

Mechanics VI: Gravitation

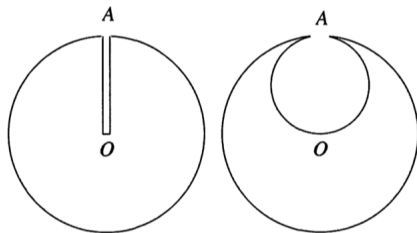
Chapters 8 and 9 of Kleppner and Kolenkow cover orbits and fictitious forces, as do chapters 8 and 11 of Wang and Ricardo, volume 1, and chapters 7 and 10 of Morin. For *much* more, see chapters 2 and 3 of *Galactic Dynamics* by Binney and Tremaine. There is a total of **86** points.

1 Computing Fields

Idea 1

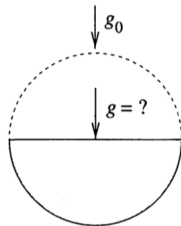
Gravitational fields obey the shell theorem and the superposition principle, which is sufficient to find the field in a variety of setups. One useful trick is to think of objects with holes as superpositions of objects without holes, and holes with negative mass.

- [3] **Problem 1** (PPP 110). A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point A on its surface to the center O of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached O, where he died on impact.



However, work continued and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter AO inside the asteroid. Then a second accident occurred: another little green man similarly fell from point A to point O, and died. Find the ratio of the impact speeds, and total times for impact, of the two little green men.

- [3] **Problem 2** (PPP 111). The titanium-devouring little green people of the previous problem continued their excavating. As a result of their environmentally destructive activity, half of the asteroid was soon used up, as shown.



What is the gravitational acceleration at the center of the circular face of the remaining hemisphere if the gravitational acceleration at the surface of the original spherical asteroid was g_0 ? (This can be done without any integrals.)

- [3] **Problem 3** (Morin 5.67). You are given a fixed volume of a moldable material, with a fixed density. Describe the shape it should take to maximize the gravitational field at the origin.

Remark

Newton's first theorem states that there is no gravitational field inside a uniform shell, while Newton's second theorem states that the gravitational field outside the shell is the same as a point mass of equal total mass at its center. In his *Principia Mathematica*, Newton noted that similarly elegant results held for the linear force law $F(r) \propto r$.

To see this, consider a spherical shell centered at the origin, with the test mass at \mathbf{r}_0 . The contribution to the force due to a mass in the shell at \mathbf{r} is $\mathbf{F} \propto \mathbf{r} - \mathbf{r}_0$. When we integrate over the shell, \mathbf{r} averages to zero, giving $\mathbf{F} \propto -\mathbf{r}_0$, which is precisely the result for a mass exactly at the center of the shell. That is, for a linear force law, Newton's first theorem doesn't work; instead the second theorem's result applies both inside and outside the shell. In fact, the inverse square force law is the *only* one for which Newton's first theorem holds inside a shell. At the time, this was a surprising result because it meant that Hell couldn't be at the center of the Earth; if it were, the fire and brimstone would be floating around.

Newton's first and second theorems, now called the shell theorems, can be proven with simple geometry. If you're as good as Newton at Euclidean geometry, you can try to prove his third theorem. A homoeoid is an elliptical shell, i.e. a three-dimensional region defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \in [1, 1 + \epsilon]$$

for some constant ϵ . Newton's third theorem states that the gravitational field inside a homoeoid of uniform mass density vanishes. (The gravitational field outside is more complicated, so there's no "fourth" theorem.)

2 Central Potentials

Idea 2: Effective Potential

A particle moving in a central potential $V(r)$ has conserved angular momentum

$$L = |\mathbf{r} \times \mathbf{p}| = mr^2\dot{\theta}.$$

Its kinetic energy can thus be written in terms of the radial velocity \dot{r} and L ,

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \left(V(r) + \frac{L^2}{2mr^2} \right).$$

By setting the time derivative of this expression to zero, we find

$$m\ddot{r} = -\frac{d}{dr} \left(V(r) + \frac{L^2}{2mr^2} \right).$$

Therefore, if we are interested in $r(t)$ alone, we can find it by treating the problem as one-dimensional, where the particle moves in the effective potential $V(r) + L^2/2mr^2$. The extra term is called the "angular momentum barrier" and repels the particle away from the

center. Once we know $r(t)$, we can find $\theta(t)$ if desired by using $\dot{\theta} = L/mr^2$.

One way of understanding the effective potential term is to think in terms of the energy methods of **M4**. From the perspective of $r(t)$ alone, any dependence on \dot{r}^2 is part of the kinetic energy, and any dependence on r is part of the potential energy. In particular, the kinetic energy of tangential motion depends only on r , because it is fixed by angular momentum conservation, so it appears as part of the potential when considering only radial motion.

Example 1: KK 9.4

For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$?

Solution

Note that circular orbits can only possibly exist if the force is attractive. This implies that A must have the same sign as n .

The effective potential is

$$U_{\text{eff}}(r) = -\frac{A}{r^n} + \frac{L^2}{2mr^2}.$$

In a circular orbit, r is constant, so the particle just sits still at a minimum of the effective potential. That is, the circular orbit radius satisfies $U'_{\text{eff}}(r) = 0$, so

$$\frac{An}{r_0^{n+1}} - \frac{L^2}{mr_0^3} = 0, \quad r_0^{2-n} = \frac{L^2}{Anm}.$$

The orbit is stable if $U''_{\text{eff}}(r) > 0$, so

$$-\frac{An(n+1)}{r_0^{n+2}} + \frac{3L^2}{m} \frac{1}{r_0^4} > 0$$

which simplifies to

$$r_0^{n-2} > \frac{m}{3L^2} An(n+1).$$

Plugging in the value of r_0 , this becomes simply $n < 2$. As expected, for inverse square forces ($n = 1$) and spring forces ($n = -2$) the orbits are stable, while, e.g. for inverse cube forces, the circular orbits are neutrally stable.

[3] Problem 4 (Morin 7.4). A particle of mass m moves in a potential $V(r) = \beta r^k$. Let the angular momentum be L .

- Find the radius r_0 of the circular orbit.
- Find the frequency of small oscillations ω_r about this radius.
- Now consider a slightly perturbed circular orbit. Explain why the orbit remains a closed curve if the ratio of the time period of small oscillations and the time period of the original circular orbit is rational, and find the integer values of k where this holds.

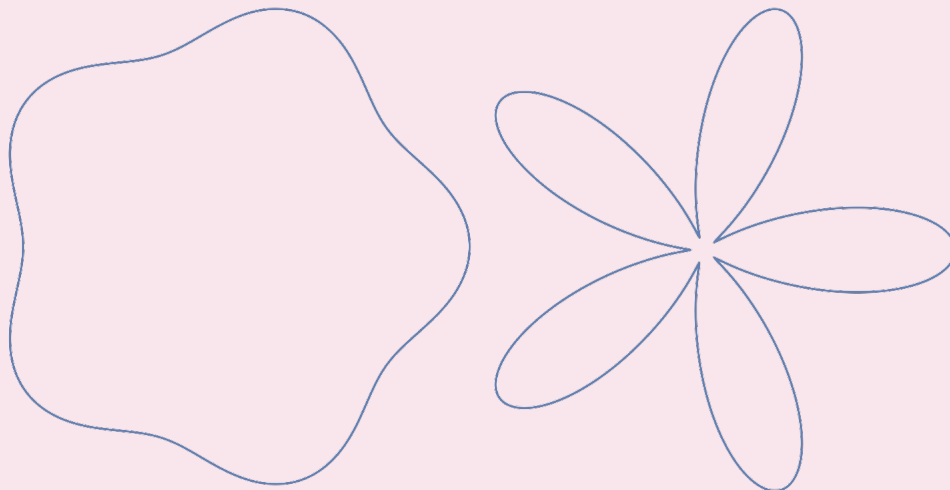
Remark: Bertrand's Theorem

In problem 4, you showed that for a certain group of potentials, all bound orbits that are nearly circular are closed. Bertrand's theorem states that the *only* central potentials for which *all* bound orbits are closed are $V(r) \propto 1/r$ and $V(r) \propto r^2$.

The idea of the proof is as follows. First, for a general potential $V(r)$, we can compute the ratio of periods of a small radial oscillation and the underlying circular orbit and demand it be rational, just like in part (c) above. However, since this ratio changes continuously as the orbit parameters are varied, it must be a constant if it is to always be rational. Using this condition, you can show that $V(r)$ must be a power law, which we had to assume above.

You found in part (c) that infinitely many power laws give closed nearly circular orbits. To rule out the other ones, we need to expand to higher orders, i.e. account for the fact that the effective potential is not perfectly simple harmonic, which is rather complicated.

However, there is a relatively simple way to rule out power laws of the form $V(r) \propto r^n$ for $n > 2$. For example, at left is a rough sketch for a nearly circular orbit when $n = 23$, in which case the ratio of periods is 5.



We see the orbit has 5 “petals”. Now assume the orbit stays closed as we make the perturbation larger. Then by continuity, the number of petals has to stay the same; we can't suddenly lose any. But as the perturbation gets larger, making the petals more eccentric, we would have to get an orbit that looks like the one at right above. And such an orbit would make no sense, because an orbit in an attractive central force must always be concave inward, while it must be concave outward to turn around. The only cases which make sense are when the ratio of periods is 1 or 2, corresponding to the cases $V(r) \propto 1/r$ and $V(r) \propto r^2$.

[4] **Problem 5.** In general relativity, the potential describing a black hole of mass M is

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mc^2 r^3}.$$

The second term is a relativistic effect which strengthens the attraction towards the black hole. (It has nothing to do with the angular momentum barrier; you still have to add that separately.)

- (a) Explain why this new term allows particles to fall to the center of the black hole, $r = 0$, and why this is impossible in Newtonian gravity.
- (b) For a fixed L , find the values of the circular orbit radii.
- (c) Find the radius of the smallest possible stable circular orbit, for any value of L . What happens if you try to orbit the black hole closer than this? (Answer: $6GM/c^2$.)
- (d) Find the closest possible approach radius of an unbound object. That is, among the set of all trajectories that start and end far away from the black hole (i.e. without falling into it), find the smallest possible minimum value of r . (Answer: $3GM/c^2$.)

For all parts, assume the particle is moving nonrelativistically.

Example 2: Binney 5.1

For over 150 years, most astronomers believed that Saturn's rings were rigid bodies, until Laplace showed that a solid ring would be unstable. The same instability plagues Larry Niven's *Ringworld*, a science fiction novel once popular among boomer nerds. Following Laplace, consider a rigid, circular wire of radius R and mass m , centered on a planet of mass $M \gg m$. The wire rotates around the planet with the Keplerian angular velocity $\omega = (GM^2/R^3)^{1/3}$. Show that this configuration is linearly unstable.

Solution

One way to understand the stability of an ordinary planetary orbit is angular momentum conservation: if you displace a planet radially inward, then it'll start moving faster tangentially, which will tend to make it go back out, even though the inward gravitational force gets stronger too. This tendency is absent for a rigid ring, because the entire ring always rotates with the same angular velocity $\omega = L/mR^2$.

Place the planet at the origin, and parametrize the ring by the angle θ along it. If the whole ring is shifted by a small distance a in the plane of the ring, the elements of the ring are at

$$r^2 = (R \cos \theta + a)^2 + (R \sin \theta)^2.$$

The total gravitational potential energy is

$$U = -GMm \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{r} = -\frac{GMm}{R} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\sqrt{1 + (2a/R) \cos \theta + a^2/R^2}}.$$

We have to be a bit careful here, remembering some lessons from **P1**. The first order term in a is going to vanish, because we started at an equilibrium point, which means we need to expand everything to second order in a . Using the Taylor series

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + O(x^3)$$

we conclude

$$U = -\frac{GMm}{R} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(1 - \frac{a}{R} \cos \theta + \frac{a^2}{R^2} \frac{3 \cos^2 \theta - 1}{2} \right) = -\frac{GMm}{R} \left(1 + \frac{a^2}{4R^2} \right).$$

The energy goes down upon a small displacement, so the configuration is unstable.

3 Kepler's Laws

Everybody knows Kepler's laws hold for circular orbits. The more general statements are below.

Idea 3

Kepler's laws for a general orbit are as follows.

1. The trajectories of planets are conic sections, with a focus at the Sun.
2. The trajectories sweep out equal areas in equal times.
3. When the orbit is a circle/ellipse, the period T and semimajor axis a obey $T^2 \propto a^3$.

Unlike the other laws, the second is valid for any central force, because the rate of area sweeping is $rv_{\perp}/2 \propto |\mathbf{r} \times \mathbf{v}| \propto |\mathbf{L}|$.

Idea 4

For a general orbit with semimajor axis a , the total energy is

$$E = -\frac{GMm}{2a}.$$

This also applies to parabolas, where a is infinity, and hyperbolas, where a is negative. This is extremely useful in situations where you only directly know the energy.

Remark: Virial Theorem

Another neat fact is the virial theorem: when time-averaged across an entire elliptical orbit,

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle.$$

In fact, the virial theorem holds for more complicated systems of particles as well, as long as they interact by a power law potential $V(r) \propto r^n$. In this case, we have

$$\langle K \rangle = \frac{n}{2}\langle V \rangle$$

where gravity corresponds to the case $n = -1$. You can check it works for two particles interacting with $n = 1$ (constant force) and $n = 2$ (linear force). In astrophysics, the virial theorem is useful because it allows us to estimate V , which can be hard to measure, given K . For an interesting discussion of the virial theorem along with applications to dark matter, see section 1.4.3 of [these notes](#). We will return to these subjects in **X3**.

[3] **Problem 6.** In this problem we'll verify some of the basic facts stated above.

- (a) Prove the statement of idea 4 for the case of elliptical orbits.
- (b) Using this result, prove the vis-viva equation

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

which is often used in rocketry.

- (c) Prove Kepler's third law. (Hint: see the area of an ellipse below.)

Remark

Kepler's third law, $a \propto T^{2/3}$, is not really specific to orbits, but rather a general scaling property of inverse square force laws. For example, suppose you had a collection of particles interacting only under gravity, and it so happened that their initial velocities were set up so the whole system scales up over time as the particles move away from each other,

$$\mathbf{r}_i(t) = a(t)\mathbf{r}_{i,0}$$

where $a(t)$ is called the scale factor. Also suppose that the system has zero total energy, so that as the particles get infinitely far away, their speeds drop to zero. Then you can show that $a(t) \propto t^{2/3}$. This describes the expansion of a matter-filled universe, which was a good description of our universe for most of its lifetime. We'll revisit cosmology in **X3**.

- [3] **Problem 7.** [A] A simple derivation of Kepler's first law is given in section 7.4 of Morin, and revolves around solving a differential equation for $1/r(\theta)$. (You can motivate this by noting that the polar form of an ellipse is quite simple, $1/r = (1 + e \cos \theta)/p$, where p is the semilatus rectum and e is the eccentricity.) However, in this problem, we'll consider an alternative approach that uses a subtle conserved quantity, which is also important in more advanced physics.

- (a) Show that the Laplace–Runge–Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}}$$

is conserved, where the star is at the origin and $\hat{\mathbf{r}}$ is the radial unit vector at the planet's position \mathbf{r} . (Hint: use the fact that $\mathbf{L} = m\mathbf{r}^2\boldsymbol{\omega}$ to evaluate the time derivative.)

- (b) We have $\mathbf{A} \cdot \mathbf{r} = Ar \cos \theta$, where θ is the angle between \mathbf{A} and \mathbf{r} . Evaluate $\mathbf{A} \cdot \mathbf{r}$ using the definition of \mathbf{A} , and the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, in order to derive an expression for r in terms of θ and constants. Then use this to show that the orbit is a conic section.

Now we'll consider some really slick problems that can be solved with pure geometry.

Idea 5

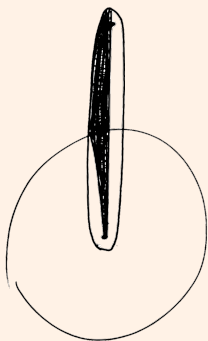
An ellipse is defined by two foci F_1 and F_2 separated by a distance $2d$. It consists of the set of points P so that $PF_1 + PF_2 = 2\ell$ is a constant. The semimajor and semiminor axes a and b of the ellipse are related by $a = \sqrt{b^2 + d^2} = \ell$, and the area is πab .

Example 3

An object is dropped from rest at a distance R above the Earth's surface, where R is the radius of the Earth. How long does it take to hit the Earth's surface?

Solution

The answer doesn't change much if we give the object a tiny horizontal velocity. In this case, the orbit becomes a part of a very thin ellipse, where $a \approx d \approx \ell$, with one focus at the center of the Earth (by the shell theorem) and the other near the starting point.



If the Earth were replaced by a point mass at its center, then the object could perform a full orbit, with total period T . The time until the object actually hits the Earth's surface is determined by the fraction of the orbit's area swept out. Referring to the diagram, this is

$$t = T \frac{\pi ab/4 + ab/2}{\pi ab} = T \left(\frac{1}{4} + \frac{1}{2\pi} \right)$$

by summing a quarter of an ellipse and a triangle. All that's left is to solve for T . Note that the semimajor axis is R . Another orbit with the same semimajor axis is simply a circular orbit around the Earth, just above its surface. This orbit has

$$\frac{v^2}{R} = \frac{GM}{R^2}$$

so $v = \sqrt{GM/R}$. Using $T = 2\pi R/v$ gives the answer,

$$t = \left(\frac{\pi}{2} + 1 \right) \sqrt{\frac{R^3}{GM}}.$$

Example 4: MPPP 39

An astronaut jumps out of the international space station directly towards the Earth. What happens afterward? In particular, will the astronaut survive?

Solution

If you've seen certain movies, you might get the impression that the astronaut spirals into the Earth, and so will surely die. But that isn't what Kepler's laws say! After the jump, the astronaut simply performs a Keplerian orbit. Since the change in energy is negligible, so is the change in semimajor axis and hence the change in period. The astronaut simply orbits in a nearly circular ellipse, with the same period as the space station.

After one rotation period of the space station, which takes time $T = 92$ min, the astronaut arrives back. They are unharmed as long as their oxygen and cooling supply lasts this long. (If you draw some pictures of the orbits, you may think the answer is $T/2$, because the orbits intersect twice. This is incorrect because while the orbits do intersect geometrically halfway through, the space station and the astronaut won't arrive at that point at the same time.)

Example 5: Wang 8.4

A particle moves in a circle of radius R , under the influence of a central force. If its minimum and maximum speeds are v_1 and v_2 , what is the period T ?

Solution

At first the problem statement might sound confusing, until you realize that the origin need not be at the center of the circle; it must be off-center. Now, it would be intractable to find the trajectory for a general central force law, but we can infer T by thinking about how quickly area is swept out, as in Kepler's second law. This works because conservation of angular momentum holds for all central force laws, not just the inverse square.

At the furthest and closest points, the distances from the origin must be r_1 and r_2 , and by conservation of angular momentum, the speeds v_1 and v_2 are achieved at these points, so

$$r_1 v_1 = r_2 v_2, \quad r_1 + r_2 = 2R, \quad \frac{dA}{dt} = \frac{1}{2} r_1 v_1.$$

Using the first two equations, we can solve for r_1 and plug it into the third, for

$$r_1 = \frac{2R}{1 + v_1/v_2}, \quad \frac{dA}{dt} = \frac{R}{1/v_1 + 1/v_2}.$$

Since $dA/dt = \pi R^2/T$, we have

$$T = \pi R \left(\frac{1}{v_1} + \frac{1}{v_2} \right).$$

- [3] **Problem 8** (PPP 88). A rocket is launched from and returns to a spherical planet of radius R so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is θ . How long does the flight take, if the period of a satellite flying around the planet just above its surface is T_0 ?
- [4] **Problem 9** (Physics Cup 2012). A cannon at the equator fires a cannonball, which hits the North pole. Neglecting air resistance and the Earth's rotation, at what angle to the horizontal should the cannonball be fired to minimize the required speed?
- [4] **Problem 10** (EFPhO 2015). An asteroid is initially stationary, a distance R from a star of mass M . The asteroid suddenly explodes into many pieces, with speed ranging from zero to v_0 . What is the set of all points that can be hit by a piece of the asteroid? (Hint: this problem requires more

geometry than the rest. For simplicity, you can begin by treating the problem as two-dimensional, but the solution you find will work just as well for three.)

Idea 6: Reduced Mass

Consider two objects of mass m_1 and m_2 with positions \mathbf{r}_1 and \mathbf{r}_2 with relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, interacting by a central potential $V(r)$. For the purposes of computing \mathbf{r} alone, we may replace this system with a single mass μ in the same central potential $V(r)$, where μ is the reduced mass, obeying

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Both systems have the same solutions for $\mathbf{r}(t)$.

Example 6

Consider two planets of mass m . If one planet is somehow fixed in place, the other can perform a circular orbit of radius R with period T . If both planets are allowed to move, they can simultaneously perform circular orbits of radius $R/2$ about their center of mass. What is the period of this motion?

Solution

First let's try an explicit solution. In the first case,

$$\frac{mv^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{Gm}{R}}.$$

In the second case, we have

$$\frac{mv^2}{R/2} = \frac{Gm^2}{R^2}, \quad v = \frac{1}{\sqrt{2}} \sqrt{\frac{Gm}{R}}.$$

The velocity in this case is a factor of $1/\sqrt{2}$ smaller, but the arc length of the orbit is a factor of 2 smaller, so the period is $T/\sqrt{2}$.

We can also handle the problem with reduced mass. Consider the relative position $\mathbf{r}_1 - \mathbf{r}_2$ in the second case, which orbits in a circle of radius R . Applying the above idea, we can work in the reduced system. In this system, there is a single mass $\mu = (1/m + 1/m)^{-1} = m/2$ in a circular orbit of radius R , experiencing the same force Gm^2/R^2 , so

$$\frac{\mu v^2}{R} = \frac{Gm^2}{R^2}, \quad v = \sqrt{\frac{2Gm}{R}}.$$

The speed is $\sqrt{2}$ bigger than in the first case, but the arc length of the orbit is the same, so the period is $T/\sqrt{2}$.

Reduced mass is a bit unintuitive, since you need to work in two very different pictures. On the other hand, some people like it because it's mathematically concrete, and can reduce problems to

one-liners. Whether you use it is up to you.

- [2] **Problem 11** (MPPP 27). Two permanent magnets are aligned on a horizontal frictionless table, separated by a distance d . The magnets are held in such a way so that the net force between them is attractive, and there are no torques generated.

If one of the magnets is held and the other is released, the two collide after time t_1 . If instead the roles are reversed, the two collide after time t_2 . If instead both magnets are released from rest, how long does it take for them to collide? (For a simpler related problem, see $F = ma$ 2019 B4.)

- [3] **Problem 12.**  USAPhO 2012, problem A4.

4 Rocket Science

So far you've done some challenging problems, but they haven't exactly been rocket science. These questions [literally are rocket science](#). Since almost all of these kinds of questions rely on the same few essential ideas, I've grouped them into a few longer questions. The 4 and 5 point problems below are representative of older IPhO problems in length.

- [2] **Problem 13.** A rocket begins at rest in empty space. The engine is turned on and exerts a constant thrust, so $P = Fv$ increases over time. After a long time, the power of the engine can become arbitrarily high, even though it's doing the same thing at all times. This is called the Oberth effect, and has real practical consequences; all else equal, a rocket should be preferentially used when the velocity is high, since it will produce extra power. Where does the extra power come from?
- [4] **Problem 14.** A rocket with a full fuel tank has a mass M and is initially stationary. The fuel is ejected at a rate σ , where σ has units of kg/s, at a relative velocity of u .

- (a) If the rocket begins in space, show that the velocity of the rocket when its total mass is M' is

$$v = u \log \frac{M}{M'}.$$

This is the Tsiolkovsky rocket equation.

- (b) Repeat part (a) for a rocket in a uniform gravitational field g . Do you get the best final velocity if σ is high or low? (Ignore gravity for the rest of this problem.)
- (c) In a multi-stage rocket, an empty fuel tank detaches from the rocket once it is used up, after which a second engine starts up. Explain why this can achieve a much higher final velocity than just firing both engines at once. (If you want a quantitative treatment of this, you can see INPhO 2016, problem 3.)
- (d) It is desired for a rocket to begin at zero speed and accelerate to speed v , to deliver a given payload. If the exhaust comes out with a relative velocity of u , how should u be chosen to minimize the fuel energy that must be spent to perform this maneuver? (Hint: let the *final* mass of the rocket be fixed, since that's the mass of the payload we want to transport. You will have to solve an equation numerically.)
- (e) If u has this value, what fraction η of the spent fuel's energy ends up in the rocket's final kinetic energy?

- (f) Now suppose u can be freely varied over time. Qualitatively, how should it be chosen to maximize η , and what is the maximum possible value of η ?

[3] **Problem 15.**  USAPhO 2015, problem B1. A basic, two-step rocket maneuver.

[5] **Problem 16.** The classic cosmic speeds. For each part, evaluate your answers numerically, using

$$M_{\text{Earth}} = 5.97 \times 10^{24} \text{ kg}, \quad M_{\text{Sun}} = 1.99 \times 10^{30} \text{ kg}, \quad R_{\text{Earth}} = 6.37 \times 10^6 \text{ m}, \quad d_{\text{Sun}} = 1.50 \times 10^{11} \text{ m}.$$

Neglect the rotation of the Earth about its own axis for all parts except for part (b).

- What is the minimum launch speed required to put a satellite into orbit around the Earth? This is the first cosmic speed. (It's useful to think in terms of speeds because the Tsiolkovsky rocket equation tells us that directly determines the amount of fuel needed. Multistage rocket maneuvers are often described in terms of their "total Δv ".)
- If you account for the rotation of the Earth, what is the new minimum speed and how should the satellite be launched?
- What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed.
- What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched? (Hint: doing this exactly is very hard; instead use the approximation $R_{\text{Earth}} \ll d_{\text{Sun}}$. To check, the answer is 16.7 km/s.)
- What is the minimum launch speed required for a rocket to hit the Sun? Assume you cannot make any adjustments to the rocket's path after launch. (To check, the answer is 31.8 km/s.)
- If subsequent adjustments are allowed, the minimum launch speed to hit the Sun can be dramatically reduced. Find the minimum launch speed required to hit the Sun if an infinitesimal adjustment later is allowed.

Remark

There's a whole science of multi-stage rocket maneuvers. For example, suppose your goal is to quickly escape the solar system. As you found in part (d) of problem 16, the minimum launch speed necessary is the third cosmic speed. However, you can also start by doing the maneuver of part (e). Once the rocket is very close to the Sun, it'll be moving extremely quickly, which means that a second impulse can provide a huge amount of energy. This is called the Oberth maneuver, as it uses the Oberth effect. Doing it this way costs more fuel, in terms of total Δv , but can allow the rocket to leave much faster.

In practice, you can only get within some distance r_{\min} of the Sun without the rocket burning up, so there's a limit to how much you can employ the Oberth effect. Thus, in some cases a three-impulse maneuver, called the Edelbaum maneuver, can be even better. In the Edelbaum maneuver, you begin with a forward impulse to get to a higher elliptical orbit, then perform a backward impulse to drop to r_{\min} . This gives a higher speed at r_{\min} , since the rocket is on an elliptical orbit with higher total energy. Then a final forward impulse can be used to escape the solar system. You can read more about these maneuvers [here](#). However, neither

the Oberth or Edelbaum maneuvers have ever been used, because the Δv requirement is too high for them to be feasible. For an authoritative reference on rocket maneuvers, see *An Introduction to the Mathematics and Methods of Astrodynamics* by Battin.

- [4] **Problem 17** (MPPP 36). Consider a solar system with two planets, in circular orbits with radii R_1 and $R_2 = xR_1$, where $x > 1$. A space probe is planned to be launched from the first planet, which we will call the Earth, and use a gravitational slingshot from the second planet to exit the solar system. The goal is to do this with the smallest fuel energy expenditure possible.
- The space probe is launched so that, after it has exited the gravitational field of the Earth, but before it has moved very far, it has speed v_0 in the Sun's frame. Furthermore, its velocity is parallel to the Earth's velocity in the Sun's frame. Explain why this direction of launch minimizes the energy needed.
 - Assume the space probe arrives near the second planet, with radial and tangential speeds v_r and v_t with respect to the Sun. Find v_r and v_t .
 - Suppose the planet have speed v_p . In terms of v_p , v_r , and v_t , what is the largest possible speed v_f of the space probe (relative to the Sun) after the gravitational slingshot ends?
 - Find the value of x that minimizes the required initial launch speed v_0 , for the probe to be able to escape the solar system. (Hint: to save space, consider nondimensionalizing variables. Unless you are very good at algebra, you will have to optimize something numerically.)
 - Which real solar system planet is closest to this ideal planet?

Remark

Above we discussed the Oberth and Edelbaum maneuvers, which use two and three impulses, respectively. In general, if you only deal with the gravity of the Sun, optimal maneuvers never require more than three impulses, so they can't get *too* complicated. But in reality, it would be impractical to exit the solar system or reach the Sun without also using gravitational slingshots. The Voyager probes used multiple slingshots off the gas giants to do the former, while the [Parker Solar Probe](#) did an incredible seven gravitational slingshots off Venus in a row to do the latter! These kinds of trajectories need to be calculated years in advance.

Still, you might be thinking, is this really the hardest stuff in the world, when it just boils down to Newtonian mechanics? Well, as Lee DuBridge, the president of Caltech once said:

I [like] to talk about space to nonscientific audiences. In the first place, they can't check up on whether what you are saying is right or not. And in the second place, they can't make head or tail out of what you are telling them anyway—so they just gasp with surprise and wonderment, and give you a big hand for being smart enough to say such incomprehensible things. And I never let on that all you have to do to work the whole thing out is to set the centrifugal force equal to the gravitational force and solve for the velocity. That's all there is to it!

I'm just being glib here – the moon landing is unquestionably one of the greatest engineering

feats in history. The physical laws at play are elementary, but their application is subtle, and the engineering required getting thousands of subtle, real-world details right.

Example 7

An object quickly flies past a star of mass M , with nearly constant speed v , so that its distance of closest approach is R . Estimate the angle by which the object is deflected.

Solution

To solve this exactly, we would write down Newton's second law in polar coordinates and solve a differential equation. However, we can get a rough estimate a simpler way. Since the object is flying quickly, its path is approximately a straight line. Most of the transverse impulse it experiences occurs when it is at a distance of order R from the star, and we can approximate this as

$$\Delta p_{\perp} = \int F_{\perp} dt \sim F_{\perp} \Delta t \sim \frac{GMm}{R^2} \frac{R}{v}.$$

The small angle of deflection is

$$\Delta\theta \approx \frac{\Delta p_{\perp}}{mv} \sim \frac{GM}{Rv^2}.$$

This is a decent approximation for the true answer, which turns out to be $2GM/Rv^2$.

In Newtonian gravity, we can think of light as consisting of massless particles moving at speed c , so we can find the deflection of light by setting $v = c$. However, in general relativity the bending of light is actually twice as large, $\Delta\theta = 4GM/Rc^2$. The observation of this factor of 2 by Eddington during a solar eclipse was one of the first tests of general relativity.

Roughly speaking, the source of the factor of 2 is that general relativity is a theory of spacetime curvature. The curvature in the “time direction” corresponds, in the Newtonian limit, to motion under a potential. For particles moving nonrelativistically, the curvature in space can be neglected because the particles move much faster through time than through space, which is why you could derive correct answers in problem 5 without any reference to it. But photons always move at the speed of light, so for them the effects are comparable.

Remark

Another famous prediction of general relativity is the perihelion precession of Mercury. As you will show in **R3**, the orbit of Mercury advances by a tiny angle $\Delta\theta$ on each cycle. However, knowing that general relativity is a relativistic theory of gravity, we can estimate this angle by dimensional analysis. The only dimensionful parameters are the strength of gravity GM , the radius R of Mercury's orbit, and the speed of light c . (Other parameters we might care about can be expressed in terms of these; for instance, the speed of Mercury is $v = \sqrt{GM/R}$.) By similar logic to the above problem, the only possible expression is

$$\Delta\theta \sim \frac{GM}{Rc^2} \sim 10^{-8}$$

which is incredibly tiny. The true answer is larger by a factor of $6\pi/(1 - e^2)$, where $e \approx 0.2$ is the eccentricity.

5 Fictitious Forces

Idea 7

Consider an inertial frame and a rotating frame with angular velocity $\boldsymbol{\omega}$. For any vector \mathbf{V} , the time derivatives of \mathbf{V} in these two frames are related by

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{V}.$$

For example, when \mathbf{V} is the position \mathbf{r} , we have the familiar result

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Applying this equation to the velocity \mathbf{v} , we find

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

The two terms on the right correspond to the Coriolis and centrifugal forces,

$$\mathbf{F}_{\text{rot}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

In the case where $\boldsymbol{\omega}$ can change, we also have the azimuthal force $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$. (If you prefer, these forces can also be derived by working in components in polar coordinates, as shown in chapter 11 of Wang and Ricardo, volume 1.)

Idea 8

Sometimes, the best way to deal with fictitious forces is to just avoid them by using an inertial frame instead. This is especially true when the Coriolis force is not small; it's straightforward to treat it approximately if it's small, but otherwise it's quite complicated. If a problem presents a situation in a rotating frame, there's no reason you have to stay in that frame! (One example of this is $F = ma$ 2018, problem B12.)

Example 8

Angular momentum conservation tells us that an ice skater increases their angular velocity as they pull their arms inward. Derive this result by working in the frame that always rotates with the skater, as the skater pulls their arms in radially. Specifically, model the skater as two point masses m a distance r from the axis. Show that balancing the Coriolis and azimuthal forces yields a result equivalent to using angular momentum conservation in an inertial frame.

Solution

Let ω be the (time-dependent) angular velocity of the skater's frame. Balancing the forces on one arm,

$$2m\omega\dot{r} = -m\dot{\omega}r$$

which is equivalent, by the product rule, to the statement that ωr^2 is constant. Then $mr^2\omega$ is constant, which is exactly the angular momentum in an inertial frame.

Example 9

A projectile is dropped from height h at the equator. Let the Earth have angular velocity ω , and let the local gravitational acceleration be g . Counting only the Coriolis force, which direction is it deflected when it hits the ground, and by about how far? Is the correction due to the centrifugal force significant?

Solution

The earth rotates from west to east, so the angular velocity points north. The velocity of the falling ball points radially inward, so the Coriolis force points east. We naturally assume the height h is much less than the radius of the Earth, so the inward gravitational acceleration is constant. The Coriolis acceleration is thus

$$a_c = 2\omega v = 2\omega gt$$

in the eastward direction, and integrating this twice gives a deflection

$$d(t) = \frac{1}{3}\omega gt^3.$$

The projectile hits the ground at $t = \sqrt{2h/g}$, giving a final eastward deflection of

$$d = \frac{\omega}{3} \sqrt{\frac{(2h)^3}{g}}.$$

This is the right answer to first order in ω . For a neat, geometric method that arrives at the same result, see the solutions to [NBPhO 2016, problem 9](#).

Remark

What if you wanted to refine the answer above? It turns out this is quite subtle, because a slew of other effects appear at higher order in ω and h , including the centrifugal force (which affects both the mass and the shape of the Earth), and the variation in g with height. If you want to explore this in detail, see problems 10.12 and 10.13 of Morin.

Example 10

Explain where the factor of 2 in the Coriolis force comes from, working in an inertial frame.

Solution

For concreteness, consider a rotating cylindrical space station of radius R with angular velocity ω . An astronaut initially stands on its rim, then jumps upward, picking up an inward radial velocity u in the space station's rotating frame. The Coriolis force implies that the astronaut will have tangential acceleration $2\omega u$.

In an inertial frame, this 2 comes from the combination of two effects of equal magnitude. Let $v = \omega R$ be the initial tangential velocity of the astronaut in this frame. As the astronaut moves radially inward, angular momentum conservation implies that their tangential velocity increases, so that after a time dt it is

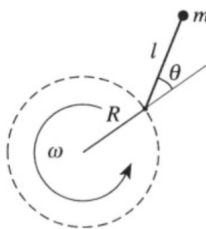
$$\omega R \frac{R}{R - u dt} = \omega R + \omega u dt.$$

At the same time, the tangential speed of the rotating frame decreases, to

$$\omega(R - u dt) = \omega R - \omega u dt.$$

The relative tangential acceleration is thus $2\omega u$, giving the desired result.

- [2] **Problem 18.** A cylindrical space station of radius R can create artificial gravity by rotating with angular velocity ω about its axis.
- For an observer rotating along with the spaceship on the rim, what gravitational acceleration g do they perceive?
 - The observer throws a ball parallel to the floor. For some launch speed v , the observer will see the ball perform a circular orbit along the spaceship, always parallel to the floor. Find v .
 - What does the motion of part (b) look like, in a frame that isn't rotating with the ship?
- [2] **Problem 19.** Every satellite in orbit around the Earth is slowly falling due to drag. Consider a satellite steadily falling, with a large tangential velocity and small inward radial velocity.
- Show that for a satellite initially in a circular orbit, losing energy U to drag *increases* the kinetic energy of the satellite. By how much is it increased?
 - The result of part (a) seems almost paradoxical. How can it be explained in an inertial frame, given that the drag force always acts to slow down the satellite?
 - Now consider a uniformly rotating frame, whose angular velocity is equal to the initial angular velocity of the satellite. In this frame, the drag force always points tangentially backwards, but the satellite ends up going tangentially forward. What force is responsible?
- [3] **Problem 20 (Grad).** A pendulum is designed for use on a gravity-free spacecraft. The pendulum consists of a mass at the end of a rod of length ℓ . The pivot at the other end of the rod is forced to move in a circle of radius R with angular frequency ω . Let θ be the angle the rod makes with the radial direction.



Show this system behaves exactly like a pendulum of length ℓ in a uniform gravitational field $g = \omega^2 R$. That is, show that $\theta(t)$ is a solution for one system if and only if it is for the other.

- [4] **Problem 21.** ⌚ IPhO 2016, problem 1B. A useful set of Coriolis force exercises.
- [3] **Problem 22.** ⌚ USAPhO 2020, problem A2. A tricky question on the Foucault pendulum. For an algebraic derivation of the final result, see section 9.9 of Taylor; it uses the complex number method introduced for a problem in **M1**. For a beautiful but more abstract geometric derivation, see section 11.5.1 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition).
- [4] **Problem 23** (Morin 10.26). A coin stands upright on a turntable rotating with angular frequency ω , and rolls without slipping so that its center is motionless in the lab frame. Thus, in the frame of the turntable, the coin rolls without slipping in a large circle with angular frequency ω .
- (a) In the lab frame, explain how $\mathbf{F} = d\mathbf{p}/dt$ and $\boldsymbol{\tau} = d\mathbf{L}/dt$ are satisfied. (This is the easy part.)
 - (b) In the frame of the turntable, verify that $\mathbf{F} = d\mathbf{p}/dt$.
 - (c) In the frame of the turntable, verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$. (This is the hard part.)

If you slogged through part (c), you'll understand why we rarely want to think about torques for extended bodies in rotating frames.

Idea 9

An object freely falling in a gravitational field will experience tidal forces, due to the spatial variation of gravitational fields. Specifically, suppose a mass M is placed at $(R, 0)$. Then the gravitational acceleration near the origin is

$$\mathbf{g} \approx \frac{GM}{R^2} \hat{\mathbf{x}} + \mathbf{g}_{\text{tidal}}, \quad \mathbf{g}_{\text{tidal}} = \frac{GM}{R^3} (2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}).$$

In particular, if a small rigid object with center of mass at the origin is released, it will have acceleration $(GM/R^2)\hat{\mathbf{x}}$. In the frame of the center of mass, the rest of the object will experience the residual gravitational acceleration $\mathbf{g}_{\text{tidal}}$. Intuitively $\mathbf{g}_{\text{tidal}}$ tends to stretch the object in the longitudinal direction and squeeze it in the transverse direction.

Example 11

Estimate the height of the tides on Earth induced by the Moon.

Solution

We will simplify the problem by treating the Earth and Moon as stationary, and the Earth as a ball of fluid. The result above tells us that the tidal acceleration due to the Moon is

$$\mathbf{g} = \frac{GM_M}{D^3}(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}})$$

where D is the distance to the Moon.

Suppose that without the Moon's presence, the Earth is a sphere of radius R_E . By integrating this, the tidal gravitational potential induced on the Earth's surface is

$$\Delta\Phi = -\frac{GM_MR_E^2}{D^3}\frac{3\cos^2\theta - 1}{2}.$$

Changing the gravitational potential on the surface by $\Delta\Phi$ means the equilibrium height of the water shifts by $\Delta h = \Delta\Phi/g$, where g is the gravitational acceleration of the Earth. Thus,

$$\Delta h \sim \frac{GM_MR_E^2/D^3}{GM_E/R_E^2} = \frac{M_M}{M_E} \left(\frac{R_E}{D}\right)^3 R_E.$$

This can also be written in terms of the densities,

$$\Delta h \sim \frac{\rho_M}{\rho_E} \left(\frac{R_M}{D}\right)^3 R_E.$$

The first term is of order one. The second term is the angular radius of the Moon in the sky, which is about a quarter of a degree, giving

$$\Delta h \sim \left(\frac{2\pi}{360 \times 4}\right)^3 R_E \sim 0.5 \text{ m}$$

which is of the right order of magnitude.

Remark

The tidal effects of the Sun and Moon are comparable, but the Moon's are somewhat larger. This can be seen by the final equation above: the Sun and Moon have comparable angular diameter, since eclipses can just barely happen, but the Moon is 2.4 times denser.

When the Moon and Sun's tidal effects reinforce, one has a larger tidal effect, called a spring tide; otherwise, one has a neap tide. Given what's been said so far, you should be able to figure out what phase(s) of the Moon and time(s) of day correspond to each (at least for an idealized "spherical cow" Earth), and also roughly how much stronger tides are at spring tides than at neap tides.

It's worth noting that the *actual* tides are far more complicated than just computing the tidal force, because they depend on the dynamical response of the water, which in turn depends on the detailed lay of the land. For some more details, see [here](#).

One should be careful when applying tidal forces, because many astronomical objects are also spinning or orbiting, which provides an additional centrifugal force. For a good discussion of tides, see section 10.3 of Morin.

- [3] **Problem 24** (Morin 10.31). A small spherical rock covered with sand falls in radially toward a planet. Let the planet have radius R and density ρ_p and let the rock have density ρ_r . When the rock gets close enough to the planet, the tidal force will pull the sand off the rock. The cutoff distance is called the Roche limit; it gives the radial distance below which loose objects can't coalesce into larger ones.

(a) Show that the Roche limit is

$$d = R \left(\frac{2\rho_p}{\rho_r} \right)^{1/3}.$$

(b) Now suppose the rock is orbiting the planet and rotating so that the same side always faces the planet. Show that the Roche limit is

$$d = R \left(\frac{3\rho_p}{\rho_r} \right)^{1/3}.$$

- [4] **Problem 25.** ⌚ IPhO 2009, problem 1. A neat problem about how the Moon has gradually slowed down the Earth's rotation. This is a fascinating subject; in the deep past, not only were the days shorter, but the weather on Earth was completely different, owing to the much higher Coriolis forces.
- [5] **Problem 26.** ⌚ IPhO 1992, problem 1. A problem on a strange propulsion mechanism, which is in some sense the reverse of the previous problem. Fill in your answers on the attached answer sheet, along with your solutions.

Remark: Discovering Gravity

In elementary school, we hear that Newton understood gravity in a flash of inspiration, after being hit on the head with an apple. Later, you might learn that it didn't quite work that way: there was an apple tree in Newton's childhood home, but an apple didn't hit him, and Newton didn't publish his ideas on gravity until decades afterward.

However, the story is an oversimplification in a much more significant way. Newton's law of gravity actually contains many independent insights. For example, you need to realize that gravitational forces occur between *pairs* of objects, rather than emanating from an object, or reflecting an object's desire to move towards its "natural" place of being. To explain the orbits, you need to understand that the force is radial, not tangential, and moreover that it is *not* balanced by any other radial force. You need to see that the force acts between *all* pairs of objects, and not just certain pairs of objects with the right qualities, like iron and magnets, that the force is proportional to mass and falls off with distance, and that it occurs "at a distance" with nothing in between.

All of these insights, which we think of as obvious today, were viewed as unintuitive or downright occult by intelligent thinkers of the time. For example, you probably think the

astrological idea that Jupiter governs blood and Venus governs phlegm is laughable, as did many 17th century astronomers, but would the idea that the Moon governs the rise and fall of water on Earth sound any more plausible, if you hadn't been told early on that it's true by people you trust? (If you flip this logic around, you can understand why so many people believe in astrology.) Or, going further back to antiquity, if you claimed then that everything is affected by gravity, how could you explain why flames go up? (Before you embark on an explanation of buoyancy, you would first have to explain how air exerts a massive yet somehow unobservable pressure on everything, why air has mass but doesn't fall, and that buoyant forces for air exist at all; the balloon wouldn't be invented for thousands of years.)

Between Galileo and Newton, there were many [incremental steps](#) towards the development of universal gravitation. For instance, Borelli proposed that Jupiter's moons obeyed Kepler's laws, and Horrocks found that Jupiter and Saturn slightly deviated from Kepler's laws because of their mutual attraction. Newton played an important role by putting everything on a solid foundation, such as by deriving Kepler's first law and the shell theorems. But as you can see from [Newton's notebooks](#), these insights came from decades of experience tinkering with concrete calculations.

Remark: Negative Mass

One intriguing difference between electrostatics and gravity is that charge can be positive or negative, while mass is apparently only positive. Could there be a "negative mass", to complete the analogy, which falls upward in a gravitational field? If general relativity is true, the answer is no: it respects the equivalence principle, which implies that particles all accelerate the same way in a gravitational field. (Every month or so, people will write clickbait articles saying that negative mass has been discovered, but most of those examples are like balloons. They don't go up in a gravitational field because they actually have negative mass, they go up because they have less mass than the air they're displacing.)

However, there's an alternative formulation of negative mass that does respect the equivalence principle: let $\mathbf{F} = m\mathbf{g}$ and $\mathbf{F} = m\mathbf{a}$ with the same negative m in both equations. Then the force on the mass points upward, but it accelerates downward anyway.

This leads to some seriously strange consequences. For example, suppose we had a positive and a negative mass interacting. Then the two would repel each other, which implies that both will start zooming off in the *same* direction! This is completely compatible with momentum and energy conservation, since the negative mass has flipped momentum and negative energy, but it's quite strange. In a universe with an equal amount of both kinds of masses, they'll both eventually end up zooming around at the speed of light, but the energy will be zero. Anyway, at the moment we don't know of any kind of matter with these weird properties.

Remark: Lagrangian Points

Earnshaw's theorem states that a point charge in an electrostatic field cannot be in stable equilibrium, i.e. it is impossible for the charge to be in equilibrium and feel a restoring

force when displaced in an arbitrary direction. The proof is simply Gauss's law: for some candidate equilibrium point P , draw a small Gaussian surface around it. Since there is no charge inside (the test charge itself doesn't count, since we're only caring about the fields of *other* charges, which act on it), the flux must be zero. But a stable equilibrium requires the field to have an inward-pointing component in all directions, which implies nonzero flux.

Earnshaw's theorem also applies to gravity; by the same logic, in space there are no gravitational potential extrema. However, you may have heard that there are stable [Lagrangian points](#) in the Earth-Sun system, where satellites can be stored. How is this possible, given that stability usually requires a potential minimum?

The first subtlety is that Lagrangian points rotate with the Earth about the Sun, so we really should be working in a rotating frame. In this frame, there is an additional centrifugal force. However, this doesn't help, because you can check that the divergence of the centrifugal force is *positive* everywhere, i.e. it causes instability rather than stability.

As usual for subtleties involving fictitious forces, the resolution comes from the Coriolis force. Near the stable Lagrangian points, the Coriolis force deflects satellites sideways. The stability of the points is not because there is an ordinary restoring force, but because of this continuous sideways deflection, which keeps particles from moving far away.

How can this be analyzed mathematically? For a typical restoring force, we would expand the force linearly in $\Delta \mathbf{r}$ and determine stability using the methods of **M4**. Here that doesn't work, because we have a velocity-dependent force, but we can still write the force as linear in the *four* coordinates $(\Delta \mathbf{r}, \mathbf{v})$. This gives a set of four linear differential equations, which can be solved in terms of normal modes as in **M4**. The motion is stable if no modes are exponentially growing. For a more complete analysis, see [here](#).