STAT 510 Homework 1
Due Date: 11:00 A.M., Wednesday, January 18

1. The very last expression on slide 4 of slide set 1 shows how to compute a matrix product as a sum of n matrices, each of dimension $m \times k$. Write down the matrices summed together to give the matrix product $\boldsymbol{AB}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ 4 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 3 & 3 & -2 & 4 \\ 5 & -1 & 2 & 3 \end{bmatrix}.$$

2. Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 2 \\ 3 & -1 & 7 \end{bmatrix}.$$

   (a) Use the algorithm on slide 11 of slide set 1 to find a generalized inverse of $\boldsymbol{A}$.

   (b) Use the `R` function `ginv` in the `MASS` package to find a generalized inverse of $\boldsymbol{A}$.

3. Use the definitions of t and F distributions to explain why the relationship between t and F distributions described on slide 33 of slide set 1 is true.

4. Imagine extending a string from $(0, 0)$, the origin in $\mathbb{R}^2$, to a random point (x, y) in $\mathbb{R}^2$, where $x \sim N(2, 1)$ independent of $y \sim N(1, 1)$. Use `R` to find the probability that the string will need to be longer than 6 units to reach from $(0, 0)$ to (x, y).

5. Suppose $z_1, z_2 \overset{iid}{\sim} N(0, 1)$. Find the distribution of the following random variables and prove that your answer is correct.

   (a) $(z_1 - z_2)^2/2$

   (b) $(z_1 + z_2)/|z_1 - z_2|$

6. Suppose $y_1, \ldots, y_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Let $\boldsymbol{y} = [y_1, \ldots, y_n]'$, and let $\bar{y}_{\cdot} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

   (a) Show that $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y}_{\cdot})^2$ can be written as $\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}$ for some matrix $\boldsymbol{B}$.

   (b) Prove that $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$ using the result on slide 27 of slide set 1.

7. The main purpose of this question is to prove several results about orthogonal project matrices stated on slide 5 of slide set 2 (*A Review of Some Key Linear Model Results*).

   (a) Prove that a matrix $\boldsymbol{A}$ is $\boldsymbol{0}$ if and only if $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{0}$. (Hint: What are the diagonal elements of $\boldsymbol{A}'\boldsymbol{A}$?)

   (b) Prove that $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B}$ if and only if $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}\boldsymbol{B}$. (Note that the "if" part of the proof, i.e.,

$$\boldsymbol{XA} = \boldsymbol{XB} \implies \boldsymbol{X}'\boldsymbol{XA} = \boldsymbol{X}'\boldsymbol{XB},$$

   holds trivially. Thus, proving the converse, i.e.,

$$\boldsymbol{X}'\boldsymbol{XA} = \boldsymbol{X}'\boldsymbol{XB} \implies \boldsymbol{XA} = \boldsymbol{XB},$$

is the challenging part. One proof starts like this

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} \quad &\Longrightarrow \quad \boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} - \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} = \boldsymbol{0} \\ &\Longrightarrow \quad \boldsymbol{X}'\boldsymbol{X}(\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{0} \end{aligned}$$

Now if you multiply on the left by the appropriate matrix, you can use the result of part (a) to help complete the proof.)

(c) Use the definition of generalized inverse and the result of part (b) to prove that

$$\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{X} = \boldsymbol{X}$$

for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

(d) Prove that if $\boldsymbol{A}$ is any symmetric matrix and $\boldsymbol{G}$ is any generalized inverse of $\boldsymbol{A}$, then it must be true that $\boldsymbol{G}'$ is also a generalized inverse of $\boldsymbol{A}$.

(e) Use the results of problems (c) and (d) to prove that

$$\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}' = \boldsymbol{X}'$$

for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

(f) Show that idempotency of $\boldsymbol{P}_{\boldsymbol{X}}$ (i.e., $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{P}_{\boldsymbol{X}}$) follows from the result of part (c) and, alternatively, from the result of part (e).

(g) Use parts (c) and (e) to prove that

$$\boldsymbol{X}\boldsymbol{G}_1\boldsymbol{X}' = \boldsymbol{X}\boldsymbol{G}_2\boldsymbol{X}'$$

for any two generalized inverses of $\boldsymbol{X}'\boldsymbol{X}$ denoted by $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$. (This says that $\boldsymbol{P}_{\boldsymbol{X}}$ is the same matrix no matter which generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$ is used to compute it.)

(h) Use (d) and (g) to prove that $\boldsymbol{P}_{\boldsymbol{X}}$ is symmetric (i.e., $\boldsymbol{P}'_{\boldsymbol{X}} = \boldsymbol{P}_{\boldsymbol{X}}$).