DTS203TC
Design and Analysis of Algorithms

Lecture 9: Dynamic Programming

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Learning outcome

- Understand the basic idea of dynamic programming
- Able to apply dynamic programming to compute Fibonacci numbers
- Able to apply dynamic programming to solve the assembly line scheduling problem

Acknowledgment: Some slides are adapted from ones by Prof. Prudence Wong

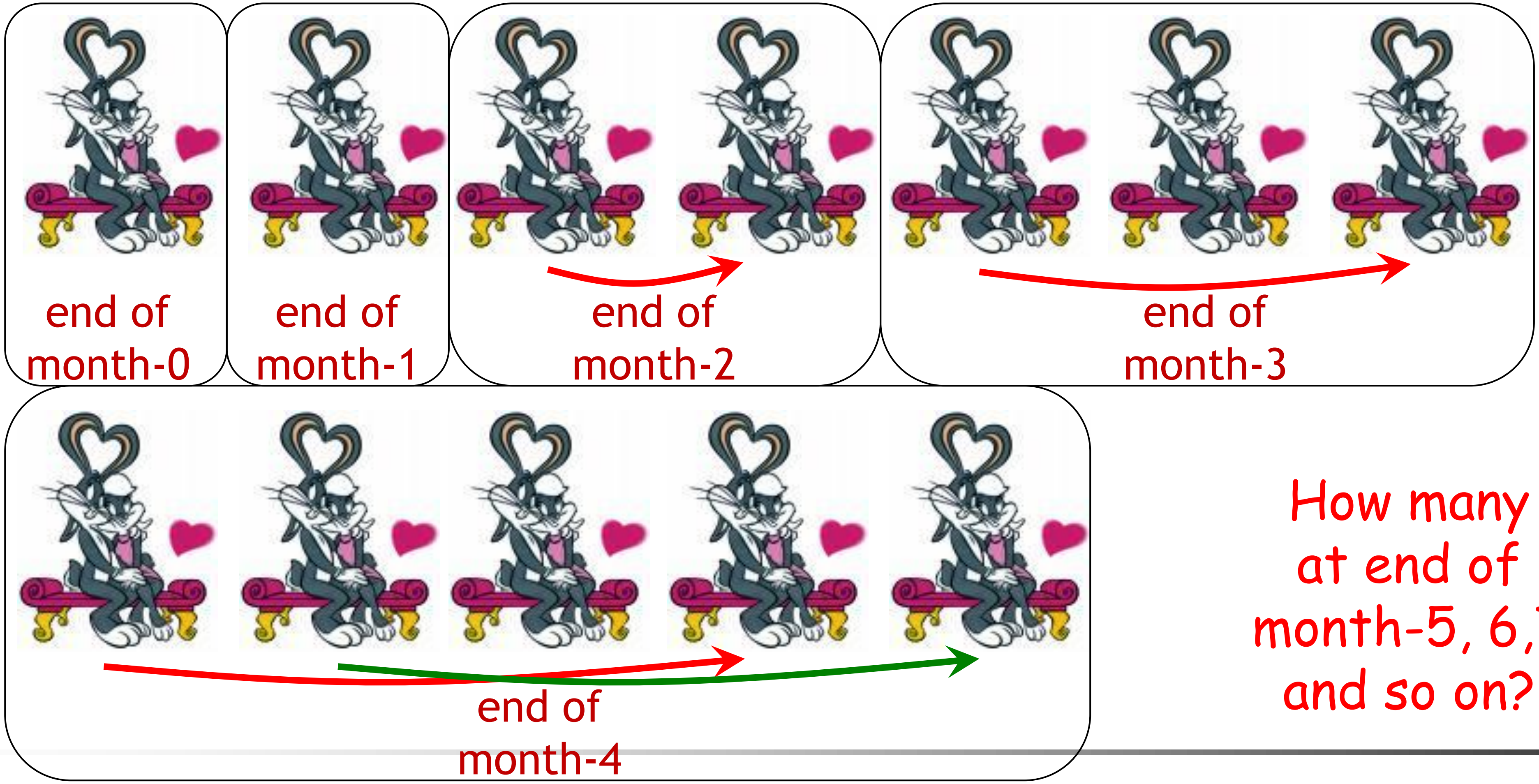
Dynamic programming

an efficient way to implement some divide and conquer algorithms

Fibonacci numbers

Fibonacci's Rabbits

A pair of rabbits, one month old, is too young to reproduce. Suppose that in their second month, and every month thereafter, they produce a new pair.



How many at end of month-5, 6,7 and so on?

Petals on flowers

1 petal:
white calla lily

2 petals:
euphorbia

3 petals:
trillium

5 petals:
columbine

8 petals:
bloodroot

13 petals:
black-eyed susan

21 petals:
shasta daisy

34 petals:
field daisy

Fibonacci Numbers in Nature

Fibonacci number

- Fibonacci number F(n)

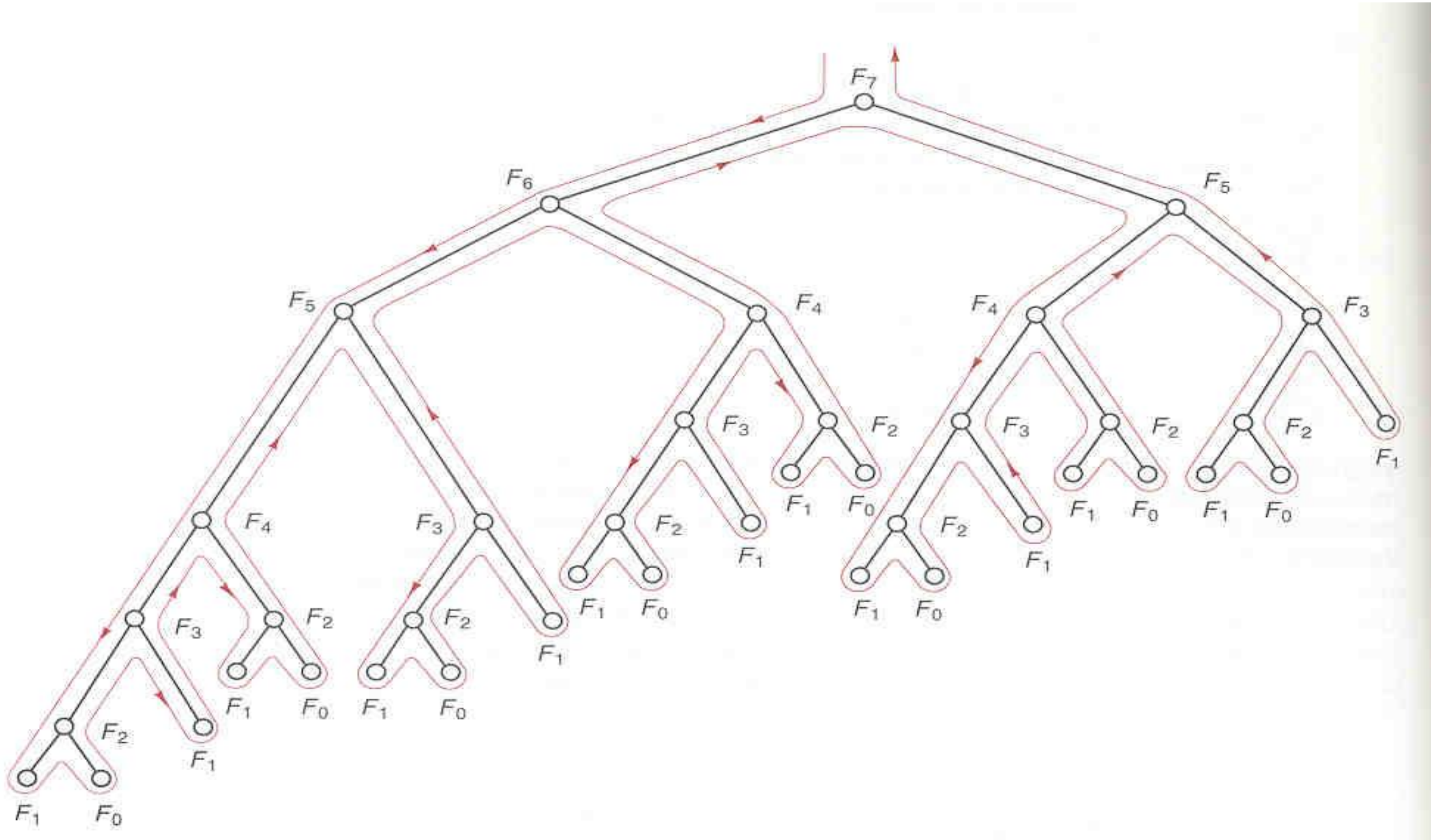
$$F(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10
F(n)	1	1	2	3	5	8	13	21	34	55	89

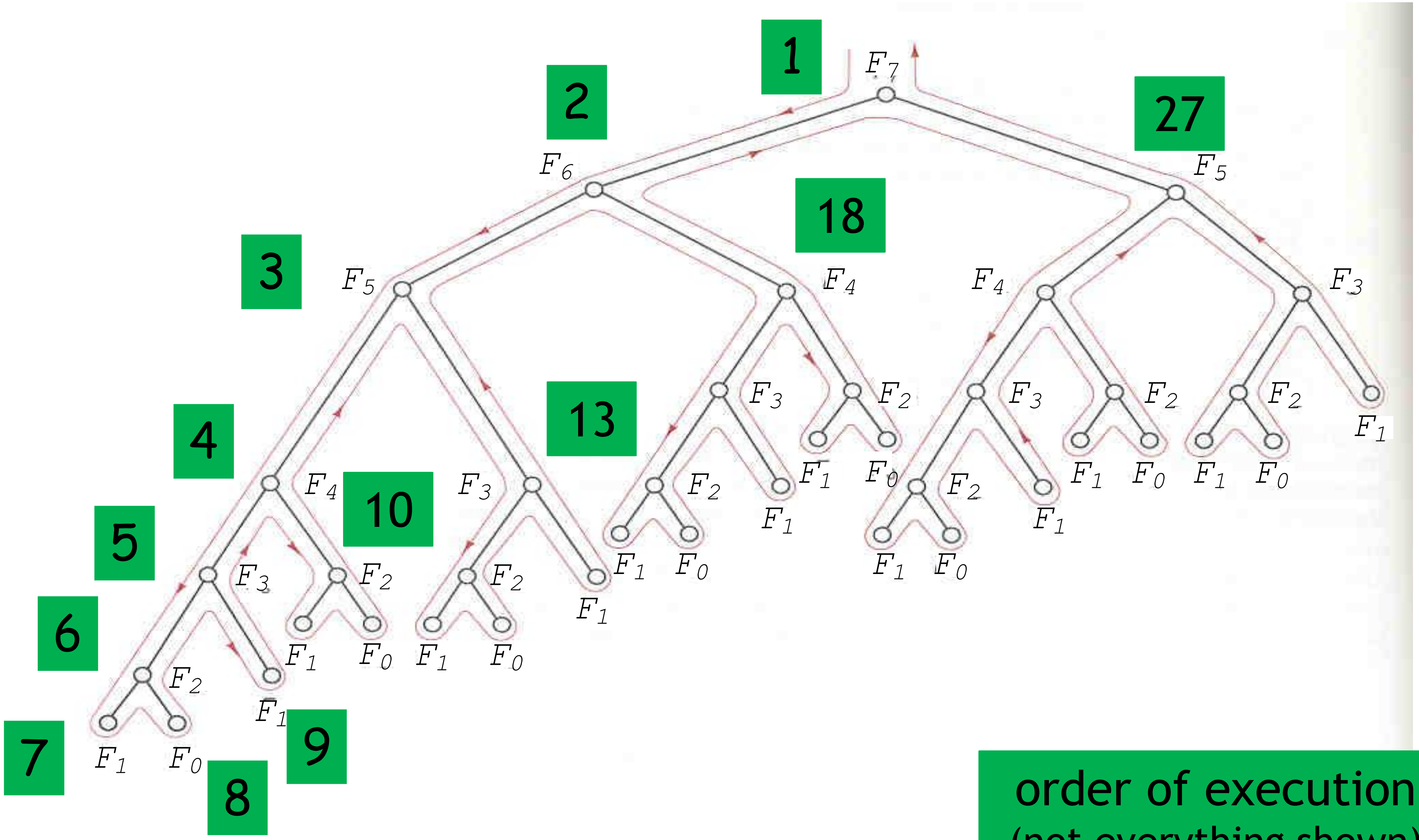
Pseudo code for the recursive algorithm:

```
Algorithm F(n)
    if n==0 or n==1 then
        return 1
    else
        return F(n-1) + F(n-2)
```

The execution of F(7)

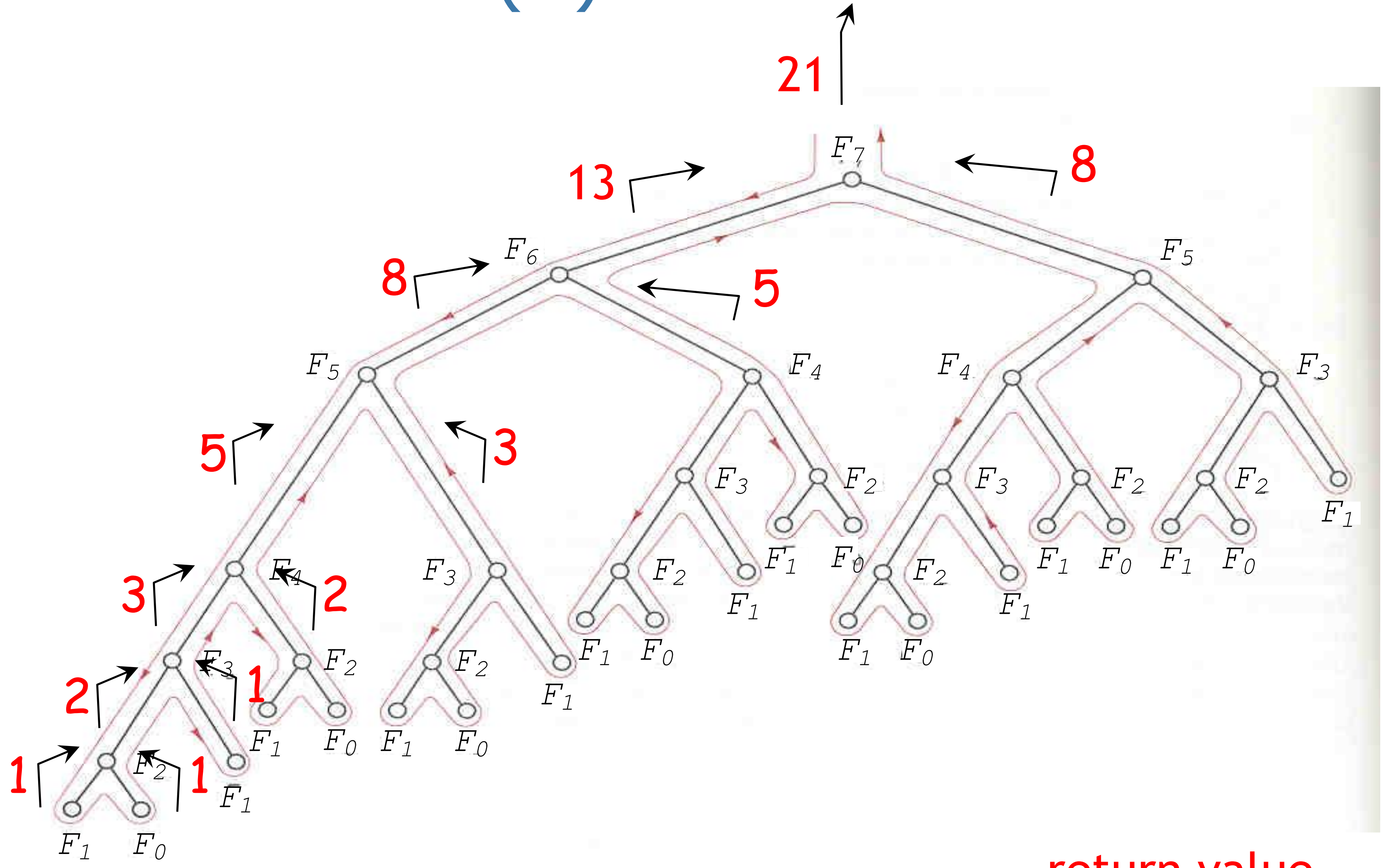


The execution of F(7)



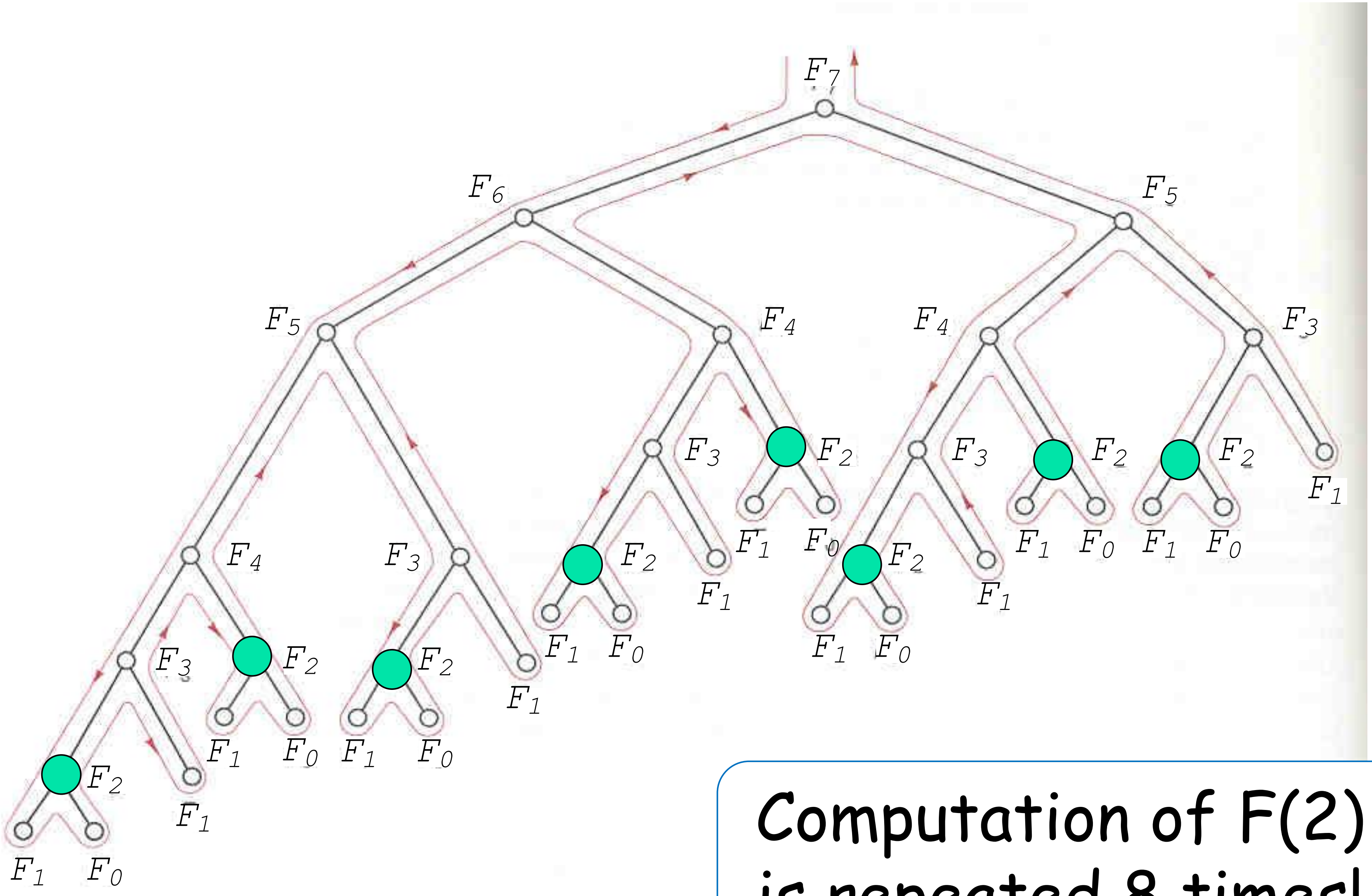
order of execution
(not everything shown)

The execution of F(7)



return value
(not everything shown)

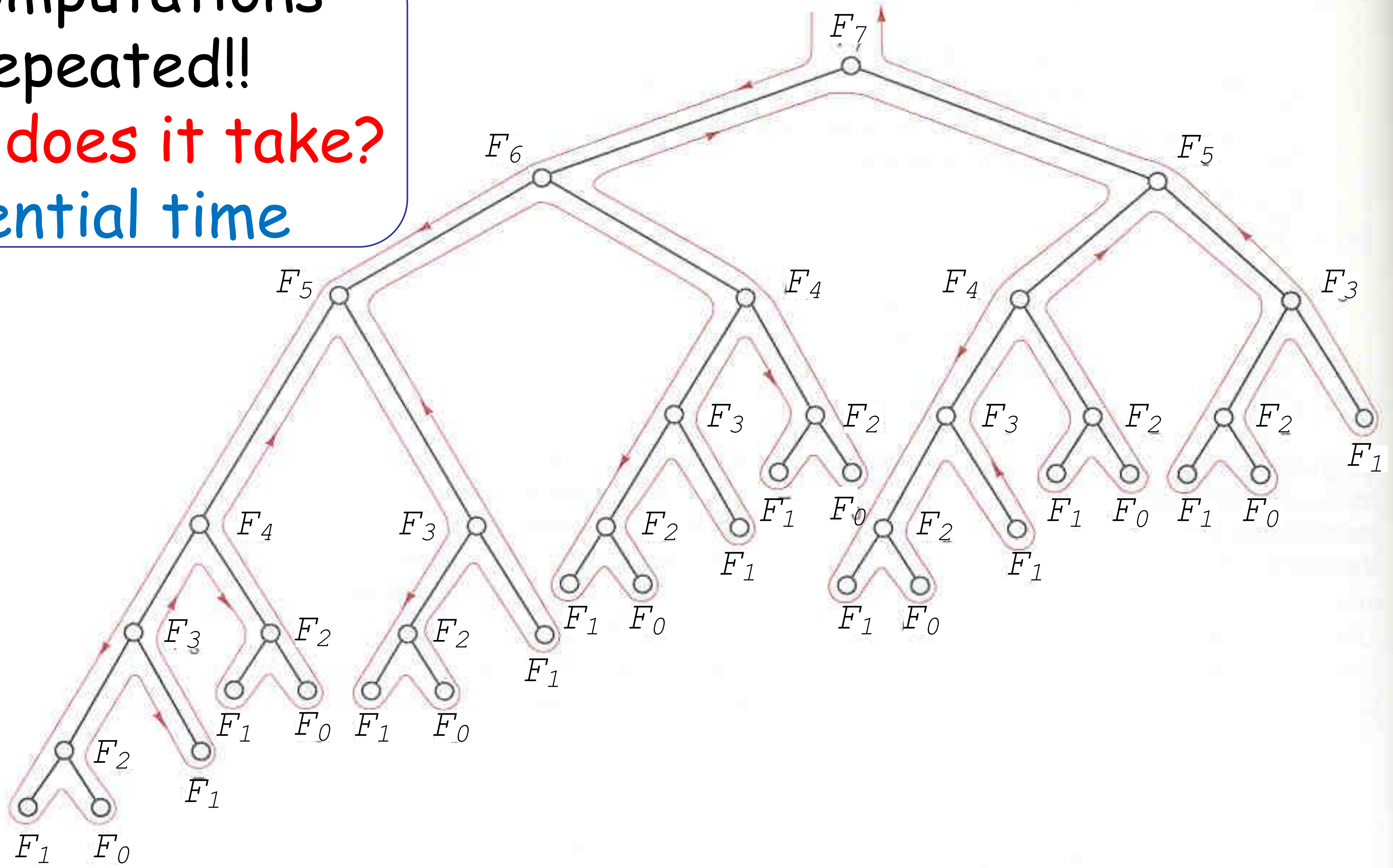
The execution of F(7)



Computation of F(2) is repeated 8 times!

The execution of F(7)

Many computations are repeated!!
How long does it take?
Exponential time



Idea for improvement

Memoization:

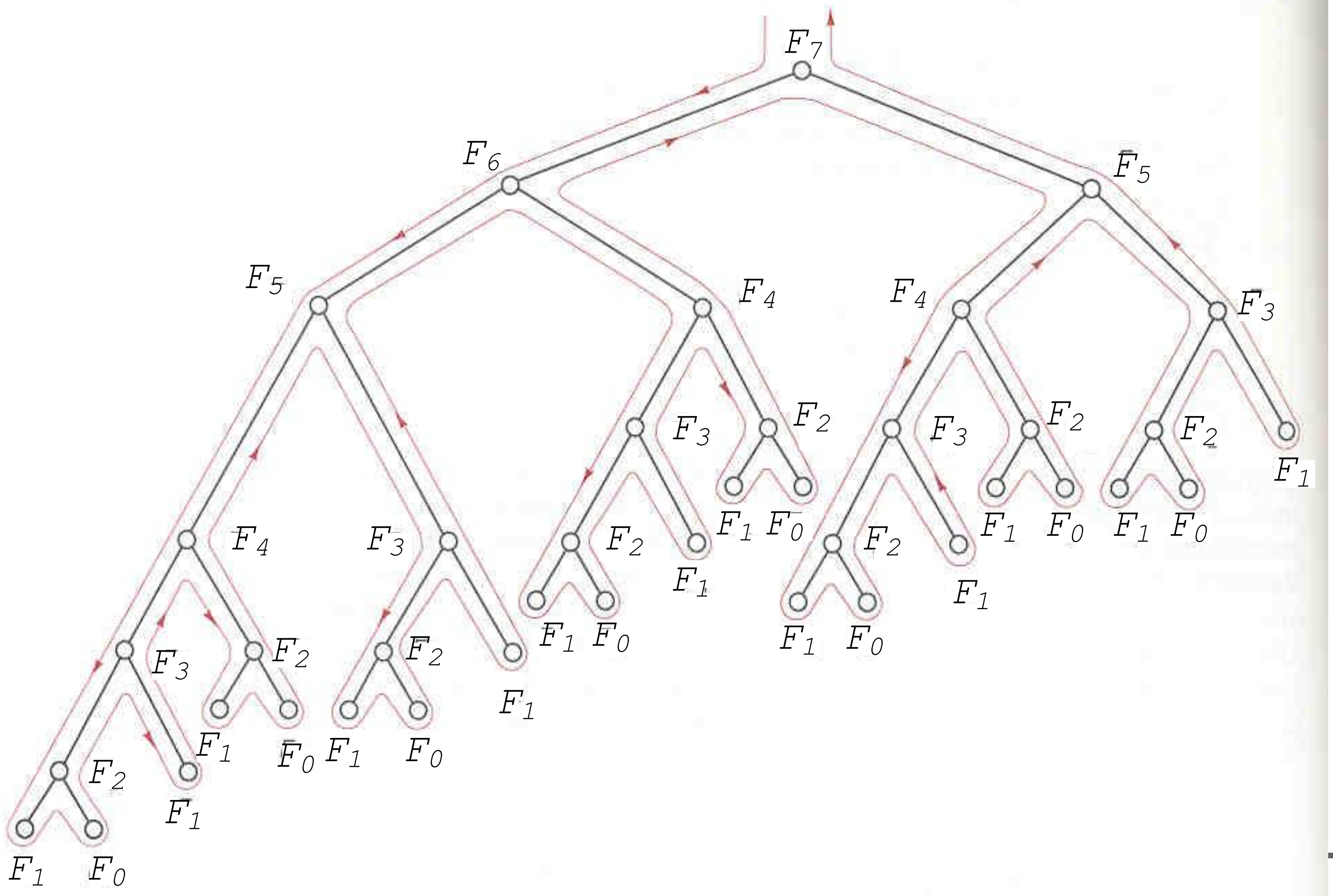
- Store F(i) somewhere after we have computed its value
- Afterward, we don't need to re-compute F(i); we can retrieve its value from our memory.

[ ] refers to array
( ) is parameter for calling a procedure

```
Procedure F(n)
  if (v[n] < 0) then
    v[n] = F(n-1)+F(n-2)
  return v[n]
```

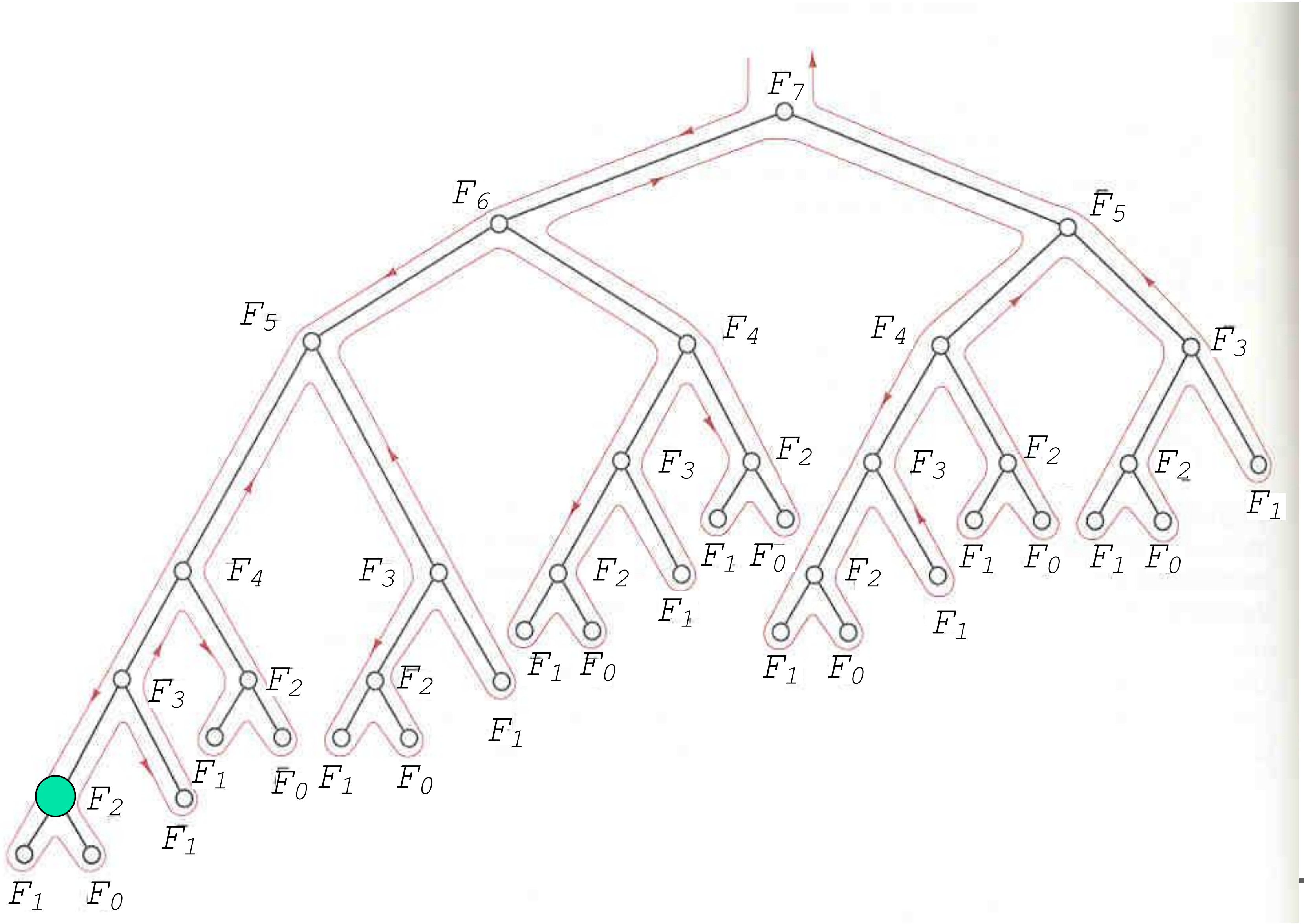
```
Main
  set v[0] = v[1] = 1
  for i = 2 to n do
    v[i] = -1
  output F(n)
```

Look at the execution of F(7)



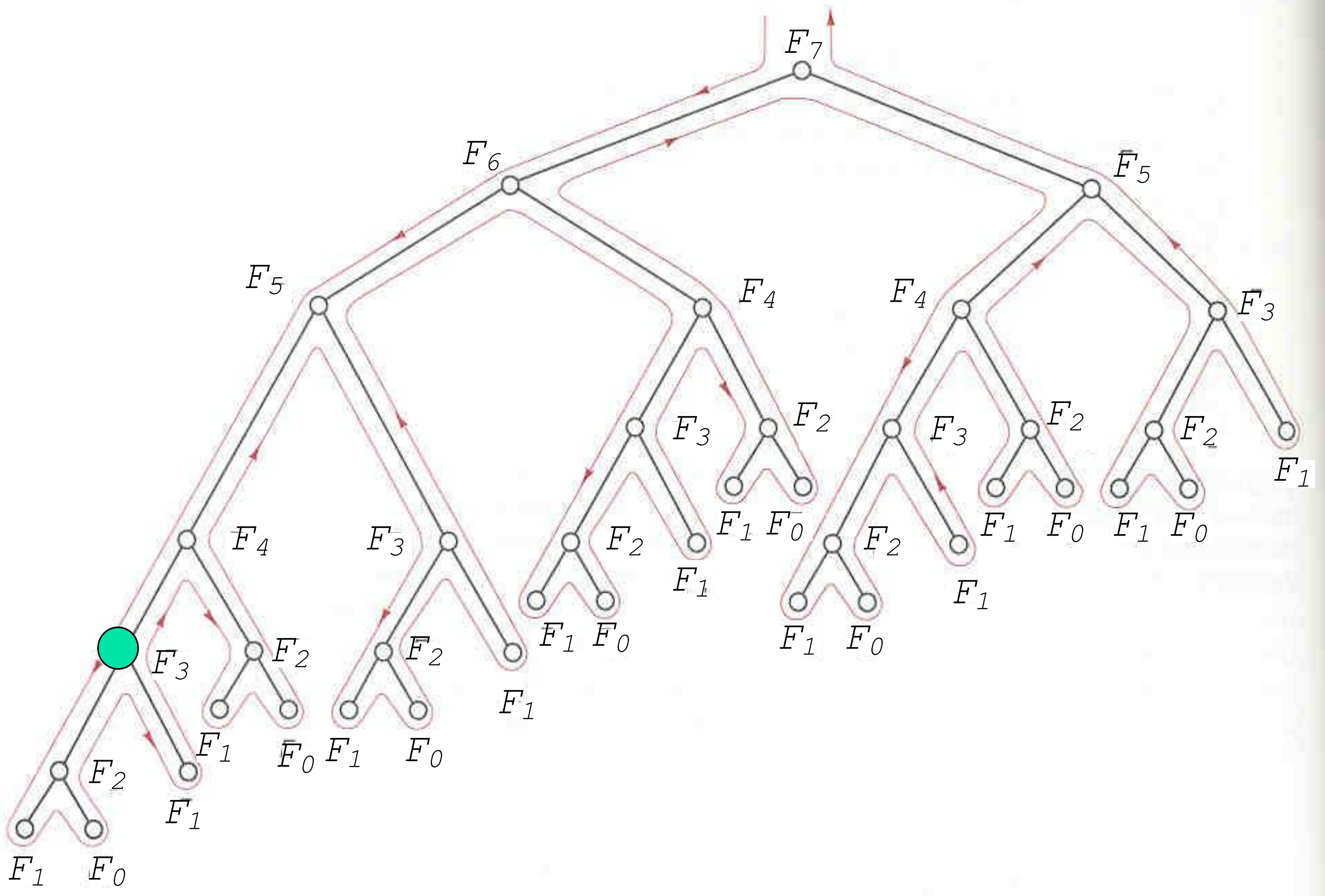
v[0]	1
v[1]	1
v[2]	-1
v[3]	-1
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



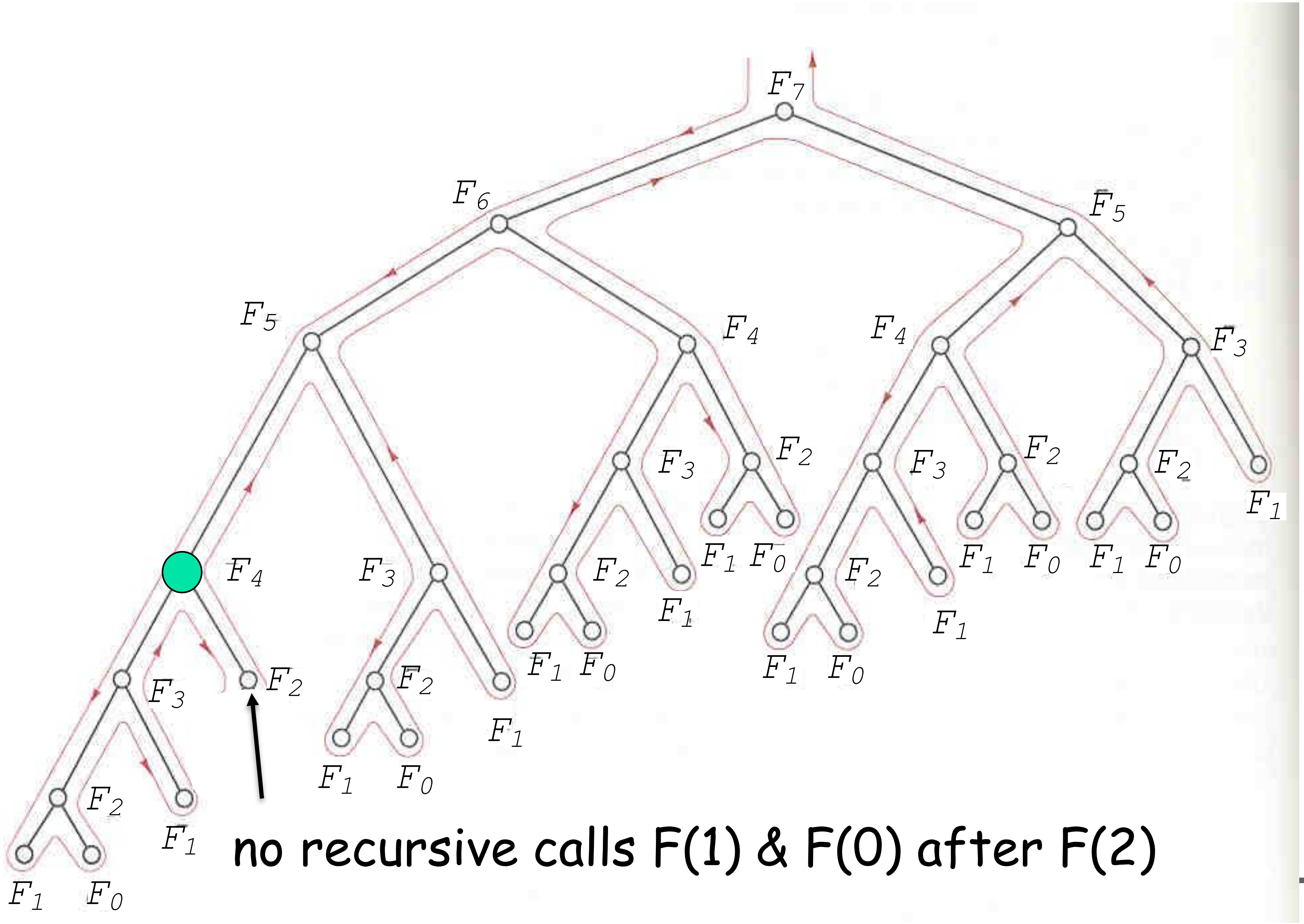
v[0]	1
v[1]	1
v[2]	2
v[3]	-1
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	-1
v[5]	-1
v[6]	-1
v[7]	-1

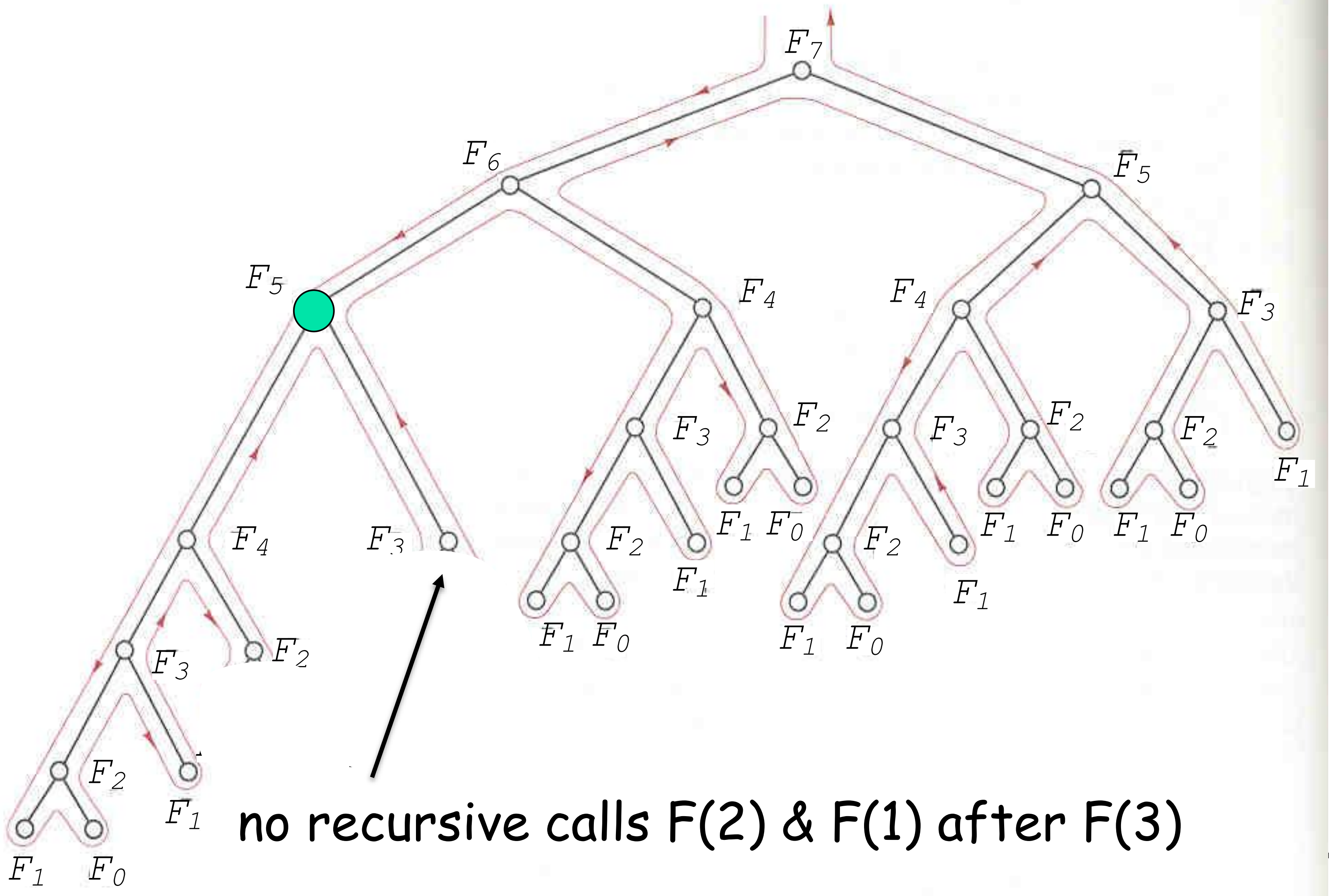
Look at the execution of F(7)



no recursive calls F(1) & F(0) after F(2)

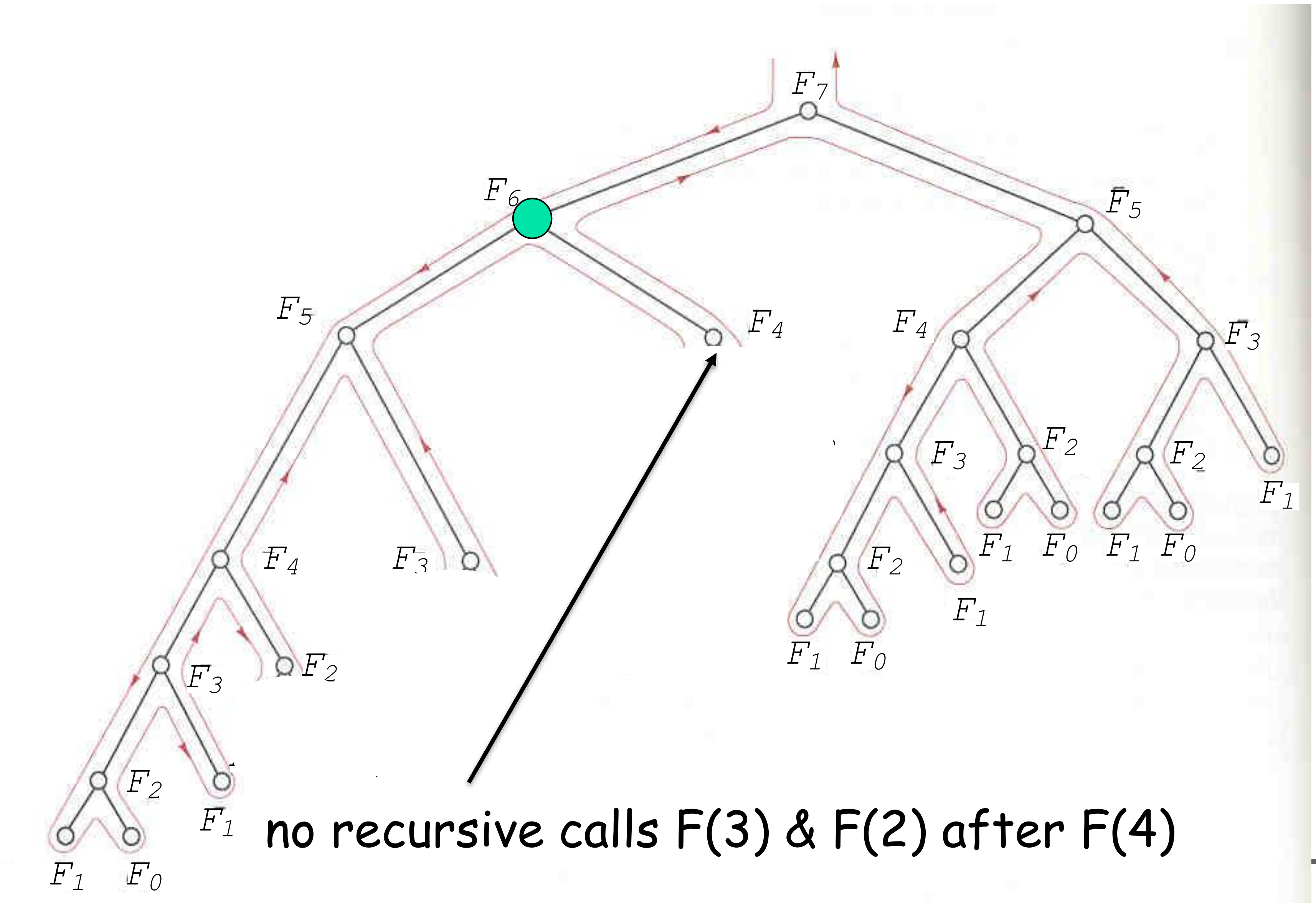
v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	-1
v[6]	-1
v[7]	-1

Look at the execution of F(7)



v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	-1
v[7]	-1

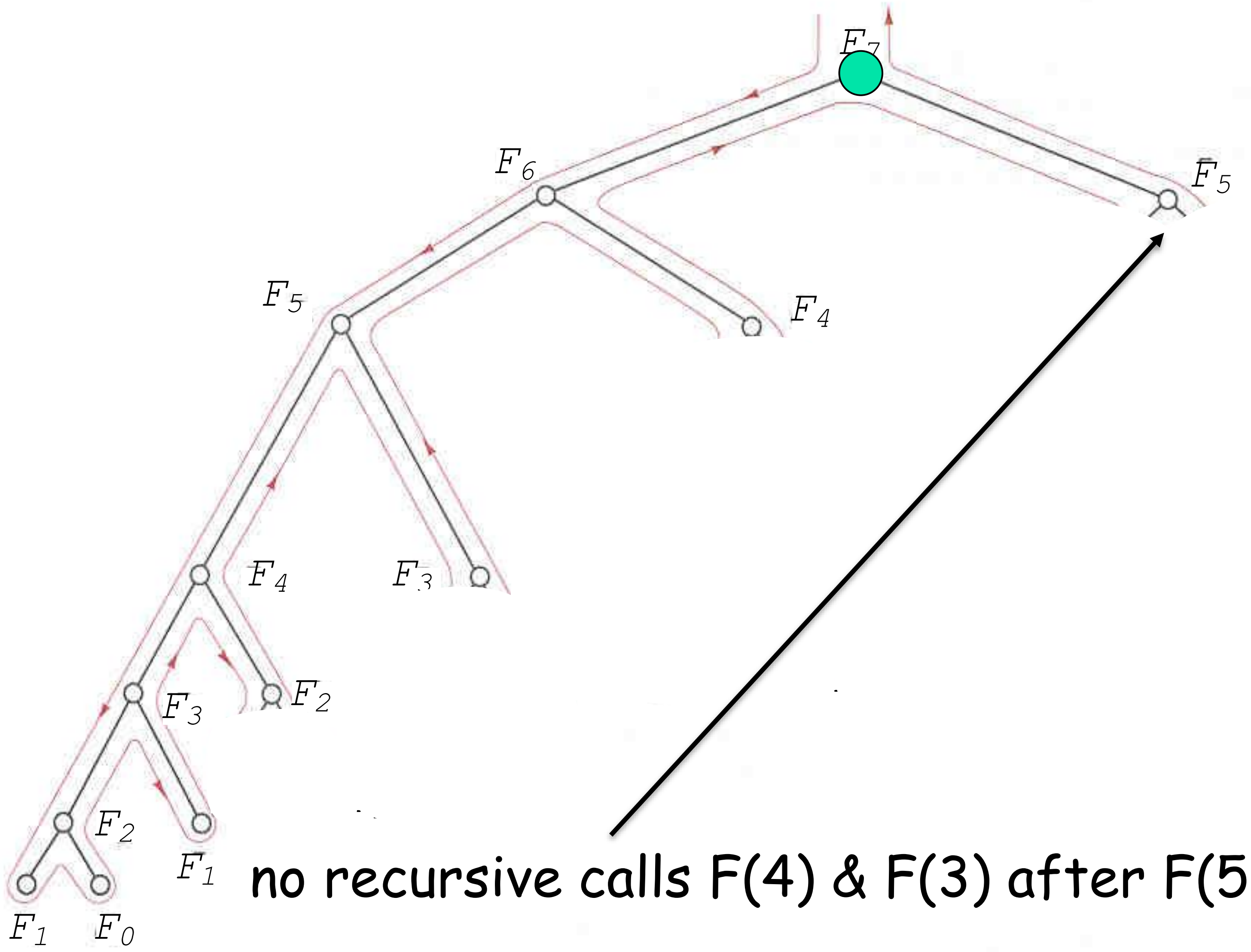
Look at the execution of F(7)



no recursive calls F(3) & F(2) after F(4)

v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	13
v[7]	-1

Look at the execution of F(7)



no recursive calls F(4) & F(3) after F(5)

v[0]	1
v[1]	1
v[2]	2
v[3]	3
v[4]	5
v[5]	8
v[6]	13
v[7]	21

Can we do better?

Observation

- The 2nd version stills make many function calls, and each wastes times in parameters passing, dynamic linking, ...
- In general, to compute F(i), we need F(i-1) & F(i-2) only

Idea to further improve

- Compute the values in bottom-up fashion.
- That is, compute F(2) (we already know F(0)=F(1)=1), then F(3), then F(4)...

```
Procedure F(n)
    Set A[0] = A[1] = 1
    for i = 2 to n do
        A[i] = A[i-1] + A[i-2]
    return A[n]
```

Recursive vs DP approach

Recursive version:

```
Procedure F(n)
   if n==0 or n==1 then
      return 1
   else
      return F(n-1) + F(n-2)
```

Too Slow! exponential

Dynamic Programming version:

```
Procedure F(n)
   Set A[0] = A[1] = 1
   for i = 2 to n do
      A[i] = A[i-1] + A[i-2]
   return A[n]
```

Efficient!
Time complexity is O(n)

Summary of the methodology

- Write down a formula that relates a solution of a problem with those of sub-problems.
  E.g. $F(n) = F(n-1) + F(n-2)$.
- Index the sub-problems so that they can be **stored** and **retrieved** easily in a table (i.e., array)
- Fill the table in some **bottom-up** manner; start filling the solution of the smallest problem.
  - This ensures that when we solve a particular sub-problem, the solutions of all the smaller sub-problems that it depends are available.

For historical reasons, we call such methodology

Dynamic Programming.

In the late 40's (when computers were rare), programming refers to the "tabular method".

Exercise

- Consider the following function

$$G(n) = \begin{cases} 1 & \text{if } 0 \le n \le 2 \\ G(n-1) + G(n-2) + G(n-3) & \text{if } n > 2 \end{cases}$$

1. Draw the execution tree of computing G(6) recursively
2. Using dynamic programming, write a pseudo code to compute G(n) efficiently
3. What is the time complexity of your algorithm?

Exercise

$$G(n) = \begin{cases} 1 & \text{if } 0 \leq n \leq 2 \\ G(n-1) + G(n-2) + G(n-3) & \text{if } n > 2 \end{cases}$$

Recursive version:

```
Procedure G(n)
    if n >= 0 and n <= 2 then
        return 1
    return G(n-1) + G(n-2) + G(n-3)
```

Dynamic Programming version:

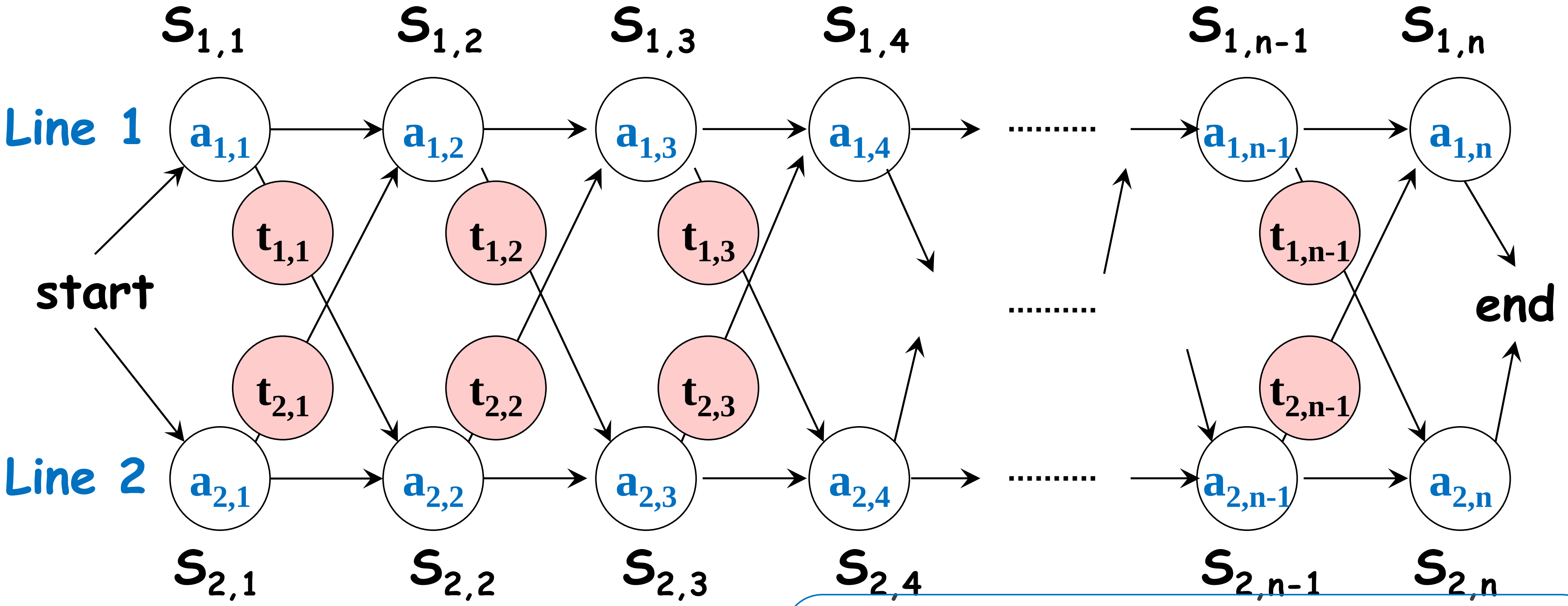
```
Procedure G(n)
    Set A[0] = A[1] = A[2] = 1
    for i = 3 to n do
        A[i] = A[i-1] + A[i-2] + A[i-3]
    return A[n]
```

O(n)

Assembly line scheduling

Assembly line scheduling

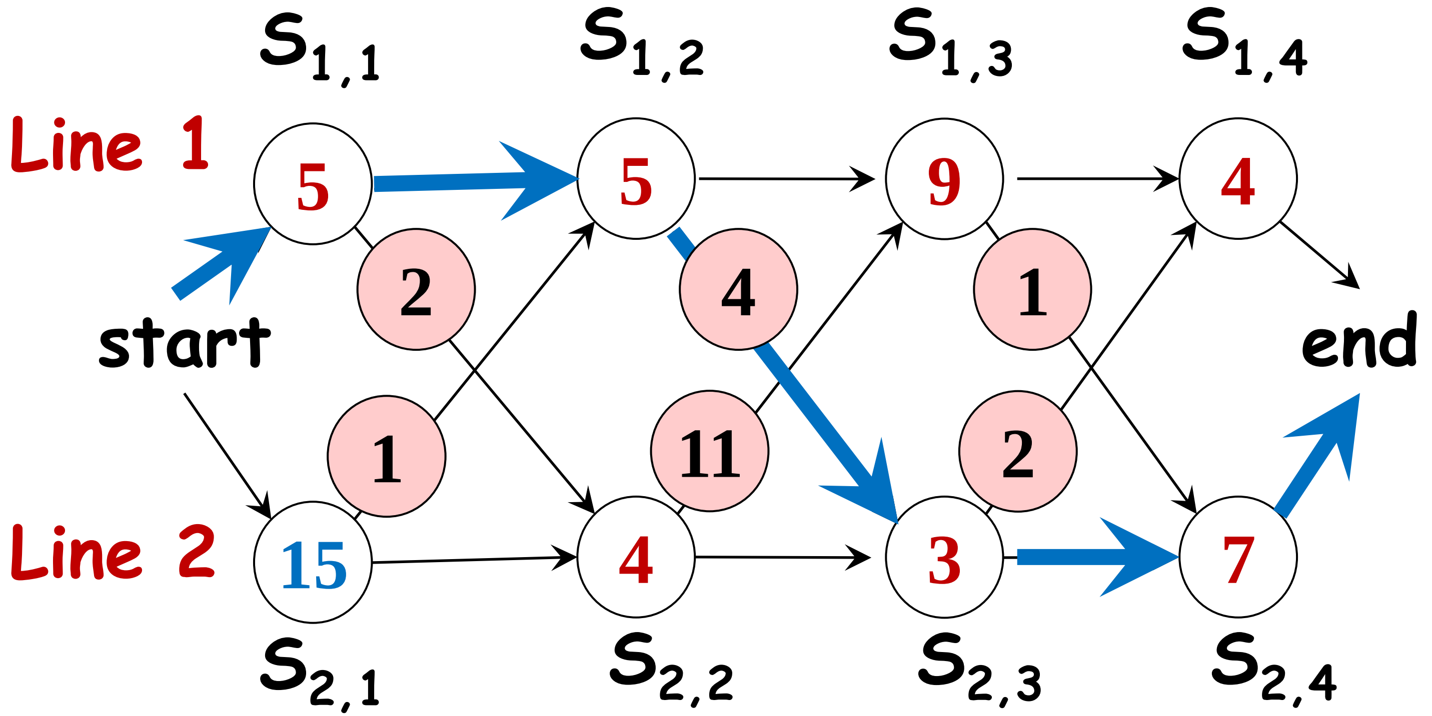
2 assembly lines, each with n stations ($S_{i,j}$: line i station j)
$S_{1,j}$ and $S_{2,j}$ perform same task but time taken is different



$a_{i,j}$: assembly time at $S_{i,j}$
$t_{i,j}$: transfer time after $S_{i,j}$

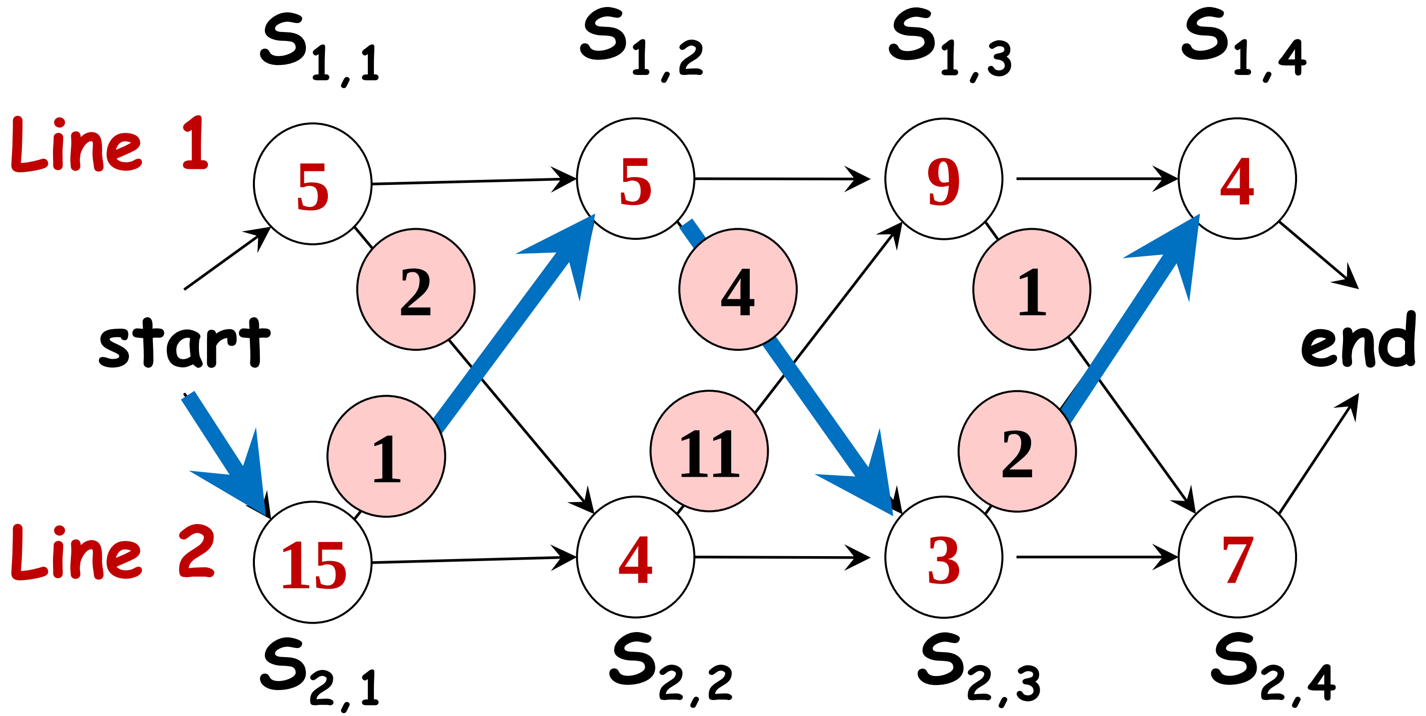
<u>Problem:</u> To determine which stations to go in order to **minimize** the total time through the n stations

Example (1)



stations chosen:	$S_{1,1}$	$S_{1,2}$		$S_{2,3}$	$S_{2,4}$		
time required:	5	5	4	3	7	= 24	

Example (2)



stations chosen:	$S_{1,1}$		$S_{1,2}$		$S_{2,3}$		$S_{2,4}$	
time required:	5		5	4	3		7	= 24

stations chosen:	$S_{2,1}$		$S_{1,2}$		$S_{2,3}$		$S_{1,4}$	
time required:	15	1	5	4	3	2	4	= 34

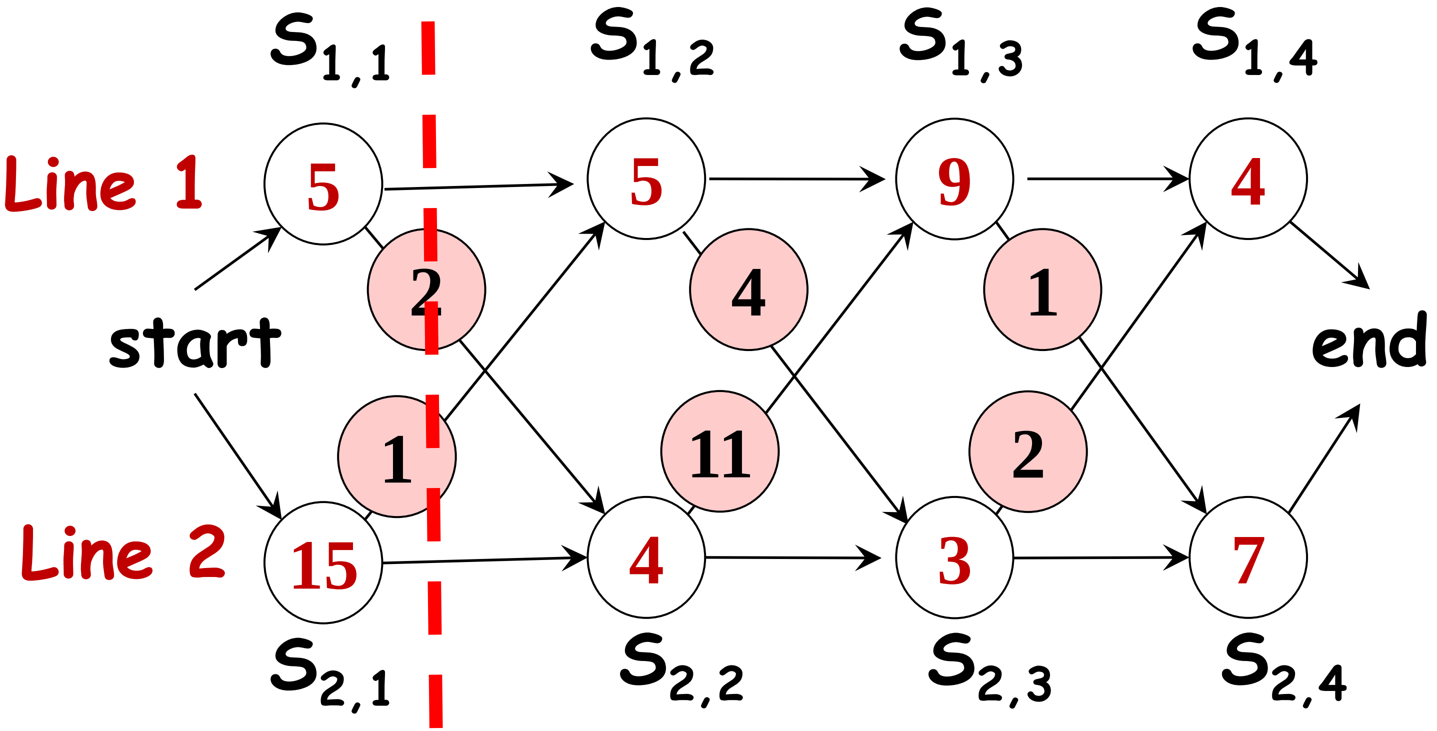
How to determine the best stations to go? There are altogether 2^n choices of stations. Should we try them all?

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Good news: Dynamic Programming

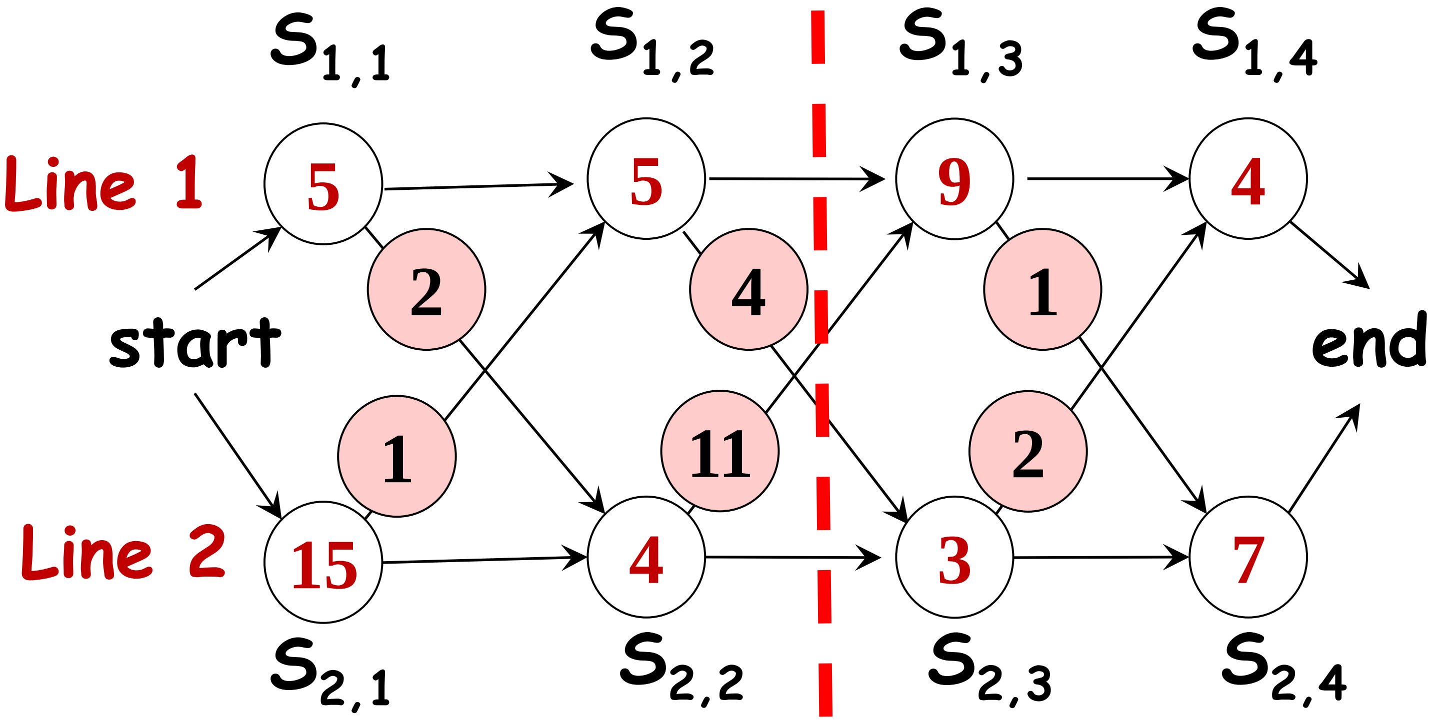
- We **don't** need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we can compute the fastest ways to get thro' station $S_{1,n}$ and $S_{2,n}$, then the faster of these two ways is the overall fastest way.
  2. To compute the fastest ways to get thro' $S_{1,n}$ (similarly for $S_{2,n}$), we need to know the fastest way to get thro' $S_{1,n-1}$ and $S_{2,n-1}$
  3. In general, we want to know the fastest way to get thro' $S_{1,j}$ and $S_{2,j}$, for all j.

Example again



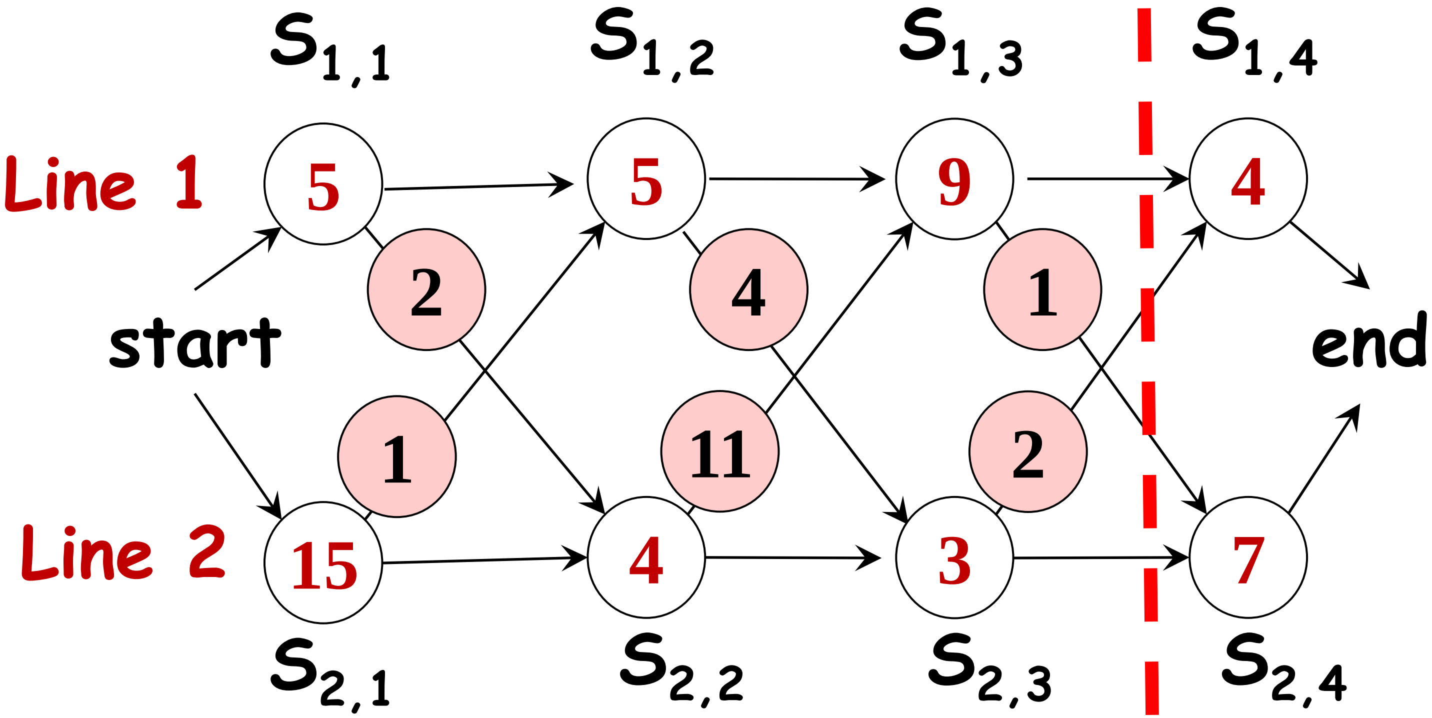
	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	s_{14}
minimum cost:	5			
	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$	$s_{2,4}$
	15			

Example again



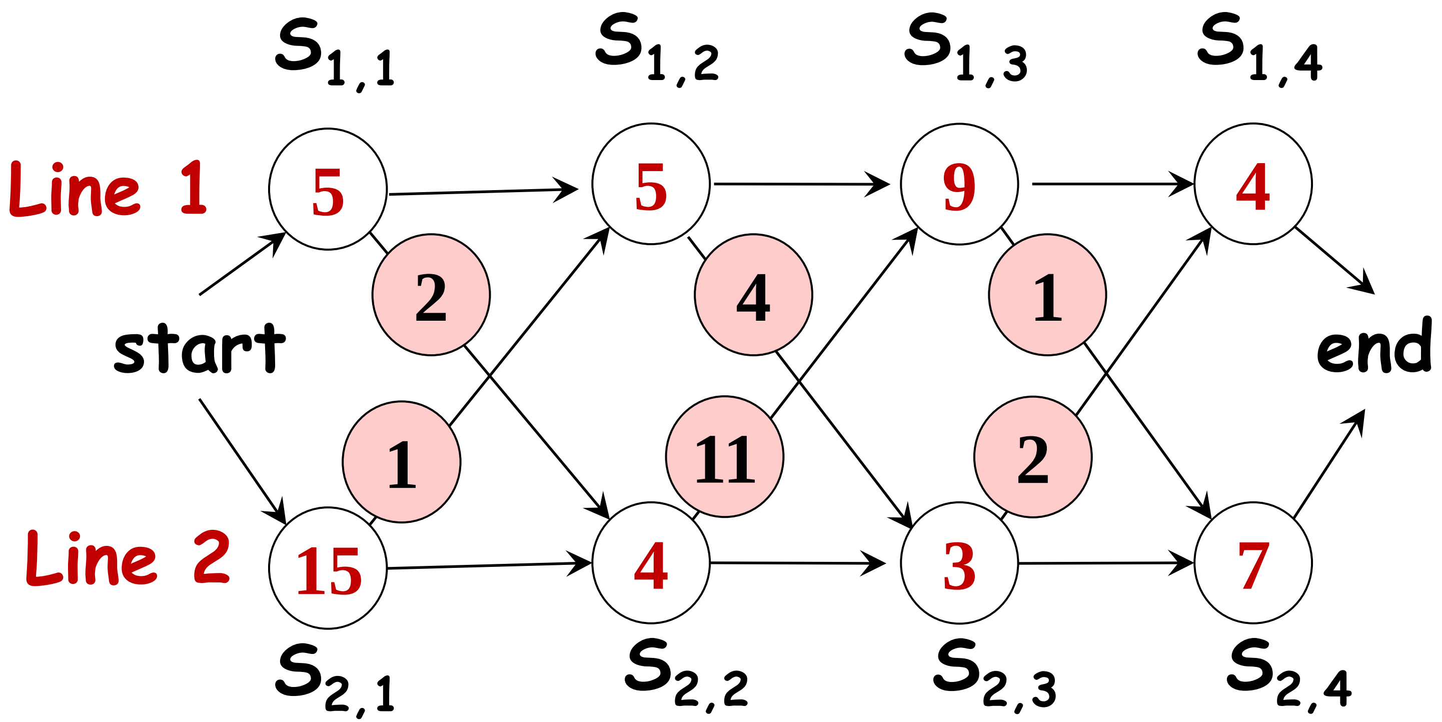
	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	S_{14}
minimum cost:	5	5+5 ~~15+1+5~~		
	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$
	15	5+2+4 ~~15+4~~		

Example again



	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	S_{14}
minimum cost:	5	5+5 ~~15+1+5~~	10+9 ~~11+11+9~~	
	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$
	15	5+2+4 ~~15+4~~	~~10+4+3~~ 11+3	

Example again



	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	S_{14}	
minimum cost:	5	5+5 ~~15+1+5~~	10+9 ~~11+11+9~~	~~19+4~~ 14+2+4	20
	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{2,4}$	
	15	5+2+4 ~~15+4~~	~~10+4+3~~ 11+3	~~19+1+7~~ 14+7	~~21~~

A dynamic programming solution

What are the sub-problems?

- given j, what is the fastest way to get thro' $S_{1,j}$
- given j, what is the fastest way to get thro' $S_{2,j}$

Definitions:

- $f_1[j]$ = the fastest time to get thro' $S_{1,j}$
- $f_2[j]$ = the fastest time to get thro' $S_{2,j}$

The final solution equals to **min { $f_1[n]$, $f_2[n]$ }**

Task:

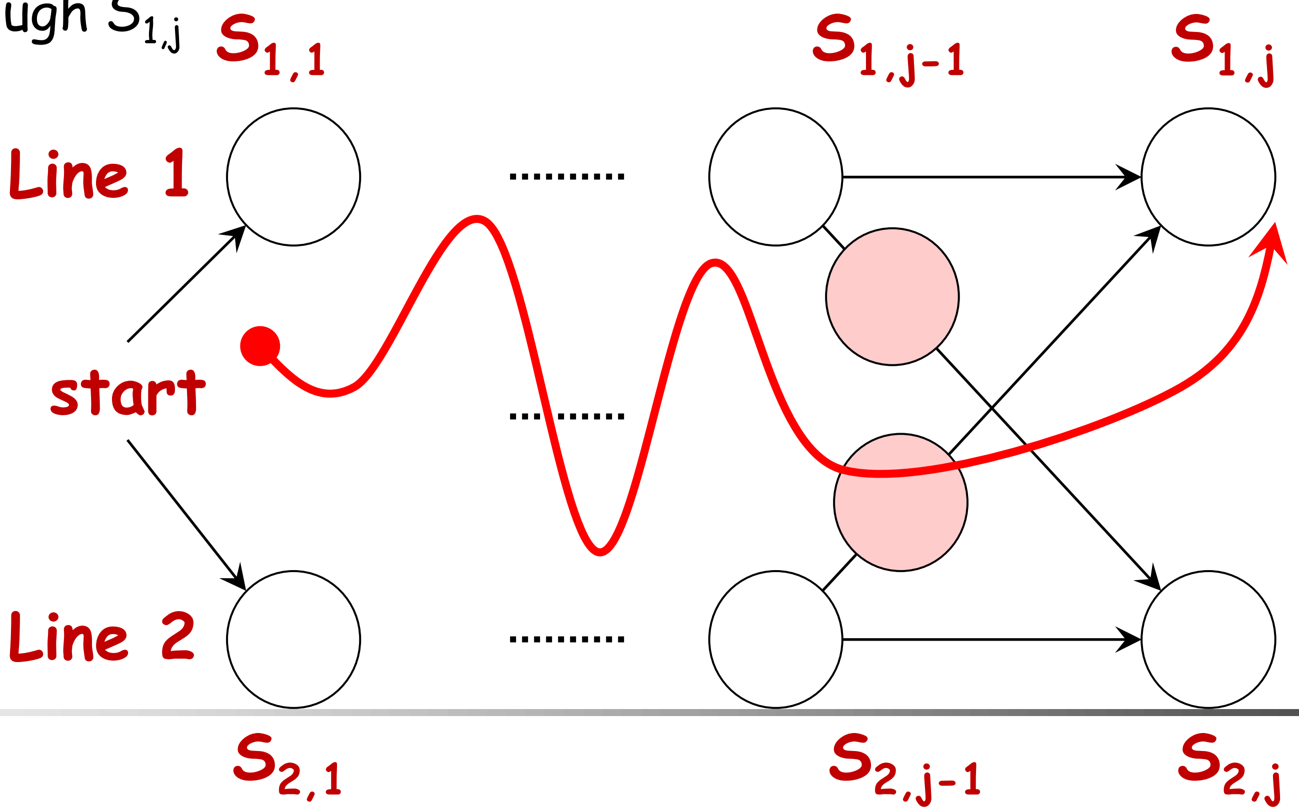
- Starting from $f_1[1]$ and $f_2[1]$,
  compute $f_1[j]$ and $f_2[j]$ incrementally

Solving the sub-problems (1)

Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then **directly** to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a **transfer** from line 2 to line 1, and then through $S_{1,j}$



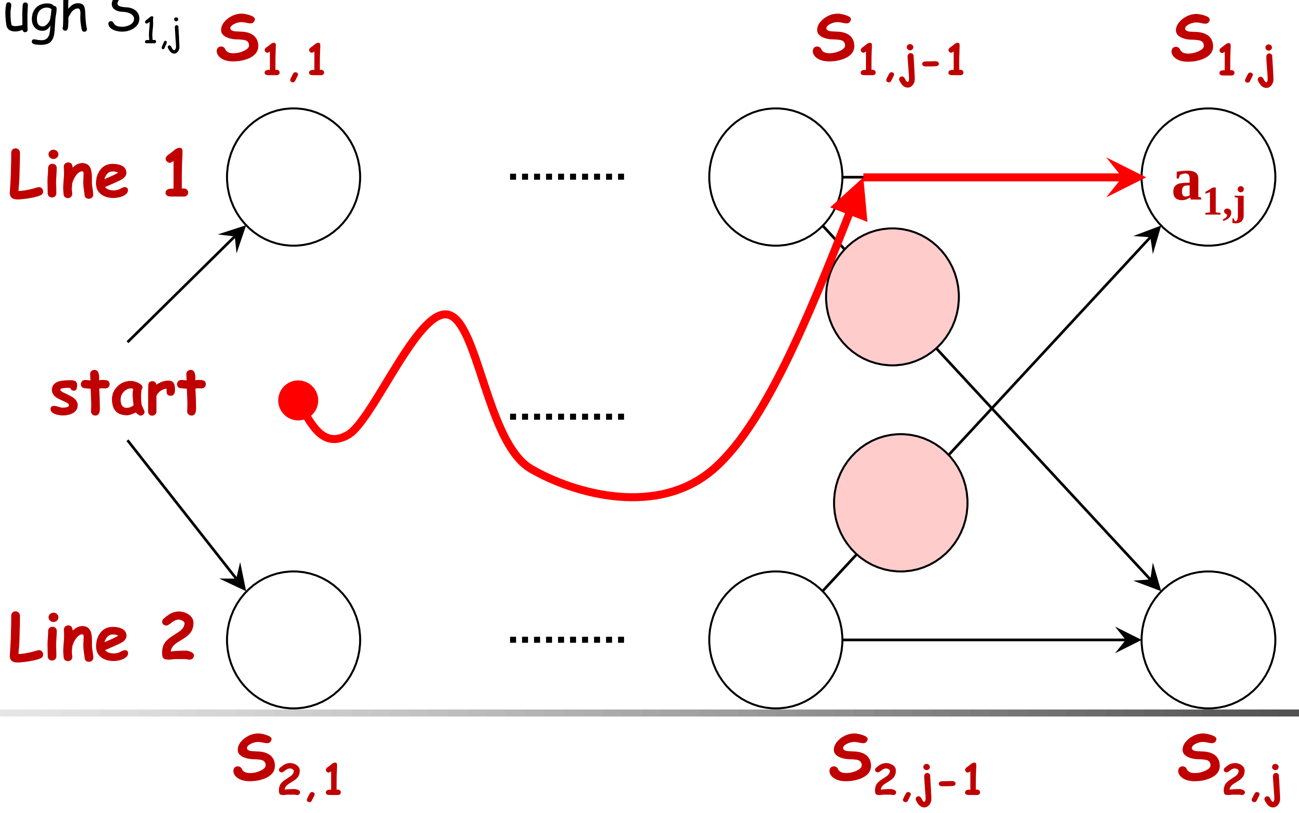
Solving the sub-problems (1)

Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then **directly** to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a **transfer** from line 2 to line 1, and then through $S_{1,j}$

Time required
$= f_1[j-1] + a_{1,j}$



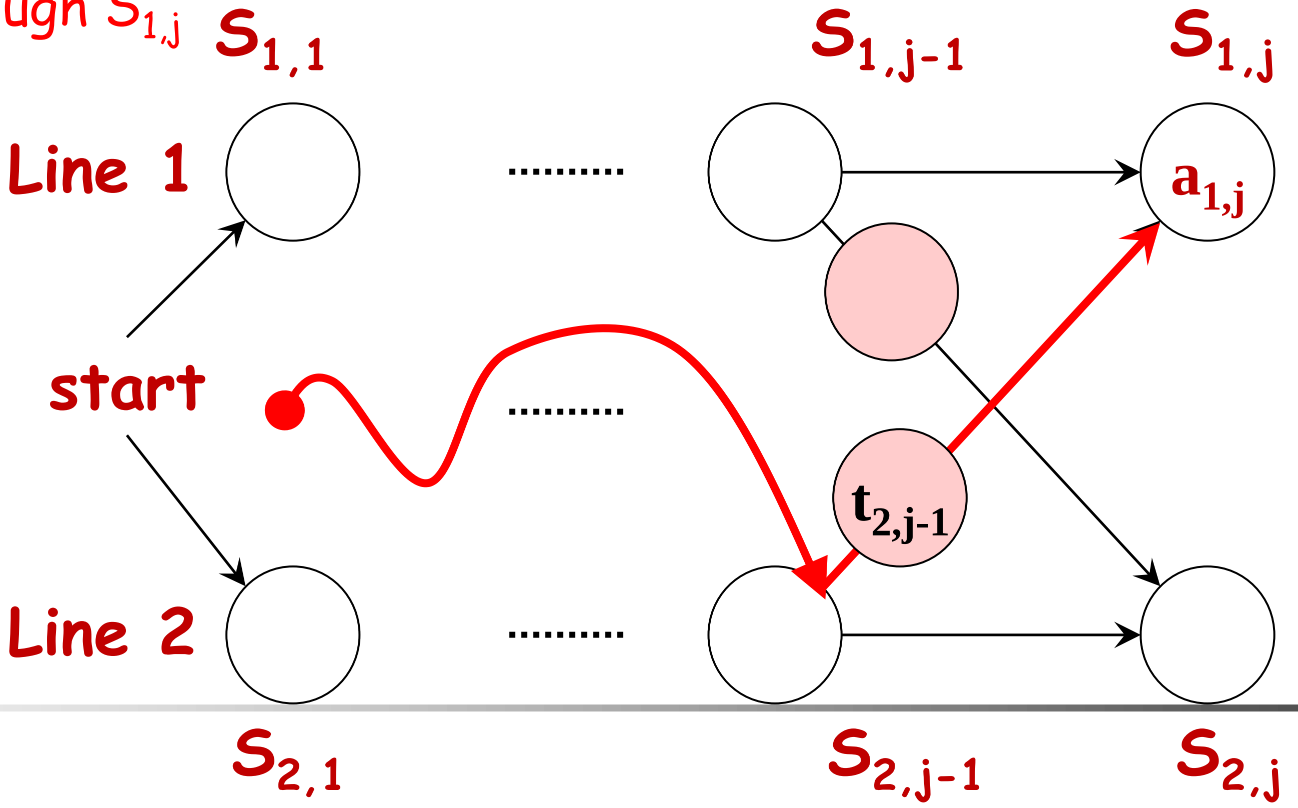
Solving the sub-problems (1)

Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then **directly** to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a **transfer** from line 2 to line 1, and then through $S_{1,j}$

Time required
$= f_2[j-1] + t_{2,j-1} + a_{1,j}$



Solving the sub-problems (1)

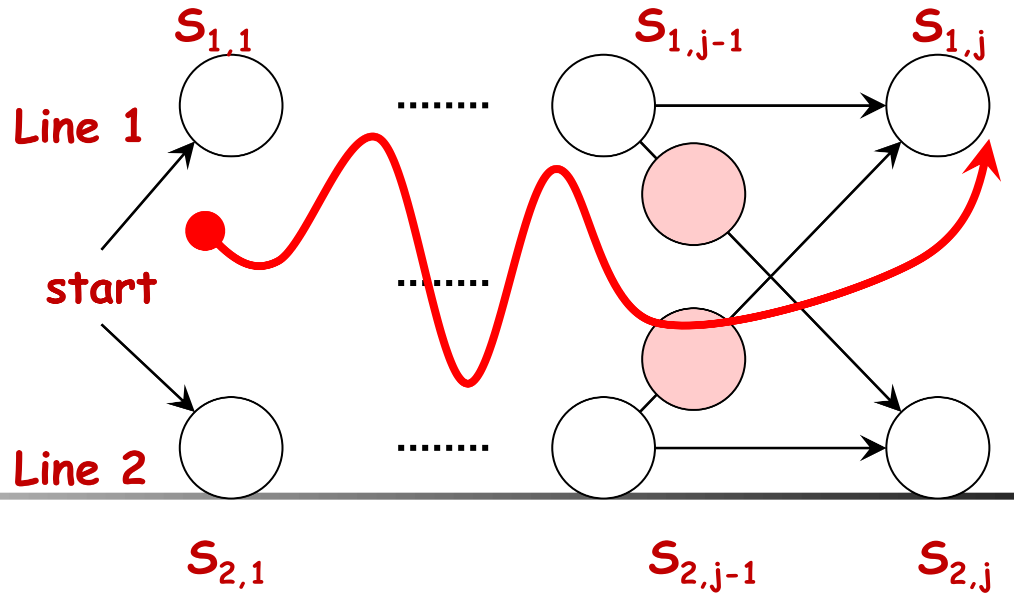
Q1: what is the fastest way to get thro' $S_{1,j}$?

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then through $S_{1,j}$

Conclusion: $\boxed{f_1[j] = \min(\, f_1[j-1]+a_{1,j}\ ,\ f_2[j-1]+t_{2,j-1}+a_{1,j}\,)}$

Boundary case: $\boxed{f_1[1] = a_{1,1}}$



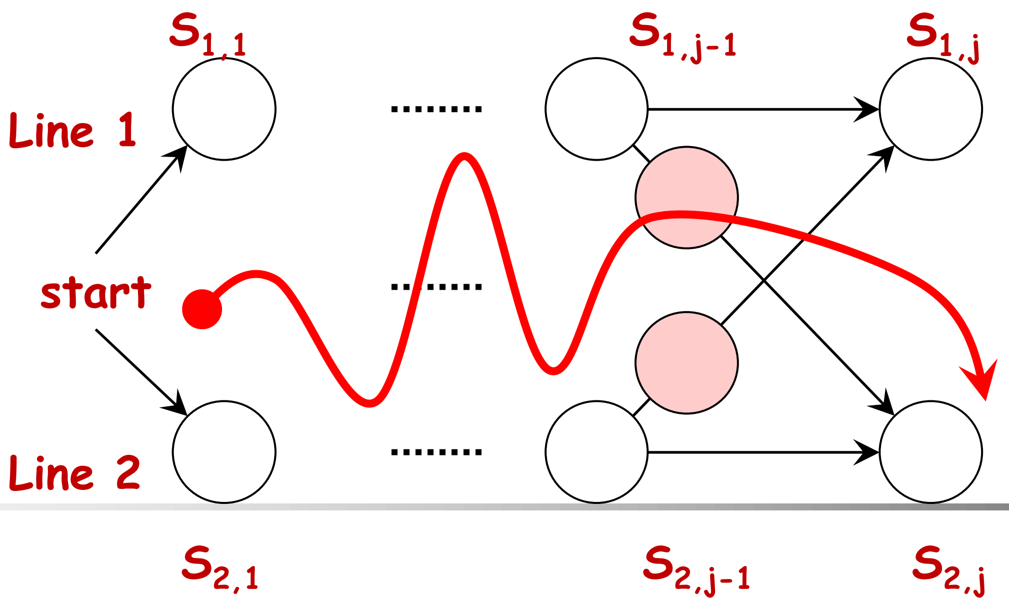
Solving the sub-problems (2)

Q2: what is the fastest way to get thro' $S_{2,j}$?

By exactly the same analysis, we obtain the formula for the fastest way to get thro' $S_{2,j}$:

$$\boxed{f_2[j] = \min(\ f_2[j\text{-}1]+a_{2,j}\ ,\ \ f_1[j\text{-}1]+t_{1,j\text{-}1}+a_{2,j}\ )}$$

Boundary case: $\boxed{f_2[1] = a_{2,1}}$

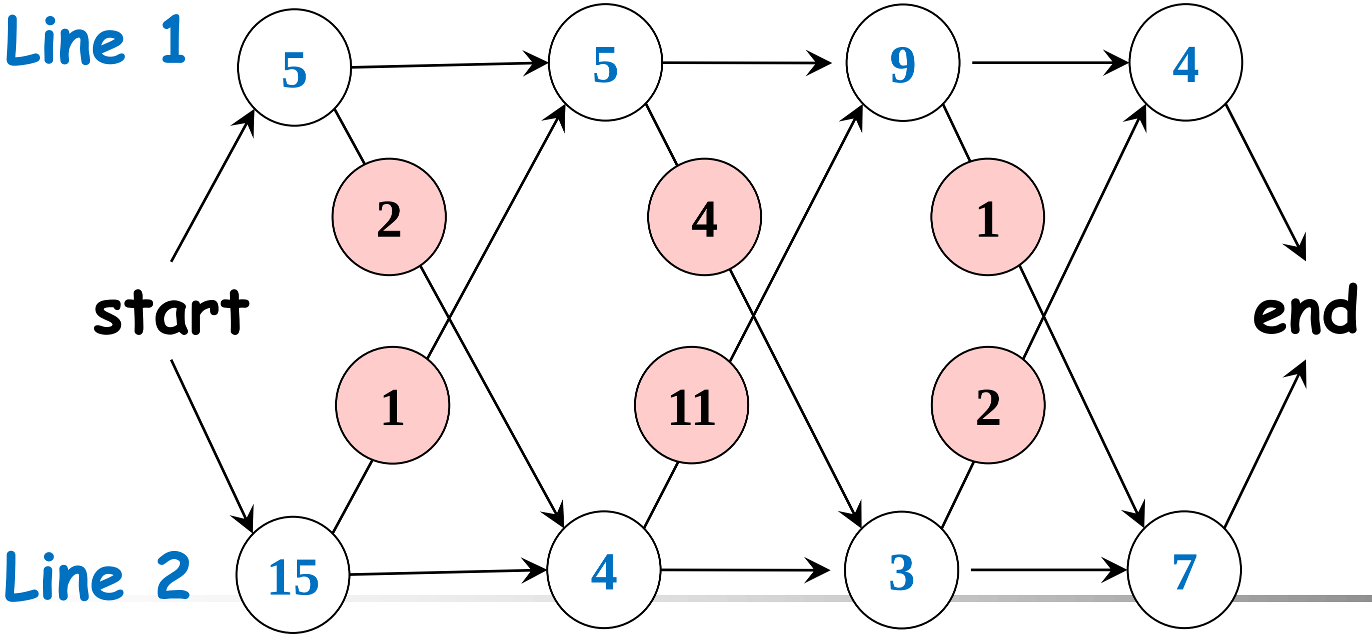


Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min(f_1[j-1]+a_{1,j}\ ,\ f_2[j-1]+t_{2,j-1}+a_{1,j}) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min(f_2[j-1]+a_{2,j}\ ,\ f_1[j-1]+t_{1,j-1}+a_{2,j}) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n]\ ,\ f_2[n])$$



j	$f_1[j]$	$f_2[j]$
1		
2		
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min\left(f_1[j-1]+a_{1,j}\ ,\ f_2[j-1]+t_{2,j-1}+a_{1,j} \right) & \text{if } j>1 \end{cases}$$

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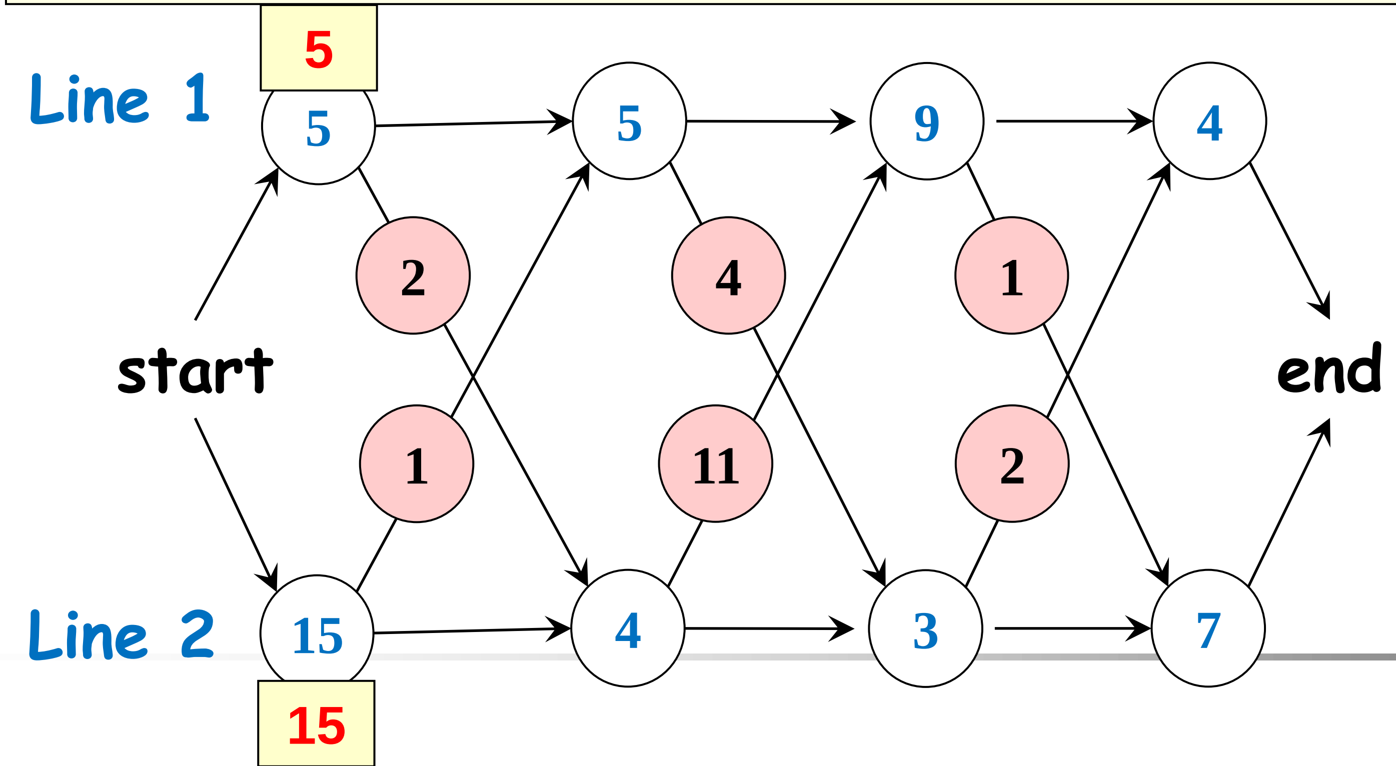


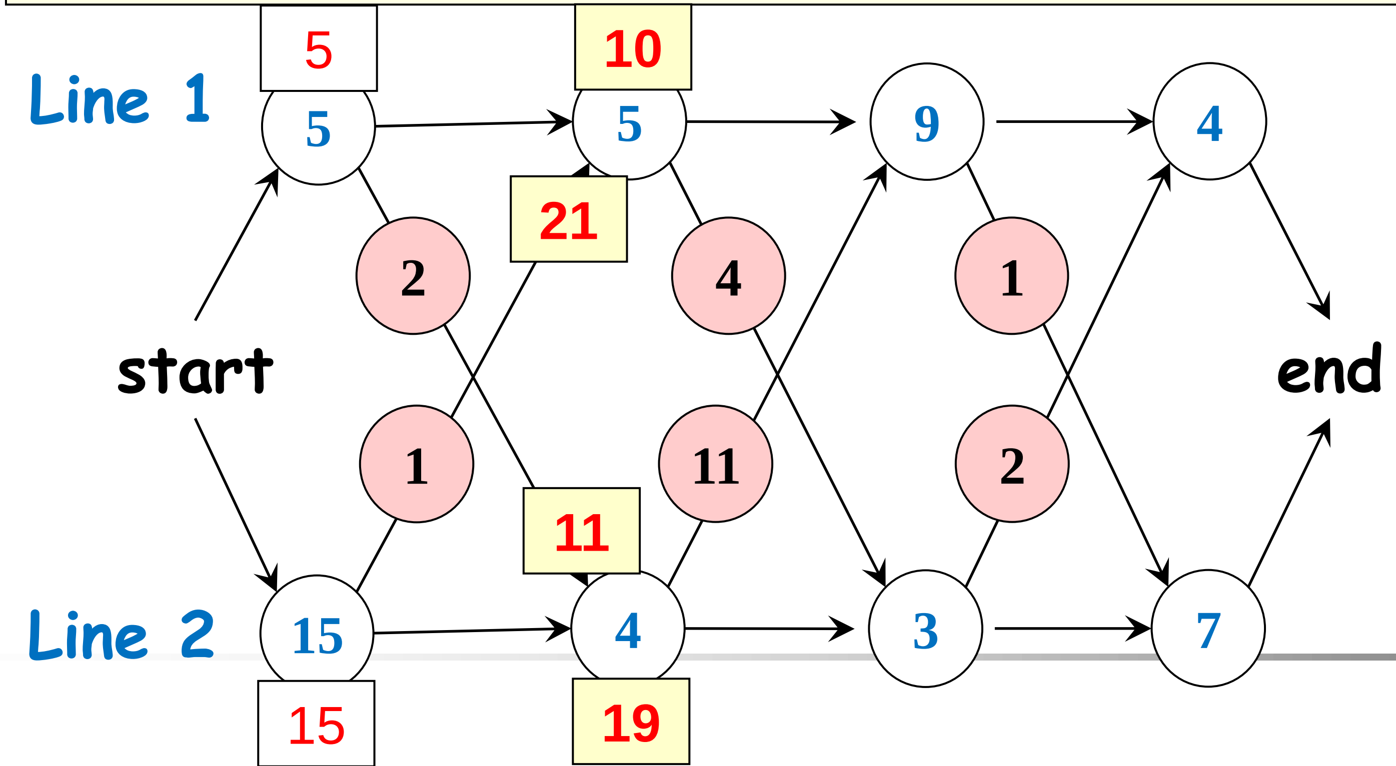
j	$f_1[j]$	$f_2[j]$
1	5	15
2		
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min(f_1[j-1]+a_{1,j},\ f_2[j-1]+t_{2,j-1}+a_{1,j}) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min(f_2[j-1]+a_{2,j},\ f_1[j-1]+t_{1,j-1}+a_{2,j}) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n],\ f_2[n])$$



j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3		
4		

Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min(f_1[j-1]+a_{1,j}\ ,\ f_2[j-1]+t_{2,j-1}+a_{1,j}) & \text{if } j>1 \end{cases}$$

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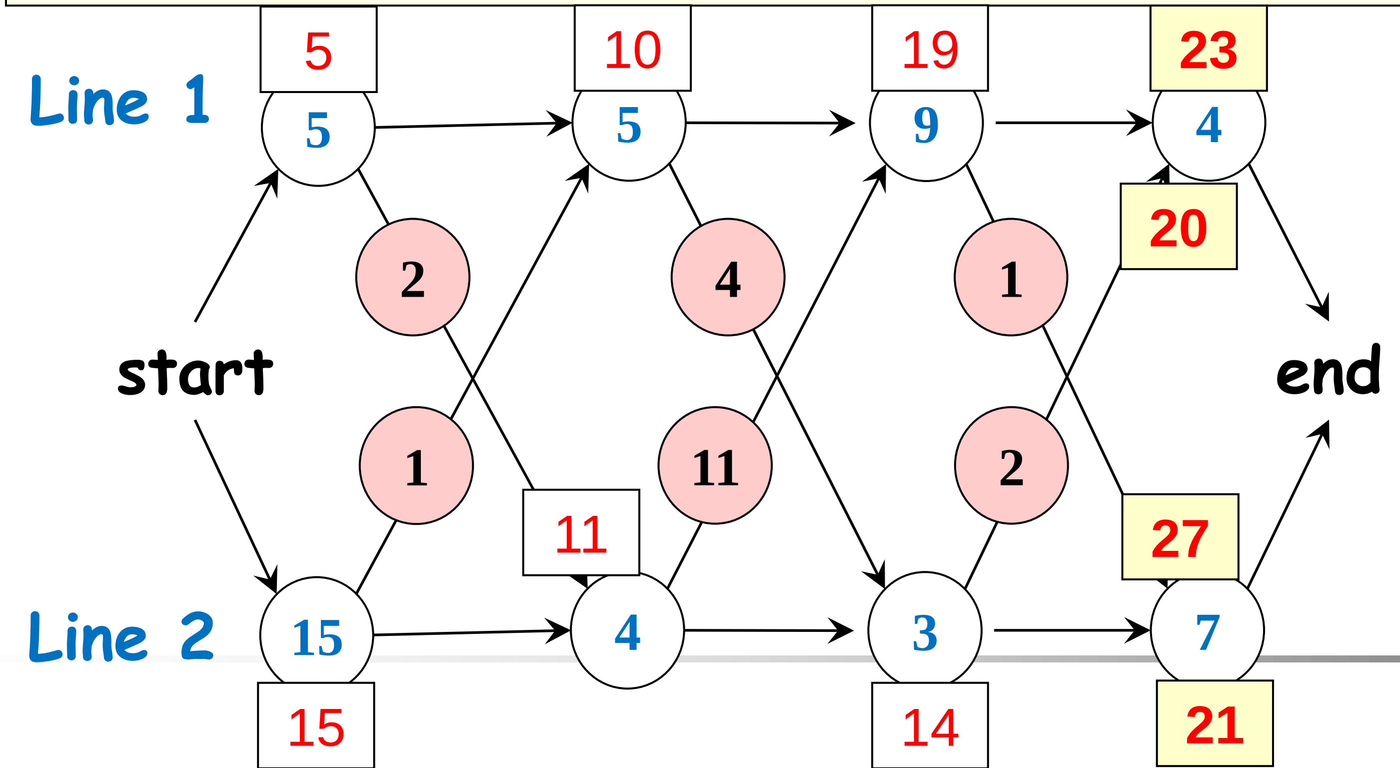
$$f^* = \min(f_1[n]\ ,\ f_2[n])$$



j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3	19	14
4	20	21

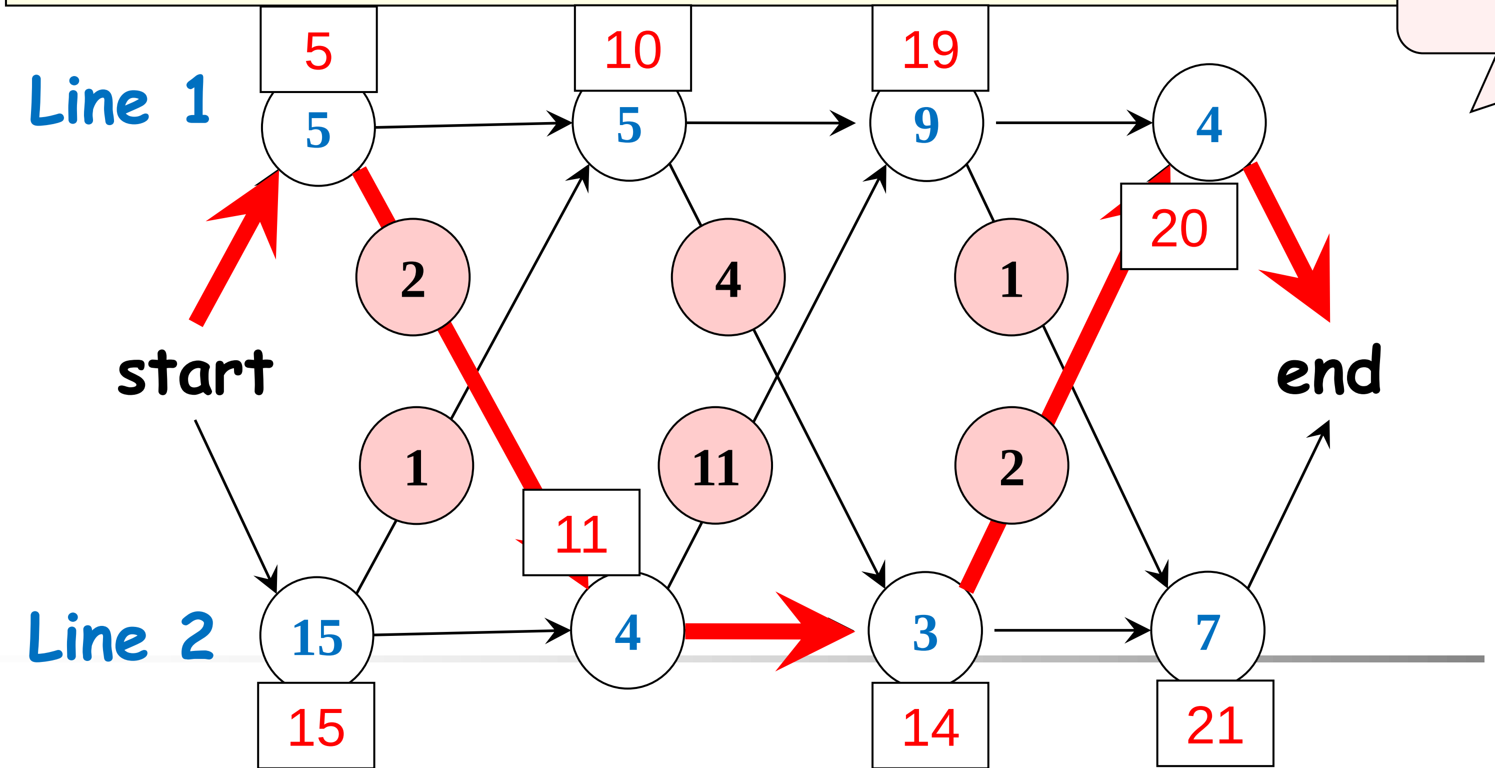
Summary

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j=1, \\ \min(f_1[j-1]+a_{1,j}\ ,\ f_2[j-1]+t_{2,j-1}+a_{1,j}) & \text{if } j>1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j=1, \\ \min(f_2[j-1]+a_{2,j}\ ,\ f_1[j-1]+t_{1,j-1}+a_{2,j}) & \text{if } j>1 \end{cases}$$

$$f^* = \min(f_1[n]\ ,\ f_2[n])$$

$f^* = 20$



j	$f_1[j]$	$f_2[j]$
1	5	15
2	10	11
3	19	14
4	20	21

Pseudo code

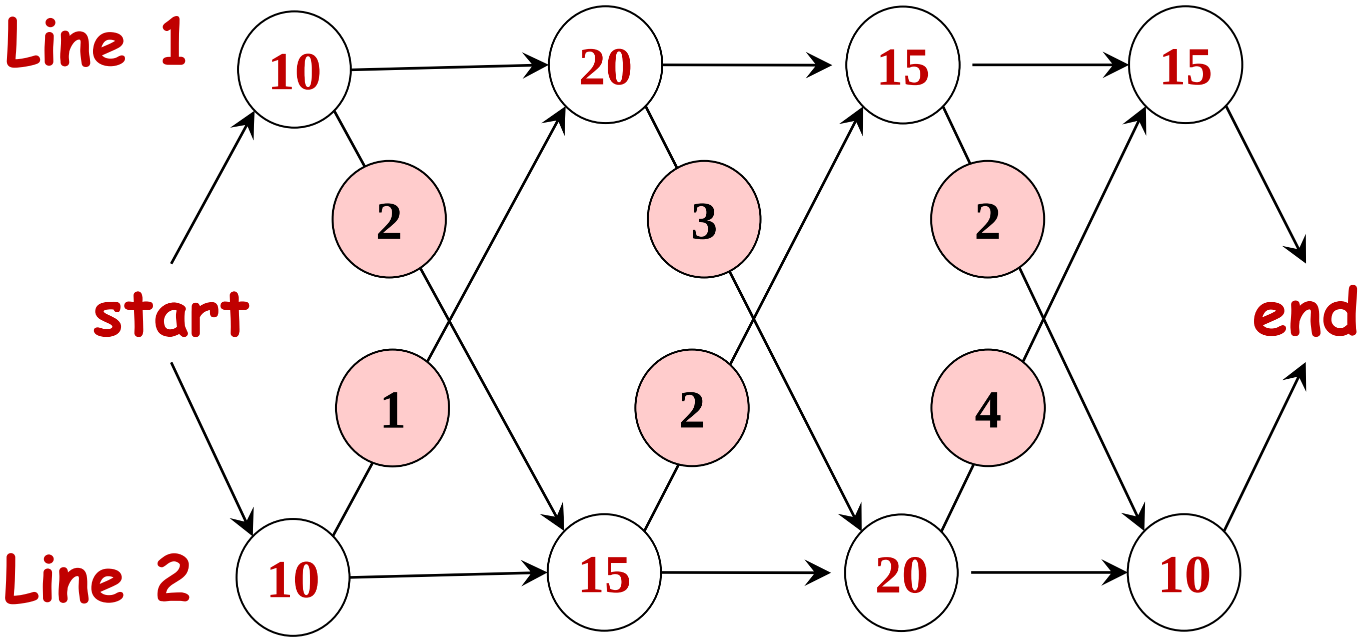
set $f_1[1] = a_{1,1}$
set $f_2[1] = a_{2,1}$
for $j = 2$ to n **do**
begin
    set $f_1[j] = \min(f_1[j-1]+a_{1,j},\ f_2[j-1]+t_{2,j-1}+a_{1,j})$
    set $f_2[j] = \min(f_2[j-1]+a_{2,j},\ f_1[j-1]+t_{1,j-1}+a_{2,j})$
end
set $f^* = \min(f_1[n], f_2[n])$

Time complexity is O(n)

How to apply Dynamic Programming?

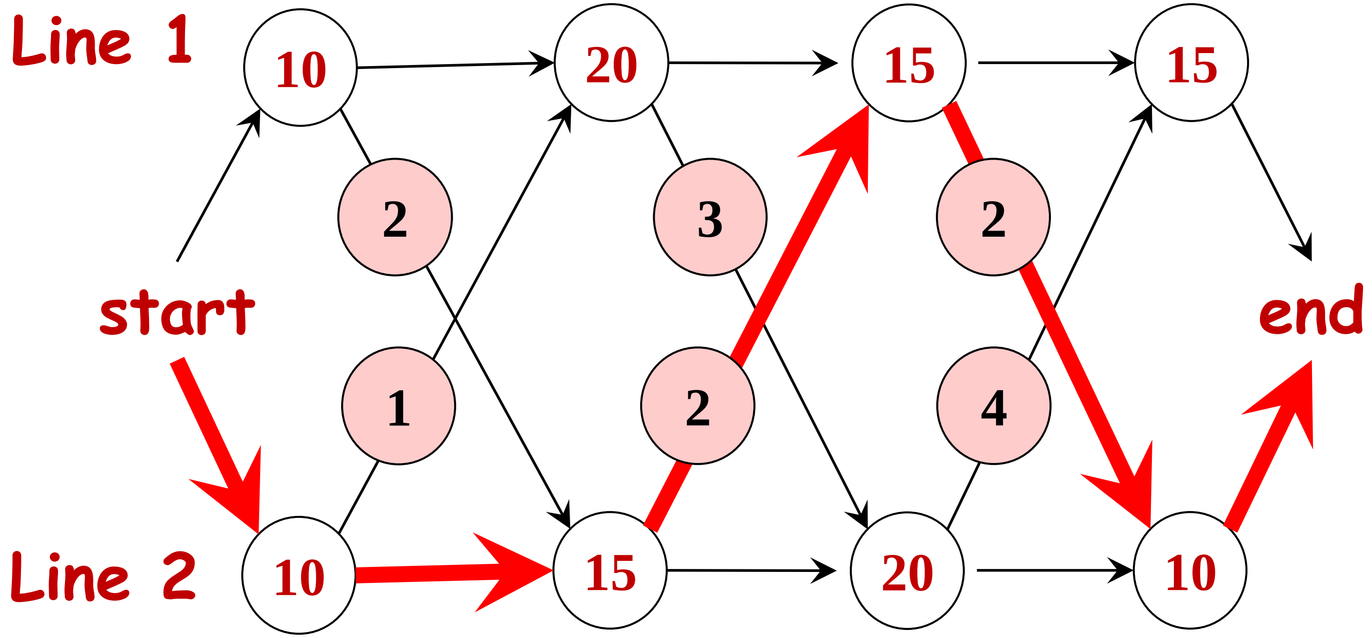
- Step 1: Identify sub-problems and optimal substructure.
- Step 2: Find a recursive formulation
- Step 3: Use dynamic programming (typically in a bottom-up fashion) to compute the value of an optimal solution
- Step 4: If needed, keep track of some additional info to get the optimal solution

Exercise



j	$f_1[j]$	$f_2[j]$
1		
2		
3		
4		

Exercise – solution



f* = 54

j	$f_1[j]$	$f_2[j]$
1	10	10
2	30	25
3	42	45
4	57	54

What if there are 3 or more lines?

- Pseudo code?
  - Calculate f[i][j]
  - Find optimal cost
  - Find optimal path

- Time complexity?

optional

Exercise

Given an $m \times n$ grid filled with integers representing "rewards". There is a robot placed at the top left cell and need to find a way to the bottom right cell. The robot can only move either down or right. What is the maximum reward the robot could collect through its path from top left to bottom right? Solve this problem with dynamic programming.

Example:

Input: grid = [[1,3,4],

[7,5,8],

[2,6,3]]

1	3	4
7	5	8
2	6	3

Output: 24

Explanation: The path 1->7->5->8->3 gives the maximum reward.

Learning outcomes

- Understand the basic idea of dynamic programming
- Able to apply dynamic programming to compute Fibonacci numbers
- Able to apply dynamic programming to solve the assembly line scheduling problem