Mechanics I: Kinematics

See chapters 3 and 4 of Morin for material on solving differential equations. For general review on kinematics, see chapter 1 of Kleppner and Kolenkow. For fun, see chapters I-1 through I-8 of the Feynman lectures. There is a total of **84** points.

1 Motion in One Dimension

Example 1

When a projectile moves slowly through air, the drag is linear in the velocity, $F = -\alpha m v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Solution

We write Newton's second law as

$$\frac{dv}{dt} = -g - \alpha v$$

and multiply through by dt. Integrating both sides from the initial condition to time t_f gives

$$\int_{v_0}^{v(t_f)} \frac{dv}{g + \alpha v} = -\int_0^{t_f} dt.$$

Performing the integrals gives

$$\frac{1}{\alpha} \log(g + \alpha v)\Big|_{v_0}^{v(t_f)} = -t_f.$$

Renaming t_f to t and solving for v yields

$$v(t) = e^{-\alpha t} v_0 + \frac{g}{\alpha}(e^{-\alpha t} - 1).$$

This renaming is necessary because we don't want to confuse t, the dummy variable that we integrating over, with t_f, the time at which we want to evaluate the velocity; t ranges from zero to t_f. Unfortunately, often people just call both of these t, so you need to watch out.

[2] **Problem 1.** Investigating some features of this solution.

(a) By using results from **P1**, verify that $v(t)$ makes sense for both small times and large times.

(b) If the projectile is then caught at the launch point, did it spend more time going up or down?

(c) Do you think the total time is longer or shorter than for a projectile without drag?

Solution. (a) For small times ($\alpha t \ll 1$), we have

$$v(t) \approx (1 - \alpha t) v_0 + \frac{g}{\alpha}(-\alpha t) = v_0 - (g + \alpha v_0) t$$

which makes sense, since it's just the result of uniform acceleration $g + \alpha v_0$, under the initial net force. For large times ($\alpha t \gg 1$), the exponentials decay away and we get $v(t) \approx -g/\alpha$, which is the terminal velocity.

(b) Let P_u be the path going up, and P_d be the path going down. Additionally, imagine the video of P_u playing in reverse. By conservation of energy, the magnitude of the velocity of the projectile in P_u will be greater than that of P_d at a given height y. Since $dt = dy/v$, and at every value of y, $|v_u| \geq |v_d|$ by conservation of energy (and the bounds of integration are the same), then the ball will spend less time going up.

(c) I'm just asking this so you can use your intuition. Intuitively, it feels like it should take longer with drag, and for the linear drag this is indeed the case. More generally, if the drag force is proportional to v^n, then it turns out that the trajectory with drag always takes longer for $n \geq 1$, but for $n < 1$ it depends on the initial speed. You can find a proof of all these statements here.

[3] **Problem 2.** Now assume quadratic drag, $F = -\alpha m v^2$, which applies for fast-moving projectiles.

(a) Integrate Newton's second law to get an implicit equation for $v(t)$ with the same initial conditions as above. That is, you don't need to solve for $v(t)$, as it'll just make things messy.

(b) Your equation will only be valid when the projectile is going up; explain why.

(c) Find $v(t)$ for an object released from rest at time $t = 0$. (It may be helpful to look up some standard integrals involving hyperbolic trigonometric functions. These will be given to you if necessary on the USAPhO, but they will be useful for several problems in this program.)

Some people only call this quadratic case drag; they call the linear case viscous resistance. This is because they behave fundamentally differently at the microscopic level, as we will explore in **M7**.

Solution. (a) Newton's second law is

$$\frac{dv}{dt} = -g - \alpha v^2.$$

By the same reasoning as before, we find

$$\int_{v_0}^{v(t)} \frac{dv'}{g + \alpha v'^2} = -\int_0^t dt' = -t.$$

By nondimensionalizing the integral as described in **P1**, the left-hand side is

$$-t = \frac{1}{\sqrt{\alpha g}} \int_{v_0\sqrt{\alpha/g}}^{v(t)\sqrt{\alpha/g}} \frac{dx}{1+x^2} = \frac{1}{\sqrt{\alpha g}} \left(\tan^{-1}\left(v(t)\sqrt{\frac{\alpha}{g}} \right) - \tan^{-1}\left(v_0 \sqrt{\frac{\alpha}{g}} \right) \right)$$

where I pulled out a factor of $1/\sqrt{\alpha g}$ to get the right overall dimensions, then used dimensional analysis again to convert the integration bounds to dimensionless numbers. (You can also do this by ordinary u-substitution if you prefer.) This is essentially the final result. It can be solved for $v(t)$, but that just makes it look worse.

(b) The reason the equation only makes sense when the projectile is going up is that the force should always opposite the direction of motion, so we really wanted to solve $F = -m\alpha|v|v$. Equivalently, the sign of α changes when the direction of the velocity changes. This means our solution really should have two separate cases.

(c) By the same reasoning, we have

$$\int_0^{v(t)} \frac{dv'}{g - \alpha v'^2} = -t$$

where the changes are the initial condition and the sign of α. The left-hand side is

$$\frac{1}{\sqrt{\alpha g}} \int_0^{v(t)\sqrt{\alpha/g}} \frac{dx}{1-x^2} = \frac{1}{\sqrt{\alpha g}} \left(\tanh^{-1} \left(v(t) \sqrt{\frac{\alpha}{g}} \right) \right).$$

If you don't know this hyperbolic trig integral, you could also derive it by expanding $1/(1-x^2)$ in partial fractions and integrating each term. You will get a bunch of logarithms, which is equivalent to the hyperbolic tangent. However, if you don't know what the hyperbolic tangent is, you should look it up now, because such functions will be useful later!

Because of the simpler initial condition, we can get an explicit solution,

$$v(t) = -\sqrt{\frac{g}{\alpha}} \tanh(\sqrt{\alpha g}\, t).$$

The speed approaches $\sqrt{g/\alpha}$ with a timescale $1/\sqrt{\alpha g}$, a fact we could also have deduced by physical intuition and dimensional analysis. Actually, another way to arrive at this result is by just substituting $\alpha \to -\alpha$ in the answer for part (a)! This will produce the tangent of an imaginary number, which is in fact how the hyperbolic tangent is defined.

[3] **Problem 3.** A projectile of mass m is dropped from a height h above the ground. It falls and bounces elastically, experiencing the same quadratic drag as in problem 2. Find the maximum height to which it subsequently rises. (Hint: don't try to use your results from problem 2.)

Solution. The reason you shouldn't try to use the results from problem 2 is that they are in terms of time. Given how complicated the implicit expressions for $v(t)$ are, the expressions for $x(t)$ would be extremely clunky. And they're not necessary, because in this problem we don't care about the time-dependence at all; we just want to know the final height.

Another way to say this is that we aren't interested in $v(t)$, we're interested in $v(x)$. While the projectile is moving downward, we can integrate dv/dx to find the speed v_0 at the moment it hits the ground. Then, when it's moving upward, we integrate dv/dx until it has zero speed again, which is its final height. This will be a lot simpler than integrating dv/dt.

For the upward and downward trajectories, Newton's second law says

$$\frac{dv}{dt} = -g \pm \alpha v^2$$

and multiplying both sides by dt/dx gives

$$\frac{dv}{dx} = -\frac{g}{v} \pm \alpha v.$$

Separating and integrating, on the way down we have

$$\int_h^0 dx = \int_0^{-v_0} \frac{dv}{\alpha v - g/v} = \frac{1}{\alpha} \int_0^{-v_0} \frac{v\, dv}{v^2 - g/\alpha}.$$

Carrying out the integral and simplifying,

$$h = -\frac{1}{2\alpha} \log(1 - \alpha v_0^2/g).$$

Now, on the way up, we have

$$\int_0^{h'} dx = \int_{v_0}^0 \frac{dv}{-g/v - \alpha v} = \frac{1}{\alpha} \int_0^{v_0} \frac{v\,dv}{v^2 + g/\alpha}$$

and carrying out the integral gives

$$h' = \frac{1}{2\alpha} \log(1 + \alpha v_0^2/g).$$

Combining the two equations gives

$$h' = \frac{1}{2\alpha} \log(2 - e^{-2\alpha h})$$

which you can check has the right limits. Also note that g drops out, as required by dimensional analysis.

Example 2

Find how the speed of a rowing boat depends on the number of rowers N.

Solution

A fast-moving boat experiences quadratic friction, so a drag force

$$F \propto v^2 A$$

where A is the submerged cross-sectional area of the boat. Since the submerged volume scales as $V \propto N$ in hydrostatic equilibrium, we have $A \propto N^{2/3}$. (This is the sketchy step of the analysis, since the scaling of A depends on how we adjust the shape of the boat as N increases.) Thus, the power the rowers need to provide scales as $P = Fv \propto v^3 N^{2/3}$, but we also have $P \propto N$. Combining gives the exceptionally weak dependence $v \propto N^{1/9}$, which agrees decently with Olympic rowing times.

Idea 1

A linear differential equation is one that contains only terms like x, its derivative v, or higher derivatives, without any products. For example, the simple harmonic oscillator $ma = -kx$ is linear. Linear differential equations have a number of wonderful properties.

- Solutions to linear differential equations obey the superposition principle: if $x_1(t)$ and $x_2(t)$ are both solutions, so is $c_1 x_1(t) + c_2 x_2(t)$. So any n independent solutions gives you an entire n-dimensional space of solutions.
- If the highest time derivative is the second, as in Newton's second law, then two parameters are needed to specify a solution: an initial position and initial velocity. So the

space is solutions is only two-dimensional, which means that once you find two different solutions $x_1(t)$ and $x_2(t)$, the general solution takes the form $c_1x_1(t) + c_2x_2(t)$, where c_1 and c_2 are determined by the boundary conditions.

- If $x(t)$ is real, we can always replace it with a complex variable $\tilde{x}(t)$, then take the real part to get a valid solution in the end. This works because if we have

$$m\tilde{a} = -k\tilde{x}$$

then we can define $x(t) = \operatorname{Re}\tilde{x}(t)$. Taking the real part of both sides shows that $ma = -kx$, as required. This doesn't work if there are nonlinear terms, like x^2.

Idea 2

Almost all linear differential equations can be solved using the same method: the basic solutions $x_i(t)$ are complex exponentials,

$$x_i(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

where the allowed ω are determined by the differential equation. (You might have learned about "guessing an exponential" in physics class and thought it was just some random trick. But this trick is actually a deep principle, because it almost always works!)

Example 3

Solve the simple harmonic oscillator, $ma + kx = 0$, using the above principles.

Solution

We'll go straight to complex solutions, suppressing the tildes. We guess $x(t) = e^{i\omega t}$. Plugging this in and using the chain rule gives

$$m(i\omega)^2 e^{i\omega t} + ke^{i\omega t} = 0$$

and canceling $e^{i\omega t}$ and solving gives two solutions,

$$\omega = \pm\omega_0, \quad \omega_0 = \sqrt{k/m}.$$

The most general solution is thus given by superposition,

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B are complex numbers. We can get a real solution by taking the real part,

$$\operatorname{Re} x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

which is indeed the familiar general sinusoidal solution.

[1] **Problem 4.** To make sure you know how to go from the complex solution to the real one, write C and D in terms of A and B.

Solution. Let $A = a_A + b_A i$ and $B = a_B + b_B i$ where a_i, b_i are real. Applying Euler's formula,

$$\operatorname{Re} x(t) = (a_A + a_B)\cos(\omega_0 t) + (-b_A + b_B)\sin(\omega_0 t)$$

from which we read off

$$C = \operatorname{Re}(A + B), \quad D = \operatorname{Im}(B - A).$$

[2] **Problem 5.** Now assume there is an additional damping force $F_d = -bv$. Using the same procedure, solve for the frequencies ω assuming b is small. See section 4.3 of Morin if you have trouble with this. We'll consider this system in more detail in **M4**.

Solution. The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Guessing an exponential, every dot/time derivative is an $i\omega$, so

$$m(i\omega)^2 + b(i\omega) + k = 0.$$

Using the quadratic formula,

$$\omega = \frac{-ib \pm \sqrt{4km - b^2}}{-2m}.$$

In other words, we have

$$\omega = \pm\sqrt{k/m - b^2/4m^2} + \frac{ib}{2m}.$$

The oscillation is slightly slowed down, as you might expect, and the frequency has an imaginary part. This corresponds to exponential decay of the solution, $e^{i(ib)t/2m} = e^{-bt/2m}$.

[3] **Problem 6.** USAPhO 2012, problem B1.

[3] **Problem 7.** The reason the exponential guess works is because the derivative of an exponential is the exponential itself; at the end of the day the exponential cancels out and we're left with a polynomial in ω, which has just the right number of roots. But you could worry about multiple roots, in which case there are fewer distinct solutions for ω and hence not enough solutions.

(a) Consider a second order differential equation with a double root ω. What is the other solution, besides $e^{i\omega t}$? (Hint: to help find a good guess, consider the case $m\ddot{x} = 0$, where $\omega = 0$ is the double root. Then generalize your guess to nonzero ω and check that it works.)

(b) This should be setting off alarm bells: the form of the solutions to the equation changes when the two roots are *exactly* equal, while it's just exponentials/sinusoids if the roots are different, no matter how small the difference is. Since no two roots are *ever* exactly equal in practice, it seems the behavior of part (a) can never actually happen in the real world. But it gets taught in applied differential equations courses. Why?

(c) [A] Generalize to an n^{th} order linear differential equation

$$\left(a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0\right) x = 0.$$

What does the general solution look like? (The answer to this part is basically the punchline of a college course on differential equations; it means *every* linear differential is "easy".)

Solution. (a) In the case of a double root $\omega = 0$, the differential equation is $\ddot{x} = 0$. The solution we get by guessing an exponential is $x(t) = e^{i(0)t} = 1$, which is a constant. The other solution is linear, $x(t) = te^{i(0)t} = t$. This leads us to guess that for a double root ω, the two independent solutions are $e^{i\omega t}$ and $te^{i\omega t}$.

(b) As two roots get closer and closer together, we can get solutions that look more and more like $(A+Bt)e^{i\omega t}$, which is what we would get if they were exactly the same. Of course, it's intuitive that we can get $e^{i\omega t}$. To get a solution that looks like $te^{i\omega t}$, note that for roots $\omega \pm \Delta\omega$,

$$e^{i(\omega+\Delta\omega)t} - e^{i(\omega-\Delta\omega)t} = e^{i\omega t}(2i\sin(\Delta\omega\, t)) \propto \sin(\Delta\omega\, t)e^{i\omega t}.$$

This is an $e^{i\omega t}$ oscillation with a slowly varying envelope $\sin(\Delta\omega\, t)$. For small times, $t \ll 1/\Delta\omega$, the envelope is just proportional to t. As the roots get closer and closer together, this linear behavior persists for longer and longer time, but nothing is ever discontinuous. One can see this kind of envelope behavior in two weakly coupled pendulums, a system which has two nearby oscillation frequencies. You'll investigate this kind of thing in more detail in **M4**.

So the point is that the closer the roots are to each other, the more the behavior can look like $(A+Bt)e^{i\omega t}$, at least for times shorter than $1/\Delta\omega$. This is more direct and intuitive than superposing two sinusoids with almost equal frequencies, so we use it in practice.

(c) Guessing $e^{i\omega t}$ gives

$$a_n(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \ldots + a_0 = 0.$$

In the case where the roots are distinct, there are n possible values for ω, and hence n parameters in our trial solution,

$$x(t) = \sum_{i=1}^{n} A_i e^{i\omega_i t}.$$

Since the differential equation has order n, there are n parameters needed to specify the solution, so this is the general solution. If ω_i is a double root, then both $e^{i\omega_i t}$ and $te^{i\omega_i t}$ are solutions. For a triple root, $t^2 e^{i\omega_i t}$ is also a solution, and so on.

2 Tricks

In this section we'll consider some kinematics problems that require cleverness, not computation.

Idea 3

Many problems can be solved by a clever choice of reference frame. It is often useful to go to the frame moving with one of the objects in the problem, or to go into a frame that makes the motion in the problem more symmetric. For the purposes of kinematics it can even be useful to use noninertial reference frames, such as a falling frame where projectiles don't accelerate, or a rotating frame, though this will introduce fictitious forces into the dynamics. It is also useful to tilt the coordinate axes to be parallel to various objects.

[1] **Problem 8** (KoMaL 2019). A cannon A is at the edge of a cliff with a 800 m drop. Cannon B is on the ground below the cliff and 600 m horizontally away from it. Cannon A shoots a cannonball directly towards cannon B at 60 m/s. Cannon B shoots a cannonball directly towards cannon A at 40 m/s. Will the two cannonballs hit each other in midair?

Solution. Work in the frame freely falling with the cannonballs. In this case, the balls have a relative velocity of $100\,\text{m/s}$ and initial separation of $1000\,\text{m}$, so it takes $10\,\text{s}$ to collide. If there were no gravity, this collision would occur at a point $(2/5)(800\,\text{m}) = 320\,\text{m}$ above the ground. However, because of gravity both balls have fallen by an extra $gt^2/2 = 500\,\text{m}$ by this time. Hence the balls hit the ground before they can hit each other in midair.

[2] **Problem 9** (Wang). Two particles are released in gravitational acceleration g with leftward and rightward speeds v_1 and v_2. Find the distance between them when their velocities are perpendicular.

Solution. After time t, the velocity vectors are $(-v_1, -gt)$ and $(v_2, -gt)$. These are perpendicular when the dot product is zero, so $v_1 v_2 = (gt)^2$, which you can also show with basic geometry. Thus,

$$t = \frac{\sqrt{v_1 v_2}}{g}.$$

To compute the distance, we can just work in the frame falling with the masses. Then it's clear that the acceleration g doesn't matter, and the distance is just

$$d = (v_1 + v_2)t = \frac{(v_1 + v_2)\sqrt{v_1 v_2}}{g}.$$

[3] **Problem 10** (Kalda). Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

Solution. Work in the frame where the circles are moving towards each other with speed $v/2$. Then by the Pythagorean theorem, the speed of the point of intersection is

$$\frac{d}{dt}\sqrt{r^2 - (a/2)^2} = \frac{av}{4\sqrt{r^2 - a^2/4}}$$

where we used $da/dt = v$. However, we're not done yet, because the speed of the point of intersection depends on the frame; we need to go back to the original frame. Using the Pythagorean theorem again, the answer is

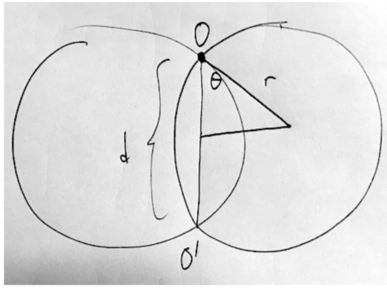
$$\sqrt{\left(\frac{av}{4\sqrt{r^2 - a^2/4}}\right)^2 + \left(\frac{v}{2}\right)^2} = \frac{v}{2}\frac{1}{\sqrt{1 - (a/2r)^2}}.$$

[2] **Problem 11** (Kalda). A mirror rotates about its center with angular speed ω. A stationary point source of light sits at a distance a from the rotation axis. What is the speed of its mirror image?

Solution. Work in the frame rotating with the mirror. Because the image is always flipped across the mirror with respect to the source, since the source rotates with angular velocity $-\omega$, the image rotates with angular velocity ω. Then the relative angular velocity of the source and image is 2ω, which holds in all frames. Thus, in the original frame the image has angular velocity 2ω and speed $2\omega a$.

[2] **Problem 12** (Kalda). Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

Solution. Remarkably, the answer does not depend on the time! Let d be the distance between the points of intersection, and work in the rotating frame where the circles rotate with angular velocities $\omega/2$ and $-\omega/2$ about O.



Since $\dot{\theta} = \omega/2$ and $\cos\theta = d/2r$, we have

$$-\frac{\omega}{2}\sin\theta = \frac{\dot{d}}{2r}, \quad \dot{d} = -r\omega\sin\theta.$$

This is the vertical velocity of O'. Now we need to go back to the original frame, which involves rotating with angular velocity $\omega/2$ about O. Then O' picks up a horizontal velocity of $(2r\cos\theta)(\omega/2)$ for a total speed of

$$v = \sqrt{r^2\omega^2\sin^2\theta + r^2\omega^2\cos^2\theta} = r\omega$$

which is constant. The geometrical reason is that the second intersection point rotates around the nonrotating circle with uniform angular velocity ω, as you can show by some angle chasing.

Idea 4

To find the minimum value of some quantity, it's often useful to think about all possible values of that quantity. This can reveal a solution using geometry or symmetry.

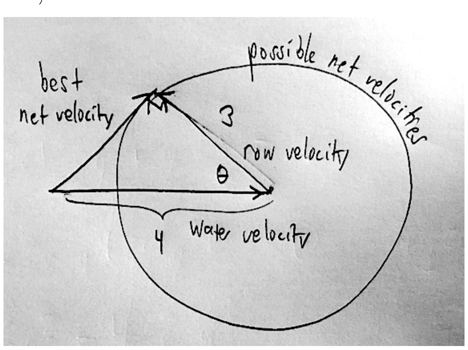
[2] **Problem 13** (PPP 3). A boat can travel a speed of $3\,\text{m/s}$ on still water. A boatman wants to cross a river while covering the shortest possible distance.

(a) In what direction should he row if the speed of the water is $2\,\text{m/s}$?

(b) How about if it is $4\,\text{m/s}$?

Solution. (a) The boatman can completely cancel out the horizontal velocity of the water. He should row an angle $\cos^{-1}(2/3)$ from the upstream direction, so that the boat moves directly across the river.

(b) The boatman cannot cancel out the horizontal velocity. Instead, the set of possible velocities forms a circle in velocity space, as shown.



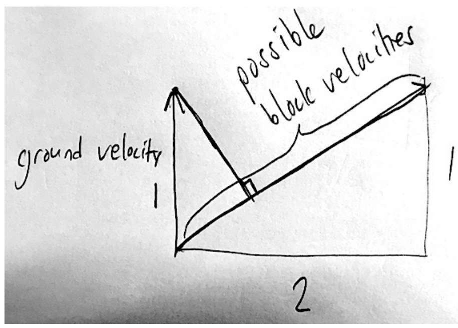
By taking the velocity with the angle closest to directly across the river, we see the boatman should row an angle $\cos^{-1}(3/4)$ from the upstream direction.

Idea 5

In problems with friction, the best reference frame to use is almost always the frame of whatever is causing the friction.

[2] **Problem 14** (Kalda)**.** A block is pushed onto a conveyor belt. The belt is moving with speed 1 m/s, and the block's initial speed is 2 m/s, with initial velocity perpendicular to that of the belt. During the subsequent motion, what is the minimum speed of the block with respect to the ground?

Solution. If the belt were not moving, the block would just decelerate in the direction of its speed, so that's what happens in the reference frame of the belt. The possible block velocities are shown in this frame.



The minimum relative speed with the ground is shown by the altitude, which has length $2/\sqrt{5}$ m/s by similar triangles.

Idea 6

For a variety of kinematics problems, it can be useful to think about the motion from a different perspective. For example, if your problem involves complicated accelerations, it can be useful to think in "velocity space", i.e. directly think about how the velocity vector evolves over time, and deal with the position later. Or, if your problem involves complicated processes occurring in time, it can be useful to think in "spacetime", meaning to visualize the process on a space where time is one of the axes. It can also be useful to parametrize motion in terms of quantities other than the usual Cartesian coordinates.

[2] **Problem 15** (Kalda)**.** A boy enters a patch of ice with a coefficient of friction μ with speed v. By running on the ice, the boy turns his velocity vector by 90° in the minimum possible time, so that his final speed is also v. What is the minimum possible time, and what kind of curve is the trajectory? Assume the normal force with the ice is constant.

Solution. The acceleration always has magnitude μg. The velocity needs to change by $v(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ if it starts at $v\hat{\mathbf{y}}$, so $v\sqrt{2} = \mu g t$. Thus, $t = \frac{v}{\mu g}\sqrt{2}$. The acceleration is constant, so the trajectory is a parabola.

[2] **Problem 16** (PPP 5)**.** Four snails travel in uniform, rectilinear motion on a plane. The velocities are chosen so that three snails never meet at once, and no two of the velocities are equal. Since time $t = -\infty$, five of the $\binom{4}{2}$ possible encounters have already occurred. Must the sixth also occur?

Solution. It's hard to visualize what's going on in the plane; instead think about what's going on in *spacetime*. The spacetime here is three-dimensional, and the paths of the worms are lines through it, called worldlines; two worms will encounter each other if their worldlines intersect. For some set of three of the snails, all possible encounters occur, so their worldlines lie in a plane in spacetime. (This means that in space, these three snails move on the same line.)

If the fourth snail's worldline lies in this plane, then it must intersect all three others. If it doesn't, it can intersect at most one. Hence if five encounters have already occurred, the sixth must also occur.

[2] **Problem 17.** Six bugs are placed at the vertices of a regular hexagon with side length s. At time $t = 0$ each bug starts moving directly towards the next with speed v. At what time do they collide?

Solution. By symmetry, the bugs always remain in a hexagon shape, but this hexagon rotates and shrinks. We want to know the time when it collapses completely.

We can first do this by considering how the distance between adjacent bugs changes in an infinitesimal time dt. The first bug moves a distance $v\,dt$ towards the second. The second moves a distance $(\sqrt{3}/2)v\,dt$ to the side, and a distance $(v/2)\,dt$ directly away from the first. The side-to-side motion doesn't contribute to the change in distance (one can use the Pythagorean theorem and binomial theorem to show it is second order, and hence negligible for infinitesimals), so we ignore it. Then the rate of change of distance between the bugs is just $v - v/2 = v/2$, so the bugs meet at $t = 2s/v$.

Another method is to note that all the bugs meet in the center of the original hexagon, so we can consider the component of velocity for each bug directed towards the center. This is always $v/2$ by the hexagonal symmetry, and the original distance from the center is s, so the bugs again meet in time $t = 2s/v$.

[3] **Problem 18.** A rabbit begins at the origin, and the fox begins at the point $(0, -a)$. The rabbit begins running with a constant speed $v\hat{\mathbf{x}}$. At the same time, the fox begins chasing the rabbit, always moving towards it with speed v.

(a) Sketch the subsequent trajectory of the rabbit and fox.

(b) Let the displacement between the rabbit and fox be
$$\mathbf{r}(t) = (x(t), y(t)).$$
Show that $r + x$ is conserved.

(c) Find the distance between the rabbit and fox after a long time.

(d) Now suppose the fox has speed $u > v$. How long does it take to catch the rabbit?

Solution. (a) Initially the fox moves up while the rabbit moves to the right. After a while, the two simply follow each other, with a constant distance between them, along the x axis.

(b) Let θ be the angle between the velocity vectors. Then
$$\frac{dr}{dt} = -v + v\cos\theta$$
because of the fox and rabbit, and
$$\frac{dx}{dt} = v - v\cos\theta$$
because of the rabbit and fox. Then $r + x$ is constant, as desired.

(c) Initially, $r + x = a + 0 = a$. After a long time, $r = x$, so $r = x = a/2$.

(d) Solving for the trajectory of the fox is extremely difficult, but we can use an extension of the idea of part (b). Now the equations of motion are

$$\frac{dr}{dt} = -u + v\cos\theta, \quad \frac{dx}{dt} = v - u\cos\theta.$$

Combining these equations, we can cancel out θ to get

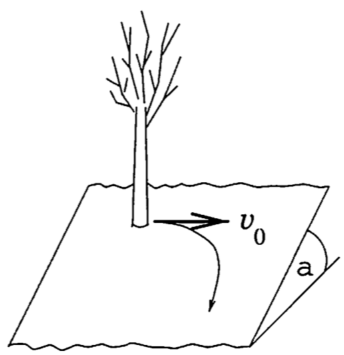
$$u\frac{dr}{dt} + v\frac{dx}{dt} = v^2 - u^2.$$

This can now easily be integrated from between the initial and final time. During this time, the change in r is $-a$, while the change in x is zero, so

$$-au = (v^2 - u^2)t, \quad t = \frac{ua}{u^2 - v^2}.$$

If you're curious what the full trajectory looks like, you can find it in this paper, which was written by a past coach of the U.S. Physics Team.

[2] **Problem 19** (PPP 85)**.** A child is standing on an icy hill, which may be modeled as an inclined plane.



The coefficient of friction $\mu_k = \mu_s$ is small enough so that, if the child gets the tiniest push, she will begin sliding down the plane. Now suppose the child gets a horizontal push, with initial speed v_0. What is the child's final speed?

Solution. This is easy because you've already solved the problem; it's just the same thing as problem 18. Specifically, the displacement between the rabbit and fox there corresponds to the velocity of the child here. At every increment of time, the velocity changes in two ways: it shrinks along its direction by friction (corresponding to the fox) and it has a constant added by gravity (corresponding to the rabbit). Hence the answer is $v_0/2$.

You should definitely not try to solve for the trajectory exactly, since it's very messy, but you can find the gory result in this paper.

3 Motion in Two Dimensions

Idea 7

Often, motion in two dimensions can be treated as two independent one-dimensional problems. A change of reference frame may be necessary first.

Idea 8

In problems involving an inclined plane, always set the angle θ to be much closer to either $0°$ or $90°$ than to $45°$. This reduces mistakes, because almost every angle will be either θ or $90° - \theta$, and you can identify which by sight.

Example 4

Consider projectile motion where wind provides a constant horizontal force F. At what angle should a projectile of mass m be launched in order to return to the thrower?

Solution

The key idea is to use tilted coordinate systems. Clearly, when the only force is downward, the projectile must be launched straight upward. Now, the horizontal force acts like an effective horizontal gravitational acceleration of F/m, so that gravity is effectively tilted an angle $\tan^{-1}(F/mg)$ away from the vertical. One must launch the projectile directly "upward" with respect to this effective gravitational field, so the launch angle is an angle $\tan^{-1}(F/mg)$ from the vertical. (For a related problem, see the infamous $F = ma$ 2014 problem 19.)

[1] **Problem 20** (Quarterfinal 2002)**.** A cart is rigged with a vertical cannon so that, when the cart is stationary on a horizontal track, the cannonball is fired straight up and lands back in the cannon. In each of the following situations, does the cannonball land back in the cannon, in front of it, or behind it?

(a) The cart is moving on a frictionless horizontal track with speed v.

(b) The cart is accelerating down a frictionless inclined track with angle θ.

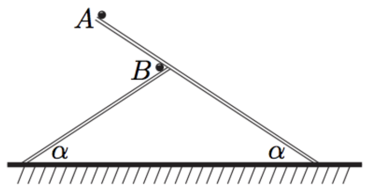
(c) The cart is accelerating down an inclined track with angle θ, and friction slows it down.

Solution. (a) The motion in the x and y directions is independent. In the x direction, both the cannonball and cart just continue moving with speed v, so the cannonball lands right back into the cannon.

(b) Work in the tilted frame where the x axis is parallel to the track. In the x direction, both the cannonball and cart start with the same speed v and accelerate with the same acceleration $g \sin\theta$, so the cannonball lands right back into the cannon, again.

(c) In this case the cart accelerates less, so the cannonball lands in front.

[2] **Problem 21** (Kalda). Two balls at points A and B are released from rest at the same moment, from the locations shown below. All surfaces are frictionless.



If it takes time t_A and t_B for the balls to hit the ground, at what time was the distance between the balls the smallest?

Solution. Both balls have a downward acceleration of $g \sin \alpha$, and they have leftward and rightward accelerations of $g' = g \cos \alpha$. Since the balls always have the same vertical speed, we can ignore the vertical motion entirely. The distance between the balls is thus smallest when their horizontal separation is zero.

Let the total horizontal distances the balls travel be d_A and d_B. Then

$$d_A = \frac{1}{2} g' t_A^2, \quad d_B = \frac{1}{2} g' t_B^2$$

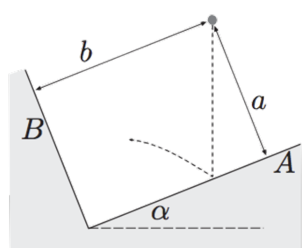
and we are looking for the time t where

$$\frac{d_A - d_B}{2} = \frac{1}{2} g' t^2.$$

Solving these equations for t gives

$$t = \sqrt{\frac{t_A^2 - t_B^2}{2}}.$$

[2] **Problem 22** (Kalda). Two planar frictionless walls are placed at right angles, where wall A makes an angle α to the horizontal. A perfectly elastic ball is released from rest at a point a distance a from wall A and b from wall B.



After a long time, what is the ratio of the number of times the ball has bounced against wall B to the number of times it has bounced against wall A?

Solution. In the coordinate system tilted by angle α, the motions in the x and y directions are independent, because collisions with wall A leave v_x unchanged and vice versa. In the y direction, the ball simply bounces up and down with uniform acceleration $g \cos \alpha$ and bounce height a, so

$$\Delta t_A = 2 \sqrt{\frac{2a}{g \cos \alpha}}.$$

By similar reasoning, in the x direction

$$\Delta t_B = 2 \sqrt{\frac{2b}{g \sin \alpha}}.$$

Thus the answer is

$$\frac{\Delta t_A}{\Delta t_B} = \sqrt{\frac{a \sin\alpha}{b\cos\alpha}}.$$

When this ratio is a rational number, the ball eventually returns to its starting point. If it isn't, it never does; instead it eventually explores all of the space permitted by energy conservation, i.e. it eventually passes arbitrarily close to any point whose height is at most the height of the starting point.

[3] **Problem 23.** USAPhO 2004, problem A4.

[3] **Problem 24** (EFPhO 2010)**.** A sprinkler can be modeled as a small hemisphere on the ground. Water shoots out from the hemisphere in all directions, with speed v perpendicular to the hemisphere.

(a) Find the total surface area of ground watered by the sprinkler.

(b) At what distance from the sprinkler does the ground get the wettest?

Solution. (a) The range of the sprinkler is maximized at 45° and is equal to v^2/g. Then the area is $\pi(v^2/g)^2 = \pi v^4/g^2$.

(b) The outermost circle, at radius v^2/g, gets by far the wettest. This is because a maximum of radius is achieved here, so a large range of launch angles gets to near this radius. (It's the same reason that balls thrown upward spend the most time near the very top of their trajectories.)

This idea is a little tricky, but very general; for instance, it's the principle behind the formation of caustics such as rainbows, as we'll see in **W3**. It is also the way in which classical mechanics emerges from quantum mechanics: classically things follow the trajectory of least action because it's a caustic of the quantum sum over all trajectories. So if you continue in physics, you'll see this beautiful little idea over and over again, in richer and richer settings! For an Olympiad problem that gives a bit more detail about caustics in optics, see here.

Example 5

A bug flies towards a light with constant speed v, always making an angle α with the radial direction. If the initial distance to the lamp is L and the radius of the lamp is R, through what total angle does it turn before hitting the lamp?

Solution

In this case we can't avoid solving differential equations, but they're not too hard. It's easiest to work in polar coordinates, with the center of the lamp at the origin. By decomposing the velocity into radial and tangential components, we have

$$\frac{dr}{dt} = -v\cos\alpha, \quad r\frac{d\theta}{dt} = v\sin\alpha.$$

We only care about the path, not the time-dependence, so we divide these equations to get

$$\frac{dr}{d\theta} = -\frac{r}{\tan\alpha}$$

where we manipulated differentials as in **P1**. Separating and integrating,

$$-\int_L^R \frac{dr}{r} = \frac{\Delta\theta}{\tan\alpha}$$

which tells us that

$$\Delta\theta = (\tan\alpha)\log\frac{L}{R}.$$

The shape traced out is a logarithmic spiral.

[2] **Problem 25.** The pilot of a supersonic jet airplane wishes to make a big noise at the origin by flying around it in a path such that all of the noise he makes is heard simultaneously at the origin. The jet travels with Mach number M, meaning that its speed is M times the speed of sound. If the pilot starts at $(r,\theta) = (a, 0)$, find the pilot's path $r(\theta)$.

Solution. In order for the sound to reach the origin simultaneously, we must have $r(t) = a - ct$, so that the sound all reaches the origin at time a/c. On the other hand, we have

$$(Mc)^2 = \dot{r}^2 + r^2\dot{\theta}^2 = c^2 + r^2\dot{\theta}^2.$$

This is a bit messy because we have two functions of time, but we can eliminate time by using

$$\dot{\theta} = \frac{d\theta}{dr}\frac{dr}{dt} = c\frac{d\theta}{dr}.$$

Plugging this in above, we have

$$M^2 - 1 = r^2\left(\frac{d\theta}{dr}\right)^2$$

and separating and integrating gives

$$\int \frac{dr}{r} = \int \frac{d\theta}{\sqrt{M^2-1}}, \quad r(\theta) = ae^{-\theta/\sqrt{M^2-1}}.$$

[4] **Problem 26.** Consider a mass m on a table attached to a spring at the origin with zero relaxed length, which exerts the force

$$\mathbf{F} = -k\mathbf{r}$$

on the mass. We will find the general solution for $\mathbf{r}(t) = (x(t), y(t))$ in two different ways.

(a) Directly write down the answer, using the fact that the x and y coordinates are independent.

(b) Sketch a representative sample of solutions. What kind of curve does the trajectory follow?

(c) Here's a more unusual way to arrive at the same answer. Go to a noninertial reference frame rotating with angular velocity ω_0 about the origin, so that the centrifugal force cancels out the spring force. In this frame, the only relevant force is the Coriolis force $-2m\boldsymbol{\omega}_0 \times \mathbf{v}$. Find the general solution in this frame, then transform back to the original frame and show that you get the same answer as in part (a). (This can get a bit messy; the easiest way is to treat the plane as the complex plane, i.e. work in terms of the variable $r = x + iy$.)

Solution. (a) We just have two separate equations for each component,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x, \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y.$$

Both describe a harmonic oscillator with frequency $\omega_0 = \sqrt{k/m}$. Then the general solution can be written as

$$x(t) = A\cos(\omega_0 t + \phi_1), \quad y(t) = B\sin(\omega_0 t + \phi_2).$$

In general, it is very rare for the x and y coordinates to be independent. Another example of this type is projectile motion in linear drag, $\mathbf{F} = -k\mathbf{v}$. In these cases the 2D or 3D problem is no harder than the 1D version, but we're rarely so lucky.

(b) In the case where $\phi_1 = \phi_2 = 0$ and $A = B$, the mass moves in a circle centered at the origin. More generally, when the angles ϕ_i are unequal, the mass can move in an ellipse with center at the origin.

(c) The centrifugal force is $m\omega_0^2\mathbf{r}$, so to cancel the spring force we need to choose $\omega_0 = \sqrt{k/m}$. Now, in the rotating frame, the Coriolis force acts just like a magnetic field: it's always perpendicular to the motion, so the solution is circular motion. The angular frequency ω_c of that circular motion satisfies

$$2m\omega_0 v = \frac{mv^2}{r} = m\omega_c v$$

from which we conclude $\omega_c = 2\omega_0$. So in complex notation,

$$r(t) = r_0 + r_1 e^{2i\omega_0 t}$$

in the rotating frame. We can return to the original frame by simply multiplying by $e^{-i\omega_0 t}$ (the sign is important, i.e. you have to keep track of the direction of $\boldsymbol{\omega}_0$), to give

$$r(t) = r_0 e^{-i\omega_0 t} + r_1 e^{i\omega_0 t}.$$

Taking real and imaginary parts and letting $r_i = a_i + ib_i$,

$$x(t) = (a_0 + a_1)\cos(\omega_0 t) + (b_0 - b_1)\sin(\omega_0 t), \quad y(t) = (b_0 + b_1)\cos(\omega_0 t) + (a_1 - a_0)\sin(\omega_0 t).$$

This is the same as our result for part (a), after you use the sine and cosine addition formulas and appropriately redefine the parameters.

By the way, there's another interesting thing that this derivation has uncovered: we have shown that elliptical motion is the superposition of two circular motions of equal period but opposite orientation. This is one of the reasons that ancient astronomy based on epicycles worked so well: you can match the predictions of modern theory using just a few of them.

4 Optimal Launching

Finally, we'll consider projectile motion questions that involve optimization. These are rare on the USAPhO, but they are quite fun problems, with occasionally very slick solutions.

Example 6

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution

Let P be the point at the top of the log. For the bug to be moving horizontally at P, energy conservation applied to the vertical motion gives an initial v_y obeying

$$\frac{1}{2}mv_y^2 = 2mgR, \quad v_y = 2\sqrt{gR}.$$

Thus, we need to find the minimum v_x for the motion to be possible. If v_x is too low, the hypothetical trajectory of the bug will instead pass through the log. At the lowest possible v_x, the bug's trajectory is not just tangent to the log at point P, but also has the same radius of curvature (i.e. the trajectory and the log's shape have the same first and second derivatives).

For uniform motion in a circle of radius r, the acceleration is $a = v^2/r$. Conversely, when an object follows a trajectory of instantaneous radius of curvature r, its acceleration component normal to the path must be $a = v^2/r$. So applying this to the bug at P gives

$$g = \frac{v_x^2}{R}, \quad v_x = \sqrt{gR}.$$

Thus, the minimum initial speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5}\,gR.$$

This radius of curvature trick doesn't come up often, but it's cool when it does.

[2] **Problem 27.** NBPhO 2020, problem 3. A nice warmup for the problems below.

Solution. See the official solutions here.

[3] **Problem 28.** An object is launched from the top of a hill, where the ground lies an angle ϕ below the horizontal. Show that the range of a projectile is maximized if it is launched along the angle bisector of the vertical and the ground.

Solution. This is a straightforward if messy problem; we'll show one of many ways to set it up. Setting the origin at the launch point and using ordinary horizontal/vertical coordinates, the object hits the hill when $\tan\phi = -y/x$. Using results for projectile trajectories from the preliminary problem set, we have

$$\frac{y}{x} = -\tan\phi = \tan\theta - \frac{gx}{2v^2\cos^2\theta}$$

where θ is the launch angle from the horizontal. Solving for x,

$$x = \frac{2v^2\cos^2\theta}{g}(\tan\theta + \tan\phi) \propto \sin\theta\cos\theta + \cos^2\theta\tan\phi.$$

To maximize the range, we want to maximize x, so setting the derivative to zero gives

$$0 = \cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\tan\phi$$

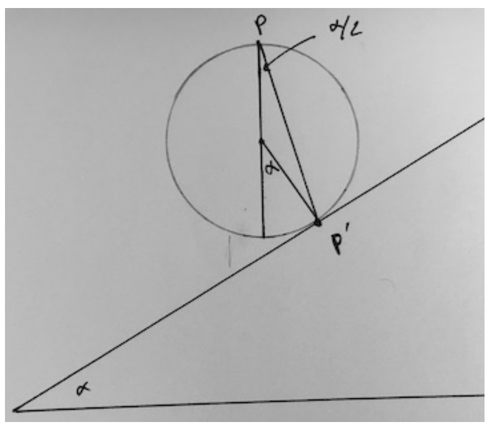
which simplifies to

$$\tan(2\theta) = \frac{1}{\tan\phi} = \tan(\phi + \pi/2), \quad \theta = \frac{\phi + \pi/2}{2}$$

as desired. This famous problem was first posed by Torricelli in the 1640s, and solved by Halley in the 1690s.

[3] **Problem 29** (PPP 35)**.** A point P is located above an inclined plane with angle α. It is possible to reach the plane by sliding under gravity down a straight frictionless wire, joining P to some point P' on the plane. Geometrically, how should P' be chosen so as to minimize the time taken? (Hint: think about the set of points that can be reached for all possible angles of the wire, after time t.)

Solution. Suppose the wire is at angle θ with respect to the vertical. Then, the distance traveled in time t is $\frac{1}{2}(g\cos\theta)t^2$. Keeping t fixed, we then see that the locus of all reached points is a circle whose topmost point is P, and whose radius is $\frac{1}{2}gt^2$. Therefore, P' is the point where one of these circles is tangent to the incline, so an $\alpha/2$ angle to the vertical.



Idea 9

Since mechanics is time-reversible, and the speed of a projectile only depends on its height and not the path taken, finding the way to reach point B from point A with the lowest possible initial speed is the same as finding the way to reach point A from point B with the lowest possible initial speed.

[4] **Problem 30.** Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the *tops* of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

Solution. It's very confusing to think about how to throw the projectile starting from the ground, because you need to figure out where to launch and at what angle, under the condition that the trajectory just touches the tops of both fences. A much better way is to imagine the projectile *starts* at the top of the higher fence; the goal is then to throw it with minimal energy so that it just touches the top of the lower fence. At some point, this projectile will then reach the ground, though we don't have to worry about where. Since mechanics is time-reversible, its speed at this point (which is found easily by energy conservation) will be the minimal possible speed.

Now there are many ways to do this problem. A very slick solution, which requires no computation at all, is presented in problem 31. However, we'll present a more direct attack for completeness. Note that if you want to hit the top of the lower fence with the minimum velocity, it's equivalent to maximizing your throwing range down an inclined plane, namely the plane that connects the tops of the two fences. Then the optimal launch angle is along the angle bisector, as we found in problem 28. Using the same starting point as the solution to that problem, we have

$$-\frac{h}{\sqrt{d^2-h^2}} = \tan\theta - \frac{g\sqrt{d^2-h^2}}{2v^2\cos^2\theta}$$

where we let $h = h_2 - h_1 > 0$. That solution gives a simple expression for $\tan 2\theta$, so we massage this equation to

$$\frac{g}{v^2} = \frac{\sin 2\theta}{\sqrt{d^2-h^2}} + \frac{h}{d^2-h^2}(1+\cos 2\theta).$$

We then plug in our previous results, which are

$$\sin 2\theta = \frac{\sqrt{d^2-h^2}}{d}, \quad \cos 2\theta = \frac{h}{d}$$

to get the result

$$v^2 = (d-h)g = (d+h_1-h_2)g.$$

By energy conservation, the speed at the ground is

$$v_0^2 = v^2 + 2h_2 g = (d+h_1+h_2)g$$

as desired.

[4] **Problem 31.** It's possible to solve problems 28 and 30 using pure geometry, with no computation. One can show that the set of points a projectile can reach with a fixed initial speed v is a parabola with a focus at the launching point. A parabola is defined as the set of points whose distance to the focus equals the distance to a line, called the directrix.

(a) Show that trajectories that touch the parabola must be tangent to it.

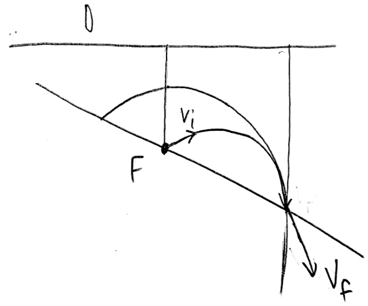
(b) Show that if a point is hit with the smallest possible initial speed, then the initial velocity must be perpendicular to the final velocity.

(c) Using the geometric definition of a parabola, recover the answers to problems 28 and 30.

Solution. (a) This is just because the parabola is defined to be the set of points you can hit. If the trajectory weren't tangent to the parabola, you would be able to hit a point outside the parabola by continuing it.

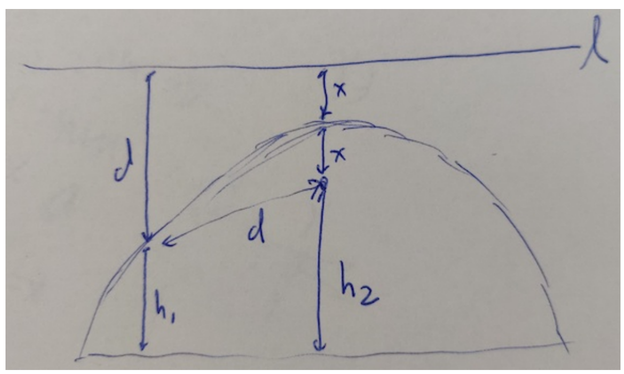
(b) Let $\mathbf{v}_i$ be the initial velocity and $\hat{v}_\perp$ be a unit vector in the perpendicular direction. If we replace the initial velocity by $\mathbf{v}_i + \epsilon\hat{v}_\perp$, where ϵ is infinitesimal, then the speed isn't changed, which implies that the new trajectory should remain inside the parabola. Now suppose the original projectile's velocity is $\mathbf{v}_f$ when it is tangent to the parabola, at position $\mathbf{r}_f$. Then at the same time, the new projectile's position is $\mathbf{r}_f + t\epsilon\hat{v}_\perp$. In order to keep this inside the parabola for all infinitesimal ϵ, both positive and negative, $t\epsilon\hat{v}_\perp$ must be tangent to the parabola at this point. Hence $\hat{v}_\perp$ is parallel to $\mathbf{v}_f$, so $\mathbf{v}_i$ is perpendicular to $\mathbf{v}_f$, as desired.

(c) A parabola is the set of points equidistant from a focus and a line, called the directrix. Refer to the diagram below.



The final velocity $\mathbf{v}_f$ is tangent to the parabola. Therefore, it points along the angle bisector between the vertical and the direction along the plane since the distance from the focus and the directrix will remain equal to each other. Now, $\mathbf{v}_i$ is perpendicular to this, which means it is along the angle bisector between the vertical and the downward direction along the plane, which is precisely the result we found in problem 28.

Assume $h_2 > h_1$. Draw the parabola of the projectile range with its focus on h_2. At the minimum launching velocity, the parabola should just touch the top of the h_1 fence. The (horizontal) directrix will be a distance $2x$ above h_2, where $x = v_2^2/2g$ and v_2 is the launching velocity from h_2. Then use the fact that the distances from a point on a parabola to the focus and directrix are the same.



From the picture, we see that $d + h_1 = 2x + h_2$. Thus the launching velocity at h_2 satisfies $v_2^2/g = d + h_1 - h_2$, and it has a total energy upon launching of

$$E/m = \frac{1}{2}v_2^2 + gh_2 = g\frac{d + h_1 - h_2}{2} + gh_2 = \frac{1}{2}mv_0^2.$$

This gives the answer with almost no computation,

$$v_0 = \sqrt{g(d + h_1 + h_2)}.$$

[3] **Problem 32.** IPhO 2012, problem 1A.

5 Reading Graphs

In some kinematics problems, you'll have to infer what's going on from a diagram. To make progress, you'll have to print out the diagram to make measurements directly on it.

[3] **Problem 33.** EFPhO 2015, problem 6.

Solution. See the official solutions here.

[3] **Problem 34.** EFPhO 2008, problem 3.

Solution. See the official solutions here.

Remark

For a ridiculously hard problem from the same genre, see EuPhO 2019, problem 3. Almost all competitors received zero points on it; you can try it for entertainment if you've finished everything else and really like kinematics. The official solutions are here.