1. (a)

$$
\begin{aligned}
EMS_{ou(xu,trt)} &= \frac{1}{df_{ou(xu,trt)}} \operatorname{E}\left(SS_{ou(xu,trt)}\right) \\
&= \frac{1}{tnm - tn} \operatorname{E}\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{ij\boldsymbol{.}})^2\right) \\
&= \frac{1}{tnm - tn} \operatorname{E}\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}([\mu + \tau_i + u_{ij} + e_{ijk}] - [\mu + \tau_i + u_{ij} + \bar{e}_{ij\boldsymbol{.}}])^2\right) \\
&= \frac{1}{tnm - tn} \sum_{i=1}^{t}\sum_{j=1}^{n} \operatorname{E}\left(\sum_{k=1}^{m}(e_{ijk} - \bar{e}_{ij\boldsymbol{.}})^2\right) \\
&= \frac{1}{tnm - tn} \sum_{i=1}^{t}\sum_{j=1}^{n} (m-1)\,\sigma_e^2 \qquad\qquad \text{since } e_{ijk} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_e^2\right) \\
&= \frac{1}{tnm - tn} tn\,(m-1)\,\sigma_e^2 \\
&= \sigma_e^2.
\end{aligned}
$$

(b) We can show this in a general case for t, n, m first. From slide 6, the sum of squares can be written as $\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P}_3)\boldsymbol{y}$, where

$$
\begin{aligned}
\boldsymbol{P}_3 &= [\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]\Big([\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]'[\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]\Big)^{-1}[\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]' \\
&= [\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]\Big(m\boldsymbol{I}_{tn\times tn}\Big)^{-1}[\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}_{m\times 1}]' \\
&= \frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}.
\end{aligned}
$$

Let

$$
\boldsymbol{A} = \frac{\boldsymbol{I} - \boldsymbol{P}_3}{tnm - tn} = \frac{\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}}{tn(m-1)}.
$$

Then, by linearity of trace,

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{A\Sigma}) &= \operatorname{tr}\left(\left[\frac{\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}}{tn(m-1)}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right)\\
&= \frac{1}{tn(m-1)}\operatorname{tr}\left(\left[\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right)\\
&= \frac{1}{tn(m-1)}\operatorname{tr}\Bigg[\boldsymbol{I}_{tnm\times tnm}\sigma_u^2(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}) + \boldsymbol{I}_{tnm\times tnm}\sigma_e^2\boldsymbol{I}_{tnm\times tnm}\\
&\qquad\qquad - \frac{1}{m}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m})(\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}) - \frac{1}{m}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m})\sigma_e^2\boldsymbol{I}_{tnm\times tnm}\Bigg]\\
&= \frac{1}{tn(m-1)}\operatorname{tr}\Bigg[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\\
&\qquad\qquad - \frac{\sigma_u^2}{m}(\boldsymbol{I}_{tn\times tn}\boldsymbol{I}_{tn\times tn})\otimes(\boldsymbol{11}'_{m\times m}\boldsymbol{11}'_{m\times m}) - \frac{\sigma_e^2}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}\Bigg]\\
&= \frac{1}{tn(m-1)}\operatorname{tr}\Bigg[\sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\\
&\qquad\qquad - \frac{\sigma_u^2}{m}\boldsymbol{I}_{tn\times tn}\otimes m\boldsymbol{11}'_{m\times m} - \frac{\sigma_e^2}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}\Bigg]\\
&= \frac{1}{tn(m-1)}\Bigg[\sigma_u^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}) + \sigma_e^2\operatorname{tr}(\boldsymbol{I}_{tnm\times tnm})\\
&\qquad\qquad - \sigma_u^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}) - \frac{\sigma_e^2}{m}\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m})\Bigg]\\
&= \frac{1}{tn(m-1)}\left[\sigma_e^2\operatorname{tr}(\boldsymbol{I}_{tnm\times tnm}) - \frac{\sigma_e^2}{m}\operatorname{tr}(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m})\right]\\
&= \frac{1}{tn(m-1)}\left[\sigma_e^2(tnm) - \frac{\sigma_e^2}{m}(tnm)\right]\\
&= \sigma_e^2 \quad \text{since } \boldsymbol{I}_{tnm\times tnm} \text{ and } \boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m} \text{ have 1's on the diagonal}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{y})'\boldsymbol{A}\,\mathrm{E}(\boldsymbol{y}) &= \mathrm{E}(\boldsymbol{y})'\left(\frac{\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}}{tn(m-1)}\right)\mathrm{E}(\boldsymbol{y})\\
&= \frac{1}{tn(m-1)}\mathrm{E}(\boldsymbol{y})'\left(\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m}\right)\mathrm{E}(\boldsymbol{y})\\
&= \frac{1}{tn(m-1)}\left(\mathrm{E}(\boldsymbol{y})'\boldsymbol{I}_{tnm\times tnm} - \frac{1}{m}\mathrm{E}(\boldsymbol{y})'(\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{11}'_{m\times m})\right)\mathrm{E}(\boldsymbol{y})\\
&= \frac{1}{tn(m-1)}\left(\mathrm{E}(\boldsymbol{y})' - \frac{1}{m}m\,\mathrm{E}(\boldsymbol{y})'\right)\mathrm{E}(\boldsymbol{y})\\
&= \frac{1}{tn(m-1)}\boldsymbol{0}'\,\mathrm{E}(\boldsymbol{y})\\
&= 0.
\end{aligned}
$$

Now,

$$\begin{aligned}
EMS_{ou(xu,trt)} &= \mathrm{E}\left(\boldsymbol{y}'\left(\frac{\boldsymbol{I}-\boldsymbol{P}_3}{tnm-tn}\right)\boldsymbol{y}\right) \\
&= \mathrm{E}\left(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}\right) \\
&= \mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \mathrm{E}(\boldsymbol{y})'\boldsymbol{A}\,\mathrm{E}(\boldsymbol{y}) && \text{by slide 19 of set 12} \\
&= \sigma_e^2 + 0 \\
&= \sigma_e^2,
\end{aligned}$$

The result also holds for the special case of $t = 2, n = 2, m = 2$, which is the same result as in part (a).

2. (a) Describe the distribution of these differences.
   Based on the model assumptions of $e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$, for each subject $j = 1, \cdots, 20$,

$$\begin{aligned}
d_j &= y_{1j} - y_{2j} \\
&= \mu_1 + u_j + e_{1j} - (\mu_2 + u_j + e_{2j}) \\
&= (\mu_1 - \mu_2) + e_{1j} - e_{2j}
\end{aligned}$$

   $E(d_j) = \mu_1 - \mu_2$, $Var(d_j) = Var(e_{1j}) + Var(e_{2j}) = 2\sigma_e^2$. Because a linear combination of independent normal distributions is still normal, we have $d_j \sim N(\mu_1 - \mu_2, 2\sigma_e^2)$.
   For any $j \neq j'$, $Cov(d_j, d_{j'}) = Cov(e_{1j} - e_{2j}, e_{1j'} - e_{2j'}) = 0$, so all d_j's are independent.
   Therefore $d_j \overset{iid}{\sim} N(\mu_1 - \mu_2, 2\sigma_e^2)$, which is a constant mean model.

   (b) Provide a formula for a test statistic (as a function of $d_1, \cdots, d_{20}$) to test $H_0 : \mu_1 = \mu_2$.
   We can rewrite the model for differences as $d_j \overset{iid}{\sim} N(\mu_d, \sigma_d^2)$ where $\mu_d = \mu_1 - \mu_2, \sigma_d^2 = 2\sigma_e^2$.
   Now the null hypothesis is equivalent to $H_0 : \mu_1 - \mu_2 = \mu_d = 0$.
   Let $\bar{d}. = \dfrac{\sum_{j=1}^{20} d_j}{20}$, and $\bar{d}. \sim N\left(\mu_d, \dfrac{\sigma_d^2}{20}\right)$. Then we can build up a t statistic

$$\begin{aligned}
t &= \frac{\bar{d}. - \mu_d}{\sqrt{\widehat{Var}(\bar{d}.)}} \\
&= \frac{\bar{d}.}{\sqrt{\hat{\sigma_d^2}/20}} && \text{under } H_0 \\
&= \frac{\bar{d}.}{\sqrt{\frac{1}{20} \cdot \frac{1}{20-1} \sum_{j=1}^{20} (d_j - \bar{d}.)^2}} && \text{MSE for constant mean model}
\end{aligned}$$

   Or use F test statistic $F = t^2 = \dfrac{380\,\bar{d}.^2}{\sum_{j=1}^{20} (d_j - \bar{d}.)^2}$

(c) Fully state the exact distribution of the test statistic provided in part (b).

$$t \sim t_{19}\left(\frac{\mu_d}{\sqrt{\sigma_d^2/20}}\right) \stackrel{d}{=} t_{19}\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_e^2/10}}\right)$$

$$F \sim F_{1,19}\left(\frac{5(\mu_1-\mu_2)^2}{\sigma_e^2}\right)$$

(d) Provide a formula for a 95% confidence interval for $\mu_1 - \mu_2$.
Given only the 40 scores of the subjects who received only drink one type, the model for these scores is simplfied to be a Markov model as

$$\boldsymbol{Y} = \underbrace{[\boldsymbol{I}_{2\times 2} \otimes \mathbf{1}_{20\times 1}]}_{\boldsymbol{X}} \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + \boldsymbol{e}$$

with $\boldsymbol{Y}$ is a vector of $[a_1, \cdots, a_{20}, b_1, \cdots, b_{20}]'$ and $\boldsymbol{e}$ is a vector of random errors $[e_{11}, \cdots, e_{1,20}, e_{11}, \cdots, e_{1,20}]'$ where $e_{ik} \stackrel{iid}{\sim} N(0, \sigma_u^2 + \sigma_e^2)$ for $i = 1, 2; k = 1, \cdots, 20$.
So the BLUE for $\mu_1 - \mu_2$ is $\bar{a}. - \bar{b}..$

$$\begin{aligned}\widehat{Var}(\bar{a}. - \bar{b}.) &= \widehat{Var(\bar{a}.) + Var}(\bar{b}.)\\ &= 2 \times \frac{1}{20}(\widehat{\sigma_u^2 + \sigma_e^2}) \qquad\qquad \text{MSE for the Markov model above}\\ &= \frac{1}{10}\cdot\frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j-\bar{a}.)^2 + \sum_{j=1}^{20}(b_j-\bar{b}.)^2\right)\end{aligned}$$

Therefore the 95% confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{a}. - \bar{b}.) + t_{38,\,0.975}\sqrt{\frac{1}{380}\left(\sum_{j=1}^{20}(a_j-\bar{a}.)^2 + \sum_{j=1}^{20}(b_j-\bar{b}.)^2\right)}$$

with $df = n - rank(\boldsymbol{X}) = 38$

(e) Provide formulas for unbiased estimators of σ_u^2 and σ_e^2

From part (b), we have $\hat{\sigma}_d^2 = 2\hat{\sigma}_e^2 = \frac{1}{20-1}\sum_{j=1}^{20}(d_j - \bar{d}.)^2$.
From part (d) we have $\widehat{\sigma_u^2 + \sigma_e^2} = \frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j-\bar{a}.)^2 + \sum_{j=1}^{20}(b_j-\bar{b}.)^2\right)$.
By solving the equations above, we can obtain

$$\begin{cases}\hat{\sigma}_e^2 = \dfrac{\sum_{j=1}^{20}(d_j-\bar{d}.)^2}{38}\\ \hat{\sigma}_u^2 = \dfrac{\left(\sum_{j=1}^{20}(a_j-\bar{a}.)^2 + \sum_{j=1}^{20}(b_j-\bar{b}.)^2\right)}{38} - \dfrac{\sum_{j=1}^{20}(d_j-\bar{d}.)^2}{38}\end{cases}$$

(f) Provide a simplified expression for the best linear unbiased estimator of $\mu_1 - \mu_2$.

Both $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ are independent unbiased estimators of $\mu_1 - \mu_2$. Thus, the BLUE of $\mu_1 - \mu_2$ is the weighted average of $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ with weights proportional to the inverse of the variances.

$$
\begin{aligned}
\widehat{\mu_1 - \mu_2} &= \frac{Var^{-1}(\bar{d}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot \bar{d}. + \frac{Var^{-1}(\bar{a}. - \bar{b}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot (\bar{a}. - \bar{b}.) \\
&= \frac{\sigma_u^2 + \sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot \bar{d}. + \frac{\sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot (\bar{a}. - \bar{b}.)
\end{aligned}
$$

3. Suppose the responses in problem 2 were sorted first by subject and then by drink, the response vector $\boldsymbol{y} = [y_{11}, y_{21}, \cdots, y_{1,20}, y_{2,20}, y_{1,21}, \cdots, y_{1,40}, y_{2,41}, \cdots, y_{1,60}]'$.
In model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$,

$$
\boldsymbol{X} = \left[\begin{array}{cc}
1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \hline
1 & \\ 1 & \\ \vdots & \\ 1 & \\ \hline
 & 1 \\ & 1 \\ & \vdots \\ & 1
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\ \vdots\\1\\1\end{array}}\right\} \text{40 rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\ \vdots\\1\end{array}}\right\} \text{20 rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\ \vdots\\1\end{array}}\right\} \text{20 rows}
\end{array}
\quad \text{and } \boldsymbol{Z} = \left[\begin{array}{cccc|c|ccc}
1 & & & & & & & \\
1 & & & & & & & \\
 & 1 & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & \\
 & & & & & & & 1 \\
 & & & & & & & \ddots \\
 & & & & & & & & 1
\end{array}\right]
$$

the Kronecker product notation for $\boldsymbol{X}$ and $\boldsymbol{Z}$ are

$$
\boldsymbol{X}_{80\times 2} = \begin{bmatrix} \mathbf{1}_{20\times 1} \otimes \boldsymbol{I}_{2\times 2} \\ \boldsymbol{I}_{2\times 2} \otimes \mathbf{1}_{20\times 1} \end{bmatrix}
$$

$$
\boldsymbol{Z}_{60\times 2} = diag(I_{20\times 20} \otimes \mathbf{1}_{2\times 1}, \boldsymbol{I}_{40\times 40})
$$

4. By slide 54 of set 12, the BLUE of μ is a weighted average of independent linear unbiased estimators, where the weights are proportional to the inverse variances of the linear unbiased estimators.

We can divide $\boldsymbol{y}$ into two independent subvectors by considering y_5 separately from $y_1, \ldots, y_4$. By the hint given, the BLUE of μ based only on $y_1, \ldots, y_4$ is $\frac{1}{4}\sum_{i=1}^4 y_i$. Clearly, the BLUE of μ based on only y_5 is y_5 itself. These two estimators are independent, with variances

$$
\begin{aligned}
\operatorname{Var}\left(\frac{1}{4}\sum_{i=1}^4 y_i\right) &= \operatorname{Var}\left(\frac{1}{4}\mathbf{1}'_{4\times 1}(y_1, y_2, y_3, y_4)'\right) \\
&= \frac{1}{16}\mathbf{1}'_{4\times 1}\operatorname{Var}\left(y_1, y_2, y_3, y_4\right)'\mathbf{1}_{4\times 1} \\
&= \frac{1}{16}\mathbf{1}'_{4\times 1}\begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 \end{pmatrix}\mathbf{1}_{4\times 1} \\
&= \frac{32}{16} \\
&= 2,
\end{aligned}
$$

$$
\operatorname{Var}(y_5) = 4.
$$

Then,

$$
\begin{aligned}
\hat{\mu}_{\text{BLUE}} &= \frac{\frac{1}{\operatorname{Var}\left(\frac{1}{4}\sum_{i=1}^4 y_i\right)}\frac{1}{4}\sum_{i=1}^4 y_i + \frac{1}{\operatorname{Var}(y_5)}y_5}{\frac{1}{\operatorname{Var}\left(\frac{1}{4}\sum_{i=1}^4 y_i\right)} + \frac{1}{\operatorname{Var}(y_5)}} \\
&= \frac{\frac{1}{2}\frac{1}{4}\sum_{i=1}^4 y_i + \frac{1}{4}y_5}{\frac{1}{2} + \frac{1}{4}} \\
&= \frac{2}{3}\left(\frac{1}{4}\sum_{i=1}^4 y_i\right) + \frac{1}{3}y_5 \\
&= \frac{1}{6}\sum_{i=1}^4 y_i + \frac{1}{3}y_5.
\end{aligned}
$$