Practise

1

Q1

CDF: $F(x) = 1 - e^{-\frac{x}{\lambda}} = 1 - e^{-\frac{x}{10000}}$

$\therefore F(X > 15000) = e^{-\frac{3}{2}}$

$\frac{1}{e^{-\frac{3}{2}}} = e^{\frac{3}{2}}$

(b): $F(X \mid X > 15000)$

$E(X \mid X \geq 15000) = 15000 + E(X) = 25000$

Q2

$E(X^3) = \int_0^\infty x^3 f(x)\,dx = \int_0^\infty \lambda x^3 e^{-\lambda x} dx = \int_0^\infty \lambda x^3 d\left(-\frac{1}{\lambda} e^{-\lambda x}\right) = \int_0^\infty -x^3 de^{-\lambda x} = -\left[[x^3 e^{-\lambda x}]_0^\infty - \int_0^\infty 3x^2 e^{-\lambda x} dx \right]$

$= \frac{3}{\lambda} \int_0^\infty x^2 e^{-\lambda x} dx = \frac{6}{\lambda^2}$

Q3: $P(M > t) = P(\min(X_1, \cdots X_n) > t) = P(X_1 > t) P(X_2 > t) \cdots P(X_n > t) = e^{(-\lambda_1 - \lambda_2 \cdots - \lambda_n) y}$

Q4: (a): by symmetric. $\frac{1}{2}$.

(b): $\frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$

2.

Q1:

$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} = \int_0^\infty \frac{1}{\lambda} e^{(t - \frac{1}{\lambda})x} = \frac{1}{\lambda} \frac{1}{t - \frac{1}{\lambda}} [e^{(t-\frac{1}{\lambda})x}]_0^\infty = \frac{1}{1 - t\lambda}$

$M'''(0) = \frac{6}{\lambda^2}$

x -

Q2:

$E(e^{t(a+bX)}) = E(e^{at+btX}) = e^{at}E(e^{t(bX)}) = e^{at}M(bt)$

Q3

$E(60U_1) = E(e^{60tU}) = [E(e^{tU})]^{60} = \frac{(e^t-1)^{60}}{t^{60}}$

Q4:

$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} P(X=k)$

$= \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \cdot e^{-\lambda} = e^{\lambda e^t - \lambda} = \sum_{j=1}^{\infty} \frac{\lambda}{j!} t^j$

About memoryless:

$P(X > x \mid X > Y) \quad (x > Y)$

$= \frac{P(X>x, X>Y)}{P(X>Y)} = \frac{P(X>x)}{P(X>Y)} = \frac{e^{-\lambda x}}{e^{-\lambda Y}} = e^{-\lambda(x-Y)}$

same as $0 \to x - Y$

$\therefore E(X \mid X > Y) = E(X - Y)$

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two student with Ex. distribution:

$Z = \min(X, Y) \qquad X \sim Expo(\lambda),\ Y \sim Expo(\lambda)$

$P(Z \le z) = 1 - P(Z > z) = 1 - P(X > z)P(Y > z) = 1 - e^{-2\lambda x}$

$\therefore Z \sim Expo(2\lambda)$, average $\frac{1}{2\lambda}$

HW

Q1: Uniform on $[0,10]$

average: 5

(b): $P(X \geq 9 | X > 6) = \frac{P(X \geq 9)}{P(X > 6)} = \frac{1}{4}$

(c) by the memoryless properties: average is still 10 min

(d):

Actually, Fred is more likely to arrive at a longer interval so he cries

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Q2 $T = T_1 + T_2 + T_3$

T_1: first students to finish, T_2: second's additional time, T_3:

$E(T_1 + T_2 + T_3) = E(T_1) + E(T_2) + E(T_3) = Ex(\frac{1}{2}) + Ex(\frac{1}{3}) + E(x)(\frac{1}{6}) = 11$

Q3:

(a): $Var(t) = Var(t-\theta) = E((t-\theta)^2) - E^2(T-\theta) = MSE(T) - b^2(T)$

(b): $E(X-c)^2 = Var(X) + (E(X)-c)^2 \geq Var(X)$

(c): $E|X-c| = \int_{-\infty}^{\infty} |x-c| f(x)\,dx = \int_{-\infty}^{c} (c-x) f(x)\,dx + \int_{c}^{\infty} (x-c) f(x)$

$= \int_c^{\infty} x f(x)\,dx - c\int_c^{\infty} f(x)\,dx$

$\frac{dE(X-c)}{dc} = 0 \therefore$ minimized, c is media

Q5:

$$f_{\Lambda|X}(\lambda|x) = \frac{P(X=x|\Lambda=\lambda)f_\Lambda(\lambda)}{P(X=x}$$

A is capital of X. $f_\Lambda(\lambda) = 3e^{-3\lambda}$

$P(X=x|\Lambda=\lambda)$ is found using Pois(2λ). numerator for $X=2$: $\frac{e^{-2\lambda}(2\lambda)^2}{2!} \cdot 3e^{-3\lambda} = 6\lambda^2 e^{-5\lambda}$

$P(X=2) = \int_0^\infty P(X=2|\Lambda=\lambda) f_\Lambda(\lambda) d\lambda = \int_0^\infty \lambda^2 e^{-5\lambda} d\lambda$ $\therefore F_{\Lambda|X}(\lambda|2) = \frac{125}{2}(\lambda^2 e^{-5\lambda})$

$$\left(\int_0^\infty \lambda^2 e^{-5\lambda} = \frac{2}{125}\right)$$

Q6: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$$E(e^{tX_n}) = (1-P_n+P_ne^t)^n = \left[1+\left(\frac{\lambda(e^t-1)}{n}\right)\right]^n \to e^{\lambda(e^t-1)} = E(e^{tX})$$

Q7: $E(Y) = E(e^X) = e^{M+\frac{\sigma^2}{2}}$

$$E(Y^2) = E(e^{2X}) = e^{2M+2\sigma^2}$$

$$Var(Y) = e^{2M+2\sigma^2} - e^{2M+\sigma^2} = E^{2M+\sigma}(e^{\sigma^2}-1)$$

$$E(Y^n) = E(e^{nX}) = e^{\frac{n^2}{2}}$$