Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For a deeper treatment that covers normal modes in more detail, see chapters 1 through 6 of French. Jaan Kalda also has short articles on using Lagrangian-like techniques and the adiabatic theorem. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of **85** points.

1 Small Oscillations

> **Idea 1**
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> If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

> **Example 1: KK 4.13**
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> The Lennard–Jones potential
>
> $$U(r) = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right)$$
>
> is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard–Jones interaction.

> **Solution**
>
> To keep the notation simple, we'll set $\epsilon = r_0 = 1$ and restore them later. The equilibrium radius is the radius where the derivative of the potential vanishes, and
>
> $$U'(r) = -12r^{-13} + 12r^{-7} = 0$$
>
> implies that that the equilibrium radius is $r = r_0$. Because the force accelerates both of the atoms, the angular frequency is
>
> $$\omega = \sqrt{\frac{U''(r)}{m/2}}$$
>
> where $m/2$ is the so-called reduced mass. At the equilibrium point, we have
>
> $$U''(r_0) = (12)(13)r_0^{-14} - (12)(7)r_0^{-8} = 72.$$
>
> Restoring the dimensionful factors, we have $U''(r_0) = 72\epsilon/r_0^2$, so
>
> $$\omega = \frac{12}{r_0} \sqrt{\frac{\epsilon}{m}}.$$

[**3**] **Problem 1** (Morin 5.13). A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

    (a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

    (b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate angular frequency of the subsequent oscillations.

    (c) Now suppose that $x \gg R$ instead. Find the period of the resulting oscillations.

    (d) Now suppose the mass begins at rest on the plane, but slightly displaced from the center. Do oscillations occur? If so, what is the approximate frequency?

Solution.   (a) Consider a ring with radius r and thickness dr. It has mass $dM = 2\pi r\sigma \, dr$. By symmetry, the net force is towards the sheet so we only take that component. Thus we multiply the total force from the ring $dF = Gm \, dM/(x^2 + r^2)$ by $x/\sqrt{x^2 + r^2}$. We integrate from R to infinity due to the hole, giving

$$F = -\int_R^\infty Gm \frac{2\pi r x\sigma \, dr}{(x^2 + r^2)^{3/2}} = -\pi Gmx\sigma \int_{x^2+R^2}^\infty \frac{du}{u^{3/2}} = -2\pi Gm\sigma \frac{x}{\sqrt{x^2 + R^2}}$$

    (b) Approximating the force at small x gives

$$F = -2\pi Gm\sigma \frac{x}{R\sqrt{1 + x^2/R^2}} \approx -2\pi Gm\sigma x/R.$$

This implies simple harmonic oscillations,

$$\ddot{x} = -\frac{2\pi G\sigma}{R}x = \omega^2 x, \quad \omega = \sqrt{\frac{2\pi G\sigma}{R}}.$$

    (c) In this case, approximating the force at large x gives

$$F = -2\pi Gm\sigma \frac{1}{\sqrt{1 + R^2/x^2}} \approx -2\pi Gm\sigma$$

with the force always directed towards the plane. This corresponds to a uniform acceleration of $g = 2\pi G\sigma$. A ball takes time $t = \sqrt{2x/g}$ to fall from a height x. The period is thus

$$T = 4\sqrt{\frac{x}{\pi G\sigma}}.$$

    (d) This situation is unstable; the mass will just accelerate further away from the center, so no oscillations occur. This is related to Earnshaw's theorem, which we cover in **E1**, which tells us that no gravitational field (or electrostatic field) in vacuum can have a point that is stable in all directions.

[**2**] **Problem 2.** Some small oscillations questions about the buoyant force.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Find the frequency of small oscillations, assuming that a face of the glacier always remains parallel to the water surface, and that the force of the water on the glacier is always given by the hydrostatic buoyant force.

(b) A ball of radius R floats in water with half its volume submerged. Find the frequency of small oscillations, making the same assumption.

(c) There are important effects that both of the previous parts neglect. What are some of them? Is the true oscillation frequency higher or lower than the one found here?

Solution.   (a) Let $V = xL^2$ be the submerged volume, and let $V_0 = L^3$. We then have that

$$F = -\rho_w V g + \rho_i V_0 g = -\rho_w L^2 g x + \text{const. Thus, } \omega = \sqrt{\rho_w L^2 g/\rho_i L^3} = \boxed{\sqrt{\frac{\rho_w}{\rho_i}\frac{g}{L}}}.$$

(b) The density of the ball is half that of water, so its mass is $(2\pi/3)\rho_w R^3$. The "spring constant" is $\pi R^2 \rho_w g$, so

$$\omega = \sqrt{\frac{\pi R^2 \rho_w g}{(2\pi/3)\rho_w R^3}} = \sqrt{\frac{3g}{2R}}.$$

(c) The most egregious omission is that we have completely neglected the motion of the water. This clearly should add extra inertia, because the water has to move around to accommodate the moving glacier or ball, and it should be a significant change since the water is more dense than these objects. This "virtual mass" effect leads to a decrease in the oscillation frequency. However, treating the motion of the water is usually mathematically involved; we'll see some examples in **M7**.

In fact, the situation is even worse. As we'll also see in **M7**, viscosity between the object and water leads to a boundary layer of water carried along with the object. But since this boundary layer builds up over time, its thickness depends on the *entire history* of the object's motion! This is called the Basset force, and it turns Newton's second law into an "integro-differential equation", one where the second derivative of the position depends on an integral over all the past positions. It has the effect of damping the oscillations. In general, nothing in fluid dynamics is easy.

[**3**] **Problem 3.** ⟳ USAPhO 1998, problem A2. To avoid some confusion, skip part (a), since there actually isn't a nice closed-form expression for it.

[**3**] **Problem 4.** ⟳ USAPhO 2009, problem A3.

[**3**] **Problem 5.** ⌣ USAPhO 2010, problem B1.

Idea 2

A useful generalization of Newton's second law is given by generalized coordinates. Let q be any number that describes the state of the system, not necessarily a Cartesian coordinate. Suppose the energy of a system can be decomposed into two parts, a potential energy that

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depends only on q and a kinetic energy that depends only on $\dot{q}$,

$$K = K(\dot{q}), \quad V = V(q).$$

Then since energy is conserved, $d(K + V)/dt = 0$, the chain rule gives

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

We call the left-hand side the rate of change of a "generalized momentum", and the right-hand side a "generalized force". When q is a Cartesian coordinate, this recovers the usual $F = ma$.

Remark

The result above is a special case of the Euler–Lagrange equation in Lagrangian mechanics, which states that if a system is described by a Lagrangian L, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

When $L = K(\dot{q}) - V(q)$, we recover the previous result. But more generally, it might not be possible to meaningfully decompose L into a "kinetic" and "potential" piece at all! We won't use this more general form below. While it is more powerful, it is also more complicated, and if you find yourself using it for an Olympiad problem, there's probably an easier way.

Example 2

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. The reason this procedure is so complicated is that we are using two coordinates when the string really ensures the system has only a single degree of freedom.

Instead, let q be a generalized coordinate that describes "how much the string has moved". In other words, $q = 0$ initially, and for some $q > 0$, the mass M has moved down by q and the mass m has moved up by q. Then

$$K = \frac{1}{2}(m + M)\dot{q}^2, \quad V = qg(m - M)$$

and applying the idea above gives

$$\ddot{q} = \frac{M - m}{M + m}g.$$

Another way of saying this is that, from the standpoint of this generalized coordinate, the

"total force" is $(M - m)g$, and the "total inertia" is $M + m$.

[1] **Problem 6.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Solution. Let q be the amount the rope has moved. The kinetic energy is $\lambda \ell \dot{q}^2/2$. The "generalized force" is the change in gravitational potential energy per change in q, so $\partial V/\partial q = \lambda g h$. Therefore the acceleration is gh/ℓ.

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

then the oscillation frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This technique allows us to avoid dealing with possibly complicated constraint forces.

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[3] **Problem 7.** Suppose a particle is constrained to move on a curve $y(x)$ with a minimum at $x = 0$. We know that if $y(x)$ is a circular arc, then the motion is not exactly simple harmonic, for the same reason that pendulum motion is not. Find a differential equation relating y' and y, so that the motion is exactly simple harmonic for arbitrary amplitudes; you don't have to solve it. (Hint: work in terms of the coordinate s, the arc length along the curve.)

Solution. The reason s is useful is because

$$K = \frac{1}{2} m \dot{s}^2$$

exactly. Thus, the condition for simple harmonic motion is $V \propto s^2$, which means $y = ks^2$ for some constant k. Taking the derivative with respect to x of both sides, we have

$$y' = 2ks \frac{ds}{dx} = 2ks\sqrt{1 + y'^2} = 2\sqrt{ky}\sqrt{1 + y'^2}.$$

Solving this for y' gives

$$y' = \sqrt{\frac{4ky}{1 - 4ky}}.$$

which is the desired differential equation. Solving this is a bit nasty, but it turns out to be a cycloid. This fact was first discovered by Huygens, who invented the cycloidal pendulum for accurate timekeeping.

[3] **Problem 8** (Grad). A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.
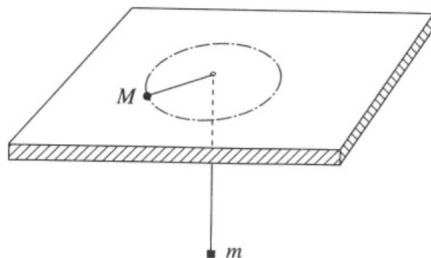
The motion is frictionless. Show that the motion is stable with respect to small changes in the height of m, and find the frequency of small oscillations.

Solution. We'll use the energy method for this one. In equilibrium we have

$$mg = \frac{Mv^2}{r} = \frac{L^2}{Mr^3}$$

where r is the radius of the circle, and L is the conserved angular momentum. If the hanging mass goes downward, then r decreases, so the tension in the string increases providing a restoring force; hence the orbit is stable.

To find the frequency of small oscillations, we'll use the energy method, with generalized coordinate r. The "kinetic energy", which is the part of the energy dependent on $\dot{r}$, is

$$T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2.$$

The "potential energy", which is the part of the energy dependent on r, is

$$V = mgr + \frac{L^2}{2Mr^2}.$$

Note that $L^2/2Mr^2$ is treated as potential energy here even though it is associated with the motion of the large mass. From the kinetic energy, we see the "effective mass" is $m_{\text{eff}} = M + m$, as one might expect. From the potential energy, we see the "effective spring constant" is

$$k_{\text{eff}} = V'' = \frac{3L^2}{Mr^4} = \frac{3mg}{R}.$$

Hence the oscillation frequency is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{3g}{R}}\sqrt{\frac{m}{m+M}}.$$

We'll see much more of this technique in **M7**.

[4] **Problem 9.** ⏱ IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

Solution. We will find the period of oscillation, we will find expressions for the kinetic energy and potential energy associated with the seiching.
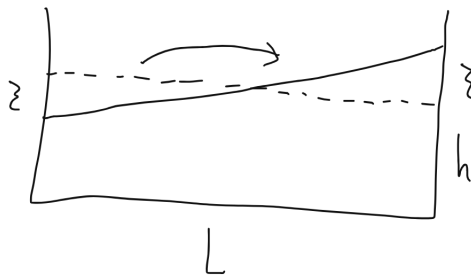
First, to find the potential energy increase when the water is displaced by ξ, note that a triangular prism of water has effectively been moved upward, as shown above. The centers of masses of these triangles are $\xi/3$ from their bases, so the center of mass of the triangle will move up a distance of $2\xi/3$. Let the width of the container by w. Then the potential energy U will be $2mg\xi/3$, where the mass of the triangular prism of water is $m = \frac{1}{2}\frac{L}{2}\xi w\rho$, so

$$U = \frac{1}{6}\rho Lwg\xi^2.$$

To find a rough estimate of the kinetic energy, consider the movement of the center of mass alone; this won't get all of the kinetic energy, but it'll get enough to get a reasonable answer. By thinking of the contribution of moving the triangle mentioned above, we have

$$\Delta x_{\text{cm}} = \frac{m(2L/3)}{M} = \frac{\frac{1}{4}L\xi w(2L/3)}{Lwh} = \frac{1}{6}\frac{L\xi}{h}.$$

$$\Delta y_{\text{cm}} = \frac{m(2\xi/3)}{M} = \frac{\xi^2}{6h}.$$

We see that Δx_{cm} dominates since ξ is small, so we focus on it. The total mass of the water is

$$M = \rho Lwh$$

and our approximation for the kinetic energy is

$$K \approx \frac{1}{2}M\dot{x}_{\text{cm}}^2 = \frac{1}{2}\rho Lwh\frac{L^2\dot{\xi}^2}{36h^2}.$$

Besides the overall side-to-side center of mass motion of the water, the water also has internal motions that can't be described just in terms of the center of mass moving. However, our result is good enough for the purposes of this problem.

Putting this together yields

$$E \approx \frac{\rho wL^3}{72h}\dot{\xi}^2 + \frac{1}{6}\rho Lwg\xi^2 = \frac{1}{2}m_{\text{eff}}\dot{\xi}^2 + \frac{1}{2}k_{\text{eff}}\xi^2$$

and thus a period of

$$T \approx 2\pi\sqrt{\frac{L^2}{12gh}}.$$

This is compatible with the data given in the problem statement, up to order-one factors. Your answer may look different, since we've made a lot of approximations throughout the problem; as long as it agrees dimensionally, with the prefactor within an order of magnitude, you can regard it as correct.

2   Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions. As a first example, we'll use the fictitious forces met in **M2**.

Example 3: PPP 79

A pendulum of length ℓ and mass m initially hangs straight downward in a train. The train begins to move with uniform acceleration a. If a is small, what is the period of small oscillations? If a can be large, is it possible for the pendulum to loop over its pivot?

Solution

The fictitious force in the train's frame due to the acceleration is equivalent to an additional, horizontal gravitational field, so the effective gravity is

$$\mathbf{g}_{\text{eff}} = -a\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

For small oscillations, we know the period is $2\pi\sqrt{L/g}$ in ordinary circumstances. By precisely the same logic, it must be replaced with

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = 2\pi\frac{\sqrt{L}}{(g^2 + a^2)^{1/4}}.$$

As a gets larger, the effective gravity points closer to the horizontal. In the limit $g/a \to 0$, the effective gravity is just horizontal, so the pendulum oscillates about the horizontal. Its endpoints are the downward and upward directions, so it never can get past the pivot.

Here's a follow-up question: if the train can decelerate quickly, how should you stop it so that the pendulum doesn't end up swinging at the end? The most efficient way is to first quickly decelerate to half speed, which, in the frame of the train, provides a horizontal impulse to the pendulum. Then wait a half-period $\pi\sqrt{L/g}$, so that the pendulum's momentum turns around, and then quickly stop, providing a second impulse that precisely cancels the pendulum's horizontal motion. This cute maneuver is useful for crane operators.

Example 4

If a spring with spring constant k_1 and relaxed length ℓ_1 is combined with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring, if the combination is in series or in parallel.

Solution

For the series combination, the new relaxed length is clearly $\ell = \ell_1 + \ell_2$. Suppose the first spring is stretched by x_1 and the second by x_2. The tensions in the springs must balance,

$$F = k_1 x_1 = k_2 x_2.$$

Thus, the new spring constant is

$$k = \frac{F}{x_1 + x_2} = \frac{k_2 x_2}{x_2(k_2/k_1 + 1)} = \frac{k_1 k_2}{k_1 + k_2}.$$

For example, if the spring is cut in half, the pieces have spring constant $2k$.

Now consider the parallel combination. In this case it's clear that the new spring constant is $k = k_1 + k_2$, since the tensions of the springs add. The new relaxed length ℓ is when the forces in the springs cancel out, so

$$k_1(\ell - \ell_1) + k_2(\ell - \ell_2) = 0$$

which implies

$$\ell = \frac{k_1 \ell_1 + k_2 \ell_2}{k_1 + k_2}.$$

[2] **Problem 10** (Morin 4.20). A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

Solution. Suppose the anchor of spring i is at $\mathbf{r}_i$. Then the force on the mass is

$$\mathbf{F} = -\sum_i k_i(\mathbf{r} - \mathbf{r}_i) = \left(\sum_i k_i\right)\mathbf{r} - \mathbf{C}$$

where $\mathbf{C}$ is some constant vector. Thus, we see that the mass undergoes simple harmonic motion with frequency $\omega = \sqrt{\frac{\sum_i k_i}{m}}$.

[3] **Problem 11** (Morin 4.22). A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Geometrically, what kind of curve is the resulting trajectory?

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

Solution.   (a) Let the anchor of the spring be the origin. Then, the force on the particle is $-k\mathbf{r} - mg\hat{\mathbf{y}} = -k(\mathbf{r} - \mathbf{r}_0)$, so it is effectively a single spring force. The motion in 2D due to a spring force is an ellipse (independent x and y oscillations of the same frequency), so the shape is a portion of an ellipse, whose center is a distance mg/k directly below the launch point.

(b) Note that the horizontal velocity takes the form $v_x(t) = (v\cos\theta)\cos(\omega t)$, because the motion in the horizontal direction is just simple harmonic. The horizontal velocity vanishes when the phase is $\pi/2$, a total of a quarter cycle.

At this time, the vertical displacement must vanish. Vertically, the motion is just simple harmonic but with an equilibrium point shifted downward by mg/k. Let the vertical velocity take the form

$$v_y(t) = v_0 \cos(\omega t + \phi).$$

The initial phase is ϕ, and just before the mass hits the ground, its vertical velocity is the opposite of the original one, so the final phase is $\pi - \phi$. So hitting the ground occurs at the same time as having a vertical velocity if the phase difference is $\pi/2$, which implies $\phi = \pi/4$.

Now, by matching the initial velocity and acceleration, we know that

$$v_0 \cos \phi = v \sin \theta, \quad -v_0 \omega \sin \phi = -g$$

Dividing these equations gives

$$\tan \phi = \frac{g}{\omega v \sin \theta}$$

and we must have $\tan \phi = 1$, so

$$v = \frac{g}{\omega \sin \theta} = \sqrt{\frac{m}{k}} \frac{g}{\sin \theta}.$$

[5] **Problem 12.** A uniform spring of spring constant k and total mass m is attached to the wall, and the other end is attached to a mass M.

(a) Show that when $m \ll M$, the oscillation frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}.$$

(b) [A] $\star$ Generalize part (a) to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses, as in the example in **M2**. It will not be possible to solve for ω in closed form, but you can get a compact implicit expression for it. Check that it reduces to the result of part (a) for small m/M, and interpret the results for large m/M. This is a challenging problem that requires almost all the techniques we've seen so far; you might want to return to it after doing section 4.)

Solution. In the $m \ll M$ case, we can assume the velocity of a piece of spring that is at position a fraction x of the total length is proportional to x. (More precisely, accounting for nonlinear stretching of the spring would contribute at higher order.) Therefore, the total kinetic energy of the spring is

$$\int_0^L \frac{1}{2}(xv_0)^2 m \, dx = \frac{1}{6} m v_0^2$$

where v_0 is the velocity of M, and L is the current length of the spring. Therefore, the total kinetic energy is $\frac{1}{2}(M + m/3)v_0^2$, so we have an effective mass of $M + m/3$. The spring is uniformly stretched at this order, so the effective spring constant is still k, giving the desired result.

Part (b) is a nice exercise in dealing with continuum systems, with many possible pitfalls – it is very easy to spend a lot of time attacking extremely complicated equations. We'll present a relatively clean approach.

First, as usual, we break the spring into pieces. Suppose the spring is made of N masses connected with small springs, and let their displacements from equilibrium be x_i. Each piece has mass m/N and each small spring has spring constant kN, as established in an earlier problem. The equation of motion for each mass is

$$\frac{m}{N}\ddot{x}_i = Nk(x_{i-1} + x_{i+1} - 2x_i).$$

We define $x_0 = 0$ and let x_N be the position of the mass M. Then its equation is different,

$$M\ddot{x}_N = Nk(x_{N-1} - x_N).$$

The spring is really continuous, so we would like to take the limit $N \to \infty$. To this end, define the displacement function $x(s, t)$ to be the continuous function with values

$$x(i/N, t) = x_i(t).$$

The argument s ranges from 0 at the left end of the spring to 1 at the right end. We'll suppress the t argument for brevity. Plugging this into the second equation above gives

$$M\ddot{x}(1) = Nk(x(1 - 1/N) - x(1)) = k\,\frac{x(1 - 1/N) - x(1)}{1/N}.$$

Upon taking the limit $N \to \infty$, the fraction on the right becomes a derivative, giving

$$M\ddot{x}(1) = -kx'(1)$$

where a prime denotes a derivative with respect to s. Similarly, in the $N \to \infty$ limit, the quantity $N^2(x_{i-1} + x_{i+1} - 2x_i)$ becomes a second derivative (check this!), so our first equation becomes

$$m\ddot{x}(i/N) = kx''(i/N).$$

Rearranging a bit and defining $\omega_0 = \sqrt{k/M}$, we have shown that

$$\frac{m}{M}\frac{\ddot{x}(s)}{\omega_0^2} = x''(s), \quad \frac{\ddot{x}(1)}{\omega_0^2} = x'(1).$$

Since we are looking for solutions where the whole spring oscillates uniformly with frequency ω, we plug in $x(s) = \cos(\omega t)f(s)$ for

$$\frac{m}{M}\frac{\omega^2}{\omega_0^2}f = -f'', \quad \frac{\omega^2}{\omega_0^2}f(1) = f'(1).$$

For simplicity we now define $\alpha = \sqrt{m/M}$ and set $\omega_0 = 1$. Solving the first equation gives

$$f(s) \propto \sin(\alpha\omega s)$$

which yields the expected nonlinear stretching of the spring. The second equation says

$$\omega^2 \sin(\alpha\omega) = \alpha\omega \cos(\alpha\omega)$$

or alternatively

$$\boxed{\tan(\alpha\omega) = \frac{\alpha}{\omega}.}$$

There are generically infinitely many solutions for ω, which correspond to the infinitely many normal modes of the spring. However, we're concerned with the lowest-frequency mode. This is the unique mode with $\alpha\omega < \pi/2$ where all the pieces of the spring are going in the same direction at the same time; it is the fundamental frequency.

The transcendental equation we have here has no closed form solution, but we can approximate it. For small α, if we Taylor expand the tangent to third order we recover the answer to the previous problem. To see this, note that

$$\alpha\omega + \frac{(\alpha\omega)^3}{3} = \frac{\alpha}{\omega}$$

which can be simplified to

$$\frac{\alpha^2}{3}\omega^4 + \omega^2 - 1 = 0.$$

If we parametrize the frequency shift by $\omega^2 = 1 + \epsilon$, then plugging in gives

$$\frac{\alpha^2}{3} + \epsilon + (\text{higher order terms}) = 0$$

which tells us that

$$\epsilon = -\frac{\alpha^2}{3} = -\frac{m}{3M}$$

which is the same result found earlier, to first order.

For large α, the right-hand side is large, so the tangent must be large. The lowest frequency mode has $\alpha\omega \approx \pi/2$. In this case it's also useful to look at all the modes, which have $\alpha\omega \approx (n + 1/2)\pi$. Restoring the units, this means

$$\omega \approx \left(n + \frac{1}{2}\right)\pi\sqrt{k/m}.$$

To understand this, note that in this limit the mass M doesn't matter; the spring acts as if it has a free end. Hence we've just found the standing wave frequencies for longitudinal waves with one fixed and one free end! The lowest frequency is the fundamental.

Jumping ahead a bit, we can compare this with some results from **W1**. The wavenumbers for these boundary conditions are

$$k_n = \left(n + \frac{1}{2}\right)\pi$$

and the wave velocity is

$$v = \sqrt{\frac{Y}{\rho}}$$

where Y is the Young's modulus, and ρ is the mass density. (If this isn't familiar, you can also derive it using dimensional analysis.) But this wave velocity can also be written as

$$v = \sqrt{\frac{kL/A}{m/LA}} = L\sqrt{\frac{k}{m}}.$$

Putting these two together using $\omega_n = vk_n$ recovers exactly the frequencies we found above! So another way of saying what we've done is that we're *derived* the expression $v = \sqrt{Y/\rho}$ for the speed of sound.

[2] **Problem 13** (PPP 77). A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

Solution. Suppose it is stretched x in the perpendicular direction, and let θ be the angle of the spring with respect to the horizontal (the original line of the springs). We see that $\tan \theta = x/L$ where L is the unstretched spring length. Now, the change in the length of one of the springs is $L(1/\cos\theta - 1) \approx L\theta^2/2$. The restoring force is then proportional to $\theta^2 \sin\theta \approx \theta^3 \propto x^3$. Therefore, the potential energy is of the form Cx^4 for some constant C. Thus, we have

$$\frac{1}{2}m\dot{x}^2 + Cx^4 = CA^4$$

where A is the amplitude of oscillation. Thus, $\dot{x} = D\sqrt{A^4 - x^4}$ for some constant D, so $dt \propto \frac{dx}{\sqrt{A^4-x^4}}$. Thus, the period is proportional to $\int_0^A \frac{dx}{\sqrt{A^4-x^4}}$, which by dimensional analysis is proportional to $1/A$. Thus, by doubling the amplitude, we halve the period, so the answer is $\boxed{T/2}$. Note that we saw a similar idea in **P1**.

[3] **Problem 14.** ⟳ USAPhO 2015, problem A3.

[3] **Problem 15.** ⟲ USAPhO 2008, problem B1.

3   Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 16.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. In this limit, show that by taking the real part, one finds an exponentially damped sinusoidal oscillation. Roughly how many oscillation cycles happen when the amplitude damps by a factor of e?

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

Solution.   (a) By setting up and solving a quadratic equation,

$$\omega_\pm = \frac{-ib \pm \sqrt{-b^2 + 4mk}}{-2m} = \frac{ib}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

(b) In this limit, we have

$$\omega_\pm \approx \frac{ib}{2m} \pm \sqrt{\frac{k}{m}}$$

in which case we have

$$e^{i\omega_\pm t} \approx e^{-bt/2m} e^{i\sqrt{k/m}\, t}$$

which is an exponentially damped oscillation. The time for a damping of a factor of e is $2m/b$, which occurs after $\sqrt{km}/\pi b$ cycles.

(c) This occurs if

$$\frac{k}{m} - \left(\frac{b}{2m}\right)^2 < 0$$

which implies $b^2 > 4mk$.

[**4**] **Problem 17.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator which experiences a driving force $F = F_0 \cos(\omega t)$. Passing to complex variables, Newton's second law is

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters, to match the initial position and velocity. Argue that it takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in problem 16.

(c) After a long time, the "transient" $A_\pm$ terms will decay away, leaving the steady state solution

$$x(t) \approx A_0 e^{i\omega t}$$

which oscillates at the same frequency as the driving. The actual position is the real part,

$$x(t) \approx |A_0| \cos(\omega t - \phi)$$

where $A_0 = |A_0| e^{-i\phi}$. Evaluate $|A_0|$ and ϕ.

(d) Sketch the amplitude $|A_0|$ and phase shift ϕ as a function of ω. Can you intuitively see they take the values they do, for ω small, $\omega \approx \sqrt{k/m}$, and ω large?

(e) There are several distinct things people mean when they speak of "resonant frequencies". Find the driving frequency ω that maximizes (i) the amplitude $|A_0|$, (ii) the amplitude of the velocity, and (iii) the average power absorbed from the driving force. (As you'll see, these are all about the same when the damping is weak, so the distinction between these isn't so important in practice.)

Solution.   (a) If we plug in $x = A_0 e^{i\omega t}$, we find the differential equation is satisfied if

$$(-m\omega^2 + ib\omega + k)A_0 = F_0,$$

which yields

$$A_0 = \frac{F_0}{(k - m\omega^2) + ib\omega}.$$

(b) This follows from linearity. If we plug this solution in, then the first term balances the driving term on the right-hand side. Then the other two terms need to satisfy the damped harmonic oscillator equation with no driving, so they're just the same as in problem 16.
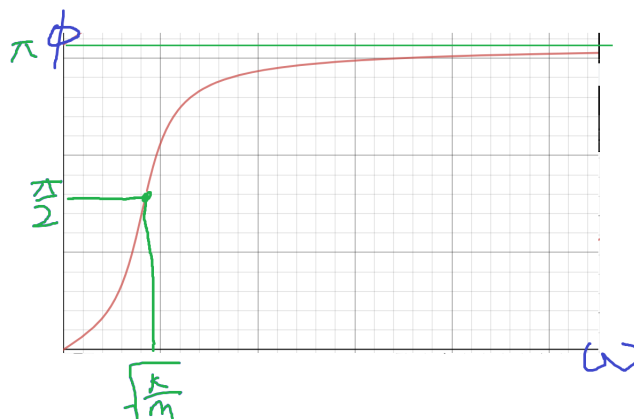
(c) The answers are
$$|A_0| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}, \quad \tan\phi = \frac{b\omega}{k - m\omega^2}.$$

(d) The amplitude is shown below, for a few values of b (called ζ here).
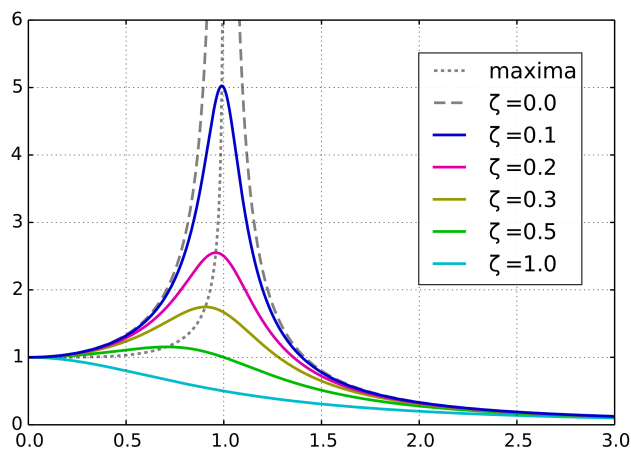
The phase shift is shown below.

This all makes physical sense. For very small frequency, we are effectively stretching the spring statically, so the amplitude approaches a constant $|A_0| = F_0/k$, and the phase shift is zero. For $\omega \approx \sqrt{k/m}$, the amplitude is high because we're driving the oscillator at the frequency it wants to oscillate at, in the absence of driving and damping. Here, a large power is absorbed from the driving force, and since $P = Fv$, that means F and v must be approximately in phase, so the phase shift between F and x is 90°. Finally, for high frequencies, the amplitude goes to zero because the mass doesn't have time to move far before the force turns around. In this case, the driving force is always the largest force acting on the mass, so F and a are in phase, so the phase shift between F and x is 180°.

(e) First, to find the maximum $|A_0|$, it suffices to minimize the square of its denominator. Setting the derivative of that quantity to zero gives
$$2b^2\omega = 2(k - m\omega^2)(2m\omega)$$
which can be solved to yield
$$\omega = \sqrt{k/m - b^2/2m^2}.$$

The amplitude of the velocity is

$$v_0 = \omega |A_0| = \frac{F_0 \omega}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} = \frac{F_0}{\sqrt{(k/\omega - m\omega)^2 + b^2}}$$

which is clearly minimized when $\omega = \sqrt{k/m}$. Finally, the rate of power dissipation is

$$P = F(t)v(t) = -F_0 v_0 \cos(\omega t) \sin(\omega t - \phi) = F_0 v_0 \cos(\omega t) \cos(\omega t + (\pi/2 - \phi)).$$

As we've just seen, v_0 is maximized at $\omega = \sqrt{k/m}$. In addition, the average value of the product of cosines is maximized when they are in phase with each other, $\phi = \pi/2$, which *also* happens when $\omega = \sqrt{k/m}$. Therefore, the maximum average power dissipation occurs at $\omega = \sqrt{k/m}$.

[3] **Problem 18.** The quality factor of a damped oscillator is defined as $Q = m\omega_0/b$, where $\omega_0 = \sqrt{k/m}$. It measures both how weak the damping is, and how sharp the resonance is.

(a) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

Then estimate Q for a guitar string.

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonance frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

For more about Q, see pages 424 through 428 of Kleppner and Kolenkow.

Solution. (a) Take $x = A\cos\omega_0 t$. In one cycle, the energy dissipated is

$$\int_0^{2\pi/\omega_0} bv \cdot v \, dt = bA^2 \omega_0 \pi,$$

so the average energy dissipated per radian is $bA^2\omega_0/2$. The average energy stored is $\frac{1}{2}m\omega_0^2 A^2$, so the ratio is $m\omega_0/b = Q$.

A typical guitar string oscillates at $\sim 100\,\text{Hz}$ for a few seconds. Hence $Q \sim 1000$.

(b) We have $|A_0| = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}$. At the edge of the range that we call the width, we have

$$m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2 = 2(b\omega_0)^2 \implies m(\omega_0^2 - \omega^2) = \pm b\omega_0,$$

so $m(\omega_0 + \omega)(\omega_0 - \omega) = \pm b\omega_0$. We have $\omega \approx \omega_0$ (to first order), so

$$2m\omega_0(\omega_0 - \omega) = \pm b\omega_0 \implies 1 - \omega/\omega_0 = \pm\frac{1}{2Q}.$$

Thus the width is approximately ω_0/Q, as desired.

The next two problems explore other ways of driving harmonic oscillators.

[2] **Problem 19.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

    (a) Move the end of the string horizontally with sinusoidal frequency f'.

    (b) Move the end of the string vertically with sinusoidal frequency f'.

    (c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just visualize it!)

Solution.   (a) In the frame of the string, this is a sinusoidal horizontal (fictitious) force, so it's just the same kind of sinusoidal driving we saw above. Resonance happens when $f' \approx f$.

    (b) In this case, there is a sinusoidal vertical force by the same reasoning. Resonance can happen when $f' \approx 2f$, in which case gravity is weaker whenever the bob is moving up and stronger whenever it is moving down. In fact, one can show that weaker resonance occurs for $f' \approx 2nf$. This is called parametric resonance, and you can find more details here.

    (c) This works as long as the impulse always comes when the object is moving to the right, i.e. in the same phase of the object's oscillation. This happens as long as the impulse's period is an integer multiple of the object's period, so $f' \approx f/n$.

[5] **Problem 20.** ☺ GPhO 2016, problem 1. Record your answers on the official answer sheet.

Solution. See the official solutions here.

4 Normal Modes

> **Idea 4: Normal Modes**
>
> A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.
>
> The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

> **Example 5**
>
> Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1 - L)$$
$$m\ddot{x}_2 = k(x_1 + L - x_2).$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u - v}{2}, \quad x_2 = \frac{u + v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega} \sin \omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega} \sin \omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega} \sin \omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 21** (Morin 4.10). Three springs and two equal masses lie between two walls, as shown.
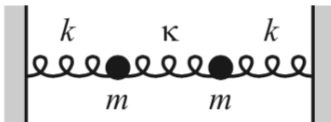
The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a \cos((\omega + \epsilon)t) \cos(\epsilon t), \quad x_2(t) \approx a \sin((\omega + \epsilon)t) \sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

Solution. The equations of motion are

$$m\ddot{x}_1 = -kx_1 - \kappa(x_1 - x_2)$$
$$m\ddot{x}_2 = -kx_2 - \kappa(x_2 - x_1).$$

Again define $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$. Adding and subtracting the two EOMs tells us that

$$m\ddot{y}_1 = -ky_1$$
$$m\ddot{y}_2 = -(k + 2\kappa)y_2.$$

The initial conditions are $y_1(0) = y_2(0) = a$ and $\dot{y}_1(0) = \dot{y}_2(0) = 0$. The solution is

$$y_1(t) = a\cos(\sqrt{k/m}\,t)$$
$$y_2(t) = a\cos(\sqrt{(k + 2\kappa)/m}\,t).$$

Solving for x_1 and x_2, we see that

$$x_1(t) = a\cos\left(\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right)\cos\left(-\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right)$$
$$x_2(t) = a\sin\left(\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right)\sin\left(-\frac{\sqrt{k/m} + \sqrt{(k+2\kappa)/m}}{2}t\right).$$

Note that

$$\frac{\sqrt{k/m} + \sqrt{(k + 2\kappa)/m}}{2} \approx \omega$$

and

$$-\frac{\sqrt{k/m} + \sqrt{(k + 2\kappa)/m}}{2} \approx \sqrt{k/m}\frac{\kappa/k}{2} = \epsilon,$$

so the result follows. We have an envelope curve of $a\cos(\epsilon t)$ and $a\sin(\epsilon t)$, and a very high frequency oscillation that matches the envelope. What this looks like is energy gradually sloshing back and forth between the masses. If the second mass begins still, it will gradually pick up energy, until the first mass begins still. Then the process repeats in reverse.

Note that without the weak spring in the middle, we would have two normal modes of equal frequency, while adding the spring causes the frequencies to split apart. This is a very common phenomenon in physics, known as "avoided crossing". For this reason, you will rarely see two acoustic modes of exactly equal frequency in a room, or two electromagnetic modes of equal frequency inside a conducting cavity, or two quantum states of the same energy, unless there's a symmetry at play.

[**3**] **Problem 22** (KK 10.11). Two identical particles are hung between three identical springs.
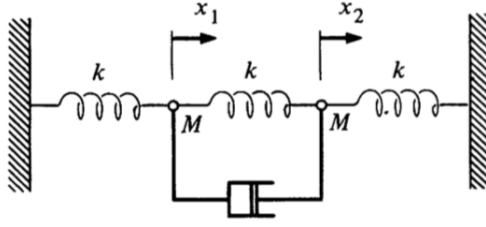
Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega} \sin \omega t$$

and evaluate ω.

Solution. The equations of motion are

$$M\ddot{x}_1 = -kx_1 - k(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2)$$
$$M\ddot{x}_2 = -kx_2 - k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1).$$

Adding the two tells us that

$$M\ddot{y}_1 = -ky_1$$

and subtracting tells us that

$$M\ddot{y}_2 = -3ky_2 - 2b\dot{y}_2.$$

Let us solve for y_1. The initial condition is $y_1(0) = 0$ and $\dot{y}_1(0) = v_0$. Thus,

$$y_1(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/M}$. After a very long time, the y_2 terms goes to 0, since it is damped. Thus, we have that $x_1 - x_2 = 0$, so $x = x_1 = x_2$, and

$$2x = \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

so

$$x_1 = x_2 = \frac{v_0}{2\omega_0} \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/M}$.

Example 6

Three identical masses are connected by three identical springs, forming an equilateral triangle in equilibrium. Describe the normal modes of the system.

Solution

Let the system be confined to the xy plane. Then there are three masses that each can move in two dimensions, giving six degrees of freedom. Since we must be able to construct the general solution by superposing normal modes, there should be six normal modes. They are:

- Uniform translation in the x or y directions. This yields two normal modes, both formally with zero frequency, since $\sin(\omega t) \propto t$ in the limit $\omega \to 0$.

- Uniform rotation about the axis of symmetry.

- A "breathing" motion where the whole triangle expands and contracts.

- A "scissoring" motion where one mass moves outward and the other two move inward.

You might think there are three scissoring normal modes, which would give us one too many. However, they are redundant: just like how the three sides of the equilateral triangle lie in a plane, these three normal modes formally lie in a plane, in the sense that you can superpose any two of them to get the third. Thus we have six normal modes, as expected. If the system can move in three-dimension space, we need three more; they are uniform translation in the z direction, and rotation about the x and y axes.

[5] **Problem 23** (Morin 4.12, IPhO 1986). N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) **[A]** ⋆ Do the same for general N. (Hint: consider the normal modes found in (a) and (b), arranged so that in each normal mode, each mass oscillates with unit amplitude but a different phase. Look at the phases and guess a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Part (c) will be useful in **X1**, where we will quantize the normal modes found here.

Solution.   (a) Let the positions of the masses along the circle be x_1 and x_2. Then

$$m\ddot{x}_1 = -k(2x_1 - 2x_2), \quad m\ddot{x}_2 = -k(2x_2 - 2x_1).$$

Adding and subtracting these equations and letting $\omega_0 = \sqrt{k/m}$ gives

$$\ddot{x}_1 + \ddot{x}_2 = 0, \quad \ddot{x}_1 - \ddot{x}_2 = -4\omega_0^2(x_1 - x_2)$$

which tells us the normal mode frequencies are zero and $2\omega_0$. These correspond to the masses uniformly rotating around the circle together, and to the two moving oppositely.

(b) Defining quantities analogously to part (a), we have

$$\ddot{x}_1 = -\omega_0^2(2x_1 - x_2 - x_3), \quad \ddot{x}_2 = -\omega_0^2(2x_2 - x_1 - x_3), \quad \ddot{x}_3 = -\omega_0^2(2x_3 - x_1 - x_2).$$

If we subtract the first two equations, we get

$$\ddot{x}_1 - \ddot{x}_2 = -3\omega_0^2(x_1 - x_2)$$

which gives a normal mode with frequency $\sqrt{3}\,\omega_0$, where the first two masses move oppositely and the third doesn't move at all. The same happens if we subtract the first and third equation, and second and third equation. Finally, if we add all three equations, we get

$$\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$$

which gives a normal mode with frequency zero. Therefore, the normal mode frequencies are zero and $\sqrt{3}\,\omega_0$.

Strangely, it seems like we have *four* normal modes even though there are only three masses! The reason is that the first three we found are redundant: if you sum any two of them, you get the third.

(c) Using the same notation as in the previous parts,

$$\ddot{x}_j = -\omega_0^2(2x_j - x_{j-1} - x_{j+1}), \quad j = 1, 2, \ldots N.$$

Solving these equations requires an inspired guess. Notice that in part (a), the zero frequency normal mode has x_1 and x_2 in phase, while the other normal mode has them π out of phase. In part (b), the zero frequency normal mode has x_1, x_2, and x_3 in phase. The other two normal modes can be expressed as having phases

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix} e^{i\omega t}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-2\pi i/3} \\ e^{-4\pi i/3} \end{pmatrix} e^{i\omega t}.$$

In other words, in each normal mode, the phase differences between adjacent masses are uniform! Therefore, we are inspired to guess

$$x_j = e^{i\omega t}\, e^{ikj}, \quad k = \frac{2\pi n}{N}$$

for integer n. Plugging this in, the differential equations simplify to

$$\omega^2 = \omega_0^2 (2 - e^{-ik} - e^{ik})$$

which simplifies to

$$\omega = 2\omega_0 \sin\left(\frac{k}{2}\right) = 2\omega_0 \sin\left(\frac{\pi n}{N}\right).$$

For $n = 0, \ldots, N - 1$, these are the normal mode frequencies.

As an aside, for $n \ll N$, note that ω_n is proportional to n, which is also proportional to the wavenumber k of the wave. This indicates that waves built out of only normal modes with $n \ll N$ travel with constant velocity, and hence satisfy the ideal wave equation. In general, systems that satisfy the ideal wave equation are often found by taking the low n limit of a system with many discrete parts. We'll see these points in more detail in **W1**.

As another aside, we could have also gotten the solution without "inspiration". As we've discussed above, guessing a complex exponential $e^{i\omega t}$ is the *general* technique when dealing with linear equations with time translation symmetry. Similarly, in this problem we considered linear equations with a discrete spatial translational symmetry, i.e. the equations stay the same upon substituting $j \to j + 1$. Thus, the general technique is to guess a complex exponential in j, which is precisely the e^{ikj} factor.

(d) When we add the one light mass, it adds a new normal mode with frequency $\sqrt{2k/m'}$, where the light mass oscillates back and forth and nothing else moves. The band of frequencies from zero to $2\omega_0$ barely changes.

(e) Naively, if we turn half the masses into light masses, we get $N/2$ modes with frequency $\sqrt{2k/m'}$, consisting of each light mass oscillating independently. But this isn't right, because we must take sinusoidal combinations of these modes to get normal modes, by the same logic as we used in the previous parts. This broadens the normal mode frequencies into a band centered around $\sqrt{2k/m'}$. Meanwhile, for the low-frequency modes, the heavy masses can't even see the light masses, so it's as if every spring has been doubled in length. We hence have a second band of normal modes with frequencies centered on $\sqrt{k/2m}$, which is nonoverlapping if $m' \ll m$.

This idea of normal mode frequencies filling dense but separated bands is crucial in solid state physics. The result of part (d) shows how "defects" in a solid can lead to isolated energy levels, outside the bands. For further discussion, see this paper.

[4] **Problem 24.** [**A**] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy K and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) The Euler–Lagrange equations for the system are

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}.$$

Using the results of part (a), write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L} A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where A is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(c) Find the normal modes and their frequencies, using the general method in section 4.5 of Morin.

Solution.   (a) To second order, the horizontal displacements of the masses are

$$x_1 = \ell\theta_1, \quad x_2 = \ell(\theta_1 + \theta_2)$$

which gives a kinetic energy of

$$K = \frac{m\ell^2}{2}((\dot{\theta}_1)^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2).$$

The vertical displacements are

$$y_1 = \ell(1 - \cos(\theta_1)), \quad y_2 = \ell(2 - \cos(\theta_1) - \cos(\theta_2))$$

and expanding the cosines to second order gives

$$y_1 = \frac{\ell}{2}\theta_1^2, \quad y_2 = \frac{\ell}{2}(\theta_1^2 + \theta_2^2)$$

which gives a potential energy of

$$V = \frac{mg\ell}{2}(2\theta_1^2 + \theta_2^2).$$

(b) The resulting Euler–Lagrange equations are

$$2\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{2g}{\ell}\theta_1, \quad \ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{\ell}\theta_2.$$

Solving the system, we find

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

straightforwardly.

(c) We must find the eigenvalues of the matrix, which obey the equation

$$(2 - \lambda)^2 - 2 = 0$$

which implies $\lambda = 2 \pm \sqrt{2}$. The normal mode amplitudes are

$$\text{high frequency:} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \quad \text{low frequency:} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

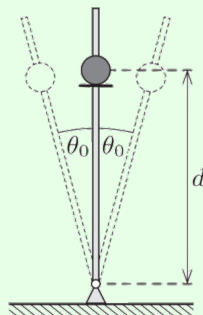and the frequencies are $\omega_\pm^2 = (g/L)(2 \pm \sqrt{2})$.

> **Remark**
>
> We mostly considered examples with two or three masses, but the techniques above work for systems with arbitrarily many degrees of freedom. However, this quickly becomes intractable unless the setup is highly symmetric, as in problem 23. Without such symmetry, a computer is generally necessary, so this sort of question won't appear on standard Olympiads. However, if you're curious, see ITPO 2016, problem 1 and Physics Cup 2021, problem 3 for examples.

5 [A] Adiabatic Change

> **Idea 5**
>
> When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

> **Example 7: MPPP 21**
>
> A small smooth pearl is threaded onto a rigid, smooth, vertical rod, which is pivoted at its base. Initially, the pearl rests on a small circular disc that is concentric with the rod, and attached to it a distance d from the rotational axis. The rod starts executing simple harmonic motion around its original position with small angular amplitude θ_0.
>
>
>
> What frequency of oscillation is required for the pearl to leave the rod?

> **Solution**
>
> The reason the pearl leaves the rod is that the normal force rapidly varies in direction, with an average upward component. If this average upward force is greater than gravity, the pearl accelerates upward and leaves the rod.
>
> In this case, the fast motion is the oscillation of the rod, while the slow motion is the rate of change of the pearl's distance from the pivot, which can be neglected during one oscillation. The pearl has horizontal displacement and acceleration
>
> $$x(t) = -d\sin\theta \approx -d\theta(t) = -\theta_0 d\sin\omega t, \quad a_x(t) = \theta\omega^2 d\sin\omega t.$$

This is supplied by the horizontal component of the normal force. The vertical component is

$$N_y = N_x \tan \theta(t) \approx m a_x(t) \theta(t) = m \theta_0^2 \omega^2 d \sin^2 \omega t.$$

Now we average over the fast motion to understand the slow motion. Since the average value of $\sin^2(\omega t)$ is $1/2$, the condition for the pearl to go up is

$$\frac{1}{2} m \theta_0^2 \omega^2 d > mg$$

which gives

$$\omega > \frac{1}{\theta_0} \sqrt{\frac{2g}{d}}.$$

Example 8

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Solution

In this case the fast motion is the oscillation of the mass, while the slow motion is the weakening of the spring. We can solve the problem by considering how the energy changes in each oscillation, due to the slight decrease in k.

Suppose that the spring constant drops in one instant by a factor of $1 - \epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1 - \epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant gradually decreases by a factor of $1 - x$ over a full cycle, with $x \ll 1$, then the energy decreases by a factor of $1 - x/2$.

The process finishes after N oscillations, where $(1 - x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1 - x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. But the energy is also kA^2, so the new amplitude is $\sqrt[4]{2} A_0$.

Amazingly, the question can also be solved in one step using a subtle conserved quantity.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular*

momentum is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k}A^2.$$

Then the final amplitude is $\sqrt[4]{2}A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But angular momentum turns out to be a special case of a more general conserved quantity, which is useful in a wide range of similar problems.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p \, dx$$

is conserved. This is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

Solution

Using conservation of energy,
$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Therefore, the curve $p(x)$ over one oscillation cycle traces out an ellipse in phase space, with semimajor and semiminor axes of $\sqrt{2mE}$ and $\sqrt{2E/k}$. The area of this ellipse is the adiabatic invariant,
$$I = \oint p \, dx = \pi\sqrt{2mE}\sqrt{2E/k} = 2\pi E\sqrt{\frac{m}{k}} \propto A^2\sqrt{km}.$$

Thus, $A \propto k^{-1/4}$ in an adiabatic change of k, recovering the answer found earlier.

Remark

The existence of the adiabatic invariant is hard to see in pure Newtonian mechanics, but it falls naturally out of the framework of Hamiltonian mechanics, which works with phase space. In fact, Hamiltonian mechanics makes a lot of theoretically useful facts easier to see.

For example, as you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually one and the same. If you'd like to learn more about Hamiltonian mechanics, see David Tong's lecture notes or chapter 15 of Morin.

[3] **Problem 25.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein replied in the affirmative. Reproduce Einstein's analysis.

Solution.   (a) Using the small angle approximation, we have

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mgL\theta^2$$

and the adiabatic invariant is

$$\oint p\,dx = L\oint p\,d\theta = mL\oint v\,d\theta.$$

On the other hand, from conservation of energy, we know that $v(\theta)$ is an ellipse with semimajor and semiminor axes $\sqrt{2E/m}$ and $\sqrt{2E/mgL}$, so

$$\oint p\,dx \propto mL\,\sqrt{E/m}\sqrt{E/mgL} = E\sqrt{\frac{L}{g}}.$$

The total energy is $E = mgL\theta_0^2/2$, so

$$\oint p\,dx \propto \theta_0^2 L^{3/2} g^{1/2}$$

which implies that when L halves, the amplitude becomes $2^{3/4}\theta_0$. Since we kept track of factors of g, this derivation also tells us what happens to the amplitude if g is slowly changed.

The most famous literary example of a pendulum with changing length appears in Edgar Allan Poe's short story, *The Pit and the Pendulum*. In the story, the narrator is strapped to a table, and sees a pendulum above him slowly moving and lengthening, bringing its razor edge toward him. Poe describes the pendulum's amplitude as initially small, but "increasing inexorably". As you can see from the solution to this problem, precisely the opposite happens. When the length goes up, the amplitude goes down, so it would just gently land on the narrator's chest.

(b) As for the case of a mass on a spring, we can add a third dimension and let the pendulum oscillate in a horizontal circle. Then the adiabatic invariant is simply

$$\oint L_z\,d\theta = 2\pi L_z \propto L_z$$

which is the angular momentum in the z-direction.

(c) Given the way we did part (a), this is immediate. The adiabatic invariant is

$$E\sqrt{\frac{L}{g}} = \frac{E}{\omega}.$$

Therefore, E/ω remains an integer multiple of $\hbar$ under adiabatic change.

[**4**] **Problem 26.** A block of mass M and velocity v_0 to the right approaches a stationary puck of mass $m \ll M$. There is a wall a distance L to the right of the puck.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by explicitly analyzing each collision. (Note that it does not suffice to just use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

(c) The adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force. Using the adiabatic theorem, infer the value of γ for a one-dimensional monatomic gas.

Solution.   (a) Let the speeds of the block and puck be v and w. Every collision, w increases by $2v$. If the block is a distance x from the wall, then a collision happens in time $2x/w$. Therefore, we have

$$\frac{\Delta w}{\Delta x} = \frac{2v}{-2xv/w} = -\frac{w}{x}.$$

Because $m \ll M$, many collisions happen. After many collisions have happened, w will be very large, so in the next collision, Δx will be small compared to x, and Δw will be small compared to w. In this case, we can approximate the finite differences with a derivative, giving

$$\frac{dw}{dx} \approx -\frac{w}{x}.$$

Separating and integrating shows that wx is conserved. We could also have arrived at this by the adiabatic theorem,

$$I = \oint p \, dx = mw(2x) \propto wx.$$

However, in the earlier collisions $(\Delta w)/w$ and $(\Delta x)/x$ aren't small, so this reasoning is invalid. For instance, wx is zero before the first collision and nonzero right after it. Thus, we must treat the first few collisions manually. Right before the second collision, we have

$$w \approx 2v_0, \quad x \approx L/3$$

by one-dimensional kinematics. Right before the third collision we have

$$w \approx 4v_0, \quad x \approx L/5$$

where for these early few collisions we are treating v as constant since $m \ll M$. It is not hard to show that right before collision $n+1$, we have $w \approx 2nv_0$ and $x \approx L/(2n+1)$, which means that after many (but not too many collisions) we have $wx \approx Lv_0$. Then, for future collisions, wx stays at this value.

The block turns around when the puck has all its energy, so

$$\frac{1}{2}Mv_0^2 = \frac{1}{2}mw^2.$$

Plugging in $wx = Lv_0$ and solving for x gives the solution, $x = L\sqrt{m/M}$.

(b) At each collision we have $\Delta w = 2v$, and energy conservation gives

$$v^2 + \frac{m}{M}w^2 = v_0^2.$$

Therefore, the number of collisions is approximately

$$n \approx \int \frac{dw}{2v} = \frac{1}{2}\int \frac{dw}{\sqrt{v_0^2 - (m/M)w^2}} = \frac{1}{2}\sqrt{\frac{M}{m}}\int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{4}\sqrt{\frac{M}{m}}.$$

Note that we didn't need to separate out the first few collisions here, even though the approximation as an integral technically doesn't work, because they're just that not important for calculating the total number of collisions, which is large. The appearance of π in this result has a nice geometric interpretation, as explained here.

(c) The analogue of pressure in one dimension is just force. The average force exerted by the puck, which we now think of as a gas molecule, is

$$F = \frac{\Delta p}{\Delta t} = \frac{2mw}{2x/w} = \frac{mw^2}{x}.$$

Meanwhile, the analogue of volume is one dimension is simply x. Then the conservation of wx says that Fx^3 is conserved, which means $\gamma = 3$. This is exactly what we would expect for a one-dimensional gas, where $C_v = kT/2$ and $C_p = 3kT/2$.

[4] **Problem 27** ($F = ma$, BAUPC). Two particles of mass m are connected by pulleys as shown.
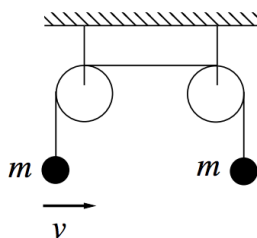
The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.

Solution. (a) The mass on the right will be higher. If the masses didn't move up or down, both would have the same average y-component of tension. But the mass on the left also has an x-component of tension, so its average magnitude of tension would be higher. This is a contradiction; to make the tension constant throughout the rope the mass on the right must rise.

(b) The tension provides the centripetal force, so

$$T = \frac{mv^2}{r} + mg\cos\theta.$$

where θ is the angle from the vertical. By energy conservation, the first term is $mgr(\cos\theta - \cos\theta_0)$ where θ_0 is the amplitude, so

$$T = (3\cos\theta - 2\cos\theta_0)mg.$$

Since the amplitude is small, we Taylor expand for

$$T = \left(1 + \theta_0^2 - \frac{3}{2}\theta^2\right)mg.$$

Again using the small angle approximation, so that the motion is approximately simple harmonic, the average value of θ^2 is $\theta_0^2/2$, so

$$\overline{T} = \left(1 + \theta_0^2 - \frac{3}{4}\theta_0^2\right)mg > mg$$

as expected.

(c) As we've seen above,

$$\overline{T} = \left(1 + \frac{1}{4}\theta^2\right)mg$$

where θ is the amplitude. Let x be the distance the right mass has risen. From the standpoint of the left mass, it is simply a pendulum whose length is being adiabatically lengthened, so by the result of problem 25, we have

$$\int \overline{T} - mg\,dx = \frac{mg}{4}\int_L^{2L} L^{3/2}\theta_0^2\,\frac{dx}{x^{-3/2}} = \frac{mg}{2}L\theta_0^2\left(1 - \frac{1}{\sqrt{2}}\right).$$

This is the increase in kinetic energy of the right mass, so setting this equal to $mv^2/2$ gives

$$v = \sqrt{gL\left(1 - \frac{1}{\sqrt{2}}\right)}\,\theta_0.$$