Lecture Notes on

String Theory

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- Henneaux and Teitelboim, *Quantization of Gauge Systems*. The most thorough book by far on the subject; if you're wondering how constrained quantization, Grassmann variables, or BRST symmetry really work, this is the place to go. Naturally, more formal than any of the other books on this list. The most relevant chapters for these notes are 1, 4, 6, and 13.

The most recent version is here; please report any errors found to kzhou7@gmail.com.

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1 Preliminaries

1.1 Miscellanea

As a first pass, we'll follow Zwiebach's text. In this first section, we establish some conventions.

- There are three fundamental dimensions: mass, length, and time. Charge is not a independent dimension; this is clearest in the cgs system, where the Coulomb force law is $F = q_1 q_2 / r^2$. That is, one can define charges purely in terms of the forces they produce. In the SI system, all occurrences of the unit of charge in measurable quantities are canceled out by ϵ_0 or μ_0.
- String theory is said to have no adjustable parameters. This means that it has no dimensionless parameters; there is a single dimensionful parameter, the string length ℓ_s. When string theory was considered as a theory of hadrons, ℓ_s was thought to be on the nuclear scale, but now it is viewed as much smaller.
- We use the $(-+++)$ metric convention, but define the interval ds^2 to measure proper time,
$$-ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$
- We will perform the quantization of the relativistic string in light cone coordinates, where
$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1).$$
The other coordinates stay the same, so in x^μ, the μ index runs over $(+, -, 2, 3)$. The light cone coordinate axes are tilted so that they lie on the light cone, e.g. a photon moving to the right has $x^- = 0$.
- The metric in light cone coordinates is
$$-ds^2 = -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2, \quad \eta_{\mu\nu} = \begin{pmatrix} & -1 & & \\ -1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$
For example, we have
$$a_+ = -a^-, \quad a_- = -a^+, \quad a^\mu b_\mu = -a^- b^+ - a^+ b^- + a^2 b^2 + a^3 b^3.$$
- None of the coordinates are true time coordinates, but we will conventionally think of x^+ as the light cone time coordinate. This isn't completely unnatural, since $dx^\pm / d\tau > 0$ for almost all particles.
- We think of dx^-/dx^+ as a 'light cone velocity'. For a particle with speed v, the light cone velocity is $(1-v)/(1+v)$. In particular, it is zero for a massless particle moving to the right, infinite for a massless particle moving to the left, and one when $v = 0$.
- Next we must define the light cone energy. Energy is the conjugate variable to time, and
$$p_\mu x^\mu = p_+ x^+ + p_- x^- + p_2 x^2 + p_3 x^3$$
which motivates us to define the light cone energy as $-p_+ = p^-$. Defining energy as a conjugate variable is useful as, e.g. the Schrodinger equation $i\,\partial\psi/\partial x^0 = p^0 \psi$ becomes $i\,\partial\psi/\partial x^+ = p^- \psi$ in lightcone coordinates.

Next, we consider the possibility of extra spatial dimensions. We maintain only one time dimension, since it is difficult to construct a consistent theory with more than one.

- The extra dimensions can be topologically nontrivial; for example, for a single dimension we may identify $x \sim x + 2\pi R$. The interval $0 \leq x < 2\pi R$ is a fundamental domain for this identification, as every point is identified with exactly one point in the fundamental domain. We then construct the space by identifying points on the boundary, getting a circle.
- Sometimes identifications have fixed points. The resulting space is not a manifold, since it is singular at the fixed point, but an orbifold. For example, identifying $x \sim -x$ gives the half-line $x \geq 0$, called the $\mathbb{R}^1/\mathbb{Z}_2$ orbifold.
- As another example, consider the identification $z \sim e^{2\pi i/N} z$ in the complex plane, giving the $\mathbb{C}/\mathbb{Z}_N$ orbifold. It is a cone which is singular at its vertex; one fundamental domain is $0 \leq \theta < 2\pi/N$.
- Physics on spaces with generic singularities is typically complicated and possibly even inconsistent. Orbifolds are interesting because they have "tractable" singularities, so we may quantize strings on them.
- For point particles, a small compactified dimension creates new energy levels, but they are very high if the dimension is small. By contrast, for a string, new low-lying states can appear if the dimension is much *smaller* than the string length, corresponding to the string wrapping around it. This is a consequence of T duality, as we'll see.
- Compactified dimensions lead to subtleties with gauge theory. It can be the case that two configurations with the same fields are not related by a gauge transformation; then we must consider the states physically distinct. It also also be the case that some configuration of fields cannot be realized by a potential, even if we allow the potential be defined on patches and related between patches with gauge transformations; such states are forbidden.

Note. A quick review of the nonrelativistic string. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2.$$

When we attempt to extremize the action, we run into a boundary term

$$\delta S \supset -T \int dt \, \left(\frac{\partial y}{\partial x}\delta y\right)\Bigg|_{x=0}^{x=a}.$$

To remove this unwanted boundary term, we need to apply boundary conditions. One acceptable boundary condition is Neumann boundary conditions (free ends), where

$$\left.\frac{\partial y}{\partial x}\right|_{x=0,a} = 0.$$

Alternatively, we could use Dirichlet boundary conditions (fixed ends),

$$\left.\frac{\partial y}{\partial t}\right|_{x=0,a} = 0$$

which ensures that y is constant on the boundaries, and hence $\delta y = 0$ there. A third alternative is that the string is closed, so that there are no boundaries at all.

In the initial days of string theory, open strings were given Neumann boundary conditions, but it was later realized that they could have Dirichlet boundary conditions if they attached to an extended object called a Dp-brane, where D stands for Dirichlet and p is the number of spatial dimensions. (One can think of Neumann boundary conditions as the special case where the D-brane fills all space.) Remarkably, it turns out that D-branes are physical objects in their own right, and arise naturally from string theory without being introduced by hand.

Finally, it will be useful later to consider the conjugate momenta,

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \frac{\partial y}{\partial t}, \quad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'} = -T \frac{\partial y}{\partial x}.$$

Here $\mathcal{P}^t$ is simply the usual momentum density in the y-direction. The equation of motion is

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0.$$

In this notation, Neumann and Dirichlet boundary conditions set $\mathcal{P}^x = 0$ and $\mathcal{P}^t = 0$ respectively.

Note. Gravity and electromagnetism in higher dimensions. We will use

$$\mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad \mathrm{vol}(B^d) = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$

In d spatial dimensions, Gauss's law continues to hold, so the field of a point charge is

$$E(r) = \frac{1}{\mathrm{vol}(S^{d-1})} \frac{q}{r^{d-1}}$$

in cgs units. As for gravitation, we have

$$g(r) = \frac{Gm}{r^2}$$

in $D = 4$ spacetime dimensions. More generally, the gravitational field is determined by

$$\nabla^2 V^{(D)} = 4\pi G^{(D)} \rho, \quad \mathbf{g} = -\nabla V$$

where $G^{(D)}$ is the gravitational constant in D dimensions. The Planck length in D dimensions is related by dimensional analysis,

$$\frac{\hbar}{c^3} = \frac{\ell_P^2}{G} = \frac{(\ell_P^{(D)})^{D-2}}{G^{(D)}}.$$

In order to relate gravitational constants in different dimensions, we need to relate densities in different dimensions. Consider a single small compactified dimension of length ℓ_C. Then for distances much greater than ℓ_C we have the coarse-grained mass density

$$\rho^{(4)} = 2\pi R \rho^{(5)}.$$

Plugging this into Poisson's equation, we conclude

$$\frac{G^{(5)}}{G} = \ell_C$$

and more generally for multiple compactified dimensions with volume V_C,

$$\frac{G^{(D)}}{G} = V_C.$$

For a D-dimensional theory where each compactified dimension has length ℓ_C,

$$(\ell_P^{(D)})^{D-2} = (\ell_P)^2 \frac{G^{(D)}}{G} = (\ell_P)^2 (\ell_C)^{D-4}, \quad \ell_C = \ell_P^{(D)} \left(\frac{\ell_P^{(D)}}{\ell_P} \right)^{\frac{2}{D-4}}.$$

Intuitively, in the presence of compactified dimensions, gravity appears much weaker than it actually is, leading to an underestimate of the fundamental Planck length $\ell_P^{(D)}$. In particular, for two compactified dimensions ($D = 6$) we could have a TeV-scale Planck energy if $\ell_C \sim 10^{-5}$ m.

In the context of string theory, the macroscopic dimensions form a D3-brane; open strings must terminate on this brane while closed strings do not. Electromagnetism and other forces are associated with open strings and hence are not affected by the extra dimensions, while gravity is associated with closed strings. Then the hypothesis of large extra dimensions could only be tested by measuring gravitational effects at short distances, which is extremely difficult. It predicts the appearance of many new, stringy excitations at a scale much lower than the conventional Planck scale.

Note. The study of higher dimensions began with Kaluza-Klein theory, which is essentially general relativity in five dimensions. Upon compactifying the fifth dimension, and restricting to four dimensions, the metric tensor g_{MN} (with $M, N = 0, \ldots, 4$) decomposes into the ordinary metric $g_{\mu\nu}$, a massless vector field $g_{\mu 4}$ interpreted as the electromagnetic field, and a massless scalar field g_{44}. This scalar field was originally thought to be problematic, but was later incorporated into Brans-Dicke theory, an extension of general relativity. In string theory, the dilaton plays a similar role.

1.2 Classical Strings

We now arrive at the classical relativistic string.

- A relativistic string traces out a two-dimensional surface in spacetime, called a worldsheet. The Nambu-Goto action is the area of the worldsheet.
- As a warmup, consider a surface in space, parametrized as $\mathbf{x}(\xi^1, \xi^2)$. Since the area spanned by two vectors is the determinant of the 2×2 matrix that contains them,

  $$A = \int d\xi^1 d\xi^2 \, \sqrt{(\partial_1 \mathbf{x} \cdot \partial_1 \mathbf{x})(\partial_2 \mathbf{x} \cdot \partial_2 \mathbf{x}) - (\partial_1 \mathbf{x} \cdot \partial_2 \mathbf{x})^2}$$

  where ∂_i is a derivative with respect to ξ^i.
- It is simpler to express this in terms of the induced metric on the worldsheet,

  $$ds^2 = g_{ij} \, d\xi^i d\xi^j, \quad g_{ij} = \partial_i \mathbf{x} \cdot \partial_j \mathbf{x}.$$

  This is the metric in the (ξ^1, ξ^2) coordinates, and the area is

  $$A = \int d\xi^1 d\xi^2 \, \sqrt{g}$$

  which is manifestly reparametrization invariant.

- Now, for a spacetime surface, we parametrize the worldsheet as $X^\mu(\tau, \sigma)$. Note that conventionally one calls both the domain and image of this map the worldsheet, though we will stick to the latter usage. The letter X is capitalized to avoid confusing the string coordinates X^μ with spacetime coordinates x^μ.

- The parameter τ is roughly related to time on the string, while σ is roughly related to position along the string. As such, we define
$$\dot{X} = \frac{\partial X}{\partial \tau}, \quad X' = \frac{\partial X}{\partial \sigma}$$
and we require $\dot{X}^0 \neq 0$. We will usually choose X' to be spacelike and $\dot{X}$ to be timelike or null. It is tempting to identify $\dot{X}$ with the velocity of a piece of the string, but this is inappropriate as none of the points on the string are distinguishable.

- By analogy with the spatial case, the area is
$$A = \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}.$$
This is an area in Lorentzian signature, which is somewhat less intuitive. There is a minus sign relative to the spatial case so the argument of the square root is positive, and the area is zero if the string is moving transverse to itself at the speed of light.

- The Nambu-Goto action is
$$S = -T_0 \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$$
where T_0 is called the string tension. The easiest way to check reparametrization invariance is again to introduce a metric on the worldsheet,
$$-ds^2 = \eta_{\mu\nu} dX^\mu dx^\nu = \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta, \quad \gamma_{\alpha\beta} = \frac{\partial X}{\partial \xi^\alpha} \cdot \frac{\partial X}{\partial \xi^\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix}.$$
Then the Nambu-Goto action can be written in the manifestly reparametrization invariant form
$$S = -T_0 \int d\tau d\sigma \sqrt{-\gamma}.$$

Now we turn to the equations of motion.

- Writing the Nambu-Goto action as the integral of a Lagrangian density as usual, we define the canonical momenta
$$P_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T_0 \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}, \quad P_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T_0 \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}.$$
Throwing away boundary terms, the Euler-Lagrange equation is
$$\frac{\partial P_\mu^\tau}{\partial \tau} + \frac{\partial P_\mu^\sigma}{\partial \sigma} = 0$$
which is quite complicated in terms of $\dot{X}$ and X'.

- As for the nonrelativistic string, the derivation only goes through if we can ignore boundary terms. The τ boundary terms vanish by fixing the initial condition $X^\mu(\tau_i, \sigma)$ and final condition $X^\mu(\tau_f, \sigma)$, leaving the σ boundary term,

$$\delta S \supset \int d\tau \, (\delta X^\mu \, P^\sigma_\mu)\Big|_0^{\sigma_1}.$$

- In the case of a closed string, these conditions are automatically satisfied by the periodicity of X^μ. For an open string, we may remove these terms by imposing Dirichlet boundary conditions,

$$\frac{\partial X^\mu}{\partial \tau}\bigg|_{\sigma=\sigma_*}$$

  or the free endpoint condition,

$$P^\sigma_\mu\Big|_{\sigma=\sigma_*} = 0.$$

  The Dirichlet boundary condition cannot be used for $\mu = 0$, so we must use the free endpoint condition there. The free endpoint condition is analogous to a Neumann boundary condition. We will show later that the Dirichlet boundary condition implies $P^\tau_\mu = 0$ at the string endpoints.

- In order to simplify the equations, we work in the static gauge, where

$$\tau = t$$

  where t is the time coordinate. This is called the static gauge because constant τ slices match constant t slices. We normalize σ by

$$\sigma \in [0, \sigma_1] \text{ for open string}, \quad \sigma \in [0, \sigma_c] \text{ for closed string}.$$

  In the static gauge we simply have

$$X' = \left(0, \frac{\partial \mathbf{X}}{\partial \sigma}\right), \quad \dot{X} = \left(1, \frac{\partial \mathbf{X}}{\partial t}\right).$$

Example. A stretched string. Suppose a string is static along the x^1 axis, with endpoints at $x^1 = 0$ and $x^1 = a$. Then $X^1(f, \sigma) = f(\sigma)$, and we have

$$(X')^1 = f'(\sigma), \quad (\dot{X})^0 = 1$$

with all other elements zero. Note that $f(\sigma)$ must assign every point on the string a unique parameter value, so $f'(\sigma) > 0$. The action is simply

$$S = -T_0 \int dt \int d\sigma \sqrt{f'^2} = -T_0 a \int_{t_i}^{t_f} dt.$$

Since generically $L = T - V$ and the kinetic energy vanishes here, this may be interpreted as a potential $T_0 a$, justifying the interpretation of T_0 as a tension. The equation of motion is

$$\frac{\partial P^\sigma_\mu}{\partial \sigma} = 0, \quad P^\sigma_\mu = -T_0 \frac{X'_\mu}{f'}$$

Then $P^\sigma_\mu = -T_0 \delta_{\mu 1}$ and hence the equation of motion is satisfied.

The longitudinal velocity of a string is not meaningful, but we can get further insight by writing the action in terms of transverse velocity.

- Let s measure length along the string, so that $\partial\mathbf{X}/\partial s$ is a spatial unit vector. We then project out the longitudinal part of the string velocity $\partial\mathbf{X}/\partial t$, defining

$$\mathbf{v}_\perp = \frac{\partial\mathbf{X}}{\partial t} - \left(\frac{\partial\mathbf{X}}{\partial t}\cdot\frac{\partial\mathbf{X}}{\partial s}\right)\frac{\partial\mathbf{X}}{\partial s}, \quad v_\perp^2 = \left(\frac{\partial\mathbf{X}}{\partial t}\right)^2 - \left(\frac{\partial\mathbf{X}}{\partial t}\cdot\frac{\partial\mathbf{X}}{\partial s}\right)^2.$$

  The Nambu-Goto action can then simply be written as

$$S = -T_0\int dt\int ds\,\sqrt{1-v_\perp^2}.$$

- This is closely analogous to the relativistic point particle action. To make the analogy even clearer, we imagine a string as starting with an infinitesimal length. It takes work $T_0 a$ to stretch it to a length a, which gives it a mass $T_0 a$. Hence T_0 is the mass per length of the string.

- A somewhat nasty calculation shows that

$$P^{\sigma\mu} = -\frac{T_0}{\sqrt{1-v_\perp^2}}\left(\left(\frac{\partial\mathbf{X}}{\partial s}\cdot\frac{\partial\mathbf{X}}{\partial t}\right)\dot{X}^\mu + \left(1-\left(\frac{\partial\mathbf{X}}{\partial t}\right)^2\right)\frac{\partial X^\mu}{\partial s}\right)$$

  as well as

$$P^{\sigma 0} = -T_0\frac{\left(\frac{\partial\mathbf{X}}{\partial s}\cdot\frac{\partial\mathbf{X}}{\partial t}\right)}{\sqrt{1-v_\perp^2}}, \quad P^{\tau 0} = T_0\frac{ds}{d\sigma}\frac{1}{\sqrt{1-v_\perp^2}}, \quad P^{\tau i} = T_0\frac{ds}{d\sigma}\frac{v_\perp^i}{\sqrt{1-v_\perp^2}}.$$

- Now consider the motion of the endpoints of an open string; these are distinguished points and hence have an unambiguous velocity. At the endpoints, P^σ_μ must vanish, and hence for $P^{\sigma 0}$ to vanish we must have $(\partial\mathbf{X}/\partial s)\cdot(\partial\mathbf{X}/\partial t) = 0$. Then the string endpoints move transverse to the string. Plugging this back into the general expression for $P^{\sigma\mu}$, we see that $(\partial\mathbf{X}/\partial t)^2 = 1$, so the endpoints move at the speed of light.

- Finally, by writing the Lagrangian as a function of $\partial_s\mathbf{X}$ and $\partial_t\mathbf{X}$ and performing a Legendre transformation with respect to the canonical momentum $\mathbf{P} = \partial\mathcal{L}/\partial(\partial_t\mathbf{X})$, we find

$$H = \int\frac{T_0\,ds}{\sqrt{1-v_\perp^2}}$$

  which is precisely the kinetic energy of the string, with rest mass T_0 per unit length. Similarly, integrating $P^{\tau i}$ gives the momentum. Note that the speed of waves on a string is $v = \sqrt{T_0/\rho}$ where ρ is the mass density, so excitations on the string travel at the speed of light.

Now we choose a useful parametrization for σ.

- One could replace the parameter σ with s, but it is more convenient to use a parameter with a fixed range. Note that in static gauge, the spatial configuration of a string is described by $\mathbf{X}(\sigma, t)$. This yields a two-dimensional spatial surface, which we'll call the string surface; this should not be confused with the worldsheet.

- We know that $\partial \mathbf{X}/\partial t$ is a unit vector on the string surface, so we choose σ so that

$$\frac{\partial \mathbf{X}}{\partial \sigma} \cdot \frac{\partial \mathbf{X}}{\partial t} = 0, \quad \mathbf{v}_\perp = \frac{\partial \mathbf{X}}{\partial t}.$$

  Since s and σ are related to a scaling, we also have $(\partial \mathbf{X}/\partial s) \cdot (\partial \mathbf{X}/\partial t) = 0$.

- As a result, our expressions for the momenta simplify,

$$P^{\tau\mu} = T_0 \frac{ds}{d\sigma} \frac{\partial X^\mu/\partial t}{\sqrt{1 - v_\perp^2}}, \quad P^{\sigma\mu} = -T_0 \sqrt{1 - v_\perp^2} \frac{\partial X^\mu}{\partial s}.$$

  In other words, the simplifications that we previously found held for the string endpoints hold everywhere on the string for this parametrization.

- Now we consider the equations of motion again. The $\mu = 0$ component of the equation of motion is simply

$$\frac{\partial P^{\tau 0}}{\partial t} = 0.$$

  However, $P^{\tau 0}$ is simply the energy density of the string, per unit σ. The conservation of this quantity makes sense because, unlike strings that actually exist, the string motion here is purely transverse, so the tension does zero work on any segment with endpoints of fixed σ. If the transverse speed of such a segment changes, the energy comes from the individual stretching or shrinking of that segment.

- The spatial components of the equation of motion read

$$\frac{\partial}{\partial \sigma} \left(T_0 \sqrt{1 - v_\perp^2} \frac{\partial \mathbf{X}}{\partial s} \right) = T_0 \frac{ds/d\sigma}{\sqrt{1 - v_\perp^2}} \frac{\partial \mathbf{v}_\perp}{\partial t}$$

  which rearranges to

$$\frac{T_0}{\sqrt{1 - v_\perp^2}} \frac{\partial \mathbf{v}_\perp}{\partial t} = \frac{\partial}{\partial s} \left(T_0 \sqrt{1 - v_\perp^2} \frac{\partial \mathbf{X}}{\partial s} \right).$$

  This is precisely the usual wave equation, if we interpret T_0 as the tension in the rest frame of a piece of string. Then the factor of $\sqrt{1 - v_\perp^2}$ on the right simply reflects the Lorentz transformation of force, while the factor of γ on the left reflects the fact that moving objects are harder to turn by a factor of γ.

- This also physically explains why the endpoints of an open string must move at the speed of light. The tension on a real open string goes to zero at endpoints because the endpoints have no mass and hence can experience no force. But the proper tension in these abstract strings is constant, so we only avoid a divergence if it Lorentz transforms to zero at the ends.

- We have still not used up all the freedom in σ parametrization. We already know that the energy of a piece of string whose endpoints have fixed σ is constant. Hence it is consistent to scale σ so the energy density is constant,

$$d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2}} = \frac{dE}{T_0}.$$

Then we have $\sigma \in [0, \sigma_1]$ where $\sigma_1 = E/T_0$. This is equivalent to

$$\left(\frac{\partial \mathbf{X}}{\partial \sigma}\right)^2 + \left(\frac{\partial \mathbf{X}}{\partial t}\right)^2 = 1.$$

- In this case, the equation of motion reduces to

$$\frac{\partial^2 \mathbf{X}}{\partial t^2} = \frac{\partial^2 \mathbf{X}}{\partial \sigma^2}.$$

  The canonical momenta and boundary conditions are now

$$P^{\tau\mu} = T_0 \frac{\partial X^\mu}{\partial t}, \quad P^{\sigma\mu} = -T_0 \frac{\partial X^\mu}{\partial \sigma}, \quad \frac{\partial \mathbf{X}}{\partial \sigma} = 0 \text{ at endpoints.}$$

  Note that we no longer have $1/\sqrt{1 - v_\perp^2}$ Lorentz factors, because we're now parametrizing by energy rather than by length.

Note. As shown in the lecture notes on General Relativity, energy density and pressure contribute equally to gravitational attraction. A string has a negative pressure due to its tension, which is equal to its mass density, so a long static string would exert no gravitational force. Cosmic strings would instead be detected by the deficit angles they produce, which would yield multiple images of distant objects.

1.3 String Motion

With the wave equation, we may now solve for the general motion of open and closed strings.

- A solution to the wave equation can always be written as a superposition of "left moving" and "right moving" solutions,

$$\mathbf{X}(t, \sigma) = \frac{1}{2}(\mathbf{F}(t + \sigma) + \mathbf{G}(t - \sigma)).$$

  The boundary condition at the $\sigma = 0$ endpoint demands that $\mathbf{F}' = \mathbf{G}'$, so they differ by a constant. Hence the general solution can be written as

$$\mathbf{X}(t, \sigma) = \frac{1}{2}(\mathbf{F}(t + \sigma) + \mathbf{F}(t - \sigma)).$$

- Next, consider the boundary condition at $\sigma = \sigma_1$, which gives

$$\mathbf{F}'(t + \sigma_1) = \mathbf{F}'(t - \sigma_1).$$

  This implies that $\mathbf{F}$ is quasi-periodic,

$$\mathbf{F}(u + 2\sigma_1) = \mathbf{F}(u) + 2\sigma_1 \mathbf{v}_0.$$

  To interpret $\mathbf{F}$, note that $\mathbf{X}(t, 0) = \mathbf{F}(t)$, so the curve $\mathbf{F}(u)$ traces out the motion of the $\sigma = 0$ endpoint, and $\mathbf{v}_0$ is its average speed.

- By adding and subtracting the results
$$\left(\frac{\partial \mathbf{X}}{\partial \sigma}\right)^2 + \left(\frac{\partial \mathbf{X}}{\partial t}\right)^2 = 1, \quad \frac{\partial \mathbf{X}}{\partial \sigma} \cdot \frac{\partial \mathbf{X}}{\partial t}$$
we have the equivalent set of two constraints
$$\left(\frac{\partial \mathbf{X}}{\partial \sigma} \pm \frac{\partial \mathbf{X}}{\partial t}\right)^2 = 1.$$
Plugging this into our general form shows that
$$\left|\frac{d\mathbf{F}}{du}\right|^2 = 1$$
so that u can be interpreted as a length parameter along the curve $\mathbf{F}(u)$. This constraint is equivalent to demanding the endpoint $\sigma = 0$ moves at the speed of light.

- We now consider the motion of a closed string. This is a bit more complicated because we can't use boundary conditions to eliminate $\mathbf{G}$. We start with the general solution
$$\mathbf{X}(t, \sigma) = \frac{1}{2}(\mathbf{F}(u) + \mathbf{G}(v)), \quad u = t + \sigma, \quad v = t - \sigma.$$
We may solve for the derivatives of $\mathbf{F}$ and $\mathbf{G}$ in terms of $\mathbf{X}$,
$$\frac{\partial \mathbf{X}}{\partial \sigma} + \frac{\partial \mathbf{X}}{\partial t} = \mathbf{F}'(u), \quad \frac{\partial \mathbf{X}}{\partial \sigma} - \frac{\partial \mathbf{X}}{\partial t} = -\mathbf{G}'(v).$$
As a result, the parametrization constraints give
$$\left|\frac{d\mathbf{F}}{du}\right|^2 = \left|\frac{d\mathbf{G}}{dv}\right|^2 = 1.$$

- Instead of a boundary condition, we have a periodicity condition $\sigma \sim \sigma + \sigma_1$, which means that $\mathbf{X}(t, \sigma + \sigma_1) = \mathbf{X}(t, \sigma)$. This is equivalent to
$$\mathbf{F}(u + \sigma_1) - \mathbf{F}(u) = \mathbf{G}(v) - \mathbf{G}(v - \sigma_1).$$
That is, both $\mathbf{F}$ and $\mathbf{G}$ are quasi-periodic with the same constant.

- One interesting generic phenomenon is the formation of cusps. Note that $\mathbf{F}'$ and $\mathbf{G}'$ are periodic functions on the unit sphere. Suppose that $\mathbf{F}'(u_0) = \mathbf{G}'(v_0)$, with corresponding coordinates (t_0, σ_0). Without loss of generality, we shift to set $t_0 = \sigma_0 = 0$ and hence $u_0 = v_0 = 0$.

- Now we Taylor expand the shape of the string about this point,
$$\mathbf{X}(0, \sigma) - \mathbf{X}(0, 0) = \sigma \frac{\partial \mathbf{X}}{\partial \sigma} + \frac{\sigma^2}{2} \frac{\partial^2 \mathbf{X}}{\partial \sigma^2} + \ldots.$$
We note that
$$\frac{\partial \mathbf{X}}{\partial \sigma} = \frac{1}{2}(\mathbf{F}' - \mathbf{G}') = \mathbf{0}.$$
Therefore, the leading term is quadratic, which means we have a cusp; the string enters in one direction and exits along the same direction. Generically cusps will appear and disappear periodically throughout the string. They are thought to be efficient sources of gravitational waves for cosmic strings.

Example. Consider a straight string of length ℓ rotating with angular velocity ω, with its midpoint at the origin of the xy plane. We can infer $\mathbf{F}(u)$ from the endpoint motion,

$$\mathbf{F}(u) = \frac{\ell}{2}(\cos\omega u, \sin\omega u).$$

Periodicity requires $\omega = \pi m/\sigma_1$ for an integer m, and

$$\mathbf{X}(0,\sigma) = \frac{\mathbf{F}(\sigma) + \mathbf{F}(-\sigma)}{2} = \frac{\ell}{2}(\cos(\pi m\sigma/\sigma_1), 0).$$

For $m > 1$, the string traces over itself multiple times, so we focus on the case $m = 1$. Since the $\sigma = 0$ endpoint moves at the speed of light,

$$\ell = \frac{2}{\omega} = \frac{2\sigma_1}{\pi} = \frac{2}{\pi}\frac{E}{T_0}$$

so the total energy is $(\pi/2)T_0\ell$. The tension provides $T_0\ell$, while the rest is due to the kinetic energy of the string. The complete solution as a function of time is

$$\mathbf{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}\left(\cos\frac{\pi t}{\sigma_1}, \sin\frac{\pi t}{\sigma_1}\right).$$

We now apply Noether's theorem to find conserved quantities.

- The results of Noether's theorem will look somewhat different. Usually, a classical field maps from spacetime, $\phi\colon \mathbb{R}^4 \to N$ where N is the field space. However, in string theory we deal with maps into spacetime, $X^\mu\colon M \to \mathbb{R}^4$ where M is the worldsheet. Hence the dynamics of a single string can be thought of as a two-dimensional field theory with a four-component field X.

- The action is
$$S = \int d\xi^i\, \mathcal{L}(\partial_i X^\mu), \quad (\xi^0, \xi^1) = (\tau, \sigma).$$
The fields X^μ have a continuous shift symmetry $\delta X^\mu = \epsilon^\mu$, which corresponds to translation in spacetime. Applying Noether's theorem, the conserved currents are
$$j^\alpha_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)} = P^\alpha_\mu.$$
These are simply the canonical momenta we defined earlier.

- The equation for current conservation is
$$\partial_\alpha P^\alpha_\mu = \frac{\partial P^\tau_\mu}{\partial\tau} + \frac{\partial P^\sigma_\mu}{\partial\sigma} = 0.$$
We know this holds on-shell, since it is precisely the equation of motion.

- The four conserved charges are found by integrating over "space" (i.e. σ) on the worldsheet,
$$p_\mu = \int_0^{\sigma_1} P^\tau_\mu(\tau,\sigma)\,d\sigma, \quad \frac{dp_\mu}{d\tau} = 0.$$

To check conservation, note that

$$\frac{dp_\mu}{d\tau} = \int_0^{\sigma_1} \frac{\partial P^\tau_\mu}{\partial \tau}\, d\sigma = -\int_0^{\sigma_1} \frac{\partial P^\sigma_\mu}{\partial \sigma}\, d\sigma = -P^\sigma_\mu \Big|_0^{\sigma_1}.$$

The right-hand side vanishes for closed strings and open strings with free endpoints. For open strings with Dirichlet boundary conditions, p_μ is not conserved, reflecting the fact that momentum can be transferred to the D-brane.

- Since p_μ it is associated with spacetime translation of the string, we expect it is the total energy-momentum of the string, and indeed in the static gauge $\tau = t$ we have $dp_\mu/dt = 0$ and p_μ is the total energy-momentum of the string, as we argued above. Since ϵ^μ is a vector, the index on p_μ is a vector index, so it transforms as expected under Lorentz transformations.
- To interpret p_μ for a general parametrization, we simply note that Noether charges $Q = \int_\Sigma (n \cdot j)$ are scalars which are independent of the surface Σ used to compute them, a fact proven in the lecture notes on General Relativity. Hence p_μ is always the conserved energy-momentum, in any parametrization. Furthermore, we need not compute it over a curve of constant τ. For an arbitrary curve γ, which is only required to wrap around the worldsheet once for a closed string or end at the endpoints for an open string, we have

$$p_\mu = \int_\gamma P_\mu \cdot dn = \int_\gamma P^\tau_\mu \, d\sigma - P^\sigma_\mu \, d\tau.$$

The relative sign is just because the normal to $(d\tau, d\sigma)$ is $dn = (d\sigma, -d\tau)$.

Next, we turn to Lorentz symmetry.

- As usual, Lorentz symmetry is generated by

$$\delta X^\mu = \epsilon^{\mu\nu} X_\nu, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}.$$

The string Lagrangian contains terms of the form $\eta_{\mu\nu}(\partial X^\mu/\partial \xi^\alpha)(\partial X^\nu/\partial \xi^\beta)$. Since X^μ is a Lorentz vector and ξ is a Lorentz scalar, the two quantities in parentheses are Lorentz vectors, so the string Lagrangian is a Lorentz scalar as expected.

- Applying Noether's theorem, the conserved currents can be packaged into a Lorentz tensor,

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu.$$

The conserved charges can be computed similarly,

$$M_{\mu\nu} = \int_\gamma \mathcal{M}^\tau_{\mu\nu} \, d\sigma - \mathcal{M}^\sigma_{\mu\nu} \, d\tau.$$

As usual, these Lorentz tensors are antisymmetric. As with the momenta, these charges might not be conserved for an open string attached to a D-brane.

- Explicitly, integrating over a curve γ with constant τ,

$$M_{\mu\nu} = \int X_\mu P^\tau_\nu - X_\nu P^\tau_\mu \, d\sigma$$

and the charges associated with boosts are

$$M^{0i} = \int tP^{\tau i} - X^i P^{\tau 0}\, d\sigma = tp^i - X_{\mathrm{CM}} E$$

where X_{CM} is the spatial center of mass of the string, and conservation of M^{0i} ensures the center of mass moves at a constant velocity.

Example. Consider the rotating string of the previous example. We have

$$\mathbf{P}^\tau = T_0 \frac{\partial \mathbf{X}}{\partial t} = T_0 \cos\frac{\pi\sigma}{\sigma_1}\left(-\sin\frac{\pi t}{\sigma_1}, \cos\frac{\pi t}{\sigma_1}\right).$$

The angular momentum is

$$J = M_{12} = \int_0^{\sigma_1} X_1 P_2^\tau - X_2 P_1^\tau\, d\sigma = \frac{\sigma_1 T_0}{\pi}\int_0^{\sigma_1}\cos^2\frac{\pi\sigma}{\sigma_1}\, d\sigma = \frac{\sigma_1^2 T_0}{2\pi}.$$

Finally, using $\sigma_1 = E/T_0$ we have the relation

$$\boxed{\frac{J}{\hbar} = \alpha' E^2, \quad \alpha' = \frac{1}{2\pi T_0 \hbar c}}$$

where we restored $\hbar$ and c. Here α' is called the slope parameter, and was important in ancient times when string theory was used to predict the masses of mesons, which lied on "Regge trajectories" with slope α'. The relation $J \sim E^2$ is quite unusual, and comes from the fact that the mass of a string scales with its length. Given α', we may also define a length scale

$$\boxed{\ell_s = \hbar c\sqrt{\alpha'}}$$

called the string length. Most modern work on string theory uses ℓ_s or α', rather than T_0. Originally, $\ell_s \sim 10^{-15}$ m to describe mesons, but this was invalidated by deep inelastic scattering experiments, which instead supported QCD. In the modern usage of string theory as a theory of quantum gravity, ℓ_s is around the Planck length.

1.4 Light Cone Gauge

We now introduce light cone gauge, which will be used to carry out the quantization of the string.

- Previously, we worked in static gauge, $X^0(\tau, \sigma) = \tau$. We now consider the more general gauge

  $$n_\mu X^\mu(\tau, \sigma) = \lambda\tau$$

  for a constant vector n_μ, which reduces to static gauge for $n_\mu = (1, 0, 0, 0)$ and $\lambda = 1$. In this gauge, the configuration of the string at worldsheet time τ is the intersection of the worldsheet and a hyperplane orthogonal to n_μ. To ensure this configuration is everywhere spacelike or null, we restrict n_μ to be spacelike or null.

- Like static gauge, this gauge is not Lorentz covariant. In a more advanced treatment, string quantization would be carried out in a Lorentz covariant gauge, but the choice made here will be simpler to understand.

- Restricting to open strings, we recall that the string momentum p^μ is conserved for free boundary conditions, as P^σ vanishes at the endpoints. For Dirichlet boundary conditions, $n \cdot p$ is conserved as long as $n \cdot P^\sigma$ vanishes at the endpoints, a weaker condition which we will assume holds.
- In either case, $n \cdot p$ is conserved, and we conventionally normalize τ to be dimensionless, with

$$n \cdot X = 2\alpha'(n \cdot p)\tau.$$

  Since both sides are proportional to n, the gauge no longer depends on the normalization of n. To check the dimensions work out, note that $[X] = -1$, $[p] = 1$, $[\tau] = 0$, and $[\alpha'] = -2$.
- Previously, we parametrized σ so that the energy density $P^{\tau 0}$ was constant. This is generalized to demanding the energy-momentum density $n \cdot P^\tau$ in the n direction is constant. Scaling so that $\sigma \in [0, \pi]$ for open strings, this means

$$n \cdot P^\tau = \frac{n \cdot p}{\pi}.$$

  That is, the left-hand side does not depend on σ or τ.
- Dotting both sides of the equation of motion with n^μ, we find

$$\frac{\partial}{\partial \tau}(n \cdot P^\tau) + \frac{\partial}{\partial \sigma}(n \cdot P^\sigma) = \frac{\partial}{\partial \sigma}(n \cdot P^\sigma) = 0$$

  However, by assumption $n \cdot P^\sigma = 0$ at the endpoints, and hence

$$n \cdot P^\sigma = 0$$

  everywhere on the string.
- For closed strings, the momentum p is conserved in all cases, so the same reasoning as above goes through, with slightly different conventions,

$$n \cdot X = \alpha'(n \cdot p)\tau, \quad \sigma \in [0, 2\pi], \quad n \cdot P^\tau = \frac{n \cdot p}{2\pi}.$$

  Since there are no endpoints, it is ambiguous where $\sigma = 0$ is; furthermore we cannot use the endpoints to show $n \cdot P^\sigma = 0$. These two problems may be solved simultaneously by choosing the curve $\sigma = 0$ so that $n \cdot P^\sigma = 0$ on it.
- More explicitly, using the explicit expression for P^σ we have

$$n \cdot P^\sigma \propto (\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X}^2)\partial_\sigma(n \cdot X) = (\dot{X} \cdot X')\partial_\tau(n \cdot X) \propto \dot{X} \cdot X'.$$

  Hence picking $n \cdot P^\sigma = 0$ is equivalent to setting $\dot{X} \cdot X' = 0$. This generalizes the condition $\dot{\mathbf{X}} \cdot \mathbf{X}' = 0$ we imposed in static gauge, as there we had $(X^0)' = 0$.

We now turn to the associated constraints and wave equations.

- Using $\dot{X} \cdot X' = 0$, the expression for P^τ simplifies to

$$P^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{X'^2 \dot{X}^\mu}{\sqrt{-\dot{X}^2 X'^2}}.$$

Dotting n into both sides and applying the gauge condition, we have

$$1 = \frac{X'^2}{\sqrt{-\dot{X}^2 X'^2}}$$

for both open and closed strings, and using $X'^2 > 0$ we have

$$\dot{X}^2 + X'^2 = 0$$

which generalizes the normalization condition $(\partial \mathbf{X}/\partial\sigma)^2 + (\partial \mathbf{X}/\partial t)^2 = 1$ in static gauge. These results together imply the induced metric on the worldsheet is conformally equivalent to the flat metric, so this gauge is also called conformal gauge.

- Using this result, the expressions for the momenta simplify considerably,

$$P^{\tau\mu} = \frac{\dot{X}^\mu}{2\pi\alpha'}, \quad P^{\sigma\mu} = -\frac{X^{\mu\prime}}{2\pi\alpha'}.$$

The equation of motion are simply wave equations,

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0.$$

- Finally, adding and subtracting $2\dot{X} \cdot X' = 0$, our two constraints are equivalent to

$$(\dot{X} \pm X')^2 = 0.$$

For an open string with free endpoints, we have the further constraint $P^{\sigma\mu} = 0$ at the endpoints, so $X^{\mu\prime}$ vanishes at the endpoints.

We will now find the general solution for the open string motion, going further than in static gauge.

- Now consider an open string with free endpoints. The general solution of the wave equation is

$$X^\mu(\tau, \sigma) = \frac{1}{2}(f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)).$$

The constraint $X' = 0$ at $\sigma = 0$ yields $f^{\mu\prime} = g^{\mu\prime}$, so the two differ by a constant, which can be absorbed in a redefinition of f^μ. Hence

$$X^\mu(\tau, \sigma) = \frac{1}{2}(f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)).$$

- The constraint $X' = 0$ at $\sigma = \pi$ yields

$$f^{\mu\prime}(\tau + \pi) - f^{\mu\prime}(\tau - \pi) = 0$$

which shows that $f^{\mu\prime}$ is periodic with period 2π, justifying our earlier normalization convention.

- So far, this is familiar from static gauge; we now go further by expanding $f^{\mu\prime}$ in a Fourier series,

$$f^{\mu\prime}(u) = f_1^\mu + \sum_{n>0} a_n^\mu \cos nu + b_n^\mu \sin nu$$

which integrates to

$$f^\mu(u) = f_0^\mu + u f_1^\mu + \sum_{n>0} A_n^\mu \cos nu + B_n^\mu \sin nu$$

and gives a general solution of

$$X^\mu(\tau, \sigma) = f_0^\mu + f_1^\mu \tau + \sum_{n>0} (A_n^\mu \cos n\tau + B_n^\mu \sin n\tau) \cos n\sigma.$$

- It is useful to change variables to

$$A_n^\mu \cos n\tau + B_n^\mu \sin n\tau = -i\frac{\sqrt{2\alpha'}}{\sqrt{n}}(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau})$$

so that the a_n^μ are dimensionless; it will turn into an annihilation operator upon quantization. Moreover, since the momentum density is

$$P^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}f_1^\mu + \text{oscillatory terms}$$

we have $f_1^\mu = 2\alpha' p^\mu$. In these variables, the solution takes the form

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\sum_{n>0}(a_n^{\mu *}e^{in\tau} - a_n^\mu e^{-in\tau})\frac{\cos n\alpha}{\sqrt{n}}.$$

This is the notation one might see in a standard string theory textbook.

- It is also useful to define the scaled variables

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu, \quad \alpha_n^\mu = a_n^\mu\sqrt{n}, \quad \alpha_{-n}^\mu = (\alpha_n^\mu)^*$$

so the sum ranges over all nonzero n,

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}\frac{\cos n\sigma}{n}.$$

The first term is called the position zero mode.

- In particular, in this notation the derivatives of X^μ are particularly simple,

$$\dot{X}^\mu = \sqrt{2\alpha'}\sum_n \alpha_n^\mu e^{-in\tau}\cos n\sigma, \quad X^{\mu\prime} = -i\sqrt{2\alpha'}\sum_n \alpha_n^\mu e^{-in\tau}\sin n\sigma.$$

This gives two nice linear combinations,

$$\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'}\sum_n \alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

Note that at this point we have not yet imposed the constraints $(\dot{X}\pm X')^2 = 0$.

Finally, we impose the constraints by specializing to light cone gauge.

- Light cone gauge is the choice

$$n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0), \quad n\cdot X = X^+, \quad n\cdot p = p^+$$

in which case the gauge conditions become

$$X^+ = \beta\alpha p^+\tau, \quad p^+ = \frac{2\pi}{\beta}P^{\tau +}, \quad \beta = \begin{cases} 2 & \text{open string,} \\ 1 & \text{closed string.}\end{cases}$$

We define all the coordinates but the first two to be transverse coordinates X^I, so the metric restricted to transverse coordinates is Euclidean.

- The constraints take the form
$$-2(\dot{X}^+ \pm X^{+\prime})(\dot{X}^- \pm X^{-\prime}) + (\dot{X}^I \pm X^{I\prime})^2 = 0.$$
In light cone gauge we have $X^{+\prime} = 0$ and $\dot{X}^+ = \beta\alpha p^+$, giving
$$\dot{X}^- \pm X^{-\prime} = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I\prime})^2.$$
Here we have assumed $p^+ \neq 0$, which holds for almost all configurations. Hence in light cone gauge, we may easily solve for the derivatives of X^-. The key reason this is easy is that in light cone coordinates, we get inner product expressions like X^+X^- rather than squares, so we avoid having to take square roots.

- We can hence solve for the derivatives of X^- in terms of X^I, so we know X^- up to an integration constant x_0^-. Note that for a closed string we also have the consistency condition
$$\int_0^{2\pi} d\sigma\, X^{-\prime} = 0.$$
Hence the full evolution of the string is determined by the X^I and the constants p^+ and x_0^-.

- Going back to our earlier solution for the motion of the open string, we have
$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^I e^{-in\tau}\frac{\cos n\sigma}{n}$$
where we restricted to the transverse coordinates. The plus component is simply
$$X^+(\tau,\sigma) = 2\alpha' p^+\tau$$
which is equivalent to setting the position zero mode x_0^+ and the α_n^+ (for $n \neq 0$) all to zero.

- Now we expand the minus component in the same way,
$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\,\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^- e^{-in\tau}\frac{\cos n\sigma}{n}.$$
Using our earlier identity for $\dot{X} \pm X'$, we may solve for the α_n^-,
$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{p^+}L_n^\perp, \quad L_n^\perp = \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$$
where $L_n^\perp$ is called the transverse Virasoro mode. We have now found the general solution for the motion of the open string.

- In particular, for $n = 0$ we have
$$\sqrt{2\alpha'}\,\alpha_0^- = 2\alpha' p^- = \frac{1}{p_+}L_0^\perp$$
which implies that
$$2p^+p^- = \frac{1}{\alpha'}L_0^\perp = \frac{1}{\alpha'}\left(\frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n>0}\alpha_n^{I*}\alpha_n^I\right) = p^Ip^I + \frac{1}{\alpha'}\sum_{n>0} n a_n^{I*}a_n^I.$$

In particular, the mass of a string is

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}\sum_{n>0} n a_n^{I*} a_n^I.$$

This confirms that $M^2 > 0$, which is actually hard to show without light cone gauge. Upon quantization, this formula will yield a discrete spectrum of masses. As a check, note that when the a_n^I all vanish, the string collapses to a point, and $M = 0$ accordingly.

1.5 Light Cone Fields

We now briefly consider the quantization of fields in light cone coordinates and light cone gauge.

- Writing the spacetime coordinates as $(x^+, x^-, \mathbf{x}_T)$, the Klein-Gordan equation is

$$(\partial^2 - m^2)\phi = \left(-2\partial_{x^+}\partial_{x^-} + \partial_{x^I}\partial_{x^I} - m^2\right)\phi = 0$$

  where I indexes over transverse coordinates.

- As usual, we can simplify this by Fourier transforming the spatial coordinates only, which we take to be x^- and $\mathbf{x}_T$. The conjugate momentum to x^- is $-p^+$, while the conjugate momentum to $\mathbf{x}_T$ is $\mathbf{p}_T$, so we define

$$\phi(x^+, x^-, \mathbf{x}_T) = \int đp^+ đ\mathbf{p}_T \, e^{-ix^-p^+ + i\mathbf{x}_T\cdot\mathbf{p}_T}\phi(x^+, p^+, \mathbf{p}_T)$$

  which converts the Klein-Gordan equation to

$$\left(i\partial_{x^+} - \frac{p^Ip^I + m^2}{2p^+}\right)\phi = 0.$$

  As usual, this equation simply enforces the mass-shell condition $p^2 + m^2 = 0$, as it implies

$$p^- = \frac{p^Ip^I + m^2}{2p^+}.$$

- As usual, we can quantize the field, constructing a creation and annihilation operator for every spatial momentum $\mathbf{p}$, where E is constrained to be positive and on the mass shell,

$$\phi = \int \frac{đ\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{ipx} + a_{\mathbf{p}}^\dagger e^{-ipx}, \quad P^\mu = \int đ\mathbf{p}\, P^\mu = \int đ\mathbf{p}\, p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}}.$$

  Similarly in light cone coordinates, we label the operators by $(p^+, \mathbf{p}_T)$, and pick out the physical half of the mass shell by $p^+ > 0$. Then we have, for example,

$$P^+ = \sum_{p^+, \mathbf{p}_T} p^+ a^\dagger_{p^+\mathbf{p}_T} a_{p^+\mathbf{p}_T}, \quad \mathbf{P}^I = \sum_{p^+, \mathbf{p}_T} \mathbf{p}^I a^\dagger_{p^+\mathbf{p}_T} a_{p^+\mathbf{p}_T}, \quad P^- = \sum_{p^+, \mathbf{p}_T} \frac{p^Ip^I + m^2}{2p^+} a^\dagger_{p^+\mathbf{p}_T} a_{p^+\mathbf{p}_T}.$$

- Now we consider massless vector fields, which have the gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon.$$

  Alternatively, for the Fourier transform of the field,

$$\delta A_\mu(p) = ip_\mu \epsilon(p), \quad A^\mu(x) = \int đp\, e^{ipx} A^\mu(p).$$

- In light cone coordinates we have $\delta A^+ = ip^+\epsilon$. As usual, neglecting the set of measure zero where $p^+ = 0$, we may use the gauge freedom to reach light cone gauge,

$$A^+(p) = 0.$$

  Just like for temporal or axial gauge, this fixes the gauge symmetry up to a set of measure zero.

- The equation of motion is $\partial_\mu(\partial \cdot A) = \partial^2 A_\mu$, so for $\mu = +$ we find $\partial \cdot A = 0$. Hence

$$p \cdot A = -p^+ A^- - p^- A^+ + p^I A^I = 0.$$

  This hence determines A^- in terms of the transverse A^I,

$$A^- = \frac{p^I A^I}{p^+}$$

  which is reminiscent of the string in light cone gauge.

- Furthermore, the field equation reduces to $\partial^2 A_\mu = 0$, so the degrees of freedom are massless. They are contained in the transverse components only, $p^2 A^I = 0$, so that a massless vector field in D dimensions has $D-2$ degrees of freedom for each momentum.

- This is consistent with group theory, as the little group is E_{D-2}. Translations must act trivially to get a finite-dimensional representation of the little group, so we only need worry about $SO(D-2) \subseteq E_{D-2}$, and evidently the states for a given momentum transform in the fundamental of $SO(D-2)$, indicating spin one.

- We can also see the degrees of freedom are massless without fixing a gauge. Suppose that $A(p)$ has support only on $p^2 \neq 0$. Then the equation of motion is

$$p^2 A_\mu = p_\mu (p \cdot A)$$

  which may be solved for A_μ,

$$A_\mu = i p_\mu \left(\frac{ip \cdot A}{p^2} \right).$$

  However, this implies A_μ is pure gauge, related to the zero field by $\epsilon = ip \cdot A / p^2$.

- The quantization of the field is similar to that of the real scalar field. The creation and annihilation operators are now labeled by p^+, $\mathbf{p}_T$, and a transverse index I. The one-photon states take the form

$$\sum_{I=2}^{D-1} \xi_I {a^I_{p^+\mathbf{p}_T}}^\dagger |\Omega\rangle.$$

Next, we consider the more complicated case of gravitational fields.

- We will linearize about flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, in which case the linearized Einstein equations for $h_{\mu\nu}$ in momentum space are

$$S^{\mu\nu}(p) \equiv p^2 h^{\mu\nu} - p_\alpha (p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}) + p^\mu p^\nu h = 0, \quad h = h^\mu_\mu.$$

  Here, indices on h are raised and lowered by the Minkowski metric.

- The equation of motion is invariant under the infinitesimal gauge transformation

$$\delta h^{\mu\nu} = ip^\mu \epsilon^\nu + ip^\nu \epsilon^\mu$$

where the gauge parameter is now a vector. Physically the gauge symmetry is reparametrization invariance.

- In light cone coordinates, the components of the metric are

$$(h^{IJ}, h^{+I}, h^{-I}, h^{+-}, h^{++}, h^{--}).$$

Note that in particular, we have

$$\delta h^{++} = 2ip^+\epsilon^+, \quad \delta h^{+-} = i(p^+\epsilon^- + p^-\epsilon^+), \quad h^{+I} = i(p^+\epsilon^I + p^I\epsilon^+).$$

Again ignoring degrees of freedom with $p^+ = 0$, we can choose ϵ^+ to set $h^{++} = 0$, we can choose ϵ^- to set $h^{+-} = 0$, and we can choose ϵ^I to set $h^{+I} = 0$. Then in light cone gauge

$$h^{++} = h^{+-} = h^{+I} = 0.$$

- Setting $\mu = \nu = +$ in the equations of motion gives $(p^+)^2 h = 0$, which implies $h = 0$, or equivalently $h^{II} = 0$. Plugging this back into the equation of motion,

$$p^2 h^{\mu\nu} - p^\mu p_\alpha h^{\nu\alpha} - p^\nu p_\alpha h^{\mu\alpha} = 0.$$

In particular, setting $\mu = +$ we have $p_\alpha h^{\nu\alpha} = 0$. Plugging this back in, we have $p^2 h^{\mu\nu} = 0$. Therefore, the equations of motion boil down to

$$h^{II} = 0, \quad p_\alpha h^{\mu\alpha} = 0, \quad p^2 h^{\mu\nu} = 0.$$

- Setting $\mu = I$ in the second condition above, we have

$$-p^+ h^{I-} + p_J h^{IJ} = 0$$

while setting $\mu = -$ gives

$$-p^+ h^{--} + p_I h^{-I} = 0.$$

These indicate that the h^{I-} and h^{--} are determined in terms of h^{IJ}.

- We thus conclude the degrees of freedom are all massless, embedded in a symmetric, traceless, transverse tensor field in $D-2$ dimensions. There are hence

$$n(D) = \frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3)$$

degrees of freedom per momentum. As with the photons, this is consistent with a little group analysis, assuming the gravitons have spin two. Of course we could have gotten this much more easy by simply subtracting D from $D(D-1)/2$, but it's good to do an explicit check.

- The one-particle states can hence be written as

$$\sum_{I,J=2}^{D-1} \xi_{IJ} a^{IJ}_{p^+\mathbf{p}_T}{}^\dagger |\Omega\rangle, \quad \xi_{II} = 0$$

where ξ_{IJ} is the graviton polarization tensor.

Example. The Kalb-Ramond field is a two-form gauge field B, with an associated field strength $H = dB$ and gauge symmetry $\delta B = d\epsilon$. Explicitly, in components,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad \delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu.$$

Note that the gauge parameter itself has a gauge symmetry, $\delta\epsilon = \partial_\mu \lambda$. By the same argument as above, we can use up this gauge symmetry to set $\epsilon^+ = 0$. The action and equation of motion are

$$S = -\frac{1}{6}\int dx\, H_{\mu\nu\rho}H^{\mu\nu\rho}, \quad \partial^\mu H_{\mu\nu\rho} = 0.$$

The degrees of freedom are in $(B^{+-}, B^{+I}, B^{-I}, B^{IJ})$ where B^{IJ} is antisymmetric, and the gauge freedom is enough to set the first two to zero. The equation of motion then simplifies to $p^2 B^{IJ} = 0$, giving $(D-2)(D-3)/2$ degrees of freedom per momentum.

1.6 The Veneziano Amplitude

In this section we'll cover the early history of string theory, when it was used as a theory of the strong interaction. This history is still regarded as important today, despite the advent of QCD, for it constitutes the first and only prediction of string theory.

- During the 1950s and 1960s, many hadrons were discovered. It was found that many particles lay on lines on a plot of M^2 versus spin J, called a Chew-Frautschi plot. These lines were called Regge trajectories. The Regge trajectory of lowest mass was parametrized as

$$J = \alpha' M^2 + \alpha(0).$$

  This was quite mysterious, and a theory that explained the presence of all of these particles was required.

- Furthermore, it was known that fundamental, high-spin particles have problematic features in the UV. Consider the scattering of scalar particles, $\phi\phi \to \phi\phi$, with four-momenta directed inward. The Mandelstam invariants are

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_4)^2, \quad u = -(p_1+p_3)^2, \quad s+t+u = \sum_i m_i^2.$$

- Now consider tree-level processes mediated by a particle of spin J. Roughly speaking, the interaction term must look like

$$\delta\mathcal{L} \sim g_J(\phi \partial_{\mu_1} \ldots \partial_{\mu_J}\phi)\sigma^{(\mu_1 \ldots \mu_J)}$$

  where the derivatives act both to the left and the right. That is, we require powers of momentum to contract with the indices. But then the t-channel exchange would yield

$$\mathcal{A} \sim \frac{g_J^2\, s^J}{t - m_\sigma^2}.$$

  In the Regge limit of fixed (negative) t and large s, the scattering amplitude grows as s^J. This is called "hard" behavior, but was not observed for the scattering of pions. It is also difficult to understand theoretically, as it violates perturbative unitarity bounds.

- Furthermore, loop amplitudes diverge very strongly. If we consider the one-loop diagram consisting of the exchange of two σ's, we have

$$\mathcal{A} \sim \int đp \, \frac{p^{4J}}{(p^2)^4}.$$

This suggests that we cannot write down a renormalizable theory involving the high-spin σ particles, which indeed is the case.

- With regard to the tree-level amplitude, we should really sum over all particles,

$$\mathcal{A}_t \sim \sum_{J=0}^{\infty} \frac{g_J^2 \, s^J}{t - m_J^2}.$$

Similarly, we can sum over all s-channel diagrams, yielding $\mathcal{A}_s$. For simplicity, we suppose the scalars are not identical, so we can ignore the u-channel.

- Dolan, Horn, and Schmid were inspired by data to guess the complete tree-level amplitude is

$$\mathcal{A} = \mathcal{A}_s = \mathcal{A}_t.$$

That is, one can sum over *either* s-channel diagrams or t-channel ones, and they will give the same result. This would be impossible for usual scattering processes, because $\mathcal{A}_s$ only has poles in s and $\mathcal{A}_t$ only has poles in t, but the infinite sum can change the analytic structure. This led to the development of "dual models".

- In 1968, Veneziano guessed that the amplitude had the form

$$\mathcal{A} = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} = B(-\alpha(s), -\alpha(t)), \quad \alpha(x) = \alpha(0) + \alpha' x$$

where B is the Euler beta function. The gamma function has simple poles at zero and negative integers n with residues $(-1)^n/n!$.

- This implies that the Veneziano amplitude has only simple poles,

$$\text{s-channel poles: } s = \frac{-\alpha(0) + n}{\alpha'}, \quad \text{t-channel poles: } t = \frac{-\alpha(0) + n}{\alpha'}, \quad n = 0, 1, 2, \ldots.$$

Since $\mathcal{A}(s, t) = \mathcal{A}(t, s)$, the Taylor expansions in s and in t are identical, e.g.

$$\mathcal{A} = -\sum_{n=0}^{\infty} \frac{(\alpha(s) + 1)(\alpha(s) + 2) \ldots (\alpha(s) + n)}{n!} \frac{1}{\alpha(t) - n}$$

with the same expansion in powers of t, hence realizing Dolan-Horn-Schmid duality. Furthermore, the poles are just in the right places for the leading Regge trajectory.

- In the following year, Virasoro and Shapiro generalized the amplitude to display duality between the s, t, and u channels. Furthermore, these amplitudes do not have the undesirable "hard" behavior. In the Regge limit we instead have

$$A_{\text{dual}} \sim s^{\alpha(t)}$$

which is "soft" since $\alpha(t) < 0$. (In fact, one only finds this behavior if s is given a small imaginary part, which allows the amplitude to "average" over many resonances. This is physically correct, because the resonances really have finite lifetimes.)

- However, at higher energies, these amplitudes were ultimately found to be too soft; by 1974 QCD was recognized as the correct theory of the strong interaction. Another problem was that the amplitudes were not clearly unitary; as we can see from our explicit field theory amplitudes above, the residues must have the correct sign to represent scattering of physical particles; residues of the opposite sign indicate ghosts.

- It was found that the Veneziano amplitude is only unitary if spacetime has 26 dimensions and $\alpha(0) = 1$, which implies the existence of a massless spin 1 particle (i.e. a gauge boson) and a tachyonic scalar particle, which indicates an unstable vacuum.

- In 1974, Scherk and Schwarz found that in the low-energy limit $s\alpha' \ll 1$, a modified Veneziano amplitude (manually removing the tachyon), one finds Maxwell scattering of photons. Similarly, for the Virasoro-Shapiro amplitude, one finds a massless spin 2 particle undergoing gravitational scattering. Hence the theory giving rise to the Veneziano-Shapiro amplitude was a quantum theory of gravity with good UV behavior. Furthermore, the theory has only one parameter α', which could be taken to be on the order of the Planck scale.

Dolan-Horn-Schmid duality can be explained by the scattering of strings.

- First consider scattering of open strings, represented by the below diagram. This can be interpreted as either a tree-level s-channel or t-channel amplitude, by squeezing the diagram vertically or horizontally, explaining the duality.

- This should be compared to the Feynman diagrams in the worldline formulation of perturbative QFT, explained in the notes on Quantum Field Theory. Unlike in those Feynman diagrams, here there is no definite spacetime point where the strings split or merge. Each individual observer can define such a point, but it is not Lorentz invariant.

- The above diagram can't be interpreted as a u-channel diagram, as it would change the connectivity of the diagram. Instead, $s/t/u$-channel duality, as in the Veneziano-Shapiro amplitude, can be explained in terms of closed string scattering. This is easy to see by imagining shrinking the four tubes to points; we can then freely move the points around on the sphere.

- In QFT, vertices are distinguished points in the Feynman diagrams, which accounts for the diversity of QFTs; we can place a wide variety of factors at the vertices. However, in string theory there are no such points. Instead, about every point the string locally appears free. Hence the form of the free theory essentially determines the interactions, and anomaly cancellation strongly constrains the free theory, making string theory essentially unique.

- Another useful feature of string perturbation theory is that there are fewer string diagrams. For the four-point amplitude, there is only one closed string diagram at each order, as shown.

- As in worldline QFT, the diagrams above should be regarded as drawn in "worldsheet space", not physical space; the shapes are determined by the worldsheet metric. This allows us to use Weyl invariance on the worldsheet to deform the diagrams to a more convenient shape. For example, the tree-level closed string amplitude above can be deformed to a sphere with four punctures corresponding to external states.

2 Quantizing the Particle

2.1 Constrained Hamiltonian Systems

To understand the issues above more generally, we take a detour into classical mechanics.

- A gauge theory can be thought of as a theory where the dynamical variables are specified with respect to a reference frame whose choice is arbitrary at every instant of time. Physical observables are independent of this choice, but evolution is not deterministic; solutions to the equations of motion may contain arbitrary functions of time. It turns out that all gauge theories are constrained Hamiltonian systems (though not vice versa), as we will see shortly.

- In the Lagrangian formalism, we have

$$S_L = \int_{t_1}^{t_2} dt\, L(q, \dot{q}), \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^n} - \frac{\partial L}{\partial q^n} = 0.$$

  Using the chain rule, the Euler-Lagrange equation is equivalent to

$$\ddot{q}^{n'} \frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n} = \frac{\partial L}{\partial q^n} - \dot{q}^{n'} \frac{\partial^2 L}{\partial q^{n'} \partial \dot{q}^n}.$$

  This shows that the accelerations $\ddot{q}^{n'}$ are determined if and only if the matrix $J = \partial^2 L / \partial \dot{q}^{n'} \partial \dot{q}^n$ can be inverted. If J is singular, the solutions to the equations of motion contain arbitrary functions of time, i.e. we are dealing with a gauge system.

- To switch to the Hamiltonian formalism, we define the canonical momenta

$$p_n = \frac{\partial L}{\partial \dot{q}^n}.$$

  Then J is just the Jacobian matrix to switch from the $\dot{q}$ to the p, and its singularity means that different $\dot{q}$ will be mapped to the same p.

- Then the p are not independent, so there must be some "primary constraints"

$$\phi_m(q, p) = 0, \quad m = 1, \ldots, M.$$

  These constraints follow purely from the "kinematics", not from the equations of motion.

- We assume for simplicity that these constraints are independent everywhere and smooth functions of the q and p. When the rank of J is constant and equal to $N - M$, and the primary constraints define a submanifold of dimension $2N - M$ in phase space, called the primary constraint surface. Each point on this surface corresponds to a submanifold of dimension M in configuration space. Furthermore, we can set up local coordinates for phase space where M of the coordinates are the ϕ_m.

- If a phase space function G vanishes on the primary constraint surface, we say it weakly vanishes and write $G \approx 0$, which implies

$$G = g^m \phi_m$$

for some phase space functions g^m. Moreover, if $\lambda_n \delta q^n + \mu^n \delta p_n = 0$ for arbitrary variations on the primary constraint surface, then

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial q^n}, \quad \mu^n = u^m \frac{\partial \phi_m}{\partial p_n}$$

for phase space functions u^m.

Next, we discuss the Hamiltonian.

- The Hamiltonian is only well-defined on the primary constraint surface, though we may extend it to the entire phase space arbitrarily. Then we expect the replacement

$$H \to H + c^m(q, p)\phi_m$$

  should make no difference, as we'll see below.

- As usual, the variation of the Hamiltonian can be written in terms of δp and δq,

$$\delta H = \dot{q}^n \delta p_n - \delta q^n \frac{\partial L}{\partial q^n}.$$

  The variations here are restricted to the primary constraint surface. The partial derivatives are a bit tricky; for functions defined on phase space, $\partial/\partial q$ keeps p constant, while for the Lagrangian, $\partial/\partial q$ keeps $\dot{q}$ constant.

- We may rewrite the equation above as

$$\left(\frac{\partial H}{\partial q^n} + \frac{\partial L}{\partial q^n}\right) \delta q^n + \left(\frac{\partial H}{\partial p_n} - \dot{q}^n\right) \delta p_n = 0.$$

  Since δq^n and δp_n are arbitrary variations on the constraint surface,

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \quad -\frac{\partial L}{\partial q^n} = \frac{\partial H}{\partial q^n} + u^m \frac{\partial \phi_m}{\partial q^n}.$$

  The first equation lets us solve for $\dot{q}$ in terms of q, p, and the new functions u, which can be thought of as coordinates on the submanifold of inverse images of a given p_n. That is, the transformation from $(q, \dot{q})$ to (q, p, u) is invertible.

- The equations of motion are now

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n}, \quad \phi_m(q, p) = 0.$$

  We recognize that these equations follow from the variational principle

$$\delta \int_{t_1}^{t_2} dt \, (\dot{q}^n p_n - H - u^m \phi_m) = 0$$

  for arbitrary variations with fixed q endpoints; here u^m acts as a Lagrange multiplier that enforces the constraints. We can now see clearly that substituting $H \to H + c^m \phi_m$ changes nothing, since it just shifts the u^m.

- The equations of motion can also be written in terms of Poisson brackets,

$$\dot{F} \approx [F, H + u^m \phi_m] \approx [F, H] + u^m [F, \phi_m], \quad [F, G] = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$$

  for phase space functions F and G.

- For consistency, the constraints must remain satisfied in time, $\dot{\phi}_m \approx 0$, which requires

$$[\phi_m, H] + u^{m'}[\phi_m, \phi_{m'}] \approx 0.$$

  This may yield further constraints on the u's, called secondary constraints. The distinction between primary and secondary constraints is not important; we let there be J in total and write a generic one as $\phi_j \approx 0$.

- In any case, all J constraints lead to restrictions on the Lagrange multipliers u^m,

$$[\phi_j, H] + u^m[\phi_j, \phi_m] \approx 0.$$

  These are linear in the u_m, and a general solution takes the form

$$u^m = U^m + V^m$$

  where U^m is a particular solution of the inhomogeneous equations, and V^m solves the homogeneous equations $V^m[\phi_j, \phi_m] \approx 0$. Then a general solution can be expanded as

$$u^m = U^m + v^a(t) V_a^m$$

  where the v^a are arbitrary functions of time and the V_a^m are a basis of homogeneous solutions. We have thus separated out the gauge freedom in the solution to the equations of motion.

- Therefore, the equation of motion on the constraint surface can be written as

$$\dot{F} \approx [F, H_T], \quad H_T = H' + v^a \phi_a = H + U^m \phi_m + v^a \phi_a, \quad \phi_a = V_a^m \phi_m$$

  where H_T is called the total Hamiltonian.

- A phase space function F is first-class if

$$[F, \phi_j] \approx 0$$

  for all j, and second-class otherwise. For example, H' is first-class, as are the ϕ_a, which give a complete set of first-class primary constraints.

- The Poisson bracket of first-class functions is first-class. To see this, note that if F and G are first-class, we may expand

$$[F, \phi_j] = f_j^{j'} \phi_{j'}, \quad [G, \phi_j] = g_j^{j'} \phi_{j'}.$$

  By the Jacobi identity,

$$[[F, G], \phi_j] = [F, [G, \phi_j]] - [G, [F, \phi_j]] = [F, g_j^{j'} \phi_{j'}] - [G, f_j^{j'} \phi_{j'}] \approx 0.$$

Next, we connect first-class constraints to gauge transformations.

- When a physical state (q, p) is specified, the time evolution is not unique, because of the arbitrary functions of time v^a. We take as a postulate that time evolution should be unique; this implies that we must identify distinct points in phase space as being the same physical state, i.e. we have a gauge redundancy.
- For example, consider a dynamical variable F with some initial condition, with either some v^a or $\tilde{v}^a$. Then an infinitesimal time δt later, the final values of F differ by

$$\delta F = (v^a - \tilde{v}^a)\delta t[F, \phi_a].$$

  Then we say the first-class primary constraints ϕ_a generate gauge transformations. It doesn't make sense to say second-class constraints do, because they take us off the constraint surface.
- Similarly, the Poisson bracket $[\phi_a, \phi_{a'}]$ of two first-class primary constraints generates a gauge transformation. This can be realized by "translating in a rectangular loop" in $(v^a, v^{a'})$ space over time.
- Finally, the Poisson bracket $[\phi_a, H']$ generates a gauge transformation; it is the difference of translating in time and incrementing v^a, or doing the same in reverse order.
- From the two previous points, we see that some secondary first-class constraints may also generate gauge transformations, if they are the results of such Poisson brackets. Dirac's conjecture says this is always the case, but it is false: there exist systems where we can get a deterministic time evolution without using all the secondary first-class constraints as gauge generators. On the other hand, from an axiomatic perspective it is useful to postulate Dirac's conjecture to be true. That is, we *define* all first-class constraints to be gauge generators. Thus from this point on we completely ignore the primary/secondary distinction.
- We denote all first-class constraints by γ and all second-class ones by χ. Then the most general time evolution allowing gauge transformations is generated by the extended Hamiltonian

$$H_E = H' + u^a \gamma_a$$

  where the u^a are arbitrary functions of time. In comparison, the total Hamiltonian H_T only included first-class primary constraints.

2.2 Dirac Brackets and Gauge Fixing

Now we turn to the interpretation of second-class constraints. For simplicity, we consider the irreducible case, i.e. the case where all the constraints are independent.

- Define the matrix $C_{jj'} = [\phi_j, \phi_{j'}]$, whose elements are phase space functions. If we split the constraints into first-class and second-class constraints in order, the matrix has the block form

$$C_{jj'} \approx \begin{pmatrix} 0 & 0 \\ 0 & C_{\beta\alpha} \end{pmatrix}, \quad C_{\beta\alpha} = [\chi_\beta, \chi_\alpha].$$

  The reduced matrix $C_{\beta\alpha}$ is invertible on the constraint surface, because if it were not, there would be a nonzero solution to $\lambda^\beta C_{\beta\alpha} \approx 0$, which would give a first-class constraint $\lambda^\beta \chi_\beta$. Since $C_{\beta\alpha}$ is antisymmetric, this implies there are an even number of second-class constraints.

- Consider the simple case where two conjugate variables q^1 and p_1 are constrained to be zero,

$$\chi_1 = q^1 \approx 0, \quad \chi_2 = p_1 \approx 0.$$

  They are second-class since $[\chi_1, \chi_2] = 1 \not\approx 0$. It is obvious in this case that the first pair of canonical variables simply plays no role.

- To make this manifest, we define the Dirac bracket, a modified Poisson bracket which does not include the first pair,

$$[F, G]^* = \sum_{n=2}^{N} \left(\frac{\partial F}{\partial q^n} \frac{\partial G}{\partial p_n} - \frac{\partial G}{\partial q^n} \frac{\partial F}{\partial p_n} \right).$$

  The Dirac bracket has the same nice properties as the Poisson bracket and still yields the time evolution, but the bracket of any phase space function with either of the χ_α is zero. Hence we can simply treat the χ_α as if they are strongly equal to zero, setting them to zero before evaluating the bracket.

- More generally, let $C^{\alpha\beta}$ be the inverse of $C_{\alpha\beta}$ and define the Dirac bracket

$$[F, G]^* = [F, G] - [F, \chi_\alpha] C^{\alpha\beta} [\chi_\beta, G].$$

  It can be verified explicitly that this bracket satisfies all the usual properties of the Poisson bracket, in addition to

$$[\chi_\alpha, F]^* = 0 \text{ for any } F.$$

- We also have

$$[F, R]^* \approx [F, R] \text{ for any first-class } F.$$

  This shows that H_E still generates time evolution under the Dirac bracket, and the γ_a still generate gauge transformations; the Dirac bracket replaces the Poisson bracket completely.

- Upon switching to the Dirac bracket, the second-class constraints effectively become strong relations between the canonical variables, so we may in principle eliminate the redundant ones. This was straightforward in our trivial example, but in practical situations it's often cleaner to keep them all.

We now turn to the question of gauge fixing.

- Similarly, for first-class constraints we may impose gauge-fixing conditions $C_b(q, p) \approx 0$ to remove the gauge freedom. Geometrically, for a complete gauge fixing, the gauge fixing surface must intersect each gauge orbit exactly once. However, in some circumstances this is impossible; this situation is called a Gribov obstruction.

- After gauge fixing, the original first-class constraints become second-class constraints, since they now take us off the new constraint surface. Conversely, one can think of all second-class constraints as arising from a gauge fixation. For example, in our trivial example with second-class constraints $q_1 = p_1 = 0$, we could regard $p_1 = 0$ as a first-class constraint which generates shifts in q_1, and $q_1 = 0$ as a gauge fixing condition. This is occasionally useful because it allows the use of Poisson brackets, which are simpler than Dirac brackets in quantization.

- In the infinite dimensional case, gauge fixing can become even more subtle. Consider the $u^a\gamma_a$ part of the extended Hamiltonian. In continuum mechanics, this becomes

$$\int dx\, u^a(x)\gamma_a(x)$$

and we must ask what function space the u^a live in; it must be large enough to impose the constraint $\gamma_a(x) \approx 0$ but no larger.

- To see what can go wrong, note that the u^a generate the gauge transformation

$$\delta F = \int dx\, u^a(x)[F, \gamma_a(x)].$$

In the case of electrodynamics, physical fields vanish at infinity; if we choose $u^a(x)$ to be constant we generate a global "charge rotation". The only states invariant under such a rotation are those of zero total charge.

- Another subtlety is the possibility of large gauge transformations, where the $u^a(x)$ are not continuously connected to the identity in function space. We may choose to regard them as proper gauge transformations, but this is an additional assumption, as everything we've done above is at the infinitesimal level.

- A classical observable F is a function on the constraint surface. It must be gauge-invariant, so

$$[F, \gamma_a]^* \approx 0.$$

Note that we also have $[F, \chi_a]^* = 0$ automatically.

Example. We consider the Lagrangian

$$L = \sum_{i=1}^{n-1} \frac{1}{2}(q_i - \dot{q}_{i+1})^2.$$

The canonical momenta are thus

$$\pi_i = \dot{q}^i - q^{i-1}, i \geq 2, \quad \pi_1 = 0, \quad H = \frac{1}{2}\sum_{i\geq 2} \pi_i^2 + \sum_{i \geq 2} \pi_i q_{i-1}.$$

The only primary constraint is $\pi_1 = 0$, so time evolution is generated by $H_T = H + u\pi_1$. Requiring $\dot{\pi}_1 = 0$ gives the secondary constraint $\pi_2 = 0$, which then gives $\pi_3 = 0$, and so on. Now all of the constraints π_i are first-class, so the extended Hamiltonian is

$$H_E = H + u^i \pi_i \approx 0$$

and the theory possesses no physical degrees of freedom. This system is equivalent to the system with Lagrangian $L = 0$, which has the same first-class constraints, except that they are all primary. This is another illustration of the fact that we need not distinguish between primary and secondary constraints.

2.3 General Covariance

Next, we consider the case of generally covariant systems, with no constraints for simplicity.

- Usually, one describes a Hamiltonian system by giving the canonical variables as a function of time t, where t is assumed to be directly physically measurable. In such cases, one can always promote t to a canonical variable by "parametrizing" the theory with a parameter τ, which then plays the same formal role that t originally did. The resulting system is generally covariant, having reparametrization invariance under τ.
- However, the interpretation of t and τ can be quite tricky. For example, general relativity is already generally covariant, as it is invariant under diffeomorphisms of spacetime, but we think of the τ–like coordinate as the physical time for some observer. For now we'll think of t as time and τ as a meaningless parameter, but will return to this point below.
- Explicitly, the action for a system with canonical variables q^i and p_i and Hamiltonian H_0 is

  $$S[q^i(t), p_i(t)] = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{dt} - H_0 \right) dt.$$

  Now we let $t = q^0$ with conjugate momentum p_0. Then an equivalent action is

  $$S[q^0(\tau), q^i(\tau), p_0(\tau), p_i(\tau), u^0(\tau)] = \int_{\tau_1}^{\tau_2} p_0 \dot{q}^0 + p_i \dot{q}^i - u^0(p_0 + H_0)\, d\tau$$

  where the dot indicates a τ derivative.
- To show this, note that varying with respect to auxiliary variables u^0 and p_0 yields

  $$\gamma_0 \equiv p_0 + H_0 = 0, \quad \dot{t} - u^0 = 0.$$

  These equations may be used to eliminate u^0 and p_0, to arrive at the action

  $$\int_{\tau_1}^{\tau_2} p_i \dot{q}^i - H_0 \dot{t}\, d\tau = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{dt} - H_0 \right) dt$$

  as before. However, this equality only holds if t is monotonic in τ. Thus the covariant formulation is more general, as it can accommodate trajectories with $\dot{t} < 0$. In fact, even in the covariant path integral for a nonrelativistic particle, one must admit trajectories with $\dot{t} < 0$.

We now consider the consequences of our result.

- There is a single constraint, $\gamma_0 \approx 0$, which is thus first-class. The extended Hamiltonian above contains only the constraint term $-u^0\gamma_0$, so the Hamiltonian itself in this formalism is *zero*. This is not completely unreasonable, because physically systems evolve in time, not in the arbitrary parameter τ. The motion itself is solely "the unfolding of a gauge transformation".
- This procedure can be practically useful in systems with complicated explicit time dependence, since it always results in a system with no explicit dependence on τ.

- In this formalism, γ_0 generates a gauge transformation which is identified with time evolutions. Note that an infinitesimal reparametrization $\tau \to \overline{\tau} = \tau - \epsilon(\tau)$ induces the changes

$$\delta q = \dot{q}\epsilon, \quad \delta p = \dot{p}\epsilon, \quad \delta u^0 = \frac{d}{d\tau}(u^0 \epsilon)$$

  where ϵ must vanish at the endpoints. This is the gauge transformation generated by γ_0, up to a trivial "equation-of-motion symmetry".

- We will always require gauge transformations to vanish at the endpoints. This is really just an artifact of keeping the limits of integration fixed. The key point is that it sets total derivatives of terms proportional to ϵ to zero.

- One can argue very generally that H must vanish. We say that q and p transform as scalars under reparametrization invariance since they obey the equations above, while u^0 transforms as a scalar density. Then all terms in the integrand of the action transform as scalar densities, making the action a scalar. If a Hamiltonian were present as well, it would have to transform as a scalar density, but it must be a scalar since it is a function of q and p.

- However, there can be systems where q and p are not scalars, in which case H need not vanish. For example, this can be achieved by performing a τ-dependent canonical transformation.

Now we turn to the interpretation of the formalism.

- General covariance may be viewed as a special case of gauge symmetry, as in either case solutions to the equation of motion may contain arbitrary functions of the time τ. This implies that *something* about the system is unphysical, such as the time τ or some of the canonical variables, but we cannot decide which from the theory alone. Instead, additional information must come from outside.

- For example, in electromagnetism, we suppose the time parameter is physical while A_μ is not. This is justified because the electromagnetic field is just a subsystem of the universe, and we know we can build clocks that measure τ independently.

- On the other hand, for a classical point particle, we suppose the canonical variables $(t, \mathbf{x})$ are physical while τ is not; that is, we treat t and $\mathbf{x}$ as the measurable quantities.

- General relativity is the best-known generally covariant theory, but in this case there is no "outside perspective" we can take. In this case, the most symmetric formulation is one where the Hamiltonian is weakly zero, and all physical questions are formulated in terms of functions with zero brackets with the constraints; these first-class functions are gauge-invariant constants of the motion.

- Such functions suffice even to ask apparently time-dependent questions. For example, for the free particle, the quantity

$$q(\tau) - \frac{p(\tau)}{m}(t(\tau) - t_0)$$

  does not depend on τ, and it is equal to the position of the particle at time t_0.

2.4 Constrained Quantization

Finally, we discuss the quantization of constrained Hamiltonian systems. There are many sophisticated quantization methods, such as BRST, but it will suffice to consider the simplest ones. First, we consider the case of second-class constraints.

- In canonical quantization, Poisson brackets are replaced with commutators. The resulting operators are then postulated to act irreducibly on a Hilbert space, allowing us to construct it. For example, a single (q, p) pair gives $\hat{q} = q$ and $\hat{p} = -i\hbar\,\partial/\partial q$ uniquely by the Stone-von Neumann theorem, leading to the Hilbert space $L^2(\mathbb{R})$.
- Second-class constraints are quantized by replacing the commutator with the Dirac bracket. For example, consider our trivial example again,

$$\chi_1 = q_1 \approx 0, \quad \chi_2 = p_1 \approx 0.$$

  Naive canonical quantization would give $[\hat{p}_1, \hat{q}_1] = -i\hbar$ which is inconsistent with the constraint $\hat{q}_1 = \hat{p}_1 = 0$. But with the Dirac bracket, $[\hat{p}_1, \hat{q}_1] = 0$, and there is no issue in imposing the operators equations $\hat{q}_1 = \hat{p}_1 = 0$.
- The disadvantage of this method is that it may be difficult to find a representation of the Dirac brackets. After using the second-class constraints to eliminate redundant degrees of freedom, we will have independent variables $\hat{y}^i$ satisfying the commutation relations

$$[\hat{y}^i, \hat{y}^j] = i\hbar\sigma^{ij}(\hat{y}^k).$$

  There is no general analogue of the Stone-von Neumann theorem that covers this case.
- However, as we've shown earlier, every second-class constraint can be turned into a first-class constraint by "undoing a gauge fixation", allowing us to return to Poisson brackets. Hence it also suffices to consider quantization of first-class constraints.

Next, we consider the quantization of first-class constraints.

- In reduced phase space quantization, we find a complete set of gauge-invariant functions and build the Hilbert space from those. For example, for a single first-class constraint $p_1 = 0$, a complete set of observables is $(q_2, p_2), \ldots, (q_N, p_N)$. All of these functions are gauge-invariant, and every function F obeying $[F, p_1] \approx 0$ is weakly equal to some function of them. Applying canonical quantization, the wavefunctions are functions of $q_2, \ldots, q_N$.
- In practice, finding such a complete set is quite difficult. Another way to carry out reduced phase space quantization is to perform a complete gauge fixing, reducing all remaining constraints to second class, which are handled with Dirac brackets. However, this has the same technical issues we saw above.
- The advantage of reduced phase space quantization is that every state in the Hilbert space is physical, and only gauge-invariant observables are realized as quantum mechanical operators. However, in practice this procedure is difficult and may destroy manifest invariance under an important symmetry, such as Lorentz symmetry. Furthermore, for field theories, the elimination of the gauge degrees of freedom generally destroys locality in space.

- In Dirac quantization, we simply naively canonically quantize everything, ignoring the constraints, then impose them by restricting to "physical states".
- Specifically, if the gauge generators are $\hat{G}_a$, then physical states should satisfy
$$e^{i\epsilon^a \hat{G}_a}|\psi\rangle = |\psi\rangle$$
or equivalently
$$\hat{G}_a|\psi\rangle = 0.$$
For example, for $p_1 = 0$, the Hilbert space contains wavefunctions $\psi(q_1, \ldots, q_N)$, and the physical state condition is $\partial\psi/\partial q_1 = 0$, equivalent to the reduced phase space result.
- At the classical level, the constraints G_a obey
$$[G_a, G_b] = C^c_{ab} G_c$$
and we expect this relation to be preserved quantum mechanically,
$$[\hat{G}_a, \hat{G}_b] = i\hbar \hat{C}^c_{ab} \hat{G}_c.$$
Then we will automatically have $[\hat{G}_a, \hat{G}_b]|\psi\rangle = 0$.
- However, it is possible that at the quantum level, there will be ordering ambiguities that make this impossible; instead we generally could have
$$[\hat{G}_a, \hat{G}_b] = i\hbar \hat{C}^c_{ab} \hat{G}_c + \hbar^2 \hat{D}_{ab}$$
and the physical states would have to obey $\hat{D}_{ab}|\psi\rangle = 0$ as well. This usually gives us far too few physical states; if we do not impose this condition, then we have a gauge anomaly: the gauge symmetry is broken at the quantum level, and the entire procedure above is not applicable.
- Similarly, at the classical level we have
$$[H_0, G_a] = V^b_a G_b$$
but at the quantum level we may have
$$[\hat{H}_0, \hat{G}_a] = i\hbar \hat{V}^b_a \hat{G}_b + \hbar^2 \hat{C}_a.$$
When $\hat{C}_a$ is nonzero, physical states are not closed under time evolution, spoiling the theory. However, the quantization may sometimes be carried out with a more advanced formalism such as BRST, where the ghosts play an essential role for consistency.

Dirac quantization can also be inconvenient because it is difficult to define a finite scalar product, as we can already see in our trivial example $p_1 = 0$ if q_1 has a noncompact range. The Dirac-Fock method avoids this issue, and works whenever there is an even number of first-class constraints. It is also called the Gupta-Bleuler method in field theory and string theory.

- We consider a system with N degrees of freedom and first-class constraints

$$p_1 = p_2 = 0.$$

  If we define

$$a = p_1 + ip_2, \quad b = -\frac{i}{2}(q^1 + iq^2)$$

  along with the conjugates a^* and b^*, then we have the Poisson brackets

$$[a, b^*] = [b, a^*] = -i$$

  with all others zero, and the constraints are equivalent to $a = a^* = 0$.

- At the quantum level, defining $a_\mu = (a, b)$, we have

$$[a_\mu, a_\nu^*] = \eta_{\mu\nu}, \quad \eta_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 9 \end{pmatrix}$$

  with all other commutators zero. Hence we have a set of two quantum harmonic oscillators with an indefinite metric.

- Defining the vacuum $|0\rangle$ to be annihilated by both a and b, we see that $a^* - b^*$ creates negative norm states while acting an odd number of times, $a^* + b^*$ creates positive norm states, and a^* and b^* each create states of zero norm. The creation operators generate an entire Fock space.

- The other physical degrees of freedom $(q^3, p_3), \ldots, (q^N, p_N)$ may be quantized as usual, giving wavefunctions $\psi(q^3, \ldots, q^N)$. A general state is the tensor product of one of these wavefunctions with a Fock state. There are hence no divergences when defining the norm of a state, but the norm may be negative.

- Next, we need to impose the constraints. Naively, we would demand

$$a|\psi\rangle = a^*|\psi\rangle = 0.$$

  However, this leaves us with no physical states at all, because the raising operator a^* has no nullspace. Instead, we take the weaker condition

$$a|\psi\rangle = 0.$$

  This is equivalent to demanding a physical state contains no b^* modes, and ensures that no negative-norm states are physical.

- We might wonder if this condition is sufficient. A general physical state may be written as

$$|\psi\rangle = f(q^3, \ldots, q^N)|0\rangle + |n\rangle$$

  where $|n\rangle$ is a "null spurious" state containing a^* excitations but no b^* excitations. The $|n\rangle$ have zero norm, and in fact zero overlap with every physical state. This is because we may always write $|n\rangle = a^*|\chi\rangle$, and for any physical state $|\psi\rangle$,

$$\langle\psi|n\rangle = \langle\psi|a^*|\chi\rangle = 0.$$

- Therefore, the null spurious states completely disappear from any physical matrix element, so one can consistently factor them out. That is, one can identify two physical states that differ by a null spurious state as the same state. In particular, each distinct physical state has a representative of the form $f(q^3, \ldots, q^N)|0\rangle$, and hence a^* is equivalent to the zero operator, imposing the other half of the constraint. The inner product on the reduced state space is positive definite.

- More generally, in the context of field theory, the analogue of imposing $a|\psi\rangle = 0$ is to impose $\hat{G}^{(-)}|\psi\rangle = 0$, where $\hat{G}^{(-)}$ is the annihilation part of $\hat{G}$. For each situation, one must check that there are physical null spurious states that decouple, to recover the second half of the gauge invariant. This requirement fixes $D = 26$ in bosonic string theory.

2.5 Classical Point Particle

To warm up for quantizing the string, we quantize a relativistic particle.

- There are generally two routes to a quantum theory: we may canonically quantize particle or field degrees of freedom. These two routes are called first and second quantization, respectively. In second quantization, one ends up with a theory of many particles, where the one-particle sector matches the result of first quantization.

- In general, string theory takes the first approach. The downside is that this approach is necessarily perturbative. The analogous second quantize formalism is called string field theory, where strings arise as excitations of a string field; little is known about this complicated subject.

- Viewing the configuration of a particle as a set of scalar fields on its worldline, the first quantized approach is formally analogous to a one-dimensional field theory. Similarly string theory is formally like a two-dimensional field theory.

- Formally, an elementary particle is a unitary irrep of the Poincare group, classified by its mass and spin. Physically, it is a particle without structure. Ignoring any internal degrees of freedom, the classical action of such a particle should hence only depend on properties of its worldline. Furthermore, dimensional analysis forbids any dependence on, e.g. the curvature of the worldline, as there are no other length scales.

- Given these assumptions, the unique relativistic particle action is the proper time,
$$S = -m \int dt \, \sqrt{1 - \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}.$$
To put time and space on an even footing, we can instead parametrize by τ,
$$S = -m \int d\tau \, \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}} = -m \int d\tau \, \sqrt{-\dot{x}^2}, \quad \dot{x} = \frac{dx}{d\tau}.$$
Here τ is an arbitrary, usually dimensionless parameter, and the action has reparametrization invariance, in the sense that $S[x'] = S[x]$ if $x'(\tau') = x(\tau)$ for any monotonic function $\tau'(\tau)$. In temporal gauge we set $\tau = t$, recovering our original action.

- The canonical momenta and equation of motion are
$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-x^2}}, \quad \frac{dp_\mu}{d\tau} = 0.$$

In particular, this yields the primary first-class constraint

$$p^2 + m^2 = 0.$$

At this point, Dirac quantization yields wavefunctions $\phi(x^\mu)$ and the constraint $p^2 + m^2 = 0$ means the wavefunctions obey the Klein-Gordan equation.

- In addition, the Hamiltonian is identically zero,

$$H = \dot{x}^\mu p_\mu - L = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} = 0$$

because the "time" variable τ has reparametrization invariance.

- Alternatively, we may completely fix the gauge, i.e. use reduced phase space quantization. Light cone gauge is the choice

$$x^+ = \frac{1}{m^2} p^+ \tau.$$

In this case, the $+$ component of the equation of motion immediately gives

$$\dot{x}^2 = -\frac{1}{m^2}$$

which simplifies the momenta and equation of motion to

$$p_\mu = m^2 \dot{x}_\mu, \quad \ddot{x}_\mu = 0.$$

Since we have removed the reparametrization invariance, the Hamiltonian no longer vanishes.

- The primary constraint may be used to solve for p^-,

$$p^- = \frac{1}{2p^+}(p^I p^I + m^2).$$

The value of p^- then determines the evolution of x^-, up to an integration constant x_0^-. Furthermore, x^+ is determined by p^+. Hence the independent dynamical variables are (x^I, p^I, x_0^-, p^+). We can straightforwardly quantize these variables because we have removed the gauge freedom and accounted for all the constraints.

- The disadvantage of this method is that we lose explicit Lorentz invariance. If we pressed on with the gauge symmetry intact, with the accompanying constraints, then we must impose the constraints at the quantum level, as we saw in the previous section. In this context, this is called covariant quantization.

- It is also useful to rewrite the point particle action with an einbein $e(\tau)$,

$$S = \frac{1}{2}\int d\tau \, e^{-1}\dot{x}^2 - em^2.$$

This action has reparametrization invariance,

$$\tau \to \tau', \quad x(\tau) \to x'(\tau') = x(\tau), \quad e \to e' = \frac{d\tau}{d\tau'} e$$

which is, infinitesimally,

$$\delta\tau = -\eta, \quad \delta x^\mu = \frac{dx^\mu}{d\tau}\eta, \quad \delta e = \frac{d}{d\tau}(\eta e)$$

where $\eta(\tau)$ is arbitrary. It is advantageous because it has no square roots, which makes it easier to handle in the path integral, and it can handle massless particles just as well as massive ones.

- Naively, if we use the reparametrization invariance to set $e = 1$, then the equation of motion for x is simply $\ddot{x} = 0$. However, this isn't quite right, because we've forgotten about the equation of motion for e, which is
$$\dot{x}^2 + e^2 m^2 = 0.$$
In the massive case, this tells us $\dot{x}$ is normalized to be the four-momentum. In the massless case, it tells us that $\dot{x}$ is null.

- To return to our original action in the massive case, we solve the equation of motion of e for e, and plug it back into the action to eliminate it; this is possible since e is an auxiliary field.

- Formally, one can think of this action as corresponding to a one-dimensional quantum gravity theory. This is easier to see if we write $e = \sqrt{-g_{\tau\tau}}$, so
$$S = -\frac{1}{2}\int d\tau\, \sqrt{-g_{\tau\tau}}(g^{\tau\tau}\dot{x}^2 + m^2).$$

- When we quantize the string, our actions here will correspond to the Nambu-Goto and Polyakov actions. The Polyakov action can be thought of in terms of two-dimensional quantum gravity on the worldsheet, and we will try to quantize it both covariantly and in light cone gauge.

We now explicitly quantize the relativistic point particle in light cone gauge.

- Starting from the Lagrangian in light cone gauge, we can show that (x^I, p^I) and (x_0^-, p^+) are conjugate variable pairs, so that in canonical quantization we have
$$[x^I, p^J] = i\eta^{IJ}, \quad [x_0^-, p^+] = i\eta^{-+} = -i.$$
In Heisenberg picture, these commutators hold when the operators are evaluated at equal times. We can then define the redundant operators
$$x^+ = \frac{p^+}{m^2}\tau, \quad x^- = x_0^- + \frac{p^-}{m^2}\tau, \quad p^- = \frac{1}{2p^+}(p^I p^I + m^2).$$
Note that p^- has no explicit τ-dependence, though in Heisenberg picture it has τ-dependence via p^+ and p^I.

- We know that H generates τ translations, and we expect p^- to generate x^+ evolution. Since these are proportional, we have
$$H = \frac{p^+}{m^2}p^- = \frac{1}{2m^2}(p^I p^I + m^2).$$
Note that unusually, H is dimensionless, because τ is.

- It's easy to check the Heisenberg equations of motion match the classical Hamilton's equations. For example, we have

$$i\frac{dp^+}{d\tau} = [p^+, H] = 0, \quad i\frac{dp^I}{d\tau} = [p^I, H] = 0, \quad i\frac{dx^I}{d\tau} = [x^I, H] = i\frac{p^I}{m^2}$$

  where the last result gives

$$x^I = x_0^I + \frac{p^I}{m^2}\tau$$

  as expected. We also have

$$i\frac{dx_0^-}{d\tau} = [x_0^-, H] = 0$$

  which is as expected, since x_0^- is a constant of the motion.

- We can choose (p^+, p^I) as a maximum commuting set and hence label the states of the point particle by their eigenvalues, as $|p^+, \mathbf{p}_T\rangle$. In this basis, the Hamiltonian is diagonal.

- For a general state $|\psi\rangle$ we may define a wavefunction by

$$|\psi\rangle = \int dp^+ d\mathbf{p}_T\, \psi(p^+, \mathbf{p}_T)|p^+, \mathbf{p}_T\rangle$$

  and the wavefunction obeys the Schrodinger equation

$$i\frac{\partial}{\partial\tau}\psi = \frac{1}{2m^2}(p^I p^I + m^2)\psi.$$

  Of course, up to rescaling this matches the equation of motion for the Klein-Gordan field, providing an example of the equivalence of first and second quantization: the equation of motion for a classical field matches the equation of motion for the one-particle wavefunction of the second quantized field, which in turn matches the Schrodinger equation in first quantization.

- Historically, this coincidence of equations led to confusion, as physicists thought the classical field that was the starting point for second quantization was the first quantized wavefunction itself, leading to the name. This is conceptually incorrect since the first quantized theory is already quantum; there is no need to quantize it again. In the modern view, the equivalence of first and second quantization is so well-known that in condensed matter, the two are introduced as slightly different ways of describing the same theory, i.e. by many-body wavefunctions or occupation numbers.

To discuss conserved quantities, it will be useful to remove the gauge fixing.

- Without the gauge fixing, we have canonical commutators

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}.$$

  This is quite different from quantization in light cone gauge. For instance, the commutator $[x^+, p^-]$ vanishes in light cone gauge but not here, where $[x^+, p^-] = i\eta^{+-} = -i$. In other words, the Poisson bracket structure in light cone gauge is not merely a restriction of the structure without gauge fixing. Conceptually, we must distinguish between objects in light cone gauge and objects merely written in light cone coordinates, which unfortunately have identical notation.

- As expected, the operators p^μ generate translations of the particle, so that

$$\delta x^\mu = [i\epsilon_\nu p^\nu, x^\mu] = \epsilon^\mu.$$

We would like to confirm the same thing occurs in light cone gauge. We may expand

$$i\epsilon_\nu p^\nu = -i\epsilon^- p^+ - i\epsilon^+ p^- + i\epsilon^I p^I.$$

It is clear that the p^I generate translations in x^I, and that p^+ generates translations in x^-.

- It is less clear that the same holds for p^-, since it is determined in terms of the other momenta. In this case we have

$$\delta x^\mu = -i\epsilon^+ [p^-, x^\mu].$$

Taking specific values for μ, we find

$$\delta x^+ = -i\epsilon^+ [p^-, x^+] = 0, \quad \delta x^I = -i\epsilon^+ [p^-, x^I] = -i\epsilon^+ \frac{1}{2p^+}(-2ip^I) = -\epsilon^+ \frac{p^I}{p^+}.$$

The trickiest commutator to compute is

$$\delta x^- = -i\epsilon^+ \left[p^-, x_0^- + \frac{p^-}{m^2}\tau\right] = -i\epsilon^+ [p^-, x_0^-] = \frac{p^I p^I + m^2}{2} \left[p^-, \frac{1}{p^+}\right].$$

To finish this evaluation, note that

$$\left[p^-, \frac{1}{p^+}\right] = \frac{1}{p^+}[p^+, x_0^-]\frac{1}{p^+} = \frac{i}{{p^+}^2}.$$

In conclusion, we have

$$\delta x^+ = 0, \quad \delta x^I = -\epsilon^+ \frac{p^I}{p^+}, \quad \delta x^- = -\epsilon^+ \frac{p^-}{p^+}.$$

- This result is very different from what we expect. The resolution is that, even though we have removed the diffeomorphism symmetry, the action retains a symmetry under τ translations, which corresponds to

$$\delta x^\mu = \lambda \dot{x}^\mu.$$

In this case, the action of p^- generates a translation in x^+ plus a translation in τ by $\lambda = -m^2\epsilon^+/p^+$. This is necessary to set $\delta x^+ = 0$, which preserves the light cone gauge condition.

- Similarly, the infinitesimal Lorentz transformations and conserved charges take the form

$$\delta x^\mu = \epsilon^{\mu\nu} x_\nu, \quad M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu.$$

It is straightforward to see the conserved charges generate the transformations in covariant quantization. In light cone gauge, we wish to construct similar operators which generate the same transformations (up to τ translations) and obey the Lorentz algebra.

- The calculation here is a bit involved, but it turns out to be possible for the point particle. However, it turns out that for the relativistic string, it is only possible if $D = 26$.

3 Quantizing the String

3.1 The Polyakov Action

We now introduce the Polyakov action, the analogue of the einbein action for strings.

- First, recall that the Nambu-Goto action can be written in terms of the worldsheet metric,

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma}.$$

  We can derive the equations of motion from this form directly, using

$$\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$$

  and the definition of $\gamma_{\alpha\beta}$, which gives

$$\partial_\alpha(\sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0.$$

- The Polyakov action removes the square root by introducing another field $g_{\alpha\beta}$ on the worldsheet,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$

  Note that here we restrict to flat spacetime, and σ conventionally stands for both worldsheet coordinates. The new field $g_{\alpha\beta}$ is a metric with signature $(-+)$, so this is a two-dimensional quantum gravity theory on the world sheet, interacting with worldsheet scalars X^μ.

- One might think of including the Einstein term $\sqrt{-g}\,R$, but this is a total derivative in $1+1$ dimensions. A cosmological constant term $\sqrt{-g}$ would ruin the Weyl symmetry, so we forbid it.

- The equation of motion for X^μ is simply

$$\partial_\alpha(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta X^\mu) = 0$$

  which resembles the Nambu-Goto equation of motion, except that $g_{\alpha\beta}$ has its own dynamics.

- The equation of motion for $g_{\alpha\beta}$ is

$$\left(\sqrt{-g}\,\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu\right)\eta_{\mu\nu} = 0$$

  which allows us to solve for the worldsheet metric,

$$g_{\alpha\beta} = 2f(\sigma)\,\partial_\alpha X \cdot \partial_\beta X, \quad f^{-1} = g^{\rho\sigma}\,\partial_\rho X \cdot \partial_\sigma X.$$

  We see that $g_{\alpha\beta}$ matches $\gamma_{\alpha\beta}$ up to a conformal factor f. However, since the Polyakov action only depends on $g_{\alpha\beta}$ by the combination $\sqrt{-g}\,g^{\alpha\beta}$, f cancels out upon substituting it back in, recovering the Nambu-Goto action; note that this cancellation only holds in two dimensions.

- Like the Nambu-Goto action, the Polyakov action has Poincare invariance,

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu.$$

Both actions also have reparametrization invariance, i.e. diffeomorphisms of the worldsheet. That is to say, the reparametrization $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ induces the transformations

$$X^\mu(\sigma) \to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma), \quad g_{\alpha\beta}(\sigma) \to \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha}\frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma)$$

which are to be regarded as gauge symmetries. The infinitesimal gauge transformation induced by $\sigma^\alpha \to \tilde{\sigma}_\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$ is

$$\delta X^\mu(\sigma) = \eta^\alpha \partial_\alpha X^\mu, \quad \delta g_{\alpha\beta}(\sigma) = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha$$

where the covariant derivative is defined by the Levi-Civita connection of the worldsheet metric.

- The Polyakov action further has Weyl invariance, special to a two-dimension worldsheet,

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma).$$

Infinitesimally, writing $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ we have

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma) g_{\alpha\beta}(\sigma).$$

We have seen above why the Polyakov action is invariant under a Weyl transformation. Because the symmetry is local (i.e. Ω is a function on the worldsheet, not a constant) we interpret it as a gauge symmetry. As we'll see below, this choice ensures that $g_{\alpha\beta}$ doesn't introduce any new degrees of freedom. Weyl invariance is quite rare and strongly constrains interaction terms that can be added to the action; at the quantum level it also constrains $D = 26$.

- Like the Nambu-Goto action, we may fix a gauge to make concrete progress. The worldsheet metric has three independent components, so using reparametrization invariance we may fix

$$g_{\alpha\beta} = e^{2\phi} \eta_{\alpha\beta}$$

which is known as conformal gauge. We can further use Weyl transformations to set $g_{\alpha\beta} = \eta_{\alpha\beta}$, making the metric flat.

- Since the curvature of the metric isn't changed by reparametrizations, we should also be able to see that a Weyl transformation alone can make the metric flat. It can be shown that under a Weyl transformation $g'_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}$ we have

$$\sqrt{g'} R' = \sqrt{g}(R - 2\nabla^2 \phi)$$

which gives a differential equation for ϕ which may be used to set $R = 0$. Since the Riemann tensor has only one degree of freedom in two dimensions, this implies the metric is flat.

- Upon setting $g_{\alpha\beta} = \eta_{\alpha\beta}$ in the Polyakov action, we simply have

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \cdot \partial^\alpha X$$

which gives the simple equation of motion

$$\partial_\alpha \partial^\alpha X^\mu = 0.$$

We've seen this equation of motion for several gauge choices in the Nambu-Goto action before.

- There are constraints due to the equation of motion for $g_{\alpha\beta}$. It is convenient to write them as

$$T_{\alpha\beta} = -\frac{2}{T_0}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = 0$$

  where T_0 is the string tension. The vanishing of the stress-energy tensor is due to reparametrization invariance, just like the vanishing of the Hamiltonian for the point particle.

- Setting $g_{\alpha\beta} = \eta_{\alpha\beta}$, we have

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X.$$

  The vanishing of the stress-energy tensor gives the constraints

$$T_{01} = T_{10} = \dot{X} \cdot X' = 0, \quad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$

  which are just what we found earlier in light cone gauge. In terms of components of $g_{\alpha\beta}$, they simply reiterate that the metric takes the required flat form; also note that Weyl invariance alone guarantees $\operatorname{tr} T = 0$ and hence $T_{00} = T_{11}$.

Next, we write down the mode expansion.

- For reference, we are taking the conventions

$$\ell^2 = 2\alpha' = \frac{1}{\pi T_0}.$$

  Later, we will set $\ell = 1$, so that $\alpha' = 1/2$.

- Ignoring the constraint $T_{\alpha\beta} = 0$ for now, for a closed string with $\sigma \in [0, \pi]$, decomposing into left-moving and right-moving solutions gives

$$X^\mu(\tau, \sigma) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma)$$

  where we conventionally define

$$X^\mu_R(u) = \frac{x^\mu}{2} + \frac{\ell^2 p^\mu}{2}u + \frac{i\ell}{2}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2inu}, \quad X^\mu_L(u) = \frac{x^\mu}{2} + \frac{\ell^2 p^\mu}{2}u + \frac{i\ell}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2inu}.$$

  Reality of X^μ implies x^μ and p^μ are real, and $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$. The string length ℓ is related to the tension by $\ell^2 = 1/\pi T_0$, and later we will set it to one.

- For an open string with Neumann boundary conditions ($X' = 0$ at endpoints), the general solution is

$$X^\mu(\tau, \sigma) = x^\mu + \ell^2 p^\mu \tau + i\ell\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in\tau}\cos(n\sigma).$$

  That is, the left-moving and right-moving waves are forced to combine into standing waves, $\alpha^\mu_n = \tilde{\alpha}^\mu_n$. For now we put aside Dirichlet boundary conditions, returning to the subject later.

- With the definition of ℓ as above, the Noether charge for translational symmetry is simply

$$P^\mu = T_0 \partial_\tau X^\mu = p^\mu.$$

Next, we consider the Noether charge for Lorentz symmetry,

$$M^{\mu\nu} = \int J^{\mu\nu}_\tau(\tau,\sigma)\, d\sigma, \quad J^{\mu\nu}_a(\tau,\sigma) = T_0(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu).$$

Evaluating this by using the above solutions, we find

$$M^{\mu\nu} = \begin{cases} \ell^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{closed} \\ \ell^{\mu\nu} + E^{\mu\nu} & \text{open} \end{cases}$$

where

$$\ell^{\mu\nu} = x^\mu p^\mu - x^\nu p^\mu, \quad E^{\mu\nu} = -i \sum_{n>0} \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), \quad \tilde{E}^{\mu\nu} = -i \sum_{n>0} \frac{1}{n}(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n).$$

- Next, we impose the constraint $T_{\alpha\beta} = 0$. This is easiest if we switch to light cone coordinates,

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma), \quad \eta_{+-} = \eta_{-+} = -\frac{1}{2}, \quad \eta^{+-} = \eta^{-+} = -2.$$

By Weyl symmetry, our general solution above automatically satisfies $T_{00} = T_{11}$, which implies $T_{-+} = T_{+-} = 0$. As for the other components,

$$T_{++} = \partial_+ X \cdot \partial_+ X, \quad T_{--} = \partial_- X \cdot \partial_- X.$$

- By translational symmetry on the worldsheet, the stress-energy tensor is conserved for our general solutions above, so they obey

$$\partial_+ T_{-+} + \partial_- T_{++} = \partial_+ T_{--} + \partial_- T_{-+} = 0.$$

Combining with the previous result, we have

$$\partial_- T_{++} = \partial_+ T_{--} = 0.$$

Thinking of the worldsheet as the complex plane, we can think of T_{++} as a holomorphic function and T_{--} as antiholomorphic.

- This result leads to an infinite number of conserved charges,

$$Q_f = \int d\sigma\, f(\sigma_+) T_{++}(\sigma_+)$$

for any function f, because $\partial_-(fT_{++}) = 0$, so

$$\frac{\partial Q_f}{\partial \tau} = \int d\sigma\, \partial_\tau(fT_{++}) = \int d\sigma\, \partial_\sigma(fT_{++}) = 0.$$

- Geometrically, the reason for these conserved quantities is that there is residual diffeomorphism invariance, namely conformal transformations whose effect on the metric can be cancelled by a Weyl rescaling. Such diffeomorphisms are generated by a vector field ξ satisfying
$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}.$$
This doesn't violate our parameter counting earlier, as this remaining gauge freedom is of "measure zero" compared to the original freedom. However, it remains infinite-dimensional, which is special to two dimensions. These symmetries are generated by the Q_f with $f \sim \xi^+$.

At this point, we impose the constraint $T_{\alpha\beta} = 0$.

- Before continuing, it is useful to compute the Poisson brackets. For closed strings, starting with
$$[\dot{X}^\mu(\sigma), X^\nu(\sigma')] = \frac{1}{T_0}\delta(\sigma - \sigma')\eta^{\mu\nu}$$
with all other Poisson brackets zero, we easily find
$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = im\delta_{m+n,0}\eta^{\mu\nu}, \quad [p^\mu, x^\nu] = \eta^{\mu\nu}$$
with all others zero. These also hold for $m, n = 0$, where we define
$$\alpha_0^\mu = \begin{cases} \ell p^\mu & \text{open} \\ \ell p^\mu/2 & \text{closed} \end{cases}, \quad \tilde{\alpha}_0^\mu = \alpha_0^\mu \text{ for closed.}$$
We see the position and momentum of the string are canonically conjugate, and the Fourier modes α_n^μ for $n \neq 0$ are harmonic oscillator coordinates with conjugate variable α_{-n}^μ. The solution for open strings has been normalized so that it obeys the same set of Poisson brackets, without the extra $\tilde{\alpha}_n^\mu$.

- Another straightforward calculation shows that the Hamiltonian is
$$H = \frac{T_0}{2}\int_0^\pi (\dot{X}^2 + X'^2)\, d\sigma = \frac{1}{2}\sum_n \begin{cases} \alpha_{-n}\cdot\alpha_n & \text{open} \\ \alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n & \text{closed} \end{cases}.$$

- The nontrivial content of the constraint $T_{\alpha\beta} = 0$ is $T_{++} = T_{--} = 0$. For closed strings, defining
$$L_m = \frac{T_0}{2}\int_0^\pi d\sigma\, e^{2im\sigma_-} T_{--}, \quad \tilde{L}_m = \frac{T_0}{2}\int_0^\pi d\sigma\, e^{2im\sigma_-} T_{++}$$
it is sufficient to show that the Fourier components L_m and $\tilde{L}_m$ all vanish. We have $T_{--} = \dot{X}_R^2$ and $T_{++} = \dot{X}_L^2$, so
$$L_m = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n, \quad \tilde{L}_m = \frac{1}{2}\sum_n \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n.$$

- For open strings, we can get a similar expression if we formally extend the range of σ to $[0, 2\pi]$, defining $X_R(\sigma + \pi) = X_L(\sigma)$ and $X_L(\sigma + \pi) = X_R(\sigma)$. In this case, open string boundary conditions imply X_R is periodic with period 2π. The constraints imply that T_{++} vanishes on $[-\pi, \pi]$, which is equivalent to the vanishing of the Fourier components
$$L_m = T_0 \int_0^\pi e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}\, d\sigma = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n.$$
The constraint for T_{--} is redundant.

- Note in particular that
$$H = \begin{cases} L_0 & \text{open} \\ L_0 + \tilde{L}_0 & \text{closed} \end{cases}.$$
The constraint $L_0 = \tilde{L}_0 = 0$ and definition $M^2 = -p_\mu p^\mu$ gives the mass shell conditions
$$M^2 = \frac{1}{\alpha'} \sum_{n>0} \begin{cases} \alpha_{-n} \cdot \alpha_n & \text{open} \\ 2(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) & \text{closed} \end{cases}$$
where the two terms in the closed case give equal contributions. At the quantum level, these results will be modified due to normal ordering effects.

- By another straightforward calculation, we find that
$$[L_m, \alpha_n^\mu] = -in\alpha_{m+n}^\mu$$
along with the Witt algebra
$$[L_m, L_n] = i(m-n)L_{m+n}.$$

- This appearance of this algebra has a simple interpretation. A complete basis for diffeomorphisms of the circle is
$$D_n = ie^{in\theta} \frac{d}{d\theta}$$
and these satisfy the Witt algebra, so it is the algebra of infinitesimal diffeomorphisms of the circle. In fact, the transformations generated by the L_n and $\tilde{L}_n$ correspond to the worldsheet diffeomorphisms generated by $e^{2in\sigma^\pm} \partial_\pm$, where the $\sigma^\pm$ behave like angular variables because solutions to the equations of motion are periodic in them.

3.2 Old Covariant Quantization

We now continue with the quantization of the string. There are several possible approaches.

- In light cone quantization, we fix all gauge symmetry by going to light cone gauge, and solve all of the constraints of the system to determine the space of physically distinct classical solutions. This is the analogue of Coulomb gauge in QED, but loses manifest Lorentz invariance.

- In old covariant quantization, one quantizes the string in conformal gauge, then imposes the constraints $T_{++} = T_{--} = 0$ at the quantum level on the operators. This is the analogue of Gupta-Bleuler quantization in Lorenz gauge in QED.

- In covariant BRST quantization, one uses the path integral instead. One must be careful to account for the diffeomorphism and Weyl gauge symmetries, which leads to Faddeev-Popov ghosts and BRST cohomology, as we saw for Yang-Mills theory.

In this section, we use old covariant quantization, focusing on the closed string.

- As usual in canonical quantization, we replace Poisson brackets with commutators. The relations in the previous section can be converted by multiplying the right-hand sides by $-i$, so
$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \quad [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$$
where $\hat{x}$ and $\hat{p}$ are Hermitian, and $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu$.

- The α_n^μ for $n > 0$ can be interpreted as annihilation operators for a harmonic oscillator, with α_{-n}^μ the corresponding creation operators. Ignoring $\hat{x}$ and $\hat{p}$ for now, we define the vacuum state $|0\rangle$ to be annihilated by all the α_n^μ for $n > 0$, and build up the Fock space by acting with α_{-n}^μ, all as usual. However, unlike in quantum field theory, the vacuum state should be interpreted as the lowest energy state of a *single* string, not the absence of any strings.

- We define the right-moving "number operator"

$$N = \sum_{k>0} \alpha_{-k} \cdot \alpha_k$$

with a similar definition for $\tilde{N}$, and we say a state is at level n if its eigenvalue of N is n. Then

$$N \left(\prod_i \alpha_{n_i}^{\mu_i} |0\rangle \right) = \sum_i n_i.$$

- Now we need to account for the zero mode associated with $\hat{x}$ and $\hat{p}$. This should be interpreted as generating the Hilbert space for a free particle. We may define the states

$$\hat{p}^\mu |p\rangle = p^\mu |p\rangle, \quad \langle p|p'\rangle = \delta(p - p')$$

and the resulting Hilbert space is just $L^2(\mathbb{R}^{1,D-1})$. The full Hilbert space is the tensor product of this with the Fock space associated with the harmonic oscillators, and we write the ground state as $|0, p\rangle$.

- The Poincare charges are promoted to the operators

$$P^\mu = \hat{p}^\mu, \quad M^{\mu\nu} = \hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu - i \sum_{n>0} \frac{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu}{n} - i \sum_{n>0} \frac{\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu}{n}$$

which obey the expected Poincare algebra. This is the benefit of working covariantly. Note that demanding $M^{\mu\nu}$ be antisymmetric eliminates the ordering ambiguity.

- The drawback is the need to impose the constraints $L_n = 0$. For $n \neq 0$, we have

$$L_n = \frac{1}{2} \sum_k \alpha_{n-k} \cdot \alpha_k, \quad L_{-n} = L_n^\dagger$$

unambiguously, but for $n = 0$ there is an ordering ambiguity. The naive ordering above is unacceptable because it yields infinity when acting on the vacuum, so a better ordering is the normal ordering

$$L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{k>0} \alpha_{-k} \cdot \alpha_k.$$

There is a similar story for the $\tilde{L}_n$.

- Because of this correction, the Witt algebra becomes the Virasoro algebra,

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{D}{12} (m^3 - m) \delta_{m+n,0}.$$

This is a central extension of the Witt algebra, and the new term is sometimes called an "anomaly". Note that the algebra of $\{L_{-1}, L_0, L_1\}$ is unmodified; these are the generators of $\mathfrak{sl}(2, \mathbb{R})$, the conformal transformations which also exist in $d > 2$.

- Further results that will be useful below are

$$[L_m, \alpha_n^\mu] = -n\alpha_{m+n}^\mu, \quad [L_m, x^\mu] = \begin{cases} -i\ell\alpha_m^\mu & \text{open} \\ (-i\ell/2)\alpha_m^\mu & \text{closed} \end{cases}$$

  which may be combined to show that for closed strings,

$$[L_m, X^\mu] = -ie^{2im\sigma_-}\partial_- X^\mu$$

  which confirms the interpretation of the L_m given above.

- A problem related to the constraints is the presence of ghosts, i.e. negative norm states. These are generated by the α_n^0 because of the indefinite sign of the metric, just like we saw in QED. There are also zero norm states, such as $(\alpha_n^0 + \alpha_n^1)|0, p\rangle$, which arise generically in gauge theories. These are the states generated by gauge transformations, which must necessarily have zero norm because gauge transformations cannot affect probabilities. As in QED, the hope is that fixing a gauge, which removes the zero norm states, will simultaneously decouple the negative norm states, so that they cannot be produced in physical processes.

Note. A very heuristic way to understand the appearance of the 1/12 is that the difference between our L_0 and the original ordering is an additive factor of $D\sum_{k>0} k$, which by zeta function regularization is $-D/12$.

At a slightly more respectable level, we must add a cosmological constant term to preserve conformal invariance at the quantum level; in a suitable regularization, this soaks up the divergent part of the sum $D\sum_{k>0} k$ but leaves behind the finite part $-D/12$.

Note. There is an annoying sign issue here: usually in canonical quantization we multiply the result of a Poisson bracket by i to get a commutator, not $-i$. The reason is that canonical momenta are naturally covectors, so the contravariant momenta p^μ pick up a relative minus sign due to our unusual $(-+++)$ metric convention, causing the sign flip.

Next, we impose the classical constraints $L_n = \tilde{L}_n = 0$.

- As for QED, imposing that $L_n = \tilde{L}_n = 0$ as an operator equation is too strong. For example, we would necessarily have $[L_m, L_n] = 0$, but then the Virasoro algebra cannot be satisfied.

- Instead, as in the Gupta-Bleuler quantization of QED, we only demand that the L_n and $\tilde{L}_n$ have vanishing matrix elements within the subspace of physical states. Since $L_n^\dagger = L_{-n}$, it is sufficient to require

$$L_n|\text{phys}\rangle = \tilde{L}_n|\text{phys}\rangle = 0, \quad n > 0.$$

  For $n = 0$, we also add a normal ordering constant

$$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0$$

  accounting for the fact that we don't know the proper ordering to define L_0 and $\tilde{L}_0$ a priori. Recall that we need to remove the timelike oscillators; since we have $L_n \sim p \cdot \alpha_n + \cdots$ where p is timelike, this procedure stands a chance of working.

- The introduction of a modifies the mass shell constraint for closed strings to

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{k>0} \alpha_{-k} \cdot \alpha_k\right) = \frac{4}{\alpha'}\left(-a + \sum_{k>0} \tilde{\alpha}_{-k} \cdot \tilde{\alpha}_k\right)$$

  which constrains the allowed values of p for a state with given oscillator excitations. It also yields the "level matching" constraint $N = \tilde{N}$.

- Note that $[L_n, M^{\mu\nu}] = 0$. This implies the physical state conditions are invariant under Lorentz transformations, so the physical states will form Lorentz multiplets.

- For simplicity, we focus on the open string case. The reasoning for the closed string is very similar, with essentially two copies of the theory (left-moving and right-moving) plus the level matching constraint. For open strings, we instead have

$$M^2 = \frac{1}{\alpha'}\left(-a + \sum_{k>0} \alpha_{-k} \cdot \alpha_k\right).$$

  At level zero, the states $|0, k\rangle$ hence have mass squared $M^2 = -a/\alpha'$.

- Also note that the string Hamiltonian is modified to

$$H = \begin{cases} L_0 - a & \text{open} \\ L_0 + \tilde{L}_0 - 2a & \text{closed} \end{cases}.$$

- Now consider the states at level one, given by

$$\zeta \cdot \alpha_{-1}|0, k\rangle$$

  for a polarization vector $\zeta^\mu(k)$. These states have $M^2 = (1-a)/\alpha'$, and the L_1 condition implies that $\zeta \cdot k = 0$, giving $D - 1$ allowed polarizations, where the norm of the state is $\zeta \cdot \zeta$.

- If $a > 1$, then these states are tachyonic, so it is possible to rotate k to have no time component. Then one of the physical states has a timelike polarization and negative norm, so we require

$$a \leq 1.$$

  When $a < 1$ the mass is positive, and we get $D - 1$ spacelike polarizations, to be interpreted as a massive vector particle.

- In the boundary case $a = 1$ the particle is massless; accordingly one of the physical polarizations is $\zeta^\mu = k^\mu$ with zero norm. As in the Gupta-Bleuler quantization of QED, this state decouples from the S-matrix, as we will see below.

Next, we define spurious states.

- In general, we define a state $|\psi\rangle$ to be spurious if

$$(L_0 - a)|\psi\rangle = 0, \quad \langle\phi|\psi\rangle = 0$$

for all physical states $|\phi\rangle$. All spurious states can be written in the form

$$|\psi\rangle = \sum_{n>0} L_{-n}|\chi_n\rangle$$

where the $|\chi_n\rangle$ satisfy

$$(L_0 - a + n)|\chi_n\rangle = 0.$$

In fact, since all L_{-n} can be constructed as commutators of L_{-1} and L_{-2}, the sum can be stopped at $n = 2$. Hence the general spurious state is

$$|\psi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle.$$

- A state can be both spurious and physical, in which case they must be null. For example, consider states of the form

  $$|\psi\rangle = L_{-1}|\tilde{\chi}\rangle, \quad L_m|\tilde{\chi}\rangle = 0 \text{ for } m > 0, \quad (L_0 - a + 1)|\tilde{\chi}\rangle = 0.$$

  The physical state conditions are automatically satisfied, except for the L_1 condition, where

  $$L_1|\psi\rangle = L_1 L_{-1}|\tilde{\chi}\rangle = 2L_0|\tilde{\chi}\rangle.$$

  This only vanishes for $a = 1$. We interpret spurious physical states as gauge equivalent to zero. For example, when $a = 1$ we have seen there is an extra massless state at level one; this is rendered unphysical since we may take $|\tilde{\chi}\rangle = |0, k\rangle$. Hence at level one we have a massless vector particle, corresponding to a gauge field.

- Now fixing $a = 1$, consider spurious states with the structure

  $$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\tilde{\chi}\rangle, \quad L_m|\tilde{\chi}\rangle \text{ for } m > 0, \quad (L_0 + 1)|\tilde{\chi}\rangle = 0.$$

  The latter condition ensures that $(L_0 - 1)|\psi\rangle = 0$. The physical state conditions $L_m|\psi\rangle = 0$ for $m > 2$ are always satisfied, so we only need impose $L_1|\psi\rangle = L_2|\psi\rangle = 0$. It turns out these are satisfied when

  $$\gamma = \frac{3}{2}, \quad D = 26$$

  so that there are many more spurious physical states in $D = 26$.

- Furthermore, it is possible to construct physical states of negative norm in $D > 26$. In fact, one can show the spectrum is ghost-free provided that $a = 1$ and $D = 26$, or $a \leq 1$ and $D \leq 25$. In the former case, there are many more zero-norm states, and the physical spectrum corresponds to 24 sets of α oscillators, while in the latter case it corresponds to $D - 1$ oscillators.

- Physically, we say the string has only transverse excitations in $D = 26$ but also longitudinal oscillations in lower dimension. Since the gauge symmetry is evidently much larger in $D = 26$, we will focus on this case. This formally contains the cases with $D < 26$ by restricting the momenta.

3.3 Computing Spectra

Now we'll use the results above to investigate the low-lying spectra of open and closed strings.

Example. The physical Hilbert space at level two for the open string with $a = 1$. We parametrize the states as

$$|g, \epsilon, p\rangle = (g_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + \epsilon_\mu \alpha^\mu_{-2})|0, p\rangle$$

where $g_{\mu\nu}$ may be taken symmetric, giving $D(D+1)/2 + D$ candidate physical states. Note that

$$L_0|0, p\rangle = \frac{1}{2}(\alpha_0 \cdot \alpha_0)|0, p\rangle = \frac{1}{2}p^2|0, p\rangle$$

where we have set $\ell = 1$. Now consider the physical state condition $(L_0 - 1)|g, \epsilon, p\rangle$. By commuting the L_0 to the right,

$$\left(\frac{1}{2}p^2 - 1 + 2\right)|g, \epsilon, p\rangle = 0$$

which shows that $m^2 = -p^2 = 2$, so the states have positive mass, and

$$L_0|0, p\rangle = -|0, p\rangle.$$

The physical state conditions $L_k|g, \epsilon, p\rangle = 0$ are trivial for $k > 2$. For $k = 1$ we find

$$\left(g_{\mu\nu}\alpha^\mu_0\alpha^\nu_{-1} + g_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_0 + 2\epsilon_\mu\alpha^\mu_{-1}\right)|0, p\rangle = 0.$$

Since α_{-1} is a raising operator, it must act on the zero state, giving the constraint

$$g_{\mu\nu}p^\nu + \epsilon_\mu = 0.$$

Next, for $k = 2$ we have

$$\left(g_{\mu\nu}\alpha^\mu_1\alpha^\nu_{-1} + g_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_1 + 2\epsilon_\mu\alpha^\mu_0\right)|0, p\rangle$$

and since $[\alpha^\mu_1, \alpha^\nu_{-1}] = \eta^{\mu\nu}$ this gives

$$g_{\mu\nu}\eta^{\mu\nu} + 2\epsilon_\mu p^\mu = 0.$$

These are the full physical state conditions, which give a total of $D + 1$ constraints on $g_{\mu\nu}$ and ϵ_μ.

Next, the most general spurious state at level two is

$$|\tilde{\epsilon}, \tilde{\gamma}, p\rangle = (L_{-1}\,\tilde{\epsilon} \cdot \alpha_{-1} + \tilde{\gamma}L_{-2})|0, p\rangle$$

by the reasoning above. The simplest way to impose the physical state condition is simply to expand the expression above in terms of oscillator modes and use our earlier result. This gives

$$|\tilde{\epsilon}, \tilde{\gamma}, p\rangle = \left[\frac{1}{2}(\tilde{\gamma}\eta_{\mu\nu} + \tilde{\epsilon}_\mu p_\nu + \tilde{\epsilon}_\nu p_\mu)\alpha^\mu_{-1}\alpha^\nu_{-1} + (\tilde{\epsilon} + \tilde{\gamma}p) \cdot \alpha{-2}\right]|0, p\rangle.$$

The two physical state conditions are

$$3\tilde{\gamma} + \tilde{\epsilon} \cdot p = 0, \quad 3\tilde{\epsilon} \cdot p + \frac{D}{2}\tilde{\gamma} - 4\tilde{\gamma} = 0$$

for $k = 1$ and $k = 2$ respectively. For $D = 26$, they are redundant, so there are D spurious physical states; for $D < 26$ we have $\tilde{\gamma} = \tilde{\epsilon} \cdot p = 0$, giving $D - 1$ spurious physical states. Therefore, accounting for the constraints and the spurious physical states, we have $D(D-1)/2 - 1$ states in $D = 26$, and $D(D-1)/2$ states otherwise. These are the number of degrees of freedom in a symmetric $SO(D-1)$ tensor, which is traceless in $D = 26$. In $D = 26$, this is to be interpreted as a massive spin 2 particle.

The analysis for the closed string is more complicated, so for variety we'll use lightcone quantization.

- In lightcone quantization, the counting is more straightforward, as there are precisely $D-2$ physical oscillator modes. Fixing $a=1$ and setting $\ell=1$ again, we have:
  - At level zero, the states $|0,k\rangle$ have mass squared $M^2=-8$.
  - At level one, there are $(D-2)^2$ massless states, which corresponds to the rank two tensor of $SO(D-2)$, the homogeneous part of the little group for massless particles.
  - At level two, the counting gets a bit more complicated, so consider only the left-moving sector. The states can be built from two α_{-1}'s or from one α_{-2}, giving
  $$\frac{1}{2}(D-2)(D-1)+(D-2)=\frac{1}{2}D(D-1)-1$$
  states of mass squared $M^2=8$. This is precisely the traceless symmetric tensor of $SO(D-1)$. The full state space at level two fits in the square of this representation.
- Furthermore, it turns out that the states at level one are only massless in lightcone quantization if $D=26$. Since $(D-2)^2$ can't fit in a representation of $SO(D-1)$, the little group for massive particles, Lorentz invariance can only be preserved if $D=26$. Thus can also be seen by attempting to compute $[\mathcal{M}^{i-},\mathcal{M}^{j-}]$, which must vanish for the Lorentz algebra to be satisfied; this only holds if $a=1$ and $D=26$.
- Next, we'll try to recover the level one result in covariant quantization. In this case the states can be written as
  $$|\Omega,p\rangle=\Omega_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0,0,p\rangle.$$
  Note that $\Omega_{\mu\nu}$ is a tensor in D-dimensional spacetime. We may decompose it as
  $$\Omega_{\mu\nu}=G_{\mu\nu}+B_{\mu\nu}+\eta_{\mu\nu}\Phi$$
  where $G_{\mu\nu}$ is traceless symmetric and $B_{\mu\nu}$ is antisymmetric.
- Applying the physical state condition, we have the constraints
  $$p^{\mu}G_{\mu\nu}=0,\quad p^{\mu}B_{\mu\nu}=0.$$
  By looking at spurious physical states, we have the identifications
  $$G_{\mu\nu}\sim G_{\mu\nu}+p_{\mu}\xi_{\nu}+p_{\nu}\xi_{\mu},\quad p\cdot\xi=0$$
  and **(are these two correct?)**
  $$B_{\mu\nu}\sim B_{\mu\nu}+p_{\mu}\xi_{\nu}+p_{\nu}\xi_{\mu}.$$
  Note that in both cases, $\xi\sim\xi+p$.
- The field Φ is trickier. The physical state condition naively is $p^{\mu}\eta_{\mu\nu}=0$, which is impossible to satisfy. It turns out that one must write the states corresponding to Φ in terms of a polarization ζ_{μ}, and the gauge symmetry will ensure all of these polarizations are equivalent.

- It is then straightforward to see the number of physical degrees of freedom at level one is $(D-2)^2$, as in lightcone quantization. However, we may go further and interpret the components of $\Omega_{\mu\nu}$. The field $G_{\mu\nu}$ is massless with spin 2 and hence must be interpreted as the graviton; note that it has the same gauge symmetries as the (linearized) metric. The field $B_{\mu\nu}$ is called a Kalb-Ramond field and has spin 1, and the Φ field is called the dilaton, with spin 0.

- We could continue to higher levels, but the massless particles at level one are far more interesting, because the higher particles are presumably too heavy to observe. As we'll see, the graviton, Kalb-Ramond field, and dilaton are common to all string theories.

- The Kalb-Ramond field is the two-form gauge field which creates the $B_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-2}|0,0,p\rangle$ states we found above. A Kalb-Ramond field can naturally couple to strings in the same way that a one-form gauge field couples to particles, and we will see examples later where strings carry Kalb-Ramond charge.

- The dilaton field, which creates the particle displayed above, turns out to be related to the value of the string coupling g by $g \sim e^{\phi}$. This string coupling is the only dimensionless parameter of string theory, but its relation with the dilaton implies it may be determined dynamically.

Note. We have used the word "spin" above casually. Properly speaking, the spin of a Lorentz representation in $D > 4$ is the highest helicity of any one of the components. Under this definition, antisymmetric tensor fields (differential forms) have spin 1, while symmetric rank n tensor fields have spin n. These correspond to the maximum possible helicities of the particles they generate.

The key property of the above definition is that it is preserved upon compactifying some dimensions to leave $D = 4$, as in Kaluza-Klein theory. For example, the states created by the Kalb-Ramond field will have helicity ± 1 or 0 in the 4D theory, depending on how the helicity was oriented in the original D dimensions. Similarly, the graviton $G_{\mu\nu}$ contains "our" graviton $g_{\mu\nu}$ in the low-energy 4D theory, along with some particles of helicity 0 or ± 1.

Finally, we give a brief previous of superstring theory.

- To pass to superstring theory, we add fermionic modes on the worldsheet. We find that the critical dimension becomes $D = 10$, there is no tachyon, and the massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ all appear.

- In type II string theory, there are both left-moving and right-moving worldsheet fermions on a closed string. The resulting spacetime theory in $D = 10$ has $\mathcal{N} = 2$ supersymmetry. There are additional massless bosonic excitations called Ramond-Ramond fields.

  - In type IIA string theory, they are a 1-form C_μ and a 3-form $C_{\mu\nu\rho}$.
  - In type IIB string theory, they are a scalar C, a 2-form $C_{\mu\nu}$, and a 4-form $C_{\mu\nu\rho\sigma}$ with a self-dual field strength.

  All of these Ramond-Ramond fields are to be interpreted as gauge fields.

- In heterotic string theory, there are only right-moving worldsheet fermions on a closed string. There is $\mathcal{N} = 1$ spacetime supersymmetry. Instead of Ramond-Ramond fields, there is a non-Abelian gauge field whose gauge group is either $SO(32)$ or $E_8 \times E_8$.

- It turns out that theories of open strings necessarily contain closed strings, as an open string can join into a closed string. In type I string theory, there are both types. In type II string theory, there are also both types, but for heterotic string theory there are only closed strings.
- It also turns out that string theories can contain Dp-branes, dynamical objects with p spatial dimensions where the endpoints of open strings can attach. In fact, type IIA string theory has stable Dp-branes with p even, and type IIB string theory has stable Dp-branes with p odd.
- Note that strings themselves are D1-branes, while particles are D0-branes. Instantons also exist in string theory, and are sometimes called D(-1)-branes.

Note. The interpretation of the tachyon in bosonic string theory. In field theory, tachyons arise as excitations of a quantum field if we expand about a field value with a negative mass squared; this indicates we are expanding about a maximum of the potential, so the theory is unstable.

In open bosonic string theory, we can think of the string end points as attached to a space-filling D25-brane; the tachyon indicates an instability of this brane. String field theory techniques have been used to show that there is indeed a minimum of the potential. Alone the journey to this minimum, the D25-brane decays into closed strings, and only closed string excitations remain at the minimum. The theory about this minimum is called vacuum string field theory, and is not well-understood. It has also been shown that Dp-branes with $p < 25$ can be thought of as coherent states of the open string tachyon.

The closed bosonic string tachyon is even less well-understood. Physically, tachyons don't appear in the superstring theories because the D-branes carry charge and are hence stable against decay. However, refinements of these theories meant to describe the real world sometimes contain tachyons.

Example. The open string with a Dp-brane. For simplicity, we take the Dp-brane to be a hyperplane. The boundary conditions are

$$\partial_\sigma X^a = 0, \quad X^I = c^I, \quad a = 0, \ldots, p, \quad I = p+1, \ldots, D-1.$$

This breaks the $SO(1, D-1)$ Lorentz group to $SO(1,p) \times SO(D-p-1)$. We recall that Neumann boundary conditions ensure $\alpha_n^\mu = \tilde{\alpha}_n^\mu$. In this case, we only have $\alpha_n^a = \tilde{\alpha}_n^a$, while for the dimensions with Dirichlet boundary conditions,

$$x^I = c^I, \quad p^I = 0, \quad \alpha_n^I = -\tilde{\alpha}_n^I.$$

As before, the right-moving and left-moving modes are not independent, and the spectrum computation goes through mostly as before, with the same conditions $D = 26$ and $a = 1$. The main difference is that the zero mode x^μ must lie on the D-brane. That is, for low-lying excitations the strings are confined to be near the brane.

At level one, we can split the excitations into those longitudinal and transverse to the brane,

$$\alpha_{-1}^a |0, p\rangle, \quad \alpha_{-1}^I |0, p\rangle$$

respectively. The longitudinal states transform as a vector of the $SO(1,p)$ Lorentz group of the brane and hence correspond to a spin 1 particle, i.e. a gauge field A_a restricted to the brane. The transverse states transform as scalars under $SO(1,p)$ and hence can be thought of as scalar fields ϕ^I living on the brane. In fact, it turns out that the brane can be thought of as a nonperturbative composite state of strings, and these transverse states correspond to fluctuations of the brane. The transverse states transform as a vector under the $SO(D-p-1)$ group, which is a global internal symmetry of a field theory living on the brane.

Note. Presumably, branes would be described by the Dirac action, a generalization of the Nambu-Goto action equal to their volume. In particular, the transverse components may be identified with the fields ϕ^I above associated with transverse excitations of the open string. However, quantizing the brane is more difficult than quantizing the string. We do not have Weyl invariance to work with. Furthermore, hypersurfaces are "more flexible" than strings, with many very different configurations having the same volume; this results in a continuous spectrum of states. This could possibly be interpreted as describing multi-particle states in the full theory.

3.4 Lightcone Quantization

4 Conformal Field Theory

4.1 Conformal Transformations

Before beginning, we need to clear up a persistent confusion over what a conformal transformation precisely is. For simplicity, we'll consider a scalar field theory.

- Given a spacetime manifold M, consider a diffeomorphism $f\colon M \to M$. Fixing a single set of coordinates, f maps the point with coordinates x to the point with coordinates x', which we write as $x \to x'$.

- As covered in the notes on General Relativity, we can interpret f either actively or passively. In the active picture, if $f(p) = q$, then we imagine the point p being physically moved to q. All other fields are transformed by applying a pushforward or inverse pullback via f, so that
$$\phi(x) \to \phi'(x') = \phi(x), \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x).$$
In the passive picture, we interpret each point p as staying in the same place, but change the coordinate description of that point from x to x'. In these new coordinates the fields are
$$\phi'(x') = \phi(x), \quad g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x).$$
We will prefer to fix one coordinate system throughout and use the active interpretation.

- A Weyl transformation is an active rescaling of the fields of the form
$$\phi(x) \to \tilde{\phi}(x) = \Omega^{-\Delta}(x)\phi(x), \quad g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x).$$
For completeness, we also have
$$\partial_\mu(x) \to \partial_\mu\tilde{\phi}(x) = \partial_\mu(\Omega^{-\Delta}(x)\phi(x)).$$
The quantity Δ is the scaling dimension of the field, which is part of the definition of the Weyl transformation.

- A conformal transformation is a special case of a diffeomorphism, where the net effect is to change the metric by a scale factor,
$$\phi'(x') = \phi(x), \quad g'_{\mu\nu}(x') = \Omega^{-2}(x) g_{\mu\nu}(x)$$

- A classical theory has a symmetry if its action remains the same after an active transformation, and *all* theories have diffeomorphism invariance, as it amounts to saying that physics is independent of the choice of coordinate system. Hence all theories trivially have conformal invariance under the definition above.

- When we speak of the conformal invariance of a theory, we usually mean the composition of a conformal transformation and the Weyl transformation that cancels the rescaling of the metric,
$$\phi(x) \to \tilde{\phi}'(x') = \Omega^{-\Delta}(x)\phi(x), \quad g_{\mu\nu}(x) \to \tilde{g}'_{\mu\nu}(x') = g_{\mu\nu}(x)$$
and
$$\partial_\mu\phi(x) \to \partial'_\mu\tilde{\phi}'(x') = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu(\Omega^{-\Delta}(x)\phi(x)).$$
Unfortunately, this is also usually called a conformal transformation, and we will do the same.

- Thus, nontrivial conformal invariance is a special case of Weyl invariance, i.e. the case where the Weyl scaling factor Ω can be derived from a diffeomorphism. Sometimes one says that conformal invariance is only a "global" Weyl invariance. Weyl invariance implies conformal invariance, but not vice versa, but most commonly encountered theories have both or neither.

- Another way of making this distinction is to say that the trivially true conformal invariance uses a "dynamical" metric, while the nontrivial conformal invariance uses a fixed "background" metric. Unfortunately, both notions come up in string theory; we will be careful to specify which is being used.

- Explicitly, conformal invariance can be checked as follows. Let

$$S = \int d^4x\, \mathcal{L}[g_{\mu\nu}(x), \phi(x), \partial_\mu \phi(x)], \quad S' = \int d^4x\, \mathcal{L}[\tilde{g}'_{\mu\nu}(x), \tilde{\phi}'(x), \partial'_\mu \tilde{\phi}'(x)].$$

  Then the theory is conformally invariant if $S = S'$. Since we know the primed fields in terms of the primed variables, the simplest way to evaluate S is to first rename x to x', then transform back from x' to x,

$$\begin{aligned} S' &= \int d^4x'\, \mathcal{L}[\tilde{g}'_{\mu\nu}(x'), \tilde{\phi}'(x'), \partial'_\mu \tilde{\phi}'(x')] \\ &= \int d^4x\, |\Omega(x)|^4 \mathcal{L}\left[g_{\mu\nu}(x), \Omega^{-\Delta}(x)\phi(x), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu (\Omega^{-\Delta}(x)\phi(x))\right] \end{aligned}$$

- We will often be concerned with the case of flat space in Cartesian coordinates. In this case, a conformal transformation leaves the metric unchanged. Intuitively, if we have a drawing on a plane, a conformal transformation picks it up off the plane, distorts it while preserving angles, then pastes it back on. In this case, the plane is the fixed background structure.

- A useful special case of a conformal transformation is a dilation transformation,

$$x \to x' = \Omega x, \quad \phi(x) \to \tilde{\phi}'(x') = \Omega^{-\Delta}\phi(x).$$

  In this case Δ is called the scaling dimension of the field. Theories symmetric under dilation transformations are called scale invariant.

Now we provide some motivation for studying conformal field theory.

- As we've seen, a classical theory has scale/conformal/Weyl invariance if its action is invariant under appropriate transformations. A quantum theory has these symmetries if its partition function is invariant under the same transformations; as a result the classical consequences of these symmetries hold as Ward identities. For example, in classically conformally invariant theories the trace of the stress-energy tensor vanishes; at the quantum level the expectation value of its trace vanishes.

- One can show that source-free electromagnetism is conformally invariant at the classical level. One manifestation of this is the use of conformally mapping to solve two-dimensional electrostatics problems. **(how about sources?)**

- Other examples of classically conformally invariant theories free massless fermions, Yang-Mills, and massless ϕ^4 theory in $d = 4$. However, the last two are not conformally invariant at the quantum level, as they have nonvanishing beta functions and hence scale dependence. Conceptually, this occurs because the path integral measure fails to be conformally invariant.
- Fixed points of Wilsonian RG flow have quantum scale invariance because they obey the appropriate Ward identities, as we showed in the notes on Quantum Field Theory. In all cases we will deal with, quantum scale invariant comes with full quantum conformal invariance, so we will treat the two as equivalent. For example, the Wilson-Fisher fixed point hence has conformal invariance at the quantum level. At this point, all particles are massless.
- String theory has Weyl invariance and diffeomorphism invariance on the worldsheet, and hence is a conformal field theory at the classical level. Since these two symmetries are gauged, they must survive at the quantum level, and requiring the Weyl anomaly to vanish fixes the spacetime dimension. Hence string theory is a 2D quantum conformal field theory.
- Conformal symmetry is thought to be the most general spacetime symmetry, unless one includes supersymmetry, in which case we have superconformal field theories (SCFTs). For example, $\mathcal{N} = 4$ super Yang-Mills (SYM) is an SCFT, both at the classical and quantum level. This is the CFT that appears in the most prominent version of the AdS/CFT correspondence.
- Formally, CFTs can be used to define quantum field theories without reference to a Lagrangian. In fact, some CFTs do not have any known Lagrangian description, such as the 6d $(2, 0)$ SCFT. In the conformal bootstrap program, one attempts to solve a theory using only conformal invariance and consistency conditions.

4.2 Elementary Aspects

We now consider some elementary aspects of conformal invariance. We begin with establishing notation and conventions.

- We will work on a Euclidean worldsheet with coordinates $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$. It is useful to work with the complex coordinates

  $$z = \sigma^1 + i\sigma^2, \quad \overline{z} = \sigma^1 - i\sigma^2$$

  which are the Euclidean analogues of the lightcone coordinates. As such, we will refer to holomorphic functions as "left-moving" and anti-holomorphic functions as "right-moving".
- The holomorphic derivatives are

  $$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\overline{z}} = \overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

  which are defined so that

  $$\partial z = \overline{\partial}\overline{z} = 0, \quad \partial\overline{z} = \overline{\partial} z = 0.$$
- We work in flat Euclidean space, with metric

  $$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz\, d\overline{z}.$$

  In components, this means

  $$g_{zz} = g_{\overline{z}\overline{z}} = 0, \quad g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2}.$$

- Note that there are two possible integration measures,

$$dz\, d\bar{z} = 2d\sigma^1\, d\sigma^2.$$

We define two delta functions with different normalization,

$$\int d^2z\, \delta(z, \bar{z}) = \int d^2\sigma\, \delta(\sigma) = 1.$$

- Vectors naturally have their indices up, with

$$v^z = v^1 + iv^2, \quad v^{\bar{z}} = v^1 - iv^2$$

and indices are lowered using the metric, giving

$$v_z = \frac{1}{2}(v^1 - iv^2), \quad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2).$$

- Note that we are heuristically treating z and $\bar{z}$ as independent complex variables. This is justified as in the notes on Quantum Field Theory. We may think of working in the larger space $\mathbb{C}^2$, and imposing the constraint that $\bar{z}$ is the conjugate of z at the end.

- In two dimensional Euclidean space, all holomorphic changes of coordinates

$$z \to z' = f(z), \quad \bar{z} \to \bar{z}' = \overline{f(z)}$$

are conformal transformations, because

$$ds^2 = dz\, d\bar{z} \to |df/dz|^2\, dz\, d\bar{z}.$$

Hence the conformal group in two dimensions is infinite-dimensional, which makes conformal symmetry much more powerful.

- Finally, it is conventional to define the stress-energy tensor as

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}}$$

in string theory. We have $\nabla^\alpha T_{\alpha\beta} = 0$, which reduces to $\partial^\alpha T_{\alpha\beta} = 0$ for a flat worldsheet. As we saw in the notes on General Relativity, conformal invariance implies the stress-energy tensor is traceless on-shell.

4.3 Free Scalar Field

4.4 The Virasoro Algebra

4.5 The State-Operator Map

5 The Polyakov Path Integral

6 String Interactions

6.1 Motivation

Now we turn to a heuristic discussion of string interactions.

- As motivated earlier, one can describe string interactions by evaluating a series of "string diagrams", where the worldsheet has nontrivial topology, and Weyl invariance on the worldsheet can be used to simplify the diagrams. For tree-level scattering of n closed strings, it can be shown that the worldsheet can be converted to a sphere with n punctures.

- At each puncture, there must appear some local operator in the worldsheet QFT with the quantum numbers of the external string state $|\Lambda\rangle$ mapped to that point, called the vertex operator V_Λ. This "state–operator correspondence" is a common idea in conformal field theories.

- Heuristically, for each particle type there should be a local operator $W_\Lambda(\sigma, \tau)$. It must be a scalar under reparametrizations of σ and τ, and have the same Lorentz quantum numbers as Λ. We can try to build W_Λ out of X^μ and its derivatives.

- Since the tachyon is a Lorentz scalar, we can simply take $W = 1$, while for the graviton the simplest spin two operator is $W^{\mu\nu} = \partial_\alpha X^\mu \partial^\alpha X^\nu$.

- However, we must also take into account spacetime translations. Under the global symmetry $X^\mu \to X^\mu + a^\mu$, the wavefunction of an external state of momentum k^μ is multiplied by $e^{ik\cdot a}$. This can be accounted for by taking a factor of $e^{ik\cdot X}$ in the vertex operator. Finally, the vertex operator should be integrated over the worldsheet, since it may appear anywhere on it, giving

$$V_\Lambda(k) = \int d^2\sigma \sqrt{h}\, W_\Lambda(\sigma, \tau) e^{ik\cdot X}.$$

  This handles both emission and absorption; by convention k is always directed inward.

- Finally, to compute scattering amplitudes, we would expect a path integral to give

$$A(\Lambda_1, k_1, \ldots, \Lambda_M, k_M) = \kappa^{M-2} \int \mathcal{D}X\mathcal{D}h\, e^{-S} \prod_{i=1}^{M} V_{\Lambda_i}(k_i)$$

  where S is the Polyakov action and κ is a coupling constant. Actually deriving this require background in string field theory and is far beyond our scope.

- In this section, we will focus on evaluating the tree-level contribution, which for closed strings gives a sphere; accounting for higher-order contributions involves a sum over Riemann surfaces of arbitrary genus.

- Focusing on the tree-level contribution, it is most convenient to stereographically project the sphere to the plane. Since we have fixed the metric h, the path integral simplifies to

$$A = \kappa^{M-2} \int \mathcal{D}X(x, y) \exp\left(-\frac{1}{2\pi} \int d^2x\, \partial_\alpha X_\mu \partial^\alpha X^\mu\right) \prod_{i=1}^{M} V_{\Lambda_i}(k_i) \equiv \kappa^{M-2} \left\langle \prod_{i=1}^{M} V_{\Lambda_i}(k_i) \right\rangle$$

  which is simply a free field theory correlator that can be directly evaluated.

- However, there are further issues involving the path integral measure. As usual, we should mod out by gauge symmetries, but our final configuration has residual diffeomorphism symmetry. This is easiest to see by adopting complex coordinates in the plane. The worldsheet metric must be in the form $ds^2 = e^\phi dz d\bar{z}$, and changing coordinates to $w(z)$ where w is analytic preserves this. Infinitesimally, we have transformations $\delta z = \epsilon(z)$ where ϵ is holomorphic.
- However, we must ensure that $\delta z = \epsilon(z)$ does not have a pole at the point at infinity, which corresponds to a point in the original sphere. Equivalently, if $\tilde{z} = 1/z$, then $\delta \tilde{z}$ cannot have a pole at zero, but
$$\delta \tilde{z} = -\frac{\epsilon(z)}{z^2}$$
which implies $\epsilon(z)$ is a quadratic polynomial, $\delta z = a + bz + cz^2$. These generate a group isomorphic to $SL(2, \mathbb{C})$.
- As an explicit example, consider the scattering of M tachyons. Then
$$A = \kappa^{M-2} \int \prod_{i=1}^{M} d^2 z_i \left\langle \prod_{i=1}^{M} e^{ik_i \cdot X(z_i)} \right\rangle.$$
The free field correlator can be evaluated by completing the square, where k_i plays a role like a source in ordinary field theory, giving
$$A = \kappa^{M-2} \int \prod_{i=1}^{M} d^2 z_i \prod_{i<j} \exp\left(\frac{1}{2}(k_i \cdot k_j) G(z_i, z_j)\right)$$
where G is the propagator of the free field X^μ, satisfying
$$\Delta_z G(z, z') = 2\pi \delta^2(z - z').$$
- Taking the inverse Fourier transform, we find
$$G(z, z') = -2\pi \int \frac{d^2 q}{4\pi^2} \frac{e^{iq\cdot(z-z')}}{q^2} = \log(\mu |z - z'|)$$
where μ is an arbitrary constant which regulates the IR divergence. **(prove this)** Therefore we have
$$A = \kappa^{M-2} \int \prod_{i=1}^{M} d^2 z_i \prod_{i<j} |z_i - z_j|^{k_i \cdot k_j / 2}.$$
- This integral diverges because we have not accounted for the $SL(2, \mathbb{C})$ gauge symmetry. This gauge symmetry is conventionally fixed by setting
$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty.$$
The terms involving z_3 then multiply to $|z_3|^{-k_3^2/2} = |z_3|^{m^2/2}$ by momentum conservation, giving a constant which can be ignored. We hence have
$$A = \kappa^{M-2} \int \prod_{\ell=4}^{M} d^2 z_\ell \prod_{j=4}^{M} |z_j|^{k_1 \cdot k_j / 2} |1 - z_j|^{k_2 \cdot k_j / 2} \prod_{4 \le i < j \le M} |z_i - z_j|^{k_i \cdot k_j / 2}.$$
In particular, for the four-point function we have
$$A = \kappa^2 \int d^2 z_4 \, |z_4|^{k_1 \cdot k_4 / 2} |1 - z_4|^{k_2 \cdot k_4 / 2}.$$

- To derive these results more properly, we must ensure the gauge symmetries are not anomalous. One manifestation of $SL(2,\mathbb{C})$ gauge symmetry is that V is $SL(2,\mathbb{C})$ invariant. However, since

$$V = \int d^2z \, e^{ik\cdot X(z)}$$

  it appears that $e^{ik\cdot X}$ must be an operator of dimension two. Since X^μ is dimensionless, $e^{ik\cdot X}$ would classically have dimension zero, but it can receive an anomalous dimension. These can appear even for free field theories, as long as we work in $1+1$ dimensions. **(what?!)**

- Concretely, for an operator Y of dimension p, we have

$$\langle Y(z)Y(0)\rangle \propto |z|^{-2p}.$$

  From our computations above, we already know that

$$\langle e^{ik\cdot X(z)} e^{-ik\cdot X(0)}\rangle = |z|^{-k^2/2}$$

  which indicates that the dimension is $k^2/4$. Since this must be equal to two, we have $k^2 = 8$, which gives the correct tachyon mass for the critical string. A similar computation for graviton scattering shows that the gravitons must be massless for consistency.

Next, we apply the same ideas to open strings.

- A tree-level open string diagram can be mapped onto a disc or a half-plane, with the vertex operators on the boundary. Therefore, the vertex operators take the form

$$V = \int d\tau \, \sqrt{h_{\tau\tau}}\, U(\tau), \quad U = We^{ik\cdot X}$$

  where τ is a parameter on the boundary of the worldsheet; invariance of V under conformal rescalings now require that U should have dimension one.

- For a spin zero particle we may take $W = 1$. Then it turns out that V has dimension $k^2/2$, and since the integral is only over the boundary, V must have dimension one. Then $k^2 = 2$, corresponding to the open string tachyon. For spin one, we may try $W = dX^\mu/d\tau$, which requires $k^2 = 0$.

- Mapping the worldsheet to the upper half-plane, the amplitude is

$$A(k_1, \ldots, k_M) = g^{M-2} \int dx_1 \ldots dx_M \left\langle \prod_{i=1}^{M} e^{ik_i\cdot X(x_i)} \right\rangle$$

  where g is the open string coupling constant. The residual gauge symmetries are conformal maps from the plane to itself, which preserve the real axis and are nonsingular at infinity; these take the form $\delta z = a + bz + cz^2$ where a, b, and c are real, and correspond to the group $SL(2,\mathbb{R})$.

- Note that conformal transformations can only produce cyclic permutations of the x_i, so a given string diagram fixes their cyclic order.

- Furthermore, it turns out that the endpoints of open strings may carry gauge charges. This may be motivated from the original purpose of string theory as a model for the strong interactions, where the string corresponds to a flux tube and quarks and antiquarks sit at the endpoints. (In fact, the first attempt to include fermions to describe baryons eventually led to the development of superstring theory.)

- One concrete realization of this is to consider a set of N space-filling D-branes. Then each endpoint of the open string can lie on any of the D-branes, so the state of the two endpoints is described by an $N \times N$ matrix.

- For gauge group $U(n)$, if we let the endpoints of the string transform in the fundamental and antifundamental, each vertex operator comes with a matrix $\lambda^i{}_j$ transforming in the adjoint. If the vertex operators come in the order $12 \ldots M$, we pick up a group theory factor of $\operatorname{tr}(\lambda_1 \ldots \lambda_M)$ as each antiquark is contracted with the next quark. This is called a Chan-Paton factor.

- We use the $SL(2, \mathbb{R})$ symmetry to fix
$$x_1 = 0, \quad x_{M-1} = 1, \quad x_M = \infty$$
so that the remaining x_i lie in $(0, 1)$. We will also need the Green's function $\tilde{G}(z, z')$ which satisfies the Neumann boundary conditions
$$\left. \frac{\partial G(x + iy, z')}{\partial y} \right|_{y=0} = 0.$$
This may be found by the method of images, which gives
$$\tilde{G}(z, z') = \log |z - z'| + \log |z - \overline{z}'|.$$
However, we are actually interested in the case where z and z' are both on the real axis, in which case $\tilde{G}(x, x') = 2 \log |x - x'|$, differing from G only by a factor of 2. This also accounts for the factor of 2 in the dimension of V above.

- Plugging everything in, for tachyon scattering we have the result
$$A = g^{M-2} \int_{0 < x_2 < \ldots < x_{M-2} < 1} dx_2 \ldots dx_{M-2} \prod_{j=2}^{M-2} |x_j|^{k_1 \cdot k_j} |1 - x_j|^{k_j \cdot k_{M-1}} \prod_{2 \leq \ell < m \leq M-2} |x_\ell - x_m|^{k_\ell \cdot k_m}.$$
This is the Koba-Nielsen M-particle generalization of the Veneziano amplitude. For $M = 4$ it simplifies to
$$A = g^2 \int_0^1 dx\, x^{k_1 \cdot k_2} (1 - x)^{k_2 \cdot k_3} = g^2 B\left(-\frac{s}{2} - 2, -\frac{t}{2} - 2 \right)$$
which is the Veneziano amplitude.

- It is also possible to have diagrams with both external open and closed strings. These can be evaluated by mapping the worldsheet to the upper half-plane, where the open string vertex operators are on the boundary and the closed string vertex operators in the bulk.

6.2 Vertex Operators

Next, we turn to a more detailed study of vertex operators, focusing on open strings.

- Consider a local operator at the string endpoint, $A(\tau) \equiv A(0, \tau)$. Since the string Hamiltonian is $L_0 - a$, we have
$$A(\tau) = e^{i\tau L_0} A(0) e^{-i\tau L_0}.$$

- We say $A(\tau)$ is a boundary primary operator with conformal dimension J if, under a change of variable $\tau \to \tau'$, we have
$$A'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^J A(\tau).$$
This is equivalent to our definition of the conformal dimension in the previous section. **(show this)** Such operators transform "nicely" and are rather special; not every operator can be expanded as a linear combination of primary operators.

- The above condition is easier to handle at the infinitesimal level. For $\delta\tau = \epsilon(\tau)$, we have
$$\delta A(\tau) = -\epsilon \frac{dA}{d\tau} - JA\frac{d\epsilon}{d\tau}$$
by the definition above. The L_m generate transformations with $\epsilon = ie^{im\tau}$, so equivalently
$$[L_m, A(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mJ\right) A(\tau).$$

- If $A(\tau)$ has an expansion in Fourier modes
$$A(\tau) = \sum_m A_m e^{-im\tau}$$
then this condition is equivalent to
$$[L_m, A_n] = (m(J-1) - n)A_{m+n}.$$

- It is straightforward to show that $X^\mu(\tau)$ has $J = 0$, as
$$X^\mu(\tau) = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}$$
and
$$[L_m, X^\mu(\tau)] = i\alpha_m - i\sum_{n\neq 0}\alpha_{m+n}^\mu e^{-in\tau} = -i\sum_n \alpha_n^\mu e^{-i(n-m)\tau} = -ie^{im\tau}\frac{d}{d\tau}X^\mu(\tau).$$
Furthermore, the momentum operator $\dot{X}^\mu(\tau)$ has $J = 1$. However, $\ddot{X}^\mu(\tau)$ is not a boundary primary operator at all. It's easiest to show these using the expansion in Fourier modes.

- We have seen earlier that vertex operators should be primaries of dimension 1. Another way to see this makes sense is that if $A(\tau)$ has dimension 1 and $|\phi\rangle$ is a physical state, then $[L_m, A_0] = 0$, and hence $A_0|\phi\rangle$ is also a physical state. This is what we expect, as an emission or absorption should map a physical state to another.

Now we consider some examples of vertex operators.

- By the same arguments as before, we expect vertex operators to contain a factor of $e^{ik\cdot x(\tau)}$, where $x(\tau)$ is the string's center of mass position,
$$x^\mu(\tau) = x^\mu + p^\mu \tau.$$
However, this is not a local operator, so we instead consider $e^{ik\cdot X(\tau)}$. This exponential requires normal ordering, giving
$$V(k,\tau) = :\, e^{ik\cdot X(0,\tau)}\, : \, = \exp\left(k\cdot\sum_{n=1}^\infty \frac{\alpha_{-n}}{n}e^{in\tau}\right) e^{ik\cdot x(\tau)} \exp\left(-k\cdot\sum_{n=1}^\infty \frac{\alpha_n}{n}e^{-in\tau}\right).$$
This differs from the unordered expression by the divergent sum $\alpha' k^2 \sum 1/n$, by the BCH formula. As motivated earlier, this is the vertex operator for a tachyon.

- Naively, products of primary operators would be primary operators whose dimension is the sum of the product factors'. Indeed, this appears to be true if we naively use commutator identities above. The problem is that some products may be undefined without a subtraction or regularization scheme. In this case, we can remove the singularities by normal ordering, but typical normal-ordered products such as $:\, X^\mu(\tau)X_\mu(\tau)\,:$ aren't primaries at all.

- To compute the dimension of $V(k,\tau)$, we note that
$$[\alpha_p^\mu, e^{k\cdot\alpha_{-n}}] = p\delta_{p,n}k^\mu e^{k\cdot\alpha_{-n}}$$
which easily leads to
$$[L_m, e^{k\cdot\alpha_{-n}}] = \frac{1}{2}n\{k\cdot\alpha_{m-n}, e^{k\cdot\alpha_{-n}}\}$$
where the right-hand side contains an anticommutator, by using the commutator product rule to split apart L_m, and then using our previous identity.

- Now we need to evaluate $[L_m, V(k,\tau)]$. By expanding in a Taylor series, we have
$$[L_m, e^{ik\cdot X(0,\tau)}] = -ie^{im\tau}\frac{d}{d\tau}e^{ik\cdot X(0,\tau)}$$
which indicates that $e^{ik\cdot X(0,\tau)}$ has dimension zero, as we'd expect. When we add normal ordering, the right-hand side remains normal ordered, but the left-hand side is not. This causes the two sides to differ, introducing an extra term that yields the conformal dimension.

- Hence we can focus on the terms in $[L_m, V(k,\tau)]$ that are not normal ordered. This computation can be done by using the commutator product rule to split apart L_m, then using our previous identity. This produces m terms with creation operators on the far left, precisely
$$\frac{1}{2}\sum_{n=1}^m k\cdot\alpha_{m-n}e^{in\tau}V(k,\tau) = \left[\frac{1}{2}\sum_{n=1}^m k\cdot\alpha_{m-n}e^{in\tau}, V(k,\tau)\right] + :\, \frac{1}{2}\sum_{n=1}^m k\cdot\alpha_{m-n}e^{in\tau}V(k,\tau)\, :\,.$$
The second term contributes to $:\, (d/d\tau)e^{ik\cdot X(0,\tau)}\,:$ as in the non-normal ordered case, while the extra commutator term is
$$\left[\frac{1}{2}\sum_{n=1}^m k\cdot\alpha_{m-n}e^{in\tau}, V(k,\tau)\right] = \frac{1}{2}\sum_{n=1}^m k^2 e^{im\tau}V(k,\tau) = \frac{1}{2}mk^2e^{im\tau}V(k,\tau)$$
which yields a conformal dimension $J = k^2/2$, in accordance with our earlier computation.

- This is indeed a vertex operator when $k^2 = 2$, which is precisely the on-shell condition for the tachyon. Hence $V(k, \tau)$ is the vertex operator for the open string tachyon.

- Next, we can consider vertex operators for the photons at level one. One candidate which could have the right dimension is

$$V_\zeta(k, \tau) = \zeta \cdot \frac{dX}{d\tau} \exp(ik \cdot X), \quad k^2 = 0.$$

  This expression is already normal ordered, provided that $\zeta \cdot k = 0$, as $e^{ik\cdot X}$ is already normal ordered for $k^2 = 0$, and commuting components of $\zeta \cdot dX/d\tau$ through it gives terms proportional to $\zeta \cdot k$. Hence this operator is a primary of dimension 1. It is the vertex operator for a photon of polarization ζ.

- We may also define vertex operators for states of zero norm. For instance,

$$V(k, \tau) = -i \frac{d}{d\tau} \exp(ik \cdot X), \quad k^2 = 0$$

  is a vertex operator for the zero norm longitudinal photon state. The fact that this state is not physical is reflected in the fact that $V(k, \tau)$ is a τ derivative, so V_0 vanishes.

- Finally, we can consider vertex operators at level two. Candidates with the right dimension are

$$\zeta^{\mu\nu} \dot{X}_\mu \dot{X}_\nu : \exp(ik \cdot X) : , \quad k^2 = -1$$

  which is free of short-distance singularities if

$$k_\mu \zeta^{\mu\nu} = \eta_{\mu\nu} \zeta^{\mu\nu} = 0.$$

  This gives a total of $D(D-1)/2 - 1$ degrees of freedom, which is just enough to account for the symmetric traceless tensor of $SO(D-1)$ that appears at level two.

There are also analogous results for the closed string.

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6.3 Bosonic Open Strings

Now we consider scattering amplitudes for open strings.

- Instead of using the path integral to evaluate a tree-level scattering amplitude, we will use a setup like that of worldline QFT, which we saw in the notes on Quantum Field Theory. Each diagram is a graph, and each edge of the graph corresponds to a worldline, which yields a propagator factor. One also writes down a factor for each vertex.

- We will take a slightly different route. Note that if we start with any tree-level open string diagram, with M external particles, we can deform it to the form of a single line on which $M-2$ external string emissions are attached. This leads to the ansatz

$$A = g^{M-2}\langle\phi_1|V_2(k_2)\Delta V_3(k_3)\ldots\Delta V_{M-1}(k_{M-1})|\phi_M\rangle$$

  where Δ is the string propagator and the V_i are vertex operators. This manifestly contains poles due to on-shell propagators, as required by tree-level unitarity. However, symmetry under cyclic permutations of the external particles (i.e. duality) is not manifest.

- For scalar field theory, where $(\partial^2 + m^2)\phi = 0$, the propagator is simply $(\partial^2+m^2)^{-1}$. For free open strings, the closest analogue of the first equation is $(L_0 - 1)|\phi\rangle = 0$, so we define the propagator to be

$$\Delta = (L_0 - 1)^{-1} = \int_0^1 z^{L_0-2}\, dz.$$

- For the vertex operators, it will be useful to define $z = e^{i\tau}$, so that

$$[L_m, V(k,z))] = \left(z^{m+1}\frac{d}{dz} + mz^m\right)V(k,z), \quad X^\mu(z) = x^\mu - ip^\mu \log z + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu z^{-n}.$$

- It is also useful to write the tachyon vertex operator as

$$V_0(k,z) \equiv\ :e^{ik\cdot X(z)}:\ = Z_0 W_0$$

  where Z_0 is the zero mode operator,

$$Z_0 = e^{ik\cdot x + k\cdot p\log z} = e^{ik\cdot x}z^{k\cdot p+1} = z^{k\cdot p-1}e^{ik\cdot x}$$

  and W_0 is the remaining factor,

$$W_0 = \exp\left(k\cdot\sum_{n=1}^\infty \frac{\alpha_n}{n}z^n\right)\exp\left(-k\cdot\sum_{n=1}^\infty\frac{\alpha_n}{n}z^{-n}\right).$$

- Similarly, the vertex operator for a massless photon is

$$V(\zeta,k,z) = \zeta\cdot\dot{X}(z)e^{ik\cdot X(z)}$$

  where a dot indicates a derivative with respect to $\tau = -i\log z$.

- Next, we need to account for the external states. These can be constructed by a similar trick to that of quantum field theory. In quantum field theory, if we start with an arbitrary state with at least some overlap with the vacuum, then after evolution in imaginary time it will evolve to the vacuum state because everything else exponentially decays away. Similarly, to get, e.g. the lowest-energy state with a given conserved charge, we can start with an arbitrary state with the same charge and do the same.

- In string theory, the operators which give the desired states are precisely the vertex operators. That is, we have

$$|\Lambda,k\rangle = \lim_{\tau\to i\infty} e^{-i\tau}V_\Lambda(k,\tau)|0,0\rangle, \quad \langle\Lambda,k| = \lim_{\tau\to-i\infty}e^{i\tau}\langle 0,0|V_\Lambda(k,\tau).$$

  Note that we do not have to actually Wick rotate to do this; the operator $V_\Lambda(k,\tau)$ with τ imaginary is defined in exactly the same way as $V_\Lambda(k,\tau)$ with τ real. This result is called the operator–state correspondence, and is an important principle in conformal field theory.

- Note that we are using the state $|0,0\rangle$. It is formally the vacuum of the 2D worldsheet QFT. However, it is not interpreted as a physical state; it is a tachyon with the momentum off-shell.
- We can check this explicitly for the tachyon vertex operator. By design, acting with W_0 on $|0,0\rangle$ will do nothing in the limit $\tau \to i\infty$, leaving

$$Z_0|0,0\rangle = e^{ik\cdot x} z^{k\cdot p+1}|0,0\rangle = z|0,k\rangle$$

since $e^{ik\cdot x}$ translates by k in momentum space. The factor of $e^{-i\tau}$ precisely cancels the extra z.
- Finally, the propagator can also be expressed as

$$\Delta = \int_0^\infty d\tau\, e^{-\tau(L_0-1)}$$

where the integrand generates evolution in imaginary time τ. The full amplitude is hence written in terms of infinite imaginary time evolution, with vertex operators V_1 and V_M inserted at infinity and the other V_i inserted at finite imaginary times; we integrate over these intermediate times.
- One can show that ghosts decouple as required, preserving tree-level unitarity. Another important formal task is to show that the amplitude has cyclic symmetry. Substituting in our results above and using the fact that $(L_0-1)|\phi_1\rangle = (L_0-1)|\phi_M\rangle = 0$, the amplitude becomes

$$A = g^{M-2}\int_0^1 \frac{dz_3\ldots dz_{M-1}}{z_3\ldots z_{M-1}}\langle\phi_1|V(k_2,1)V(k_3,z_3)\ldots V(k_{M-1},z_3\ldots z_{M-1})|\phi_M\rangle.$$

The ordering can be made more explicit by changing variables to $y_i = z_3 z_4 \ldots z_i$, giving

$$A = g^{M-2}\int_0^1 \left(\prod_{i=3}^{M-1}\theta(y_{i-1}-y_i)\frac{dy_i}{y_i}\right)\langle\phi_1|V(k_2,y_2)\ldots V(k_{M-1},y_{M-1})|\phi_M\rangle.$$

Now, applying the operator-state correspondence, the expectation value becomes

$$\lim_{y_1\to\infty}\lim_{y_M\to 0}\frac{y_1}{y_M}\langle 0,0|V(k_1,y_1)\ldots V(k_M,y_M)|0,0\rangle.$$

- We must now evaluate an expectation value in the "unphysical" vacuum state $|0,0\rangle$. Now, string theory can be thought of as a two-dimensional field theory with an enormous symmetry group, and $SL(2,\mathbb{R})$ generated by L_{-1}, L_0, L_1 is the non-anomalous part of the Virasoro algebra. The vacuum is indeed $SL(2,\mathbb{R})$ invariant, $L_1|0,0\rangle = L_0|0,0\rangle = L_1|0,0\rangle = 0$. This is, of course, the same residual $SL(2,\mathbb{R})$ symmetry we encountered earlier.
- Infinitesimally, these $SL(2,\mathbb{R})$ transformations generate the change

$$y \to y' = y + \lambda_{-1} + \lambda_0 y + \lambda_1 y^2.$$

One might wonder why the 2×2 matrices in $SL(2,\mathbb{R})$ have a natural nonlinear action on the line; it turns out y transforms like v_1/v_2 for a vector $(v_1,v_2)^T$. The general transformation is

$$y \to y' = \frac{ay+b}{cy+d}$$

conventionally scaled so that $ad-bc=1$. This is called a Mobius transformation.

- Mobius transformations give us the freedom to move the points at y_1 and y_M. As for the path integral, we must quotient out by the volume of $SL(2,\mathbb{R})$ to avoid overcounting. After some calculation, we find that the measure for the integral is

$$d\mu_M(y) = \delta(y_A - y_A^0)\delta(y_B - y_B^0)\delta(y_C - y_C^0)(y_A - y_B)(y_A - y_C)(y_B - y_C)\prod_{i=2}^{M}\theta(y_{i-1} - y_i)\prod_{j=1}^{M} dy_j$$

  where we have fixed three of the y_i's, namely y_A, y_B, and y_C, to specific values. This establishes cyclic symmetry in the integration variables y_i. Finally, cyclic symmetry in the product of vertex operators can also be shown, though we omit this calculation.

With all of this setup, we are now finally ready for some examples.

Example. The three-tachyon amplitude. This is

$$A = g\langle 0, -k_1|V_0(k_2,\tau)|0, k_3\rangle$$

where all momenta are directed inward and τ is an arbitrary constant which we do not need to integrate over, following the arguments above. To verify the value of τ does not matter, note that

$$A = g\langle 0, -k_1|e^{i\tau L_0}V_0(k_2,0)e^{-i\tau L_0}|0, k_3\rangle = g\langle 0, -k_1|V_0(k_2,0)|0, k_3\rangle$$

where we used the physical state condition $(L_0 - 1)|0, k\rangle = 0$. Plugging in the form of the tachyon vertex operator, the oscillator part W_0 does nothing, since there are no excitations to annihilate, so

$$A = g\langle 0, -k_1|Z_0|0, k_3\rangle = g\langle 0, -k_1|e^{ik_2\cdot x}|0, k_3\rangle = g\langle 0, -k_1|0, k_2 + k_3\rangle = g\delta(k_1 + k_2 + k_3).$$

The momentum-conserving delta function is generic and we will omit it below.

Example. The tachyon-tachyon-photon amplitude, where the photon has polarization ζ. Then

$$g\langle 0, -k_1|V(\zeta, k_2)|0, k_3\rangle = g\langle 0, -k_1|\zeta\cdot\dot{X}(1)V_0(k_2,0)|0, k_3\rangle$$

where we simply set $\tau = 0$ in the vertex operator, following the discussion above.

6.4 Bosonic Closed Strings