1.
```
> d = read.delim("http://dnett.github.io/S510/LeafArea.txt")
> library(lme4)
>
> o = lmer(LeafArea ~ Dose + (1 + Dose | ResearchStation), data = d)
> summary(o)
Linear mixed model fit by REML ['lmerMod']
Formula: LeafArea ~ Dose + (1 + Dose | ResearchStation)
   Data: d

REML criterion at convergence: 1333.905

Random effects:
 Groups          Name        Variance  Std.Dev. Corr
 ResearchStation (Intercept) 1.049e+01 3.238634
                 Dose        5.623e-05 0.007499 0.06
 Residual                    3.949e+00 1.987147
Number of obs: 300, groups: ResearchStation, 15

Fixed effects:
             Estimate Std. Error t value
(Intercept) 13.857667   0.859498   16.12
Dose         0.051900   0.003779   13.73

Correlation of Fixed Effects:
     (Intr)
Dose -0.131
>
> u=ranef(o)$ResearchStation
> b=fixef(o)
> vcov(o)
2 x 2 Matrix of class "dpoMatrix"
              (Intercept)          Dose
(Intercept)  0.7387375451 -0.0004269892
Dose        -0.0004269892  0.0000142787
```

   (a) $\hat{\sigma}_e^2 = 3.949$

   (b)
$$\hat{\mathbf{\Sigma}}_{\boldsymbol{b}} = \begin{bmatrix} 10.49 & 0.06 * \sqrt{10.49 * 0.00005623} \\ 0.06 * \sqrt{10.49 * 0.00005623} & 0.00005623 \end{bmatrix}$$

   (c)
```
> plot(d$Dose[d$ResearchStation == 7], d$LeafArea[d$ResearchStation == 7],
+      xlab = "Dose", ylab = "Leaf Area")
> lines(c(0,100), b[1] + (b[2]) * c(0,100))
```
   See actual figure after part (f).

   (d)
```
> #eBLUP of intercept for research station 7.
```

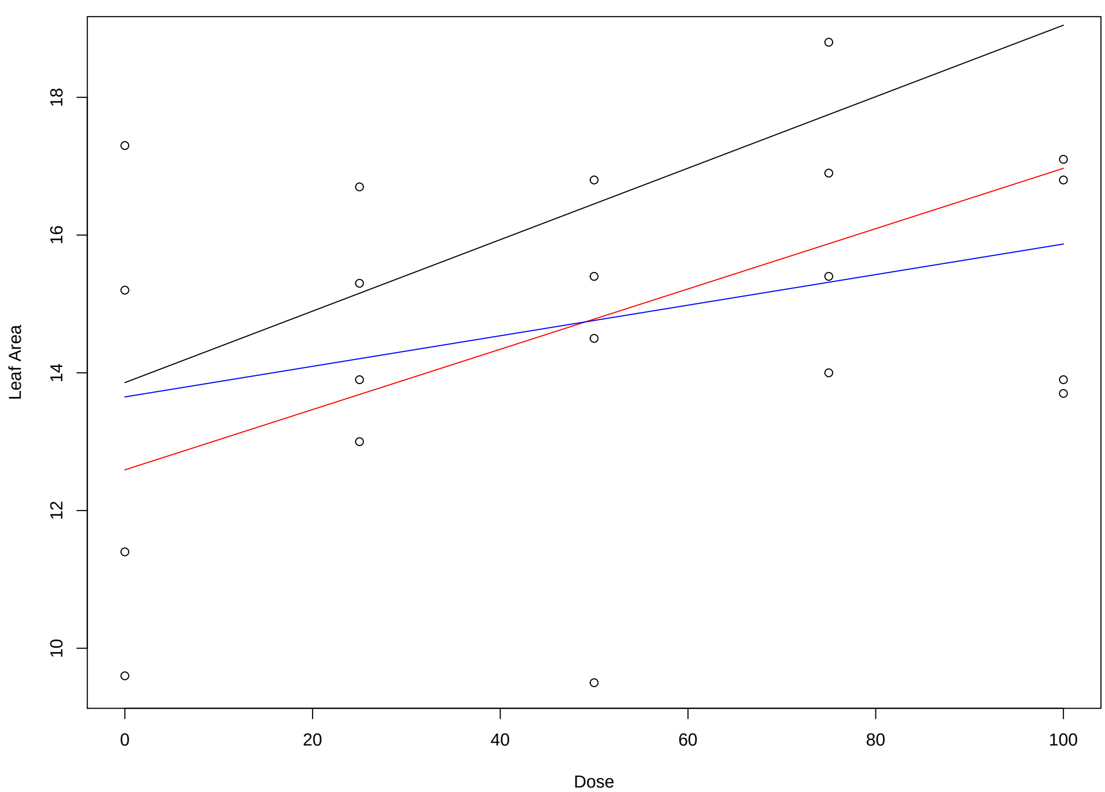
```
>
> b[1] + u[7,1]
(Intercept)
   12.59071
>
> #eBLUP of slope for research station 7.
>
> b[2] + u[7,2]
      Dose
0.04378323
```

$12.59071 + 0.04378323x$

(e)
```
> #Simple linear regression for research station 7
>
> o7 = lm(d$LeafArea[d$ResearchStation == 7] ~ d$Dose[d$ResearchStation == 7])
> b7 = coef(o7)
> b7
                   (Intercept) d$Dose[d$ResearchStation == 7]
                       13.6500                          0.0222
```

$13.6500 + 0.0222x$

(f)
```
> lines(c(0,100), b[1] + u[7,1] + (b[2] + u[7,2]) * c(0,100), col = "red")
> lines(c(0,100), b7[1] + (b7[2]) * c(0,100), col = "blue")
```



Note that, relative to the blue simple linear regression fit, the red eBLUP regression line is rotated towards the population regression line in black. The slopes of the red and black lines are similar because the small estimated variance for the slope random effects means there will be strong shrinkage of the eBLUP predicted slope towards the slope of the black line.

(g) To get the correct likelihood ratio statistic, we need to be sure to ask R to use ML rather than REML to estimate parameters in this case. It doesn't make sense to compare REML likelihoods here because the two models have different models for the mean of $\boldsymbol{y}$.

```
> null.model = lmer(LeafArea ~ 1 + (1 + Dose | ResearchStation),
+                   data = d, REML = F)
> alt.model = lmer(LeafArea ~ Dose + (1 + Dose | ResearchStation),
+                  data = d, REML = F)
>
> anova(null.model, alt.model, test = "Chisq")
Data: d
Models:
null.model: LeafArea ~ 1 + (1 + Dose | ResearchStation)
alt.model: LeafArea ~ Dose + (1 + Dose | ResearchStation)
           Df    AIC    BIC  logLik deviance  Chisq Chi Df Pr(>Chisq)
null.model  5 1376.1 1394.6 -683.06   1366.1
alt.model   6 1338.0 1360.3 -663.02   1326.0 40.078      1   2.44e-10 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Based on the code and output above, the likelihood ratio statistic is 40.078.

(h)
```
> ofull = lmer(LeafArea ~ Dose + (1 + Dose | ResearchStation),
+              data = d, REML = F)
> AIC(ofull)
[1] 1338.042
```

See below for AIC computed using REML.

(i)
```
> o1slope = lmer(LeafArea ~ Dose + (1 | ResearchStation),
+                data = d, REML = F)
> AIC(o1slope)
[1] 1334.575
```

See below for AIC computed using REML.

(j)
```
> o1int1slope = lm(LeafArea ~ Dose,
+                  data = d)
> AIC(o1int1slope)
[1] 1650.107
```

Verification that the above is indeed the likelihood version of AIC – and thus comparable to the calculations in parts (h) and (i) – comes from the following calculations.

$$
\begin{aligned}
\ell(\hat{\boldsymbol{\theta}}) &= -\frac{1}{2}\log|\hat{\boldsymbol{\Sigma}}| - \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) - \frac{n}{2}\log(2\pi) \\
&= -\frac{1}{2}\log\left|\frac{\text{SSE}}{n}\boldsymbol{I}\right| - \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\left(\frac{\text{SSE}}{n}\boldsymbol{I}\right)^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) - \frac{n}{2}\log(2\pi) \\
&= -\frac{n}{2}\log\left(\frac{\text{SSE}}{n}\right) - \frac{1}{2\frac{\text{SSE}}{n}}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) - \frac{n}{2}\log(2\pi) \\
&= -\frac{n}{2}\log\left(\frac{\text{SSE}}{n}\right) - \frac{n}{2\text{SSE}}\text{SSE} - \frac{n}{2}\log(2\pi) \\
&= -\frac{n}{2}\log\left(\frac{\text{SSE}}{n}\right) - \frac{n}{2} - \frac{n}{2}\log(2\pi)
\end{aligned}
$$

```
> #AIC for likelihood following R convention for AIC
>
> SSE = sum(residuals(o1int1slope)^2)
>
> logMLlike = -0.5 * 300 * log(SSE / 300) -
+             0.5 * 300  -
+             (300 / 2) * log(2*pi)
>
> -2 * logMLlike + 2 * 3
[1] 1650.107
>
> #Alternatively, using the R function logLik
>
> -2 * logLik(o1int1slope) + 2 * 3
[1] 1650.107
```

The answers for AIC computed using REML are as follows.

```
> ofull = lmer(LeafArea ~ Dose + (1 + Dose | ResearchStation),
+             data = d)
> AIC(ofull)
[1] 1345.905
>
> o1slope = lmer(LeafArea ~ Dose + (1 | ResearchStation),
+               data = d)
> AIC(o1slope)
[1] 1342.693
>
> #According to my reading of R help files, the following should give AIC
> #for REML with the lm function.
>
> o1int1slope = lm(LeafArea ~ Dose, data = d)
>
> - 2 * logLik(o1int1slope, REML = T) + 2 * 3
'log Lik.' 1659.678 (df=3)
```

(k) Whether we use ML or REML to get AIC, the model from part (i) is preferred because it has the lowest AIC.

2. Model (1) from problem 1 can be written as $\boldsymbol{y} = \boldsymbol{X\beta}+\boldsymbol{Zu}+\boldsymbol{\epsilon}$. Define $\boldsymbol{X}, \boldsymbol{\beta}, \boldsymbol{Z}, \boldsymbol{u}, \boldsymbol{G} = Var(\boldsymbol{u})$ and $\boldsymbol{R} = Var(\boldsymbol{\epsilon})$:

$$\boldsymbol{X} = \mathbf{1}_{15\times 1} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 25 \\ 1 & 50 \\ 1 & 75 \\ 1 & 100 \end{bmatrix} \otimes \mathbf{1}_{4\times 1}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

$$\boldsymbol{Z} = \boldsymbol{I}_{15} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 25 \\ 1 & 50 \\ 1 & 75 \\ 1 & 100 \end{bmatrix} \otimes \mathbf{1}_{4\times 1}, \quad \boldsymbol{u} = (b_{11}, b_{21}, \cdots, b_{1,15}, b_{2,15})',$$

$\boldsymbol{G} = \boldsymbol{I}_{15} \otimes \boldsymbol{\Sigma_b}$ where $\boldsymbol{\Sigma_b}$ is a 2×2 variance matrix for $(b_{1i}, b_{2i})'$ for any $i = 1, \cdots, 15$, and $\boldsymbol{R} = \sigma_e^2 \cdot \boldsymbol{I}_{300}$.

3. (a) Specify matrix $\boldsymbol{X}$:

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{n_1\times 1} \otimes \boldsymbol{I}_t & & \\ & \mathbf{1}_{n_2\times 1} \otimes \boldsymbol{I}_t & \\ & & \mathbf{1}_{n_3\times 1} \otimes \boldsymbol{I}_t \end{bmatrix}$$

(b) Specify matrix $Var(\boldsymbol{y}) = \boldsymbol{\Sigma}$ in terms of $\boldsymbol{W}$:

$$\boldsymbol{\Sigma} = \boldsymbol{I}_{(n_1+n_2+n_3)} \otimes \boldsymbol{W}$$

(c) Compute $(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}$:

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{I}_{(n_1+n_2+n_3)} \otimes \boldsymbol{W}^{-1}$$

$$\boldsymbol{X}' = \begin{bmatrix} \mathbf{1}_{1\times n_1} \otimes \boldsymbol{I}_t & & \\ & \mathbf{1}_{1\times n_2} \otimes \boldsymbol{I}_t & \\ & & \mathbf{1}_{1\times n_3} \otimes \boldsymbol{I}_t \end{bmatrix}$$

$$\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X} = \begin{bmatrix} (\mathbf{1}_{1\times n_1} \cdot \boldsymbol{I}_{n_1} \cdot \mathbf{1}_{n_1\times 1}) \otimes (\boldsymbol{I}_t \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{I}_t) & & \\ & (\mathbf{1}_{1\times n_2} \cdot \boldsymbol{I}_{n_2} \cdot \mathbf{1}_{n_2\times 1}) \otimes (\boldsymbol{I}_t \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{I}_t) & \\ & & (\mathbf{1}_{1\times n_3} \cdot \boldsymbol{I}_{n_3} \cdot \mathbf{1}_{n_3\times 1}) \otimes (\boldsymbol{I}_t \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{I}_t) \end{bmatrix}$$

$$= \begin{bmatrix} n_1\boldsymbol{W}^{-1} & & \\ & n_2\boldsymbol{W}^{-1} & \\ & & n_3\boldsymbol{W}^{-1} \end{bmatrix}$$

therefore

$$(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1} = \begin{bmatrix} \frac{\boldsymbol{W}}{n_1} & & \\ & \frac{\boldsymbol{W}}{n_2} & \\ & & \frac{\boldsymbol{W}}{n_3} \end{bmatrix}$$

(d) Compute $(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}$:

$$\begin{aligned}
\boldsymbol{X}'\boldsymbol{\Sigma}^{-1} &= \begin{bmatrix} (\mathbf{1}_{1\times n_1}\cdot \boldsymbol{I}_{n_1})\otimes(\boldsymbol{I}_t\cdot \boldsymbol{W}^{-1}\cdot) & & \\ & (\mathbf{1}_{1\times n_2}\cdot \boldsymbol{I}_{n_2})\otimes(\boldsymbol{I}_t\cdot \boldsymbol{W}^{-1}\cdot) & \\ & & (\mathbf{1}_{1\times n_3}\cdot \boldsymbol{I}_{n_3})\otimes(\boldsymbol{I}_t\cdot \boldsymbol{W}^{-1}\cdot) \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{1}_{1\times n_1}\otimes \boldsymbol{W}^{-1} & & \\ & \mathbf{1}_{1\times n_2}\otimes \boldsymbol{W}^{-1} & \\ & & \mathbf{1}_{1\times n_3}\otimes \boldsymbol{W}^{-1} \end{bmatrix}
\end{aligned}$$

so

$$(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{n_1}\mathbf{1}_{1\times n_1}\otimes \boldsymbol{I}_t & & \\ & \frac{1}{n_2}\mathbf{1}_{1\times n_2}\otimes \boldsymbol{I}_t & \\ & & \frac{1}{n_3}\mathbf{1}_{1\times n_3}\otimes \boldsymbol{I}_t \end{bmatrix}$$

(e) Compute $(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y}$:

$$\begin{aligned}
(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} &= \begin{bmatrix} \frac{1}{n_1}\mathbf{1}_{1\times n_1}\otimes \boldsymbol{I}_t & & \\ & \frac{1}{n_2}\mathbf{1}_{1\times n_2}\otimes \boldsymbol{I}_t & \\ & & \frac{1}{n_3}\mathbf{1}_{1\times n_3}\otimes \boldsymbol{I}_t \end{bmatrix}\cdot \boldsymbol{y} \\
&= \begin{bmatrix} \frac{1}{n_1}\sum_{j=1}^{n_1}\boldsymbol{y}_{1j} \\ \frac{1}{n_2}\sum_{j=1}^{n_2}\boldsymbol{y}_{2j} \\ \frac{1}{n_3}\sum_{j=1}^{n_3}\boldsymbol{y}_{3j} \end{bmatrix}
\end{aligned}$$

(f) Give the BLUEs of $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3$:

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_1 &= (\boldsymbol{I}_t, \mathbf{0}_{t\times t}, \mathbf{0}_{t\times t})\hat{\boldsymbol{\beta}}_{\text{OLS}} \\
&= (\boldsymbol{I}_t, \mathbf{0}_{t\times t}, \mathbf{0}_{t\times t})(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\boldsymbol{y}_{1j}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_2 &= (\mathbf{0}_{t\times t}, \boldsymbol{I}_t, \mathbf{0}_{t\times t})\hat{\boldsymbol{\beta}}_{\text{OLS}} \\
&= (\mathbf{0}_{t\times t}, \boldsymbol{I}_t, \mathbf{0}_{t\times t})(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \\
&= \frac{1}{n_2}\sum_{j=1}^{n_2}\boldsymbol{y}_{2j} \\
\hat{\boldsymbol{\mu}}_3 &= (\mathbf{0}_{t\times t}, \mathbf{0}_{t\times t}, \boldsymbol{I}_t)\hat{\boldsymbol{\beta}}_{\text{OLS}} \\
&= (\mathbf{0}_{t\times t}, \mathbf{0}_{t\times t}, \boldsymbol{I}_t)(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \\
&= \frac{1}{n_3}\sum_{j=1}^{n_3}\boldsymbol{y}_{3j}
\end{aligned}$$