Electromagnetism V: Induction

Chapter 7 of Purcell covers induction, as does chapter 7 of Griffiths. For magnetism, see section 6.1 of Griffiths; for applications, see the end-of-chapter technology reviews in each chapter of Purcell, along with chapters II-16 and II-17 of the Feynman lectures. For a qualitative introduction to superconductivity, see appendix I of Purcell. Aim to solve or attempt at least **45/88** points.

1 Motional EMF

Idea 1

If $\mathbf{F}$ is the force on a charge q, then the emf about a loop C is

$$\mathcal{E} = \frac{1}{q}\oint_C \mathbf{F} \cdot d\mathbf{s}.$$

For a moving closed loop in a time-independent magnetic field, the emf through the loop is

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where Φ is the magnetic flux through the loop. The direction of the emf produces a current that opposes the change in flux.

Example 1

A wire is bent into an arbitrary planar shape, so that its two ends are separated by a distance R, and the wire is rotated inside the xy plane with angular velocity ω. There is a constant magnetic field $B\,\hat{\mathbf{z}}$. Find the emf across the wire.

Solution

The emf is motional emf due to the magnetic force, so

$$\mathcal{E} = \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}.$$

The main point of this problem is to get you acquainted with some methods for manipulating vectors. First, we'll use components. Placing the origin along the axis of rotation, we have

$$\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega} = (x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \times \omega\hat{\mathbf{z}} = \omega(y\hat{\mathbf{x}} - x\hat{\mathbf{y}})$$

for a point on the wire at $\mathbf{r}$. Evaluating the cross product with the magnetic field,

$$\mathbf{v} \times \mathbf{B} = \omega B(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}) \times \hat{\mathbf{z}} = -\omega B(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = -\omega B\mathbf{r}.$$

Therefore, we have

$$\mathcal{E} = -\omega B \int \mathbf{r} \cdot d\mathbf{r} = -\frac{\omega B}{2}\int_0^R d(r^2) = -\frac{\omega B R^2}{2}$$

which is completely independent of the wire's shape.

Now let's solve the question again without components. Here it's useful to apply the double cross product, or "BAC-CAB" rule,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

If you want to show this for yourself, note that both sides are linear in $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, so it's enough to prove it for all combinations of unit vectors they could be; this just follows from casework. We can now simplify the emf integrand as

$$(\mathbf{r} \times \boldsymbol{\omega}) \times \mathbf{B}) = \mathbf{B} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\mathbf{B} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{B} \cdot \boldsymbol{\omega}).$$

The first term is zero since $\mathbf{r}$ lies in the xy plane, while the second term is $-\omega B \mathbf{r}$. The rest of the solution follows as with the component method.

For problems that are essentially two-dimensional, there's not much difference in efficiency between the two methods, so you should use whatever you're more comfortable with. On the other hand, for problems with three-dimensional structure, components tend to get clunky.

[2] **Problem 1** (Purcell). [A] Prove this result using the Lorentz force law as follows.

(a) Let the loop be C and let $\mathbf{v}$ be the velocity of each point on the loop. Argue that after a time dt, the change in flux is

$$d\Phi = \oint_C \mathbf{B} \cdot ((\mathbf{v}\, dt) \times d\mathbf{s}).$$

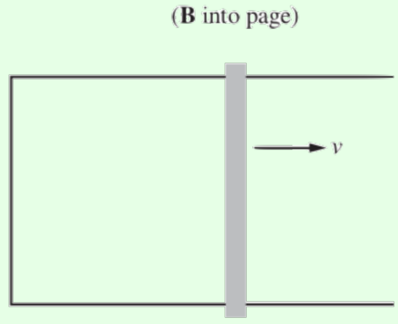
(b) Using the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$, show that

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}$$

and use this to conclude the result.

Example 2: Purcell 7.2

A conducting rod is pulled to the right at speed v while maintaining a contact with two rails. A magnetic field points into the page.



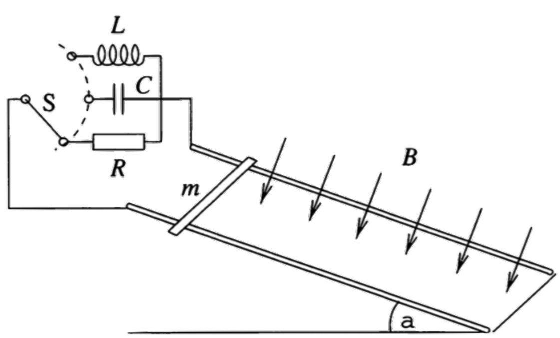
An induced emf will cause a current to flow in the counterclockwise direction around the loop. Now, the magnetic force $q\mathbf{u} \times \mathbf{B}$ is perpendicular to the velocity $\mathbf{u}$ of the moving charges, so it can't do work on them. However, the magnetic force certainly looks like it's doing work. What's going on here? If the magnetic force doing work or not? If not, then what is? There is definitely something doing work because the wire will heat up.

Solution

A perfectly analogous question is to imagine a block sliding down a ramp with friction, at a constant velocity. Heat is produced, so something is certainly doing work. We might suspect it's the normal force, because it has a horizontal component along the block's direction of horizontal travel. However, it also has a vertical component opposite the block's direction of vertical travel, so it of course performs no work. All it does is redirect the block's velocity; the ultimate source of energy is gravity.

Similarly, in this case, the current does not flow vertically (along the page), but also has a horizontal component because it is carried along with the rod. Just like the normal force in the ramp example, the magnetic force is perpendicular to the velocity, and does no work. It simply redirects the velocity created by whatever is pulling the rod to the right, which is the ultimate source of energy.

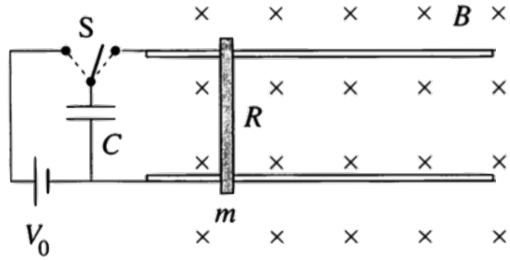
[3] **Problem 2** (PPP 167)**.** A homogeneous magnetic field $\mathbf{B}$ is perpendicular to a track inclined at an angle α to the horizontal. A frictionless conducting rod of mass m and length ℓ straddles the two rails as shown.



How does the rod move, after being released from rest, if the circuit is closed by (a) a resistor of resistance R, (b) a capacitor of capacitance C, or (c) a coil of inductance L? In all cases, neglect the self-inductance of the closed loop formed, i.e. neglect the flux that its current puts through itself.

[3] **Problem 3.** USAPhO 2006, problem B1.

[3] **Problem 4** (PPP 168)**.** One end of a conducting horizontal track is connected to a capacitor of capacitance C charged to voltage V_0. The inductance of the assembly is negligible. The system is placed in a homogeneous, vertical magnetic field B, as shown.



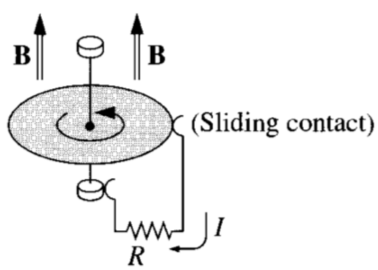
A frictionless conducting rod of mass m, length ℓ, and resistance R is placed perpendicularly onto the track. The capacitor is charged so that the rod is repelled from the capacitor when the switch is turned. This arrangement is known as a railgun. Neglect self-inductance throughout this problem.

(a) What is the maximum velocity of the rod?

(b) What is the maximum possible efficiency?

Not all motional emfs can be found using $\mathcal{E} = -d\Phi/dt$. Sometimes, for more complex geometries where there is no clear "loop", we need to go back to the Lorentz force law.

[3] **Problem 5** (Griffiths)**.** A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform magnetic field $\mathbf{B}$ pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk.



This system is a Faraday disk, also called a homopolar generator.

(a) Find the current in the resistor. (Hint: do not try to find the current distribution in the disk. This is an extremely complicated set of eddy currents. In particular, there is no clear "loop" that any charge travels on.)

(b) If we replace the resistor with a battery of voltage V, with the positive end towards the sliding contact, we get a homopolar motor, which makes the disk rotate starting from rest. Neglecting friction, which way will it rotate, and roughly what is the maximum possible angular speed?

[5] **Problem 6.** IPhO 1990, problem 2. A neat problem which also reviews **M6**.

2 Faraday's Law

Idea 2

Faraday's law states that even for a time-dependent magnetic field, we still have

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

In the case where the loop isn't moving but the magnetic field is changing, the emf is entirely provided by the electric field,

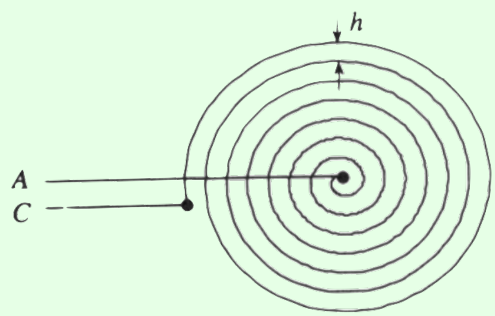
$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{s}.$$

Electric fields in the presence of changing magnetic fields can thus be nonconservative, i.e. they can have a nonzero closed line integral, a situation we haven't seen in any previous problem set. The differential form of Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Example 3

A flat metal spiral, with a constant distance h between coils, and $N \gg 1$ total turns is placed in a uniformly growing magnetic field $B(t) = \alpha t$ perpendicular to the plane of the spiral.



Find the emf induced between points A and C.

Solution

In theory, you can imagine connecting A and C and finding the flux through the resulting loop, but this is hard to visualize. A better way is to imagine turning the spiral into N concentric circles, connected in series. Then the emf is the sum of the emfs through each,

$$\mathcal{E} = \sum_{k=1}^{N} \pi (kh)^2 \alpha \approx \pi h^2 \alpha \int_0^N dk\, k^2 = \frac{\pi}{3} h^2 N^3 \alpha.$$

To see why this deformation is valid, remember that the emfs are due to a nonconservative electric field, integrated along the length of the loop. Deforming it into a bunch of concentric circles doesn't significantly change $\mathbf{E} \cdot d\mathbf{s}$ along it, so it doesn't change the answer.

Remark: EMF vs. Voltage

We mentioned earlier in **E3** that we often care about electromotive forces, which just mean any forces that act on charges to push them around a circuit. The force due to a non-conservative electric field is another example.

In such cases, the idea of "voltage" really breaks down entirely, because you can't define it consistently. However, electrical engineers use a more pragmatic definition of voltage: to them, voltage is just whatever a voltmeter displays. In other words, what they call voltage is what we call electromotive force. This can lead to rather nonintuitive results, as you'll see below. For example, the "voltage" can be different for different voltmeters even if they are connected at the same points!

Despite this trouble, we'll go along with the standard electrical engineer nomenclature and refer to these emfs as voltages in later problem sets. For example, Kirchoff's loop rule should properly say that the sum of the voltage drops along a loop is not zero, but rather $d\Phi/dt$. But it is conventional to move $d\Phi/dt$ to the other side and call it a "voltage drop" of $-d\Phi/dt$.

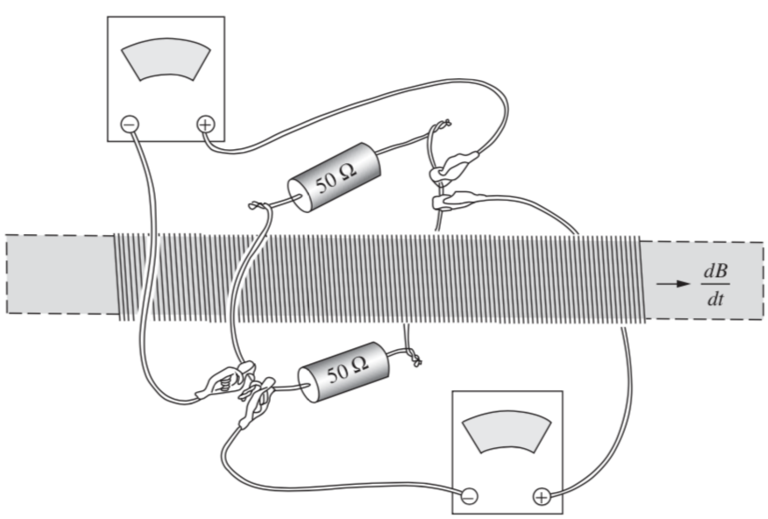
Remark

When we apply Faraday's law, we often use Ampere's law (without the extra displacement current term) to calculate the magnetic field. This is not generally valid, but works if the currents are in the slowly changing "quasistatic" regime, which means radiation effects are negligible. All the problems below assume this, but we'll see more subtle examples in **E7**.

[2] **Problem 7** (Purcell 7.6)**.** An infinite cylindrical solenoid has radius R and n turns per unit length. The current grows linearly with time, according to $I(t) = Ct$. Find the electric field everywhere.

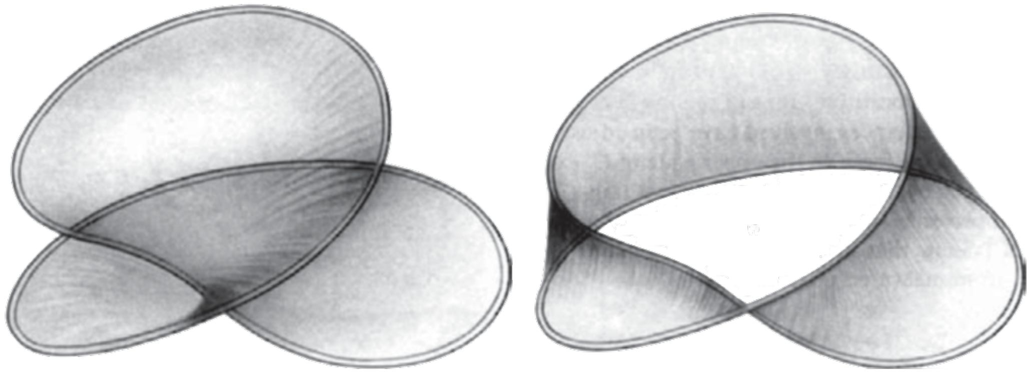
We'll begin with some conceptual questions about applying Faraday's law.

[2] **Problem 8** (Purcell 7.4)**.** Two voltmeters are attached around a solenoid as shown.



Find the readings on the two voltmeters in terms of $d\Phi/dt$.

[2] **Problem 9** (Purcell 7.28). [A] Consider the loop of wire shown below.

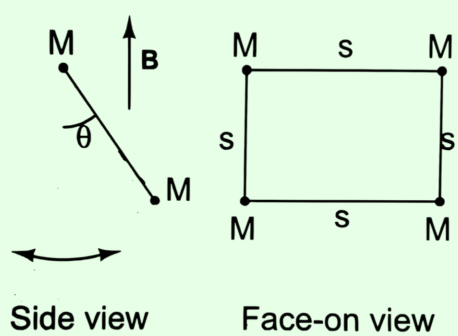


Suppose we want to calculate the flux of **B** through this loop. Two surfaces bounded by the loop are shown above. Which, if either, is the correct surface to use? If each of the two turns in the loop are approximately circles of radius R, then what is the flux? Generalize to an N-turn coil.

Now we move onto more physical questions.

Example 4

A square, rigid loop of wire has resistance R, sides of length s, and negligible mass. Point masses of mass M are attached at each corner. The top edge of the square loop is mounted so it is horizontal, and the loop may rotate as a frictionless pendulum about a fixed axis passing through this edge. Initially the pendulum is at rest at $\theta = 0$, and a uniform magnetic field **B** points horizontally through the loop. The magnetic field is then quickly rotated to the vertical direction, as shown.



Describe the subsequent evolution.

Solution

The rotation of the magnetic field provides a sharp impulse that causes the pendulum to start swinging. Letting ϕ be the angle of the field to the horizontal,

$$\mathcal{E} = -\frac{d(B_x s^2)}{dt} = -Bs^2\,\frac{d(\cos\phi)}{dt}$$

and the torque about the axis of rotation is

$$\tau = (IsB_y)s = -\frac{s^4B^2}{R}\sin\phi\,\frac{d(\cos\phi)}{dt}.$$

The total impulse delivered is

$$L = \int \tau \, dt = \frac{s^4 B^2}{R} \int_0^{\pi/2} \sin^2 \phi \, d\phi = \frac{\pi}{4} \frac{s^4 B^2}{R}$$

which causes an initial angular velocity $\omega = L/(2Ms^2)$.

After the pendulum begins swinging, the presence of the magnetic field causes an effective drag force. To see this, note that now we have

$$\mathcal{E} = -Bs^2 \frac{d(\sin\theta)}{dt}$$

which implies

$$\tau = Is^2 B \cos\theta = -\frac{s^4 B^2}{R} \cos^2\theta \, \frac{d\theta}{dt}.$$

Therefore, the $\tau = I\alpha$ equation is

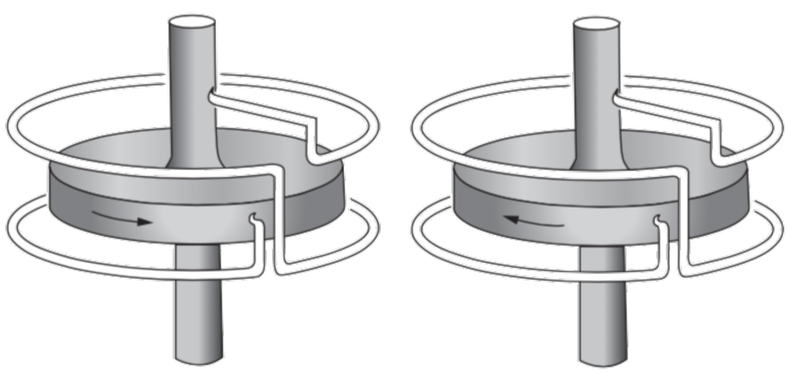
$$2Ms^2 \frac{d^2\theta}{dt^2} = -2Mgs \sin\theta - \frac{B^2 s^4}{R} \cos^2\theta \, \frac{d\theta}{dt}.$$

If we take the small angle approximation, then we recover ordinary damped harmonic oscillations, as covered in **M4**.

[3] **Problem 10.** USAPhO 2009, problem A1.

[3] **Problem 11.** USAPhO 1999, problem B2.

[3] **Problem 12** (Purcell)**.** A dynamo is a generator that works as follows: a conductor is driven through a magnetic field, inducing an electromotive force in a circuit of which that conductor is part. The source of the magnetic field is the current that is caused to flow in that circuit by that electromotive force. An electrical engineer would call it a self-excited dynamo. One of the simplest dynamos conceivable is shown below.



It has only two essential parts. One part is a solid metal disk and axle which can be driven in rotation. The other is a two-turn "coil" which is stationary but is connected by sliding contacts, or "brushes", to the axle and to the rim of the revolving disk.

(a) One of the two devices pictured is, at least potentially, a dynamo. The other is not. Which is the dynamo?

A dynamo like the one above has a certain critical speed ω_0. If the disk revolves with an angular velocity less than ω_0, nothing happens. Only when that speed is attained is the induced $\mathcal{E}$ enough to make the current enough to make the magnetic field enough to induce an $\mathcal{E}$ of that magnitude. The critical speed can depend only on the size and shape of the conductors, the conductivity σ, and the constant μ_0. Let d be some characteristic dimension expression the size of the dynamo, such as the radius of the disk in our example.

(b) Show by a dimensional argument that ω_0 must be given by a relation of the form $\omega_0 = K/\mu_0\sigma d^2$ where K is some dimensionless numerical factor that depends only on the arrangement and relative size of the parts of the dynamo.

(c) Demonstrate this result again by using physical reasoning that relates the various quantities in the problem (R, $\mathcal{E}$, E, I, B, etc.). You can ignore all numerical factors in your calculations and absorb them into the constant K.

For a dynamo of modest size made wholly of copper, the critical speed would be practically unattainable. It is ferromagnetism that makes possible the ordinary DC generator by providing a magnetic field much stronger than the current in the coils, unaided, could produce. For an Earth-sized dynamo, however, the critical speed is much smaller. The Earth's magnetic field is produced by a nonferromagnetic dynamo involving motions in the fluid metallic core.

[5] **Problem 13.** APhO 2009, problem 2. This problem analyzes a dynamo in more detail, completing the rough analysis made above. It's a worthwhile problem that gives you practice with electromagnetic systems.

[2] **Problem 14** (MPPP 178)**.** In general, a magnet moving near a conductor is slowed down by induction effects. Suppose that inside a long vertical, thin-walled, brass tube a strong permanent magnet falls very slowly due to these effects, taking a time t to go from the top to the bottom.

(a) Let the magnet have mass m, and let the tube have resistivity ρ, thickness r, and length L. Suppose both the magnet and tube have radius approximately R, and let the magnet's length also be of order R. Let the typical magnetic fields produced at the magnet's surface have magnitude B_0. Find an estimate for t, to the nearest order of magnitude.

(b) If the experiment is repeated with a copper tube of the same length but a larger diameter, the magnet takes a time t' to fall through. How long does it take for the magnet to fall through the tubes if they are fitted inside each other? Neglect the mutual inductance of the tubes.

3 Inductance

Idea 3: General Inductance

Consider a set of loops with fluxes Φ_i and currents I_i. By linearity, they are related by

$$\Phi_i = \sum_j L_{ij} I_j$$

where the L_{ij} are called the coefficients of inductance. It can be shown that $L_{ij} = L_{ji}$. By

Faraday's law, we have

$$\mathcal{E}_i = \sum_j L_{ij} \dot{I}_j.$$

In contrast with capacitance, we're usually concerned with the self-inductance $L_i = L_{ii}$ of single loops; these inductors provide an emf of $L\dot{I}$ each. However, mutual inductance effects can also impact how circuits behave, as we'll see in **E6**.

Remark

The inductance coefficients are similar to the capacitance coefficients in **E2**, but more useful. For capacitors, we are typically interested in configurations with one positive and one negative plate, and the capacitance of this object is related to all of the capacitance coefficients in a complicated way, as we saw in **E2**. But most inductors just use self-inductance, so the inductance we care about is simply one of the coefficients, L_{ii}. Moreover, the "mutual inductance" coefficients L_{ij} are also in the right form to be directly used, since they tell us how current changes in one part of the circuit impact emfs elsewhere.

A more general way to describe the difference is that $\mathcal{E}$ and I are directly measurable and controllable quantities, while the Q and V (i.e. the voltage relative to infinity) that the capacitance coefficients relate are less so.

Idea 4

The energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0} \int B^2 \, dV$$

which implies the energy stored in an inductor is

$$U = \frac{1}{2} L I^2$$

where L is the self-inductance.

Example 5

Compute the self-inductance of a cylindrical solenoid of radius R, length $H \gg R$, and n turns per length.

Solution

One straightforward way to do this is to use the magnetic field energy. We have

$$U = \frac{1}{2\mu_0} (\mu_0 n I)^2 (\pi R^2 H)$$

and setting this equal to $LI^2/2$ gives

$$L = \pi\mu_0 n^2 R^2 H = \mu_0 N^2 \frac{\pi R^2}{H}$$

where N is the total number of turns.

We can also try to use the definition of inductance directly, $\Phi = LI$. But it's hard to imagine a surface bounded by the solenoid wires; as we saw in problem 9, even the case $N = 2$ is tricky! Instead it's better to use the form $\mathcal{E} = L\dot{I}$. We can then compute the emf across each turn of the solenoid individually, then add them together.

To compute the emf across one turn, we can replace it with a circular loop; this is valid because the emf ultimately comes from the local electric field, which shouldn't change too much if we deform the loop in this way. Then

$$|\mathcal{E}_{\text{loop}}| = \frac{d\Phi}{dt} = (\mu_0 n\dot{I})(\pi R^2).$$

The inductance is hence

$$L = \frac{N\mathcal{E}_{\text{loop}}}{\dot{I}} = (\mu_0 nN)(\pi R^2) = \mu_0 N^2 \frac{\pi R^2}{H}$$

as expected.

Example 6

Find the outward pressure at the walls of the solenoid in the previous example.

Solution

An outward pressure exists because of the Lorentz force of the the axial magnetic field of the solenoid acting on the circumferential currents at the walls. The force per length acting on a wire is IB, and the pressure is this quantity times the turns per length, so naively

$$P = (\mu_0 nI)(nI).$$

However, this is off by a factor of 2. To see why, consider a small Amperian rectangle that straddles the surface of the solenoid. The currents near this rectangle contribute axial magnetic fields of $\mu_0 nI/2$ inside and $-\mu_0 nI/2$ outside. Thus, the currents due to the entire rest of the solenoid contribute $\mu_0 nI/2$ both inside and outside.

Since a wire can't exert a force on itself, only the latter field matters. So the true answer is

$$P = \frac{1}{2}\mu_0 n^2 I^2.$$

The factor of $1/2$ here is essentially the same as the one we saw for the pressure on a conductor's surface due to electrostatic forces, in **E1**.

[3] **Problem 15.** Consider a toroidal solenoid with a rectangular cross section of height h and width w, N turns, and inner radius R.

(a) Find the self-inductance by considering the magnetic flux.

(b) Now suppose the current increases at a constant rate dI/dt. Find the magnitude of the electric field at a height z above the center of the solenoid, assuming $h, w \ll R$. (Hint: write down the divergence and curl of $\mathbf{E}$ in terms of $\dot{\mathbf{B}}$ in general, and notice the similarities to the equations for $\mathbf{B}$ in terms of $\mathbf{J}$. This allows us to use the ideas of **E3** by analogy.)

(c) Verify that the two formulas for energy given in idea 4 are consistent.

Remark

In electromagnetism, we often have issues with divergences when we take idealized point sources. For example, the voltage near a point charge can become arbitrarily high. Similarly, the magnetic field diverges as you approach an idealized, infinitely-thin wire, which causes the self-inductance of wire loops to diverge. Of course, the fix is that you don't actually get an infinite magnetic field as you approach a wire. A real wire has finite thickness, and its magnetic field instead goes to zero as you approach its center. (Note that we didn't run into this problem for the solenoids, because we modeled their wires as a uniform sheet of current, whose magnetic field isn't singular at all.)

[2] **Problem 16.** A wire of length L is bent into a long "hairpin" shape, with two parallel straight edges of length $L/2$ separated by a distance $d \ll L$.

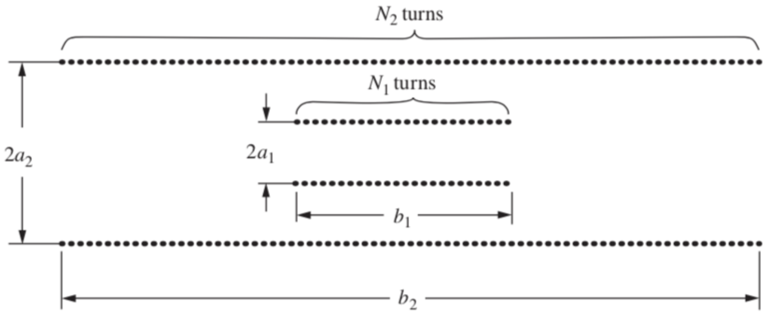
(a) Write down an integral expression for the self-inductance, neglecting the curved parts, and show that it diverges.

(b) The reason the inductance doesn't diverge in reality is that the wire has finite thickness. Once you get inside the wire, the magnetic field does not continue to increase, diverging at the center, but rather smoothly goes to zero. Find a rough estimate for the self-inductance by taking the wire to have radius $r \ll d$ and ignoring any flux through the wire itself.

If a problem does involve inductance and a wire loop, it'll usually circumvent this issue by just telling you what the self-inductance of the loop is.

In general, computing mutual inductance is a hard and importnat problem (there have been whole books written on the subject). But the next few problems demonstrate some cases where it can be calculated relatively elegantly.

[2] **Problem 17** (Purcell 7.9)**.** Find the mutual inductance of the two cylindrical solenoids shown in cross-section below.



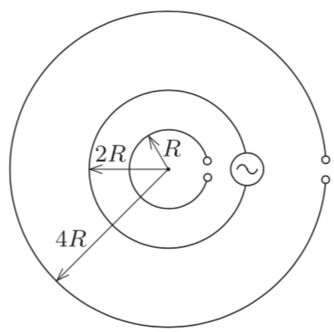
[3] **Problem 18.** Consider two concentric rings of radii r and $R \gg r$.

(a) Compute the mutual inductance by considering a current through the larger ring.

(b) Compute the mutual inductance by considering a current through the smaller ring, and verify your results agree. (Hint: this can be done with no messy integrals.)

(c) Suppose that an increasing current is passed through the outer ring. What forces does the inner ring experience?

[2] **Problem 19** (MPPP 181)**.** Three nearly complete circular loops, with radii R, $2R$, and $4R$ are placed concentrically on a horizontal table, as shown.



A time-varying electric current is made to flow in the middle loop. Find the voltage induced in the largest loop at the moment when the voltage between the terminals of the smallest loop is V_0.

4 Magnetism

In this section we'll dip a little into atomic physics and the origin of magnetism. However, a proper understanding of this subject requires quantum mechanics, as we'll cover in **T3** and **X3**.

[3] **Problem 20.** A spinning charged object carries both a magnetic dipole moment and angular momentum. If the object's mass and charge distributions are proportional to each other, then $\boldsymbol{\mu}$ and $\mathbf{J}$ point in the same direction, and their ratio is the gyromagnetic ratio.

(a) Compute the gyromagnetic ratio for a thin uniform donut of charge Q, mass M, and radius R rotating about its axis.

(b) Show that the gyromagnetic ratio for a uniform spinning sphere is the same.

By similar reasoning, any axially symmetric charge/mass distribution would give the same gyromagnetic ratio. Unfortunately, the result is wrong by a factor of 2 when applied to electrons. This factor of 2 cannot be explained without using quantum mechanics.

(c) Suppose the magnetic moment of an iron atom is due to a single unpaired electron, with angular momentum of order $\hbar$. The atoms are separated by a distance of order 10^{-10} m. Estimate the maximum magnetic field an iron magnet can produce. (Don't worry about the 2, since this is just a rough estimate.)

[3] **Problem 21.** USAPhO 2007, problem B2.

[5] **Problem 22.** APhO 2013, problem 3. A solid question involving classical magnetic moments, which gives some intuition for the quantum behavior.

5 Superconductors

There are many rather tricky Olympiad problems involving superconductors. Superconductors can be a bit intimidating at first, but they actually obey simple rules.

Idea 5

An ideal conductor has zero resistivity, which implies that the magnetic flux through any loop in the conductor is constant: attempting to change the flux instantly produces currents that cancel out the change. However, the flux can be nonzero.

A superconductor is an ideal conductor with the additional property that the magnetic field in the body of the superconductor is exactly zero, no matter what the initial conditions are; once an object becomes superconducting it forces all the existing flux out. This is known as the Meissner effect. It further implies that all the current in a superconductor is confined to its surface, and that the normal component of the magnetic field $B_\perp$ is zero on the surface. Many problems involving superconductors don't even use the Meissner effect, so they would also work for ideal conductors.

Example 7: PPP 153

A superconducting uniform spring has N turns of radius R, relaxed length x_0, and spring constant k. The two ends of the spring are connected by a wire, and a small, steady current I is made to flow through the spring. At equilibrium, what is the change in its length?

Solution

This question really is about ideal conductors, not just superconductors. The additional superconductivity property would tell us about the field *inside* the wires themselves (not the loops that the wires form), and thereby about some small screening currents on the surfaces of the wires. This is not important because the wires are thin compared to the spring as a whole.

In order to find the equilibrium length x_{eq}, we can use the principle of virtual work. We compute how the energy changes if we slightly perturb the system. At equilibrium, this change in energy should be zero.

We have $B = \mu_0 NI/x$, so the magnetic field energy is

$$U = \frac{B^2}{2\mu_0}V = \frac{AI^2}{x}, \quad A = \frac{\mu_0 \pi R^2 N^2}{2}.$$

Naively, this means the magnetic field energy decreases as x increases, so the spring would like to stretch. But this makes no sense, because we know that parallel currents attract, squeezing the spring. We have to recall that the spring is an ideal conductor, so when it is stretched or squeezed, the current changes to keep the flux the same. The flux is

$$\Phi_B = N(\pi R^2)B \propto \frac{I}{x}$$

so we have

$$I(x) = I\frac{x}{x_{\text{eq}}}, \quad U(x) = \frac{AI^2}{x_{\text{eq}}^2}x.$$

The other energy contribution is $k(x - x_0)^2/2$, so setting the derivative of energy to zero,

$$\frac{AI^2}{x_{\text{eq}}^2} = k(x_0 - x_{\text{eq}}).$$

Since the current is small, $x_0 \approx x_{\text{eq}}$, so we can replace x_{eq} with x_0 on the left-hand side, giving the answer,

$$x_{\text{eq}} = x_0 - \frac{AI^2}{x_0^2 k}.$$

As a sidenote, the original formulation of this question involved a battery forcing the current I to be constant. However, in this case using energy conservation is more subtle because one has to account for the work done by the battery. Here we used a superconductor, which keeps the flux constant, so that the spring can be thought of as an isolated system. The final answers are the same, since they can also be found from the magnetic forces, which match.

Example 8

A long, thin cylinder of radius R is placed in a magnetic field B_0 parallel to its axis. The cylinder originally carries no current on its surface, and it is cooled until it reaches the superconducting state. Find the resulting distribution of current on its surface. Now suppose the external magnetic field is turned off; what is the new current distribution?

Solution

Solving this question requires using both properties. The Meissner effect tells us there is no magnetic field within the body of the cylinder itself (i.e. the region from $r = R$ to $r = R + dr$). The ideal conducting property tells us that the flux through a cross-section of the cylinder (i.e. the region from $r = 0$ to $r = R$) is constant, and hence equal to $\pi R^2 B_0$.

When the cylinder becomes superconducting, the Meissner effect kicks in, and the field within the body of the cylinder can be cancelled by a uniform surface current on the outer surface. By the same logic as we used to compute the field of a cylindrical solenoid, it is

$$K_{\text{out}} = -B_0/\mu_0.$$

To keep the flux constant, a compensating opposite current must appear on the inner surface,

$$K_{\text{in}} = B_0/\mu_0.$$

When we turn off the external magnetic field, the two properties imply

$$K_{\text{out}} = 0, \quad K_{\text{in}} = B_0/\mu_0$$

which you should check if you're not sure.

[3] **Problem 23** (MPPP 182)**.** Two identical superconducting rings are initially very far from each other. The current in the first is I_0, but there is no current in the other. The rings are now slowly brought closer together. Find the current in the first ring when the current in the second is I_1.

[4] **Problem 24** (PPP 182)**.** A thin superconducting ring of radius r, mass m, and self-inductance L is supported by a piece of plastic just above the top of a long, cylindrical solenoid of radius $R \gg r$ and n turns per unit length. The ring and solenoid are coaxial. When the current in the solenoid is I_s, the magnetic field near the end of the solenoid is

$$B_z = B_0(1 - \alpha z), \quad B_r = B_0 \beta r$$

where we put the origin at the very top of the solenoid.

(a) Find an expression for B_0. (Keep your answers below in terms of B_0 to avoid clutter.)

(b) Find a relationship between α and β, as well as their signs.

(c) Let I be the current through the ring. Suppose that initially $I_s = I = 0$. Find the value I_c of I_s when the ring lifts off the plastic.

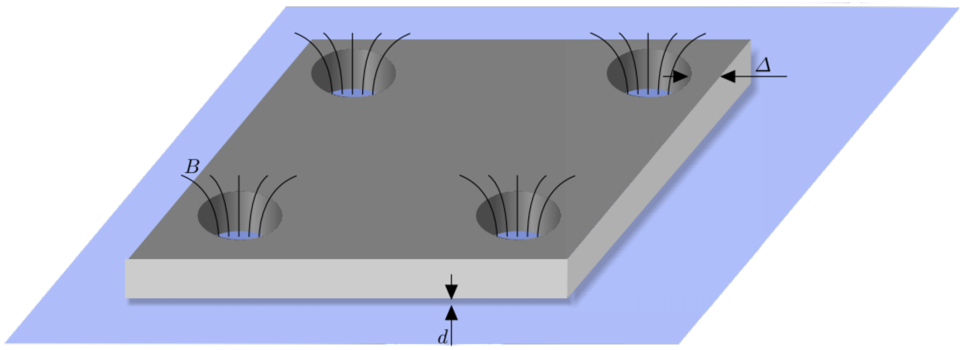
(d) Now the piece of plastic is removed and the ring is return to the same position. Initial conditions are set up so that $I_s = I_c$ and $I = 0$. (How could you do this in practice?) The ring is released from rest; find its subsequent motion, assuming that the motion is confined to regions where the expressions for B_z and B_r above hold. Express your final answers in terms of only α and g.

[4] **Problem 25.** IPhO 2012, problem 1C. A delightfully tricky problem that uses the properties of superconductors in a subtle way.

[4] **Problem 26** (EuPhO 2017)**.** Consider a mesh made from a flat superconducting sheet by drilling a dense grid of small holes into it. Initially the sheet is in a non-superconducting state, and a

magnetic dipole of dipole moment m is at a distance a from the mesh pointing perpendicularly towards the mesh. Now the mesh is cooled so that it becomes superconducting. Next, the dipole is displaced perpendicularly to the surface of the mesh so that its new distance from the mesh is b. Find the force between the mesh and the dipole. The spacing of the grid of holes is much smaller than both a and b, and the linear size of the sheet is much larger than both a and b.

[4] **Problem 27** (Physics Cup 2013)**.** A rectangular superconducting plate of mass m has four identical circular holes, one near each corner, a distance Δ from the plate's edges. Each hole carries a magnetic flux Φ. The plate is put on a horizontal superconducting surface. The magnetic repulsion between the plate and the surface balances the weight of the plate when the width of the air gap beneath the plate is $d \ll \Delta$, and d is much smaller than the radii of the holes. The frequency of small vertical oscillations is ω_0.



Next, a load of mass M is put on the plate, so that the load lays on the plate, and the plate levitates above the support. What is the new frequency of small oscillations?

[5] **Problem 28.** IPhO 1994, problem 2. This problem tests your intuition for induction, and is good preparation for **E6**.