An s-shaped three-parameter (S3) traffic stream model with consistent car following relationship

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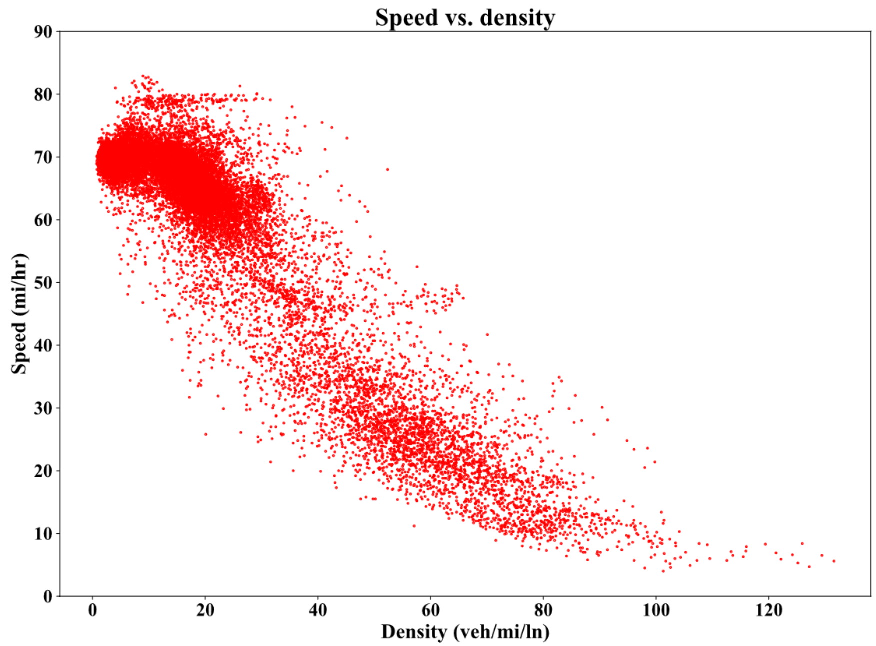
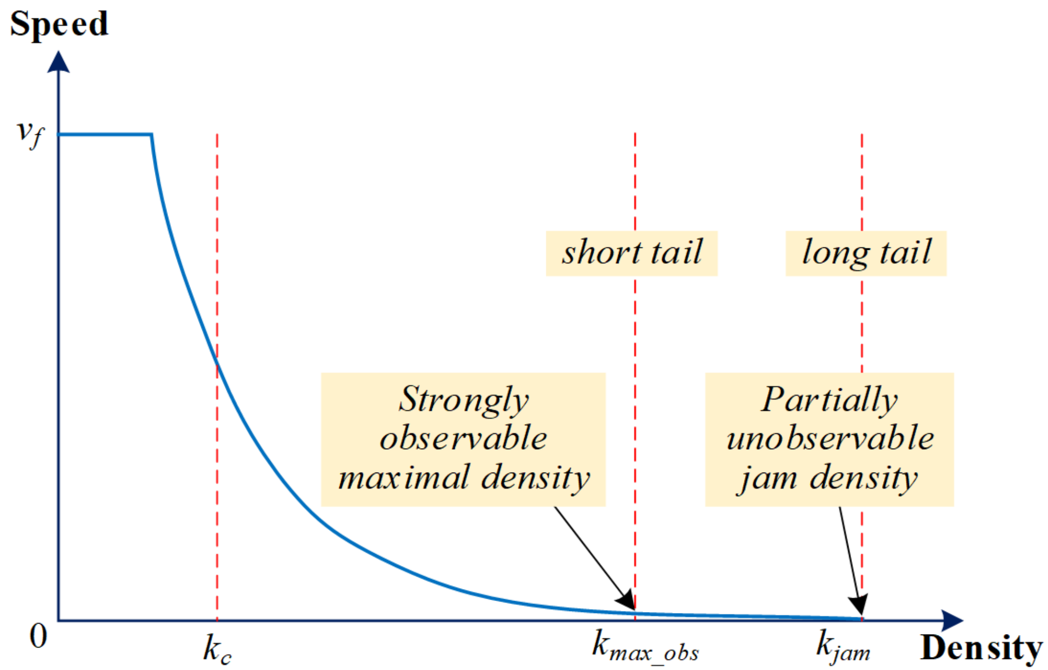
Observations on the Speed-density Relationships

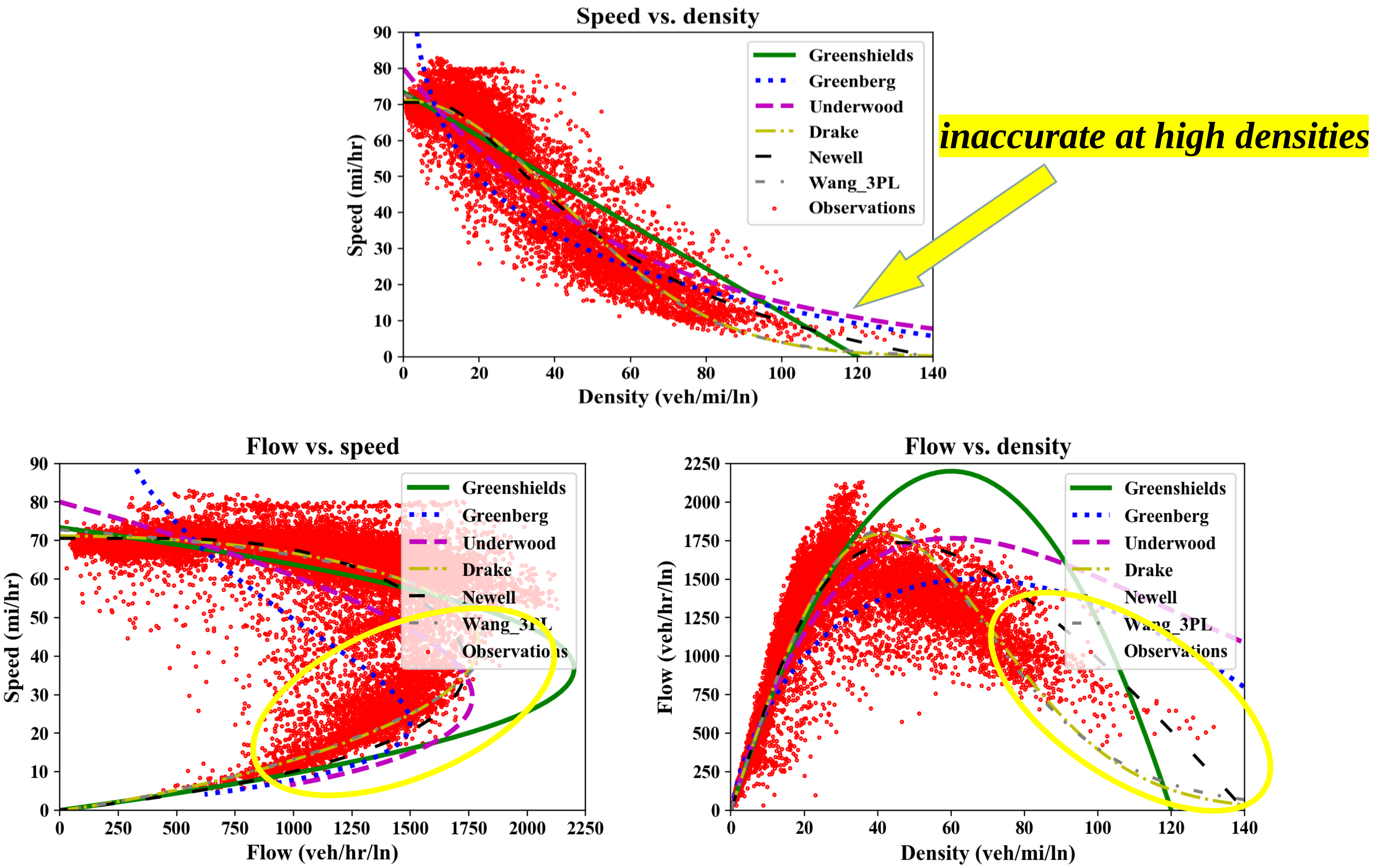




Characteristics for a well-defined single-regime speed-density function:

1) the speed is a **strictly and monotonically decreasing function** with respect to density;
2) the speed keeps **relatively stable at the free flow state** (or in a low-density range);
3) the speed keeps **relatively slow, but not reach zero, under heavily congested condition** (or in a high-density range);
4) it can smoothly capture transition between different traffic states;
5) describe the speed-density relationships with **as few parameters as possib**le; and
6) maintain important **consistency across planes** of flow-density, speed-flow based on the conservation condition of $q=kv$.

Speed-density Relationships in the Literature



S-shaped Three-parameter (S3) Fundamental Diagram

$$v = \frac{v_f}{\left[1 + \left(k/k_c\right)^m\right]^l} \qquad q = \frac{k \cdot v_f}{\left[1 + \left(k/k_c\right)^m\right]^l}$$

where v_f is the free flow speed, k_c is the critical density which can result in a maximal traffic flow rate, and m and l represent the flatness of the curve (FoC) to be determined.

$q(k)$ is continuously differentiable. When $k=k_c$, we have $q=q_{max}$.

$$\left.\begin{aligned} &\frac{dq}{dk} = \frac{v_f}{\left[1 + \left(k/k_c\right)^m\right]^l} - \frac{k \cdot v_f}{k_c} \cdot \frac{l \cdot m \cdot \left(k/k_c\right)^{m-1}}{\left[1 + \left(k/k_c\right)^m\right]^{l+1}} \\ &\left.\frac{dq}{dk}\right|_{k=k_c} = 0 \Rightarrow \frac{v_f}{2^l} - v_f \cdot \frac{l \cdot m}{2^{l+1}} = 0 \Rightarrow l = \frac{2}{m} \end{aligned}\right\} \Rightarrow \begin{cases} v = \dfrac{v_f}{\left[1 + \left(k/k_c\right)^m\right]^{2/m}} \\ q = \dfrac{k \cdot v_f}{\left[1 + \left(k/k_c\right)^m\right]^{2/m}} \end{cases}$$

S-shaped Three-parameter (S3) Fundamental Diagram

Averaged maximal flow rate (or capacity):

$$k = k_c \Rightarrow q_{\max} = \frac{k_c \cdot v_f}{2^{2/m}}$$

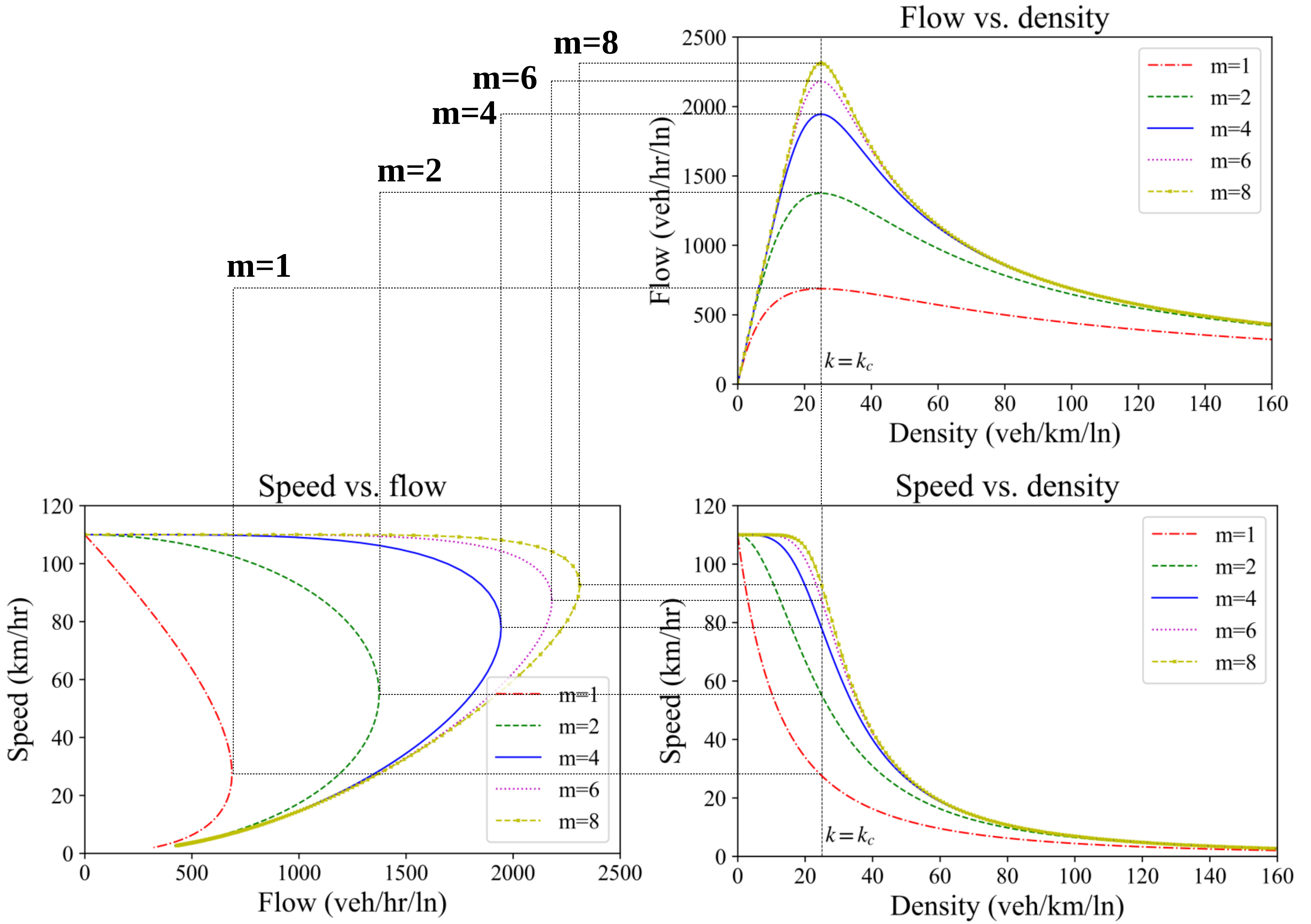
Explanation of the FoC parameter *m*:

$$v_c = \frac{v_f}{\left[1+\left(k_c/k_c\right)^m\right]^{2/m}} \Rightarrow v_c = \frac{v_f}{2^{2/m}} \Rightarrow m = \frac{2\ln 2}{\ln\left(v_f/v_c\right)}$$

where v_c is the speed at the critical density.

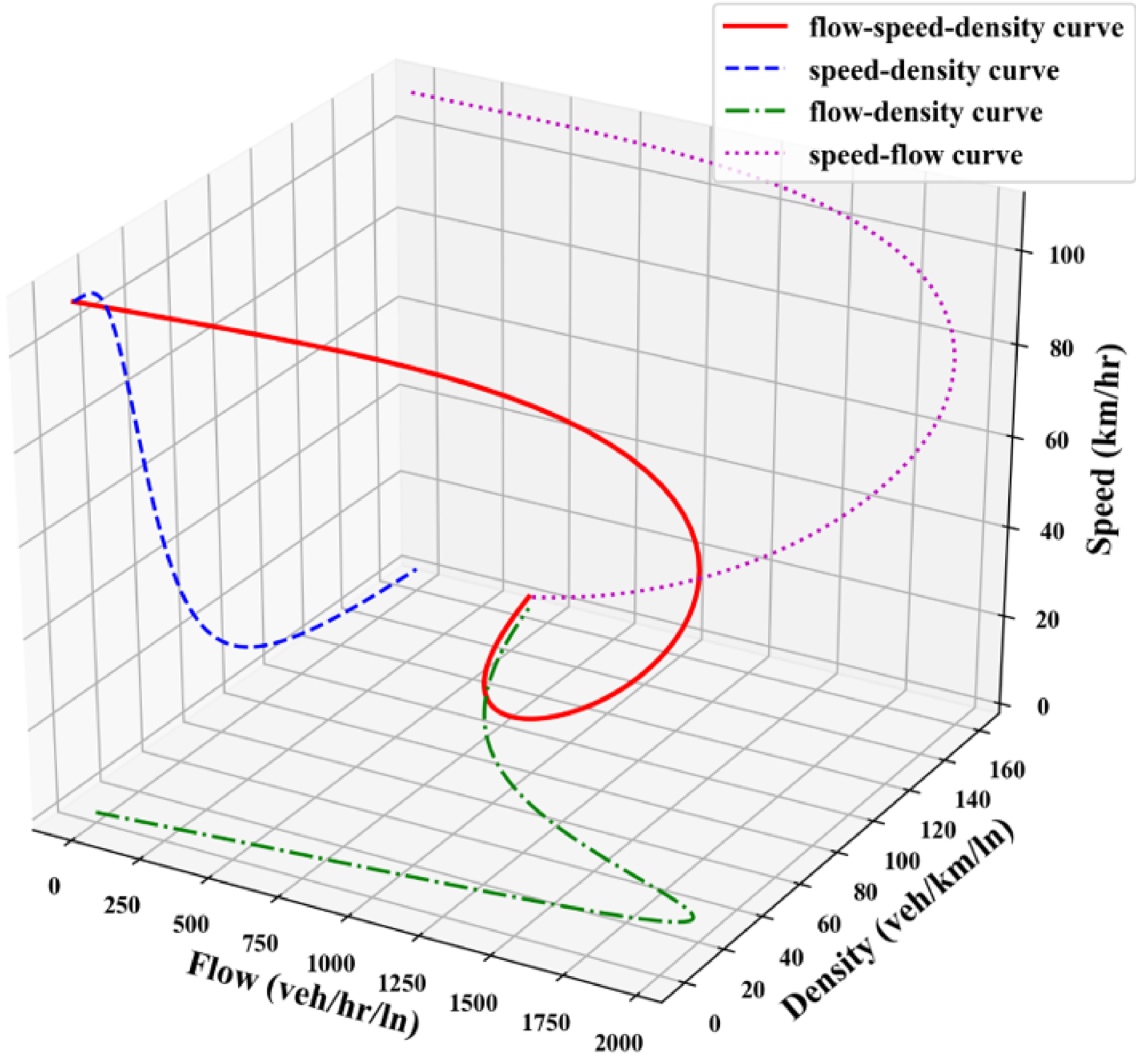
S-shaped Three-parameter (S3) Fundamental Diagram

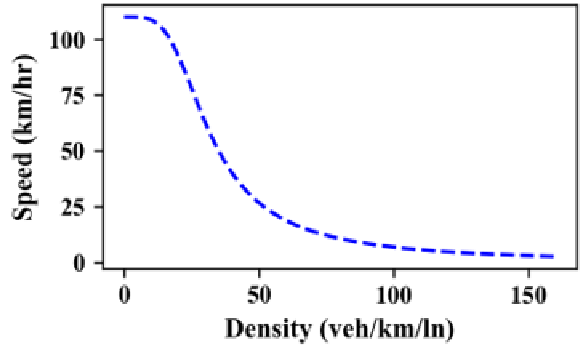
Projections to the 2D planes:

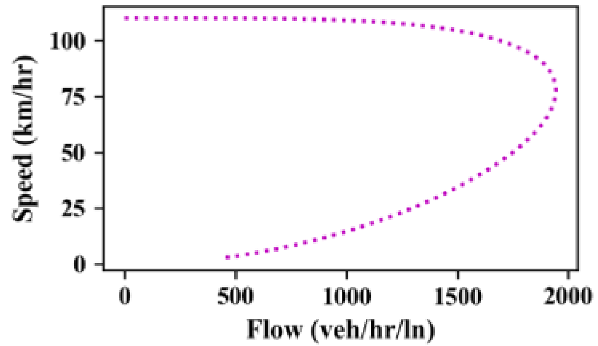


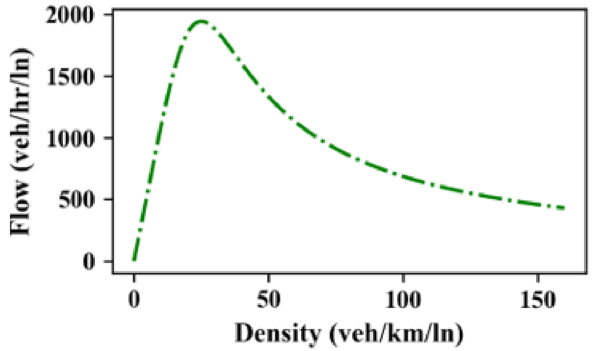
S-shaped Three-parameter (S3) Fundamental Diagram

3D Space when *m*=4



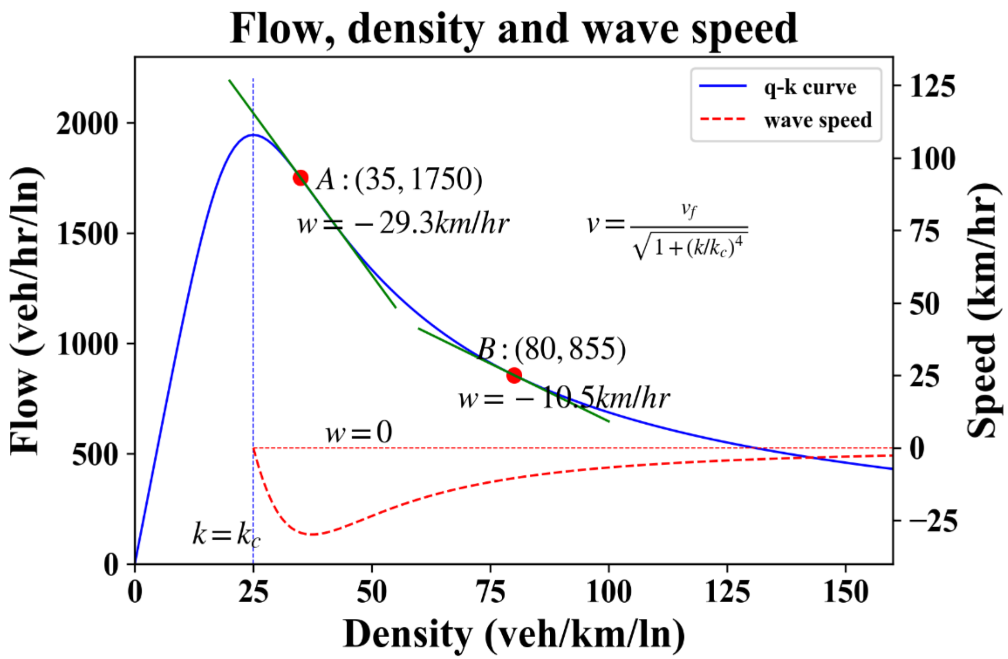


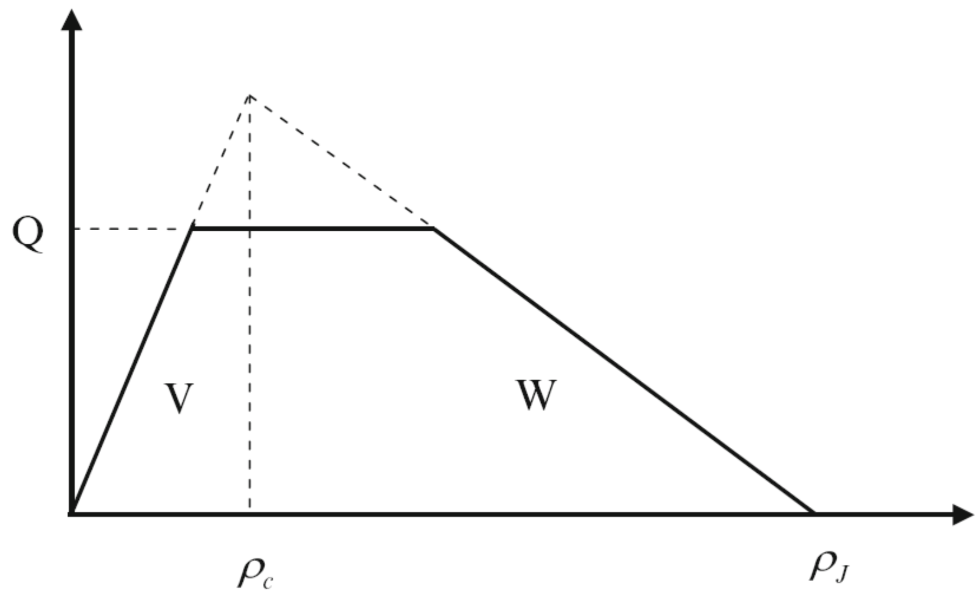




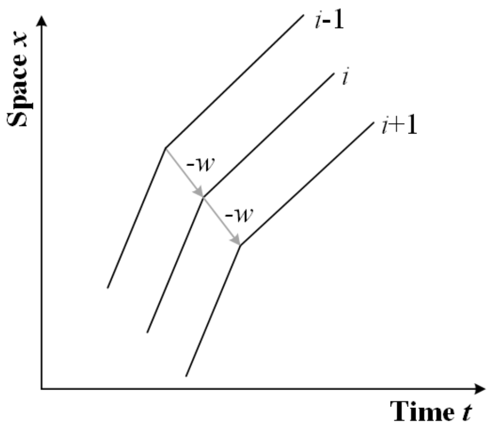
S-shaped Three-parameter (S3) Fundamental Diagram

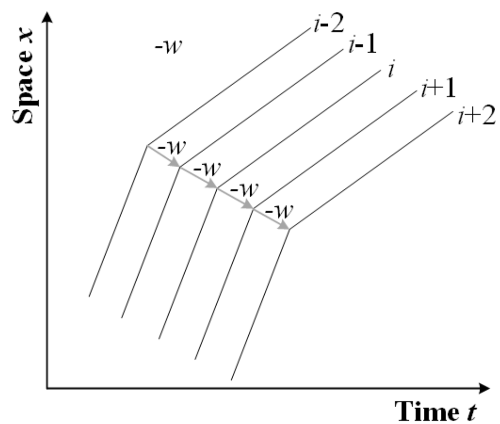
Relationship to the backward wave speed





Triangle fundamental diagram used in CTM





(b) trajectories at point A with a low density (c) trajectories at point B with a high density

From the perspective of simplified car following model, our model indicates that the ***reaction time could become longer*** if we assume a ***constant minimum spacing***. However, this explanation should be further verified with real-world traffic data in the future.

Deriving Macro-to-Micro Car-following Models

Macroscopic speed-density $\Leftrightarrow$ **Microscopic car-following**

$$v = \frac{v_f}{\left[1+\left(k/k_c\right)^m\right]^{2/m}} \Leftrightarrow a_n(t+\tau_n) = \frac{\alpha \cdot \Delta x_n(t) \cdot \Delta v_n(t)}{\left[1+\left(\beta \cdot \Delta x_n(t)\right)^m\right]^{1+\frac{2}{m}}}$$

FoC	Speed-density function	Car-following model	Connection
$m=1$	$v=\frac{v_f}{\left[1+\left(k/k_c\right)\right]^2}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)\right]^3}\cdot\Delta v$	
$m=2$	$v=\frac{v_f}{\left[1+\left(k/k_c\right)^2\right]}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)^2\right]^2}\cdot\Delta v$	
$m=3$	$v=\frac{v_f}{\left[1+\left(k/k_c\right)^3\right]^{2/3}}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)^3\right]^{5/3}}\cdot\Delta v$	$\alpha=2v_f\beta^2$
$m=4$	$v=\frac{v_f}{\sqrt{1+\left(k/k_c\right)^4}}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)^4\right]^{3/2}}\cdot\Delta v$	$\beta=k_c$
$m=5$	$v=\frac{v_f}{\left[1+\left(k/k_c\right)^5\right]^{2/5}}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)^5\right]^{7/5}}\cdot\Delta v$	
$m=6$	$v=\frac{v_f}{\left[1+\left(k/k_c\right)^6\right]^{1/3}}$	$a=\frac{\alpha\cdot\Delta x}{\left[1+\left(\beta\cdot\Delta x\right)^6\right]^{4/3}}\cdot\Delta v$	

Stability Analysis on the Car-following models

Laplace transform based approach to analyze the stability

The proposed car-following model is stable if and only if the real parts of the solutions in the following equation are all negative.

$$se^{s\tau_n} + f_{\Delta v} = 0$$

$$f_{\Delta v} = \frac{\alpha \cdot h}{\left[1+(\beta \cdot h)^m\right]^{1+\frac{2}{m}}}$$

where h is the headway space at stable state, s is the complex variable in the Laplace transform.

It should be noted that, although the above functional form seems to be relatively simple, solving it is nontrivial. Mathematically, the type of the above equation is called **Lambert *W* function or omega function**, and one can solve it numerically to obtain the stability criterion.

Calibration for the S3 model as a nonlinear regression

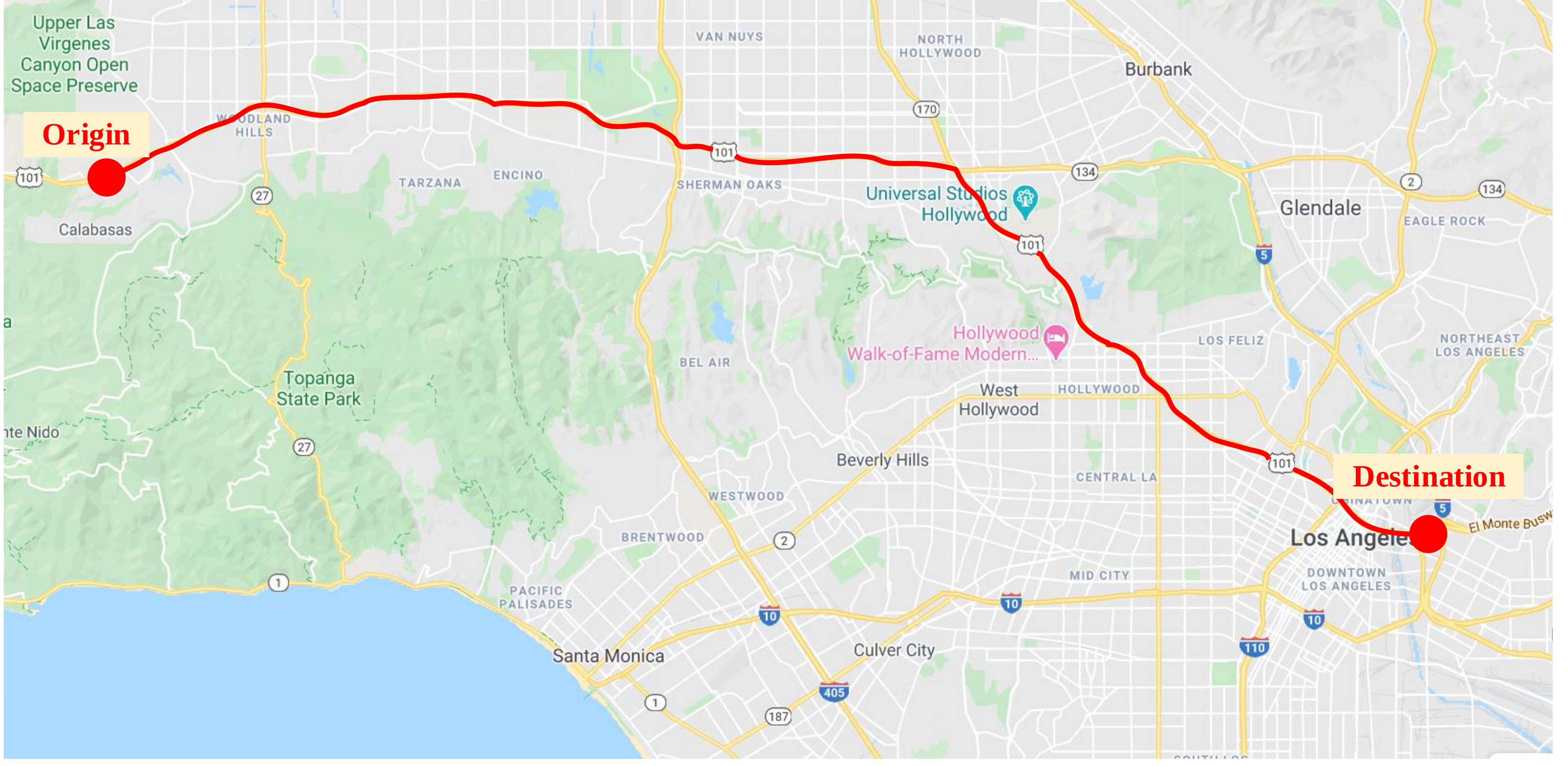
[M1] $$\min\ Z_1 = \sum_{i=1}^{P}\left(v_i - \hat{v}(k_i; v_f, k_c, m)\right)^2$$

$$\begin{cases} \dfrac{\partial Z_1}{\partial v_f} = -2\sum_{i=1}^{P}\left(v_i - \hat{v}(k_i; v_f, k_c, m)\right)\dfrac{\partial \hat{v}(k_i; v_f, k_c, m)}{\partial v_f} = 0 \\ \dfrac{\partial Z_1}{\partial k_c} = -2\sum_{i=1}^{P}\left(v_i - \hat{v}(k_i; v_f, k_c, m)\right)\dfrac{\partial \hat{v}(k_i; v_f, k_c, m)}{\partial k_c} = 0 \\ \dfrac{\partial Z_1}{\partial m} = -2\sum_{i=1}^{P}\left(v_i - \hat{v}(k_i; v_f, k_c, m)\right)\dfrac{\partial \hat{v}(k_i; v_f, k_c, m)}{\partial m} = 0 \end{cases}$$

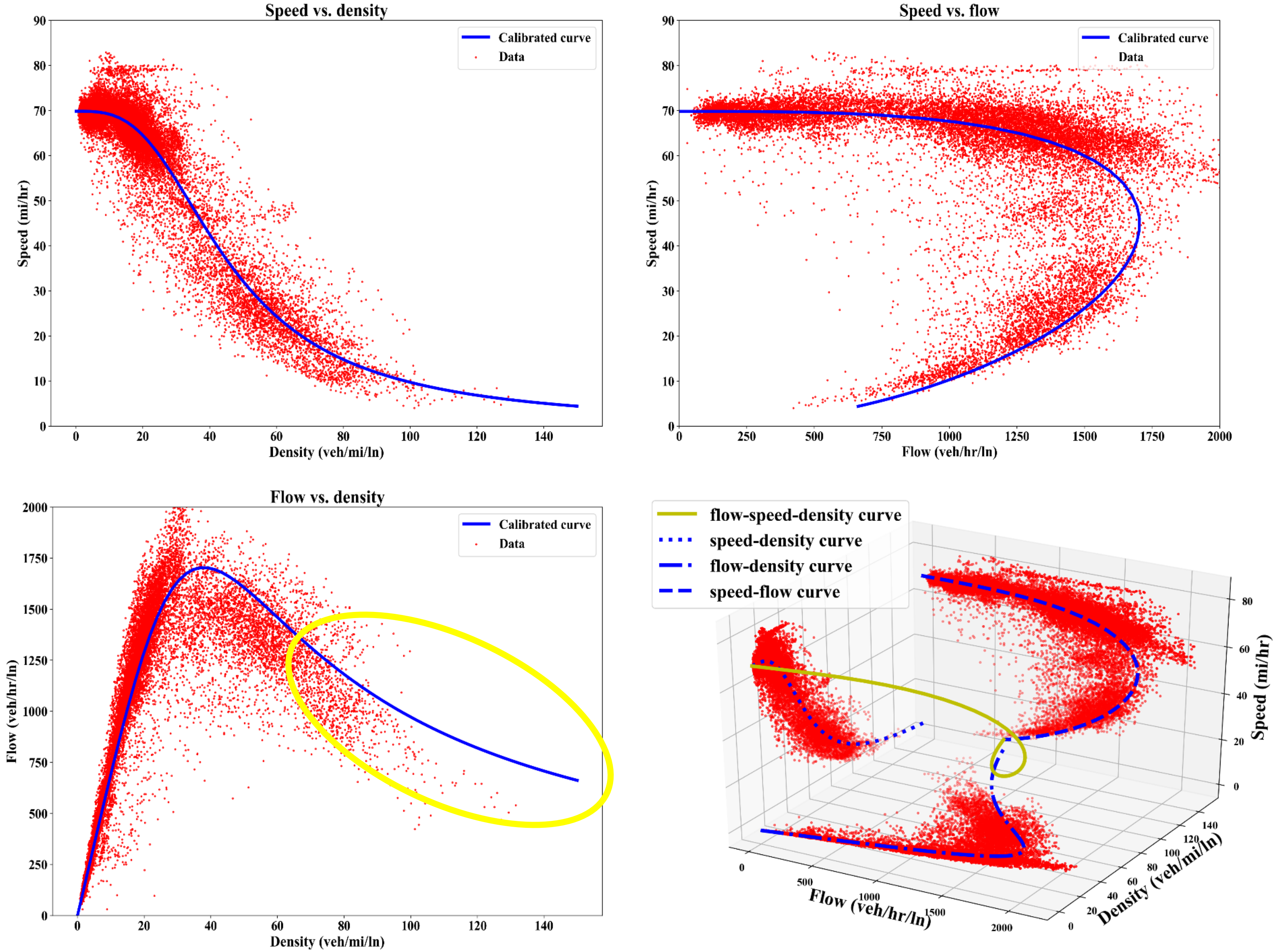
[M2] $$\min\ Z_1' = \sum_{i=1}^{P}\left[\left(v_i - \hat{v}(k_i; v_f, k_c, m)\right)^2 + \delta \cdot \left(q_i - \hat{q}(k_i; v_f, k_c, m)\right)^2\right]$$

Calibration for the S3 model (study area)

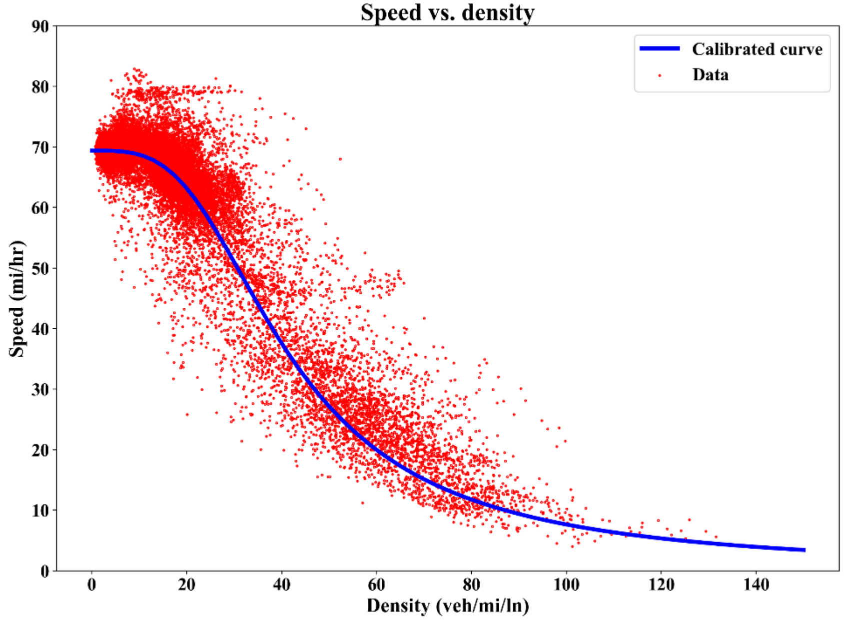
The loop detector data is collected on July 1st (Monday), 2019 from Absolute Postmile (Abs) 1.79 to Abs 30 on the freeway of US101-S in Los Angeles, USA from the open-access PeMS database, including the speed, flow, and occupancy, etc.

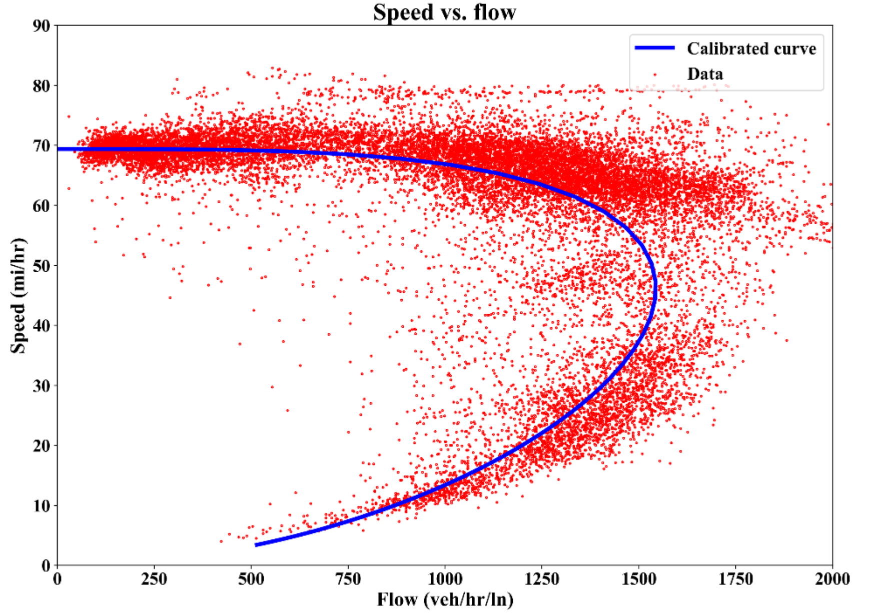


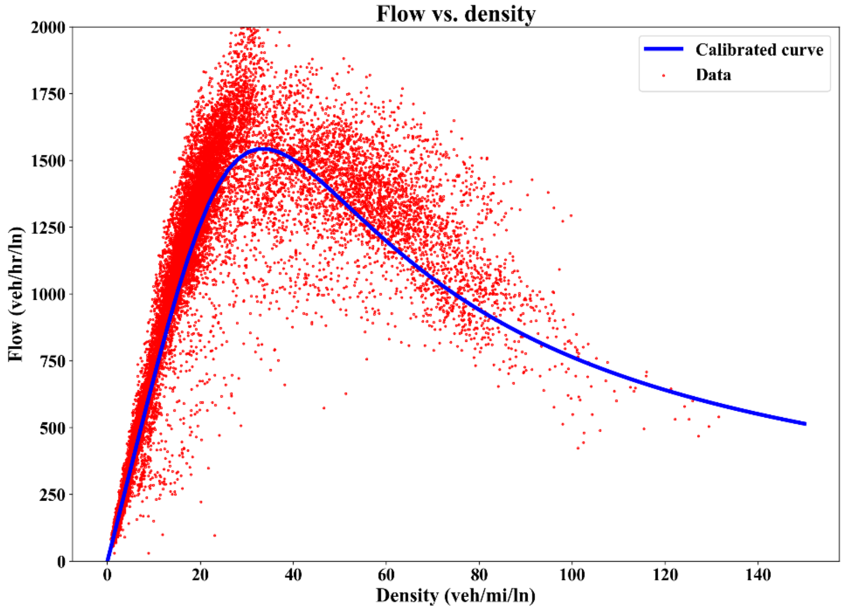
Calibration for the S3 model (Results_M1)

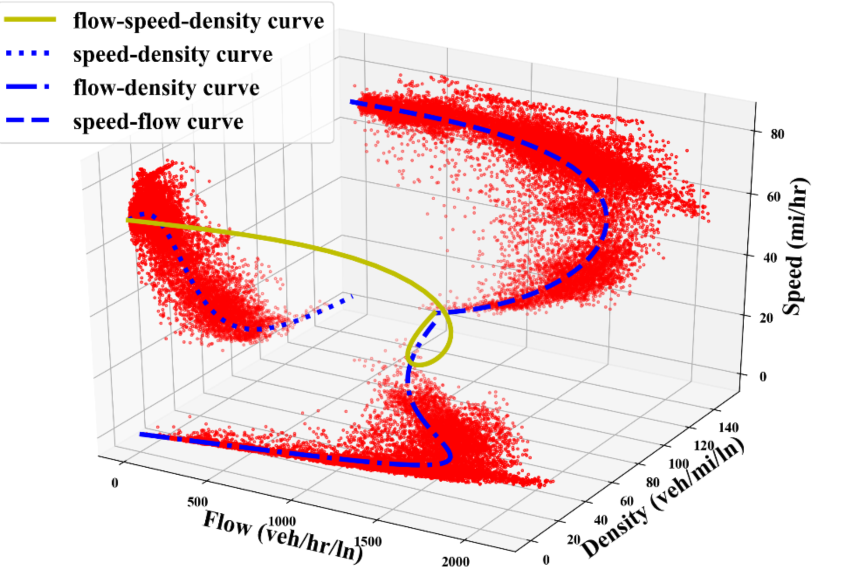


Calibration for the S3 model (Results_M2)









Calibration for the S3 model (Comparison)

Comparison of MRE on the estimated speed with only speed information in the objective function

Literature	Density range (veh/mi)											
	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	>100	Avg	Std
GS	3.59%	6.69%	9.43%	22.30%	40.92%	53.27%	60.08%	78.51%	76.80%	55.59%	69.70%	43.35%	28.46%
GB	22.34%	17.09%	25.32%	23.93%	19.32%	18.07%	23.32%	36.90%	43.60%	48.46%	62.35%	30.97%	14.82%
UW	7.22%	9.02%	14.51%	18.99%	23.94%	28.24%	34.01%	52.19%	60.34%	65.41%	86.86%	36.43%	26.12%
NW	2.96%	5.10%	9.00%	18.82%	25.00%	25.68%	26.92%	35.73%	34.36%	25.27%	33.74%	22.05%	11.68%
DK	3.15%	5.20%	8.41%	20.02%	26.53%	22.62%	21.29%	24.31%	29.83%	49.23%	65.00%	25.05%	18.44%
PP: M1	6.15%	7.37%	10.96%	20.33%	35.52%	46.74%	54.96%	76.71%	81.98%	73.35%	68.49%	43.87%	29.32%
PP: M2	6.76%	6.43%	10.76%	19.26%	28.93%	32.63%	33.58%	41.93%	36.43%	22.31%	41.67%	25.52%	13.27%
DR	6.15%	7.37%	10.96%	20.33%	35.52%	46.74%	54.96%	76.71%	81.98%	73.35%	68.49%	43.87%	29.32%
KK	3.65%	5.34%	8.46%	20.42%	27.41%	22.73%	21.23%	24.79%	31.82%	49.92%	62.19%	25.27%	17.92%
JA	6.76%	6.43%	10.76%	19.26%	28.93%	32.63%	33.58%	41.93%	36.43%	22.31%	41.67%	25.52%	13.27%
VA	220.45 %	22.06%	26.10%	24.76%	17.60%	13.02%	13.17%	12.22%	10.32%	11.20%	10.97%	34.72%	61.87%
MN	2.81%	5.11%	8.28%	19.33%	23.61%	20.44%	21.18%	24.80%	22.87%	24.95%	28.02%	18.31%	8.71%
3PL	3.65%	5.34%	8.46%	20.42%	27.41%	22.73%	21.23%	24.79%	31.82%	49.92%	62.19%	25.27%	17.92%
4PL	3.18%	5.21%	8.24%	20.01%	24.16%	19.59%	21.15%	24.64%	23.51%	24.69%	54.18%	20.78%	13.70%
5PL	2.71%	5.13%	8.20%	18.68%	21.59%	19.45%	21.51%	28.18%	29.45%	29.67%	49.13%	21.25%	13.22%
S3 model	2.71%	5.11%	8.22%	18.96%	22.26%	19.66%	21.28%	26.54%	25.31%	22.49%	26.48%	18.09%	8.64%

Calibration for the S3 model (Comparison)

Comparison of MRE on the estimated flow with only speed information in the objective function

Literature	Density range (veh/mi)												
	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	>100	Avg	Std
GS	13.83%	14.74%	12.10%	23.79%	45.81%	59.84%	71.92%	86.28%	84.49%	65.96%	83.46%	51.11%	30.23%
GB	28.29%	24.31%	27.31%	18.94%	10.98%	11.56%	19.18%	34.64%	45.07%	57.44%	94.43%	33.83%	24.45%
UW	15.59%	17.04%	16.80%	15.16%	20.84%	28.81%	38.56%	55.27%	65.73%	78.63%	124.66%	43.37%	34.93%
NW	13.42%	12.22%	11.49%	17.12%	22.85%	25.00%	27.11%	33.22%	31.92%	28.63%	33.85%	23.35%	8.53%
DK	13.55%	12.62%	10.98%	19.90%	25.49%	19.65%	12.25%	13.52%	22.93%	42.79%	59.01%	22.97%	15.04%
PP: M1	15.03%	15.46%	13.47%	20.29%	38.53%	52.27%	65.73%	84.20%	90.29%	88.19%	85.19%	51.70%	32.36%
PP: M2	15.23%	14.49%	13.26%	18.07%	28.96%	34.78%	37.97%	42.08%	34.98%	21.68%	36.49%	27.09%	10.76%
DR	15.03%	15.46%	13.47%	20.29%	38.53%	52.27%	65.73%	84.20%	90.29%	88.19%	85.19%	51.70%	32.36%
KK	13.78%	12.95%	11.07%	20.70%	26.90%	19.73%	11.50%	14.26%	25.71%	43.56%	55.47%	23.24%	14.32%
JA	15.23%	14.49%	13.26%	18.07%	28.96%	34.78%	37.97%	42.08%	34.98%	21.68%	36.49%	27.09%	10.76%
VA	231.77 %	25.06%	26.64%	19.18%	10.53%	8.36%	8.57%	8.97%	8.14%	6.89%	15.19%	33.57%	66.11%
MN	13.38%	12.21%	10.73%	18.44%	20.58%	15.47%	12.12%	14.49%	12.85%	14.68%	16.67%	14.69%	2.95%
3PL	13.78%	12.95%	11.07%	20.70%	26.90%	19.73%	11.50%	14.26%	25.71%	43.56%	55.47%	23.24%	14.32%
4PL	13.59%	12.60%	10.79%	19.89%	21.58%	13.17%	10.17%	13.69%	15.01%	26.34%	85.26%	22.01%	21.55%
5PL	13.32%	11.99%	10.49%	16.98%	16.68%	13.65%	13.91%	21.09%	25.13%	34.83%	78.60%	23.33%	19.64%
S3	13.31%	12.00%	10.58%	17.61%	18.08%	13.96%	12.81%	17.85%	18.60%	21.53%	43.12%	18.13%	8.95%

Calibration for the S3 model (Comparison)

Comparison of MRE on the estimated speed with both speed and flow information in the objective function

Literature	Density range (veh/mi)											
	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	>100	Avg	Std
GS	4.55%	10.30%	12.29%	20.24%	35.55%	46.43%	53.00%	70.37%	68.70%	49.05%	66.35%	39.71%	24.59%
GB	25.07%	15.00%	23.46%	22.89%	19.27%	18.86%	24.49%	39.34%	46.48%	51.82%	66.54%	32.11%	16.53%
UW	6.95%	10.37%	16.32%	19.53%	21.28%	23.31%	27.95%	42.54%	47.78%	50.86%	65.23%	30.19%	18.67%
NW	4.75%	5.23%	10.22%	18.49%	21.10%	20.43%	22.52%	29.09%	27.43%	21.47%	32.28%	19.36%	9.18%
DK	3.55%	5.15%	8.51%	19.43%	24.13%	20.14%	21.59%	26.36%	36.91%	57.45%	72.02%	26.84%	21.36%
PP: M1	5.01%	11.63%	15.28%	19.05%	27.81%	35.47%	42.33%	60.74%	64.40%	56.87%	55.64%	35.84%	21.49%
PP: M2	7.96%	7.11%	12.51%	18.69%	22.70%	22.53%	23.10%	27.42%	23.67%	29.68%	52.66%	22.55%	12.45%
DR	5.01%	11.63%	15.28%	19.05%	27.81%	35.47%	42.33%	60.74%	64.40%	56.87%	55.64%	35.84%	21.49%
KK	3.86%	5.43%	8.77%	19.59%	24.67%	20.31%	21.71%	26.57%	36.31%	54.36%	65.76%	26.12%	19.48%
JA	7.96%	7.11%	12.51%	18.69%	22.70%	22.53%	23.10%	27.42%	23.67%	29.68%	52.66%	22.55%	12.45%
VA	219.45 %	22.08%	26.26%	24.81%	17.56%	13.01%	13.24%	12.26%	10.39%	11.34%	11.09%	34.68%	61.55%
MN	3.45%	5.08%	8.43%	18.55%	21.05%	18.39%	21.42%	24.32%	24.00%	29.10%	31.00%	18.62%	9.25%
3PL	3.86%	5.43%	8.77%	19.59%	24.67%	20.31%	21.71%	26.57%	36.31%	54.36%	65.76%	26.12%	19.48%
4PL	4.08%	5.11%	8.37%	18.98%	21.20%	18.16%	22.64%	24.61%	22.89%	24.15%	52.81%	20.27%	13.21%
5PL	2.69%	5.30%	8.33%	18.11%	19.75%	17.82%	21.25%	24.72%	23.09%	21.90%	23.36%	16.94%	7.78%
S3 model	2.83%	5.17%	8.25%	18.34%	20.12%	17.88%	21.54%	24.46%	22.76%	22.33%	21.85%	16.87%	7.69%

Calibration for the S3 model (Comparison)

Comparison of MRE on the estimated flow with both speed and flow information in the objective function

Literature	Density range (veh/mi)											
	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	>100	Avg	Std
GS	13.89%	18.08%	14.72%	20.07%	38.56%	51.89%	63.37%	77.02%	75.31%	57.90%	77.97%	46.25%	26.15%
GB	30.66%	22.48%	25.51%	17.57%	10.86%	13.29%	21.95%	38.10%	48.96%	61.68%	99.75%	35.53%	26.27%
UW	15.48%	18.25%	18.52%	14.62%	15.78%	21.28%	28.87%	42.68%	50.69%	60.42%	98.43%	35.00%	26.30%
NW	14.07%	12.24%	12.67%	14.86%	15.54%	15.89%	17.39%	23.08%	21.83%	18.04%	26.78%	17.49%	4.61%
DK	13.69%	12.47%	11.05%	18.62%	21.50%	14.41%	10.42%	16.92%	32.25%	52.07%	67.28%	24.61%	18.69%
PP: M1	14.60%	19.32%	17.52%	16.18%	27.08%	38.53%	49.89%	65.69%	70.40%	67.70%	68.46%	41.40%	23.56%
PP: M2	15.91%	15.21%	14.91%	15.34%	18.76%	19.83%	19.15%	20.20%	14.28%	19.08%	45.81%	19.86%	8.88%
DR	14.60%	19.32%	17.52%	16.18%	27.08%	38.53%	49.89%	65.69%	70.40%	67.70%	68.46%	41.40%	23.56%
KK	13.87%	13.13%	11.37%	18.94%	22.43%	14.70%	10.49%	17.26%	31.54%	48.57%	59.69%	23.82%	16.30%
JA	15.91%	15.21%	14.91%	15.34%	18.76%	19.83%	19.15%	20.20%	14.28%	19.08%	45.81%	19.86%	8.88%
VA	230.74 %	25.16%	26.83%	19.18%	10.46%	8.25%	8.46%	8.87%	8.08%	6.74%	15.06%	33.44%	65.82%
MN	13.62%	11.90%	10.85%	16.56%	15.50%	10.69%	10.03%	13.04%	14.14%	18.57%	19.29%	14.02%	3.16%
3PL	13.87%	13.13%	11.37%	18.94%	22.43%	14.70%	10.49%	17.26%	31.54%	48.57%	59.69%	23.82%	16.30%
4PL	13.91%	12.29%	10.86%	17.62%	15.82%	9.67%	10.91%	13.19%	13.54%	23.96%	83.43%	20.47%	21.25%
5PL	13.29%	11.46%	10.34%	15.09%	12.25%	9.61%	10.02%	13.95%	13.68%	15.47%	36.88%	14.73%	7.61%
S3 model	13.38%	11.61%	10.49%	16.03%	13.22%	9.56%	10.01%	13.35%	12.82%	14.10%	26.51%	13.74%	4.65%

Calibration for the Car-following Model

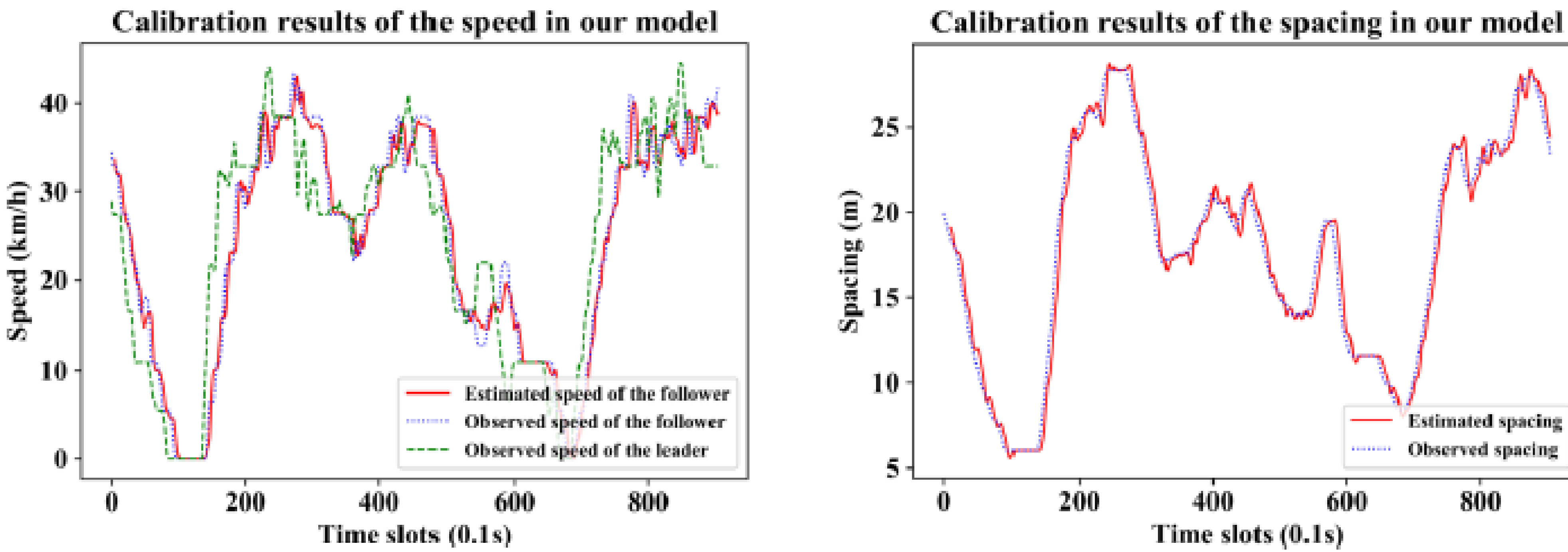
$$a_n(t+\tau_n) = \frac{\alpha \cdot \Delta x_n(t) \cdot \Delta v_n(t)}{\left[1+\left(\beta \cdot \Delta x_n(t)\right)^m\right]^{1+\frac{2}{m}}}$$

Where $\alpha = 2v_f\beta^2, \beta = k_c$

$$\min\ Z_2 = \frac{\sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(v_{n+1}(t) - \hat{v}_{n+1}(t;\alpha,\beta)\right)^2}}{\sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(v_{n+1}(t)\right)^2} + \sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(\hat{v}_{n+1}(t;\alpha,\beta)\right)^2}}$$

$$+ \frac{\sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(\Delta x_{n+1}(t) - \Delta \hat{x}_{n+1}(t;\alpha,\beta)\right)^2}}{\sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(\Delta x_{n+1}(t)\right)^2} + \sqrt{\frac{1}{M \cdot T_{n+1}} \cdot \sum_{n=1}^{M} \sum_{t=1}^{T_{n+1}} \left(\Delta \hat{x}_{n+1}(t;\alpha,\beta)\right)^2}}$$

Calibration for the Car-following Model



(a) calibration result of the speed

(b) calibration result of the spacing

Figure 9: Calibration results of the speed and spacing

Conclusion

- With a particular emphasis on the inaccurate estimation issue of the existing speed-density models **especially at high density ranges**, this paper is motivated by the need to establish an **empirically accurate and mathematically elegant** speed-density relationship which can describe the speed-density relationship under all possible traffic states.
- It has been evident that over the past decades, continuing advances in the traffic flow fundamental diagram intrinsically depend on the two-dimensional representation to describe the speed-density relationship, while other relationships between speed and flow or flow and density are ignored. This paper proposes an enhanced “s-shape” speed-density function. One can easily represent the traffic flow fundamental diagram **in a three-dimensional modeling space**.
- With the macroscopic S3 model, a consistent macro-to-micro **car-following model** is derived. By utilizing the Laplace transform approach, the **stability criterion** of the car-following model is also derived.
- The study presented in this paper is an example to represent the traffic flow fundamental diagram with a smooth and non-convex-non-concave s-shaped curve. In the future, it is possible to analyze the state transitions due to **the existence of inflection point** in the non-convex-non-concave curve. As for the calibration of the S3 model, it is expected to calibrate the parameters considering heterogeneity on the vehicle types and road grades in future work.

Future Applications

- Analyze the traffic state transitions with smooth analytical forms
- Provide analytical formulas for the MFD to better control traffic flows in urban networks
- Higher order traffic flow models
- Calibrate traffic flow model at merge and weaving segments since the capacity, free-flow speed, critical density, and the car-following model can be synthetically calibrated

Thank you!