Algebraic Number Theory-ELTE

2024 autumn semester, exam

22th November

There are **total 60 points** in the exam and you are encouraged to attempt as many questions as possible, although the maximal points you could get from the exam is **30 points**. You have to finish the exam **within two hours**.

You may need the Minkowski bound: $(\frac{4}{\pi})^s \frac{n!}{n^n}\sqrt{|d_K|}$.

1. (10 points)
   (a) Let $K = \mathbb{Q}[\sqrt{-1}]$, determine the integral basis, discriminant, unit group and ideal class group.
   (b) By projection from the rational point $P = (1, 1)$ or otherwise, find all rational solutions of the quadratic equation $x^2 + y^2 = 1$.
   (c) Show that norm map is a group homomorphism from $K^\times$ to $\mathbb{Q}^\times$, and determine the kernel and cokernel.
   (d) Show that every prime ideal of $\mathbb{Q}[\sqrt{-5}]$ is unramified in $\mathbb{Q}[\sqrt{-1}, \sqrt{-5}]$
2. (15 points)Let $K = \mathbb{Q}[\sqrt[3]{2}]$ and $L = \mathbb{Q}[\sqrt[3]{2}, \zeta_3]$ where ζ_3 is a primitive 3-th root of unity.
   (a) For K, determine an integral basis, discriminant, unit group and ideal class group.
   (b) Show that a rational prime p is ramified in L if and only if $p = 2, 3$
   (c) Determine the Galois group of the Galois extension $L/\mathbb{Q}$.
   (d) Let $\mathfrak{P}$ be a prime of L such that $\mathfrak{P}|2$ or $\mathfrak{P}|3$, determine the inertia group and decomposition group of $\mathfrak{P}$
3. (5 points) Let R be a domain and K its fractional field. An R-submodule I of K is called **fractional ideal** if there exists nonzero $r \in R$ such that $rI \subset R$. Let J_K denote the set of all nonzero fractional ideal of R. We define $I \cdot J := \{\sum a_k b_k |\ a_k \in I \text{ and } b_k \in J\}$
   (a) Show that $I \cdot J$ is a fractional ideal and J_K is an abelian monoid with respect to this operation.
   (b) Show that J_K is a group if and only if R is Dedekind domain.(Hint: you can use the fact that R is Dedekind domian if and only if R is hereditary, i.e. every ideal is projective module)
4. (5 points) Assume $K = \mathbb{Q}[\sqrt{d}]$, d is squarefree. Define $\delta_K^{-1} := \{x \in K \mid \mathrm{Tr}(xy) \in \mathbb{Z}, \forall y \in \mathcal{O}_K\}$, determine δ_K^{-1} and show that δ_K^{-1} is a fractional ideal. Let δ_K denote its inverse, called **different** of K. Calculate δ_K and deduce that $N(\delta_K) = |d_K|$.
5. (5 points) Let $K/\mathbb{Q}$ be a finite extension. Show that there are infinitely many primes p that split completely in $\mathcal{O}_K$. Deduce that for any positive integer n, there are infinitely many primes p such that $p \equiv 1 \mod n$, which is a special case of Dirichlet theorem.

6. (5 points)

   (a) Let (a_n) be a sequence in $\mathbb{Q}_p$, show that the sequence converges if and only if $\lim(a_{n+1} - a_n) = 0$.

   (b) Let $n \geq 1$ be an integer and $n = a_0 + a_1 p + \cdots + a_r p^r$ in base p $(0 \leq a_i < p)$. Further put $s = a_0 + a_1 + \cdots + a_r$. Verify that $v_p(n!) = \frac{n-s}{p-1}$.

   (c) Deduce that the formal power series $\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}$ is convergent at $x \in \mathbb{Q}_p$ if and only if $|x| < |p|^{\frac{1}{p-1}}$

7. (5 points)Let $\hat{\mathbb{Q}}$ be a completion of $\mathbb{Q}$ with respect to a nontrivial valuation, i.e. $\hat{\mathbb{Q}} = \mathbb{R}$ or $\mathbb{Q}_p$, for some p.

   (a) Show that $\hat{\mathbb{Q}}$ is not algebraic closed field and $\overline{\hat{\mathbb{Q}}}$ is complete if and only if $\hat{\mathbb{Q}} = \mathbb{R}$.

   (b) Show that $\mathrm{Gal}(\hat{\mathbb{Q}}/\mathbb{Q}) = 1$ but $\mathrm{Gal}(\hat{K}/K) \neq 1$ where $K = \overline{\mathbb{Q}_p}$

   (c) Show that any two different completions of $\mathbb{Q}$ are not isomorphic as fields.

8. (10 points) Let p be a rational prime number and ζ a primitive p-th root of unity.

   (a) show that $\pi := \zeta - 1$ is a uniformizer of $\mathbb{Q}_p(\zeta)$ and $|\pi| = |p|^{\frac{1}{p-1}}$.

   (b) Write down the π-adic expansion of ζ^n and show that all minors of the Vandermonde determinant $(\zeta^{ij})_{0 \leq i,j \leq p-1}$ are not zero.