beta} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{X\beta} \\ \boldsymbol{0} \end{bmatrix}.
\end{aligned}$$

The variance of (1), since $\boldsymbol{P_X}$ is symmetric and idempotent, is

$$\begin{aligned}
\operatorname{Var}\left(\begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\boldsymbol{y}\right) &= \begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\operatorname{Var}(\boldsymbol{y})\begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}' \\
&= \begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\operatorname{Var}(\boldsymbol{y})\left[\boldsymbol{P_X'},\ (\boldsymbol{I}-\boldsymbol{P_X})'\right] \\
&= \begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\sigma^2\boldsymbol{I}\left[\boldsymbol{P_X'},\ (\boldsymbol{I}-\boldsymbol{P_X})'\right] \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X}\\ \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\left[\boldsymbol{P_X'},\ (\boldsymbol{I}-\boldsymbol{P_X})'\right] \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X}\boldsymbol{P_X'} & \boldsymbol{P_X}(\boldsymbol{I}-\boldsymbol{P_X})' \\ (\boldsymbol{I}-\boldsymbol{P_X})\boldsymbol{P_X'} & (\boldsymbol{I}-\boldsymbol{P_X})(\boldsymbol{I}-\boldsymbol{P_X})'\end{bmatrix} \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X}\boldsymbol{P_X'} & \boldsymbol{P_X}(\boldsymbol{I}'-\boldsymbol{P_X'}) \\ (\boldsymbol{I}-\boldsymbol{P_X})\boldsymbol{P_X'} & (\boldsymbol{I}-\boldsymbol{P_X})(\boldsymbol{I}'-\boldsymbol{P_X'})\end{bmatrix} \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X}\boldsymbol{P_X} & \boldsymbol{P_X}(\boldsymbol{I}-\boldsymbol{P_X}) \\ (\boldsymbol{I}-\boldsymbol{P_X})\boldsymbol{P_X} & (\boldsymbol{I}-\boldsymbol{P_X})(\boldsymbol{I}-\boldsymbol{P_X})\end{bmatrix} \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X}\boldsymbol{P_X} & \boldsymbol{P_X}\boldsymbol{I}-\boldsymbol{P_X}\boldsymbol{P_X} \\ \boldsymbol{I}\boldsymbol{P_X}-\boldsymbol{P_X}\boldsymbol{P_X} & \boldsymbol{I}\boldsymbol{I}-\boldsymbol{I}\boldsymbol{P_X}-\boldsymbol{P_X}\boldsymbol{I}+\boldsymbol{P_X}\boldsymbol{P_X}\end{bmatrix} \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X} & \boldsymbol{P_X}-\boldsymbol{P_X} \\ \boldsymbol{P_X}-\boldsymbol{P_X} & \boldsymbol{I}-\boldsymbol{P_X}-\boldsymbol{P_X}+\boldsymbol{P_X}\end{bmatrix} \\
&= \sigma^2\begin{bmatrix}\boldsymbol{P_X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}.
\end{aligned}$$

As a linear transformation of a multivariate normal random variable, it follows that

$$\begin{bmatrix}\hat{\boldsymbol{y}}\\ \boldsymbol{y}-\hat{\boldsymbol{y}}\end{bmatrix}\sim\mathcal{N}\left(\begin{bmatrix}\boldsymbol{X\beta}\\ \boldsymbol{0}\end{bmatrix},\ \sigma^2\begin{bmatrix}\boldsymbol{P_X} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}-\boldsymbol{P_X}\end{bmatrix}\right).$$

<u>Comments</u>: it is important to notice that the expected values of $\boldsymbol{y}$ and $\hat{\boldsymbol{y}}$ depend on the parameter $\boldsymbol{\beta}$ and not the estimate $\hat{\boldsymbol{\beta}}$, which is a random variable. Similarly, the variances do not depend on the estimate $\hat{\sigma}^2$. Additionally, some students did not give the covariances or only stated that they were zero without any derivation (the distribution of a multivariate normal random variable is not fully specified without the covariances).

(b) Again using the fact that $\boldsymbol{P_X}$ is symmetric and idempotent, we see that $\hat{\boldsymbol{y}}'\hat{\boldsymbol{y}}$ is a quadratic form:

$$\begin{aligned}
\hat{\boldsymbol{y}}'\hat{\boldsymbol{y}} &= [\boldsymbol{P_X}\boldsymbol{y}]'\boldsymbol{P_X}\boldsymbol{y} \\
&= \boldsymbol{y}'\boldsymbol{P_X'}\boldsymbol{P_X}\boldsymbol{y} \\
&= \boldsymbol{y}'\boldsymbol{P_X}\boldsymbol{P_X}\boldsymbol{y} \\
&= \boldsymbol{y}'\boldsymbol{P_X}\boldsymbol{y},
\end{aligned}$$

where $\boldsymbol{y}\sim\mathcal{N}(\boldsymbol{X\beta},\sigma^2\boldsymbol{I})$. Recall the results about quadratic forms in the first slide set ("Preliminaries") on slide 18. To apply these, want to find a symmetric matrix $\boldsymbol{A}$ such that $\boldsymbol{A\Sigma}$ is idempotent for $\boldsymbol{\Sigma}\equiv\operatorname{Var}(\boldsymbol{y})$. We can't use $\boldsymbol{P_X}$ as our $\boldsymbol{A}$ matrix because the σ^2 doesn't cancel:

$$\boldsymbol{P_X}\operatorname{Var}(\boldsymbol{y})=\boldsymbol{P_X}\sigma^2\boldsymbol{I}=\sigma^2\boldsymbol{P_X}.$$

Instead using $\boldsymbol{A} = \frac{\boldsymbol{P_X}}{\sigma^2}$, which is symmetric, gives

$$\boldsymbol{A\Sigma} = \boldsymbol{A}\,\mathrm{Var}(\boldsymbol{y}) = \frac{\boldsymbol{P_X}}{\sigma^2}\sigma^2\boldsymbol{I} = \boldsymbol{P_X},$$

which we know is idempotent.

It is easy to verify that $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{I}$ is positive definite since $\sigma^2 > 0$ (see homework 1 solutions). We can use the result of problem 2(b) to determine the rank of $\boldsymbol{A}$:

$$\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{P_X}/\sigma^2) = \mathrm{rank}(\boldsymbol{P_X}) = \mathrm{rank}(\boldsymbol{X}).$$

Then,

$$\frac{1}{\sigma^2}\hat{\boldsymbol{y}}'\hat{\boldsymbol{y}} = \boldsymbol{y}'\frac{\boldsymbol{P_X}}{\sigma^2}\boldsymbol{y} \sim \chi^2_{\mathrm{rank}(\boldsymbol{X})}\big([\boldsymbol{X\beta}]'\boldsymbol{AX\beta}/2\big),$$

where the noncentrality parameter simplifies to

$$\begin{aligned}
[\boldsymbol{X\beta}]'\boldsymbol{AX\beta}/2 &= [\boldsymbol{X\beta}]'\frac{\boldsymbol{P_X}}{\sigma^2}\boldsymbol{X\beta}\frac{1}{2} \\
&= \frac{1}{2\sigma^2}\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{P_X}\boldsymbol{X\beta} \\
&= \frac{1}{2\sigma^2}\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta}.
\end{aligned}$$

Therefore, we end up with a scaled non-central chi-square random variable on rank($\boldsymbol{X}$) degrees of freedom:

$$\hat{\boldsymbol{y}}'\hat{\boldsymbol{y}} \sim \sigma^2\chi^2_{\mathrm{rank}(\boldsymbol{X})}\left(\frac{\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta}}{2\sigma^2}\right).$$

Comments: some students used results that apply to the sum of squared *independent* normal random variables with variance one (e.g., the result on slide 15 of set 1; this assumes $\boldsymbol{A} = \boldsymbol{\Sigma} = \boldsymbol{I}$). Others tried to find the distribution of the quadratic form $\hat{\boldsymbol{y}}\frac{\boldsymbol{I}}{\sigma^2}\hat{\boldsymbol{y}}$ rather than simplifying to $\boldsymbol{y}\frac{\boldsymbol{P_X}}{\sigma^2}\boldsymbol{y}$; unfortunately this doesn't work because $\mathrm{Var}(\hat{\boldsymbol{y}}) = \sigma^2\boldsymbol{P_X}$ may not be positive definite.

2. For problem 2 and 4, please see the solution of exam 1 in 2016.
   Available at: https://dnett.github.io/S510/exam1sol2016.pdf.

3. (a) Simplify $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$.

$\boldsymbol{X} = \begin{bmatrix}\mathbf{1}, & \boldsymbol{x}\end{bmatrix}$ and $\boldsymbol{X}' = \begin{bmatrix}\mathbf{1}' \\ \boldsymbol{x}'\end{bmatrix}$,then

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\hat{\beta}_0,\, \hat{\beta}_1)' \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} \\
&= \left(\begin{bmatrix}\mathbf{1}' \\ \boldsymbol{x}'\end{bmatrix}\begin{bmatrix}\mathbf{1}, & \boldsymbol{x}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{1}' \\ \boldsymbol{x}'\end{bmatrix}\boldsymbol{y} \\
&= \begin{bmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\boldsymbol{x} \\ \boldsymbol{x}'\mathbf{1} & \boldsymbol{x}'\boldsymbol{x}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1}'\boldsymbol{y} \\ \boldsymbol{x}'\boldsymbol{y}\end{bmatrix} \\
&= \begin{bmatrix} n & \sum\limits_{i=1}^{n} x_i \\ \sum\limits_{i=1}^{n} x_i & \sum\limits_{i=1}^{n} x_i^2\end{bmatrix}^{-1}\begin{bmatrix}\sum\limits_{i=1}^{n} y_i \\ \sum\limits_{i=1}^{n} x_i y_i\end{bmatrix} \\
&= \frac{1}{n\sum\limits_{i=1}^{n} x_i^2 - (\sum\limits_{i=1}^{n} x_i)^2}\begin{bmatrix}\sum\limits_{i=1}^{n} x_i^2 & -\sum\limits_{i=1}^{n} x_i \\ -\sum\limits_{i=1}^{n} x_i & n\end{bmatrix}\begin{bmatrix}\sum\limits_{i=1}^{n} y_i \\ \sum\limits_{i=1}^{n} x_i y_i\end{bmatrix} \\
&= \frac{1}{n\sum\limits_{i=1}^{n} x_i^2 - (\sum\limits_{i=1}^{n} x_i)^2}\begin{bmatrix}\sum\limits_{i=1}^{n} x_i^2 \sum\limits_{i=1}^{n} y_i - \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} x_i y_i \\ -\sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i + n\sum\limits_{i=1}^{n} x_i y_i\end{bmatrix}
\end{aligned}$$

Therefore,

$$\hat{\beta}_0 = \frac{\sum\limits_{i=1}^{n} x_i^2 \sum\limits_{i=1}^{n} y_i - \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} x_i y_i}{n\sum\limits_{i=1}^{n} x_i^2 - (\sum\limits_{i=1}^{n} x_i)^2}$$

$$\hat{\beta}_1 = \frac{-\sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i + n\sum\limits_{i=1}^{n} x_i y_i}{n\sum\limits_{i=1}^{n} x_i^2 - (\sum\limits_{i=1}^{n} x_i)^2}$$

(b) Find a matrix $\boldsymbol{B}$ so that $\boldsymbol{X}\boldsymbol{B}^{-1} = \boldsymbol{W} = \begin{bmatrix}\mathbf{1}, & \boldsymbol{x} - \bar{x}.\mathbf{1}\end{bmatrix}$.

Notice that $\boldsymbol{B}$ is a 2×2 matrix,

$$\underset{n\times 2}{\boldsymbol{X}}\ \underset{2\times 2}{\boldsymbol{B}}^{-1} = \underset{n\times 2}{\boldsymbol{W}} \iff \boldsymbol{X} = \boldsymbol{W}\boldsymbol{B} \text{ for non-singular } \boldsymbol{B}$$

Take $\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$\begin{aligned}
\boldsymbol{X} = \boldsymbol{W}\boldsymbol{B} &\iff \begin{bmatrix}\mathbf{1}, & \boldsymbol{x}\end{bmatrix} = \begin{bmatrix}\mathbf{1}, & \boldsymbol{x} - \bar{x}.\mathbf{1}\end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&\iff \begin{cases} \mathbf{1} = b_{11}\mathbf{1} + b_{21}(\boldsymbol{x} - \bar{x}.\mathbf{1}) \\ \boldsymbol{x} = b_{12}\mathbf{1} + b_{22}(\boldsymbol{x} - \bar{x}.\mathbf{1}) \end{cases} \\
&\iff \begin{cases} b_{11} - b_{21}\bar{x}. = 1 \\ b_{21} = 0 \\ b_{12} - b_{22}\bar{x}. = 0 \\ b_{22} = 1 \end{cases} \\
&\iff \begin{cases} b_{11} = 1 \\ b_{21} = 0 \\ b_{12} = \bar{x}. \\ b_{22} = 1 \end{cases}
\end{aligned}$$

Therefore $\boldsymbol{B} = \begin{bmatrix} 1 & \bar{x}. \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & -\bar{x}. \\ 0 & 1 \end{bmatrix}$.

(c) Derive expressions for the least squares estimators of α_0 and α_1 using $\hat{\boldsymbol{\alpha}} = (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{y}$.

$$
\begin{aligned}
\hat{\boldsymbol{\alpha}} &= (\hat{\alpha_0},\ \hat{\alpha_1})' \\
&= (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{y} \\
&= \left(\begin{bmatrix} \mathbf{1}' \\ \boldsymbol{x}' - \bar{x}.\mathbf{1}' \end{bmatrix} \begin{bmatrix} \mathbf{1}, & \boldsymbol{x} - \bar{x}.\mathbf{1} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{1}' \\ \boldsymbol{x}' - \bar{x}.\mathbf{1}' \end{bmatrix} \boldsymbol{y} \\
&= \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'(\boldsymbol{x} - \bar{x}.\mathbf{1}) \\ (\boldsymbol{x}' - \bar{x}.\mathbf{1}')\mathbf{1} & (\boldsymbol{x}' - \bar{x}.\mathbf{1}')(\boldsymbol{x} - \bar{x}.\mathbf{1}) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{1}'\boldsymbol{y} \\ \boldsymbol{x}'\boldsymbol{y} - \bar{x}.\mathbf{1}'\boldsymbol{y} \end{bmatrix} \\
&= \begin{bmatrix} n & \sum_{i=1}^{n} x_i - n\bar{x}. \\ \sum_{i=1}^{n} x_i - n\bar{x}. & \sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i - \bar{x}. \sum_{i=1}^{n} y_i \end{bmatrix} \\
&= \begin{bmatrix} n & 0 \\ 0 & \sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} (x_i - \bar{x}.) y_i \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} (x_i - \bar{x}.) y_i \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{i=1}^{n} y_i}{n} \\ \frac{\sum_{i=1}^{n} (x_i - \bar{x}.) y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2} \end{bmatrix}
\end{aligned}
$$

(d) Multiply $\hat{\boldsymbol{\alpha}}$ from part (c) by $\boldsymbol{B}^{-1}$ from part (b) to obtain expressions for $\hat{\beta}_0$ and $\hat{\beta}_1$.

$$
\begin{aligned}
(\hat{\beta}_0,\ \hat{\beta}_1)' &= \boldsymbol{B}^{-1}\hat{\boldsymbol{\alpha}} \\
&= \begin{bmatrix} 1 & -\bar{x}. \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sum_{i=1}^{n} y_i}{n} \\ \frac{\sum_{i=1}^{n} (x_i - \bar{x}.) y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2} \end{bmatrix} \\
&= \begin{bmatrix} \bar{y}. - \bar{x}. \frac{\sum_{i=1}^{n} (x_i - \bar{x}.) y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2} \\ \frac{\sum_{i=1}^{n} (x_i - \bar{x}.) y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2} \end{bmatrix}
\end{aligned}
$$

(e) Show that your answer to part (a) matches your answer to part (d).

Notice that $\sum_{i=1}^{n} y_i = n\bar{y}.$ and $\sum_{i=1}^{n} x_i = n\bar{x}..$

$$\text{In part (d), } \hat{\beta}_0 = \bar{y}. - \bar{x}. \frac{\sum_{i=1}^{n}(x_i - \bar{x}.)y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2}$$

$$= \frac{\bar{y}.(\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2) - \bar{x}.(\sum_{i=1}^{n} x_i y_i - \bar{x}. \sum_{i=1}^{n} y_i)}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i \cancel{- \bar{x}_.^2 \sum_{i=1}^{n} y_i} - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i \cancel{+ \bar{x}_.^2 \sum_{i=1}^{n} y_i}}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}(\sum_{i=1}^{n} x_i)^2}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} = \hat{\beta}_0 \text{ in part (a).}$$

$$\text{In part (d), } \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x}.)y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}_.^2}$$

$$= \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}(\sum_{i=1}^{n} x_i)^2}$$

$$= \frac{-\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i + n\sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} = \hat{\beta}_1 \text{ in part (a).}$$