

Algebraic Number Theory

Problem sheet 10

1. (1+2+2 points) Let K be a nonarchimedean valuation field, $f(X) := \sum_{n \geq 0} a_n X^n$ a power series over K . We define **the radius of convergence** of f as follows:

$$r_f := \sup\{r \in \mathbb{R}_{\geq 0} \mid |a_n| r^n \rightarrow 0\}$$

- (a) show that $r_f = \sup\{r \in \mathbb{R}_{\geq 0} \mid |a_n| r^n \text{ is bounded}\}$
 - (b) Let L be an extended valuation field of K , and assume L is complete. Show that for any $x \in L$, if $|x| < r_f$, then $f(x)$ is convergent; if $|x| > r_f$, then $f(x)$ is divergent.
 - (c) Show that **Hadamard's formula**: $r_f = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$
2. (2 points) Show that the radius of convergence of $\log(1+X) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n}$ is 1 and show that $\log(1+X)$ is divergent at x , with $|x| = 1$.
3. (2 points) Show that the radius of convergence of $\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}$ is $|p|^{\frac{1}{p-1}}$ and show that $\exp(X)$ is divergent at x , with $|x| = |p|^{\frac{1}{p-1}}$.
4. (3 points) Let K/\mathbb{Q}_p be a finite extension with absolute ramification index e and maximal ideal \mathfrak{p} . Show that $\exp: \mathfrak{p}^n \rightarrow U^{(n)} = 1 + \mathfrak{p}^n$ and $\log: U^{(n)} \rightarrow \mathfrak{p}^n$ establish an isomorphism between the additive group \mathfrak{p}^n and the multiplicative group $U^{(n)}$ assuming $n > \frac{e}{p-1}$. In particular, when n is large enough, $U^{(n)}$ is torsion-free as an abelian group.
5. (3 points) Let $|K : \mathbb{Q}_p| = d$ and $\pi \in K$ be a prime element. Show the isomorphism $K^\times \cong \pi^{\mathbb{Z}} \oplus Z_{q-1} \oplus Z_{p^a} \oplus Z_p^d$ for a suitable integer $a \geq 0$ where q denotes the cardinality of the residue field.
6. (2 points) For any $z \in \mathbb{Z}_p$, show that the radius of convergence of the binomial series $(1+X)^z = \sum_{n=0}^{\infty} \binom{z}{n} X^n$ is greater than or equal to $|p|^{\frac{1}{p-1}}$. Moreover, if $|x| < |p|^{\frac{1}{p-1}}$, then we have $(1+x)^z = \exp(z \log(1+x))$ in this case.
7. (3 points) Let K/\mathbb{Q}_p be finite. Show that a subgroup of K^\times is open if and only if it has finite index.
8. (3 points) Let K/\mathbb{Q}_p be a finite extension, $\pi \in K$ be a prime element, and v_π be the normalized valuation (ie. $v_\pi(\pi) = 1$). Since K is a locally compact abelian group, it admits a translation invariant Haar measure dx , that is unique if we assume $\int_{\mathcal{O}_K} dx = 1$. Show that $|a|_\pi = \int_{a\mathcal{O}_K} dx$ where the absolute value $|\cdot|$ is normalized so that $|a|_\pi := q^{-v_\pi(a)}$ where $q = |\mathcal{O}_K/(\pi)|$ is the cardinality of the residue field. Further show that $\frac{dx}{|x|_\pi}$ is a translation invariant Haar measure on the multiplicative group K^\times .
9. (3 points) Let K/\mathbb{Q}_p be finite. Show that the slopes of the Newton polygon of a polynomial $f(x) \in K[x]$ are exactly the π -adic valuations of its roots (with multiplicity) where $\pi \in K$ is a prime element (and the Newton polygon is constructed using the π -adic valuation).
10. (3 points) Let K/\mathbb{Q}_p be finite. Assume that $f(x) \in K[x]$ is irreducible. Show that the Newton polygon of f consists of a single segment. Give examples that this segment may or may not contain a lattice point apart from the end points.