Algebraic Number Theory

Problem sheet 8

1. (3 points)(**Ostrowski**) Let K be a complete field with respect to an archimedean valuation $|\cdot|$. Show that K is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ such that the isomorphism induces a homeomorphism in the topology of K induced by $|\cdot|$.

2. (3 points) Let $K/\mathbb{Q}$ be a finite extension. Show that any nontrivial absolute value on K is equivalent to either the $\mathfrak{p}$-adic absolute value for some prime ideal $\mathfrak{p} \lhd \mathcal{O}_K$ (nonarchimedean case) or to the restriction of the usual absolute value of $\mathbb{C}$ via an embedding $\tau \colon K \to \mathbb{C}$ (archimedean case). Two such absolute values are equivalent if and only if both are archmimedean and the two corresponding embeddings $\tau_1, \tau_2 \colon K \to \mathbb{C}$ are the complex conjugate of each other.

3. (1 points) Write -1 as a p-adic integer $-1 = \sum_{i=0}^{\infty} a_i p^i$ ($a_i \in \{0, 1, \dots, p-1\}$).

4. (2 points) Write $2/3$ and $-2/3$ in 5-adic form.

5. (4 points) Verify that a p-adic number of the form $\sum_{i=-m}^{\infty} a_i p^i$ ($a_i = 0, 1, \dots, p-1$, $i \geq -m$) is rational if and only if the sequence of its digits is eventually periodic. Deduce that $\sum p^{n!}$ is not a rational number.

6. (3 points) Solve the equation $x^2 = 2$ in $\mathbb{Z}_7$.

7. (3 points) Let K be a completion of $\mathbb{Q}$ with respect to a valuation, i.e. $K = \mathbb{R}$ or $\mathbb{Q}_p$, for some p. Show that $\mathrm{Gal}(K/\mathbb{Q}) = 1$.

8. (2+2 points) Let $n \geq 1$ be an integer and $n = a_0 + a_1 p + \cdots + a_r p^r$ in base p ($0 \leq a_i < p$). Further put $s = a_0 + a_1 + \cdots + a_r$. Verify that $v_p(n!) = \frac{n-s}{p-1}$. Deduce that $\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}$ is convergent at $x \in \mathbb{Q}_p$ if and only if $|x| < |p|^{\frac{1}{p-1}}$

9. (2 points) Show that the sequence $1, 1/10, 1/100, \dots, 1/10^n, \dots$ does not converge in $\mathbb{Q}_p$ for any prime p.

10. (3 points) Let $\varepsilon \in 1 + p\mathbb{Z}_p$ and $\alpha = a_0 + a_1 p + a_2 p^2 + \dots$ be a p-adic integer with nth partial sum $s_n = a_0 + a_1 p + \cdots + a_n p^n \in \mathbb{Z}$. Show that the limit $\varepsilon^\alpha := \lim_{n\to\infty} \varepsilon^{s_n}$ exists in $\mathbb{Z}_p$ making $1 + p\mathbb{Z}_p$ a $\mathbb{Z}_p$-module (written multiplicatively).

11. (3 points) Assume $(a, p) = 1$ ($a \in \mathbb{Z}$). Show that the sequence a^{p^n} converges in $\mathbb{Q}_p$.

12. (2 points) Show that for primes $p \neq q$, $\mathbb{Q}_p \not\cong \mathbb{Q}_q$ and $\mathbb{R} \not\cong \mathbb{Q}_p$ as fields.

13. (1+2 points) Verify that the algebraic closure of $\mathbb{Q}_p$ is an infinite extension of $\mathbb{Q}_p$. Show that $\overline{\mathbb{Q}_p}$ is not complete.

14. (5 points) Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n + \cdots \in \mathbb{Z}_p[[X]]$ be a formal power series with coefficients in the ring of p-adic integers. Verify that f is convergent on the p-adic open unit disc $\{|X|_p < 1\}$ and has at most finitely many roots therein.