Algebraic Number Theory

Problem sheet 5

1. (2+2 points) Decompose 7 and 11 in $\mathcal{O}_K$, where
   (a) $K = \mathbb{Q}[\sqrt[3]{2}]$
   (b) $K = \mathbb{Q}[\theta]$, where $\theta^3 - \theta - 4 = 0$.

2. (2+2 points) Determine the ideal class group Cl_K of a number field K, where
   (a) $K = \mathbb{Q}[\sqrt{-14}]$,
   (b) $K = \mathbb{Q}[\sqrt[3]{2}]$

3. (2+2 points) Determine the unit group $\mathcal{O}_K^\times$ of a number field K, where
   (a) $K = \mathbb{Q}[\sqrt{-3}]$,
   (b) $K = \mathbb{Q}[\sqrt{3}]$

4. (*a*) (2 points) Verify directly that an odd prime p ramifies in $\mathbb{Q}(\sqrt{d})$ if and only if $p \mid d$ (d squarefree). Further, it is inert if $\left(\frac{d}{p}\right) = -1$ and splits into the product of two distinct primes if $\left(\frac{d}{p}\right) = 1$.
   (*b*) (2 points) What happens if $p = 2$?

5. (3+1 points) Let f_n be the n-th element of the Fibonacci sequence ($f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_{n-1} + f_n$).
   (a) Let $p \neq 2, 5$ be a prime number. Show that
   $f_{p-1} \equiv \frac{1-\left(\frac{p}{5}\right)}{2} \pmod{p}$, $f_p \equiv \left(\frac{p}{5}\right) \pmod{p}$, and $f_{p+1} \equiv \frac{1+\left(\frac{p}{5}\right)}{2} \pmod{p}$.
   (b) show that for any prime number p, there is exactly one of f_{p-1}, f_p, f_{p+1}, which is divisible by p.

6. (2 points) Let $I, J \lhd A$ be ideals in the Dedekind domain A and let B be the integral closure of A in the finite separable field extension L/K where K is the field of fractions of A. Verify that $I = IB \cap A$ and $I \mid J \Leftrightarrow IB \mid JB$.

   Remark. This is, in general, false for integral extension.

7. (3 points) Give an example of a pair $\mathbb{Q} \leq K_1, K_2$ of finite extensions and prime p such that p totally ramifies in both extensions K_1 and K_2 (ie. $r = 1 = f_1$) but not in the composite extension K_1K_2. (Hint: You may even choose quadratic extensions.)

8. (*a*) (3 points) Assume $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$ and let $p \in \mathbb{Z}$ be a prime. Show that there are at most p prime ideals $\mathfrak{p}_i \lhd \mathcal{O}_K$ dividing p with $f_i = 1$ (ie. $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_p$).
   (*b*) (2 points) Show that 3 splits completely in $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$. Deduce that there does not exist $\alpha \in K$ with $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
   (*c*) (3 points) Find an integer $0 < N \in \mathbb{Z}$ and an element $\alpha \in \mathcal{O}_K$ such that $\mathcal{O}_K[1/N] = \mathbb{Z}[1/N][\alpha]$ and decompose all primes $p \mid N$ as a product of prime ideals in $\mathcal{O}_K$.

9. (4 points) Let $\mathbb{Q} \leq K$ be a finite extension, L and L' are two extension of K, and $\mathfrak{p} \lhd \mathcal{O}_K$ a prime ideal. Show that $\mathfrak{p}$ splits completely in L and in L' if and only if it splits completely in the composite extension LL'.

10. (3 points) Let $\mathbb{Q} \leq K \leq L$ be finite extensions and $K \leq L \leq F$ be the Galois closure of L/K. Show that a prime $\mathfrak{p} \lhd \mathcal{O}_K$ splits completely in L if and only if it splits completely in F.

11. (3 points) Show that no primes of $\mathbb{Q}(\sqrt{-5})$ ramify in the extension $\mathbb{Q}(\sqrt{-1}, \sqrt{-5})/\mathbb{Q}(\sqrt{-5})$.