Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, including damped and driven oscillators in detail. Also see chapter 10 of Kleppner and Kolenkow. For more on normal modes, see any book on waves, such as *Vibrations and Waves* by A.P. French. A wonderful series of examples using the Lagrangian-like techniques below is given by Jaan Kalda here, and some discussion of the adiabatic theorem is given here. Aim to solve or attempt at least **50/89** points.

1 Small Oscillations

First we'll consider some situations involving small oscillations.

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements.

It is important to keep in mind that there can be situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.

[2] **Problem 1** (KK 4.13)**.** The Lennard-Jones potential

$$U = \epsilon\left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right)$$

is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard-Jones interaction.

[3] **Problem 2** (Morin 5.13)**.** A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

(a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

(b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate frequency of the subsequent oscillations.

(c) Repeat the previous part for $x \gg R$.

[2] **Problem 3.** Some small oscillations questions about the buoyant force. For these questions, all you need to know is that the buoyant force is equal to the weight of the water displaced; see chapter 15 of Halliday for more details. We'll return to this in much more detail in **M8**.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Assuming a face of the glacier always remains parallel to the water surface, find the frequency of small oscillations.

(b) A ball of radius R floats in water with half its volume submerged. Find the frequency of small oscillations.

[3] **Problem 4.** USAPhO 1998, problem A2. Note that due to an oversight by the exam writers, part (a) can't be solved explicitly in a reasonable time, so don't bother trying!

[3] **Problem 5.** USAPhO 2009, problem A3.

[3] **Problem 6.** USAPhO 2010, problem B1.

Idea 2

A useful generalization of Newton's second law is given by generalized coordinates. Let q be any number that describes the state of the system, not necessarily a Cartesian coordinate. Suppose the energy of a system can be decomposed into two parts, a potential energy that depends only on q and a kinetic energy that depends only on $\dot{q}$,

$$K = K(\dot{q}), \quad V = V(q).$$

Then the equation of motion is

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

The left-hand side is the rate of change of a "generalized momentum" and the right-hand is a "generalized force". This can be shown rigorously using Lagrangian mechanics; in fact, the freedom to use generalized coordinates is one of the main benefits of Lagrangian mechanics.

More generally, Lagrangian mechanics tells us that if $L = K - V$, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

where K and V can both depend on both q and $\dot{q}$. We won't use this more general form below. While it is more powerful, it is somewhat more complicated, and if you find yourself using this form for an Olympiad problem, there is almost always a much easier way.

Example 1

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. The reason this procedure is so complicated is that we are using two coordinates when the string really ensures the system has only a single degree of freedom.

Instead, let q be a generalized coordinate that describes "how much the string has moved".

In other words, $q = 0$ initially, and for some $q > 0$, the mass M has moved down by q and the mass m has moved up by q. Then

$$K = \frac{1}{2}(m+M)\dot{q}^2, \quad V = qg(m-M)$$

and applying the idea above gives

$$\ddot{q} = \frac{M-m}{M+m}\,g.$$

[1] **Problem 7.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

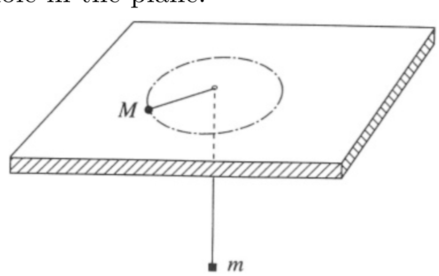
$$K = \frac{1}{2}m_{\text{eff}}\dot{q}^2, \quad V = \frac{1}{2}k_{\text{eff}}q^2$$

then the oscillation frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This technique allows us to avoid dealing with possibly complicated constraint forces.

[3] **Problem 8** (Grad)**.** A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The motion is frictionless. Show that the orbit is stable with respect to small changes in the radius, and find the frequency of small oscillations.

[4] **Problem 9.** IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions.

[2] **Problem 10.** In this problem we cover some fundamental facts about springs.

(a) Show that when a spring is cut in half, its spring constant doubles.

(b) If a spring with spring constant k_1 and relaxed length ℓ_1 is placed in series with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring.

(c) Do the same for the springs attached in parallel.

(d) Show that if a mass m is hung on a vertical spring, the resulting system behaves exactly the same as a horizontal spring system, except that the relaxed length of the spring is increased by mg/k.

[2] **Problem 11** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

[2] **Problem 12** (Morin 4.22)**.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Describe the shape of the resulting trajectory geometrically.

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

[2] **Problem 13** (HRK)**.** A mass M oscillates on a spring with spring constant k and mass m. When unstretched, the spring has uniform density. Show that when $m \ll M$, the oscillation frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}$$

in the case of small oscillations.

[5] **Problem 14.** [A] Generalize the previous problem to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses. This is a challenging problem that requires almost all the techniques we've seen so far, so feel free to ask for more hints.)

[3] **Problem 15** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

[3] **Problem 16.** USAPhO 2015, problem A3.

[3] **Problem 17.** USAPhO 2008, problem B1.

3 Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 18.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. Show that by taking the real part, one finds an exponentially damped sinusoidal oscillation.

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

[4] **Problem 19.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator, which experiences force $F = -bv - kx + F_0 e^{i\omega t}$. Show that Newton's second law can be written as

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters. Argue that the general solution takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in the previous question.

(c) The $A_\pm$ are set by initial conditions. After a long time they will decay away, leaving

$$x(t) \approx A_0 e^{i\omega t}.$$

Recalling that the physical position is just the real part, we actually have

$$x(t) \approx |A_0| \cos(\omega t + \phi), \quad \phi = \arg A_0.$$

Evaluate $|A_0|$ and ϕ.

(d) Find the driving frequency ω that maximizes the amplitude $|A_0|$.

(e) Find the driving frequency ω that maximizes the amplitude of the velocity. You should find that this condition is a bit simpler than that of the previous part.

(f) Sketch the phase shift ϕ as a function of ω. Can you intuitively see why ϕ takes the values it does, for ω small, $\omega \approx \sqrt{k/m}$, and ω large? (You can try it out using a pencil, ruler, or any similar object!)

Since the answers to (d) and (e) differ, it's really ambiguous what we mean when we say driving is "at resonance". In practice, it doesn't matter, because strong resonance is only noticeable when the damping is weak, and in that case the answers are both approximately equal to $\sqrt{k/m}$.

[3] **Problem 20** (KK 10.9)**.** The quality factor of an oscillator is defined as $Q = m\omega_0/b$. It measures how weak the damping is, but also how sharp the resonance is.

(a) Show that for a lightly damped oscillator near resonance,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonance frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

(c) Estimate Q for a guitar string.

If you want more information, see pages 424 through 428 of Kleppner and Kolenkow.

There are other ways to drive a harmonic oscillator, which contain some cool physics; the next two problems explore this.

[2] **Problem 21.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

(a) Move the end of the string horizontally with sinusoidal frequency f'.

(b) Move the end of the string vertically with sinusoidal frequency f'.

(c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just think!)

[5] **Problem 22.** GPhO 2016, problem 1.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Example 2

Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1 - L)$$
$$m\ddot{x}_2 = k(x_1 + L - x_2).$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u - v}{2}, \quad x_2 = \frac{u + v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

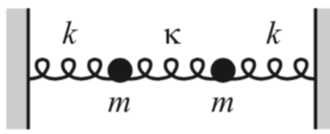
$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega}\sin\omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega}\sin\omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega}\sin\omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 23** (Morin 4.10). Three springs and two equal masses lie between two walls, as shown.

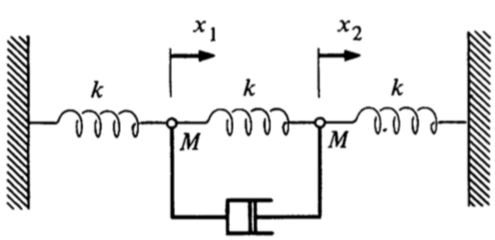


The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

[3] **Problem 24** (KK 10.11). Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega} \sin \omega t$$

and evaluate ω.

[5] **Problem 25** (Morin 4.12, IPhO 1986). N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) Do the same for general N. (Hint: it's easiest to do this by promoting the displacements to complex numbers. Do this to your answers to (a) and (b), making the normal modes as symmetric as possible, and try to identify a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Believe it or not, this strange problem will be useful in **X1**, where we will quantize the normal modes found here.

[4] **Problem 26.** [A] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy T and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) It can be shown that in general, we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial L}{\partial \theta_i}, \quad L = T - V.$$

This is called the Euler-Lagrange equation. Show that it is equivalent to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}$$

in this particular case. This is a generalization of the equation $dp/dt = F$, but in terms of angles rather than positions. Evaluate these equations.

(c) Write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L} K \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where K is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(d) Find the normal modes and their frequencies, using the general method in section 4.5 of Morin.

5 Adiabatic Change

Occasionally, tricky Olympiad problems will ask you to analyze slow change.

Example 3

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Before looking at the solution, you should stop and think about how you would solve this problem. It's very much unlike anything you've seen before: it's unclear what, if anything, is conserved, and the differential equation is nonlinear in a tricky way.

Solution

We can solve the problem by considering the energy. Suppose that the spring constant drops in one instant by a factor of $1 - \epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1 - \epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant drops by a factor of $1 - x$ over a full cycle, with $x \ll 1$, then the energy drops by a factor of $1 - x/2$.

The process finishes after N oscillations, where $(1 - x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1 - x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. But the energy is also kA^2, so the new amplitude is $\sqrt[4]{2}A_0$.

In fact, such questions can be solved instantly using a conservation law.

Example 4

Solve the previous problem with a conservation law.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x

component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular momentum* is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k} A^2.$$

Then the final amplitude is $\sqrt[4]{2} A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But there is a more general conserved quantity that is useful in a wide range of similar problems.

Idea 5: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p\, dx$$

is conserved. This is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

The existence of the adiabatic invariant is very hard to see in pure Newtonian mechanics, but it falls naturally out of the framework of Hamiltonian mechanics, which works with phase space. In fact, Hamiltonian mechanics makes a lot of theoretically useful facts easier to see.

For example, as you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually one and the same. And in **W4**, we will use phase space for light rays to understand some fundamental limits on optical systems, which show why lenses and mirrors can't violate thermodynamics. If you'd like to learn more about Hamiltonian mechanics, see David Tong's lecture notes or chapter 15 of Morin.

[1] **Problem 27.** Verify that the adiabatic theorem solves the spring problem.

[3] **Problem 28.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein relied in the affirmative. Reproduce Einstein's analysis.

As covered in **T1**, the adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force.

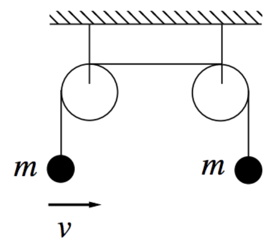
[4] **Problem 29.** [A] A block of mass M and velocity v to the right approaches a stationary ball of mass $m \ll M$. There is a wall a distance L to the right of the ball.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by analyzing each collision. (Note that it does not suffice to simply use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

(c) Using the adiabatic invariant, infer the value of γ for a one-dimensional monatomic gas.

[4] **Problem 30 ($\boldsymbol{F = ma}$, BAUPC).** Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.