Electromagnetism V: Induction

Chapter 7 of Purcell covers induction, as does chapter 7 of Griffiths. For magnetism, see section 6.1 of Griffiths; for applications, see the end-of-chapter technology reviews in each chapter of Purcell, along with chapters II-16 and II-17 of the Feynman lectures. For a qualitative introduction to superconductivity, see appendix I of Purcell. Aim to solve or attempt at least **45/89** points.

1 Motional EMF

> **Idea 1**
>
> If $\mathbf{F}$ is the force on a charge q, then the emf about a loop C is
> $$\mathcal{E} = \frac{1}{q} \oint_c \mathbf{F} \cdot d\mathbf{s}.$$
> For a moving loop in a time-independent magnetic field, the emf through the loop is
> $$\mathcal{E} = -\frac{d\Phi}{dt}$$
> where Φ is the magnetic flux. The direction of the emf produces a current that opposes the change in flux.

[2] **Problem 1** (Purcell). [A] Prove this result using the Lorentz force law as follows.

(a) Let the loop be C and let $\mathbf{v}$ be the velocity of each point on the loop. Argue that after a time dt, the change in flux is
$$d\Phi = \oint_C \mathbf{B} \cdot ((\mathbf{v}\, dt) \times d\mathbf{s}).$$

(b) Prove the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$.
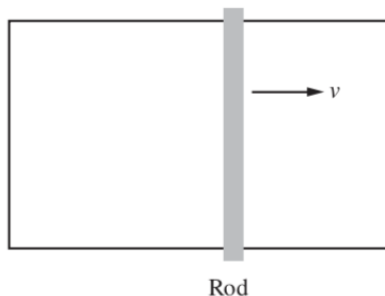
(c) Use this result to show that
$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}$$
and use this to conclude the result.

[2] **Problem 2** (Purcell 7.2). A conducting rod is pulled to the right at speed v while maintaining a contact with two rails. A magnetic field points into the page.

Rod

We know an induced emf will cause a current to flow in the counterclockwise direction around the loop. Now, the magnetic force $q\mathbf{u} \times \mathbf{B}$ is perpendicular to the velocity $\mathbf{u}$ of the moving charges, so it can't do work on them. However, the magnetic force $\mathbf{f}$ certainly looks like it's doing work. What's going on here? If the magnetic force doing work or not? If not, then what is? There is definitely something doing work because the wire will heat up.

[3] **Problem 3** (PPP 167). A homogeneous magnetic field $\mathbf{B}$ is perpendicular to a track inclined at an angle α to the horizontal. A frictionless conducting rod of mass m and length ℓ straddles the two rails as shown.
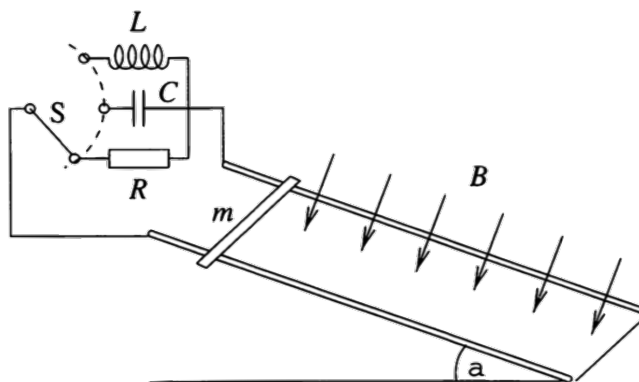

How does the rod move, after being released from rest, if the circuit is closed by

(a) a resistor of resistance R,

(b) a capacitor of capacitance C,

(c) or a coil of inductance L?

[3] **Problem 4.** ⌣ USAPhO 2006, problem B1.

[3] **Problem 5** (PPP 168). One end of a conducting horizontal track is connected to a capacitor of capacitance C charged to voltage V_0. The inductance of the assembly is negligible. The system is placed in a homogeneous, vertical magnetic field B, as shown.
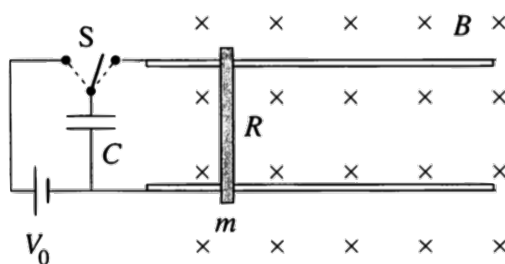

A frictionless conducting rod of mass m, length ℓ, and resistance R is placed perpendicularly onto the track. The capacitor is charged so that the rod is repelled from the capacitor when the switch is turned. This arrangement is known as a railgun.

(a) What is the maximum velocity of the rod?

(b) What is the maximum possible efficiency?

Not every problem can be solved by using the form $\mathcal{E} = -d\Phi/dt$. Sometimes, for more complex geometries where there is no clear "loop", we need to go back to the Lorentz force law.

[2] **Problem 6** (Griffiths). A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform magnetic field $\mathbf{B}$ pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk.
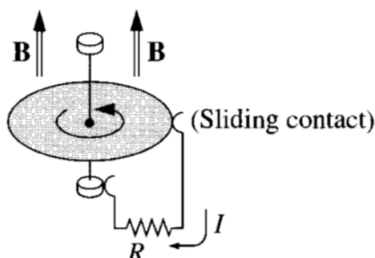
(a) Find the current in the resistor. (Hint: do not try to find the current distribution in the disk. This is an extremely complicated set of eddy currents.)

(b) Explain why an induced emf exists, even though $\mathbf{B}$ is constant and hence $d\Phi_B/dt$ is zero for any surface.

This system is a Faraday disk, also called a homopolar generator.

[5] **Problem 7.** ⏱ IPhO 1990, problem 2. A neat problem which also reviews **M7**.

2   Faraday's Law

> **Idea 2**
>
> Faraday's law states that even for a time-dependent magnetic field, we still have
> $$\mathcal{E} = -\frac{d\Phi}{dt}.$$
> In the case where the loop isn't moving but the magnetic field is changing, the emf is entirely provided by a non-conservative electric field,
> $$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{s}.$$
> The differential form of this result is
> $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

> **Remark: EMF vs. Voltage**
>
> We mentioned earlier in **E3** that we often care about electromotive forces, which are forces that act on charges, more general than just electrostatic forces. The force due to a non-conservative electric field is another example.

In such cases, the idea of "voltage" really breaks down entirely, because you can't define it consistently. However, electrical engineers use a more pragmatic definition of voltage: to them, voltage is just whatever a voltmeter displays. In other words, what they call voltage is what we call electromotive force. This can lead to rather nonintuitive results, as you'll see below. For example, the voltage can be different for different voltmeters even if they are connected at the same points!

Despite this trouble, we'll go along with the standard electrical engineer nomenclature and refer to these emfs as voltages in later problem sets. For example, Kirchoff's loop rule should properly say that the sum of the voltage drops along a circuit is not zero, but rather $d\Phi/dt$. But it is conventional to move $d\Phi/dt$ to the other side and call it a "voltage drop" of $-d\Phi/dt$.

Remark

When we apply Faraday's law, we often use Ampere's law (without the extra displacement current term) to calculate the magnetic field. This is not generally valid, but works if the currents are in the slowly changing "quasistatic" regime, which means radiation effects are negligible. All the problems in this problem set are in this regime, but we'll see more subtle examples in **E7**.

[**2**] **Problem 8** (Purcell 7.6). An infinite cylindrical solenoid has radius R and n turns per unit length. The current grows linearly with time, according to $I(t) = Ct$. Find the electric field everywhere.

[**2**] **Problem 9** (Purcell 7.4). Two voltmeters are attached around a solenoid as shown.
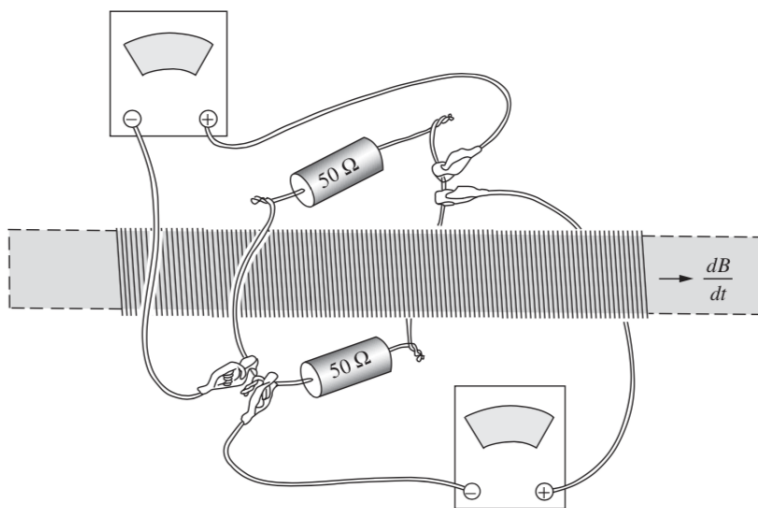
Find the readings on the two voltmeters in terms of $d\Phi/dt$.

[**2**] **Problem 10** (Purcell 7.28). [**A**] Consider the loop of wire shown below.
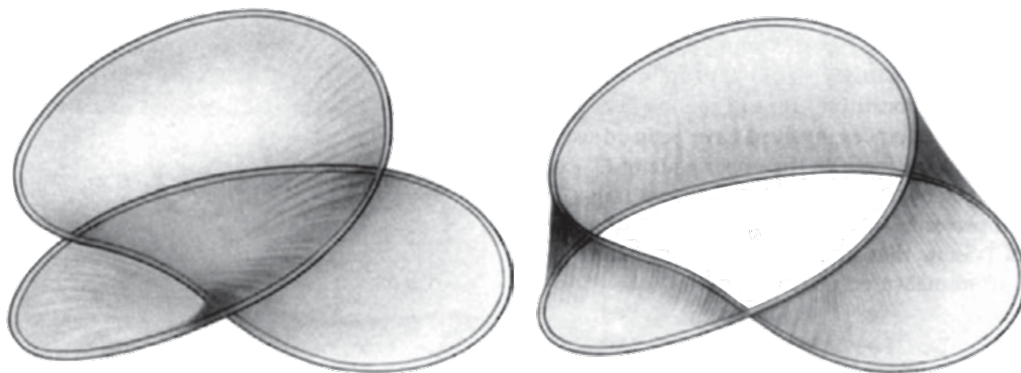
Suppose we want to calculate the flux of **B** through this loop. Two surfaces bounded by the loop are shown above. Which, if either, is the correct surface to use? If each of the two turns in the loop are approximately circles of radius R, then what is the flux? Generalize to an N-turn coil.

[3] **Problem 11.** 🕒 USAPhO 2009, problem A1.

[3] **Problem 12.** 🕐 USAPhO 1999, problem B2.

3   Inductive Systems

[3] **Problem 13** (Purcell). A dynamo is a generator that works as follows: a conductor is driven through a magnetic field, inducing an electromotive force in a circuit of which that conductor is part. The source of the magnetic field is the current that is caused to flow in that circuit by that electromotive force. An electrical engineer would call it a self-excited dynamo. One of the simplest dynamos conceivable is shown below.
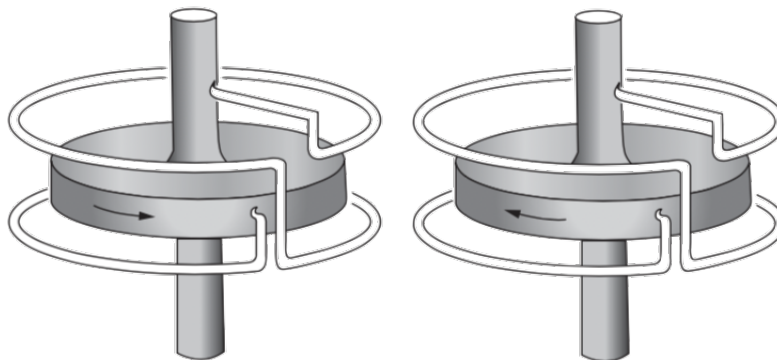
It has only two essential parts. One part is a solid metal disk and axle which can be driven in rotation. The other is a two-turn "coil" which is stationary but is connected by sliding contacts, or "brushes", to the axle and to the rim of the revolving disk.

(a) One of the two devices pictured is, at least potentially, a dynamo. The other is not. Which is the dynamo?

A dynamo like the one above has a certain critical speed ω_0. If the disk revolves with an angular velocity less than ω_0, nothing happens. Only when that speed is attained is the induced $\mathcal{E}$ enough to make the current enough to make the magnetic field enough to induce an $\mathcal{E}$ of that magnitude. The critical speed can depend only on the size and shape of the conductors, the conductivity σ, and

the constant μ_0. Let d be some characteristic dimension expression the size of the dynamo, such as the radius of the disk in our example.

(b) Show by a dimensional argument that ω_0 must be given by a relation of the form $\omega_0 = K/\mu_0 \sigma d^2$ where K is some dimensionless numerical factor that depends only on the arrangement and relative size of the parts of the dynamo.

(c) Demonstrate this result again by using physical reasoning that relates the various quantities in the problem (R, $\mathcal{E}$, E, I, B, etc.). You can ignore all numerical factors in your calculations and absorb them into the constant K.

For a dynamo of modest size made wholly of copper, the critical speed would be practically unattainable. It is ferromagnetism that makes possible the ordinary DC generator by providing a magnetic field much stronger than the current in the coils, unaided, could produce. For an Earth-sized dynamo, however, the critical speed is much smaller. The Earth's magnetic field is produced by a nonferromagnetic dynamo involving motions in the fluid metallic core.

[5] **Problem 14.** ⏱ APhO 2009, problem 2. This problem analyzes a dynamo in more detail, completing the rough analysis made above. It's a worthwhile problem that gives you practice with electromagnetic systems.

[2] **Problem 15** (MPPP 178). In general, a magnet moving near a conductor is slowed down by induction effects. Suppose that inside a long vertical, thin-walled, brass tube a strong permanent magnet falls very slowly due to these effects, taking a time t to go from the top to the bottom.

(a) Let the magnet have mass m, and let the tube have resistivity ρ, thickness r, and length L. Suppose both the magnet and tube have radius approximately R, and let the magnet's length also be of order R. Let the typical magnetic fields produced at the magnet's surface have magnitude B_0. Find an estimate for t, to the nearest order of magnitude.

(b) If the experiment is repeated with a copper tube of the same length but a larger diameter, the magnet takes a time t' to fall through. How long does it take for the magnet to fall through the tubes if they are fitted inside each other? Neglect the mutual inductance of the tubes.

4 Inductance

> **Idea 3: General Inductance**
>
> Consider a set of loops with fluxes Φ_i and currents I_i. By linearity, they are related by
>
> $$\Phi_i = \sum_j L_{ij} I_j$$
>
> where the L_{ij} are called the coefficients of inductance. It can be shown that $L_{ij} = L_{ji}$. By Faraday's law, we have
>
> $$\mathcal{E}_i = \sum_j L_{ij} \dot{I}_j.$$
>
> In contrast with capacitance, we're usually concerned with the self-inductance $L_i = L_{ii}$ of single loops; these inductors provide an emf of $L\dot{I}$ each. However, mutual inductance effects can also impact how circuits behave, as we'll see in **E6**.

Remark

The inductance coefficients are similar to the capacitance coefficients in **E2**, but more useful. For capacitors, we are typically interested in configurations with one positive and one negative plate, and the capacitance of this object is related to all of the capacitance coefficients in a complicated way, as we saw in **E2**. But most inductors just use self-inductance, so the inductance we care about is simply one of the coefficients, L_{ii}. Moreover, the "mutual inductance" coefficients L_{ij} are also in the right form to be directly used, since they tell us how current changes in one part of the circuit impact emfs elsewhere.

A more general way to describe the difference is that $\mathcal{E}$ and I are directly measurable and controllable quantities, while the Q and V (i.e. the voltage relative to infinity) that the capacitance coefficients relate are less so.

Idea 4

The energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0} \int B^2 \, dV$$

which implies the energy stored in an inductor is

$$U = \frac{1}{2}LI^2$$

where L is the self-inductance.

Example 1

Compute the self-inductance of a cylindrical solenoid of radius R, length H, and n turns per length.

Solution

One straightforward way to do this is to use the magnetic field energy. We have

$$U = \frac{1}{2\mu_0}(\mu_0 n I)^2 (\pi R^2 H)$$

and setting this equal to $LI^2/2$ gives

$$L = \pi\mu_0 n^2 R^2 H = \mu_0 N^2 \frac{\pi R^2}{H}$$

where N is the total number of turns.

We can also try to use the definition of inductance directly, $\Phi = LI$. But it's hard to imagine a surface bounded by the solenoid wires; as we saw in problem 10, even the case $N = 2$ is tricky! Instead it's better to use the form $\mathcal{E} = L\dot{I}$. We can then compute the emf across each

turn of the solenoid individually, then add them together.

To compute the emf across one turn, we can replace it with a circular loop; this is valid because the emf ultimately comes from the local electric field, which shouldn't change too much if we deform the loop in this way. Then

$$|\mathcal{E}_{\text{loop}}| = \frac{d\Phi}{dt} = (\mu_0 n \dot{I})(\pi R^2).$$

The inductance is hence

$$L = \frac{N\mathcal{E}_{\text{loop}}}{\dot{I}} = (\mu_0 n N)(\pi R^2) = \mu_0 N^2 \frac{\pi R^2}{H}$$

as expected.

[**3**] **Problem 16.** Consider a toroidal solenoid with a rectangular cross section of height h and width w, N turns, and inner radius R.

(a) Find the self-inductance by considering the magnetic flux.

(b) [**A**] Now suppose the current increases at a constant rate dI/dt. Find the electric field at a height z above the center of the solenoid, assuming $h, w \ll R$. (Hint: think of the differential equations that govern **B** in this case. Can you draw an analogy to the equations that govern some other field?)

(c) Verify that the two formulas for energy given above are consistent.

> **Remark**
>
> In electromagnetism, we often have issues with divergences when we take idealized point sources. For example, the voltage near a point charge can become arbitrarily high. Similarly, the magnetic field diverges as you approach an idealized, infinitely-thin wire, which causes the self-inductance of wire loops to diverge. Of course, the fix is that you don't actually get an infinite magnetic field as you approach a wire. A real wire has finite thickness, and its magnetic field instead goes to zero as you approach its center. (Note that we didn't run into this problem for the solenoids, because we modeled their wires as a uniform sheet of current, whose magnetic field isn't singular at all.)

[**2**] **Problem 17.** A wire of length L is bent into a long "hairpin" shape, with two parallel straight edges of length $L/2$ separated by a distance $d \ll L$.

(a) Write down an integral expression for the self-inductance, neglecting the curved parts, and show that it diverges.

(b) The reason the inductance doesn't diverge in reality is that the wire has finite thickness. Once you get inside the wire, the magnetic field does not continue to increase, diverging at the center, but rather smoothly goes to zero. Find a rough estimate for the self-inductance by taking the wire to have radius $r \ll d$ and ignoring any flux through the wire itself.

If a problem does involve inductance and a wire loop, it'll usually circumvent this issue by just telling you what the self-inductance of the loop is.

[2] **Problem 18** (Purcell 7.9). Find the mutual inductance of the two cylindrical solenoids shown in cross-section below.
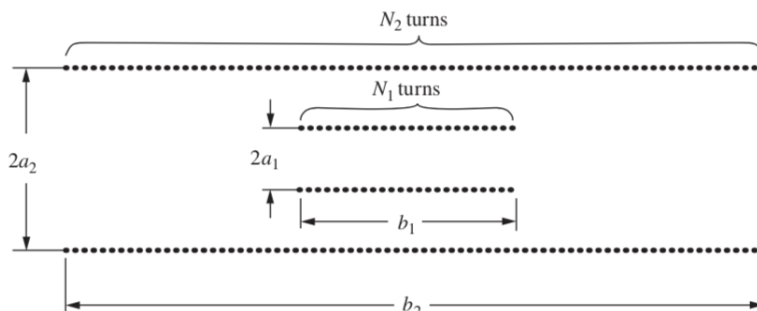
[3] **Problem 19.** Consider two concentric rings of radii r and $R \gg r$.

(a) Compute the mutual inductance by considering a current through the larger ring.

(b) Compute the mutual inductance by considering a current through the smaller ring, and verify your results agree.

(c) Suppose that an increasing current is passed through the outer ring. What forces does the inner ring experience?

[2] **Problem 20** (MPPP 181). Three nearly complete circular loops, with radii R, $2R$, and $4R$ are placed concentrically on a horizontal table, as shown.
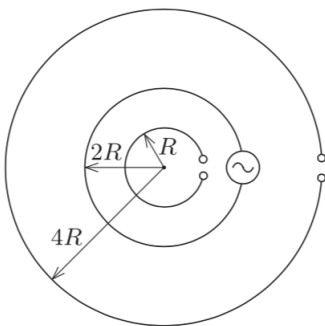
A time-varying electric current is made to flow in the middle loop. Find the voltage induced in the largest loop at the moment when the voltage between the terminals of the smallest loop is V_0.

5 Magnetism

In this section we'll dip a little into atomic physics and the origin of magnetism. However, a proper understanding of this subject requires quantum mechanics, as we'll cover in **T3** and **X3**.

[3] **Problem 21.** A spinning charged object carries both a magnetic dipole moment and angular momentum. If the object's mass and charge distributions are proportional to each other, then $\boldsymbol{\mu}$ and $\mathbf{J}$ point in the same direction, and their ratio is the gyromagnetic ratio.

(a) Compute the gyromagnetic ratio for a thin uniform donut of charge Q, mass M, and radius R rotating about its axis.

(b) Show that the gyromagnetic ratio for a uniform spinning sphere is the same.

By similar reasoning, any axially symmetric charge/mass distribution would give the same gyromagnetic ratio. Unfortunately, the result is wrong by a factor of 2 when applied to atoms. This factor of 2 cannot be explained, except by quantum mechanics.

[3] **Problem 22.** ⏱ USAPhO 2007, problem B2.

[5] **Problem 23.** ⏱ APhO 2013, problem 3. A solid question involving classical magnetic moments, which gives some intuition for the quantum behavior.

6 Superconductors

There are many rather tricky Olympiad problems involving superconductors. Superconductors can be a bit intimidating at first, but they actually obey simple rules.

> **Idea 5**
>
> An ideal conductor has zero resistivity, which implies that the magnetic flux through any loop in the conductor is constant: attempting to change the flux instantly produces currents that cancel out the change. Often this constant flux is equal to zero, but it doesn't have to be.
>
> A superconductor is an ideal conductor with the additional property that the magnetic field in the body of the superconductor is exactly zero, no matter what the initial conditions are; once an object becomes superconducting it forces all the existing flux out. As a result, all the current in a superconductor is confined to its surface. This second property is deeper, since it can only be derived using quantum mechanics.

Not all problems involving superconductors actually use this second property; such problems are really just about ideal conductors. Generally, the true superconductivity property is useful only when you really care about what's going on at the superconductor's surface.

> **Example 2: PPP 153**
>
> A superconducting uniform spring has N turns of radius R, relaxed length x_0, and spring constant k. The two ends of the spring are connected by a wire, and a small, steady current I is made to flow through the spring. At equilibrium, what is the change in its length?

> **Solution**
>
> This question really is about ideal conductors, not just superconductors. The additional superconductivity property would tell us about the field *inside* the wires themselves (not the loops that the wires form), and thereby about some small screening currents on the surfaces of the wires. But this is not important because the wires are thin.

In order to find the equilibrium length x_{eq}, we can use the principle of virtual work. We compute how the energy changes if we slightly perturb the system. At equilibrium, this change in energy should be zero.

We have $B = \mu_0 N I / x$, so the magnetic field energy is

$$U = \frac{B^2}{2\mu_0} V = \frac{1}{2} \frac{\mu_0 \pi R^2 N^2 I^2}{x}.$$

To save some space, we'll just let $A = \mu_0 \pi R^2 N^2 / 2$, so

$$U(x) = \frac{AI^2}{x}.$$

Naively, this means the magnetic field energy decreases as x increases, so the spring would like to stretch. But this makes no sense, because we know that parallel currents attract, squeezing the spring. We have to recall that the spring is an ideal conductor, so when it is stretched or squeezed, the current changes to keep the flux the same. The flux is

$$\Phi_B = N(\pi R^2) B \propto \frac{I}{x}$$

so we have

$$I(x) = I \frac{x}{x_{\text{eq}}}, \quad U(x) = \frac{AI^2}{x_{\text{eq}}^2} x.$$

The other energy contribution is $k(x - x_0)^2/2$, so setting the derivative of energy to zero,

$$\frac{AI^2}{x_{\text{eq}}^2} = k(x_0 - x_{\text{eq}}).$$

Since the current is small, $x_0 \approx x_{\text{eq}}$, so we can replace x_{eq} with x_0 on the left-hand side, giving the answer,

$$x_{\text{eq}} = x_0 - \frac{AI^2}{x_0^2 k}.$$

[3] **Problem 24** (MPPP 182). Two identical superconducting rings are initially very far from each other. The current in the first is I_0, but there is no current in the other. The rings are now slowly brought closer together. Find the current in the first ring when the current in the second is I_1.

[4] **Problem 25** (PPP 182). A thin superconducting ring of radius r, mass m, and self-inductance L is supported by a piece of plastic just above the top of a long, cylindrical solenoid of radius $R \gg r$ and n turns per unit length. The ring and solenoid are coaxial. When the current in the solenoid is I_s, the magnetic field near the end of the solenoid is

$$B_z = B_0(1 - \alpha z), \quad B_r = B_0 \beta r$$

where we put the origin at the very top of the solenoid.

(a) Find an expression for B_0. (Keep your answers below in terms of B_0 to avoid clutter.)

(b) Find a relationship between α and β, as well as their signs.

(c) Let I be the current through the ring. Suppose that initially $I_s = I = 0$. Find the value I_c of I_s when the ring lifts off the plastic.

(d) Now the piece of plastic is removed and the ring is return to the same position. Initial conditions are set up so that $I_s = I_c$ and $I = 0$. (How could you do this in practice?) The ring is released from rest; find its subsequent motion, assuming that the motion is confined to regions where the expressions for B_z and B_r above hold.

[4] **Problem 26.** 🕐 IPhO 2012, problem 1C. A delightfully tricky problem that uses the properties of superconductors in a subtle way.

[4] **Problem 27** (EuPhO 2017). Consider a mesh made from a flat superconducting sheet by drilling a dense grid of small holes into it. Initially the sheet is in a non-superconducting state, and a magnetic dipole of dipole moment m is at a distance a from the mesh pointing perpendicularly towards the mesh. Now the mesh is cooled so that it becomes superconducting. Next, the dipole is displaced perpendicularly to the surface of the mesh so that its new distance from the mesh is b. Find the force between the mesh and the dipole. The spacing of the grid of holes is much smaller than both a and b, and the linear size of the sheet is much larger than both a and b.

[4] **Problem 28** (Physics Cup 2013). A rectangular superconducting plate of mass m has four identical circular holes, one near each corner, a distance Δ from the plate's edges. Each hole carries a magnetic flux Φ. The plate is put on a horizontal superconducting surface. The magnetic repulsion between the plate and the surface balances the weight of the plate when the width of the air gap beneath the plate is $d \ll \Delta$, and d is much smaller than the radii of the holes. The frequency of small vertical oscillations is ω_0.
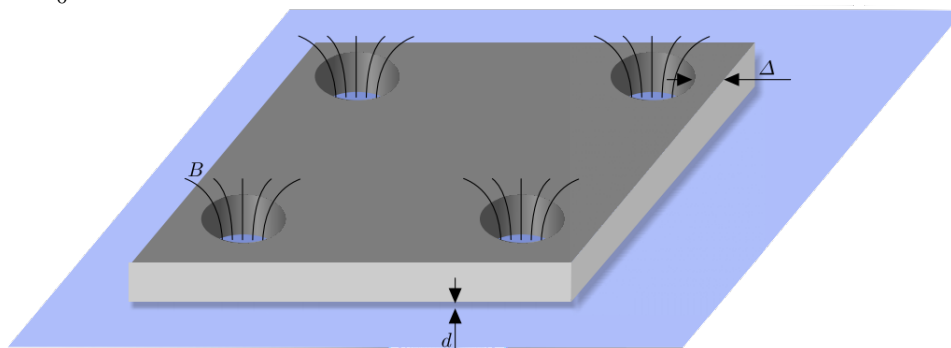
Next, a load of mass M is put on the plate, so that the load lays on the plate, and the plate levitates above the support. What is the new frequency of small oscillations?

[5] **Problem 29.** 🕐 IPhO 1994, problem 2. This problem tests your intuition for induction, and is good preparation for **E6**.