Algebraic Number Theory

Problem sheet 6

1. (3 points) Assume that L/K is a finite Galois extension of number fields ($K/\mathbb{Q}$ finite) with non-cyclic Galois group. Show that there are at most finitely many primes $\mathfrak{p}$ in K with only one prime divisor in L.
2. (3 points) Let $K/\mathbb{Q}$ be a Galois extension with non-abelian Galois group. Show that no primes p are inert in $\mathcal{O}_K$.
3. (5 points) Let $K/\mathbb{Q}$ be a finite extension. Show that there are infinitely many primes p that split completely in $\mathcal{O}_K$.
4. (3 points) Let L/K be a finite extension of number fields and let $L \leq F$ be the Galois closure of L. Put $G = \mathrm{Gal}(F/K)$, $H = \mathrm{Gal}(F/L)$ and $G_P \leq G$ for the decomposition subgroup of a prime $P \lhd \mathcal{O}_F$ dividing $\mathfrak{p} \lhd \mathcal{O}_K$. Establish a natural bijection between the primes in L above $\mathfrak{p}$ and the double cosets $H\backslash G/G_P$. Using this give a new proof of the fact that a prime splits completely in a finite extension if and only if it splits completely in its Galois closure. (+3points)
5. (5 points) Let L/K be a – not necessarily Galois – solvable extension of number fields of degree p where p is a prime (ie. the Galois group of the Galois closure of the extension is solvable). Assume further that the prime $\mathfrak{p} \lhd \mathcal{O}_K$ does not ramify in L and has at least two distinct prime divisors in L of inertia degree 1. Show that $\mathfrak{p}$ splits completely in L. (Hint: you may use without proof Galois's theorem stating that whenever G is a transitive solvable permutation group of prime degree then any element of G different from the identity has at most 1 fixed point.)
6. (4 points) Let A be a finite abelian group. Verify that there exists a finite Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \cong A$.

   Remark. The statement is true for any finite solvable group (a theorem of Shafarevich) but open for general finite groups.
7. (3 points) Let n be odd. Describe all quadratic extensions of $\mathbb{Q}$ contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$.
8. (3 points) Let $d \in \mathbb{Z}$ be squarefree. Show that there exists a positive integer n such that $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_n)$.

   Remark. This is a special case of Kronecker-Weber theorem, which we will prove finally in the class.
9. (3 points) Show that for $q \geq 3$ the quadratic subfields in $\mathbb{Q}(\zeta_{2^q})$ are exactly $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(i\sqrt{2})$.