

# Algebraic Number Theory

## Problem sheet 5

1. (2+2 points) Decompose 7 and 11 in  $\mathcal{O}_K$ , where
  - (a)  $K = \mathcal{O}[\sqrt[3]{2}]$
  - (b)  $K = \mathbb{Q}[\theta]$ , where  $\theta^3 - \theta - 4 = 0$ .
2. (2+2 points) Determine the ideal class group  $Cl_K$  of a number field  $K$ , where
  - (a)  $K = \mathbb{Q}(\sqrt{-14})$ ,
  - (b)  $K = \mathbb{Q}(\sqrt[3]{2})$
3. (a) (2 points) Verify directly that an odd prime  $p$  ramifies in  $\mathbb{Q}(\sqrt{d})$  if and only if  $p \mid d$  ( $d$  squarefree). Further, it is inert if  $\left(\frac{d}{p}\right) = -1$  and splits into the product of two distinct primes if  $\left(\frac{d}{p}\right) = 1$ .  
(b) (2 points) What happens if  $p = 2$ ?
4. (3 points) Let  $f_n$  be the  $n$ -th element of the Fibonacci sequence ( $f_0 = 0$ ,  $f_1 = 1$ ,  $f_{n+1} = f_{n-1} + f_n$ ). Show  $f_p \equiv \left(\frac{p}{5}\right) \pmod{p}$  for all prime numbers  $p \neq 2, 5$ .
5. (2 points) Let  $I, J \triangleleft A$  be ideals in the Dedekind domain  $A$  and let  $B$  be the integral closure of  $A$  in the finite separable field extension  $L/K$  where  $K$  is the field of fractions of  $A$ . Verify that  $I = IB \cap A$  and  $I \mid J \Leftrightarrow IB \mid JB$ .
6. (3 points) Give an example of a pair  $\mathbb{Q} \leq K_1, K_2$  of finite extensions and prime  $p$  such that  $p$  totally ramifies in both extensions  $K_1$  and  $K_2$  (ie.  $r = 1 = f_1$ ) but not in the composite extension  $K_1 K_2$ . (Hint: You may even choose quadratic extensions.)
7. (a) (3 points) Assume  $\mathcal{O}_K = \mathbb{Z}[\alpha]$  for some  $\alpha \in \mathcal{O}_K$  and let  $p \in \mathbb{Z}$  be a prime. Show that there are at most  $p$  prime ideals  $\mathfrak{p}_i \triangleleft \mathcal{O}_K$  dividing  $p$  with  $f_i = 1$  (ie.  $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_p$ ).  
(b) (2 points) Show that 3 splits completely in  $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$ . Deduce that there does not exist  $\alpha \in K$  with  $\mathcal{O}_K = \mathbb{Z}[\alpha]$ .  
(c) (3 points) Find an integer  $0 < N \in \mathbb{Z}$  and an element  $\alpha \in \mathcal{O}_K$  such that  $\mathcal{O}_K[1/N] = \mathbb{Z}[1/N][\alpha]$  and decompose all primes  $p \mid N$  as a product of prime ideals in  $\mathcal{O}_K$ .
8. (4 points) Let  $\mathbb{Q} \leq K$  be a finite extension. Show that a prime  $\mathfrak{p} \triangleleft \mathcal{O}_K$  splits completely in  $(K \leq)L$  and in  $(K \leq)L'$  then it splits completely in the composite extension  $LL'$  as well.
9. (3 points) Let  $\mathbb{Q} \leq K \leq L$  be finite extensions and  $K \leq L \leq F$  be the Galois closure of  $L/K$ . Show that a prime  $\mathfrak{p} \triangleleft \mathcal{O}_K$  splits completely in  $L$  if and only if it splits completely in  $F$ .
10. (3 points) Show that no primes of  $\mathbb{Q}(\sqrt{-5})$  ramify in the extension  $\mathbb{Q}(\sqrt{-1}, \sqrt{-5})/\mathbb{Q}(\sqrt{-5})$ .