

Algebraic Number Theory

Problem sheet 1

1. (2 points) Let p be an odd prime number, show that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$. (Hint: consider $\zeta_8 + \zeta_8^{-1}$ and mimic the proof of quadratic reciprocity)
2. (2+2+1 points)
 - (a) Prove that UFD is integrally closed.
 - (b) Prove that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed. Deduce that it is not UFD.
3. (3 points) Let $K = \mathbb{Q}[\sqrt[3]{2}]$. Show that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$.

Remark. More generally, let $K = \mathbb{Q}[\sqrt[3]{m}]$, where m is cubefree, then one may ask if $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{m}]$ holds. The answer is no. In fact, there is a clear classification for such K , see Marcus' book Page 28. For example, one can prove that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{m}]$ if and only if m is squarefree and $m \not\equiv \pm 1 \pmod{9}$

4. (3 points) Let $A \subseteq B$ integral domains and let $\beta \in B$ be an invertible element. Show that each element in $A[\beta] \cap A[\beta^{-1}]$ is integral over A . (Hint: For $\alpha \in A[\beta] \cap A[\beta^{-1}]$ find a finitely generated A -submodule $M \subseteq B$ such that $\alpha M \subseteq M$.)
5. (1+1+2+2 points) The goal here is to show that an integral domain R is integrally closed if and only if $R[x]$ is integrally closed.
 - (a) One side is easy: Prove that if $R[x]$ is integrally closed, then so is R .
 - (b) For the other side, first let us reduce the statement to showing that $R[x]$ is integrally closed in $K[x]$ where K is the field of fractions of R . (Hint: $K[x]$ is contained in the field of fractions of $R[x]$ and it is integrally closed.)
 - (c) Let $f, g \in K[x]$ be *monic* polynomials such that fg lies in $R[x]$. Show that both f and g are in $R[x]$. (Hint: write both polynomials as a product of linear factors over a bigger field.)
 - (d) If $f \in K[x]$ is the root of a monic polynomial of degree k with coefficients in $R[x]$ then $f + x^N$ is also the root of another monic polynomial $g_N \in R[x][y]$ of degree k (in the variable y). Increase N so that the constant term of g_N can be written as a product of two monic polynomials (in $R[x]$) one of which is $f + x^N$.

Remark. Recall that, in algebraic geometry, a normal variety is a variety whose section over affine open subset is integrally closed. Now the exercise tells us that a variety X is normal if and only if $X \times \mathbf{P}^n$ is normal for every n .

6. (1 point) What is the trace and the norm of $1 + \sqrt{2}$ in the extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$?
 7. (2 points) Consider the extension $\mathbb{Q}(i)/\mathbb{Q}$. This has a Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, in particular it is cyclic. What is the norm of an element of the form $a + bi$? What does Hilbert 90 tell us in this special case on Pythagorean triples?
 8. (3 points) Let K be a field containing a primitive n th root of unity and L/K be a Galois extension with Galois group $\text{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z}$. Show that $L = K(\sqrt[n]{\alpha})$ for some α in K . (Hint: Use Hilbert's Theorem 90.)
 9. (3 points) Let $f(x) \in \mathbb{Z}[x]$ be an irreducible monic polynomial. Assume that the Galois group of the splitting field of f over \mathbb{Q} is abelian and there is an α in \mathbb{C} such that $f(\alpha) = 0$ and $|\alpha| = 1$. Show that all the other roots of f (in \mathbb{C}) have absolute value 1.
- Remark.** It is possible that some conjugates have absolute value 1 while the other do not have.
10. (4 points) Let α be an algebraic integer, prove that α is a root of unity if and only if all of its Galois conjugates have absolute value 1.
 11. (3+3 points) Let $K \leq L \leq M$ be finite extensions. Show that $N_{M/K} = N_{L/K} \circ N_{M/L}$ and $\text{Tr}_{M/K} = \text{Tr}_{L/K} \circ \text{Tr}_{M/L}$
 12. (3 points) Let L/K be a non-separable extension. Show that $\text{Tr}_{L/K}$ is identically 0. (Hint: using the transitivity of the trace reduce the problem to the case when you are adjoining the p th root of an element to a field K of characteristic p .)
 13. (2+2+2+2+2 points) Let L/K be a Galois extension. The goal of this problem is to prove the normal basis theorem: There exists an $\gamma \in L$ such that the elements $\{\sigma(\gamma) \mid \sigma \in \text{Gal}(L/K)\}$ are linearly independent over K ie. they form a basis of L as a K -vector space (bases of this form are called *normal bases*).
- (a) Let $f(x) \in K[x]$ be a separable monic polynomial that splits over L as a product $f(x) = \prod_{i=1}^n (x - \alpha_i)$. Put $g_i(x) := \frac{f'(x)}{f'(x)(x - \alpha_i)} \in L[x]$. Verify (i) $\sum_{i=1}^n g_i(x) = 1$ (partial fraction decomposition of $1/f(x)$) and

$$(ii) \ g_i(x)g_j(x) \equiv \begin{cases} 0 \pmod{f(x)} & \text{if } i \neq j \\ g_i(x) \pmod{f(x)} & \text{if } i = j \end{cases}.$$
 - (b) Let L/K be a Galois extension as above and pick α such that $L = K(\alpha)$ and denote by $f \in K[x]$ the minimal polynomial of α . Put $\text{Gal}(L/K) = \{\text{id} = \sigma_1, \dots, \sigma_n\}$ and $\alpha_i = \sigma_i(\alpha) \in L$. Let $A \in L[x]^{n \times n}$ be the matrix with j th entry in the i th row being $\sigma_i(\sigma_j(g_1(x))) \in L[x]$. Using part (a) show that $A^T A \equiv I \pmod{f(x)}$ (where I is the identity matrix).
 - (c) Assume K is infinite. Using part (b) show that there is a $\beta \in K$ with $\det(A(\beta)) = \det(\sigma_i \sigma_j(g_1(\beta)))_{i,j} \neq 0$. In particular, $\{\sigma_1(\gamma), \dots, \sigma_n(\gamma)\}$ is a normal basis for $\gamma = g_1(\beta)$.

- (d) Assume $K \cong \mathbb{F}_q$ is finite and let $n = |L/K|$ be the degree. Use Dedekind's Lemma and the fact that $\text{Gal}(L/K)$ is cyclic of order n generated by the Frobenius Frob_q to determine the minimal polynomial of $\text{Frob}_q: L \rightarrow L$ as a K -linear map.
- (e) Using the theorem of elementary divisors (or otherwise) show that $L \cong K[x]/(x^n - 1)$ as modules over $K[x]$ where x acts on L via Frob_q . Let $\gamma \in L$ be the element corresponding to $1 + (x^n - 1)$ under this isomorphism.