Mechanics II: Statics

For review, read chapter 2 of Morin or chapter 2 of Kleppner and Kolenkow. Statics is covered in more detail in chapter 7 of Wang and Ricardo, volume 1. Surface tension is covered in detail in chapter 5 of *Physics of Continuous Matter* by Lautrup, which is an upper-division level introduction to fluids in general. There is a total of **82** points.

1 Balancing Forces

Idea 1

In principle, you can always solve every statics problem by balancing forces on every individual particle in the setup, but often you can save on effort by considering appropriate systems.

Idea 2

Any problem where everything has a uniform velocity is equivalent to a statics problem, by going to the reference frame moving with that velocity. Any problem where everything has a uniform acceleration $\mathbf{a}$ is also about statics, by going to the noninertial frame with acceleration $\mathbf{a}$, where there is an extra effective gravitational acceleration $-\mathbf{a}$. The same principle applies to uniform rotation, where a centrifugal force appears in the rotating frame, acting like an effective gravitational acceleration $\omega^2\mathbf{r}$.

Example 1

Six blocks are attached in a horizontal line with rigid rods, and placed on a table with coefficient of friction μ. The blocks have mass m and the leftmost block is pulled with a force F so the blocks slide to the left. Find the tension force in the rod in the middle.

Solution

There are six objects here and five rods, each with a different tension, so a direct analysis would involve solving a system of six equations. Instead, first consider the entire set of six blocks as one object; we can do this because the rigid rods force them to move as one. The total mass is $6m$, and applying Newton's second law gives

$$F - 6mg\mu = 6ma, \quad a = \frac{F}{6m} - \mu g.$$

Next, consider the rightmost three blocks as one object. Their total mass is $3m$, and their acceleration is the same acceleration a we computed above. This system experiences two horizontal force: tension and friction. Newton's second law gives

$$T - 3mg\mu = 3ma$$

and solving for T gives

$$T = \frac{F}{2}.$$

This is intuitive, because the differences of any two adjacent tension forces are the same; that's the amount of tension that needs to be spent to accelerate each block. So the middle rod, which has to accelerate only half the blocks, has half the tension.

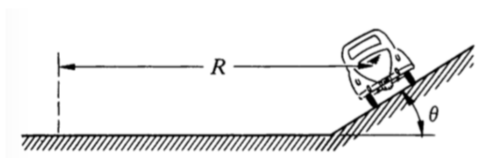
The reason we could ignore the tension forces in the other four rods is that the only thing they do is ensure the blocks move with the same acceleration. Once we assume this is the case, the specific values of the tensions don't matter; we can just zoom out and forget them. It's just like how *within* each block there is also an internal tension which keeps it together, but we rarely need to worry about its details.

Idea 3

To handle a problem where something is just about to slip on something else, set the frictional force to the maximal value μN and assume slipping is not yet occurring, so the two objects move as one. The same idea holds for problems which ask for the minimal force needed to make something move, or the minimal force needed to keep something from moving.

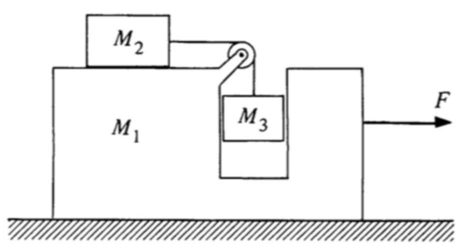
[1] **Problem 1** (KK 2.7)**.** A block of mass M_1 sits on a block of mass M_2 on a frictionless table. The coefficient of friction between the blocks is μ. Find the maximum horizontal force that can be applied to (a) block 1 or (b) block 2 so that the blocks will not slip on each other.

[2] **Problem 2** (KK 2.28)**.** An automobile enters a turn of radius R.



The road is banked at angle θ, and the coefficient of friction between the wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

[2] **Problem 3** (KK 2.19)**.** A "pedagogical machine" is illustrated in the sketch below.



All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

[3] **Problem 4.** USAPhO 2017, problem A1.

2 Balancing Torques

Idea 4

A static rigid body will remain static as long as the total force on it vanishes, and the total torque vanishes, where the torque about the origin is

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{r}_i$ is the point of application of force $\mathbf{F}_i$. If the total force vanishes, the total torque doesn't depend on where the origin is, because shifting the origin by $\mathbf{a}$ changes the torque by

$$\Delta\boldsymbol{\tau} = \sum_i \mathbf{a} \times \mathbf{F}_i = \mathbf{a} \times \left(\sum_i \mathbf{F}_i\right) = 0.$$

The origin should usually be chosen to set as many torques as possible to zero.

[1] **Problem 5.** The line of a force is defined to the line passing through its point of application parallel to its direction; then the torque of the force about any point on that line vanishes. Suppose a body is static and has three forces acting on it. Show that in two dimensions, the lines of these forces must either be parallel or concurrent. This will be useful for several problems later.

Idea 5

The center of mass $\mathbf{r}_{\text{CM}}$ of a set of masses m_i at locations $\mathbf{r}_i$ with total mass M satisfies

$$M\mathbf{r}_{\text{CM}} = \sum_i m_i \mathbf{r}_i.$$

If a system experiences no external forces, its center of mass moves at constant velocity.

Idea 6

A uniform gravitational field exerts no torque about the center of mass. Thus, for the purposes of applying torque balance on an *entire* object, the gravitational force $M\mathbf{g}$ can be taken to act entirely at its center of mass. (This is a formal substitution; of course, the actual gravitational force remains distributed throughout the object.)

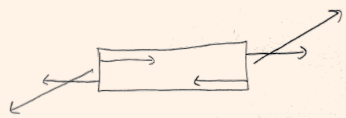
Torque balance works in noninertial frames, as long as one accounts for the torques due to fictitious forces. Thus, for an accelerating frame, the $-M\mathbf{a}$ fictitious force can be taken to act at the center of mass. In a uniformly rotating frame, the total centrifugal force is $M\omega^2\mathbf{r}_{\text{cm}}$, and for the purposes of balancing torques, can be taken to act entirely at the center of mass.

Example 2

Show that the tension in a completely flexible static rope, massive or massless, points along the rope everywhere in the rope.

Solution

Consider a tiny segment $d\ell$ of the rope. Since the rope is static, the tension forces on both ends balance, so they are opposite. Let them both be at an angle θ to the rope direction. Then the net torque on the segment is $(T\,d\ell)\sin\theta$. Since this must vanish for static equilibrium, we must have $\theta = 0$ and hence the tension is along the rope. In other words, flexible ropes can transmit force, but they can't transmit torque.



It's important to note that the argument above doesn't work for a rigid rod, because the internal forces in a rigid object can look like the picture above. In other words, there can be extra shear forces from the adjacent pieces of the rod that provide the compensating torque. If one tried to set up forces like this in a rope, it would flex instead.

In general, the force distribution within a massless rigid rod can be quite complicated, but if we zoom out, we can replace it with a single tension which does not necessarily point along the rod. This transmits both a force and a torque through the rod, in the sense that a torque is eventually exerted by whatever holds the end of the rod in place. Note that if the rod's supports are free to rotate, then they can't absorb torque, so the rod acts just like a rope, with tension always along it.

Remark

Sometimes, problem writers will intentionally not introduce any variables that are irrelevant to the answer. This can occur in two ways. First, the variables might just cancel out, as one can often see by dimensional analysis. Second, the specific values of the variables might not matter in the limit when they are very large or small. For instance, if a problem simply states a mass is "very heavy" but doesn't give it a name like m, it is asking for the answer in the limit $m \to \infty$.

Idea 7

To handle problems where an object is just about to tip over, note that at this moment, the entire normal force will often be concentrated at a point. (For example, when you're about to fall forward, all your weight goes on your toes.) That often means it's a good idea to take torques about this point.

Example 3: Povey 5.6

Suppose that on level ground, a car has a distance d between its left and right tires, and its center of mass is a height h above the ground. Now suppose the car turns as in problem 2, but in the extreme case $\theta = 90^\circ$, with speed v. For what v is this motion possible?

Solution

Again working in the noninertial frame of the car, force balance gives

$$f_{\text{fric}} = mg, \quad N = \frac{mv^2}{R}$$

where f_{fric} and N are the total friction and normal forces on the four tires. Since $f_{\text{fric}}/N \leq \mu$,

$$v \geq \sqrt{gR/\mu}$$

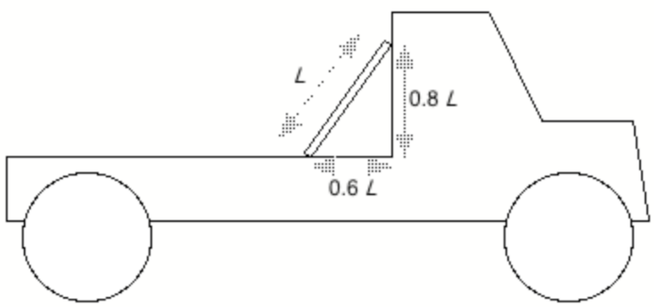
which matches the general solution to problem 2. But in that problem, we only considered force balance. In this extreme situation, we also have to consider torque balance, i.e. the possibility that the car might topple over. When the car is about to topple over, all the normal and friction force is on the bottom tires. About this point, we have only torques from gravity and the centrifugal force, giving

$$mgh = \frac{mv^2}{R}\frac{d}{2}$$

and solving for v gives $v = \sqrt{2gRh/d}$. Toppling is less likely the higher v is, so the answer is

$$v \geq \sqrt{gR}\,\max(1/\sqrt{\mu}, \sqrt{2h/d}).$$

[2] **Problem 6** (Quarterfinal 2004.3). A uniform board of length L is placed on the back of a truck.



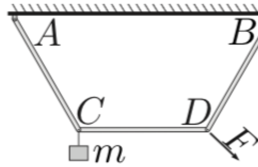
There is no friction between the top of the board and the vertical surface of the truck. The coefficient of static friction between the bottom of the board and the horizontal surface of the truck is $\mu_s = 0.5$. The truck always moves in the forward direction.

(a) What is the maximum starting acceleration the truck can have if the board is not to slip or fall over?

(b) What is the maximum stopping acceleration the truck can have if the board is not to slip or fall over?

(c) For what value of stopping acceleration is the static frictional force equal to zero?

[2] **Problem 7** (Kalda). Three identical rods are connected by freely rotating hinges.



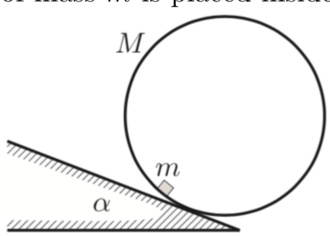
The rods are arranged so that CD is parallel to AB, and $\overline{AB} = 2\,\overline{CD}$. A mass m is hung on hinge C. What is the minimum force that must be exerted at hinge D to keep the system stationary?

Idea 8

An extended object supported at a point may be static if its center of mass lies directly above or below that point. More generally, if the object is supported at a set of points, it can be static if its center of mass lies above the convex hull of the points.

[2] **Problem 8.** N identical uniform bricks of length L are stacked on top of each other on the edge of a table. What is the maximum possible length the top brick can protrude over the edge of the table? What is the limit of this length as N goes to infinity?

[2] **Problem 9** (Kalda)**.** A cylinder with mass M is placed on an inclined slope with angle α so that its axis is horizontal. A small block of mass m is placed inside it.



The coefficient of friction between the block and cylinder is μ. Find the maximum α so that the cylinder can stay at rest, assuming that the coefficient of friction between the cylinder and slope is high enough to keep the cylinder from slipping.

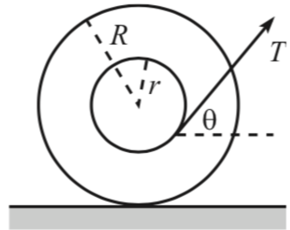
[2] **Problem 10** (PPP 11)**.** A sphere is made of two homogeneous hemispheres stuck together, with different densities. Is it possible to choose the densities so that the sphere can be placed on an inclined plane with incline 30° and remain in equilibrium? Assume the coefficient of friction is sufficiently high so that the sphere cannot slip.

[3] **Problem 11.** An object of mass m lies on a uniform floor, with coefficient of static friction μ.

(a) First, suppose the object is a point mass. What is the minimum force required to make the object start moving, if you can apply the force in any direction?

(b) Now suppose the object is a thin, uniform bar. What is the minimum force required to make the object start moving, if the force can only be applied horizontally? Assume the normal pressure on the floor remains uniform.

[3] **Problem 12** (Morin 2.17)**.** A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground.



A thread is wrapped around the axle and is pulled with tension T at an angle θ with the horizontal.

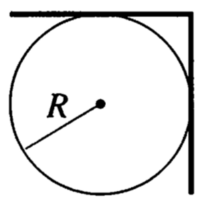
(a) Which way does the spool move if it is pulled with $\theta = 0$?

(b) Given R and r, what should θ be so that the spool doesn't move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(c) Given R, r, and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest?

(d) Given R and μ, what should r be so that you can make the spool slip from the static position with as small a T as possible? That is, what should r be so that the upper bound on T in part (c) is as small as possible? What is the resulting value of T?

[3] **Problem 13** (PPP 44)**.** A plate, bent at right angles along its center line, is placed on a horizontal fixed cylinder of radius R as shown.



How large does the coefficient of static friction between the cylinder and plate need to be if the plate is not to slip off the cylinder?

3 Trickier Torques

Idea 9

Sometimes, a clever use of torque balance can be used to remove any need to have explicit force equations at all. Rarely, the same situation can occur in reverse.

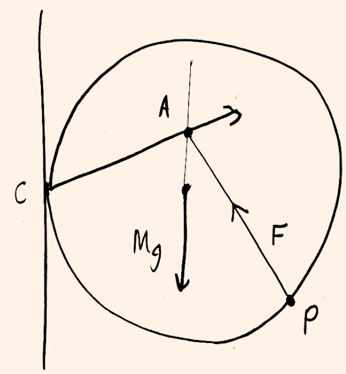
Example 4: EFPhO 2010.4

A spherical ball of mass M is rolled up along a vertical wall, by exerting a force F to some point P on the ball. The coefficient of friction is μ. What is the minimum possible force F, and in this case, where is the point P?

Solution

Following the logic of idea 3, when the minimum possible force is used, the frictional force with the wall must be maximal, $f = \mu N$, and directed upward. (If friction weren't pushing the ball up as hard as possible, we could get by using a smaller force F.) So even though we don't know the magnitude of the normal or the frictional force, we know the direction of the sum of these two forces, so we'll consider them as one combined force.

This reduces the number of independent forces in the problem to three: gravity (acting at the center of mass), the force F (acting at P), and the combined normal and friction forces (acting at the point of contact C with the wall). Therefore, by the result of problem 5, the lines of these forces must all intersect at some point A, as shown.



This ensures that the torques will balance, when taken about point A.

Next, we need to incorporate the information from force balance. Doing this directly will lead us to some nasty trigonometry, but there's a better way. There are in principle two force balance equations, for horizontal and vertical forces. However, one of these equations is just going to tell us the magnitude of the normal/frictional force, which we don't care about. So in reality, we just need one equation, which preferably doesn't involve that force.

The trick is to use torque balance *again*, about the point C, which says that the torques due to gravity and F must cancel. Now you might ask, didn't we already use torque balance? We did, but recall from idea 4 that taking the torque about a different point can give you a different equation if the forces don't balance. So by demanding the torque vanish about two different points, we actually are using force balance! (Specifically, we are using the linear combination of the horizontal and vertical force balance equations that *doesn't* involve the normal/friction force, which we don't need to find anyway.)

When taking the torque about C, we see that F is minimized if P is chosen to maximize the lever arm of the force. This occurs when $CA \perp PA$, in which case the lever arm is $R\sqrt{1+\mu^2}$, where R is the radius of the ball. So we have

$$MgR = FR\sqrt{1+\mu^2}, \quad F = \frac{Mg}{\sqrt{1+\mu^2}}$$

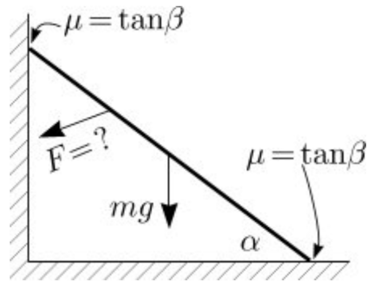
and P is determined as described above.

[2] **Problem 14.** NBPhO 2020, problem 4, parts (i) and (ii).

[3] **Problem 15.** EFPhO 2012, problem 3. The problem statement is missing some information: both the bars and rod have diameter d.

[3] **Problem 16.** EFPhO 2006, problem 6. You will need to print out the problem to make measurements on the provided figure.

[4] **Problem 17** (Physics Cup 2012)**.** A thin rod of mass m is placed in a corner so that the rod forms an angle α with the floor. The gravitational acceleration is g, and the coefficient of friction with the wall and floor is $\mu_s = \tan\beta$, which is not large enough to keep the rod from slipping.

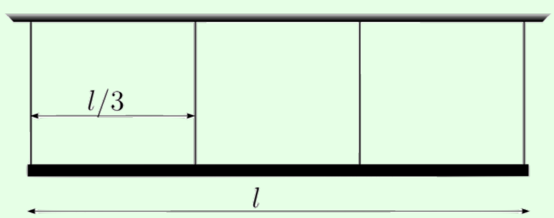


What is the minimum additional force F needed to keep the rod static?

We've now covered some really mathematically elegant problems, but it's important to remember the real-world limitations of this kind of analysis. We discuss two examples below.

Example 5

A uniform bar with mass m and length ℓ hangs on four equally spaced identical light wires. Initially, all four wires have tension $mg/4$.



Find the tensions after the leftmost wire is cut.

Solution

This illustrates a common issue with setups involving rigid supports: there are often more normal forces than independent equations, so there is not a unique solution. In the real world, the result is determined by imperfect characteristics of the wires. For example, if one of the wires was slightly longer than the others, it would go slack, reducing the number of normal forces by one and yielding a solution.

A reasonable assumption, if you aren't given any further information, is to assume that the supports are identical, very stiff springs. In equilibrium, the bar will tilt a tiny bit, so that the length of the middle wire will be the average of the lengths of the other two. By Hooke's law, the force in that wire will than be the average of the other two, so the tensions are

$mg/3 - x$, $mg/3$, and $mg/3 + x$. Applying torque balance yields $7mg/12$, $mg/3$, and $mg/12$.

The general point here is that concepts like rigid bodies or strings characterized by a single tension force are abstractions, made for the idealized problems we study in mechanics classes. A real civil engineer designing a structure would instead use a sophisticated computer program which simulates the complex internal forces, torques, and strains throughout the material.

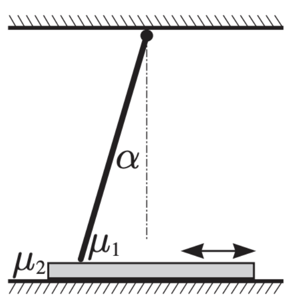
Remark: Subtleties of Friction

Statics problems involving friction can also get quite elegant, but it's important to remember that at the end of the day, they're just a decent approximation for the real world. In reality, laws like $F_{\text{fric}} = -\mu_k N$ are only approximately true for some regimes of behavior of some materials. In fact, the particular law $F_{\text{fric}} = -\mu_k N$ can sometimes yield equations with no solutions, a phenomenon called the Painleve paradox!

The source of the paradox is the discontinuous dependence of the direction of the friction force on the direction of the velocity. If you assume an object slips left, it is possible that after you do all the calculations (assuming a rightward friction force) you instead find the object slips right. And the reverse happens if you assume the object slips right, which makes the actual slipping direction completely indeterminate.

You won't have to worry about this paradox for Olympiads, which never contain setups that trigger it. But it's worth keeping in mind that friction is not just a simple law, but the subject of an entire field of study called tribology, which is essential for engineering. In these paradoxical cases, you would need to use a more refined model of friction to figure out what actually happens.

[2] **Problem 18** (Kalda). A rod is hinged to the ceiling, so that it makes an angle α with the vertical.



Underneath, a thin board is being dragged on the floor. The coefficient of (static and kinetic) friction is μ_1 between the board and rod, and μ_2 between the board and floor. The rod is meant to stop the board from being dragged to the right, no matter how hard or how quickly it is pulled. Is this possible? If so, what are the conditions on the parameters that allow this to occur?

We conclude with some questions that train three-dimensional thinking.

[2] **Problem 19** (PPP 10). In Victor Hugo's novel *les Miserables*, the main character Jean Valjean, an escaped prisoner, was noted for his ability to climb up the corner formed by the intersection of two

vertical perpendicular walls. Suppose for simplicity that Jean has no feet. Let μ be the coefficient of static friction between his hands and the walls. What is the minimum force that Jean had to exert on each hand to climb up the wall? Also, for what values of μ is this feat possible at all?

[3] **Problem 20** (PPP 69)**.** A homogeneous triangular plate has threads of length h_1, h_2, and h_3 fastened to its vertices. The other ends of the string are fastened to a common point on the ceiling. Show that the tension in each thread is proportional to its length. (Hint: with the origin at the point on the ceiling, let the vertices be at positions $\mathbf{r}_i$ and express everything in vector form.)

[4] **Problem 21** (KoMaL 2019, BAUPC 1998)**.** Two identical uniform solid cylinders are placed on a level tabletop next to each other, so that they are touching. A third identical cylinder is placed on top of the other two.

(a) Find the smallest possible values of the coefficients of static friction between the cylinders, and between a cylinder and the table, so that the arrangement can stay at rest.

(b) Repeat part (a) for spheres. That is, put three uniform solid spheres next to each other, with their centers forming an equilateral triangle, and put a fourth sphere on top.

(c) Now return to part (a), and suppose the setup is frictionless. A force is applied directly to the right on the leftmost cylinder, causing the entire setup to accelerate. Find the minimum and maximum accelerations so that all three cylinders remain in contact with each other.

Parts (a) and (b) demonstrate an interesting point: it is possible for a collection of objects to resist some force, even though a single one of those objects would begin moving even with an infinitesimal applied force! This is a simple example of how granular materials, like sand, can give rise to emergent phenomena that are hard to predict from analyzing individual grains alone. Understanding these materials is a whole field of applied research.

4 Extended Bodies

Idea 10

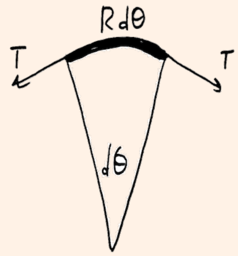
To deal with extended bodies, one can consider an infinitesimal piece of the body, or consider a well-chosen symmetrical piece of it, or consider the whole body as a system. Often, multiple approaches are needed.

Example 6

Find the tension in a circular rope of radius R spinning with angular velocity ω and mass per length λ.

Solution

Consider an infinitesimal segment of the rope, spanning an angle $d\theta$.



The mass of this segment is $dm = R\lambda\, d\theta$. The total force is downward, with magnitude

$$dF = 2T \sin\frac{d\theta}{2} \approx T\, d\theta$$

where we used the small angle approximation. This is the centripetal force, so

$$dF = (dm)\,\omega^2 R.$$

Combining these results yields $T = R^2\omega^2\lambda$.

Example 7

Find the distance d of the center of mass of a uniform semicircle of radius R to its center. (Note that a semicircle is half of a circle, not half of a disc.)

Solution

This can be done by taking the setup of the previous problem, and taking a subsystem comprising exactly half of the rope. In this case the net tension force is simply

$$F = 2T.$$

The total mass is $m = \pi R\lambda$, and the force must provide the centripetal force, so

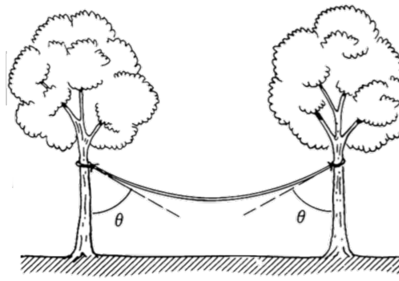
$$F = (\pi R\lambda)(\omega^2 d)$$

But we also know that $T = R^2\omega^2\lambda$ as before, so plugging this in gives

$$d = \frac{2}{\pi}R.$$

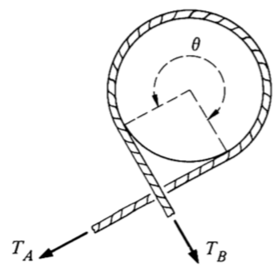
Alternatively, we could have worked in the frame rotating with the rope. The equations would be the same, but instead we would say the tension balances the centrifugal force.

[1] **Problem 22** (KK 2.22)**.** A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees.



Find the tension at either end of the rope, and the tension at the middle of the rope.

[3] **Problem 23** (KK 2.24)**.** A device called a capstan is used aboard ships to control a rope which is under great tension.



The rope is wrapped around a fixed drum with coefficient of friction μ, usually for several turns. The load on the rope pulls it with a force T_A. Ignore gravity.

(a) Show that the minimum force T_B needed to hold the other end of the rope in place is $T_A e^{-\mu\theta}$, an exponential decrease.

(b) How does this result depend on the shape of the capstan, if we fix the angle θ between the initial and final tension forces? Would the answer be the same for an oval, or a square?

(c) If $\theta = \pi$, explain why the total normal and friction force of the rope on the drum is $T_A + T_B$.

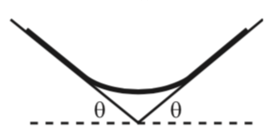
[2] **Problem 24** (***F*** $=$ ***ma*** 2018 B20)**.** A massive, uniform, flexible string of length L is placed on a horizontal table of length $L/3$ that has a coefficient of friction $\mu_s = 1/7$, so equal lengths $L/3$ of string hang freely from both sides of the table. The string passes over the edges of the table, which are smooth frictionless curves, of size much less than L. Now suppose that one of the hanging ends of the string is pulled a distance x downward, then released at rest. Neither end of the string touches the ground.

(a) Find the maximum value of x so that the string does not slip off of the table.

(b) For the case $x = 0$, draw a free body diagram for the string, indicating only the *external* forces on the entire string. Do the forces balance?

(c) Would the answer change significantly if the table's small edges had friction as well?

[3] **Problem 25** (Morin 2.25)**.** A rope rests on two platforms that are both inclined at an angle θ.



The rope has uniform mass density, and the coefficient of friction between it and the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum fraction?

Example 8

A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.

Solution

First, we note that the horizontal component of the tension T_x is constant throughout the chain; this just follows from balancing horizontal forces on any piece of it. Moreover, by similar triangles, we have $T_y = T_x y'$ everywhere.

Now consider a small segment of chain with horizontal projection Δx. The length of the piece is $\Delta x\sqrt{1+y'^2}$ which determines its weight, and this be balanced by the difference in vertical tensions. Thus

$$\Delta T_y = \lambda g \sqrt{1+y'^2}\,\Delta x.$$

For infinitesimal Δx, we have $\Delta T_y = T_x d(y') = T_x y''\,dx$, so we get the differential equation

$$y'' = \frac{\lambda g}{T_x}\sqrt{1+y'^2}.$$

Usually nonlinear differential equations with second derivatives are very hard to solve, but this one isn't because there is no direct dependence on y, just its derivatives. That means we can treat y' as the independent variable first, and the equation is effectively first order in y'.

Writing $y'' = d(y')/dx$ and separating, we have

$$\int \frac{dy'}{\sqrt{1+y'^2}} = \int \frac{\lambda g}{T_x}\,dx.$$

Integrating both sides gives

$$\sinh^{-1}(y') = \frac{\lambda g x}{T_x} + C.$$

Choosing $x = 0$ to be the lowest point of the chain, the constant C is zero, and

$$y' = \sinh\left(\frac{\lambda g x}{T_x}\right).$$

Integrating both sides again gives the solution for y,

$$y = \frac{T_x}{\lambda g}\cosh\left(\frac{\lambda g x}{T_x}\right)$$

where we suppressed another constant of integration. This curve is called a catenary.

[4] **Problem 26** (MPPP)**.** A slinky is a uniform spring with negligible relaxed length, with mass m and spring constant k.

(a) Find the shape of a slinky hung from two points on the ceiling separated by distance d. (Hint: to begin, consider the mass and tension of a small piece of the spring with horizontal and vertical extent dx and dy. Don't forget that the slinky's density won't be uniform.)

(b) Suppose a slinky's two ends are fixed, separated by distance d, and rotating uniformly with angular frequency ω like a jump rope in zero gravity. Find the values of ω for which this motion is possible, and the shape of the slinky in this case.

Example 9

A uniform spring of spring constant k, mass m, and relaxed length L is hung from the ceiling. Find its length in equilibrium, as well as its center of mass.

Solution

Problems like this contain subtleties in notation. For example, if you talk about "the piece of the slinky at z", this could either mean the piece that's actually at this position in equilibrium, or the piece that was originally at this place in the absence of gravity. Talking about it the first way automatically tells you where the piece is now, but talking about it the second way makes it easier to keep track of, because then the z of a specific piece of the spring stays the same no matter where it goes.

In fluid dynamics, these are known as the Eulerian and Lagrangian approaches, respectively. If you don't use one consistently, you'll get nonsensical results, and it's easy to mix them up.

There are many ways to solve this problem, but I'll give one that reliably works for me. We're going to use the Lagrangian approach, and avoid confusion with the Eulerian approach by breaking the spring into discrete pieces. Let the spring consist of $N \gg 1$ pieces, of masses m/N, spring constants Nk, and relaxed lengths L/N.

The i^{th} spring from the bottom has tension $(i/N)mg$, and thus is stretched by

$$\Delta L_i = \frac{1}{kN} \frac{i}{N} mg = \frac{mg}{kN^2} i.$$

The total stretch is

$$\sum_{i=1}^{N} \Delta L_i = \frac{mg}{kN^2} \int_0^N i \, di = \frac{mg}{2k}.$$

This makes sense, since the average tension is $mg/2$. To find the center of mass, note that the j^{th} spring is displaced downward by a distance

$$\Delta y_j = \sum_{i=j}^{N} \Delta L_i = \frac{mg}{2k} \left(1 - \frac{j^2}{N^2} \right)$$

downward from its position in the absence of gravity. The center of mass displacement is

$$\Delta y_{\mathrm{CM}} = \frac{1}{N}\sum_{j=1}^{N} \Delta y_j \propto \frac{1}{N}\sum_{j=1}^{N}\left(1 - \frac{j^2}{N^2}\right) = \frac{1}{N^3}\int_0^N N^2 - j^2\,dj = \frac{2}{3}$$

so restoring the proportionality constant gives

$$\Delta y_{\mathrm{CM}} = \frac{mg}{3k}.$$

If you want to test your understanding of slinkies, you can also try doing this problem with the Eulerian approach. This would be best done without discretization. The first steps would be finding a relation between the density $\rho(z)$ and tension $T(z)$ from Hooke's law, and finding out how to write down local force balance as a differential equation.

Remark

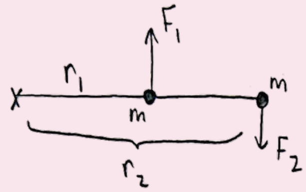
In this problem set, we've given some examples involving static, continuous, one-dimensional objects such as strings and ropes. The general three-dimensional theory of elasticity is mathematically quite complicated, but extremely important in engineering. For more about this subject, which requires comfort with tensors, see chapters 6 through 11 of Lautrup. It is also covered in chapters II-31, II-38, and II-39 of the Feynman lectures.

Remark: Why Use Torque?

Here's a seemingly naive question. Why is the idea of torque so incredibly useful in physics problems, even though in principle, everything can be derived from $F = ma$ alone? Why is it almost impossible to solve any nontrivial problem without referring to torques, and how would a student who's never heard of torque come up with it in the first place?

We don't need torque to analyze the statics of a single, featureless point particle. Torque only became useful in this problem set when we started analyzing rigid bodies with spatial extent. The reason we couldn't reduce torque balance to force balance easily is because the internal forces in these bodies, which maintain their rigidity, are generally very complicated.

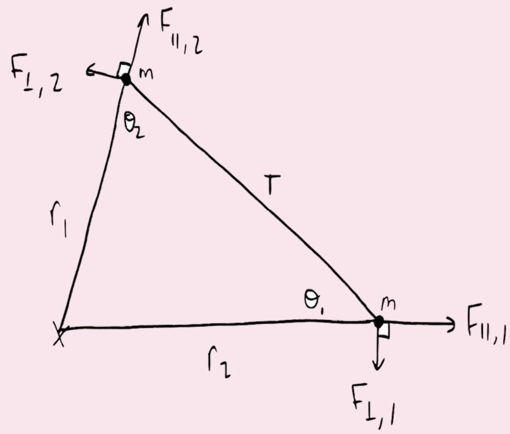
It's possible to derive torque balance from force balance in simple cases, but it's subtle. For example, suppose we try to make a rigid body by attaching two ideal springs of infinite spring constant to masses m and a pivot, as shown.



If forces F_1 and F_2 are applied, we would like to show that the setup if static only if the torques are balanced, $r_1 F_1 = r_2 F_2$, using only force balance on the masses. But our derivation fails at the very first step: the forces of ideal springs only point along the direction of the spring. Therefore, they can't balance the forces at all!

The problem is that this setup isn't actually rigid; it will bend at the first mass. We could make the setup rigid by replacing the spring with a rod, because, as we saw in example 2, a rigid rod can exert downward shear forces on the first mass and upward shear forces on the second. But in that case, it becomes nontrivial to decide when the *rod* can remain in equilibrium. You would need to decompose it microscopically, to see what is responsible for the shear forces in the first place, and balance forces on all those microscopic pieces. We tried to make a simple rigid body out of two masses, but now we need to keep track of the rod's infinitely many degrees of freedom! (To be clear, in practice you could summarize all these internal forces as a combination of a net internal force (tension) and a net internal torque (bending moment). But we're trying to avoid explicitly using torque here.)

Luckily, there's a simple modification of our original setup that works. Consider a triangle whose sides are made of ideal massless springs, pivoted at one vertex, as shown.



This actually *is* a rigid body, and the internal forces are simple: each spring just carries some tension. Let T be the tension in the spring between the two masses, opposite the pivot. The forces on each mass can be decomposed into a component parallel to the sides r_1 and r_2, and a component perpendicular to those sides. The former components can be balanced by adjusting the tensions in the other two springs. The perpendicular components can only be balanced by the tension T, from which we conclude

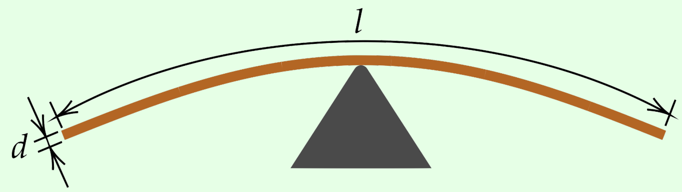
$$F_{\perp,1} = T\sin\theta_1, \quad F_{\perp,2} = T\sin\theta_2.$$

Upon eliminating T and using the law of sines, we conclude that force balance on the masses is only possible if $r_2 F_{\perp,1} = r_1 F_{\perp,2}$, which is precisely the statement of torque balance.

Tricky, right? This is the simplest example of a rigid body I know of, and in general, the internal forces can be much more complicated. The miracle of torque is that it automatically takes care of all those details for us, no matter what they are.

Example 10: IPhO 2022 3A

A thin piece of spaghetti of diameter d is balanced horizontally from its middle.



It can have a length $\ell \gg d$ before it snaps under its own weight. How does ℓ scale with d?

Solution

Let the spaghetti rod have density ρ, and consider its right half. There must be a vertical normal force $F \sim \rho d^2 \ell$ to balance the weight. This vertical force is transmitted through the rod by a shear stress (i.e. an internal force per area, perpendicular to the rod) of order $\sigma_s \sim F/A \sim \rho \ell$. Each piece of the rod exerts such a shear stress on its neighbors, just like how pieces of a string exert tensions on their neighbors.

Now consider torques on the right half of the rod, about the pivot point. The torque $\tau \sim \rho d^2 \ell^2$ of the rod's weight has to be balanced by forces from the other half of the rod. Vertical forces don't work, since they don't provide any torque about the pivot. Instead, the torque is supplied by a horizontal compression force at the bottom, and a horizontal tension force at the top, which cancel out to maintain horizontal force balance. This combination of forces, which produces no net force but does produce a net torque, is a bending moment.

Let the associated normal stresses be of order $\pm\sigma_n$. Then the net compression and tension forces are of order $\pm d^2 \sigma_n$, and the lever arm is of order d, so balancing torques gives

$$\rho d^2 \ell^2 \sim \sigma_n d^3$$

which implies $\sigma_n \sim \rho \ell^2 / d$. This is much greater than σ_s, because of the miserably small lever arm, which is why thin rods usually break by snapping, not by shearing or pulling apart. Given a fixed maximum σ_n, we conclude the maximum length scales as $\ell \sim \sqrt{d}$.

[3] **Problem 27.** USAPhO 2022, problem A1. A practical bending moment problem.

5 Pressure and Surface Tension

Example 11

A sphere of radius R contains a gas with a uniform pressure P. Find the total force exerted by the gas on one hemisphere.

Solution

The pressure provides a force per unit area orthogonal to the sphere's surface, so the straightforward way to do this is to integrate the vertical component of the pressure force over a hemisphere. However, there's a neat shortcut in this case.

Momentarily forget about the sphere and just imagine we have a sealed hemisphere of gas at pressure P. The net force of the gas on the hemisphere must be zero, or else it would just begin shooting off in some direction, violating conservation of momentum. So the force on the curved face must balance the force on the flat face, which is $\pi R^2 P$. The same logic must hold for the sphere, since the forces on the curved face are the same, so the answer is $\pi R^2 P$.

This trick works whenever one has a uniform outward pressure on a surface, and it'll come in handy for several future problems. For example, it's the quick way to do $F = ma$ 2018 B24.

Idea 11

The surface of a fluid carries a surface tension γ. If one imagines dividing the surface into two halves, then γ is the tension force of one half on the other per length of the cut. Specifically,

$$d\mathbf{F} = \gamma \, d\mathbf{s} \times \hat{\mathbf{n}}$$

which means the tension acts along the surface and perpendicular to the cut.

Example 12

A soap bubble of radius R and surface tension γ is in air with pressure P, and contains air with pressure $P + \Delta P$. Compute ΔP.

Solution

We use the result of the previous problem to conclude that the force of one hemisphere on another is $\pi R^2 \Delta P$. This must be balanced by the surface tension force. By imagining cutting the surface of the bubble in half, the surface tension force is γL where L is the total length of the surface connecting the hemispheres.

At this point, we can write $L = 2\pi R$, giving

$$\Delta P = \frac{2\gamma}{R}.$$

This is called the Young–Laplace equation. However, in this particular case, this is not the right answer. The reason is that we should actually take $L = 4\pi R$ because the surface tension is exerted at both the inside and outside surfaces of the bubble wall, and thus the answer is

$$\Delta P = \frac{4\gamma}{R}.$$

The increased pressure inside balances the surface tension, which wants to collapse the bubble.

If you're confused about why $L = 4\pi R$, you can also think about it in terms of energy. Surface tension arises from the fact that it costs energy to take soapy water and stretch it out into a surface, because this breaks some of the attractive intermolecular bonds. The Young–Laplace equation would give the correct answer for a *ball* of soapy water. But for a *bubble* of soapy water, twice as much soapy water/air surface is created. So the energy cost is double, and the force is double.

[2] **Problem 28.** One can also derive the Young–Laplace equation by just considering energy. Suppose the bubble radius changes by dr. The energy of the bubble changes for two reasons: first, there is net $\Delta P\, dV$ work from the two pressure forces, and there is the $\gamma\, dA$ surface tension energy cost. At equilibrium, the energy must be at a minimum, so it should not change at all under an infinitesimal displacement. Using this idea, rederive the Young–Laplace equation.

Remark

The general idea used in the above problem, of thinking about how energy would change in order to find an unknown force, is known as the principle of virtual work. The principle works for all kinds of forces. For example, if the bubble was charged, it would grow due to electrostatic repulsion, and the new equilibrium radius could be found using virtual work.

Remark

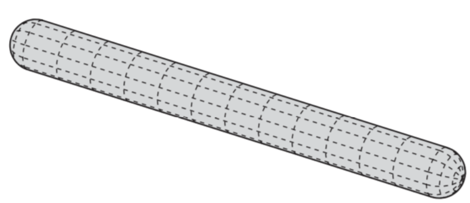
Note that the Young–Laplace equation we gave above only holds for spherical surfaces. More generally, a surface has two principle radii of curvature R_1 and R_2 at each point. These are both equal to R for a sphere of radius R, but for, e.g. a cylinder of radius R, one is equal to R and the other is infinity. For general surfaces, the Young–Laplace equation is

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where the R_i can each be positive or negative, depending on the direction of curvature.

[2] **Problem 29** (Kalda)**.** Consider two soap bubbles which have stuck together. The part of the soap film that separates the interior of the first bubble from the outside air has radius of curvature R. The part that separates the interior of the second bubble from the outside air has radius of curvature $2R$. What is the radius of curvature of the part which separates the bubbles from each other?

[3] **Problem 30** (MPPP 67)**.** When a long, straight sausage is cooked, it always splits "lengthwise" and never "across". The two modes of splitting are shown as dotted lines below.



Explain this observation, assuming the thickness of the sausage skin is uniform, and hence can support a constant surface tension before breaking. (Hint: model the sausage as a cylinder of length L capped by hemispheres of radius $R \ll L$, and consider the surface tension needed to prevent the two modes of splitting mentioned, once an excess pressure P builds up inside the sausage.)

[4] **Problem 31.** Two coaxial rings of radius R are placed a distance L apart from each other in vacuum. A soap film with surface tension γ connects the two rings.

(a) Derive a differential equation for the shape $r(z)$ of the film, and solve it.

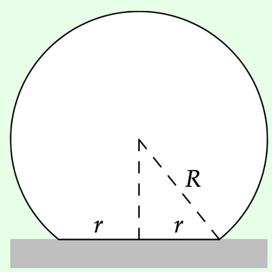
(b) Show that for sufficiently large L, there are no solutions. If L is increased to this value, what happens to the film?

(c) Using a computer or calculator, find the largest possible value of L.

We'll consider surface tension in more detail in **T3**.

Example 13

A solid ball of radius R, density ρ, and Young's modulus Y rests on a hard table. Because of its weight, it deforms slightly, so that the area in contact with the table is a circle of radius r.



Estimate r, assuming that it is much smaller than R.

Solution

Recall from **P1** that the Young's modulus is defined by

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}$$

and has dimensions of pressure. By dimensional analysis, you can show that

$$r = R\, f(\rho g R/Y)$$

but dimensional analysis alone can't tell us anything more about f. Moreover, an exact analysis using forces would be very difficult, because different parts of the ball are compressed in different amounts, and in different directions; there's little symmetry here. Instead, we resort to an order of magnitude approach. To balance gravity, we need a typical stress of

$$\text{stress} \sim Mg/r^2 \sim \rho R^3 g/r^2$$

in the deformed region. This deformed region has typical length r. To estimate the strain, note that if the ball were not deformed at all, it would be standing a height $\delta \sim r^2/R$ taller, as you can show by the Pythagorean theorem. Thus,

$$\text{strain} \sim \delta/r \sim r/R.$$

Using the definition of the Young's modulus, we conclude

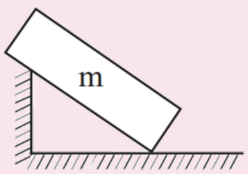
$$r \propto R \left(\frac{\rho g R}{Y}\right)^{1/3}.$$

By the way, there's a whole field of study devoted to figuring out how the normal and other forces behave for realistic, deformable solids, known as contact mechanics, which is essential in engineering. For an authoritative reference, see *Contact Mechanics and Friction* by Popov.

[4] **Problem 32.** EFPhO 2006, problem 5. A tough problem on a deforming object.

Remark

Unfortunately, some Olympiad problem writers don't really understand contact forces, and they'll end up writing questions like the one below, which is taken from a real book.



Assuming there's no friction, which way do the contact forces on the block point? Obviously, this is not well-defined, because there are three different normal directions at the bottom contact point, and at least two at the other contact point (depending on what the diagram means by the wall suddenly ending). Smart students often get seriously confused on questions like these, because there doesn't seem to be any sensible rule that works for all cases.

The resolution, of course, is that there *is* no rule. What happens in reality depends on the exact shape of the block and wall, and how deformable it is. For example, suppose the block was perfectly rigid, but actually had slightly rounded corners (not shown in the diagram). Then there's a definite normal direction at the bottom contact point, pointing up. Similarly, we could suppose that at the other contact point, the wall actually ends in a step with a rounded corner, in which case the normal direction points directly into the block.

Alternatively, suppose the block and step weren't rounded, but could deform. Then the answer depends on the relative hardness of the materials, and how they were placed in contact. For instance, if we suppose the block is much softer, then it could squash at the bottom contact point, again leading to a common upward normal direction. But then we would expect the step to dig into the block at the other contact point, which yields two separate normal forces at that point. Or perhaps the step is made of a softer material than the floor, so that it's the step rather than the block that deforms. Or maybe both deform!

There's just no way to extract a definite answer without knowing more about the situation.

As a result, you certainly won't see this kind of thing on thoroughly vetted competitions, such as the IPhO, APhO, and EuPhO, or large national Olympiads such as those in America or China. I'll never assign such a dysfunctional problem, but they're depressingly common in online resources, homework assignments, regional competitions, and national Olympiads in smaller countries. If you personally encounter such a problem, your best bet is to attempt to read the question writer's mind; that is, simply start guessing and go with whatever gives you tractable results. If you encounter this sort of thing often, in a book or competition, then it's not worth your time.