Algebraic Number Theory

Problem sheet 9

1. (3+3+2 points) Show that the direct limit is exact, the inverse limit is left exact, moreover, the inverse limit is exact on compact Hausdorff abelian groups. More precisely, the left exactness in case of inverse limits means the following: Let I be a right-filtered poset and $(A_i)_{i\in I}$, $(B_i)_{i\in I}$, $(C_i)_{i\in I}$ be inverse systems of abelian groups. Assume moreover, that for each index i there is a short exact sequence
$$0 \to A_i \overset{\alpha_i}{\to} B_i \overset{\beta_i}{\to} C_i \to 0$$
that is compatible with the connecting maps, ie. for all $i \leq j \in I$ the diagram
$$\begin{tikzcd} 0 \arrow[r] & A_j \arrow[r, "\alpha_j"] \arrow[d, "f_{ij}"'] & B_j \arrow[r, "\beta_j"] \arrow[d, "g_{ij}"'] & C_j \arrow[r] \arrow[d, "h_{ij}"] & 0 \\ 0 \arrow[r] & A_i \arrow[r, "\alpha_i"] & B_i \arrow[r, "\beta_i"] & C_i \arrow[r] & 0 \end{tikzcd}$$
is commutative where f_{ij}, g_{ij}, h_{ij} denote the connecting homomorphisms. Then the sequence
$$0 \to \varprojlim A_i \overset{\varprojlim \alpha_i}{\to} \varprojlim B_i \overset{\varprojlim \beta_i}{\to} \varprojlim C_i$$
is exact, too. Moreover, if A_i, B_i, C_i are compact Hausdorff abelian groups and the above homomorphisms ar all continuous then the sequence
$$0 \to \varprojlim A_i \overset{\varprojlim \alpha_i}{\to} \varprojlim B_i \overset{\varprojlim \beta_i}{\to} \varprojlim C_i \to 0$$
is exact, too.

2. Let M_i, $i \in I$ (resp. N_j, $j \in J$) ba a direct (resp. inverse) system of abelian groups and let M, N be arbitrary abelian groups. Under what conditions can Hom, respectively $\otimes$ be interchanged with $\varinjlim$, resp. with $\varprojlim$? For example is it true that $\mathrm{Hom}(\varprojlim_i N_i, M) \cong \varprojlim_i \mathrm{Hom}(N_i, M)$ or $\varinjlim_i (M_i \otimes N) \cong (\varinjlim_i M_i) \otimes N$? Positive statements are 3 points each, counterexamples 2 points each. (If needed, you may assume the groups are finitely generated or finite, etc.)

In the following problems we construct (p-typical) Witt rings of perfect rings of characteristic p. Let A be a commutative ring with identity in which $p = \underbrace{1 + \cdots + 1}_{p} \in A$ is not a zero divisor and the natural map $A \to \varprojlim_n A/p^n A$ is an isomorphism (ie. A is *p-adically complete*). Assume further that $R := A/pA$ is a *perfect ring of characteristic* p, ie. raising to the p-th power ("Frobenius") is bijective: for all $x \in R$ there exists a unique $x^{p^{-1}} := y \in R$ such that $y^p = x$. We call such rings A strict p-rings. For example $A = \mathbb{Z}_p$ is a strict p-ring.

3. (2 points) Show that the p-Frobenius endomorphism (ie. raising to the power p) is always injective on a field k of characteristic p and it is surjective if and only if k is perfect (ie. no irreducible polynomials over k have multiple roots). Give an example of an imperfect field of characteristic p.

4. (2 points) Show that the p-Frobenius endomorphism on a commutative ring R (with identity) of characteristic p is injective if and only if R is reduced (ie. has no nonzero nilpotent elements).

5. (3 points) Let A be a strict p-ring, in particular $R = A/pA$ is a perfect ring of characteristic p. Denote by $\hat{x} \in A$ a fixed arbitrary lift of an element $x \in R$ (ie. $x = \hat{x} + pA$). Show that the limit $[x] := \lim_{n\to\infty}(\widehat{x^{p^{-n}}})^{p^n}$ exists in the p-adic topology on A and does not depend on the choices of the lifts. Further show that $[xy] = [x][y]$. The element $[x]$ is called the multiplicative (or Teichmüller-) representative of x.

Our next goal is to construct a strict p-ring $W(R)$ for any perfect ring R of characteristic p such that $R \cong W(R)/pW(R)$. The ring $W(R)$ is called the Witt ring of R. The elements of $W(R)$ are formal power series of the form $\sum_{i=0}^{\infty} p^i[x_i]$ in the "variable" p where $x_i \in R$. The formal expressions $[x_i]$ are the multiplicative representatives of x_i. For the definition of addition and multiplication on such formal sums one first needs to construct the Witt ring of the free perfect ring of characteristic p generated by countably many elements and investigate the addition and multiplication therein. Let $X_0, X_1, \dots, Y_0, Y_1, \dots$ be two infinite series of formal variables and p be a fixed prime. For all $0 \leq n$ and $0 \leq i$ consider the formal p^nth root $X_i^{p^{-n}}$, resp. $Y_i^{p^{-n}}$ of X_i, resp. of Y_i (ie. these are formal variables, as well, but we quotient out by the relations $(X_i^{p^{-n}})^p = X_i^{p^{-n+1}}$, resp. $(Y_i^{p^{-n}})^p = Y_i^{p^{-n+1}}$ in the polynomial ring). Put

$$\begin{aligned}
\mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0] &:= \bigcup_n \mathbb{Z}_p[X_i^{p^{-n}}, Y_i^{p^{-n}} \mid i \geq 0] \; ; \\
S &:= \varprojlim_n \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]/(p^n) \; .
\end{aligned}$$

6. (3 points) Verify that S is a strict p-ring. In particular, there exist polynomials $S_i, P_i \in S/pS = \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]$ whose multiplicative representatives satisfy

$$\begin{aligned}
\left(\sum_{i=0}^{\infty} p^i X_i\right) + \left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [S_i] \\
\left(\sum_{i=0}^{\infty} p^i X_i\right)\left(\sum_{i=0}^{\infty} p^i Y_i\right) &= \sum_{i=0}^{\infty} p^i [P_i] \; .
\end{aligned}$$

7. (2 points) Determine the polynomials $S_0, S_1, P_0, P_1 \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}} \mid i \geq 0]$.

8. (3 points) Let R be an arbitrary perfect ring of characteristic p and put $W(R) = \{r = (r_0, r_1, \dots) \mid r_i \in R, i \geq 0\} = R^{\mathbb{N}}$ as a set. Consider the following operations: $(r+s)_n := S_n(r_0, r_1, \dots, s_0, s_1, \dots)$ and $(rs)_n := P_n(r_0, r_1, \dots, s_0, s_1, \dots)$. Show that these make $W(R)$ into a strict p-ring.

9. (3 points) Verify the following universal property of $W(R)$: if A is any strict p-ring and $\varphi\colon R \to A/pA$ is a (unital) ring homomorphism then there exists a unique homomorphism $\tilde{\varphi}\colon W(R) \to A$ lifting φ. In particular, W is a functor. Note: the p-Frobenius homomorphism $\mathrm{Frob}_p\colon R \to R$ can also be lifted to $W(R)$-be. We call this unique homomorphism the Frobenius lift.

10. (3 points) Verify that the functors $R \mapsto W(R)$ and $A \mapsto A/pA$ are quasi-inverses of each other, ie. for all perfect ring R of characteristic p there exists a natural isomorphism $\Phi_R\colon R \to W(R)/pW(R)$ and for all strict p-ring A there exists a natural isomorphism $\Psi_A\colon A \to W(A/pA)$. Naturality (eg. in case of R) means here that whenever $f\colon R_1 \to R_2$ is a ring homomorphism of perfect rings of characteristic p then the diagram

$$\begin{array}{ccc}
R_1 & \xrightarrow{f} & R_2 \\
{\scriptstyle \Phi_{R_1}}\downarrow & & \downarrow{\scriptstyle \Phi_{R_2}} \\
W(R_1)/pW(R_1) & \xrightarrow[f_*]{} & W(R_2)/W(R_2)
\end{array}$$

commutes. So the category of strict p-rings is equivalent to the category of perfect rings of characteristic p.