Algebraic Number Theory

Problem sheet 10

1. (1+2+2 points) Let$K$ be a nonarchimedean valuation field, $f(X) := \sum_{n\geq 0} a_n X^n$ a power series over K. We define **the radius of convergence** of f as follows:

$$r_f := \sup\{r \in \mathbb{R}_{\geq 0} |\ |a_n| r^n \to 0\}$$

   (a) show that $r_f = \sup\{r \in \mathbb{R}_{\geq 0} |\ |a_n| r^n \text{ is bounded}\}$
   (b) Let L be a extended valuation field of K, assume L is complete. Show that for any $x \in L$, if $|x| < r_f$, then $f(x)$ is convergent; if $|x| > r_f$, then $f(x)$ is divergent.
   (c) Show that **Hadamard's formula**: $r_f = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$

2. (2 points) Show that the radius of convergence of $\log(1+X) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n}$ is 1 and show that $\log(1+X)$ is divergent at x,with $|x| = 1$.

3. (2 points) Show that the radius of convergence of $\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}$ is $|p|^{\frac{1}{p-1}}$ and show that $\exp(X)$ is divergent at x,with $|p|^{\frac{1}{p-1}}$.

4. (3 points) Let $K/\mathbb{Q}_p$ be a finite extension with absolute ramification index e and maximal ideal $\mathfrak{p}$. Show that $\exp\colon \mathfrak{p}^n \to U^{(n)} = 1 + \mathfrak{p}^n$ and $\log\colon U^{(n)} \to \mathfrak{p}^n$ establish an isomorphism between the additive group $\mathfrak{p}^n$ and the multiplicative group $U^{(n)}$ assuming $n > \frac{e}{p-1}$. In particular, $U^{(n)}$ is torsion-free as an abelian group.

5. (3 points) Let $|K : \mathbb{Q}_p| = d$ and $\pi \in K$ be a prime element. Show the isomorphism $K^\times \cong \pi^{\mathbb{Z}} \oplus Z_{q-1} \oplus Z_{p^a} \oplus \mathbb{Z}_p^d$ for a suitable integer $a \geq 0$ where q denotes the cardinality of the residue field.

6. (2 points) For any $z \in \mathbb{Z}_p$, show that the radius of convergence of the binomial series $(1+X)^z = \sum_{n=0}^{\infty} \binom{z}{n} X^n$ is greater than or equal to $|p|^{\frac{1}{p-1}}$ Moreover, if $|x| < |p|^{\frac{1}{p-1}}$, then we have $(1+x)^z = \exp(z\log(1+x))$ in this case.

7. (3 points) Let $K/\mathbb{Q}_p$ be finite. Show that any finite index subgroup of $K^\times$ is open (hence closed, too).

8. (3 points) Let $K/\mathbb{Q}_p$ be a finite extension, $\pi \in K$ be a prime element, and v_π be the normalized valuation (ie. $v_\pi(\pi) = 1$). Since K is a locally compact abelian group, it admits a translation invariant Haar measure dx, that is unique if we assume $\int_{\mathcal{O}_K} dx = 1$. Show that $|a|_\pi = \int_{a\mathcal{O}_K} dx$ where the absolute value $|\cdot|$ is normalized so that $|a|_\pi := q^{-v_\pi(a)}$ where $q = |\mathcal{O}_K/(\pi)|$ is the cardinality of the residue field. Further show that $\frac{dx}{|x|_\pi}$ is a translation invariant Haar measure on the multiplicative group $K^\times$.

9. (3 points) Let $K/\mathbb{Q}_p$ be finite. Show that the slopes of the Newton polygon of a polynomial $f(x) \in K[x]$ are exactly the π-adic valuations of its roots (with multiplicity) where $\pi \in K$ is a prime element (and the Newton polygon is constructed using the π-adic valuation).

10. (3 points) Let $K/\mathbb{Q}_p$ be finite. Assume that $f(x) \in K[x]$ is irreducible. Show that the Newton polygon of f consists of a single segment. Give examples that this segment may or may not contain a lattice point apart from the end points.