Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, including damped and driven oscillators in detail, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For physical examples and more on normal modes, see chapters 1 through 6 of French. A nice series of examples using the Lagrangian-like techniques below is given by Jaan Kalda here (also see examples of $F = ma$ problems solved with this approach here), and some discussion of the adiabatic theorem is given here. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of **85** points.

1 Small Oscillations

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

Example 1: KK 4.13

The Lennard–Jones potential

$$U(r) = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right)$$

is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard–Jones interaction.

Solution

To keep the notation simple, we'll set $\epsilon = r_0 = 1$ and restore them later. The equilibrium radius is the radius where the derivative of the potential vanishes, and

$$U'(r) = -12r^{-13} + 12r^{-7} = 0$$

implies that that the equilibrium radius is $r = r_0$. Because the force accelerates both of the atoms, the angular frequency is

$$\omega = \sqrt{\frac{U''(r)}{m/2}}$$

where $m/2$ is the so-called reduced mass. At the equilibrium point, we have

$$U''(r_0) = (12)(13)r_0^{-14} - (12)(7)r_0^{-8} = 72.$$

Restoring the dimensionful factors, we have $U''(r_0) = 72\epsilon/r_0^2$, so

$$\omega = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}}.$$

[3] **Problem 1** (Morin 5.13)**.** A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

(a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

(b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate frequency of the subsequent oscillations.

(c) Repeat the previous part for $x \gg R$.

(d) Now suppose the mass begins at rest on the plane, but slightly displaced from the center. Do oscillations occur? If so, what is the approximate frequency?

[2] **Problem 2.** Some small oscillations questions about the buoyant force.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Find the frequency of small oscillations, assuming that a face of the glacier always remains parallel to the water surface, and that the force of the water on the glacier is always given by the hydrostatic buoyant force.

(b) A ball of radius R floats in water with half its volume submerged. Find the frequency of small oscillations, making the same assumption.

(c) There are important effects that both of the previous parts neglect. What are some of them? Is the true oscillation frequency higher or lower than the one found here?

[3] **Problem 3.** USAPhO 1998, problem A2. To avoid some confusion, skip part (a), since there actually isn't a nice closed-form expression for it.

[3] **Problem 4.** USAPhO 2009, problem A3.

[3] **Problem 5.** USAPhO 2010, problem B1.

Idea 2

A useful generalization of Newton's second law is given by generalized coordinates. Let q be any number that describes the state of the system, not necessarily a Cartesian coordinate. Suppose the energy of a system can be decomposed into two parts, a potential energy that depends only on q and a kinetic energy that depends only on $\dot{q}$,

$$K = K(\dot{q}), \quad V = V(q).$$

Then since energy is conserved, $d(K+V)/dt = 0$, the chain rule gives

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

We call the left-hand side the rate of change of a "generalized momentum", and the right-hand side a "generalized force". When q is a Cartesian coordinate, this recovers the usual $F = ma$.

Remark

The result above is a special case of the Euler–Lagrange equation in Lagrangian mechanics, which states that if a system is described by a Lagrangian L, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

When $L = K(\dot{q}) - V(q)$, we recover the previous result. But more generally, it might not be possible to meaningfully decompose L into a "kinetic" and "potential" piece at all! We won't use this more general form below. While it is more powerful, it is also more complicated, and if you find yourself using it for an Olympiad problem, there's probably an easier way.

Example 2

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. The reason this procedure is so complicated is that we are using two coordinates when the string really ensures the system has only a single degree of freedom.

Instead, let q be a generalized coordinate that describes "how much the string has moved". In other words, $q = 0$ initially, and for some $q > 0$, the mass M has moved down by q and the mass m has moved up by q. Then

$$K = \frac{1}{2}(m+M)\dot{q}^2, \quad V = qg(m-M)$$

and applying the idea above gives

$$\ddot{q} = \frac{M-m}{M+m}\,g.$$

Another way of saying this is that, from the standpoint of this generalized coordinate, the "total force" is $(M-m)g$, and the "total inertia" is $M+m$.

[1] **Problem 6.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

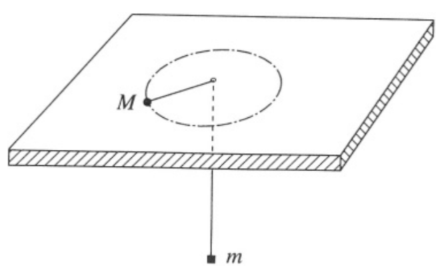
then the oscillation frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This technique allows us to avoid dealing with possibly complicated constraint forces.

[3] **Problem 7.** Suppose a particle is constrained to move on a curve $y(x)$ with a minimum at $x = 0$. We know that if $y(x)$ is a circular arc, then the motion is not exactly simple harmonic, for the same reason that pendulum motion is not. Find a differential equation relating y' and y, so that the motion is exactly simple harmonic for arbitrary amplitudes; you don't have to solve it. (Hint: work in terms of the coordinate s, the arc length along the curve.)

[3] **Problem 8** (Grad)**.** A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The motion is frictionless. Show that the motion is stable with respect to small changes in the height of m, and find the frequency of small oscillations.

[4] **Problem 9.** IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions. As a first example, we'll use the fictitious forces met in **M2**.

Example 3: PPP 79

A pendulum of length ℓ and mass m initially hangs straight downward in a train. The train begins to move with uniform acceleration a. If a is small, what is the period of small oscillations? If a can be large, is it possible for the pendulum to loop over its pivot?

Solution

The fictitious force in the train's frame due to the acceleration is equivalent to an additional, horizontal gravitational field, so the effective gravity is

$$\mathbf{g}_{\text{eff}} = -a\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

For small oscillations, we know the period is $2\pi\sqrt{L/g}$ in ordinary circumstances. By precisely the same logic, it must be replaced with

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = 2\pi\frac{\sqrt{L}}{(g^2+a^2)^{1/4}}.$$

As a gets larger and larger, the effective gravity points closer and closer to the horizontal. In the limit $g/a \to 0$, the effective gravity is just horizontal, so the pendulum oscillates about the horizontal. Its endpoints are the downward and upward directions, so it never can get past the pivot.

Example 4

If a spring with spring constant k_1 and relaxed length ℓ_1 is combined with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring, if the combination is in series or in parallel.

Solution

For the series combination, the new relaxed length is clearly $\ell = \ell_1 + \ell_2$. Suppose the first spring is stretched by x_1 and the second by x_2. The tensions in the springs must balance,

$$F = k_1x_1 = k_2x_2.$$

Thus, the new spring constant is

$$k = \frac{F}{x_1+x_2} = \frac{k_2x_2}{x_2(k_2/k_1+1)} = \frac{k_1k_2}{k_1+k_2}.$$

For example, if the spring is cut in half, the pieces have spring constant $2k$.

Now consider the parallel combination. In this case it's clear that the new spring constant is $k = k_1 + k_2$, since the tensions of the springs add. The new relaxed length ℓ is when the forces in the springs cancel out, so

$$k_1(\ell-\ell_1) + k_2(\ell-\ell_2) = 0$$

which implies
$$\ell = \frac{k_1\ell_1 + k_2\ell_2}{k_1 + k_2}.$$

[2] **Problem 10** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

[3] **Problem 11** (Morin 4.22)**.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Geometrically, what kind of curve is the resulting trajectory?

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

[5] **Problem 12.** A uniform spring of spring constant k and total mass m is attached to the wall, and the other end is attached to a mass M.

(a) Show that when $m \ll M$, the oscillation frequency is approximately
$$\omega = \sqrt{\frac{k}{M + m/3}}.$$

(b) [A] Generalize part (a) to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses, as in the example in **M2**. It will not be possible to solve for ω in closed form, but you can get a compact implicit expression for it. Check that it reduces to the result of part (a) for small m/M, and interpret the results for large m/M. This is a challenging problem that requires almost all the techniques we've seen so far, so feel free to ask for more hints.)

[2] **Problem 13** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

[3] **Problem 14.** USAPhO 2015, problem A3.

[3] **Problem 15.** USAPhO 2008, problem B1.

3 Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 16.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. In this limit, show that by taking the real part, one finds an exponentially damped sinusoidal oscillation. Roughly how many oscillation cycles happen when the amplitude damps by a factor of e?

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

[4] **Problem 17.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator, which experiences force $F = -bv - kx + F_0 e^{i\omega t}$. Show that Newton's second law can be written as

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters. Argue that the general solution takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in the previous question.

(c) The $A_\pm$ are set by initial conditions. After a long time they will decay away, leaving

$$x(t) \approx A_0 e^{i\omega t}.$$

Recalling that the physical position is just the real part, we actually have

$$x(t) \approx |A_0| \cos(\omega t - \phi), \quad \phi = \arg A_0.$$

Evaluate $|A_0|$ and ϕ.

(d) Sketch the amplitude $|A_0|$ as a function of frequency, marking the driving frequency that maximizes it.

(e) Find the driving frequency ω that maximizes the amplitude of the velocity. You should find that this condition is a bit simpler than that of the previous part.

(f) Sketch the phase shift ϕ as a function of ω. Can you intuitively see why ϕ takes the values it does, for ω small, $\omega \approx \sqrt{k/m}$, and ω large? (You can try it out by taking a pencil, ruler, or any similar object, and shaking one end back and forth.)

Since the answers to (d) and (e) differ, it's really ambiguous what we mean when we say driving is "at resonance". In practice, it doesn't matter, because strong resonance is only noticeable when the damping is weak, and in that case the answers are both approximately equal to $\sqrt{k/m}$.

[3] **Problem 18** (KK 10.9)**.** The quality factor of an oscillator is defined as $Q = m\omega_0/b$. It measures how weak the damping is, but also how sharp the resonance is.

(a) Show that for a lightly damped oscillator near resonance,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonance frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

(c) Estimate Q for a guitar string.

If you want more information, see pages 424 through 428 of Kleppner and Kolenkow.

There are other ways to drive a harmonic oscillator, which contain some cool physics; the next two problems explore this.

[2] **Problem 19.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

(a) Move the end of the string horizontally with sinusoidal frequency f'.

(b) Move the end of the string vertically with sinusoidal frequency f'.

(c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just think!)

[5] **Problem 20.** GPhO 2016, problem 1. Record your answers on the official answer sheet.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Example 5

Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1 - L)$$
$$m\ddot{x}_2 = k(x_1 + L - x_2).$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u - v}{2}, \quad x_2 = \frac{u + v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

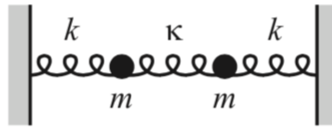
$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega} \sin \omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega} \sin \omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega} \sin \omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 21** (Morin 4.10)**.** Three springs and two equal masses lie between two walls, as shown.

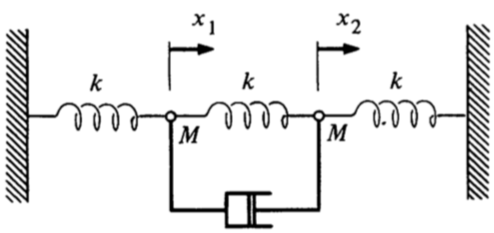


The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

[3] **Problem 22** (KK 10.11)**.** Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega}\sin\omega t$$

and evaluate ω.

[5] **Problem 23** (Morin 4.12, IPhO 1986)**.** N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) [A] Do the same for general N. (Hint: consider the normal modes found in (a) and (b), arranged so that in each normal mode, each mass oscillates with unit amplitude but a different phase. Look at the phases and guess a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Part (c) will be useful in **X1**, where we will quantize the normal modes found here.

[4] **Problem 24.** [A] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy K and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) As explained above, the Euler–Lagrange equations for the system are

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}.$$

Using the results of part (a), write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L} A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where A is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(c) Find the normal modes and their frequencies, using the general method in section 4.5 of Morin.

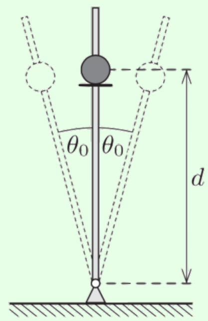
5 [A] Adiabatic Change

Idea 5

When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

Example 6: MPPP 21

A small smooth pearl is threaded onto a rigid, smooth, vertical rod, which is pivoted at its base. Initially, the pearl rests on a small circular disc that is concentric with the rod, and attached to it a distance d from the rotational axis. The rod starts executing simple harmonic motion around its original position with small angular amplitude θ_0.



What frequency of oscillation is required for the pearl to leave the rod?

Solution

The reason the pearl leaves the rod is that the normal force rapidly varies in direction, with an average upward component. If this average upward force is greater than gravity, the pearl accelerates upward and leaves the rod.

In this case, the fast motion is the oscillation of the rod, while the slow motion is the rate of change of the pearl's distance from the pivot, which can be neglected during one oscillation. The pearl has horizontal displacement and acceleration

$$x(t) = -d\sin\theta \approx -d\theta(t) = -\theta_0 d \sin\omega t, \quad a_x(t) = \theta\omega^2 d\sin\omega t.$$

This is supplied by the horizontal component of the normal force. The vertical component is

$$N_y = N_x \tan\theta(t) \approx m a_x(t)\theta(t) = m\theta_0^2\omega^2 d\sin^2\omega t.$$

Now we average over the fast motion to understand the slow motion. Since the average value of $\sin^2(\omega t)$ is $1/2$, the condition for the pearl to go up is

$$\frac{1}{2}m\theta_0^2\omega^2 d > mg$$

which gives

$$\omega > \frac{1}{\theta_0}\sqrt{\frac{2g}{d}}.$$

Example 7

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Solution

In this case the fast motion is the oscillation of the mass, while the slow motion is the weakening of the spring. We can solve the problem by considering how the energy changes in each oscillation, due to the slight decrease in k.

Suppose that the spring constant drops in one instant by a factor of $1-\epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1-\epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant gradually decreases by a factor of $1-x$ over a full cycle, with $x \ll 1$, then the energy decreases by a factor of $1-x/2$.

The process finishes after N oscillations, where $(1-x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1-x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. But the energy is also kA^2, so the new amplitude is $\sqrt[4]{2}A_0$.

Amazingly, the question can also be solved in one step using a subtle conserved quantity.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular momentum* is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k} A^2.$$

Then the final amplitude is $\sqrt[4]{2}A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But angular momentum turns out to be a special case of a more general conserved quantity, which is useful in a wide range of similar problems.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p\, dx$$

is conserved. This is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

Solution

Using conservation of energy,

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Therefore, the curve $p(x)$ over one oscillation cycle traces out an ellipse in phase space, with semimajor and semiminor axes of $\sqrt{2mE}$ and $\sqrt{2E/k}$. The area of this ellipse is the adiabatic invariant,

$$I = \oint p\, dx = \pi\sqrt{2mE}\sqrt{2E/k} = 2\pi E\sqrt{\frac{m}{k}} \propto A^2\sqrt{km}.$$

Thus, $A \propto k^{-1/4}$ in an adiabatic change of k, recovering the answer found earlier.

Remark

The existence of the adiabatic invariant is hard to see in pure Newtonian mechanics, but it falls naturally out of the framework of Hamiltonian mechanics, which works with phase space. In fact, Hamiltonian mechanics makes a lot of theoretically useful facts easier to see.

For example, as you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually one and the same. If you'd like to learn more about Hamiltonian mechanics, see David Tong's lecture notes or chapter 15 of Morin.

[3] **Problem 25.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein relied in the affirmative. Reproduce Einstein's analysis.

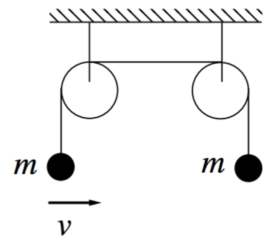
[4] **Problem 26.** A block of mass M and velocity v_0 to the right approaches a stationary puck of mass $m \ll M$. There is a wall a distance L to the right of the puck.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by explicitly analyzing each collision. (Note that it does not suffice to just use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

(c) The adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force. Using the adiabatic theorem, infer the value of γ for a one-dimensional monatomic gas.

[4] **Problem 27 ($\boldsymbol{F = ma}$, BAUPC).** Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.