Electromagnetism I: Electrostatics

Reading: chapters 1-3 of Purcell. For a separate introduction to vector calculus, see the resources mentioned in the syllabus, or chapter 1 of Griffiths' *Introduction to Electrodynamics*. Electrostatics is covered in more mathematical detail in chapter 2 of Griffiths.

1 Coulomb's Law and Gauss's Law

We'll begin with some basic problems involving Coulomb's law, which don't require calculus; all problems can be solved by simple symmetry arguments.

Idea 1

Gauss's law is written in integral form as

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}.$$

In practice, you will only apply Gauss's law to situations with spherical symmetry ($E = Q/4\pi\epsilon_0 r^2$), cylindrical symmetry ($E = \lambda/2\pi\epsilon_0 r$), or to infinite planes ($E = \sigma/2\epsilon_0$).

Exercise 1. N point charges q are placed at the vertices of a regular N-gon of side length a. If a single charge is removed, what is the electric field at the center?

Exercise 2. A charge q sits at the corner of a cube, just inside the cube.

(a) Find the electric flux through one of the opposite sides.

(b) Find the electric flux through one of the adjacent sides.

Exercise 3. To obey Gauss's law, what should Coulomb's law be in n spatial dimensions? (You can ignore an overall constant factor.) Does the shell theorem still hold?

Problem 1. Some questions about electrostatic equilibrium.

(a) Prove Earnshaw's theorem, which states that a point charge in the electrostatic fields of other point charges cannot be in stable equilibrium, i.e. it is impossible for the charge to be in equilibrium and feel a restoring force when displaced in an arbitrary direction.

(b) Earnshaw's theorem also applies to gravitational fields. However, you may have heard that there are stable Lagrangian points in the Earth–Sun system. How is this possible?

(c) Prove that when a system of point charges is in equilibrium, the total potential energy of the system is zero.

(d) Show that for a positive point charge in the electric fields of fixed, positive point charges, there is a path along which the charge can "escape to infinity", along which the potential only decreases.

Problem 2 (Griffiths 2.18)**.** Two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that they partially overlap. Call the vector from the positive center to the negative center $\mathbf{d}$. Find the field in the overlap region.

Idea 2

Gauss's law is written in differential form as

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

The divergence of a vector field $\mathbf{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$

in Cartesian coordinates, where ∂_x stands for $\partial/\partial x$, and so on.

Exercise 4. Show that the divergence of $\mathbf{F}$ at a point $\mathbf{x}$ is the flux per unit volume out of a small cube centered at $\mathbf{x}$. To do this, you should explicitly compute the flux coming out of such a cube, expanding $\mathbf{F}$ by a Taylor series to first order.

Problem 3. Consider a vector field expressed in polar coordinates, $\mathbf{F} = F_r\hat{r} + F_\theta\hat{\theta}$ where $\hat{r}$ and $\hat{\theta}$ are unit vectors in the radial and tangential directions. By considering the flux per unit area out of a small region bounded by r and $r + dr$, and θ and $\theta + d\theta$, derive the expression for the divergence in polar coordinates.

Problem 4. Suppose that all space is filled with uniform charge density ρ.

(a) Show that $\mathbf{E} = (\rho/\epsilon_0)x\hat{\imath}$ obeys Gauss's law.

(b) Show that $\mathbf{E} = (\rho/\epsilon_0)z\hat{k}$ also obeys Gauss's law.

(c) Argue that by spherical symmetry, we have $\mathbf{E} = 0$. Show this does not obey Gauss's law.

(d) What's going on? Which, if any, is the actual field?

Idea 3

A simple but occasionally subtle idea is to use Newton's third law. It may be easier to calculate the force of Y on X than the force of X on Y.

Exercise 5 (Purcell 1.28)**.** Consider a point charge q located inside a sphere of radius R. Show that the average electric field over the surface of the sphere is zero.

Problem 5. A hemispherical shell of radius R has uniform charge density σ and is centered at the origin. Find the electric field at the origin. (Hint: use the previous idea, along with the fact that the integral $\int d\mathbf{S}$ over a surface with a fixed boundary is independent of the surface.)

Problem 6. A point charge q is placed a distance $a/2$ above the center of a square of charge density σ and side length a. Find the force of the square on the point charge.

Problem 7 (Griffiths 2.43, PPP 113)**.** Consider a uniformly charged sphere of radius R and total charge Q.

(a) Find the net force that the southern hemisphere exerts on the northern hemisphere.

(b) Generalize the result to the case where the sphere is split into two parts by a plane whose minimum distance to the sphere's center is h.

It may be useful to refer to exercise 16.

Problem 8 (MPPP 140)**.** $\star$ Generalize the previous equation so that the two hemispheres have charges Q_1 and Q_2, and radii R_1 and R_2.

2 Continuous Charge Distributions

These problems involve reasoning about continuous charge distributions, and some naively require multiple integration.

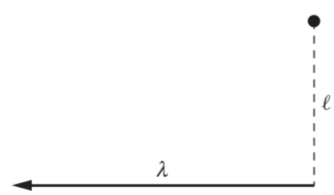
Idea 4

In almost all cases in Olympiad physics, there will be sufficient symmetry to reduce any multiple integral to a single integral.

Idea 5

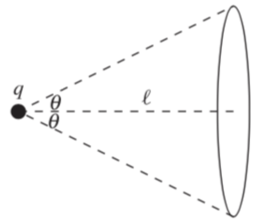
Remember that when using Gauss's law, the gaussian surface may be freely deformed as long as it doesn't pass through any charges.

Exercise 6 (Purcell 1.10)**.** A half-infinite line has linear charge density λ. Find the electric field at a point that is "even" with the end, a distance ℓ from it, as shown below.



You should find the direction of the field is independent of ℓ. Explain why.

Exercise 7 (Purcell 1.15)**.** A point charge q is located at the origin. Compute the electric flux through a circle a distance ℓ from q, subtending an angle 2θ as shown below.



This calculation becomes much easier if you deform the Gaussian surface.

For problems requiring integrals over circles or spheres, the law of cosines is often useful.

Problem 9 (Purcell 1.8)**.** A ring with radius R has uniform positive charge density λ. A particle with positive charge q and mass m is initially located in the center of the ring and given a tiny kick. If the particle is constrained to move in the plane of the ring, show that it exhibits simple harmonic motion and find the frequency.

Problem 10 (Purcell 1.12)**.** Consider the setup of problem 5. Find the electric field at an arbitrary point on the z-axis.

Exercise 8. In this problem we consider infinitely long insulating rods.

(a) Find the potential $V(r)$ for one rod of linear charge density λ. Note that you cannot set the potential to be zero at infinity.

(b) Now consider two parallel rods with linear charge density λ and $-\lambda$. Find the shapes of the equipotential surfaces.

Problem 11 (MPPP 132)**.** $\star$ An insulating rod with uniform charge density is bent into a triangle. What is the point in the triangle where the electric field vanishes? (Hint: show that the electric field due to a single side is the same as the electric field due to an appropriately chosen arc of a uniformly charged circle.)

Idea 6: Electric Dipoles

The dipole moment of two charges q and $-q$ separated by $\mathbf{d}$ is $\mathbf{p} = q\mathbf{d}$. More generally, the dipole moment of a charge configuration is

$$\mathbf{p} = \int \rho(\mathbf{x})\mathbf{x}\, dV.$$

For an overall neutral charge configuration, the leading contribution to its electric potential far away is the dipole potential,

$$\phi(r, \theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where θ is the angle of $\mathbf{r}$ to $\mathbf{p}$.

Exercise 9. Derive the dipole potential for a dipole made of a pair of point charges.

Exercise 10. Differentiate this result to find the dipole field,

$$\mathbf{E}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\;\hat{\theta}).$$

The most important facts about dipole-dipole interactions are illustrated by the following problem.

Problem 12. USAPhO 2009, problem B2.

Idea 7

The energy of a set of point charges is

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2} \sum_i q_i V(\mathbf{r}_i).$$

Note that we sum over $i \neq j$ to avoid computing the energy of a single point charge due to its interaction with itself, which would be infinite. For a continuous distribution of charge, we don't have this problem, and instead find

$$U = \frac{1}{2} \int \rho(\mathbf{x}) V(\mathbf{x})\, dV = \frac{\epsilon_0}{2} \int E^2\, dV.$$

Note that unlike any of the other quantities we've considered, energy does *not* obey the superposition principle.

Exercise 11. Compute the potential energy of a uniformly charged ball of total charge Q and radius R. (Hint: assemble the ball out of successive spherical shells.)

Exercise 12. Show that the potential energy of two point charges of charge $Q/2$ separated by radius R is lower than the result you found in exercise 11. Hence it appears that it is energetically favorable to compress the ball of charge into two point charges. Is that right?

3 Conductors

Idea 8

In electrostatic conditions, $\mathbf{E} = 0$ inside a conductor, which implies the conductor has constant electric potential V. By Gauss's law, the conductor has $\rho = 0$ everywhere inside, so all charge resides on the surface.

Exercise 13. Argue that just outside a conductor, $\mathbf{E}$ is perpendicular to the surface.

Exercise 14. Is the charge density at the surface of a charged conductor generally greater at regions of higher or lower radius of curvature? To answer this question, you could consider the extreme case of a conductor made of two spheres of different radii, connected by a very long rod.

Exercise 15. Is it possible for a connected, completely isolated conductor with a positive charge to have a negative surface charge density at any point? If not, prove it. If so, sketch an example.

Exercise 16. Consider a point on the surface of a conductor with surface charge density σ.

(a) Find the electric field just outside and just inside the conductor due to the surface charge near this point.

(b) Find the electric field just outside and just inside the conductor due to all of the other surface charge.

(c) Show that the outward pressure on the charges on this patch of surface is $\sigma^2/2\epsilon_0$.

(d) More generally, show that a surface of charge density σ with electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ to its left and right experiences a force $\sigma(\mathbf{E}_1 + \mathbf{E}_2)/2$.

Idea 9: Existence and Uniqueness

In a system of conductors where the charges or potentials of each conductor are specified, there (obviously) exists a charge configuration that satisfies these boundary conditions. Moreover, this is the *unique* charge configuration.

The existence and uniqueness theorem is useful because in many cases it is difficult to directly derive the charge distributions or fields. Instead, one can simply insightfully guess an answer; then it is the correct answer by uniqueness.

Exercise 17. Basic applications of existence and uniqueness.

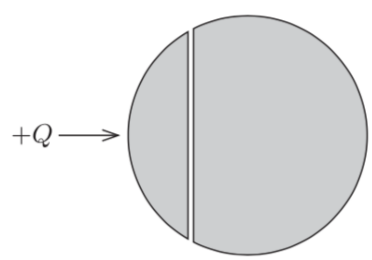
(a) Consider a conductor with an empty cavity inside. Show that the electric field is zero in the cavity.

(b) Consider a spherical conducting shell with an arbitrary charge distribution inside. Find the electric field outside the shell.

For detailed discussion, see section 2.5 of Griffiths.

Problem 13. USAPhO 2014, problem A4.

Problem 14 (MPPP 150)**.** $\star$ A solid metal sphere of radius R is divided into two parts by a planar cut, so that the outer surface area of the smaller piece is πR^2. The cut surfaces are coated with a negligibly thin insulating layer, and the two parts are put together again, so that the original shape of the sphere is restored. Initially the sphere is electrically neutral.



The smaller part of the sphere is now given a small positive electric charge Q, while the larger part of the sphere remains neutral. Find the charge distribution throughout the sphere, and the electrostatic interaction force between the two pieces of the sphere.