proof of $\boldsymbol{x}_0$ on *Lec06* page 11

Diagonal covariance matrix with **equal** elements, *which means* $\Sigma = \sigma^2 \mathrm{I}$, I *is the D-dimensional indenty matrix*, and

$$
\begin{aligned}
g_k(\boldsymbol{x}) &= \boldsymbol{w}_k^T \boldsymbol{x} + w_{k0} = \boldsymbol{\mu}_k^T \Sigma^{-1} \boldsymbol{x} + w_{k0} \\
&= \frac{1}{\sigma^2} \boldsymbol{\mu}_k^T \boldsymbol{x} - \frac{1}{2} \frac{||\boldsymbol{\mu}_k||^2}{\sigma^2} + \ln P(\omega_k)
\end{aligned}
$$

note

$$\boldsymbol{a}^T \boldsymbol{I} \boldsymbol{a} = \boldsymbol{a}^T \boldsymbol{a} = ||\boldsymbol{a}||^2$$

where I is Identity matrix.
To get the decision hyperplanes,

$$g_{ij}(\boldsymbol{x}) = g_i(\boldsymbol{x}) - g_j(\boldsymbol{x}) = 0$$

then,

$$\frac{1}{\sigma^2} (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{x} - \frac{1}{2\sigma^2} (||\boldsymbol{\mu}_i||^2 - ||\boldsymbol{\mu}_j||^2) + \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) = 0 \tag{1}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{x} - \frac{1}{2} (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \sigma^2 \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) = 0 \tag{2}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{x} - \frac{1}{2} (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \sigma^2 \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) \frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{||\boldsymbol{\mu}_i - \boldsymbol{\mu}_j||^2} = 0 \tag{3}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \{ \boldsymbol{x} - \frac{1}{2} (\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \sigma^2 \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) \frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{||\boldsymbol{\mu}_i - \boldsymbol{\mu}_j||^2} \} = 0 \tag{4}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \{ \boldsymbol{x} - [\frac{1}{2} (\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \sigma^2 \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) \frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{||\boldsymbol{\mu}_i - \boldsymbol{\mu}_j||^2}] \} = 0 \tag{5}$$

$$\boldsymbol{w}^T (\boldsymbol{x} - \boldsymbol{x}_0) = 0 \tag{6}$$

Therefore, we have

$$\boldsymbol{x}_0 = \frac{1}{2} (\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \sigma^2 \ln \left(\frac{P(\omega_i)}{P(\omega_j)} \right) \frac{\boldsymbol{\mu}_i - \boldsymbol{\mu}_j}{||\boldsymbol{\mu}_i - \boldsymbol{\mu}_j||^2}$$

proof of $\boldsymbol{x}_0$ on *Lec06* page 15

$$\begin{aligned} g_k(\boldsymbol{x}) &= \boldsymbol{w}_k^T\boldsymbol{x} + w_{k0} \\ &= \boldsymbol{\mu}_k^T\Sigma^{-1}\boldsymbol{x} - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma^{-1}\boldsymbol{\mu}_k + \ln P(\omega_i) \end{aligned}$$

To get the decision hyperplanes,

$$g_{ij}(\boldsymbol{x}) = g_i(\boldsymbol{x}) - g_j(\boldsymbol{x}) = 0$$

then,

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}\boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}_i^T\Sigma^{-1}\boldsymbol{\mu}_i - \boldsymbol{\mu}_j^T\Sigma^{-1}\boldsymbol{\mu}_j) + \ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right) = 0 \tag{7}$$

since Σ is symmetric, we can prove,

$$\boldsymbol{\mu}_i^T\Sigma^{-1}\boldsymbol{\mu}_i - \boldsymbol{\mu}_j^T\Sigma^{-1}\boldsymbol{\mu}_j = (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) \tag{8}$$

So, from equation (7), we have

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}\boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right) = 0 \tag{9}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}\boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right)\frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)} = 0 \tag{10}$$

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}\{\boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) + \ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right)\frac{\boldsymbol{\mu}_i - \boldsymbol{\mu}_j}{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}\} = 0 \tag{11}$$

$$\boldsymbol{w}^T(\boldsymbol{x} - \boldsymbol{x}_0) = 0 \tag{12}$$

Therefore, we have

$$\boldsymbol{x}_0 = \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \ln\left(\frac{P(\omega_i)}{P(\omega_j)}\right)\frac{\boldsymbol{\mu}_i - \boldsymbol{\mu}_j}{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T\Sigma^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)} = 0 \tag{13}$$