Relativity III: Fields

Relativity in electromagnetism is covered in chapter 5 of Purcell and then sprinkled in throughout the rest of the book, notably in sections 6.7 and 9.7, and appendix H. For a more advanced discussion, see chapter 3 of Schutz for tensors, and chapter 12 of Griffiths and chapters I-34, II-13, and II-25 through II-28 of the Feynman lectures for relativistic electromagnetism. For a brief taste of general relativity, see chapter 14 of Morin, and chapter II-42 of the Feynman lectures. For a great, accessible introduction to tests of general relativity, see *Was Einstein Right?* by Will. There is a total of **74** points.

1 Electromagnetic Field Transformations

Idea 1: Field Transformations

If the electromagnetic field is $(\mathbf{E}, \mathbf{B})$ in one reference frame, then in a reference frame moving with velocity $\mathbf{v}$ with respect to this frame, the components of the field parallel to $\mathbf{v}$ are

$$E'_\parallel = E_\parallel, \quad B'_\parallel = B_\parallel$$

while the components perpendicular are

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp), \quad \mathbf{B}'_\perp = \gamma \left(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp \right).$$

As alluded to in **R2**, this is the transformation rule for the components of a rank 2 antisymmetric tensor.

Under these transformations, Maxwell's equations remain true in all inertial frames, and the Lorentz force transforms properly as well. Furthermore, a Lorentz transformation does not change the total amount of charge in a system, where total charge is defined by Gauss's law via the electric flux through a surface containing the system.

Remark

There are many ways of deriving the field transformations. The tensor method alluded to above is the mathematically cleanest, but the conceptually clearest is to think about how some simple setups must Lorentz transform, if Maxwell's equations are to remain true. For example, boosting a capacitor increases the charge density on the plates because of length contraction, which is why $\mathbf{E}'_\perp$ contains $\gamma \mathbf{E}_\perp$. (Further examples are given in chapter 5 of Purcell, which is essential reading for this section.) Another method is to demand that the Lorentz force obeys the transformation of three-force derived in **R2**.

[4] **Problem 1.** Basic facts about the electric and magnetic fields of a moving charge.

(a) Show that the field of a point charge q at the origin moving with constant velocity v is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2}{(1 - v^2 \sin^2\theta)^{3/2}} \hat{\mathbf{r}}$$

in units where $c = 1$, and θ is the angle from $\mathbf{v}$. In particular, the field is still radial.

(b) Verify that the charge of this moving charge is still q. It may be useful to consult the integral table in appendix K of Purcell.

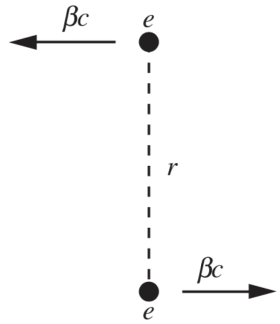
(c) Argue that the magnetic field of this point charge must be exactly

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

(d) Verify that the previous result is correct in nonrelativistic electromagnetism (i.e. using Coulomb's law and the Biot–Savart law).

The result of part (a), first found by Heaviside in 1888, implies that the field lines of a moving charge "length contract" like they were rigid rods. In fact, this was one of the inspirations for the Lorentz–Fitzgerald length contraction in the first place.

[3] **Problem 2** (Purcell 5.29)**.** Two protons are moving antiparallel to each other, along lines separated by a distance r, with the same speed v in the lab frame, as shown.



Consider the moment the protons are a distance r apart.

(a) Show that the three-force experienced by each proton due to the electric field of the other is

$$F = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}.$$

(b) Compute the three-force experienced by one of the protons by transforming to its rest frame, computing the force there, then transforming back to the lab frame. In particular, show that this is not equal to the result of part (a).

(c) Show that the discrepancy is resolved if the magnetic three-force is also included.

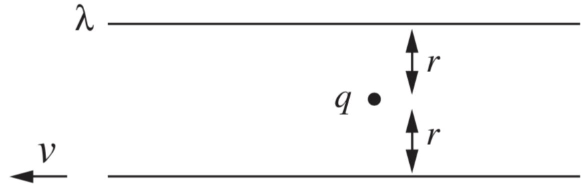
Recall from **R2** that the Lorentz three-force is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. You will also have to use the three-force transformation laws you derived there.

[3] **Problem 3.** USAPhO 2014, problem B2. This isn't the clearest of problems, but it introduces and justifies the Galilean field transformations we first saw in **E4**.

[3] **Problem 4** (Purcell 5.30)**.** Consider an infinite wire oriented along $\hat{\mathbf{x}}$ with linear charge density λ and current I. Show that under a Lorentz boost along $\hat{\mathbf{x}}$, (λ, I) transforms like (ct, x).

[2] **Problem 5** (Purcell 6.22)**.** A neutral wire carries current I. A stationary charge q is nearby; the Lorentz force on this charge is zero. Verify this remains true in a frame moving parallel to the wire with velocity $\mathbf{v}$, by using the Lorentz transformations of the fields.

[3] **Problem 6** (Purcell 6.69). Two very long sticks each have uniform linear proper charge density λ. One stick is stationary in the lab frame, while the other moves to the left with speed v, as shown.



They are $2r$ apart, and a stationary point charge q lies midway between them. Find the Lorentz three-force on the charge in the lab frame, and also in the frame of the bottom stick, and verify the forces relate properly.

[3] **Problem 7.** The vectors $\mathbf{E}$ and $\mathbf{B}$ cannot go into four-vectors, as they transform among each other, but rather fit together into an antisymmetric rank two tensor. As a result, there is a different set of associated invariant quantities.

(a) Show that under the relativistic field transformations, the quantities $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are both invariant. (Hint: this can be done using vector notation, using $\mathbf{E}_\perp \cdot \mathbf{E}_\parallel = \mathbf{B}_\perp \cdot \mathbf{B}_\parallel = 0$.) These are the two basic invariants, out of which all other invariants can be constructed.

(b) Suppose that in an inertial frame, $\mathbf{E}$ is zero at a given point and $\mathbf{B}$ is nonzero. Is it possible to find an inertial frame where $\mathbf{B}$ is zero at that point?

(c) Recall from **E7** that, in units where $\epsilon_0 = \mu_0 = 1$, the energy density of the electromagnetic field is $\mathcal{E} = E^2/2 + B^2/2$, and the Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{B}$. Show that $\mathcal{E}^2 - |\mathbf{S}|^2$ is invariant. (Hint: don't use the field transformations for this part.)

Remark: Is Magnetism Real?

Purcell's electromagnetism textbook is exceptional because it shows that a force like magnetism must exist, if one believes Coulomb's law and relativity. The idea is simple. We know how forces transform between frames, and given some reasonable assumptions, can also deduce how electric fields transform between frames. If electric fields were all there were, then electric forces would have to transform just like three-forces, but they don't. So there must be some other force to make up the difference, and it turns out to be precisely the magnetic force. We saw an example of this in problem 2.

It is important not to misunderstand this beautiful idea. Many people, upon reading Purcell, believe that magnetism "doesn't exist" because it's all "just electric fields". Sometimes people even say that magnetic forces are a "mistake" caused by "forgetting about" relativistic corrections. This is all totally backwards. Sometimes time dilation in one frame can be explained in terms of length contraction in another, but that doesn't mean that length contraction doesn't exist, or is a mistake – it's perfectly real in that particular frame. (Not to mention that there are plenty of situations where you can't get rid of the magnetic field in any frame, as we saw in problem 7!)

The real lesson of relativity isn't that magnetic fields are a mistake, it's that electric and magnetic fields are as intertwined as space and time, as you can see from their transformation properties. Just as space and time combine into a four-vector, electric and magnetic fields combine, in an equal footing, into the electromagnetic field tensor.

Remark: Electromagnetism in Covariant Form

Problem 4 is a first step to showing that $J^\mu = (\rho, \mathbf{J})$ is a four-vector, where ρ is the charge density and $\mathbf{J}$ is the current density. Note that the continuity equation for charge, as mentioned in **T2**, can be simply written in four-vector notation as

$$\partial_\mu J^\mu = 0.$$

As another example, you can show that the four-current of a single charged particle q is $J^\mu = qu^\mu$. We can go even further and write the whole of electromagnetism in terms of four-vectors and tensors. Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

The invariant quantities found in problem 7 can be written in terms of the field strength tensor as $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi–Civita symbol. These are the only two ways to "contract all the indices" to get a scalar.

Remark: Elegant Notation

Sometimes people dislike the index notation above because of all the little Greek letters floating around. If you *only* want to deal with vectors, vector notation is often better. It hides all the indices, at the cost of requiring you to introduce special symbols like $\cdot$ and $\times$ to specify the vector operations you want to do. The reason we don't use a vector-like notation for tensors is because there are too many operations you can do with them (e.g. "contract the 3rd index of a rank 4 tensor with the 1st index of a rank 2 tensor") to define separate symbols for each one; indices are just more efficient. On the other hand, if you only work with totally antisymmetric tensors, then there are only a few possible operations, and one can use the elegant, index-free "differential form" notation. In this notation, Maxwell's equations are

$$d \star F = J$$

where d is called the exterior derivative, $\star$ is the Hodge dual, and the fact that the electromagnetic fields are derivatives of potentials is expressed as

$$F = dA.$$

So is this the *best*, *most true* formulation of Maxwell's equations? Well, as Feynman once pointed out, you can easily do better. For example, you can define the "unworldliness"

$$U = |\mathbf{F} - m\mathbf{a}|^2 + (\nabla \cdot \mathbf{E} - \rho/\epsilon_0)^2 + \ldots.$$

Then *all* physical laws can be expressed in terms of the amazingly simple equation

$$\boxed{\boxed{\boxed{U = 0.}}}$$

But this doesn't actually help, because to use the equation for anything, you need to plug in the definition of U, and then you're back to where you were before. In general, more elegant notation is often more brittle: it only works well in a smaller set of situations. (For example, with differential form notation, you just can't write down the stress-energy tensor of the electromagnetic field, because that's symmetric rather than antisymmetric.) Index notation is great, because it works as long as indices are contracted in pairs, which holds as long as you're dealing with laws that are independent of coordinate system. In general, there's no need to be ideological about notation; we should just use the best tool for the job at hand.

[4] **Problem 8.** Consider an electromagnetic wave of the form

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = B_0 \cos(kz - \omega t)\hat{\mathbf{y}}.$$

As usual, you may work in units where $c = 1$.

(a) What do Maxwell's equations imply about the relation between E_0 and B_0, and k and ω?

(b) Now consider a frame moving with velocity v along the $\hat{\mathbf{z}}$ direction. Show that the electromagnetic wave continues to have the same basic functional form for $\mathbf{E}'(z', t')$ and $\mathbf{B}'(z', t')$, but with new parameters E_0', B_0', k', and ω'. Using these results, show that the energy density of the wave is smaller by a factor of $(1 - v)/(1 + v)$.

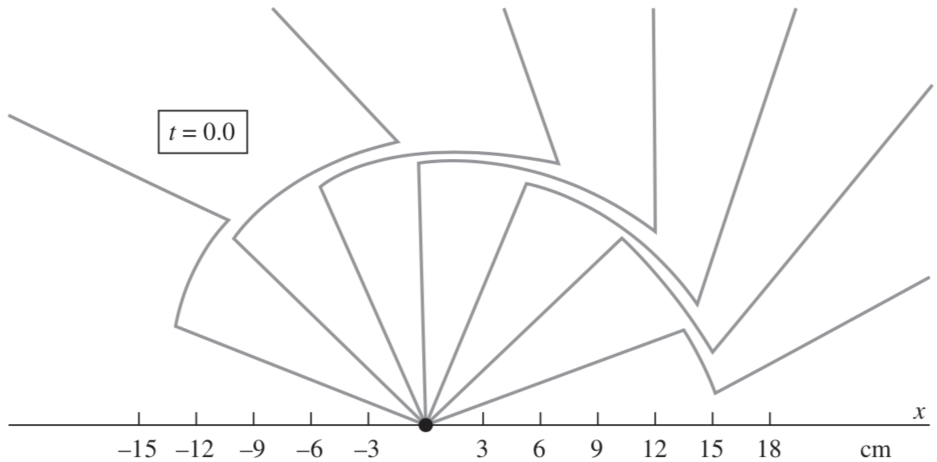
(c) The energy of a photon in an electromagnetic wave of frequency ω is $E = \hbar\omega$. Show that for a finite-sized electromagnetic wave, the initial and boosted frames agree on the number of photons. This was one of the hints Einstein used to conclude light was made of photons.

(d) Now consider another question Einstein pondered: what does the light wave look like if we try to "catch up" with it, taking $v \to c$? Is this consistent with the invariants of problem 7?

Idea 2

If a uniformly moving point charge suddenly stops moving, then the field outside a spherical shell, centered at the charge when it stopped moving, expanding at speed c, is precisely that calculated in problem 1. The same occurs if the point charge suddenly changes its velocity; information about the change only propagates at c.

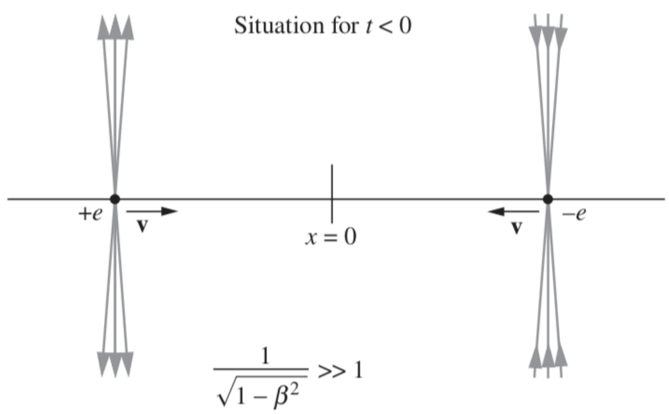
[1] **Problem 9** (Purcell 5.18)**.** In the figure below, you see an electron at time $t = 0$ and the associated electric field at that instant.



(a) Describe what has been going on, as quantitatively as you can.

(b) Where was the electron at the time $t = -0.75\,\text{ns}$?

[2] **Problem 10** (Purcell 5.19)**.** The figure below shows two highly relativistic particles with opposite charge approaching the origin.



They collide at the origin at time $t = 0$ and remain there as a neutral entity. Sketch the field lines at some time $t > 0$.

[3] **Problem 11.** Work through the derivation of the Larmor formula in Appendix H of Purcell.

[3] **Problem 12** (Purcell H.4)**.** The Larmor formula only applies to particles moving nonrelativistically. To get a result valid for faster particles, we can simply transform into an inertial frame F' where the particle is nonrelativistic, apply the Larmor formula, then transformed back to the lab frame.

(a) Consider an relativistic electron moving perpendicularly to a magnetic field $\mathbf{B}$. Defining the radiation power as $P_{\text{rad}} = dE/dt$, find P'_{rad}, the power in a frame instantaneously comoving with the electron.

(b) Argue that in this context, $P_{\text{rad}} = P'_{\text{rad}}$, and conclude that

$$P_{\text{rad}} = \frac{\gamma^2 v^2 e^4 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

Thus, the power increases rapidly as $v \to c$. Incidentally, a "relativistic" way to write the general result is

$$P_{\text{rad}} = \frac{q^2}{6\epsilon_0 c^3} \left(\frac{1}{m} \frac{dp^\mu}{d\tau} \right)^2$$

which clearly reduces to the Larmor formula in the nonrelativistic limit.

(c) This radiation is also called synchrotron radiation. Qualitatively, how does its angular distribution differ from radiation from an accelerating nonrelativistic charge?

Remark: Gravitoelectromagnetism

As mentioned in **E1**, there's a close analogy between electrostatic fields, which are sourced by charge density ρ_e, and gravitational fields, which are sourced by energy density ρ. Therefore, if you apply the analogy and run the same arguments as in Purcell, you would expect there to be a "gravitomagnetic" field, which is sourced by momentum density $\mathbf{J} = \rho \mathbf{v}$. That's indeed correct! In the theory of gravitoelectromagnetism, the force on a point mass is

$$\mathbf{F} = m(\mathbf{E} + 4\mathbf{v} \times \mathbf{B})$$

where the gravitoelectric and gravitomagnetic fields $\mathbf{E}$ and $\mathbf{B}$ satisfy

$$\nabla \cdot \mathbf{E} = 4\pi G \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \times \mathbf{B} = 4\pi G \mathbf{J} + \dot{\mathbf{E}}.$$

From this you can draw some interesting conclusions. For example:

- Two masses moving parallel to each other will have an extra attraction due to the gravitomagnetic force.
- A rotating object will produce a gravitomagnetic field which can cause gyroscopes to precess; this is called the Lense–Thirring, or frame dragging effect, which has been measured by satellites such as Gravity Probe B. (There is also a significantly larger "geodetic" effect caused by the curvature of spacetime around the Earth, but this isn't captured within gravitoelectromagnetism.)
- A mass at rest, inside a cylinder which suddenly starts to rotate, will pick up a small angular velocity in the same direction due to the induction of a gravitoelectric field.
- Gravitational waves are generated by accelerating masses and carry energy, just like electromagnetic radiation.

Now you might be puzzled by two things: first, how does gravitoelectromagnetism relate to general relativity, and second, why is there an extra 4 in one of the equations above? Well, the truth is that Purcell's arguments don't really work for gravity. These arguments crucially depend on electric charge $Q = \int \rho_e \, d\mathbf{x}$ being Lorentz invariant, which in our more sophisticated language was necessary to ensure $j^\mu = (\rho_e, \rho_e \mathbf{v})$ is a four-vector. However, the total energy $E = \int \rho \, d\mathbf{x}$ is not Lorentz invariant – instead it's itself a component of a four-vector. Thus, $(\rho, \rho\mathbf{v})$ isn't a four-vector, so none of the arguments really work: the theory of gravitoelectromagnetism is just not Lorentz invariant at all.

Instead, gravitoelectromagnetism is properly derived as a limiting case of general relativity, valid when all the masses involved are moving slowly, $v \ll c$. The fact that general relativity is a theory of a rank 2 tensor field, the metric $g_{\mu\nu}$, is responsible for the extra factors of 2 above. Even though it's only approximately true, gravitoelectromagnetism is a very useful tool for analyzing precision tests of general relativity, since it's much easier to calculate with.

On the other hand, there's also a lot of nonsense written about it by people who don't understand it. For example, a lot of internet luminaries are certain that it can be used to replace dark matter, even though, using just the basic equations above, you can see that the gravitomagnetic force is $(v/c)^2$ times smaller than the usual gravitational force. That makes it about 10^6 times too small to explain the anomalous rotation of galaxies.

In fact, now is a good time to issue a warning. There's a concept called Lizardman's constant, which is the fact that in any survey, no matter how it's designed, about 3% of the answers will be complete nonsense. 3% of people will enthusiastically tell you that they were born on Mars, that the Moon landing was faked, or that the Earth is run by lizardmen. That's because there's an irreducible fraction of people that are mistaken, crazy, or just plain trolling.

The internet is a wonderful place to learn introductory physics, because it's relatively straightforward, so the sincere and competent outnumber the crazy. But as you go to more advanced topics, the fraction of people who know what's going on, and who have the time and energy to tell you, rapidly drops, while the 3% stays just as large. Now that you're at the end of this curriculum, you're also at the point where the *majority* of internet commentators on the topics you're learning are completely wrong. Fortunately, you're also learning what sources are good, and developing the knowledge needed to check things for yourself. As you continue learning tougher subjects, these skills will keep you on the right track.

2 Charges in Fields

Now we consider some problems in the spirit of **E4**, using more advanced tools.

[3] **Problem 13** (Purcell 5.24)**.** In the rest frame of a particle with charge q, another particle with charge q is approaching with relativistic velocity $\mathbf{v}$. Assume that both particles are extremely massive, and hence their velocities are nearly constant. The second particle passes a minimum distance b from the first.

(a) Show that the impulse acquired by each particle is perpendicular to $\mathbf{v}$ with magnitude $q^2/2\pi\epsilon_0 vb$. (Hint: you can avoid doing a nasty integral by using Gauss's law.)

(b) If the particles have mass m, roughly how large does m have to be for the above result to be a good approximation?

[2] **Problem 14.** When we consider conservation of energy for a particle of charge q, we always include, along with the kinetic energy $mv^2/2$, the potential energy $q\phi$. Similarly, when we consider conservation of momentum, we must consider not only the ordinary momentum $M\mathbf{v}$, but also the "potential momentum" $q\mathbf{A}$. The sum of the two is called the canonical momentum.

Conservation laws are associated with symmetries. When the ϕ and $\mathbf{A}$ a particle moves in are time-independent, the energy $Mv^2/2 + q\phi$ is conserved. And when they are time-independent and space-independent, the canonical momentum $M\mathbf{v} + q\mathbf{A}$ is conserved. Unfortunately, this is an extremely restrictive condition which forces the electric and magnetic fields to be *zero*, rendering the full conservation law basically useless in practice. However, in some cases a single component of this momentum can be conserved, which can be useful.

(a) Check that a vector potential appropriate for a constant field $\mathbf{B}$ in the $-\hat{\mathbf{z}}$ direction is

$$\mathbf{A} = \frac{B}{2}(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}).$$

(b) Now consider a charge tied to the end of a string, executing horizontal uniform circular motion about the origin with radius r. Suppose a magnetic field $-B\hat{\mathbf{z}}$ is turned on. Show that

$$\mathbf{L} = \mathbf{r} \times (M\mathbf{v} + q\mathbf{A})$$

is conserved. This is the conserved quantity due to rotational symmetry about the z-axis.

[3] **Problem 15** (Cahn)**.** A charged particle is orbiting in a uniform magnetic field of magnitude B_0 in a circular orbit of radius R_0. Assume the particle moves much slower than the speed of light, and that all fields are cylindrically symmetric about the *original* axis of rotation of the particle. (For concreteness, the particle could originally be orbiting about the axis of symmetry of a big solenoid.)

(a) The field is slowly changed to B_1. What is the new radius R_1 of the orbit? (Hint: you can solve this problem in many ways. For example, you can directly use the Lorentz force, or you can use conservation of the quantity in problem 14, or you can use the adiabatic theorem, where the momentum is $m\mathbf{v} + q\mathbf{A}$.)

(b) The field is suddenly changed back to B_0. What is the final radius R_2?

For a more challenging problem along these lines, see Physics Cup 2017, problem 3.

[5] **Problem 16.** APhO 2001, problem 2. This tough, rather mathematical problem covers generalized momentum with vector notation, extending the results of problem 14.

[5] **Problem 17.** IPhO 1991, problem 2. A problem on a subtle relativistic effect.

[5] **Problem 18.** Physics Cup 2021, problem 1. A really tough electromagnetism question.

3 Gravitational Fields

Idea 3

In classical mechanics, you've seen that a uniform gravitational field behaves a lot like the fictitious force due to a uniform acceleration. The equivalence principle states that the two behave exactly identically, in all possible contexts; it was one of the key ideas that led to the development of general relativity.

[4] **Problem 19.** In this problem, we given one of the classic justifications for gravitational redshift, the fact that photons redshift when moving against a gravitational field. Suppose that point B is a height h above point A, in a gravitational field g. A set of electrons and positrons with total rest mass M are converted into photons of frequency f at point A. The photons fly upward to point B, where they are converted back into electrons and positrons. Assume throughout that g is small.

(a) Find the total mass M' at point B.

(b) Find the frequency f' of the photons measured at point B.

(c) Since the frequencies of photons can be used as a clock, the result of part (b) shows that gravitational fields cause time dilation, which applies to everything, not just photons. Show that your result in part (b) is equivalent to the statement that times are dilated by a factor of $1 + \phi/c^2$, where ϕ is the gravitational potential and $\phi/c^2 \ll 1$.

We should also be able to understand part (b) using the equivalence principle. To confirm this, suppose that two observers C and D begin at rest, with D a distance h to the right of C. At a certain moment, both observers begin accelerating to the right with a small acceleration a.

(d) If C emits light of frequency f (in C's rest frame), show that D observes light of frequency f', where f' matches your answer to part (b).

(e) The predicted frequency shift was observed in the 1959 Pound–Rebka experiment, where gamma rays were transmitted from the top to the bottom of a tower of height $h = 22.5\,\text{m}$. What is the fractional change in energy of the photons?

(f) Gamma ray photons of energy $14\,\text{keV}$ were used in the Pound–Rebka experiment. According to the energy-time uncertainty principle, what is the minimum time needed to detect the effect?

Remark

You might be a little worried that the result of part (c) above does not seem to be invariant under a large, constant shift of ϕ, even though in Newtonian mechanics we can always do this. In fact, in that case the same analysis is essentially valid, but the "extra" gravitational time dilation is canceled out by other effects, which unfortunately can't be explained without full general relativity. In other words, the analysis above is only valid when ϕ is small.

If you find this confusing, you're not alone. In 2018, there was some excitement as researchers claimed to explain a long-standing anomaly in particle physics, making a mistake precisely along these lines. (A rebuttal is given here.)

[3] **Problem 20.** In this problem we consider the effects of relativity on a clock on the surface of the Earth, which has mass M and radius R. It rotates about its axis in time T, as measured by an observer at infinity who is at rest relative to the center of the planet

(a) Consider a clock C that lies on the surface of the planet at a point on the equator. Compute the time measured by the clock C after a single rotation of the planet, incorporating both special relativity and gravitational time dilation. Which effect is bigger?

(b) Repeat part (a) for a clock C' on a satellite orbiting the planet, in a circular orbit a height h above the equator.

(c) Using the numbers $M = 5.97 \times 10^{24}\,\text{kg}$, $R = 6.4 \times 10^6\,\text{m}$, and $h = 2 \times 10^7\,\text{m}$, estimate the difference in time elapsed per day for the two clocks, counting only the effect of special relativity, or only the effect of gravitational time dilation.

[5] **Problem 21.** APhO 2014, problem 3. Gravitational fields bend light; this problem is about the geometry of gravitational lensing. Print out the official answer sheets and record your answers on them.

[5] **Problem 22.** IPhO 1995, problem 1. This problem is about the applications of gravitational redshift, and also serves as a nice review of **R2**.

Remark: Visualizing Relativity

You've probably heard that in general relativity, gravity is explained by the curvature of spacetime. In other words, freely falling objects always move in straight lines through spacetime; they only look like they're accelerating downward because we are constantly being accelerated upward. This is nicely illustrated here and explained in greater detail in this paper.

There is a common analogy for this involving picturing space as a distorted rubber sheet. It's a very bad analogy, because things will only accelerate towards the valleys in the sheets if you have gravity pointing down the sheet. In other words, the analogy tries to explain gravity by assuming you have spatial curvature *and* gravity. This misses the beautiful key point of relativity, which is that the gravity can be explained by *spacetime* curvature alone.

The fact that freely falling objects move in straight lines means that an object sitting on the surface of the Earth is actually being constantly accelerated. But this leads to a common followup question: in this picture, the surfaces of America and India are constantly accelerated in opposite directions, so why doesn't the Earth tear itself apart? Indeed, in special relativity this would make no sense. It's only possible because of spacetime curvature.

This can be explained with a spatial curvature analogy. Consider two people walking east, side by side, with one just north of the equator and the other south. In order to stay a *constant* distance apart, the person walking on the north will constantly have to bear to the right, while the person walking on the south will have to bear to the left, because the Earth's surface is spatially curved. Similarly, in a situation with spacetime curvature, America and India need constant opposite accelerations to maintain the same distance.

There's a dramatically different way to visualize this situation called the "river model", which is illustrated here. The basic idea is that we think of space as a river that is constantly flowing towards the center of the Earth. Observers in America and India constantly need to paddle in opposite directions against the river to stay in place. This is also a good way to think about the event horizon of a black hole, which is where the river starts to flow faster than light.

In this remark I've given three analogies about spacetime, so which of them is "correct"? None, really. The analogies don't tell us what spacetime is. They're just different ways of verbally describing what the equations of general relativity say. They each imperfectly describe some aspects of the equations, and fail to capture others. (Any simple analogy *must* fail to capture the content of a theory, because if it really were simpler and just as valid, then that analogy would be the theory instead!) There is no actual spacetime rubber or river; those are just stories we tell ourselves to make the mathematics more appealing to our animal-descended minds. Of course, philosophers debate over whether the attitude I've expressed in this paragraph is right. It's called "anti-realism", and I wrote about it here.