

1. (a) The model in this problem is  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$  with  $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \beta_1, \beta_2)'$  and

$$\mathbf{X} = \begin{pmatrix} \mathbf{1}_{20 \times 1}, \begin{bmatrix} \mathbf{1}_{10 \times 1} \\ \mathbf{0}_{10 \times 1} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{10 \times 1} \\ \mathbf{1}_{10 \times 1} \end{bmatrix} \begin{bmatrix} \mathbf{1}_{2 \times 1} \\ \mathbf{0}_{8 \times 1} \\ \mathbf{1}_{6 \times 1} \\ \mathbf{0}_{4 \times 1} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{2 \times 1} \\ \mathbf{1}_{8 \times 1} \\ \mathbf{0}_{6 \times 1} \\ \mathbf{1}_{4 \times 1} \end{bmatrix} \end{pmatrix}$$

Because  $\text{rank}(\mathbf{X}) = 3$ , one possible full rank matrix is

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} \mathbf{1}_{20 \times 1}, \begin{bmatrix} \mathbf{1}_{10 \times 1} \\ \mathbf{0}_{10 \times 1} \end{bmatrix} \begin{bmatrix} \mathbf{1}_{2 \times 1} \\ \mathbf{0}_{8 \times 1} \\ \mathbf{1}_{6 \times 1} \\ \mathbf{0}_{4 \times 1} \end{bmatrix} \end{pmatrix}$$

We can compute orthogonal columns using Gram-Schmidt Orthogonalization method:

$$\mathbf{w}_1 = \mathbf{x}_1;$$

$$\mathbf{w}_2 = (\mathbf{I} - \mathbf{P}_{\mathbf{w}_1})\mathbf{x}_2 = \frac{1}{2} \begin{bmatrix} \mathbf{1}_{10 \times 1} \\ -\mathbf{1}_{10 \times 1} \end{bmatrix}$$

$$\mathbf{w}_3 = (\mathbf{I} - \mathbf{P}_{[\mathbf{w}_1, \mathbf{w}_2]})\mathbf{x}_3 = \begin{bmatrix} 0.8 \cdot \mathbf{1}_{2 \times 1} \\ -0.2 \cdot \mathbf{1}_{8 \times 1} \\ 0.4 \cdot \mathbf{1}_{6 \times 1} \\ -0.6 \cdot \mathbf{1}_{4 \times 1} \end{bmatrix}$$

So a model matrix  $\mathbf{W}$  with orthogonal columns is  $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ .

- (b) Compute  $\mathbf{P}_{\mathbf{W}}\mathbf{y}$ :

$$\begin{aligned} (\mathbf{W}'\mathbf{W})^{-1} &= \begin{bmatrix} \begin{pmatrix} \mathbf{w}'_1 \\ \mathbf{w}'_2 \\ \mathbf{w}'_3 \end{pmatrix} (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{w}'_1\mathbf{w}_1 & & \\ & \mathbf{w}'_2\mathbf{w}_2 & \\ & & \mathbf{w}'_3\mathbf{w}_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 20 & & \\ & 5 & \\ & & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{20} & & \\ & \frac{1}{5} & \\ & & \frac{1}{4} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{W}'\mathbf{y} &= \begin{pmatrix} \mathbf{w}'_1 \\ \mathbf{w}'_2 \\ \mathbf{w}'_3 \end{pmatrix} \cdot \mathbf{y} = \begin{bmatrix} \mathbf{w}'_1\mathbf{y} \\ \mathbf{w}'_2\mathbf{y} \\ \mathbf{w}'_3\mathbf{y} \end{bmatrix} \\ &= \begin{bmatrix} \sum_i \sum_j \sum_k y_{ijk} \\ \frac{1}{2}(\sum_j \sum_k y_{1jk} - \sum_j \sum_k y_{2jk}) \\ 0.8 \cdot \sum_k y_{11k} - 0.2 \cdot \sum_k y_{12k} + 0.4 \cdot \sum_k y_{21k} - 0.6 \cdot \sum_k y_{22k} \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ -4 \\ 6.4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\mathbf{P}_W \mathbf{y} &= \mathbf{W}(\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{y} \\
&= (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \begin{bmatrix} \frac{1}{20} & & \\ & \frac{1}{5} & \\ & & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6.4 \end{bmatrix} \\
&= 5\mathbf{w}_1 - 0.8\mathbf{w}_2 + 1.6\mathbf{w}_3 \\
&= \begin{bmatrix} 5.88 \cdot \mathbf{1}_{2 \times 1} \\ 4.28 \cdot \mathbf{1}_{8 \times 1} \\ 6.04 \cdot \mathbf{1}_{6 \times 1} \\ 4.44 \cdot \mathbf{1}_{4 \times 1} \end{bmatrix}
\end{aligned}$$

- (c) The Type II sum of squares for factor B is  $S(B|1, A) = \mathbf{y}'(\mathbf{P}_3 - \mathbf{P}_2)\mathbf{y}$  where  $\mathbf{P}_3\mathbf{y} = \mathbf{P}_W\mathbf{y}$  and from part (b)

$$\begin{aligned}
\mathbf{P}_2\mathbf{y} &= \mathbf{P}_{[\mathbf{w}_1, \mathbf{w}_2]}\mathbf{y} \\
&= (\mathbf{w}_1, \mathbf{w}_2) \begin{bmatrix} \frac{1}{20} & \\ & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 100 \\ -4 \end{bmatrix} \\
&= 5\mathbf{w}_1 - 0.8\mathbf{w}_2 \\
&= \begin{bmatrix} 4.6 \cdot \mathbf{1}_{10 \times 1} \\ 5.4 \cdot \mathbf{1}_{10 \times 1} \end{bmatrix}
\end{aligned}$$

so

$$\begin{aligned}
S(B|1, A) &= \mathbf{y}'(\mathbf{P}_3 - \mathbf{P}_2)\mathbf{y} = \mathbf{y}'\mathbf{P}_3\mathbf{y} - \mathbf{y}'\mathbf{P}_2\mathbf{y} = \|\mathbf{P}_3\mathbf{y}\|^2 - \|\mathbf{P}_2\mathbf{y}\|^2 \\
&= (5.88^2 \times 2 + 4.28^2 \times 8 + 6.04^2 \times 6 + 4.44^2 \times 4) - (4.6^2 \times 10 + 5.4^2 \times 10) \\
&= 10.24
\end{aligned}$$

2. (a) For the  $j$ th woman treated with the  $i$ th drug,

$$\begin{aligned}
\mathbf{W} &= \text{Var}(y_{ij1}, y_{ij2}, y_{ij3}, y_{ij4})' \\
&= \begin{bmatrix} \sigma_w^2 + \sigma_e^2 & \sigma_w^2 & \sigma_w^2 & \sigma_w^2 \\ \sigma_w^2 & \sigma_w^2 + \sigma_e^2 & \sigma_w^2 & \sigma_w^2 \\ \sigma_w^2 & \sigma_w^2 & \sigma_w^2 + \sigma_e^2 & \sigma_w^2 \\ \sigma_w^2 & \sigma_w^2 & \sigma_w^2 & \sigma_w^2 + \sigma_e^2 \end{bmatrix} \\
&= \sigma_w^2 \mathbf{1}\mathbf{1}'_{4 \times 4} + \sigma_e^2 \mathbf{I}_{4 \times 4}.
\end{aligned}$$

We know that  $\text{Var}(\mathbf{y})$  is block diagonal with blocks  $\mathbf{W}$ . There are a total of  $3 \cdot 5 = 15$  blocks, so that

$$\text{Var}(\mathbf{y}) = \mathbf{I}_{15 \times 15} \otimes \mathbf{W} = \mathbf{I}_{15 \times 15} \otimes (\sigma_w^2 \mathbf{1}\mathbf{1}'_{4 \times 4} + \sigma_e^2 \mathbf{I}_{4 \times 4}).$$

- (b) The null hypothesis of no drug-by-time interactions is  $H_0 : \mu_{ij} - \mu_{ij*} = \mu_{i*j} - \mu_{i**}$  for all  $i \neq i^*$  and  $j \neq j^*$ . The test statistic  $F = 7.12$  on  $(6, 36)$  degrees of freedom with  $p < 0.001$ . We reject the null hypothesis and conclude that there is significant evidence for drug-by-time interactions on heart rate.

- (c) The null hypothesis for testing the same mean heart rate 15 minutes after treatment for all three drugs is  $H_0 : \mu_{14} = \mu_{24} = \mu_{34}$ . The test statistic  $F = 1.11$  on  $(2, 17.1)$  degrees of freedom with  $p = 0.352 > 0.05$ . We fail to reject the null hypothesis and conclude that there is no significant evidence for the same mean heart rate 15 minutes after treatment for all three drugs.
- (d) An approximate 95% confidence interval for  $\mu_{14} - \mu_{24}$  is  $(-13.77, 2.57)$  with  $df = 17.1$  by the SAS code below.

Note: for part (c-d),  $df = 17.1$  was computed by Cochran-Satterthwaite since it is for the difference between simple effects with different whole-plot factors. The easiest way to get this is to use SAS with `method=satterthwaite` option.

SAS code:

```
proc import datafile="./HeartRate.txt"
  dbms=TAB replace out=d;
run;
proc print data=d (obs=14);
run;
proc mixed;
class woman drug time;
model y=drug time drug*time /ddfm = satterthwaite;
random woman(drug);
contrast "same mean for drug A B C at 15 min"
  drug 1 -1 0 drug*time 0 0 0 1 0 0 0 -1 0 0 0 0 ,
  drug 1 0 -1 drug*time 0 0 0 1 0 0 0 0 0 0 0 -1 ;
estimate "drug A - drug B at 15 min"
  drug 1 -1 0 drug*time 0 0 0 1 0 0 0 -1 0 0 0 0 /cl;
run;
```

R code:

```
library(MASS)
library(nlme)
library(dplyr)

> d <- read.table("http://dnett.github.io/S510/HeartRate.txt", header = T)
> d$time <- factor((d$time + 5) / 5) # Levels of time.
> fit <- lme(y ~ drug * t, random = ~ 1 | woman, data = d)
> anova(fit)
```

	numDF	denDF	F-value	p-value
(Intercept)	1	36	2900.7782	<.0001
drug	2	12	1.3517	0.2955
t	3	36	10.2159	0.0001
drug:t	6	36	7.1153	<.0001

```
>
> # Function from Dr. Nettletons Notes
> test=function(lmout,C,d=0,df){
+   b=fixed.effects(lmout)
+   V=vcov(lmout)
+   dfn=nrow(C)
+   Cb.d=C %*% b - d
+   Fstat=drop(t(Cb.d)%*%solve(C%*%V%*%t(C))%*%Cb.d/dfn)
+   pvalue=1-pf(Fstat,dfn,df)
+ }
```

```

+   cbind(Fstat=Fstat,pvalue=pvalue)
+ }
> C1 <- matrix(c(0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
+               0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1), byrow=T,nrow = 2)
> test(fit,C1,df=17.1) # df computed by Cochran-Satterthwaite via SAS.
      Fstat      pvalue
[1,] 1.109573 0.3523058
>
> # Function from Dr. Nettletons Notes
> ci <- function(lmeout, C, df, a = 0.05) {
+   b = fixed.effects(lmeout)
+   V = vcov(lmeout)
+   Cb = C %*% b
+   se = sqrt(diag(C %*% V %*% t(C)))
+   tval = qt(1 - a / 2, df)
+   low = Cb - tval * se
+   up = Cb + tval * se
+   m = cbind(C, Cb, se, low, up)
+   dimnames(m)[[2]] = c(paste("c", 1 : ncol(C), sep = ""),
+                         "estimate", "se", paste(100 * (1 - a), "% Conf.", sep = ""), "limits")
+   return(m)
+ }
> C2 <- matrix(c(0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0), nrow = 1)
> ci(fit, C2, 17.1) # df computed by Cochran-Satterthwaite via SAS.
      c1 c2 c3 c4 c5 c6 c7 c8 c9 c10 c11 c12 estimate      se
[1,]  0 -1  0  0  0  0  0  0  0  0  0 -1  0      -5.6 3.872564
      95% Conf.  limits
[1,] -13.76676 2.566758

```

3. (a) Under a compound symmetry assumption,

$$\mathbf{W} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix},$$

where the REML estimates for the heart rate data are  $\hat{\sigma} = 6.12$  and  $\hat{\rho} = 0.777$ .  
 $(\hat{\sigma}_s^2 = 29.13, \hat{\sigma}_e^2 = 8.36)$

(b) Using R, AIC = 317.92 and BIC = 344.12.

Using SAS, AIC = 293.9 and BIC = 295.3.

(c) Under an AR(1) assumption,

$$\mathbf{W} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix},$$

where the REML estimates for the heart rate data are  $\hat{\sigma} = 6.00$  and  $\hat{\rho} = 0.828$ .

- (d) Using R, AIC = 313.94 and BIC = 340.14.  
Using SAS, AIC = 289.9 and BIC = 291.4.

- (e) Under a general symmetry assumption,

$$\mathbf{W} = \sigma^2 \begin{bmatrix} 1 & \rho_{12}\delta_2 & \rho_{13}\delta_3 & \rho_{14}\delta_4 \\ \rho_{12}\delta_2 & \delta_2^2 & \rho_{23}\delta_2\delta_3 & \rho_{24}\delta_2\delta_4 \\ \rho_{13}\delta_3 & \rho_{23}\delta_2\delta_3 & \delta_3^2 & \rho_{34}\delta_3\delta_4 \\ \rho_{14}\delta_4 & \rho_{24}\delta_2\delta_4 & \rho_{34}\delta_3\delta_4 & \delta_4^2 \end{bmatrix},$$

where the REML estimates for the heart rate data are

$$\begin{aligned} \hat{\sigma} &= 6.10, \\ \hat{\delta}_2 &= 1.08, \quad \hat{\delta}_3 = 0.995, \quad \hat{\delta}_4 = 0.928, \\ \hat{\rho}_{12} &= 0.850, \quad \hat{\rho}_{13} = 0.889, \quad \hat{\rho}_{14} = 0.625, \\ \hat{\rho}_{23} &= 0.870, \quad \hat{\rho}_{24} = 0.631, \quad \hat{\rho}_{34} = 0.794. \end{aligned}$$

- (f) Using R, AIC = 322.85 and BIC = 364.01.  
Using SAS, AIC = 298.8 and BIC = 305.9.
- (g) The model with an AR(1) correlation structure has the smallest AIC and BIC of the three (regardless of whether you used R or SAS). Consequently, the AR(1) correlation structure is preferred for this dataset.
- (h) There are several ways to find a 95% confidence interval for  $\mu_{14} - \mu_{24}$  using the model with an AR(1) correlation structure. In question 2 (f), we used a split-plot design to get a confidence interval, for which it is clear that we should compute the degrees of freedom using Cochran-Satterthwaite. However, it is less clear for the model using AR(1). Regardless of which degrees of freedom you use, you should have  $\widehat{\mu_{14} - \mu_{24}} = -5.6$  with  $\sqrt{\widehat{\text{Var}(\mu_{14} - \mu_{24})}} = 3.795$ .

We can use the Cochran-Satterthwaite method in SAS by specifying the “ddfm = satterthwaite” option, which gives the interval  $(-13.54, 2.34)$  based on  $\text{df} = 19.2$ . The default “ddfm” method in SAS uses  $\text{df} = 36$ , which gives the interval  $(-13.30, 2.10)$ .

In R, gls computes  $\text{df} = n - \text{rank}(X) = 48$ , which leads to the interval  $(-13.23, 2.03)$ .

Of these intervals, I would prefer the one where the degrees of freedom are computed by Cochran-Satterthwaite because it is the widest and hence the most conservative in terms of inference about the value of  $\mu_{14} - \mu_{24}$ .

SAS code:

```
proc mixed;
  class woman drug time;
  model y = drug time drug*time;
```

```

        repeated time / subject = woman type = cs r rcorr;
run;
proc mixed;
    class woman drug time;
    model y = drug time drug*time / ddfm = satterthwaite;
    repeated time / subject = woman type = ar(1) r rcorr;
    estimate 'drug A - drug B at 15 minutes'
        drug 1 -1 0 drug * time 0 0 0 1    0 0 0 -1    0 0 0 0 / cl;
run;
proc mixed;
    class woman drug time;
    model y = drug time drug*time;
    repeated time / subject = woman type = un  r rcorr;
run;

```

R code:

```

attach(d)
woman <- as.factor(woman)
drug <- as.factor(drug)
time <- as.factor(time)
model.cs <- gls(y ~ drug * time,
    correlation = corCompSymm(form = ~1 | woman),
    method = "REML")
model.ar <- gls(y ~ drug * time,
    correlation = corAR1(form = ~1 | woman),
    method = "REML")
model.sy <- gls(y ~ drug * time,
    correlation = corSymm(form = ~1 | woman),
    weight = varIdent(form = ~1 | time),
    method = "REML")
summary(model.cs)
getVarCov(model.cs)
summary(model.ar)
getVarCov(model.ar)
summary(model.sy)
getVarCov(model.sy)
ci.gls <- function(lmeout, C, df, a = 0.05) {
    b = coef(lmeout)
    V = vcov(lmeout)
    Cb = C %*% b
    se = sqrt(diag(C %*% V %*% t(C)))
    tval = qt(1 - a / 2, df)
    low = Cb - tval * se
    up = Cb + tval * se
    m = cbind( Cb, se, low, up)
    dimnames(m)[[2]] = c("estimate", "se", paste(100 * (1 - a), "% Conf.", sep = ""), "limits")
    return(m)
}
ci.gls(model.ar, C2, 19.2) # Cheated and took Cochran-Satterthwaite df value from SAS.
ci.gls(model.ar, C2, 36) # Default df method in SAS.
ci.gls(model.ar, C2, 48) # Default df method in R.

```

4. The model we used in this problem is  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$  where

$$\mathbf{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{31} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{21} \\ \epsilon_{31} \end{bmatrix} \text{ and } \boldsymbol{\epsilon} \sim N \left( \mathbf{0}_{4 \times 1}, \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \right)$$

(a) The expected total weight gained by a calf from birth to 12 weeks of age is  $\mu_1 + \mu_2$  (which is estimable because  $E(y_{11} + y_{12}) = \mu_1 + \mu_2$ ).

The BLUE of  $\mu_1 + \mu_2 = (1, 1)\boldsymbol{\beta}$  is

$$\begin{aligned} (1, 1)\hat{\boldsymbol{\beta}}_{\text{BLUE}} &= (1, 1)(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y} \\ &= (1, 1) \left( \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \right)^{-1} \\ &\quad \times \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \mathbf{y} \\ &= (1, 1) \begin{pmatrix} \frac{151}{3} \\ \frac{161}{3} \end{pmatrix} \\ &= 104 \end{aligned}$$

(b) According to the model and the correlation description,

$$\begin{bmatrix} \mathbf{y} \\ y_{22} \\ y_{32} \end{bmatrix} \sim N \left( \begin{bmatrix} \mathbf{X}\boldsymbol{\beta} \\ \mu_2 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right) \text{ where } \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} = \left[ \begin{array}{cccc|cc} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 & 2 \\ \hline 0 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 \end{array} \right]$$

The BLUP of  $\mathbf{u} = (y_{22}, y_{32})$  is the BLUE of  $E(\mathbf{u}|\mathbf{y})$ .

$$\begin{aligned} E(\mathbf{u}|\mathbf{y}) &= \begin{bmatrix} \mu_2 \\ \mu_2 \end{bmatrix} + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \begin{bmatrix} \mu_2 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \left( \mathbf{y} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_1 \\ \mu_1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \mu_2 \\ \mu_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} y_{21} - \mu_1 \\ y_{31} - \mu_1 \end{bmatrix} \end{aligned}$$

From part (a), the BLUE of  $\boldsymbol{\beta} = (\mu_1, \mu_2)'$  is  $\hat{\boldsymbol{\beta}}_{\text{BLUE}} = \begin{bmatrix} \hat{\mu}_2 \\ \hat{\mu}_2 \end{bmatrix} = \begin{pmatrix} \frac{151}{3} \\ \frac{161}{3} \end{pmatrix}$ , so

$$\begin{aligned} \hat{E}(\mathbf{u}|\mathbf{y}) &= \begin{bmatrix} \hat{\mu}_2 \\ \hat{\mu}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} y_{21} - \hat{\mu}_1 \\ y_{31} - \hat{\mu}_1 \end{bmatrix} \\ &= \begin{pmatrix} 52.5 \\ 54.5 \end{pmatrix} \end{aligned}$$

Therefore the BLUP of  $y_{22}$  and  $y_{32}$  are 52.5 and 54.5 respectively.