Algebraic Number Theory

Problem sheet 4

1. (1+2+2 points) Let $\mathfrak{a}$ be a fractional ideal of K, then $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1}$ for some $\mathfrak{b}, \mathfrak{c} \subseteq \mathcal{O}_K$ integral ideals, define the norm of $\mathfrak{a}$ as follows:

$$N(\mathfrak{a}) := \tfrac{N(\mathfrak{b})}{N(\mathfrak{c})}$$

   (a) Prove that the norm of fractional ideal is well-defined and $N : J_K \mapsto \mathbb{R}_{>0}$ is a homomorphism.

   (b) Show that $|d(\mathfrak{a})| = |d_K| \cdot N(\mathfrak{a})^2$.

   (c) Define $\delta_K^{-1} := \{x \in K \mid \mathrm{Tr}(xy) \in \mathbb{Z}, \forall y \in \mathcal{O}_K\}$, show that δ_K^{-1} is a fractional ideal. Let δ_K denote its inverse, called **different** of K. Show that $N(\delta_K) = |d_K|$.

2. (3 points)(**Hermite**) For any integer d, prove that there are only finitely many number fields with discriminant d. (Recall also that we have seen that, if $d = d_K$, then $d \equiv 0, 1(4)$).

3. (2 points)(**Universal Mapping Property**) Let R and R' be commutative rings and $0 \notin S \subset R$ be a multiplicatively closed subset. Assume further that $\varphi\colon R \to R'$ is a ring homomorphism such that $\varphi(S) \subseteq R'^{\times}$. Show that φ extends uniquely to a ring homomorphism $\tilde{\varphi}\colon RS^{-1} \to R'$.

   Remark. It is easy to see that the result still holds when R' is noncommutative.

4. (3 points) Let R be a commutative ring and $0 \notin S \subset R$ be a multiplicatively closed subset. Show that RS^{-1} is a flat R-module, ie. $RS^{-1} \otimes_R \cdot$ is an exact functor.

5. (3 points) Let R be a commutative ring and M a module, prove that TFAE:

   (a) $M = 0$;

   (b) $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \lhd R$;

   (c) $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \lhd R$.

6. (3 points) Let R be a commutative ring and let $f\colon M \to N$ be an R-module homomorphism between R-modules M, N. Show that TFAE:

   (*i*) f is injective (resp. surjective);

   (*ii*) $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective) for all prime ideals $\mathfrak{p} \lhd R$;

   (*iii*) $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for all maximal ideals $\mathfrak{m} \lhd R$.

   Where $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$, $M_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R M$, $N_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R N$, and $N_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R N$.

7. (3 points) Let R be an integral domain and $0 \notin S \subset R$ be a multiplicatively closed subset. Assume RS^{-1} is integral over R. Show that $RS^{-1} = R$.

8. (3 points) (Nakayama's Lemma) Let R be a local ring with maximal ideal $\mathfrak{m} \lhd R$ and let M be a finitely generated R-module. Further let $N \leq M$ be a submodule such that $M = N + \mathfrak{m}M$. Show that $M = N$.

9. (3 points) Let K be a field and $v\colon K^{\times} \to \mathbb{Z}$ be a surjective group homomorphism and let $v(0) = \infty$. Assume $v(x + y) \geq \min(v(x), v(y))$. Verify that $\mathcal{O} := \{\alpha \in K \mid v(\alpha) \geq 0\}$ is a DVR.

10. (3 points) Prove that a ring R is a DVR if and only if R is Dedekind and local ring.

11. (3 points) Let $\mathcal{O}$ be a Dedekind domain and let k be a positive integer. Let $\mathfrak{p}_i \lhd \mathcal{O}$ be a prime ideal for all $1 \leq i \leq k$, $x_i \in K$ (where K is the field of fractions of $\mathcal{O}$) and $n_i \in \mathbb{Z}$. Show that there exists an element $x \in K$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ $(i = 1, \ldots, k)$ and $v_{\mathfrak{p}}(x) \geq 0$ for all primes $\mathfrak{p} \neq \mathfrak{p}_i$.

12. Let $K/\mathbb{Q}$ be a number field and S be a finite set of prime ideals in $\mathcal{O}_K$. Put $X = \mathrm{Spec}(\mathcal{O}_K) \setminus S$.

    (*a*) (6+1 points) Verify that the sequence

$$1 \to \mathcal{O}^\times \to \mathcal{O}(X)^\times \to \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}) \setminus X} K^\times/\mathcal{O}_{\mathfrak{p}}^\times \to Cl(\mathcal{O}) \to Cl(\mathcal{O}(X)) \to 1 \ .$$

    is exact (construction of the maps is part of the problem). Further show that $K^\times/\mathcal{O}_{\mathfrak{p}}^\times \cong \mathbb{Z}$ for all primes $0 \neq \mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$.

    (*b*) (2 points) Verify that $Cl(\mathcal{O}_K(X))$ is finite and $\mathcal{O}_K(X)^\times \cong \mu(K) \times \mathbb{Z}^{|S|+r+s-1}$ where r (resp. s) is the number of real embeddings (resp. of pairs of complex embeddings) of K.

13. (3 points) (Newton polygon) Let $\mathcal{O}$ be a DVR with field of fractions K and valuation $v \colon K^\times \to \mathbb{Z}$. The Newton polygon of a polynomial $f(x) = \sum_{i=0}^n a_i x^i \in K[x]$ of degree n is the lower convex hull of the points $(-i, v(a_i))$ in the plain (ie. a convex broken line connecting $(-n, v(a_n))$ with $(0, v(a_0))$ whose vertices are among the points $(-i, v(a_i))$ such that none of these points are below the broken line). Further denote by $S(f)$ the multiset of slopes of the Newton polygon, ie. $S(f)$ is a multiset with n elements such that the multiplicity of the rational number s in $S(f)$ is the length of the projection of line in the Newton polygon with slope s. Show that $S(fg) = S(f) \cup S(g)$. In particular, if the Newton polygon of the polynomial $f(x) \in K[x]$ consists of a single segment with no lattice points apart from the two endpoints then f is irreducible.