# Algebraic Number Theory

## Problem sheet 9

In this sheet, let us fix a prime number $p$.

1. (3+3+2 points) Show that the direct limit is exact, the inverse limit is left exact, moreover, the inverse limit is exact on compact Hausdorff abelian groups. More precisely, the left exactness in case of inverse limits means the following: Let $I$ be a right-filtered poset and $(A_i)_{i\in I}$, $(B_i)_{i\in I}$, $(C_i)_{i\in I}$ be inverse systems of abelian groups. Assume moreover, that for each index $i$ there is a short exact sequence
$$0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$$
that is compatible with the connecting maps, ie. for all $i \leq j \in I$ the diagram
$$\begin{tikzcd} 0 \arrow[r] & A_j \arrow[r, "\alpha_j"] \arrow[d, "f_{ij}"'] & B_j \arrow[r, "\beta_j"] \arrow[d, "g_{ij}"'] & C_j \arrow[r] \arrow[d, "h_{ij}"] & 0 \\ 0 \arrow[r] & A_i \arrow[r, "\alpha_i"] & B_i \arrow[r, "\beta_i"] & C_i \arrow[r] & 0 \end{tikzcd}$$
is commutative where $f_{ij}$, $g_{ij}$, $h_{ij}$ denote the connecting homomorphisms. Then the sequence
$$0 \to \varprojlim A_i \xrightarrow{\varprojlim \alpha_i} \varprojlim B_i \xrightarrow{\varprojlim \beta_i} \varprojlim C_i$$
is exact, too. Moreover, if $A_i, B_i, C_i$ are compact Hausdorff abelian groups and the above homomorphisms ar all continuous then the sequence
$$0 \to \varprojlim A_i \xrightarrow{\varprojlim \alpha_i} \varprojlim B_i \xrightarrow{\varprojlim \beta_i} \varprojlim C_i \to 0$$
is exact, too.

2. (3+3 points)Let $M_i$, $i \in I$ (resp. $N_j$, $j \in J$) ba a direct (resp. inverse) system of abelian groups and let $M, N$ be arbitrary abelian groups. Show that Hom, respectively $\otimes$, preserves $\varprojlim$, resp. $\varinjlim$. In other words, show that
$$\mathrm{Hom}(M, \varprojlim_j N_j) \cong \varprojlim_j \mathrm{Hom}(M, N_j)$$
$$\mathrm{Hom}(\varinjlim_i M_i, N) \cong \varprojlim_i \mathrm{Hom}(M_i, N)$$
$$\varinjlim_i (M_i \otimes N) \cong (\varinjlim_i M_i) \otimes N.$$
**Remark.** In fact, every left adjoint preserves colimit and every right adjoint preserves limit.

In the following problems we construct ($p$-typical) Witt rings of perfect rings of characteristic $p$. Let $A$ be a commutative ring with identity in which $p \in A$ is not a zero divisor and the natural map $A \to \varprojlim_n A/p^nA$ is an isomorphism (ie. $A$ is *$p$-adically complete*). Assume further that $R := A/pA$ is a *perfect ring of characteristic* $p$, i.e. the Frobenius map $R \to R$, $x \mapsto x^p$ is bijective. We call such rings $A$ **strict $p$-ring**. For example $A = \mathbb{Z}_p$ is a strict $p$-ring.

3. (2 points) Let $k$ be a field of characteristic $p$, show that the Frobenius map $x \mapsto x^p$ is injective, prove that it is surjective if and only if $k$ is perfect field, i.e. every irreducible polynomial over $k$ is separable. Give an example of an imperfect field.

4. (2 points) Show that the Frobenius endomorphism on a commutative ring $R$ of characteristic $p$ is injective if and only if $R$ is reduced.

5. (3 points) Let $A$ be a strict $p$-ring, in particular $R = A/pA$ is a perfect ring of characteristic $p$. Denote by $\hat{x} \in A$ a fixed arbitrary lift of an element $x \in R$ (ie. $x = \hat{x} + pA$). Show that the limit $[x] := \lim_{n\to\infty}(\widehat{x^{p^{-n}}})^{p^n}$ exists in the $p$-adic topology on $A$ and does not depend on the choices of the lifts. Further show that $[xy] = [x][y]$. The element $[x]$ is called the multiplicative (or **Teichmüller**-) representative of $x$.

Our next goal is to construct a strict $p$-ring $W(R)$ for any perfect ring $R$ of characteristic $p$ such that $R \cong W(R)/pW(R)$. The ring $W(R)$ is called **the Witt ring** of $R$. The elements of $W(R)$ are formal power series of the form $\sum_{i=0}^{\infty} p^i[x_i]$ in the "variable" $p$ where $x_i \in R$. The formal expressions $[x_i]$ are the multiplicative representatives of $x_i$. For the definition of addition and multiplication on such formal sums one first needs to construct the Witt ring of the free perfect ring of characteristic $p$ generated by countably many elements and investigate the addition and multiplication therein. Let $X_0, X_1, \dots, Y_0, Y_1, \dots$ be two infinite series of formal variables. For all $0 \leq n$ and $0 \leq i$ consider the formal $p^n$th root $X_i^{p^{-n}}$, resp. $Y_i^{p^{-n}}$ of $X_i$, resp. of $Y_i$ (ie. these are formal variables, as well, but we quotient out by the relations $(X_i^{p^{-n}})^p = X_i^{p^{-n+1}}$, resp. $(Y_i^{p^{-n}})^p = Y_i^{p^{-n+1}}$ in the polynomial ring). Put

$$\begin{aligned}
\mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0] &:= \bigcup_n \mathbb{Z}_p[X_i^{p^{-n}}, Y_i^{p^{-n}} \mid i \geq 0] \; ; \\
S &:= \varprojlim_n \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]/(p^n) \; .
\end{aligned}$$

6. (3 points) Verify that $S$ is a strict $p$-ring. In particular, there exist polynomials $S_i, P_i \in S/pS = \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]$ whose multiplicative representatives satisfy

$$\begin{aligned}
\left(\sum_{i=0}^{\infty} p^i X_i\right) + \left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [S_i] \\
\left(\sum_{i=0}^{\infty} p^i X_i\right)\left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [P_i] \; .
\end{aligned}$$

7. (2 points) Determine the polynomials $S_0, S_1, P_0, P_1 \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]$.

8. (3 points) Let $R$ be an arbitrary perfect ring of characteristic $p$ and put $W(R) = \{r = (r_0, r_1, \dots) \mid r_i \in R, i \geq 0\} = R^{\mathbb{N}}$ as a set. Consider the following operations: $(r+s)_n := S_n(r_0, r_1, \dots, s_0, s_1, \dots)$ and $(rs)_n := P_n(r_0, r_1, \dots, s_0, s_1, \dots)$. Show that these make $W(R)$ into a strict $p$-ring.

9. (3 points) Verify the following universal property of $W(R)$: if $A$ is any strict $p$-ring and $\varphi\colon R \to A/pA$ is a (unital) ring homomorphism then there exists a unique homomorphism $\tilde{\varphi}\colon W(R) \to A$ lifting $\varphi$. In particular, $W$ is a functor. Note: the Frobenius homomorphism $\mathrm{Frob}\colon R \to R$ can also be lifted to $W(R)$-be. We call this unique homomorphism **the Frobenius lift**.

10. (3 points) Show that the functor $R \mapsto W(R)$ is an equivalence of categories between the category of strict $p$-rings and the category of perfect rings of characteristic $p$ with quasi-inverse $A \mapsto A/pA$.