Electromagnetism V: Induction

Reading: chapter 7 of Purcell covers induction, as does chapter 7 of Griffiths. For magnetism, see section 6.1 of Griffiths; for applications, see the nice end-of-chapter technology reviews in each chapter of Purcell.

1 Motional EMF

Idea 1

If $\mathbf{F}$ is the force on a charge q, then the emf about a loop C is

$$\mathcal{E} = \frac{1}{q} \oint_c \mathbf{F} \cdot d\mathbf{s}.$$

For a moving loop in a time-independent magnetic field, the emf through the loop is

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where Φ is the magnetic flux. The direction of the emf produces a current that opposes the change in flux.

Exercise 1 (Purcell)**.** Prove this result using the Lorentz force law as follows.

(a) Let the loop be C and let $\mathbf{v}$ be the velocity of each point on the loop. Argue that after a time dt, the change in flux is

$$d\Phi = \oint_C \mathbf{B} \cdot ((\mathbf{v}\, dt) \times d\mathbf{s}).$$

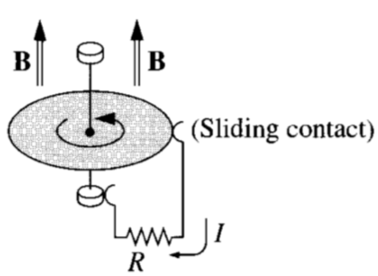
(b) Prove the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$.

(c) Use this result to show that

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}$$

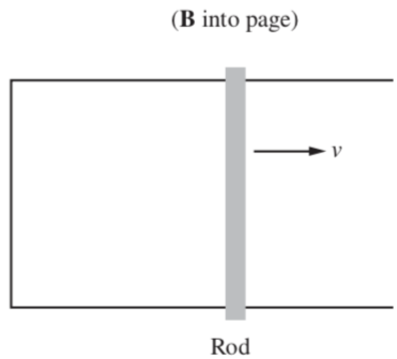
and use this to conclude the result.

Exercise 2 (Griffiths)**.** A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform magnetic field $\mathbf{B}$ pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk.



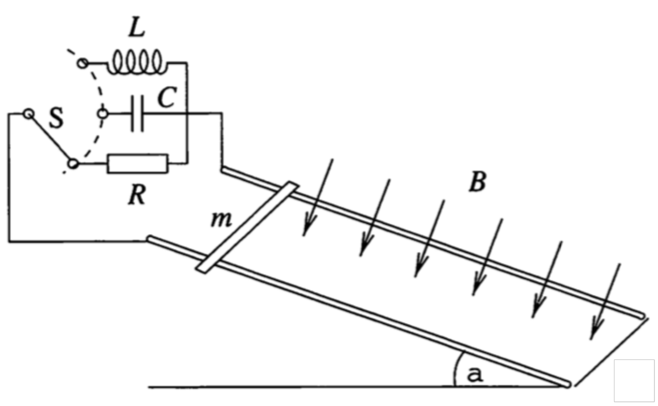
Find the current in the resistor. (Hint: do not try to find the current distribution in the disk. In general, this is an extremely complicated configuration of eddy currents.) This is a Faraday disk, a simple type of generator.

Exercise 3 (Purcell 7.2)**.** A conducting rod is pulled to the right at speed v while maintaining a contact with two rails. A magnetic field points into the page.



We know an induced emf will cause a current to flow in the counterclockwise direction around the loop. Now, the magnetic force $q\mathbf{u}\times\mathbf{B}$ is perpendicular to the velocity $\mathbf{u}$ of the moving charges, so it can't do work on them. However, the magnetic force $\mathbf{f}$ certainly looks like it's doing work. What's going on here? If the magnetic force doing work or not? If not, then what is? There is definitely something doing work because the wire will heat up.

Problem 1 (PPP 167)**.** A homogeneous magnetic field $\mathbf{B}$ is perpendicular to a track inclined at an angle α to the horizontal. A frictionless conducting rod of mass m and length ℓ straddles the two rails as shown.



How does the rod move, after being released from rest, if the circuit is closed by

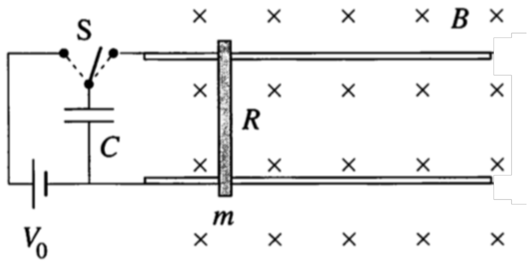
(a) a resistor of resistance R,

(b) a capacitor of capacitance C,

(c) or a coil of inductance L (see section 3 below)?

Problem 2. USAPhO 2006, problem B1.

Problem 3 (PPP 168)**.** One end of a conducting horizontal track is connected to a capacitor of capacitance C charged to voltage V_0. The inductance of the assembly is negligible. The system is placed in a homogeneous, vertical magnetic field B, as shown.



A frictionless conducting rod of mass m, length ℓ, and resistance R is placed perpendicularly onto the track. The capacitor is charged so that the rod is repelled from the capacitor when the switch is turned. This arrangement is known as a railgun.

(a) What is the maximum velocity of the rod?

(b) What is the maximum possible efficiency?

2 Faraday's Law

Idea 2

Faraday's law states that even for a time-dependent magnetic field, we have

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

In the case where the loop isn't moving but the magnetic field is changing, the emf is entirely provided by a non-conservative electric field,

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{s}.$$

The differential form of this result is

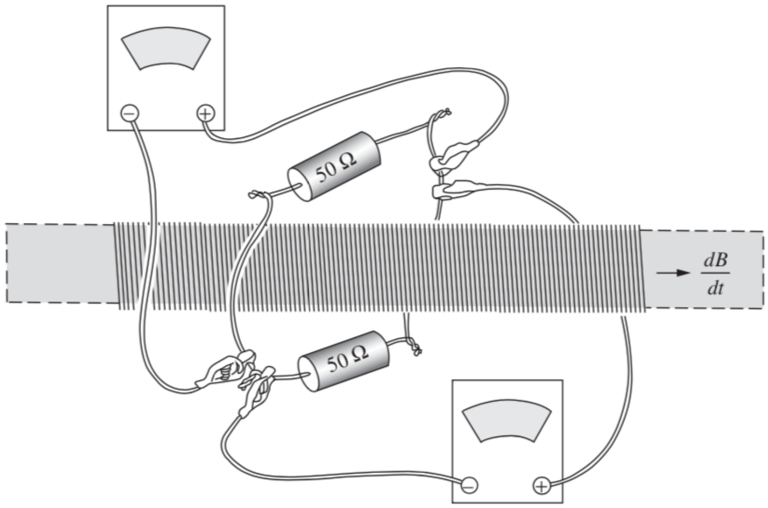
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Idea 3

When we apply Faraday's law, we often use Ampere's law (without the extra displacement current term) to calculate the magnetic field. This is not generally valid, but works if the currents are in the slowly changing "quasistatic" regime, which means radiation effects are negligible. All the problems in this problem set are in this regime, but we'll see more subtle examples in **E7**.

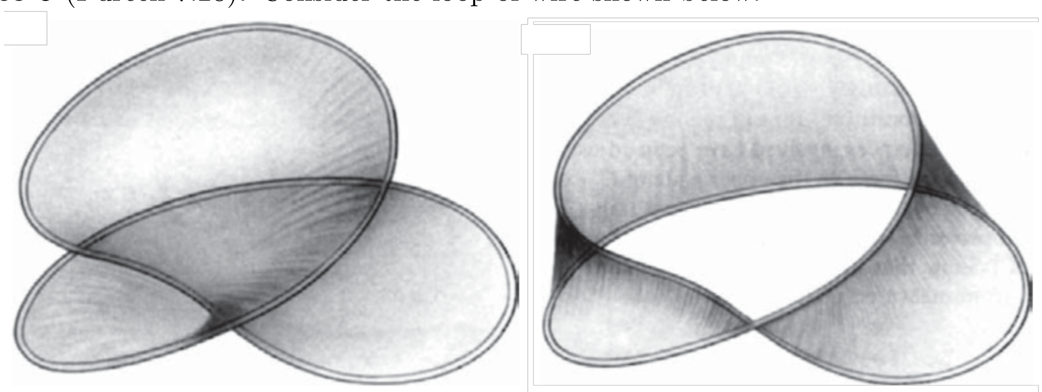
Exercise 4 (Purcell 7.6)**.** An infinite cylindrical solenoid has radius R and n turns per unit length. The current grows linearly with time, according to $I(t) = Ct$. Find the electric field.

Exercise 5 (Purcell 7.4)**.** Two voltmeters are attached around a solenoid as shown.



Find the readings on the two voltmeters in terms of $d\Phi/dt$.

Exercise 6 (Purcell 7.28)**.** Consider the loop of wire shown below.

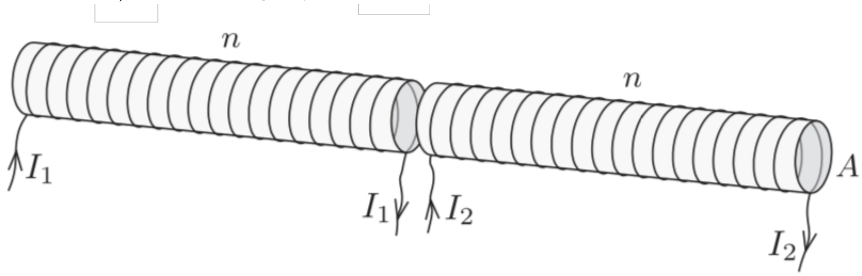


Suppose we want to calculate the flux of $\mathbf{B}$ through this loop. Two surfaces bounded by the loop are shown above. Which, if either, is the correct surface to use? If each of the two turns in the loop are approximately circles of radius R, then what is the flux? Generalize to an N-turn coil.

Problem 4. USAPhO 2009, problem A1.

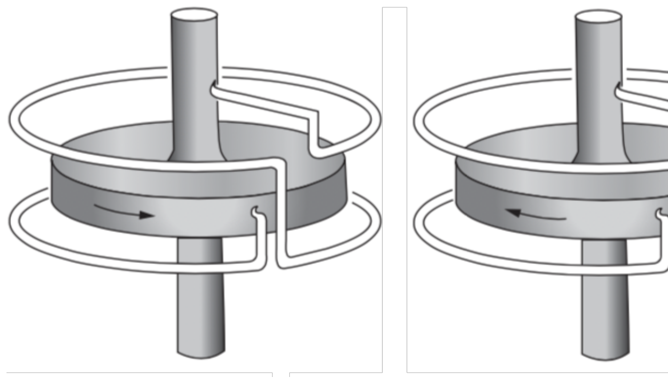
Problem 5. USAPhO 1999, problem B2.

Problem 6 (MPPP 162)**.** Two long, cylindrical solenoids are placed end to end, as shown.



The solenoids are identical, with cross-sectional area A and n turns per unit length. The currents flowing through them are I_1 and I_2. What is the magnetic force between them?

Problem 7 (Purcell 7.47)**.** A dynamo is a generator that works as follows: a conductor is driven through a magnetic field, inducing an electromotive force in a circuit of which that conductor is part. The source of the magnetic field is the current that is caused to flow in that circuit by that electromotive force. An electrical engineer would call it a self-excited dynamo. One of the simplest dynamos conceivable is shown below.



It has only two essential parts. One part is a solid metal disk and axle which can be driven in rotation. The other is a two-turn "coil" which is stationary but is connected by sliding contacts, or "brushes", to the axle and to the rim of the revolving disk. One of the two devices pictured is, at least potentially, a dynamo. The other is not. Which is the dynamo?

Problem 8 (Purcell 7.19)**.** A dynamo like the one in problem 7 has a certain critical speed ω_0. If the disk revolves with an angular velocity less than ω_0, nothing happens. Only when that speed is attained is the induced $\mathcal{E}$ enough to make the current enough to make the magnetic field enough to induce an $\mathcal{E}$ of that magnitude. The critical speed can depend only on the size and shape of the conductors, the conductivity σ, and the constant μ_0. Let d be some characteristic dimension expression the size of the dynamo, such as the radius of the disk in our example.

(a) Show by a dimensional argument that ω_0 must be given by a relation of the form $\omega_0 = K/\mu_0\sigma d^2$ where K is some dimensionless numerical factor that depends only on the arrangement and relative size of the parts of the dynamo.

(b) Demonstrate this result again by using physical reasoning that relates the various quantities in the problem (R, $\mathcal{E}$, E, I, B, etc.). You can ignore all numerical factors in your calculations and absorb them into the constant K.

For a dynamo of modest size made wholly of copper, the critical speed would be practically unattainable. It is ferromagnetism that makes possible the ordinary DC generator by providing a magnetic field much stronger than the current in the coils, unaided, could produce. For an Earth-sized dynamo, however, the critical speed is much smaller. The Earth's magnetic field is produced by a nonferromagnetic dynamo involving motions in the fluid metallic core.

Problem 9. In this problem we'll be a little more quantitative with the dynamo considered in problem 7. Suppose the disk has radius r, and the circuit forms a solenoid with resistance R and n turns per unit length. As you know from the results above, the steady state current is zero for $|\omega| < \omega_c$, and infinite otherwise. However, in reality the situation is a bit less drastic, because of the Earth's magnetic field. Suppose this field $\mathbf{B}_0$ points straight through the solenoid. Find the steady state current as a function of ω.

Problem 10 (MPPP 178). In general, a magnet moving near a conductor is slowed down by induction effects. Suppose that inside a long vertical, thin-walled, brass tube a strong permanent magnet falls very slowly due to these effects, taking a time t_1 to go from the top to the bottom. If the experiment is repeated with a copper tube of the same length but a larger diameter, the magnet takes a time t_2 to fall through. How long does it take for the magnet to fall through the tubes if they are fitted inside each other? Neglect the mutual inductance of the tubes.

3 Inductance

Idea 4: General Inductance

Consider a set of loops with fluxes Φ_i and currents I_i. By linearity, they are related by

$$\Phi_i = \sum_j L_{ij} I_j$$

where the L_{ij} are called the coefficients of inductance. It can be shown that $L_{ij} = L_{ji}$. By Faraday's law, we have

$$\mathcal{E}_i = \sum_j L_{ij} \dot{I}_j.$$

In contrast with capacitance, we're usually concerned with the self-inductance of single loops; these inductors provide an emf of $L\dot{I}$ each. However, mutual inductance effects can also impact how circuits behave.

Idea 5

The energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0} \int B^2 \, dV$$

which implies the energy stored in an inductor is

$$U = \frac{1}{2} L I^2.$$

Problem 11. Prove that $L_{ij} = L_{ji}$ as follows.

(a) Show that the flux through a loop C_1 is

$$\Phi_1 = \oint_{C_1} \mathbf{A} \cdot d\mathbf{s}_1.$$

(b) Show that the vector potential produced by a loop C_2 is

$$\mathbf{A} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|}.$$

(c) Show that L_{12} is symmetric in the two loops, so it equals L_{21}.

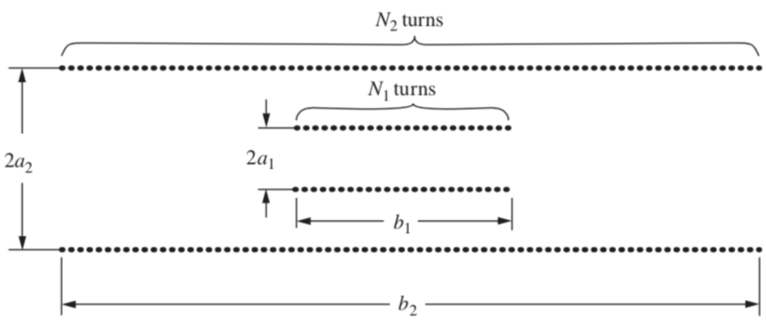
Exercise 7. Consider a toroidal solenoid with a rectangular cross section of height h and width w, N turns, and inner radius R.

(a) Find the self-inductance.

(b) Now suppose the current increases at a constant rate dI/dt. Find the electric field at a height z above the center of the solenoid, assuming $h, w \ll R$.

Exercise 8. A wire of length L is bent into a long "hairpin" shape, with two parallel straight edges of length $L/2$ separated by a distance $d \ll L$. Compute the self-inductance of the wire loop. You will run into a common problem, which may be fixed by taking the wire to have radius $r \ll d$ and ignoring any flux through the wire itself.

Exercise 9 (Purcell 7.9)**.** Find the mutual inductance of the two cylindrical solenoids shown in cross-section below.



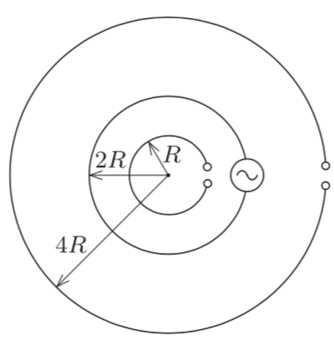
Exercise 10. Consider two concentric rings of radii r and $R \gg r$.

(a) Compute the mutual inductance by considering a current through the larger ring.

(b) Compute the mutual inductance by considering a current through the smaller ring, and verify your results agree.

(c) Suppose that an increasing current is passed through the outer ring. What forces does the inner ring experience?

Problem 12 (MPPP 181)**.** Three nearly complete circular loops, with radii R, $2R$, and $4R$ are placed concentrically on a horizontal table, as shown.



A time-varying electric current is made to flow in the middle loop. Find the voltage induced in the largest loop at the moment when the voltage between the terminals of the smallest loop is V_0.

4 Magnetism

In this section we'll dip a little into atomic physics and the origin of magnetism. However, a proper understanding of this subject requires quantum mechanics, as we'll cover in **X2**.

Exercise 11. A spinning charged object carries both a magnetic dipole moment and angular momentum. If the object's mass and charge distributions are proportional to each other, then $\boldsymbol{\mu}$ and $\mathbf{J}$ point in the same direction, and their ratio is the gyromagnetic ratio.

(a) Compute the gyromagnetic ratio for a thin uniform donut of charge Q, mass M, and radius R rotating about its axis.

(b) Show that the gyromagnetic ratio for a uniform spinning sphere is the same.

By similar reasoning, any axially symmetric charge/mass distribution would give the same gyromagnetic ratio. Unfortunately, the result is wrong by a factor of 2 when applied to atoms. This factor of 2 cannot be explained, except by quantum mechanics.

Problem 13. USAPhO 2007, problem B2.

5 Superconductors

There are many rather tricky Olympiad problems involving superconductors. Superconductors can be a bit intimidating at first, but they actually obey simple rules.

Idea 6

A perfect conductor has zero resistivity, which implies that the magnetic flux through any loop in the conductor is constant: attempting to change the flux instantly produces currents that cancel out the change. A superconductor is a perfect conductor with the additional property that the magnetic field in the body of the superconductor is exactly zero. (Not all problems involving superconductors actually use this second property.)

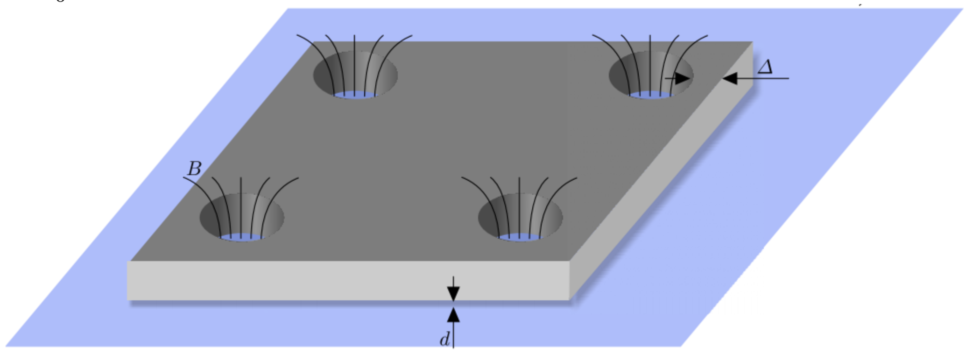
Problem 14 (PPP 153)**.** A superconducting uniform spring has N turns of radius R, relaxed length x_0, and spring constant k. The two ends of the spring are connected by a wire, and a small current I is made to flow through the spring. What is the change in its length?

Problem 15 (MPPP 182)**.** Two identical superconducting rings are initially very far from each other. The current in the first is I_0, but there is no current in the other. The rings are now slowly brought closer together. Find the current in the first ring when the current in the second is I_1.

Problem 16. IPhO 2012, problem 1C.

Problem 17 (EuPhO 2017)**.** Consider a mesh made from a flat superconducting sheet by drilling a dense grid of small holes into it. Initially the sheet is in a non-superconducting state, and a magnetic dipole of dipole moment m is at a distance a from the mesh pointing perpendicularly towards the mesh. Now the mesh is cooled so that it becomes superconducting. Next, the dipole is displaced perpendicularly to the surface of the mesh so that its new distance from the mesh is b. Find the force between the mesh and the dipole. The spacing of the grid of holes is much smaller than both a and b, and the linear size of the sheet is much larger than both a and b.

Problem 18 (Physics Cup 2013)**.** A rectangular superconducting plate of mass m has four identical circular holes, one near each corner, a distance Δ from the plate's edges. Each hole carries a magnetic flux Φ. The plate is put on a horizontal superconducting surface. The magnetic repulsion between the plate and the surface balances the weight of the plate when the width of the air gap beneath the plate is $d \ll \Delta$, and d is much smaller than the radii of the holes. The frequency of small vertical oscillations is ω_0.



Next, a load of mass M is put on the plate, so that the load lays on the plate, and the plate levitates above the support. What is the new frequency of small oscillations?

Problem 19. $\star$ IPhO 1994, problem 2.