Relativity II: Dynamics

Chapter 12 of Morin covers relativistic dynamics, as does chapter 13 of Kleppner, or chapter 12 of Wang and Ricardo, volume 2. For four-vectors in relativistic dynamics, finish chapter 13 of Morin, or chapter 14 of Kleppner. For a deeper explanation of four-vectors, see chapter 2 of Schutz. There is a total of **81** points.

1 Energy and Momentum

Idea 1

The relativistic generalizations of energy and momentum are

$$E = \gamma mc^2, \quad \mathbf{p} = \gamma m\mathbf{v}.$$

These quantities are conserved, and m is defined as the rest mass. Note that m is *not* conserved in inelastic processes, while E *is* conserved; this is precisely the opposite of what happens nonrelativistically. The relativistic energy E automatically counts all contributions to the energy, including internal energy and rest energy mc^2.

[5] **Problem 1.** A few useful facts about energy and momentum, for future reference.

(a) Recalling the definition of the four-velocity from **R1**, show that

$$(E/c, \mathbf{p}) = mu^\mu$$

where u^μ is the four-velocity. Setting $c = 1$, this establishes $p^\mu = (E, \mathbf{p})$ is a four-vector.

(b) Suppose a particle is at rest in frame S'. Confirm explicitly that the components of the four-momentum p^μ transform as expected when going to frame S.

(c) Setting $c = 1$ for all future parts, show that the norm of the four-momentum is

$$p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2.$$

This is a very useful result that can simplify the solutions to many problems below, especially ones that simply ask for a final mass m. In this case one can often compute a single four-momentum and find its norm to get the answer.

(d) The expressions in idea 1 for E and $\mathbf{p}$ don't work for photons, since γ is infinite and m is zero. Instead, show that for a photon we have $p^\mu = \hbar k^\mu$.

(e) For a system of particles with total energy and momentum E and $\mathbf{p}$, find the velocity of the center of mass, i.e. the velocity of the frame where the total momentum is zero.

(f) In Newtonian mechanics, the kinetic energy K of an object with fixed mass m satisfies $dK = \mathbf{v} \cdot d\mathbf{p}$. Show that this also holds in relativity, assuming the rest mass m is fixed.

(g) As we'll discuss in more detail below, the force three-vector is defined as $\mathbf{F} = d\mathbf{p}/dt$ in relativistic mechanics. Show that $dK = \mathbf{F} \cdot d\mathbf{x}$, continuing to assume that m is fixed.

Idea 2

In relativistic dynamics problems, it is almost always better to work with energy and momentum than velocity; one typically shouldn't even mention velocities unless the problem asks for or gives them.

Example 1: Morin 12.2

Two photons each have energy E. They collide at an angle θ and create a particle of mass M. What is M?

Solution

The total four-momentum is

$$p^\mu = (2E, E(1+\cos\theta), E\sin\theta).$$

The mass is just the norm of the four-momentum, so

$$M = \sqrt{4E^2 - E^2(1+\cos\theta)^2 - E^2\sin^2\theta} = E\sqrt{2-2\cos\theta} = 2E\sin(\theta/2)/c^2$$

where we restored the factors of c at the end.

[1] **Problem 2** (KK 13.5)**.** A particle of rest mass m and speed v collides and sticks to a stationary particle of mass M. Find the final speed of the composite particle.

[2] **Problem 3.** USAPhO 2012, problem A1.

[3] **Problem 4.** USAPhO 2002, problem A2.

Example 2: Woodhouse 7.5

A particle of rest mass m moves with velocity $\mathbf{u}$ and collides elastically with a second particle, also of rest mass m, which is initially at rest. After the collision, the particles have velocities $\mathbf{v}$ and $\mathbf{w}$. Show that if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\cos\theta = \frac{(1-\sqrt{1-v^2})(1-\sqrt{1-w^2})}{vw}.$$

Solution

First, a remark: in Newtonian mechanics, you learn that in an inelastic collision, the kinetic energy is dissipated into microscopic thermal motion. This often leads students to ask: if we keep track of the motion of all particles in detail, then are all collisions actually perfectly elastic? According to particle physics, the answer is no. You really can lose kinetic energy by converting it to mass-energy, in collisions which change the identity of the particles or produce new particles. Therefore, at particle colliders, we say a collision is elastic if the particles that come out are precisely the same as the ones that came in. For this example, that means the final particles still have rest mass m.

Conservation of energy and momentum imply

$$1 + \gamma_u = \gamma_v + \gamma_w, \quad \gamma_u \mathbf{u} = \gamma_v \mathbf{v} + \gamma_w \mathbf{w}.$$

To get an expression with $\cos\theta$, we take the norm squared of the momentum equation,

$$\gamma_u^2 u^2 = \gamma_v^2 v^2 + \gamma_w^2 w^2 + 2\gamma_v \gamma_w vw \cos\theta.$$

This can be substantially simplified by noting that $\gamma_u^2 u^2 = \gamma_u^2 - 1$, giving

$$2vw\gamma_v\gamma_w \cos\theta = \gamma_u^2 - \gamma_v^2 - \gamma_w^2 + 1.$$

The appearance of so many squares motivates us to square both sides of the energy equation,

$$1 + 2\gamma_u + \gamma_u^2 = \gamma_v^2 + \gamma_w^2 + 2\gamma_v\gamma_w.$$

Using this to simplify the right-hand side of the previous equation,

$$2vw\gamma_v\gamma_w \cos\theta = 2\gamma_v\gamma_w - 2\gamma_u = 2(\gamma_v\gamma_w - \gamma_v - \gamma_w + 1) = 2(\gamma_v - 1)(\gamma_w - 1)$$

where in the second step we used conservation of energy. After solving for $\cos\theta$, we get the desired result. This was a bit of a slog, but it's representative of the hardest calculations you'll ever have to do for special relativity problems.

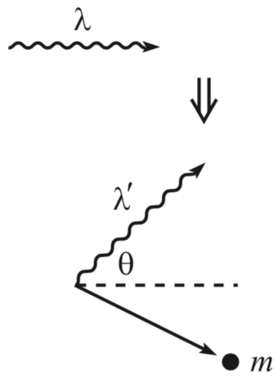
As a check on that result, note that in the nonrelativistic limit we get $\cos\theta = 0$, indicating a 90° angle, which you saw in **M3**. At relativistic speeds, the opening angle gets smaller, which is a manifestation of the "beaming" effect you saw in **R1**. This is a familiar effect, commonly observed in particle physics experiments.

[3] **Problem 5** (Morin 12.6)**.** A ball of mass M and energy E collides head-on elastically with a stationary ball of mass m. Show that the final energy of mass M is

$$E' = \frac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}.$$

This problem is a little messy, but you can save yourself some trouble by noting that $E' = E$ must be a root of the equation you get for E'.

[3] **Problem 6** (Morin 12.7)**.** In Compton scattering, a photon collides with a stationary electron.



(a) If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

(b) While Compton scattering can occur for photons of any frequency, it is usually used in reference to X-rays, which have very high frequencies. Why?

[3] **Problem 7.** USAPhO 2017, problem A4. However, to make it a little harder, solve part (a) without assuming E_b is small.

2 Relativistic Systems

Idea 3

The truly nonintuitive part of the result $E = mc^2$ is that changes in internal energy cause changes in mass. As a simple example, if you take a box of gas and heat it up, it'll have more mass than before, in every sense: the system will have more inertia, it'll have more momentum and kinetic energy when moving, it'll be heavier, and it'll exert more gravitational force on other objects. Some of the questions below illustrate how this can occur.

[3] **Problem 8.** The facts that $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$ are conserved are fundamentally new results of relativity, so the logically cleanest way to set up the theory is to simply make these postulates, without any further justification. But this certainly isn't the most *convincing* way, if you don't already believe that relativity is true.

The most striking new result is the huge rest energy $E = mc^2$. Throughout his life, Einstein came up with many derivations of this result, starting from more familiar postulates. In this problem, we'll cover Einstein's 1946 derivation of $E = mc^2$. Specifically, we will prove that when the energy content of a body at rest decreases by ΔE, its mass decreases by $\Delta E/c^2$. The result then follows if one assumes that a zero-mass object has no rest energy.

Consider an object of mass M at rest, and suppose it emits photons with equal and opposite momenta p_γ upward and downward simultaneously. Let m be the final mass of the object.

(a) Now consider the same process in a frame moving with speed $v \ll c$ to the left. By using conservation of momentum in the x direction, show that

$$M = m + \frac{2p_\gamma}{c}.$$

Don't use the relativistic momentum formula here, because we're trying to work from first principles. Just use the fact that at $v \ll c$ the Galilean formula works.

(b) Using energy conservation, conclude the desired result.

(c) The derivation also works if one considers a frame moving upward with speed $v \ll c$. Carry out this analysis.

(d) The physicist Hans Ohanian has claimed that all of Einstein's derivations of $E = mc^2$, including this one, were circular. What do you think?

[3] **Problem 9.** Consider a completely black cube of density ρ and side length L sitting in free space. In some particular frame, plane electromagnetic waves of intensity I (in units of $\mathrm{W/m^2}$) approach the cube from the left and right, striking two faces of it head on. Neglect any radiation from the cube. If the cube has an initial velocity $v \ll c$ in this frame, find its displacement after a long time. (Hint: solving the problem exactly will be very messy; it's better to approximate early, since we only want an answer correct in the limit $v/c \to 0$.)

[4] **Problem 10.** A rocket of initial mass M_0 starts from rest and propels itself forward along the x axis by emitting photons backward.

(a) Show that the final velocity of the rocket relative to the initial frame is

$$\frac{v}{c} = \frac{x^2 - 1}{x^2 + 1} = \tanh(\log x), \quad x = \frac{M_0}{M_f}$$

where M_f is the final rest mass of the rocket. (Hint: for this part, no integration is needed.)

(b) More generally, show that if the rocket fuel comes out at a speed u relative to the rocket,

$$\frac{v}{c} = \frac{x^{2u/c} - 1}{x^{2u/c} + 1} = \tanh((u/c)\log x)$$

where x is defined as above. (Hint: to avoid nasty differential equations, relate dm and dv.)

(c) Show that this reduces to the nonrelativistic rocket equation in the limit $u/c \to 0$.

(d) Show that in the limit $v/c \to 0$, the result of part (a) *also* reduces to the nonrelativistic rocket equation with exhaust speed c. Why does this work, given that photons are the most relativistic possible things?

[3] **Problem 11** (Grad)**.** An empty box of total mass M and perfectly reflecting walls is at rest in the lab frame. Then N photons are introduced into the box, each with frequency ω_0 in a standing wave configuration; one can think of these photons as continually bouncing back and forth with velocity $\pm c\,\hat{\mathbf{x}}$, with zero total momentum.

(a) State what the rest mass M_{tot} of the system will be when the photons are present.

(b) Consider the momentum of the system in an inertial frame moving along the x axis with speed $v \ll c$. Using the first order Doppler shift and assuming that at any moment, half the photons are moving left and half the photons are moving right, show that $p = M_{\text{tot}} v$. This provides a dynamical explanation of exactly how photons contribute to the inertia of an object.

(c) $\star$ Unfortunately, it is *not* true that half the photons are moving right at any given time. Show that the fraction of photons moving to the right is modified by an amount of order v/c, and find the total momentum accounting for this effect.

The analysis of part (b) is nice and neat, and you can sometimes find it in textbooks. But part (c) shows that this simple analysis is wrong! The official solution will show how to resolve the paradox, but it requires using the stress-energy tensor, which is well beyond the scope of Olympiad physics.

Remark

In Newtonian mechanics, we know that for an isolated system, $\mathbf{p}_{\text{tot}} = M_{\text{tot}} \mathbf{v}_{\text{CM}}$. In relativity, however, the idea of a "center of mass" no longer makes any sense. For example, suppose a particle with mass m decays into two photons. Each of the photons has no mass, so the center of mass is no longer defined! You can always define the mass of an overall system as $\sqrt{E_{\text{tot}}^2 - p_{\text{tot}}^2}$, and this quantity remains equal to m, but it's no longer the sum of the masses of the individual parts. Since you can't break the mass of the system into parts, you can't sum over the parts to define a center of mass.

However, you can still define a "center of energy",

$$\mathbf{x}_{\text{CE}} = \frac{\sum_i \mathbf{x}_i E_i}{\sum_i E_i}$$

where E_i is the energy of particle i. It turns out that in relativity, we always have

$$\mathbf{p}_{\text{tot}} = \frac{E_{\text{tot}}}{c^2} \mathbf{v}_{\text{CE}}$$

which is called the "center of energy theorem". (Specifically, it comes from applying Noether's theorem to the symmetry of Lorentz boosts.) Of course, this reduces to $\mathbf{p}_{\text{tot}} = M_{\text{tot}} \mathbf{v}_{\text{CM}}$ in the nonrelativistic limit, since in that case almost all the energy is rest energy, $E = mc^2$.

3 Optimal Collisions

These collision problems are conceptually simple, but somewhat more mathematically challenging.

Idea 4

The minimum energy configuration of a system of particles with fixed total momentum is the one where they all move with the same velocity. This is easiest to show by boosting to the center of mass frame (i.e. the frame with zero total momentum) and then boosting back.

Example 3: KK 14.3

A high energy photon (γ ray) collides with a proton at rest. A neutral pi meson is produced according to the reaction

$$\gamma + p \to p + \pi^0.$$

What is the minimum energy the γ ray must have for this reaction to occur? The rest mass of a proton is 938 MeV and the rest mass of a neutral pion is 135 MeV.

Solution

The total four-momentum is $(E + m_p, E)$ where E is the energy of the γ ray in the lab frame. This four-momentum has norm $2Em_p + m_p^2$. Crucially, the norms of four-momenta don't change upon changing frames, so the total four-momentum in the center of mass frame is

$$\left(\sqrt{2Em_p + m_p^2}\,, 0\right)$$

because the total spatial momentum vanishes by definition. On the other hand, we also know that the reaction can just barely happen when both the proton and pion are produced at rest in the center of mass frame, with a final four-momentum of $(m_p + m_\pi, 0)$. Hence we have

$$\sqrt{2Em_p + m_p^2} = m_p + m_\pi$$

and plugging in the numbers gives $E = 145\,\text{MeV}$. As expected, this is a little bit more than the mass-energy of the pion, because the final system inevitably has some kinetic energy too.

Example 4

Two photons of frequencies ω_1 and ω_2 collide head-on. Under what conditions can an electron-positron pair be created?

Solution

The naive answer is to say the energy present must exceed the rest energy,

$$\hbar\omega_1 + \hbar\omega_2 \geq 2m_e.$$

However, this is incorrect because the electron and positron will inevitably have kinetic energy, since the photons initially have a net momentum. The lowest total kinetic energy is achieved when the electron and positron come out with the same velocity, which is the velocity of the center of mass frame of the photons.

The total four-momentum of the photons is

$$(\hbar(\omega_1 + \omega_2), \hbar(\omega_1 - \omega_2))$$

in the lab frame, and $(E_{\text{cm}}, 0)$ in the center of mass frame. Therefore,

$$E_{\text{cm}}^2 = \hbar^2((\omega_1 + \omega_2)^2 - (\omega_1 - \omega_2)^2) = 4\hbar^2\omega_1\omega_2.$$

In the center of mass frame, the electron and positron can be produced at rest, so the condition is $E_{\text{cm}} \geq 2m_e$, which means

$$\hbar\sqrt{\omega_1\omega_2} \geq m_e.$$

[3] **Problem 12.** In a particle collider, a proton of mass m is given kinetic energy E and collided with an initially stationary proton.

(a) What is the minimum E required to produce a proton-antiproton pair, $p + p \to p + p + p + \overline{p}$?

(b) How about N proton-antiproton pairs, where $N = 1$ in part (a)?

The scaling behavior of the answer you found in part (b) is the reason many particle colliders use two beams going in opposite directions, even though managing two beams precisely enough to collide them at the desired points is technically challenging.

[3] **Problem 13** (MPPP 196)**.** Two ultrarelativistic particles with negligible rest mass collide with oppositely directed momenta p_1 and p_2 elastically. Find the minimum possible angle between their velocities after the collision.

[3] **Problem 14.** IPhO 2003, problem 3A.

[4] **Problem 15.** APhO 2007, problem 3B. A comprehensive relativistic dynamics problem.

4 Relativistic Dynamics

In this section we'll consider some genuinely dynamic situations involving relativistic particles.

Idea 5

In relativity, the force four-vector is defined as

$$f^\mu = \frac{dp^\mu}{d\tau} = ma^\mu.$$

There's a bit of a subtlety here. In relativity, the invariant mass of a system can change when it absorbs energy, even if it doesn't exchange any particles with its environment. For example, putting a system on the stove gives it energy but not momentum, thereby changing $m = \sqrt{E^2 - p^2}$. That's a perfectly valid four-force, but it feels strange to call it a "force". Therefore, we often restrict to four-forces that don't change the invariant mass, and since

$$\frac{dm^2}{d\tau} = \frac{d}{d\tau}(p \cdot p) = 2mu \cdot f$$

that corresponds to demanding $f \cdot u = 0$.

Idea 6

There's also a second way to define force in special relativity, with three-vectors. The first subtlety here is that you could define it as $d\mathbf{p}/dt$ or $m\mathbf{a}$, but the two differ in relativity. Since accelerations transform in a rather nasty way, as we saw in **R1**, the usual choice is to define

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The second subtlety is that, whenever we define forces as three-forces, we usually implicitly assume that they fix the invariant mass m, i.e. we automatically rule out "put it on a stove" forces. Otherwise, there wouldn't be any way to tell how the energy changes over time.

[4] **Problem 16.** In this problem, we'll derive some properties of the three-force and four-force. For reference, see section 12.5 of Morin.

(a) Show that for a particle traveling along the $\hat{\mathbf{x}}$ direction,

$$\mathbf{F} = m(\gamma^3 a_x, \gamma a_y, \gamma a_z).$$

This is the relativistic three-vector analogue of $\mathbf{F} = m\mathbf{a}$, but it implies that force is no longer parallel to acceleration, which will be important in the problems below.

(b) Now let S' be the momentary rest frame of that particle. In this frame, since the particle is at rest, the nonrelativistic expression $\mathbf{F}' = m\mathbf{a}'$ holds. By using the transformation of acceleration derived in **R1**, show that

$$\mathbf{F} = (F'_x, F'_y/\gamma, F'_z/\gamma).$$

That is, transverse forces are redshifted in relativity, while longitudinal forces are unchanged.

(c) Show that the components of the four-force are

$$f^\mu = \left(\gamma \frac{dE}{dt}, \gamma \mathbf{F}\right).$$

Use the relativistic transformation of the four-force to rederive the result of part (b).

(d) The four-impulse is defined as

$$\Delta p^\mu = \int f^\mu \, d\tau.$$

But you can also consider the Lorentz scalar

$$\int f^\mu \, dx_\mu.$$

This ought to be something nice and simple that you already know about. What is it?

Remark

In popular science books and some older textbooks, relativistic dynamics is introduced using the idea of relativistic mass, $m_r = \gamma m$. This definition implies the simple results $E = m_r c^2$ and $\mathbf{p} = m_r \mathbf{v}$, so these books often say that relativistic dynamics is just like ordinary dynamics, except that moving objects have more mass. This picture is misleading because it breaks down once you go beyond one dimension: in problem 16, you showed that $\mathbf{F}$ is not even parallel to $\mathbf{a}$, so there's no definition of mass that recovers Newtonian mechanics. You instead need separate "transverse" and "longitudinal" relativistic masses,

$$\mathbf{F} = m_\perp \mathbf{a}_\perp + m_\parallel \mathbf{a}_\parallel, \quad m_\perp = \gamma m, \quad m_\parallel = \gamma^3 m.$$

I think this picture is honestly more confusing than helpful, though. It's better to avoid talking about mass and acceleration too much, and focus more on momentum and energy.

You'll also see arguments that relativistic mass is useful when thinking about gravity. In general relativity, all energy produces gravity equally. If you have a box with n particles bouncing around, which all have Lorentz factor γ and rest mass m, then the energy of the box is the same as that of n particles at rest, with mass m_r. So it looks like the gravity sourced by the particles is described by their relativistic mass. Unfortunately, this argument is *also* wrong, because in general relativity pressure also produces gravity. In the limit $\gamma \to \infty$, describing a gas of ultrarelativistic particles, the pressure contribution means we get twice as much gravitational attraction as would be predicted from the energy alone.

Example 5

A circular pendulum consists of a mass m attached to a string of length ℓ, with the other end fixed. Suppose the mass rotates in a small circle of radius $r \ll \ell$. Find the angular frequency of the oscillations in the lab frame, and in a frame where the entire setup moves vertically with a relativistic speed v.

Solution

In the lab frame, this is a standard rotational mechanics problem. By the small angle approximation, the horizontal component of the three-force is $F_\perp = mgr/L$. This is equal to

$$F_\perp = ma_\perp = m\omega^2 r$$

from which we immediately conclude $\omega = \sqrt{g/L}$. We can use the results of problem 16 to find the answer in the other frame. The two effects are that the transverse force is redshifted, and the force's relation with acceleration is different,

$$F_\perp = \frac{mgr}{\gamma L}, \quad F_\perp = \gamma m a_\perp = \gamma m \omega^2 r.$$

Combining these results, we find

$$\omega = \frac{1}{\gamma}\sqrt{\frac{g}{L}}.$$

Of course, γ is just the usual time dilation factor. We knew this had to be the answer, because time dilation follows directly from the postulates of relativity, but now we can explicitly show this is the right answer in this specific example. (With similar reasoning, you can show that a mass-spring system oscillates slower, too.)

Remark

It's important not to misunderstand the meaning of the above example. Like many old physicists, Oleg Jefimenko decided one day that relativity had to be completely wrong. His argument was along the lines of the previous example: he showed that length contraction and time dilation could be derived dynamically in some simple cases, without the need to switch frames. Therefore, they can't be "real".

This argument doesn't make sense. It's like saying that energy can't be real because you can solve many mechanics problems with just $F = ma$, without needing to invoke energy conservation. (Though amazingly, some people actually do spend years arguing whether force or energy is "more real", in a debate that resembles rival high school cheerleading squads, when it's better to realize that they're both wonderful tools with complementary uses.)

Furthermore, it actually turns out to be *extremely* difficult to derive the core results of relativistic dynamics (such as the "transverse" and "longitudinal" masses, already measured by the turn of the 20^{th} century) without using relativistic assumptions. In the early 1900s, many physicists tried to explain the dynamics of the electron solely in terms of its electromagnetic fields. Since the field energy and field momentum of a moving point charge are infinite, it was necessary to take a model of the electron with finite size, but there were many possibilities, leading to many different expressions for the transverse mass, as well as persistent issues like the 4/3 problem mentioned in **E7**.

Relativity circumvents all of these issues. If you accept the postulates of relativity, you don't need to care whether the electron is shaped like a sphere, an ellipsoid, a torus, or a dumbbell: as long as its dynamics obey Lorentz symmetry, its four-momentum is a four-vector, and the usual results follow. And that's just as well, because with the advent of quantum mechanics, we learned that the electron is not like *any* of these classical models. But the relativistic result still holds, because our quantum theories obey the postulates of relativity too. This flexibility comes about because, like thermodynamics, relativity isn't so much a physical theory, as it is a framework within which many theories can be formulated.

Example 6

In the preceding example, how large of a force does the pendulum bob experience?

Solution

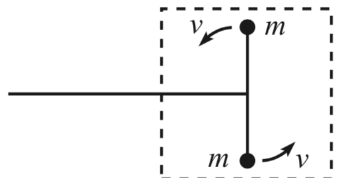
Here, we aren't referring to the three-force of the bob in the lab frame, which we already know is $m\omega^2 r$. We're referring to the proper force, i.e. the three-force in the bob's own frame.

We could calculate this by Lorentz transforming into that frame, but there's an easier way. Note that $f \cdot f$ is a Lorentz scalar which, in the bob's frame, is $-|\mathbf{F}|^2$. Therefore, the proper force is $\sqrt{-f \cdot f}$, and evaluating this in the lab frame gives

$$\sqrt{(\gamma F_\perp)^2} = \gamma^2 m \omega^2 r.$$

That's a factor of γ^2 larger than the nonrelativistic expectation, so quite violent!

[3] **Problem 17** (Morin 12.8)**.** Consider a dumbbell made of two equal masses, m. The dumbbell spins around, with its center pivoted at the end of a stick.



If the speed of the masses is v, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what's going on inside.) Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the "transverse" position shown in the figure) and showing that $F = dp/dt = Ma$.

Idea 7

The Lorentz force is a three-force as defined in problem 16. That is, we have

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{d\mathbf{p}}{dt}$$

and the force keeps the invariant mass fixed.

Example 7

A point charge q of mass m is initially at rest, and experiences a uniform electric field E. What time t does it take the object to move a distance x?

Solution

In **R1**, we found $x(t)$ for a uniformly accelerated rocket, which assumed a constant three-force in the momentarily comoving frame. By contrast, here we have a constant three-force $F = qE$ in the *lab* frame. However, we showed in problem 16 that forces along the direction of motion are the same in both frames, so these two problems are actually identical!

So we already know the answer to the problem, but it turns out that in the lab frame perspective, there's a slick alternative derivation that yields the result in one step. Consider the energy and momentum. Recall from problem 1 that the three-force F obeys $F = dp/dt$

and $F = dE/dx$. Therefore, when the object reaches its destination,

$$E = m + Fx, \quad p = Ft.$$

But we also know that $E^2 = p^2 + m^2$, so plugging the results in and solving for t gives

$$t(x) = \sqrt{x^2 + \frac{2mx}{F}}$$

which is compatible with our earlier expression for $x(t)$. The reason this was so easy is that momentum and energy behave simply in relativity, while position and velocity don't.

Example 8

The LHC accelerates protons to an energy of $E = 7\,\text{TeV}$, and is a tunnel of radius $R = 4.3\,\text{km}$. If the protons are kept in a circular orbit in the tunnel by a magnetic field of magnitude B, find the required value of B. If the value of B is kept constant, what would be the radius of a future collider which accelerates protons to an energy of $20\,\text{TeV}$?

Solution

The centripetal force required is

$$F = \left| \frac{d\mathbf{p}}{dt} \right| = \omega p$$

where ω is the angular velocity. The speed of the protons is very close to c, so the angular velocity is $\omega \approx c/R$, and the momentum is $p \approx E/c$. The deflecting force is $qvB \approx qcB$, so

$$qcB \approx \omega p \approx \frac{E}{R}.$$

Therefore, we have

$$B = \frac{E}{qcR} = \frac{7 \times 10^{12}}{(3 \times 10^8)(4.3 \times 10^3)}\,\text{T} = 5.4\,\text{T}.$$

This is slightly lower than what is actually used, because magnets don't take up the entire tunnel. Since $R \propto E$, the future collider would need a radius of

$$R' = \frac{20\,\text{TeV}}{7\,\text{TeV}} R = 12\,\text{km}.$$

Remark

You might be wondering how to write the Lorentz force as a four-force. It certainly should be possible, since we know electromagnetism is compatible with relativity (indeed, it led us to relativity in the first place), but it seems challenging because electromagnetism is so naturally written in terms of three-vectors. It turns out that the proper way to express the electromagnetic field in relativity is to join the electric and magnetic fields together, making them the components of an antisymmetric rank 2 tensor,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

called the field strength tensor. Then the four-force is

$$f^\mu = q u_\nu F^{\mu\nu}$$

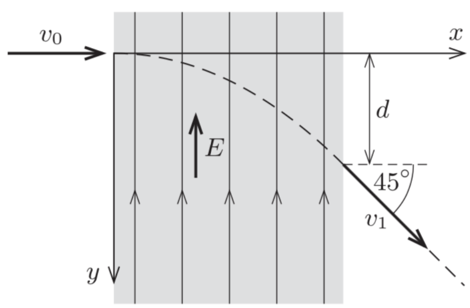
where u_ν is the four-velocity. Note that this ensures the rest mass of the particle is fixed, as

$$f \cdot u = q u_\mu u_\nu F^{\mu\nu} = -q u_\mu u_\nu F^{\nu\mu} = -f \cdot u$$

using the antisymmetric property, so $f \cdot u = 0$. (In fact, the requirement to keep the rest mass fixed is quite restrictive, so this is one of the simplest relativistic force laws.)

[2] **Problem 18.** USAPhO 2013, problem A3. A warmup question using the above facts.

[3] **Problem 19** (MPPP 192)**.** An electron moving with speed $v_0 = 0.6c$ enters a homogeneous electric field that is perpendicular to its velocity.



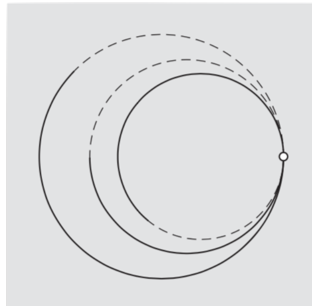
When the electron leaves the field, its velocity makes an angle 45° with its initial direction.

(a) Find the speed v_1 of the electron after it has crossed the electric field.

(b) Find the distance d shown above, if the strength of the electric field is $E = 510\,\text{kV/m}$.

Note that the rest energy of an electron is $510\,\text{keV}$.

[3] **Problem 20** (MPPP 194)**.** The trajectories of charged particles, moving in a homogeneous magnetic field, can be seen by observing the tracks they leave in cloud chambers. Because the particles are moving quickly, it is impossible to see the tracks being formed; instead, one must infer what happened from the shapes of the tracks. Is it possible that, when a charged particle decays into two other charged particles, the trail segments close to the decay point (before the particles have started to slow down significantly) are arcs of circles that touch each other, as shown?



If so, identify which track belongs to the original particle. If not, explain why not.

[3] **Problem 21.** USAPhO 2006, problem A4.

[3] **Problem 22.** USAPhO 2022, problem B2. A nice problem on deriving the time dilation formula for an electrostatic "clock".

[3] **Problem 23.** Consider a particle at the origin at time $t = 0$, with initial x-momentum p_0 and total energy E_0. A constant three-force F acts on the particle in the $-y$ direction.

(a) Calculate $y(t)$. (Hint: don't write down any equations containing γ, because it depends on $v_x(t)$, which we don't know yet.)

(b) Calculate $x(t)$.

(c) Combine these results to get $y(x)$. This is the path of a relativistic projectile.

[5] **Problem 24.** IPhO 1994, problem 1. Print out the custom answer sheets before starting.

Remark

The setup of problem 24 is a nice model for mesons, particles composed of two quarks. And it's not just something made up for an Olympiad; it is a simple version of the MIT "bag model", which was one of the most important advances in the field in the 1970s. In fact, if you look at the original paper, which has thousands of citations, you'll find the answer to the IPhO question in figure 3!

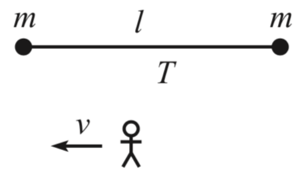
Idea 8

In string theory, strings carry a constant tension T, in the sense that the force $\mathbf{F} = d\mathbf{p}/dt$ exerted on one piece of string by its neighbors is T in the momentary rest frame of that piece. The strings may stretch or shrink freely, and have zero mass when they have zero length.

[3] **Problem 25** (Morin 12.16)**.** A simple exercise involving relativistic string.

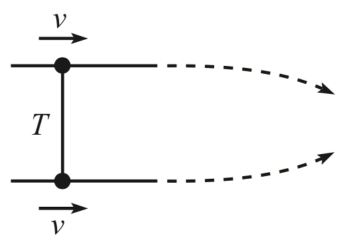
(a) Two masses m are connected by a string of length ℓ and constant tension T. The masses are released simultaneously, and they collide and stick together. What is the mass, M, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left at speed v.



The energy of the resulting blob must be $\gamma M c^2$. Show that you obtain the same result by computing the work done on the two masses.

[3] **Problem 26** (Morin 12.37)**.** Two equal masses are connected by a relativistic string with tension T. The masses are constrained to move with speed v along parallel lines, as shown.



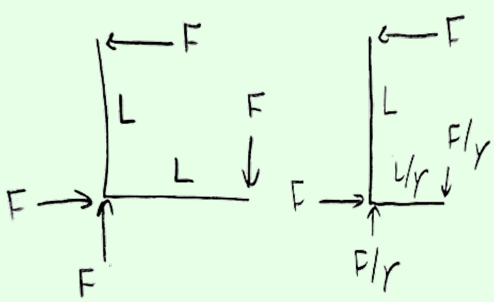
The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct?

> The forces on the masses point in the y direction. Therefore, there is no change in the momentum of the masses in the x direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collide with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x direction.

If your answer is "no," exactly what's wrong about the reasoning above?

Example 9: Right Angle Lever Paradox

In 1909, Lewis and Tolman found one of the first relativistic paradoxes. Consider a rigid lever in static equilibrium, with both arms of length L, experiencing the forces shown at left.



In a frame where the lever moves to the right with speed v, one of the lever arms will be contracted to L/γ, as shown at right. In addition, by the results of problem 16, the vertical external forces will be redshifted to F/γ. This implies a net torque of

$$\tau = FL - \frac{F}{\gamma}\frac{L}{\gamma} = FLv^2.$$

The paradox is, given that $\boldsymbol{\tau} = d\mathbf{L}/dt$, why doesn't the lever rotate?

Solution

The resolution is that, in the frame shown at right, the angular momentum of the lever is constantly increasing. The horizontal forces are continually doing equal and opposite work on the lever, resulting in a upward flow of energy of rate Fv in the vertical arm. As explained in **E7** and earlier in this problem set, in relativity, energy flow is equal to momentum density, so the total upward momentum in the vertical arm is FLv. Therefore,

$$\frac{dL}{dt} = \frac{dx}{dt}\,(FLv) = FLv^2$$

exactly as expected.

Remark

The resolution of the right angle lever paradox is very controversial, with dozens of papers written on the subject, so we should discuss what it even means to "resolve" a paradox. As long as we believe relativity is self-consistent, we already know what's going to happen: the lever won't rotate. Everything the lever does is determined by $\mathbf{F} = d\mathbf{p}/dt$ alone, so if it looks like angular momentum considerations give a different answer, that just means we haven't formulated the latter correctly. The reason there are so many different resolutions out there is just that people choose different ways to define torque and angular momentum.

The solution above is the standard one, and its implicit definition of angular momentum can be motivated by Noether's theorem. That's a reasonable choice, since it's a specific output of a useful and general theorem, and we thereby know for sure that it's conserved for isolated systems. Unfortunately, explaining the definition takes some advanced math.

We define the angular momentum density tensor

$$M^{\mu\nu\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x)$$

where the right-hand side contains the stress-energy tensor, from the solution to problem 11. The total angular momentum density is an antisymmetric rank 2 tensor,

$$J^{\mu\nu}(t) = \int d\mathbf{x}\, M^{\mu\nu\rho}(x).$$

Noether's theorem states that it is this quantity that is conserved for an isolated system, due to symmetry under rotations and boosts. (More specifically, the spatial components correspond to what we call angular momentum in nonrelativistic physics, $J^{xy} = L^z$, while the J^{0i} components have to do with the center of mass motion.) If there is an external four-force per unit proper volume $f^\mu(x)$, which in terms of the stress-energy tensor implies $\partial_\mu T^{\mu\nu} = f^\nu$, the rate of change of angular momentum is

$$\frac{dJ^{\mu\nu}}{dt} = \tau^{\mu\nu}, \quad \tau^{\mu\nu} = \int d\mathbf{x}\, x^\mu f^\nu(x) - x^\nu f^\alpha(x)$$

which looks quite similar to the Newtonian expression. The component of this equation relevant to this paradox is $dJ^{xy}/dt = \tau^{xy}$, where

$$J^{xy} = \int d\mathbf{x}\, xT^{y0} - yT^{x0}, \quad \tau^{xy} = \sum_k x^{(k)} F_y^{(k)} - y^{(k)} F_x^{(k)}$$

where the index k sums over the four forces, and the T^{i0} stand for the density of momentum in the i direction. From this point on, the solution proceeds as above.

There is something a bit strange here, though. In the lever's rest frame, the angular momentum is zero, so if $J^{\mu\nu}$ were a tensor, it would have to be zero in all frames, but instead it rises to arbitrarily high values in the other frame. The reason is that when there are external torques, $J^{\mu\nu}$ isn't a tensor at all, just like how the four-momentum wasn't a four-vector in the solution to problem 11. That's one of the reasons there's a controversy: there just doesn't exist any definition that has all the nice properties one might want.