Mechanics VII: Gravitation

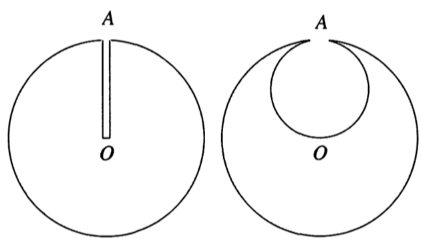
Reading: chapters 8 and 9 of Kleppner and Kolenkow cover orbits and fictitious forces. Also see chapters 7 and 10 of Morin, which go into a bit more mathematical depth.

1 Computing Fields

Idea 1

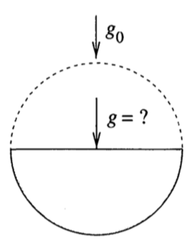
Gravitational fields obey the shell theorem and the superposition principle, which is sufficient to find the field in a variety of setups. One useful trick is to think of objects as holes as superpositions of objects without holes, and holes with negative mass.

Exercise 1 (PPP 110)**.** A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point A on its surface to the center O of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached O, where he died on impact.



However, work continued and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter AO inside the asteroid. Then a second accident occurred: another little green man similarly fell from point A to point O, and died. Find the ratio of the impact speeds of the two little green men.

Problem 1 (PPP 111)**.** The titanium-devouring little green people of the previous problem continued their excavating. As a result of their environmentally destructive activity, half of the asteroid was soon used up, as shown.



What is the gravitational acceleration at the center of the circular face of the remaining hemisphere if the gravitational acceleration at the surface of the original spherical asteroid was g_0?

Exercise 2. You are given a fixed volume of a moldable material, with a fixed density. Describe the shape it should take to maximize the gravitational field at the origin.

2 Central Potentials

Idea 2: Effective Potential

A particle moving in a central potential $V(r)$ has conserved angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, so that $L = mr^2\dot{\theta}$. The kinetic energy is

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

and substituting in the expression for L gives the total energy

$$E = \frac{1}{2}m\dot{r}^2 + \left(V(r) + \frac{L^2}{2mr^2}\right).$$

We may find $r(t)$ by treating the problem as one-dimensional, where the particle moves in the effective potential $V(r) + L^2/2mr^2$. Once we know $r(t)$, we can find $\theta(t)$ if desired by using $\dot{\theta} = L/mr^2$, and then combine to find $r(\theta)$.

Exercise 3 (KK 9.4)**.** For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$, where A has the same sign as n?

Exercise 4 (Morin 7.4)**.** A particle of mass m moves in a potential $V(r) = \beta r^k$. Let the angular momentum be L.

(a) Find the radius r_0 of the circular orbit.

(b) Find the frequency of small oscillations ω_r about this radius.

(c) Compute the ratio ω_r/ω where $\omega = \dot{\theta}$. For what values of k is the ratio rational, indicating the path closes back in on itself?

In fact, it turns out the *only* potentials for which all bound orbits are stable orbits are the cases $k = -1$ and $k = 2$ here. This is Bertrand's theorem. The idea of the proof is that we require ω_r/ω to be always rational, which by continuity means it must be constant. The only potentials for which this ratio is constant are of the form here, $V(r) \propto r^k$. Finally, to eliminate the other possible values of k we must expand to higher order to capture the corrections to simple harmonic motion.

Exercise 5. In general relativity, the potential describing a black hole of mass M is

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mr^3}.$$

Given L, find the values of the circular orbit radii. Are the orbits stable?

Problem 2 (IPhO 1989)**.** $\star$ Three non-collinear points P_1, P_2, and P_3, with masses m_1, m_2, and m_3 interact with one another gravitationally; they are isolated in free space. Let σ denote the axis going through the center of mass of the three masses, and perpendicular to the triangle $P_1P_2P_3$. What conditions should the angular velocities ω of the system (about the axis σ) and the distances

$$P_1P_2 = a_{12}, \quad P_2P_3 = a_{23}, \quad P_1P_3 = a_{13}$$

fulfill to allow the shape and size of the triangle $P_1P_2P_3$ to be unchanged during the motion of the system? Rigorously justify your answer.

3 Kepler's Laws

Everybody knows Kepler's laws holds for circular orbits. The more general statements are below.

Idea 3

Kepler's laws for a general orbit are as follows.

1. The trajectories of planets are conic sections, with a focus at the Sun.
2. The trajectories sweep out equal areas in equal times.
3. When the motion is periodic, the period T and semimajor axis a are related by $T^2 \propto a^3$.

Exercise 6. Which of these results would hold for any central potential $V(r)$, and which are specific to gravity, $V(r) \propto 1/r$?

Exercise 7. For circular orbits, prove Kepler's third law. What would it become for gravity in d spatial dimensions?

Besides Kepler's laws and conservation laws, the next most useful fact is the one below.

Idea 4

For a general orbit with semimajor axis a, the total energy is

$$E = -\frac{GMm}{2a}.$$

This also applies to parabolas, where a is infinity, and hyperbolas, where a is negative.

Exercise 8. Prove this statement for the case of elliptical orbits.

Exercise 9. A slight rewriting of this result is the vis-viva equation,

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

which is often used in rocketry. Prove this result.

Problem 3. Prove Kepler's first law for elliptical orbits of angular momentum L, total energy E, and mass m as follows.

(a) Combine the expressions for E and L to find an expression for $dr/d\theta$ in terms of r and the parameters.

(b) Change variables from r to $y = 1/r$, to find an expression for $dy/d\theta$.

(c) Show the equation is solved by

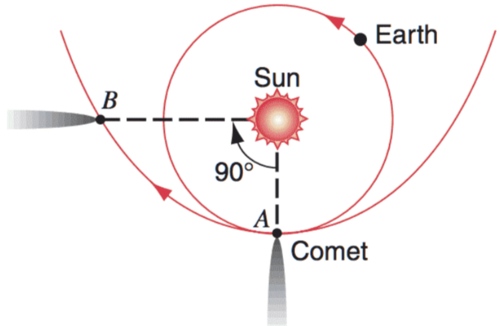
$$y(\theta) = A(1 + \epsilon \cos\theta).$$

Show that this describes an ellipse. The parameter ϵ is called the eccentricity.

If you have trouble carrying out the steps, see section 7.4 of Morin for reference.

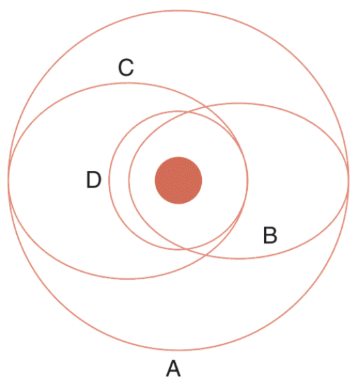
Exercise 10. Prove Kepler's third law for elliptical orbits. (Hint: use the formula $A = \pi ab$ for the area of an ellipse, and Kepler's second law.)

Exercise 11 (HRK)**.** A comet passes by Sun as shown, in a parabolic path.



How long, in years, does the comet take to get from point A to point B?

Problem 4 (HRK)**.** Several possible elliptical orbits of a satellite are shown below.



(a) Which orbit has the largest angular momentum?

(b) Which orbit has the largest total energy?

(c) On which orbit is the largest speed acquired?

Problem 5 (Physics Cup 2012)**.** A cannon at the equator fires a cannonball, which hits the North pole. Neglecting the Earth's rotation, at what angle to the horizontal should the cannonball be fired to minimize the required speed?

Problem 6 (MPPP 39)**.** The international space station orbits the Earth with a radius $R = 6700\,\text{km}$ with an orbital period of $T = 92\,\text{min}$. An astronaut stands outside the space station and jumps directly towards the Earth, with an initial velocity of $v_0 = 1\,\text{m/s}$. Find the astronaut's subsequent maximum separation from the space station. If the astronaut has no auxiliary jet pack, how long does their oxygen supply need to last for them to survive?

Idea 5: Reduced Mass

Consider two objects of mass m_1 and m_2 with positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, interacting by a central potential $V(r)$. For the purposes of computing $\mathbf{r}$ alone, we may replace this system with a single mass μ in the same central potential $V(r)$, where μ is the reduced mass, obeying

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Both systems have the same solutions for $\mathbf{r}(t)$.

Exercise 12. Consider two planets of mass m. If one planet is magically fixed in place, the other can perform a circular orbit of radius R with period T. If both planets are allowed to move, they can simultaneously perform circular orbits of radius $R/2$ about their center of mass. What is the period of this motion?

Problem 7 (MPPP 27)**.** Two permanent magnets are aligned on a horizontal frictionless table, separated by a distance d. The magnets are held in such a way so that the net force between them is attractive, and there are no torques generated.

If one of the magnets is held and the other is released, the two collide after time t_1. If instead the roles are reversed, the two collide after time t_2. If instead both magnets are released from rest, how long does it take for them to collide?

Problem 8. USAPhO 2012, problem A4.

4 Rocket Science

So far you've done some challenging problems, but they haven't exactly been rocket science. These questions literally are rocket science.

Exercise 13. A rocket begins at rest in empty space. The engine is turned on and exerts a constant thrust, so $P = Fv$ increases over time. After a long time, the power of the engine can become arbitrarily high, even though it's doing the exact same thing at all times. Where does this extra power come from?

Exercise 14. A rocket with a full fuel tank has a mass M and is initially stationary. The fuel is ejected at a rate σ, where σ has units of kg/s, at a relative velocity of u.

(a) If the rocket begins in space, show that the velocity of the rocket when its total mass is M' is

$$v = u \log \frac{M}{M'}.$$

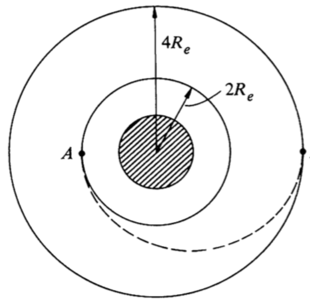
This is the Tsiolkovsky rocket equation.

(b) If the rocket begins on the ground in a gravitational field g, repeat the calculation. Do you get the best final velocity if σ is high or low?

(c) In a multi-stage rocket, an empty fuel tank detaches from the rocket once it is used up, after which a second engine starts up. This can achieve a much higher final velocity than just firing both engines at once; why?

(d) After launch, a rocket has a remaining auxiliary fuel tank. If the rocket is to perform a gravitational slingshot around Mars, when should the auxiliary fuel be burned to maximize the rocket's total final velocity? This is called the Oberth effect.

Exercise 15 (KK 9.12). A space vehicle is in a circular orbit about the Earth of radius $2R_e$ with speed v_0. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$.



An efficient way to accomplish this is by a Hohmann transfer orbit, as shown. In terms of v_0, what velocity changes are required at points A and B?

Problem 9 (PPP 88). A rocket is launched from and returns to a spherical planet of radius R so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is θ. How long does the flight take, if the period of a satellite flying around the planet just above its surface is T_0?

Problem 10. The classic cosmic speeds. For each part, evaluate your answers numerically. Neglect the rotation of the Earth about its own axis for all parts except part (b).

(a) Ignoring the rotation of the Earth, what is the minimum launch speed required to put a satellite into orbit around the Earth? This is the first cosmic speed.

(b) If you account for the rotation of the Earth, what is the new minimum speed and how should the satellite be launched?

(c) What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed.

(d) What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched?

(e) What is the minimum launch speed required for a rocket to hit the Sun?

As a check, the correct answer to part (d) is 16.7 km/s. To save time plugging numbers in, note that many answers can be written in terms of previous answers.

Problem 11 (MPPP 36). ⋆ Consider a solar system with two planets, in circular orbits with radii R_1 and $R_2 > R_1$. A space probe is planned to be launched from the first planet, and use a gravitational slingshot from the second planet to exit the solar system. What value of R_2 minimizes the initial velocity needed? Does such a planet exist in our solar system?

5 Fictitious Forces

Idea 6

Consider an inertial frame and a rotating frame with angular velocity $\boldsymbol{\omega}$. For any vector $\mathbf{V}$, the time derivatives of $\mathbf{V}$ in these two frames are related by

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{V}.$$

For example, when $\mathbf{V}$ is the position $\mathbf{r}$, we have the familiar result

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Applying this equation to the velocity $\mathbf{v}$, we find

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}.$$

The two terms on the right correspond to the Coriolis and centrifugal forces,

$$\mathbf{F}_{\text{rot}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}.$$

In the case where $\boldsymbol{\omega}$ can change, we also have the azimuthal force $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$.

Idea 7

Sometimes, the best way to deal with fictitious forces is to just go to an inertial frame instead.

Exercise 16. Early in 2018, the Tiangong-1 satellite crashed back to Earth. This occurs to satellites as friction reduces their energy.

(a) Show that for a satellite initially in a circular orbit, losing energy U to friction *increases* the kinetic energy of the satellite. By how much is it increased?

(b) In the uniformly rotating frame initially rotating with the satellite, what force is speeding up the satellite?

Exercise 17. A cylindrical space station of radius R rotates with angular velocity ω, providing artificial gravity to those standing on the floor. A child stands on the floor and throws a ball with speed v. With what angle to the vertical should the ball be thrown so that the child can catch it without moving?

Exercise 18. A projectile is dropped from height h at the equator. Counting only the Coriolis force, which direction is it deflected when it hits the ground, and by about how far? How would your answer change if you accounted for the centrifugal force?

Exercise 19. Angular momentum conservation tells us that an ice skater increases their angular velocity as they pull their arms inward. Now consider working in the skater's frame, as the skater pulls their arms in radially. Modeling the arms as point masses m a distance r from the axis, show that balancing the Coriolis and azimuthal forces yields a result equivalent to using angular momentum conservation in an inertial frame.