

Algebraic Number Theory

Problem sheet 6

1. (3 points) Assume that L/K is a finite Galois extension of number fields (K/\mathbb{Q} finite) with non-cyclic Galois group. Show that there are at most finitely many primes \mathfrak{p} in K with only one prime divisor in L .
2. (3 points) Let K/\mathbb{Q} be a Galois extension with non-abelian Galois group. Show that no primes p are inert in \mathcal{O}_K .
3. (5+2 points) Let K/\mathbb{Q} be a finite extension. Show that there are infinitely many primes p that split completely in \mathcal{O}_K . Deduce that for any positive integer n , there are infinitely many primes p such that $p \equiv 1 \pmod{n}$, which is a particular case of Dirichlet theorem.

Remark. In fact, the set of such primes has density $\frac{1}{n}$ if K/\mathbb{Q} is Galois extension and $n = [K : \mathbb{Q}]$, which is a special case of Chebotarev's density theorem.

4. (3 points) Let L/K be a finite extension of number fields and let $L \leq F$ be the Galois closure of L . Put $G = \text{Gal}(F/K)$, $H = \text{Gal}(F/L)$ and $G_P \leq G$ for the decomposition subgroup of a prime $P \triangleleft \mathcal{O}_F$ dividing $\mathfrak{p} \triangleleft \mathcal{O}_K$. Establish a natural bijection between the primes in L above \mathfrak{p} and the double cosets $H \backslash G / G_P$. Using this give a new proof of the fact that a prime splits completely in a finite extension if and only if it splits completely in its Galois closure. (+3points)
5. (5 points) Let L/K be a – not necessarily Galois – solvable extension of number fields of degree p where p is a prime (ie. the Galois group of the Galois closure of the extension is solvable). Assume further that the prime $\mathfrak{p} \triangleleft \mathcal{O}_K$ does not ramify in L and has at least two distinct prime divisors in L of inertia degree 1. Show that \mathfrak{p} splits completely in L . (Hint: you may use without proof Galois's theorem stating that whenever G is a transitive solvable permutation group of prime degree then any element of G different from the identity has at most 1 fixed point.)
6. (4 points) Let A be a finite abelian group. Verify that there exists a finite Galois extension L/\mathbb{Q} such that $\text{Gal}(L/\mathbb{Q}) \cong A$.

Remark. The statement is true for any finite solvable group (a theorem of Shafarevich) but open for general finite groups.

7. (3 points) Let n be odd. Describe all quadratic extensions of \mathbb{Q} contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$.
8. (3 points) Let $d \in \mathbb{Z}$ be squarefree. Show that there exists a positive integer n such that $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_n)$.

Remark. This is a special case of Kronecker-Weber theorem, which we will prove finally in the class.

9. (3 points) Show that for $q \geq 3$ the quadratic subfields in $\mathbb{Q}(\zeta_{2^q})$ are exactly $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(i\sqrt{2})$.