Lecture Notes on

Group Theory

Kevin Zhou

kzhou7@gmail.com

These notes cover group theory as used in particle physics, ranging from the elementary applications of isospin to grand unified theories. The main focus is on practical computations; many core statements are not proven, nor are many algorithms proven to work. Nothing in these notes is original; they have been compiled from a variety of sources. The primary sources were:

- Georgi, *Lie Algebras in Particle Physics*. The standard text on the subject for particle physicists, from one of the pioneers of grand unification. This thin book crams an enormous amount of content into its 300 pages, squeezing the Cartan classification into just 5, and gives many practical computation algorithms, at the cost of omitting proofs for most statements. The emphasis is on grand unified theories; finite groups receive little coverage.
- Wu-Ki Tung, *Group Theory in Physics*. An exceptionally clear group theory textbook that covers the material that no introductory book teaches, but every advanced book assumes you already know, such as Wigner's classification, the Wigner-Eckart theorem, and Young tableaux. More rigorous and formal than most group theory books for physicists; less applications.
- Zee, *Group Theory in a Nutshell for Physicists*. A very readable and easygoing book showing how group theory is applied by example, spending significant time on finite groups and characters, as well as applications in quantum mechanics. This is a good first book to get the idea of how group theory is used in physics.
- Sternberg, *Group Theory and Physics*. A more formal book that focuses on applications in quantum mechanics. Uses differential geometry and bundles freely throughout.
- Nick Dorey's Symmetries, Fields, and Particles lectures as transcribed by Josh Kirklin. Introduces Lie algebras and Lie groups formally and performs the Cartan classification rigorously.
- Fuchs and Schweigert, *Symmetries, Lie Algebras, and Representations*. Covers the standard material rigorously and goes far beyond; an important reference for physics graduate students working in the field.

These notes were written during the Part III course *Symmetries, Fields, and Particles* as lectured in 2017/2018, but contain substantial extra material. The material covered in the course is marked with a star. The most recent version is here; please report any errors found to kzhou7@gmail.com.

Contents

1 Introduction

In this course, we will investigate symmetries in physical laws and their mathematical representations. First, we consider how symmetries arise in nature.

- In general, we will consider a symmetry to be a mapping of the physical states of a system which leaves the dynamics invariant.
- Solutions of variational problems are often symmetrical. For example, soap bubbles are spherical because they minimize their area, and hence have $SO(3)$ symmetry. Similarly, in the Lagrangian formulation, valid paths $\mathbf{x}(t)$ for a rotationally symmetric Lagrangian remain valid paths when rotated.
- Exact symmetries can arise from redundancies in our description, e.g. gauge symmetries.
  - Coordinate transformations contain rotations $SO(3)$, or more generally Lorentz transformations $SO(3,1)$, or even more generally Poincare transformations. In general relativity we will also include arbitrary diffeomorphisms. These symmetries are collectively called spacetime symmetries; all others are called internal symmetries.
  - We also have gauge symmetries from the $U(1)$, $SU(2)$, and $SU(3)$ gauge groups. These are not truly 'symmetries' in the sense we have defined since they map a physical state to a different description of the exact same state.
  - The Coleman-Mandula theorem says that spacetime and internal symmetries cannot be combined nontrivially in relativistic quantum field theory. The famous exception is supersymmetry.
- Approximate symmetries can arise from neglecting the difference between two things. For example, isospin symmetry holds for up and down quarks if we neglect their charges and the difference in their masses, giving an $SU(2)$ symmetry. Adding the strange quark gives $SU(3)$, realized in the Eightfold Way.

As shown above, many of the groups relevant in physics are Lie groups.

- A Lie group is a group G which is also a smooth manifold, where the group operation is compatible with the smooth structure.
- The above definition is strongly constraining. A Lie group is nearly determined by its tangent space at the identity $T_e(G)$, which defines a Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ when equipped with a bracket operation. In this course we will work with matrix Lie groups ($G \subset GL(n, \mathbb{F})$) for which the bracket is the matrix commutator.
- The Cartan classification states that all finite-dimensional semi-simple Lie algebras over $\mathbb{C}$ belong to four infinite families, A_n, B_n, C_n, and D_n (for positive integer n) and the five exceptional cases E_6, E_7, E_8, G_2, and F_4.
- In classical mechanics, Lie groups and Lie algebras correspond to finite and infinitesimal symmetry transformations, which yield conserved quantities by Noether's theorem.
  - In Lagrangian mechanics, symmetries preserve the action, and the conserved quantity can be read off from the Euler-Lagrange equations.

  - In Hamiltonian mechanics, symmetries are phase space flows which preserve the Hamiltonian H, and they are generated by quantities $G(q, p)$ which are conserved. Mathematically, we require the Poisson bracket of G and H to vanish.

- Quantum mechanics is similar to Hamiltonian mechanics; groups and algebras correspond to unitary and Hermitian operators which commute with the Hamiltonian, and the latter are exactly the conserved physical quantities.
- The crucial feature is the antisymmetry of the commutator/Poisson bracket. This tells us that rotating a configuration don't change the energy if and only if time evolution doesn't change the angular momentum. This is how symmetry and conservation are generically related.
- Mathematically, the commutator and Poisson bracket make the set of infinitesimal transformations into a Lie algebra. Using the Jacobi identity shows that the set of infinitesimal symmetries is closed under the bracket and is hence a Lie subalgebra.

The link between symmetry and conservation can be somewhat subtle.

- There are discrete symmetries such as parity, charge conjugation, and time reversal. These only yield conserved quantities in quantum mechanics (i.e. the value of the symmetry operator itself), and we won't consider them in this course. Intuitively, this is because every discrete symmetry is continuous in quantum mechanics by interpolating through superpositions.
- There can be conserved quantities not associated with a continuous symmetry. For example, if a space is disconnected, the connected component a particle is in is conserved, but there is no associated continuous symmetry because the components are topologically invariant. Conserved quantities of this type are called topological charges.
- There can be algebras of symmetries that do not exponentiate to groups of symmetries; these appear in the study of supersymmetry and quantum groups. However, we still find conserved quantities, as Noether's theorem only requires infinitesimal symmetries, not finite ones.

Note. In quantum mechanics, a continuous symmetry is said to give an 'additive quantum number' (e.g. the spin and momenta of particles add together) while a discrete symmetry gives a 'multiplicative quantum number' (e.g. the parities of particles multiply). The reason is that we define the parity to be the eigenvalue of the parity operator, a discrete symmetry. But in the case of continuous symmetries, we define the associated quantum numbers to be the eigenvalues of the infinitesimal symmetries. This 'takes the logarithm', converting multiplication to addition.

More rigorously, we should think of group elements as abstract objects, not as matrices; in physics we deal with specific realizations of the abstract groups. In particular, in quantum mechanics the group elements become linear operators, giving a representation of the group.

- In Hamiltonian mechanics, a symmetry flow connects phase space points with the same energy. Similarly, in quantum mechanics, states in irreps all have the same energy.
- More generally, in quantum mechanics we would like to explain all degeneracies with symmetries; generally we should see no 'accidental' degeneracy.
- Since the irreps of abelian groups are one-dimensional, such degeneracy can only arise from nonabelian symmetries, such as rotations, as we see in spin systems with $[H, \mathbf{L}] = 0$.

- As another example, there was an approximate eight-fold degeneracy in the baryon spectrum, which was explained as an irrep of an approximate $SU(3)$ flavor symmetry.

Symmetries in quantum mechanics are a bit more subtle because quantum states are only defined up to phase factors.

- Wigner's theorem states that symmetries which preserve the norms of inner products must be either unitary linear or unitary anti-linear; the latter appears for time reversal symmetry.
- Even for unitary operators, we can have projective representations $U(g)$, which satisfy

$$U(g_i)U(g_j) = e^{i\gamma(g_i,g_j)}U(g_ig_j).$$

- In the special case where

$$\gamma(g_i, g_j) = \alpha(g_ig_j) - \alpha(g_i) - \alpha(g_j)$$

  then we can remove the phase factor by redefining $U(g)$ as $e^{i\alpha(g)}U(g)$. If the phase factors are 'nontrivial' (in some topological sense) they cannot be removed and give an entirely new representation.
- In many scenarios in physics, finding the projective representations of G is equivalent to finding the linear (i.e. normal) representations of the universal cover $\widetilde{G}$. **(when does this fail?)** This leads to the introduction of spinors, representations of $SU(2)$, since $SU(2)$ is the universal cover of $SO(3)$.

2 Finite Groups

2.1 Fundamentals

We begin by recalling the fundamentals of finite groups.

- Let H be a subgroup of G, written $H \subset G$. Then the coset gH of $g \in G$ is the set of elements of G that can be written as gh for some $h \in H$.
- All cosets satisfy $|gH| = |H|$ and they are disjoint. This yields Lagrange's theorem, which states that $|H|$ divides $|G|$. As an application, all groups of prime order are cyclic, because if any element generated a proper subgroup, it would give a number dividing a prime.
- Given groups G and K, we can build the direct product group $G \times K$. We will write elements of this group in the form 'gk'.

Example. We will present groups using generators and relations. For example, the dihedral group is the symmetry group of a regular n-gon and has presentation

$$D_n = \langle a, b | a^n = b^2 = e, bab^{-1} = a^{-1} \rangle$$

which states that n rotations of $2\pi/n$ give the identity, two flips give the identity, and a flipped rotation is a backwards rotation. The rank of the group is the number of generators.

Example. The tetrahedral group is the group of rotational symmetries of a tetrahedron. We have the identity, two nontrivial rotations of each of the four faces, and three order-two rotations that swap pairs of edges, giving a total of 12 elements. Labeling the vertices as 1, 2, 3, and 4, we recognize this group as the set of even permutations of four symbols, A_4.

Example. The permutation group S_n is the set of permutations of n objects. We will label the objects as 1 through n and represent the permutations using cycle notation. For example,

$$(12)(345)$$

denotes a permutation with two cycles, $1 \to 2 \to 1$ and $3 \to 4 \to 5 \to 3$. Generally, by picking an arbitrary element and following it until it comes back to itself, and repeating, we can write any permutation as the product of disjoint cycles. Cycles can be written as products of transpositions; permutations with an even number of transpositions form the alternating group A_n.

Consider a permutation on n objects with α_i i-cycles. We must have

$$n = \sum_j j\alpha_j$$

and the number of such permutations is

$$\frac{n!}{\prod_j j^{\alpha_j} \alpha_j!}.$$

Cayley's theorem states that every group G can be thought of as a subgroup of S_n for $n = |G|$, by letting the group elements be the symbols and thinking of group multiplication as a permutation. In particular, if we think of the group elements as basis vectors in $\mathbb{R}^n$, then we have a representation of G, called the regular representation.

We now recall some more fundamental concepts.

- A normal subgroup is one that is closed under conjugation. For example, A_n is normal in S_n because conjugation preserves the cycle structure, merely relabeling the symbols. (However, note that elements with the same cycle structure are not necessarily conjugate!)

- Conjugation by a group element yields an automorphism of the group, called an inner automorphism. All other automorphisms are called outer automorphisms. Intuitively, conjugation represents a 'basis change' and a normal subgroup 'looks the same from all perspectives'.

- If H is normal in G, then the coset space G/H has a group structure because

$$(g_1H)(g_2H) = g_1g_2HH = g_1g_2H.$$

  Groups without normal subgroups are called simple groups.

- Simple groups have been classified. There are the cyclic groups of prime order, the alternating groups A_n for $n \geq 5$, infinite families of groups of Lie type, and 26 sporadic groups.

- Groups can also be combined with the semidirect product. Let $H \subset \mathrm{Aut}(G)$, so that h acts on g by $g \to \varphi_h(g)$, such that

$$\varphi_h(g_1)\varphi_h(g_2) = \varphi_h(g_1g_2), \quad \varphi_{h_1}(\varphi_{h_2}(g)) = \varphi_{h_1h_2}(g), \quad \varphi_e(g) = g.$$

  Then we define the semidirect product group $H \ltimes G$ with the multiplication rule

$$(h, g)(h', g') = (hh', g\varphi_h(g')).$$

  Note that the subgroup $(g, e) \cong G$ is always normal, so $H \cong (H \ltimes G)/G$.

Example. The dihedral group may be written as a semidirect product,

$$D_n = \mathbb{Z}_2 \ltimes \mathbb{Z}_n.$$

If we let b be the reflection, the $\varphi_b(g) = g^{-1}$.

Example. The cycle types in S_n can be written in terms of Young tableaux. We sort the cycles by size, and draw a diagram where the first column contains a number of boxes equal to the size of the largest cycle, and so on. Then the number of boxes in the i^{th} row is the number of cycles with size at least i, and there are n boxes in total.

2.2 Representations

We now review the basics of group representation theory.

- A representation of a group G is an action of G on a vector space V by linear transformations, where the element g corresponds to the linear operator $D(g)$. The representation is faithful if $D(g)$ is distinct for every distinct g. Sometimes, we call V the representation instead. The dimension of the representation is the dimension of V.

- Two representations $D(g)$ and $D'(g)$ are equivalent if they are related by a basis change, $D'(g) = SD(g)S^{-1}$. In mathematics, S is called an intertwiner.

- All groups have the regular representation,

$$D(g)|g'\rangle = |gg'\rangle$$

  which has dimension $|G|$. Then each representation matrix has exactly one 1 in each row and column, with all other elements zero. As another example, taking the determinant of every $D(g)$ always gives a one-dimensional representation.

- A representation is reducible if there is a subspace $U \subset V$ so that $D(g)$ keeps U inside itself. Equivalently, a representation is reducible if the $D(g)$ can be brought into block upper-triangular form. A representation is completely reducible if the $D(g)$ can be brought into block-diagonal form, with each piece irreducible (an irrep).

- As a simple example, all completely reducible representations of abelian groups have only one-dimensional irreps, because all the $D(g)$ can be simultaneously diagonalized.

Note. Not all representations are completely reducible. Consider the representation of $\mathbb{Z}$,

$$D(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

The representation is reducible, but not completely reducible because the $D(n)$ do not have a complete basis of eigenvectors for $n \neq 0$.

Lemma (Schur)**.** If $D_1(g)$ and $D_2(g)$ are two irreps, then

$$SD_1(g) = D_2(g)S$$

for all g implies either $S = 0$ or the irreps are equivalent. Also, for an irrep $D(g)$, if

$$SD(g) = D(g)S$$

for all g then $S \propto I$.

Proof. For the first part, note that $\mathrm{Im}(S)$ and $\mathrm{Ker}(S)$ are both invariant subspaces; this is only possible if S is invertible, but then the equation becomes $SD_1(g)S^{-1} = D_2(g)$, which shows the representations are equivalent. For the second part, note that $D(g)$ preserves eigenspaces of S. But since $D(g)$ is irreducible, each eigenspace must be the whole space, so $S \propto I$.

Note. Suppose G represents some physical symmetry, such as rotational symmetry. Then the quantum states in an irrep of G may have different rotational quantum numbers (i.e. different L_z). However, Schur's lemma states that for any other symmetry which commutes with rotational symmetry, all the states in the irrep must have the same quantum numbers. For example, if H is rotationally invariant, all the states in the irrep have the same energy. Also, all the states in the irrep have the same P and T^2 eigenvalue.

Definition. Suppose $H \subset G$ and we have a representation $v \to D(h)v$ of H for $v \in V$. The induced representation on G acts on $V \times (G/H)$, whose elements are written as v_i where i indexes the coset. For each coset we choose a representative element g_i so that the coset is $g_i H$. Then the action of a group element $g = g_i^{-1} g_j h$ is

$$v_i \to D(h)v_j.$$

That is, we act as H would have, and also shift the coset space accordingly. Note that if $D(h)$ is irreducible, so is the induced representation. But on the other hand, restricting an irrep of G down to H generally does not give an irrep of H.

Definition. A unitary representation is one where every matrix is unitary,

$$D(g)^\dagger = D(g^{-1}) = D(g)^{-1}.$$

Then the $D(g)$ preserve the scalar product. All unitary representations are completely reducible, because if a subspace W is preserved, then $W^\perp$ must be preserved as well.

Prop. For a finite group, all representations are equivalent to unitary representations.

Proof. We use the 'group indexing trick'. Define

$$S = \sum_i D(g_i)^\dagger D(g_i).$$

Then we have

$$SD(g)^{-1} = \sum_i D(g_i)^\dagger D(g_i g^{-1}) = \sum_i D(g_i g)^\dagger D(g_i) = D(g)^\dagger \sum_i D(g_i)^\dagger D(g_i) = D(g)^\dagger S$$

where we reindexed $g_i \to g_i g^{-1}$ in the second step. Now, S is positive definite and thus has a well-defined square root, and the above shows that $D'(g) = S^{1/2} D(g) S^{-1/2}$ give a unitary representation. Another perspective is that we keep the matrices the same but change the definition of the inner product, using S as the 'metric'.

Note. We make some remarks about this result.

- The construction works for real representations as well, replacing 'unitary' with 'orthogonal'.
- The key intuition comes from the one-dimensional case. Since all elements have finite order, they must be represented by pure phases, which are unitary. Similarly, to have finite order in higher dimensions, we can only have rotations, not scalings.
- Essentially the same construction can be used for a compact continuous group, with integration over a 'group measure' replacing the sum. The theorem fails for noncompact groups, where the sum/integral doesn't exist (e.g. $\mathbb{Z}$ above). Intuitively, a noncompact group like $\mathbb{R}$ must be represented by a scaling, because a rotation would have finite order. It is also possible to have a unitary but infinite-dimensional representation, because with infinite dimensions the rotation angles can 'never get into phase again'.
- Conversely, all irreps of a compact group are finite-dimensional, though this requires substantial formal machinery to prove.
- We will use this result to make every representation considered below unitary.

Corollary (Maschke)**.** Every representation of a finite group is completely reducible.

2.3 Characters and Orthogonality

Character theory will help us identify irreps and decompose general representations. We begin by proving a powerful orthogonality theorem. Whenever we deal with multiple representations, we index the representation with a superscript.

Theorem (Great Orthogonality)**.** Given a d-dimensional irrep $D(g)$ of a finite group G,

$$\sum_g D(g^{-1})_{ij} D(g)_{k\ell} = \frac{|G|}{d} \delta^i_\ell \delta^k_j$$

Moreover, given two inequivalent irrepss $D^r(g)$ and $D^s(g)$,

$$\sum_g D^r(g^{-1})_{ij} D^s(g)_{k\ell} = 0.$$

Proof. To prove the first part, we use the reindexing trick. For an arbitrary X, let

$$A = \sum_g D(g^{-1}) X D(g).$$

By reindexing the sum, we find

$$A = D(g^{-1}) A D(g) \text{ for all } g \in G$$

which implies that $A = \lambda I$ by Schur's lemma. Taking the trace of both sides gives $\lambda = (|G|/d) \operatorname{tr} X$. The theorem follows by setting X to be the matrix with $X_{jk} = 1$ and zeroes everywhere else. The second part of the theorem uses the other half of Schur's lemma; taking X to be an arbitrary matrix with the right dimensions, we have $A = 0$, and setting X as before gives the result.

Note. The intuition for this result is that all 'orientational' information is washed out by averaging over the group. (This is even more explicit when we integrate over, say, the continuous group of rotations.) Thus the only possible results for the sum are the 'invariant tensors' built from δ. In the case of inequivalent representations, we cannot form any δ's at all because the indices ij and $k\ell$ live in 'different spaces' and cannot be contracted, so the result must be zero.

This theorem allows us to work with the characters of a representation.

- Given a representation $D^r(g)$, the character is

  $$\chi^r(g) = \operatorname{tr} D^r(g).$$

  Since trace is invariant under conjugation, the character is constant on conjugacy classes.

- We define the product of two representations $D^r(g)$ and $D^s(g)$ by

  $$D^{r\otimes s}(g) = D^r(g) \otimes D^s(g).$$

  Then the character multiplies,

  $$\chi^{r\otimes s}(g) = \chi^r(g)\chi^s(g).$$

  Generally, the product of two irreps will not be an irrep.

- Setting $i = j$ and $k = l$ in the orthogonality theorem gives the orthogonality theorem for characters. In particular, we choose $D^r(g)$ and $D^s(g)$ to be unitary, so that the taking the inverse just conjugates the character. Then

  $$\langle \chi_r, \chi_s \rangle \equiv \sum_g (\chi^r(g))^* \chi^s(g) = \sum_c n_c (\chi^r(c))^* \chi^s(c) = |G| \delta^{rs}$$

  where c is a conjugacy class representative and n_c is the size of the class.

- One example is the orthogonality of the representations of $\mathbb{Z}_n$. In this case, the orthogonality of characters tells us the Fourier modes are orthogonal.
- More generally, suppose a reducible representation contains the irrep r, n_r times. Then the 'norm' of the character is $\langle \chi, \chi \rangle = |G| \sum n_r^2$, so a representation is reducible only if the norm of its character is $|G|$. We compute the numbers n_r by computing $\langle \chi, \chi_r \rangle$.
- Suppose there are $N(C)$ conjugacy classes and $N(R)$ irreps. Then character orthogonality implies we must have $N(R) \leq N(C)$.
- Going back to the orthogonality theorem, we can think of $D(g)_{ij}$ for fixed i and j as a vector in a $|G|$-dimensional complex vector space. If we work with unitary representations, then

  $$\sum_g D(g)^*_{ji} D(g)_{k\ell} = \frac{|G|}{d} \delta^i_\ell \delta^k_j$$

  which tells us that these vectors are all orthogonal. There are $\sum_r d_r^2$ vectors in total, giving the bound $\sum_r d_r^2 \leq |G|$.
- To prove equality, consider the regular representation. Then the character of all classes besides the identity vanishes, and carrying out the sum shows that the regular representation contains every irrep D^r, d_r times. Counting dimensions gives $\sum_r d_r^2 = |G|$.
- This tells us that the $|G|$ 'vectors' $D^r(g)_{ij}$ form a complete set, so

  $$\sum_{r,i,j} d_r \, D^r(g)_{ij} D^{r\dagger}(g')_{ji} = |G| \delta_{gg'}.$$

  This is the completeness relation that complements the great orthogonality theorem.

Theorem. The character table is square: $N(R) = N(C)$. Then the characters form an orthonormal basis for the set of functions on conjugacy classes.

Proof. For variety, we prove this result using the class algebra; it also follows more easily from the completeness relation above. Define the group algebra to be the set of formal linear combinations $\sum_i a_i g_i$ for $a_i \in \mathbb{R}$. These objects are added in the usual way and multiplied using the group law. For each conjugacy class c, we define the class average

$$K(c) = \frac{1}{n_c} \sum_i g_i^{(c)}.$$

Class averages can be multiplied together, and it can be shown that they form a linear combination of class averages. Then the class averages themselves form an algebra, called the class algebra, with

$$K(c)K(d) = \sum_e \Gamma(c, d; e) K(e)$$

where the Γ's are positive integers.

The group and class algebra can be realized in representations, where the addition operation is simply matrix addition. For a representation $D(g)$, the class average is

$$\mathcal{D}(c) = \frac{1}{n_c} \sum_{g \in c} D(g).$$

By the reindexing trick, we have $D(g'^{-1})\mathcal{D}(c)D(g') = \mathcal{D}(c)$, so by Schur's lemma

$$\mathcal{D}(c) = \frac{\chi(c)}{\chi(I)} I.$$

Now, we also have

$$\mathcal{D}(c)\mathcal{D}(d) = \sum_e \Gamma(c,d;e)\mathcal{D}(e)$$

which translates to

$$\chi(c)\chi(d) = \chi(I) \sum_e \Gamma(c,d;e)\chi(e).$$

Now we restore the irrep index r and sum over all irreps, giving

$$\sum_r \chi^r(c)\chi^r(d) = \sum_e \Gamma(c,d;e) \sum_r \chi^r(I)\chi^r(e) = \Gamma(c,d;I)|G|$$

where the last step follows by using the fact that all elements of the regular representation besides the identity are traceless.

If c is a conjugacy class, then so is $\overline{c}$, the set of inverses of elements of c. Moreover, $\Gamma(c,d;I)$ is only nonzero for $d = \overline{c}$, giving

$$\sum_r \chi^r(c)\chi^r(d) \propto \delta^{d\overline{c}}.$$

Using the fact that the representations are unitary, $\chi^r(d) = \chi^r(\overline{d})^*$, so reindexing gives

$$\sum_r \chi^r(c)^*\chi^r(c') \propto \delta^{cc'}.$$

Thus, the rows of the character table are orthogonal, while before we showed that the columns are orthogonal. This is only possible if $N(R) = N(C)$.

2.4 Examples and Applications

Example. Consider an abelian group G. There are $|G|$ conjugacy classes, so $|G|$ irreps, each of which must have dimension one. Our results above essentially say the character table is a unitary matrix; every irrep simply assigns a phase to every group element which adds under group multiplication. This generalizes the Fourier modes from characters of $\mathbb{Z}_n$.

Groups and their irreps appear often in quantum mechanics.

- Let the Hamiltonian H have a symmetry group G, represented on the Hilbert space by a unitary representation $D(g)$ where $[H, D(g)] = 0$ for all g. Then if $|v\rangle$ is an energy eigenstate with energy E, so is $D(g)|v\rangle$. Therefore, the Hilbert space breaks into irreps of G containing degenerate vectors.

- In the case where G is abelian, no degeneracy can result since all irreps are one-dimensional, but it can still be useful because time evolution will keep states within one class of irrep by Schur's lemma. For example, for a $\mathbb{Z}_2$ parity symmetry, we may decompose the Hilbert space into odd and even parts; an even state remains even for all times.

- We may also have 'accidental degeneracies' where distinct irreps may have equal energies. However, this is often due to fine tuning of parameters, or a consequence of a larger symmetry group we haven't taken into account.

Example. For a discrete translational symmetry, the irreps are Bloch wavefunctions, which are periodic up to a fixed phase factor per translation. This does not yield any degeneracy, but if we have both translational symmetry and parity symmetry, there are two-dimensional irreps corresponding to the degeneracy of the $\pm k$ states. This also holds for a free particle, where it explains the seemingly trivial degeneracy of e^{ikx} and e^{-ikx}.

Example. Using group theory to find normal modes. A set of N masses in d dimensions connected by springs has equation of motion

$$H\mathbf{x} = \omega^2 \mathbf{x}$$

where $\mathbf{x}$ is a vector with Nd entries containing all the positions concatenated together. Now suppose the system has a group of symmetries G. This furnishes an Nd-dimensional representation $D(g)$, and H is invariant under conjugation by $D(g)$. Therefore, by Schur's lemma, H must be block diagonal when D is broken into irreps, i.e. every irrep provides a set of normal modes with the same frequency.

As a very simple example, consider two molecules connected in one dimension. The molecules can be exchanged, giving the symmetry group S_2 with character table

class c	n_c	1	$\bar{1}$
e	1	1	1
(12)	1	1	-1.

This tells us the normal modes are either even or odd under reflection, corresponding to the 1 and $\bar{1}$ representation. Less trivially, consider an equilateral triangle, where there are six normal modes and the symmetry group is S_3, with character table

class c	n_c	1	$\bar{1}$	2	phys
e	1	1	1	2	6
(123)	2	1	1	-1	0
(12)	3	1	-1	0	2.

Here, the last column denotes the physical representation. We know that $\chi(e) = 6$ since it is six-dimensional, and $\chi((123)) = 0$ since all atoms are moved by the transformation. However, (12) fixes the third atom, giving $\chi((12)) = 2$. Taking the inner product gives phys $= 1 \oplus 1 \oplus 2 \oplus 2$. Thus, there are two two-fold degeneracies.

We can say more about the normal modes. The normal modes in 1 must be rotationally invariant, while the normal modes in each 2 must transform into each other under rotation. Hence the two uniform translations form a 2 with frequency zero, while uniform rotation and 'breathing' are the 1's. The other 2 contains the 'scissoring' modes.

Next, we turn to the Wigner-Eckart theorem, first establishing notation. From this point on, we always take representations to be unitary.

- Consider the tensor product of two irreps, indexed by μ and ν, whose vectors are indexed by i and j. The resulting vector space has a basis $\{|\alpha\lambda\ell\rangle\}$, where $|\alpha\lambda\ell\rangle$ is vector ℓ in occurrence λ of irrep α. The two bases are related by 'Clebsch-Gordan coefficients',
$$|\alpha\lambda\ell\rangle = |ij\rangle\langle ij|\alpha\lambda\ell\rangle$$
where the summation convention is used. The coefficients implicitly depend on μ and ν.

- The two bases transform as
$$U(g)|ij\rangle = D^\mu(g)_{i'i}D^\nu(g)_{j'j}|i'j'\rangle, \quad U(g)|\alpha\lambda\ell\rangle = D^\lambda(g)_{\ell'\ell}|\alpha\lambda\ell'\rangle.$$
By inserting some copies of the identity, we have
$$D^\mu(g)_{i'i}D^\nu(g)_{j'j} = \langle i'j'|\alpha\lambda\ell'\rangle D^\lambda(g)_{\ell'\ell}\langle\alpha\lambda\ell|ij\rangle.$$
Conceptually, this shows how the tensor product representation matrices are decomposed. Applying it in reverse allows us to construct the matrices for larger irreps from smaller ones.

- Next, consider a general representation in a vector space containing two irreps $|\mu i\rangle$ and $|\nu j\rangle$, where the two irreps are not equivalent. Then we claim the vectors are orthogonal. First,
$$\langle\nu j|\mu i\rangle = \langle\nu j|U^\dagger(g)U(g)|\mu i\rangle = \langle\nu k|\mu\ell\rangle D^{\nu\dagger}(g)_{jk}D^\mu(g)_{\ell i}.$$
Now we average over the group and apply the orthogonality theorem, giving
$$\langle\nu j|\mu i\rangle = \frac{1}{|G|}\langle\nu k|\mu\ell\rangle\sum_g D^{\nu\dagger}(g)_{jk}D^\mu(g)_{\ell i} = d_\mu^{-1}\langle\nu k|\mu\ell\rangle\delta^\mu_\nu\delta^j_i\delta^k_\ell = 0$$
since $\mu \neq \nu$ and d_μ is the dimension of μ. In the case where the irreps are equivalent, but not identically the same, we can apply a change of basis to restore orthogonality.

- This result is a generalization of the fact that eigenvectors of a Hermitian operator with different eigenvalues are orthogonal. In the case of rotational symmetry, we already knew that states with distinct values of ℓ are orthogonal because they have different L^2 eigenvalues. But the fact we've shown above applies much more generally.

- A set of operators O^μ_i is said to be a set of irreducible operators if
$$U(g)O^\mu_i U(g)^{-1} = O^\mu_j D^\mu(g)_{ji}.$$
They are also sometimes called irreducible tensors.

- Note that the vectors $O^\mu_i|\nu j\rangle$ transform under the direct product representation,
$$U(g)O^\mu_i|\nu j\rangle = U(g)O^\mu_i U(g)^{-1}U(g)|\nu j\rangle = D^\mu(g)_{ki}D^\nu(g)_{\ell j}O^\mu_k|\nu\ell\rangle.$$
Therefore, we may decompose the vectors as
$$O^\mu_i|\nu j\rangle = |\alpha\lambda\ell\rangle\langle\alpha\lambda\ell|ij\rangle.$$
Acting with $\langle\lambda\ell|$, using orthogonality, and reindexing, we have the Wigner-Eckart theorem
$$\langle\lambda\ell|O^\mu_i|\nu j\rangle = \sum_\alpha\langle\alpha\lambda\ell|ij\rangle\left(\frac{1}{d_\lambda}\sum_k\langle\lambda k|\alpha\lambda k\rangle\right).$$
The quantity in parentheses is called a reduced matrix element; crucially, it only depends on the on the irrep indices. The remaining factor is purely group-theoretic.

- Note the labeling conventions here: λ indexes irreps in the full space, with different λ values possibly corresponding to equivalent but distinct irreps, while $\alpha\lambda$ indexes irreps in the direct product representation, where different λ values automatically mean inequivalent irreps. These schemes coincide when no irreps appear multiple times, as happens for simple $\mathfrak{su}(2)$ setups.
- In many practical contexts, we instead work 'infinitesimally', i.e. with representations of $\mathfrak{su}(2)$ instead of $SU(2)$. Much of the reasoning above goes through unchanged, with conjugation replaced with commutation with generators. Computationally, the Wigner-Eckart theorem is natural, as it just says we can move around an irrep by applying raising and lowering operators to both sides.

Example. An explicit $\mathfrak{su}(2)$ example. Letting α and β be irrep indices, let

$$\langle 1/2, 1/2, \alpha | r_3 | 1/2, 1/2, \beta \rangle = A, \quad \langle 1/2, 1/2, \alpha | r_1 | 1/2, -1/2, \beta \rangle = B.$$

Here, the states $r_i | 1/2, m, \beta \rangle$ form the reducible representation $1/2 \times 1 = 1/2 + 3/2$, and our goal is to relate A and B. To begin, we need to change the r_i basis. Since $[J_3, r_3] = 0$, we know that r_3 carries $m = 0$, so

$$r^0 = r_3.$$

We can then find the $m = 1$ and $m = -1$ operators by raising and lowering,

$$r^1 = -\frac{r_1 + i r_2}{\sqrt{2}}, \quad r^{-1} = \frac{r_1 - i r_2}{\sqrt{2}}.$$

The operators $\{r^{-1}, r^0, r^1\}$ are irreducible tensors, as desired. Thus to complete the problem we just need to know a Clebsch-Gordan coefficient. Alternatively, for this simple case we can explicitly compute them using raising and lowering operators. We start with

$$|3/2, 3/2\rangle = r^1 |1/2, 1/2, \beta\rangle$$

and lower both sides to yield

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}} r^0 |1/2, 1/2, \beta\rangle + \sqrt{\frac{1}{3}} r^1 |1/2, -1/2, \beta\rangle.$$

Finally, using the fact that $\langle 1/2, 1/2, \alpha | 3/2, 1/2\rangle = 0$, we find

$$0 = \sqrt{\frac{2}{3}} \langle 1/2, 1/2, \alpha | r^0 | 1/2, 1/2, \beta\rangle + \sqrt{\frac{1}{3}} \langle 1/2, 1/2, \alpha | r^1 | 1/2, -1/2, \beta\rangle.$$

We thus conclude that $A = B$.

Note. Note that in the above example, we couldn't have found $|1/2, 1/2\rangle$ in terms of the β states by orthogonality with $|3/2, 1/2\rangle$, because we don't know the norms of the states $r^i |1/2, m, \beta\rangle$. Instead, we can find $|1/2, 1/2\rangle$ by demanding that it be annihilated by J^+. The same goes for putting operators in the irreducible tensor basis; we simply use the algebra, without the 'crutch' of orthogonality.

2.5 Real and Complex Representations

In mathematics, a complex representation is one over a complex vector space. However, in physics, almost all representations are complex, so we introduce new definitions for 'physically' real and complex representations.

- Given a representation $D(g)$, its conjugate representation is $D(g)^*$. Note that this conjugates the character. A representation is said to be complex if it is not similar to its conjugate; then every representation with a non-real character is complex.

- Now suppose a given irrep is not complex. Then $D(g)^* = SD(g)S^{-1}$. Transposing and using unitarity gives
$$D(g^{-1}) = S^{-1^T} D(g)^T S^T$$
and plugging this equation into itself gives
$$D(g) = (S^{-1}S^T)^{-1} D(g)(S^{-1}S^T).$$
By Schur's lemma, this implies $S^{-1}S^T = \eta I$, which gives $S^T = \pm S$. If S is symmetric, we say the representation is real; if S is antisymmetric, we say the representation is pseudoreal, or quaternionic.

- Note that taking the transpose of an antisymmetric $n \times n$ matrix keeps its determinant the same, but also multiplies the determinant by $(-1)^n$. Therefore pseudoreal representations can only exist in even n since S must be invertible.

- The matrix S can always be chosen unitary. Solving for S above gives
$$S = D(g)^T S D(g), \quad S^\dagger = D(g)^\dagger S^\dagger D(g)^*.$$
Multiplying these gives
$$S^\dagger S = D(g)^\dagger S^\dagger S D(g)$$
so that $S^\dagger S$ is proportional to the identity, again by Schur's lemma. Some more playing around shows that the proportionality constant is real. Hence we can always scale S by a constant so $S^\dagger S = I$.

- The matrices in a real representation are always equivalent to real matrices. First, note that since S is unitary and symmetric, we have $S = e^{iH}$ where H is symmetric. Letting $W = e^{iH/2}$, we have $S = W^2$ where W is also unitary and symmetric. Now
$$W^2 D(g) W^{-2} = D(g)^*$$
which implies
$$WD(g)W^{-1} = W^{-1}D(g)^*W = (WD(g)W^{-1})^*.$$
Then the matrices $WD(g)W^{-1}$ are real as desired.

- Equivalently, we may work at the level of Lie algebra representations, letting $D(e^{iX}) = e^{id(X)}$ for a generator X. In this case the conjugate representation of d is the negative conjugate. For real representations, $d(X)$ can be made pure imaginary, so $D(g)$ is real. For pseudoreal representations, $d(X)$ can be made real.

- Physically, the conjugate representation is important because antiparticles transform in the conjugate representation. On the other hand, any representation that arises from physical rotations of coordinates is real.

Next, we build a 'reality checker' for general irreps.

- If an irrep is real or pseudoreal, then $y^T S x$ is an invariant bilinear, as
$$y^T S x \to y^T D(g)^T S D(g) x = y^T S D(g)^\dagger S^{-1} S D(g) x = y^T S x.$$
Conversely, the existence of such an invariant bilinear shows the irrep is real or pseudoreal.
- Note that $y^\dagger x$ is always trivially an invariant since the $D(g)$ are unitary; the difference here is that we have y^T rather than $y^\dagger$. In the simplest case where the matrices of $D(g)$ are already real, the invariant bilinear is just $y^T x$, i.e. the $D(g)$ are orthogonal.
- To construct S, we use the averaging trick. Define
$$S = \sum_g D(g)^T X D(g)$$
for arbitrary X. Then $D(g)^T S D(g) = S$, giving an invariant bilinear as desired. Since we can always define S this way, we must have $S = 0$ for a complex representation. If we suppose that $X_{i\ell} = 1$ with all other entries zero, we have
$$\sum_g D(g)^{ij} D(g)^{\ell k} = 0.$$
Finally set $j = \ell$ and $i = k$ to find
$$\sum_g \chi(g^2) = 0.$$
This is our test for a complex representation.
- On the other hand, for a pseudoreal representation, $S^T = \eta S$, which gives
$$\sum_g D(g)^T X^T D(g) = \eta \sum_g D(g)^T X D(g).$$
Performing the same procedure above, we find
$$\sum_g \chi(g^2) = \eta \sum_g \chi(g)^2 = \eta |G|$$
where we used the fact that the character is real. The quantity $\sum_g \chi(g^2)$ is called the Frobenius-Schur indicator.

Example. The fundamental representation of $\mathfrak{su}(2)$. The representations of the generators are the Pauli matrices. They clearly cannot be made pure imaginary by a basis change, since there is only one antisymmetric pure imaginary 2×2 matrix. The representation is pseudoreal, with
$$\sigma_a = -\sigma_2 \sigma_a^* \sigma_2$$
where σ_2 is indeed antisymmetric as expected. The fact that we use σ_2 to do the transformation is purely a matter of convention; the usual phase conventions make σ_2 alone non-real.

2.6 The Group Algebra

We return to the group algebra, to prepare for classifying the representations of S_n.

- We can promote a group G into an algebra $\widetilde{G}$ by allowing formal linear combinations of the group elements, with group multiplication compatible with this operation. This makes the group into both a ring and a vector space.

- Note that every group element can also be regarded as a linear operator on this vector space. To avoid confusion, we will write a group algebra element as $|r\rangle$ when we think of it as a vector, and as r when we think of it as an operator.

- For example, e is the identity operator, and if the group elements are g_i, then $|g_i\rangle$ is a basis. We reserve the letter g for group elements, while r and s stand for arbitrary algebra elements.

- Representations of the group are promoted to representations of the group algebra by linearity. For example, the group algebra $\widetilde{G}$ is itself the vector space for the regular representation D^R defined by $g|h\rangle = |gh\rangle$.

- We know that the regular representation decomposes into irreps as

$$\widetilde{G} = \bigoplus_{\mu} L^\mu, \quad L^\mu = \bigoplus_{a=1}^{n_\mu} L^\mu_a, \quad n_\mu = \dim D^\mu$$

  where μ indexes the distinct irreps. As such, we can find all of the distinct irreps by decomposing the regular representation.

- An subrepresentation L of D^R is spanned by a basis $|r\rangle$ so that $p|r\rangle = |pr\rangle \in L$ for all p. Thinking of the group algebra as a ring, subrepresentations are left ideals, and irreps are minimal left ideals. Therefore, finding all minimal left ideals will give all inequivalent irreps.

- We define projection operators P^μ_a onto the minimal left ideals L^μ_a by

$$P^\mu_a \widetilde{G} = L^\mu_a, \quad P^\mu_a = \text{identity on } L^\mu_a, \quad P^\mu_a P^\nu_b = \delta^{\mu\nu}\delta_{ab} P^\mu_a.$$

  Note that the projection operators commute with all of $\widetilde{G}$,

$$P^\mu_a r = r P^\mu_a.$$

  This can be shown by acting on an arbitrary element with decomposition $\sum_{\mu,a} |s^\mu_a\rangle$. Intuitively, it's because group multiplication can't take an element in or out of a left ideal.

- Note that this definition doesn't require the ideals to be minimal. For example, we can also define projection operators P^μ onto the non-minimal left ideals L^μ, where $P^\mu = \sum_a P^\mu_a$.

Next, we construct the projection operators more explicitly. We suppress the a index temporarily.

- Decompose the identity element e as

$$e = \sum_\mu e_\mu, \quad e_\mu \in L^\mu.$$

Then we claim the projection operators are given by right-multiplication by e_μ,

$$P^\mu|r\rangle = |re_\mu\rangle.$$

Using right-multiplication is useful, because P^μ automatically commutes with any r since left-multiplication and right-multiplication commute.

- The trick here is that we can always multiply by the identity on the right. Consider an arbitrary element $r = \sum_\mu r_\mu$. Then

  $$r = re = \sum_\mu re_\mu$$

  but the fact that the L^μ are left-ideals means that $r_\mu = re_\mu$ by taking the μ component of both sides. Hence P^μ is indeed a projector.

- Finally, note that

  $$e_\nu = e_\nu e = \sum_\mu e_\nu e_\mu$$

  and taking the μ component of both sides gives

  $$e_\nu e_\mu = \delta_{\mu\nu} e_\mu.$$

  In particular $e_\mu e_\mu = e_\mu$. Any algebra element satisfying this relation is called an idempotent and can be used to define a projection operator onto a left ideal.

- A similar decomposition can be performed for the L^μ_a. The difference is that the e^a_μ are primitive idempotents, i.e. they generate minimal left ideals, while the e_μ are not, as they can be further decomposed into the e^a_μ.

- An idempotent e_i is primitive if and only if $e_i r e_i = \lambda_r e_i$ for all $r \in \widetilde{G}$, where λ_r is a scalar.

  To prove the forward direction, note that e_i generates a minimal left ideal L, and right-multiplication by $e_i r e_i$ is also a projection operator onto L, which commutes with all elements of $\widetilde{G}$. Then by Schur's lemma, $e_i r e_i$ is proportional to the identity on L, giving the result.

  To show the converse, suppose that $ere = \lambda_r e$ and that e decomposes into idempotents as $e = e' + e''$ where $e'e'' = 0$. Then $ee'e = e'e = e'$ and hence $e' = \lambda e$. Then e' generates exactly the same left ideal as e does, so e is primitive.

- Two primitive idempotents e_1 and e_2 generate equivalent irreps if and only if $e_1 r e_2 \neq 0$ for some $r \in \widetilde{G}$.

  Let e_1 and e_2 generate the minimal left ideals L_1 and L_2. To prove the backward direction, note that $|q\rangle \to |qe_1 r e_2\rangle$ is a nonzero linear transformation from L_1 to L_2 which commutes with (left-multiplication by) any element of $\widetilde{G}$. Then by Schur's lemma, L_1 and L_2 are equivalent. The forward direction is similar.

Example. The reduction of the regular representation of C_3. Let the generator be a. The idempotent for the identity representation is always just the average of the group elements,

$$e_1 = \frac{1}{3}(e + a + a^{-1}).$$

Next, suppose another idempotent is $e_2 = xe + ya + za^{-1}$. We have the constraints

$$e_1 e_2 = 0, \quad e_2 e_2 = e_2$$

which yield the three solutions

$$e' = \frac{1}{3}(2e - a - a^{-1}), \quad e_+ = \frac{1}{3}(e + \omega a + \omega^{-1} a^{-1}), \quad e_- = \frac{1}{3}(e + \omega^{-1} a + \omega a^{-1}), \quad \omega = e^{2\pi i/3}.$$

One can see that e' is not primitive since it is the sum of e_+ and e_-. To check that e_+ and e_- are idempotent, we manually compute $e_\pm a e_\pm$ and $e_\pm a^{-1} e_\pm$ and apply the above theorem; this is simple since $e_\pm$ just pick up phase shifts upon multiplication by any group element. We do a similar check to show that e_+ and e_- generate inequivalent representations, giving the three irreps of C_3.

2.7 Representations of S_n

We now use our machinery to find the irreps of S_n. We warm up with the one-dimensional irreps.

- All transpositions are conjugate, so we must assign them all the same number. Since transpositions have order 2, the number must be ± 1. Since transpositions generate the entire group, there are only two distinct one-dimensional representations.
- We denote an arbitrary permutation by p and its sign by $(-1)^p$. Then the two one-dimensional representations map p to one, and p to $(-1)^p$.
- The corresponding primitive idempotents are

  $$s = \sum_p p, \quad a = \sum_p (-1)^p p.$$

  They are 'essentially idempotents', i.e. they are idempotents up to a constant scaling factor.
- To check this, note that for any q,

  $$qs = sq = s, \quad qa = aq = (-1)^q a, \quad sa = as = 0.$$

  Then we have $sqs = ss = n!s$ and $aqa = (-1)^q aa = (-1)^q n!a$, so both are indeed primitives. The corresponding irreps have basis vectors $|qs\rangle$ and $|qa\rangle$ for some arbitrary q. To check the irreps are not equivalent, note that $sqa = sa = 0$ for any q.

For higher-dimensional irreps, it is useful to introduce Young diagrams.

- A partition $\lambda = \{\lambda_1, \ldots, \lambda_r\}$ of n is a sequence of positive integers λ_i satisfying

  $$\lambda_i \geq \lambda_{i+1}, \quad \sum_{i=1}^{r} \lambda_i = n.$$

  We say $\lambda > \mu$ if the first nonzero number in the sequence $\lambda_i - \mu_i$ is positive.
- A partition λ is represented by a Young diagram, which consists of n squares arranged in r rows, where row i contains λ_i squares.

- Partitions of n are in one-to-one correspondence with possible cycle structures of permutations, where elements that are left alone are regarded as 1-cycles. Then there is one conjugacy class and hence one irrep for every Young diagram. For example, the identity element is $1+1+\ldots+1$, and a transposition is $2+1+\ldots+1$.

- A Young tableau is obtained by labeling the squares of a Young diagram with the numbers 1 through n, using each number once. The normal Young tableau Θ_λ associated with λ has the numbers in left-to-right, top-to-bottom order. A standard Young tableau is one where the numbers always increase when going down or to the right.

- A permutation acts on a Young tableau by permuting the numbers. Then an arbitrary Young tableau can be written as $p\Theta_\lambda = \Theta_\lambda^p$.

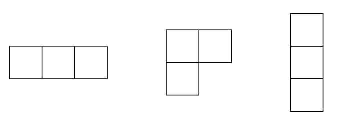
- Given a Young tableau Θ_λ^p, the horizontal permutations h_λ^p are the ones which only permute numbers within rows, and the vertical permutations v_λ^p are the ones which only permute numbers within columns.

- The symmetrizer s_λ^p, antisymmetrizer a_λ^p, and irreducible symmetrizer or Young symmetrizer e_λ^p associated with the Young tableau Θ_λ^p are defined as

$$s_\lambda^p = \sum_h h_\lambda^p, \quad a_\lambda^p = \sum_v (-1)^{v_\lambda} v_\lambda^p, \quad e_\lambda^p = s_\lambda^p a_\lambda^p = \sum_{h,v} (-1)^{v_\lambda} h_\lambda^p v_\lambda^p.$$

  Our main result will be that e_λ^p is a primitive idempotent. Note that since the Young symmetrizer acts on the right, the object it acts on is first symmetrized in rows, then antisymmetrized in columns.

Example. The simple example of S^3. There are three Young diagrams, shown below.



Labeling the corresponding normal Young tableaux Θ_1, Θ_2, and Θ_3, we have

- $s_1 = s$, $a_1 = e$, $e_1 = s$.

- $s_2 = e + (12)$, $a_2 = e - (13)$, $e_2 = e + (12) - (13) - (321)$.

- $s_3 = e$, $a_3 = a$, $e_3 = a$.

- The fourth standard Young tableau, $\Theta_2^{(23)}$, has $s_2^{(23)} = e + (13)$, $a_2^{(23)} = e - (12)$, and $e_2^{(23)} = e + (13) - (12) - (123)$.

We note the following features of the result and state they hold generally without proof.

- The horizontal permutations form a subgroup and s_λ is its symmetrizer. Then by similar logic to before, s_λ is idempotent with $s_\lambda s_\lambda = |\{h_\lambda\}| s_\lambda$. Similarly, a_λ is idempotent.

- Generally, neither s_λ nor a_λ are primitive, but e_λ is. Moreover, e_λ^p generates an irrep distinct from but isomorphic to that of e_λ. Moreover, all distinct irreps correspond to exactly one e_λ.

- The four minimal left ideals generated by the idempotents of the standard Young tableaux span the entire group algebra space S_3. Explicitly, $e = (1/6)(e_1 + 2e_2 + 2e_2^{(23)} + e_3)$.

Essentially, normal Young tableaux correspond to inequivalent irreps, while standard Young tableaux decompose the regular representation.

Example. We give a few more details. First, we explicitly construct the irrep generated by e_2.

- The irrep certainly contains e_2 itself, i.e. $|e\rangle + |(12)\rangle - |(13)\rangle - |(321)\rangle$. It is convenient to write the vectors in terms of the image of the string "123" under the corresponding permutations, giving $|123\rangle + |213\rangle - |321\rangle - |312\rangle$.
- The irrep also contains all the vectors $|pe_2\rangle$, which can be written in our notation by applying a permutation p to the numbers in the states. For example, for $p = (12)$, we get

$$|123\rangle + |213\rangle - |321\rangle - |312\rangle \quad \to \quad |213\rangle + |123\rangle - |312\rangle - |321\rangle.$$

  while for $p = (13)$ we get

$$|123\rangle + |213\rangle - |321\rangle - |312\rangle \quad \to \quad |321\rangle + |231\rangle - |123\rangle - |132\rangle.$$

  Computing the other permutations, we find a two-dimensional irrep. Generally, it's hard to tell the dimension in advance; for larger Young tableaux, the situation gets very complicated. We can check it's an irrep by acting with permutations on the left.

- The ordering is a bit confusing. Given a permutation p, we apply the Young symmetrizer on the right and apply an element $g \in S_n$ on the left, giving

$$p \to gps_\lambda a_\lambda.$$

  Thus, in the notation above where we consider how the resulting permutation acts on $|123\rangle$, we put $|123\rangle$ on the right and act right-to-left, so the antisymmetrization comes first. But if we want to think about $ps_\lambda a_\lambda$ as one symmetrized object, the symmetrization comes first.

- Also note that the second symmetrization/antisymmetrization partially destroys the first antisymmetrization/symmetrization. For example, the quantity e_2 above is not symmetric in 1 and 2. Also, the states in the ket notation above are not antisymmetric in 1 and 3.
- Since a d-dimensional irrep occurs d times in the regular representation, the dimension of the irrep corresponding to a Young diagram is the number of standard Young tableaux that can be built from it.
- For each box in a Young diagram, a hook is a right angle with vertex in that box, which opens downward and to the right; the length of the hook h_i is the number of boxes it intersects. The hook length formula states that the number of standard Young tableaux for that Young diagram is $n!/\prod_i h_i$.
- A nice fake argument is that for every hook, the only constraint is that the box with the vertex has the smallest number, which occurs with probability $1/h_i$, and multiplying these probabilities gives the result. But this isn't right because these events are not independent.

Example. Symmetry classes of tensors. Given an m-dimensional real vector space V_m, the tensor product space V_m^n of rank n tensors is acted on by $GL(m, \mathbb{R})$, and we would like to decompose V_m^n into irreps of $GL(m, \mathbb{R})$. The symmetric group enters because S_n acts on $GL(m, \mathbb{R})$ by permuting the indices, and this commutes with the action of $GL(m, \mathbb{R})$, which does the same thing to each index. Therefore, irreps of $GL(m, \mathbb{R})$ must have definite symmetry.

Specifically, one can show the irreps take the form

$$\{re_\lambda|\alpha\rangle \,|\, r \in S^n\}$$

for any λ and $|\alpha\rangle$. For example, the case e_2 considered above gives rank 3 tensors that are symmetric in the first two arguments and antisymmetric in the first and third arguments. Explicitly, if $m = 2$ and the basis vectors for V_m are $|\pm\rangle$, then we have a two-dimensional irrep containing

$$e_2|++-\rangle = 2|++-\rangle - |-++\rangle - |+-+\rangle, \quad (23)e_2|++-\rangle = 2|+-+\rangle - |-++\rangle - |++-\rangle$$

where the antisymmetrization acts on the state first. Note that the full decomposition requires non-standard Young tableaux; that is, there will be possible tensor symmetries that cannot be written in standard form. It is always possible to permute so that rows are increasing or columns are increasing, but fixing the columns might break the rows, and vice versa.

Note. We've focused on the states, but we can also do the same for the tensor components; on these the symmetrization acts first. Then the tensor components are antisymmetric in the columns but not symmetric in the rows. We couldn't have done better; for instance, if a rank 3 tensor is symmetric in the first two indices and antisymmetric in the last two, it is zero, as

$$T_{abc} = T_{bac} = -T_{bca} = -T_{cba} = T_{cab} = T_{acb} = -T_{abc}.$$

Example. Consider the symmetry properties of the Riemann tensor. We know that

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}, \quad R_{[abcd]} = 0.$$

The first two identities show that 13 and 24 have to come in columns, giving

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}.$$

The third identity rules out the second option, while the fourth rules out the third. Then the Riemann tensor is a $GL(n, \mathbb{R})$ irrep. Restricting to a subgroup breaks apart irreps, and indeed restricting to the Lorentz transformations $SO(n-1, 1)$, the Riemann tensor decomposes into the Ricci scalar, Ricci tensor, and Weyl tensor.

3 Lie Groups

3.1 $*$ Matrix Groups

We review some basic examples of matrix Lie groups.

- Let $\mathrm{Mat}(n, \mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F}$, which will be $\mathbb{R}$ or $\mathbb{C}$. It is not a group, because not all matrices are invertible, so we define $GL(n, \mathbb{F})$ to be the subset of invertible matrices.

- We define the special linear group $SL(n, \mathbb{F})$ to be the subset of $GL(n, \mathbb{F})$ with unit determinant. In general, 'special' stands for 'unit determinant'. Considering these groups as real manifolds,

  $$\dim GL(n, \mathbb{R}) = n^2, \quad \dim GL(n, \mathbb{C}) = 2n^2, \quad \dim SL(n, \mathbb{R}) = n^2 - 1, \quad \dim SL(n, \mathbb{C}) = 2n^2 - 2.$$

  The unit determinant constraint $\det M = 1$ is one constraint over $\mathbb{R}$ and two constraints over $\mathbb{C}$, as it sets $\det M = 1 + 0i$.

- A Lie group is a group that is a smooth manifold, where the group operations are smooth; we'll skip the proof that the groups above are Lie groups. We define a Lie subgroup to be a subgroup of a Lie group that is also a smooth submanifold; one can show that Lie subgroups are Lie groups in themselves.

- The orthogonal group is

  $$O(n) = \{M \in GL(n, \mathbb{R}) \,|\, M^T M = 1\}$$

  which implies $\det M = \pm 1$. The subset $SO(n)$ is called the proper rotations, and

  $$\dim O(n) = \dim SO(n) = \frac{n(n-1)}{2}.$$

  To show this, note that $M^T M$ is symmetric, so subtracting the constraints gives $n^2 - n(n+1)/2$. Alternatively, note that $n(n-1)/2$ is the number of independent planes in n dimensions.

- We consider the possible eigenvalues of $M \in O(n)$. Since M is real, eigenvalues come in complex conjugate pairs. Moreover, since M preserves lengths, they have norm 1. Thus for $n = 2$ we have eigenvalues $e^{\pm i\theta}$ and for $n = 3$ we have eigenvalues 1 and $e^{\pm i\theta}$. The eigenvector with $\lambda = 1$ specifies the axis of rotation.

- The general group element of $SO(3)$ can be written as

  $$M(\hat{\mathbf{n}}, \theta)_{ij} = \cos\theta\, \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta\, \epsilon_{ijk} n_k.$$

  This is redundant, since $M(\hat{\mathbf{n}}, 2\pi - \theta) = M(-\hat{\mathbf{n}}, \theta)$. To remove this we restrict $\theta \in [0, \pi]$ and identify $(\hat{\mathbf{n}}, \pi)$ with $(-\hat{\mathbf{n}}, \pi)$. Then the $SO(3)$ group manifold is obtained by taking the ball B_3 and identifying antipodal points on the boundary. The manifold is compact, since it is closed and bounded, has no boundary, and is connected but not simply connected.

- More generally, define $O(p, q)$ as the subset of $GL(n, \mathbb{R})$, with $n = p + q$, where

  $$M^T \eta M = \eta, \quad \eta = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

  For example, the Lorentz group is $O(3, 1)$. It is noncompact, and splits into four components since we also have time reversal. The dimension is still $n(n-1)/2$.

* Define the unitary groups as

$$U(n) = \{U \in GL(n, \mathbb{C}) \,|\, U^\dagger U = 1\}.$$

Unitary matrices preserve length and have determinants with unit norm. We have

$$\dim U(n) = n^2, \quad \dim SU(n) = n^2 - 1.$$

To derive this, note that we start with $2n^2$ degrees of freedom, and the matrix $U^\dagger U = 1$ is Hermitian. Then it contains $n(n-1)$ constraints on the off-diagonal, but only n constraints on the diagonal, because the diagonal is automatically real. Switching to the special unitary group gives one constraint since $\det U$ is a phase.

* As a simple example, $U(1) \cong SU(2)$. This is the first of a few 'accidental' Lie group isomorphisms which we'll understand later in terms of the classification of Lie algebras.

* The elements of $SU(2)$ can be parametrized as

$$U = a_0 1 + i\mathbf{a} \cdot \boldsymbol{\sigma}, \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$$

where the $\boldsymbol{\sigma}$ are the Pauli matrices. Thus $SU(2) \cong S^3$ as a manifold.

* The centers of these groups only contain elements proportional to the identity, by Schur's lemma and the fact that the fundamental representation is irreducible. For example, the center of $U(n)$ is $\{e^{i\theta} I\} \cong U(1)$, while the center of $SU(n)$ is $\{e^{2\pi i k/n} I\} \cong \mathbb{Z}_n$.

Note. Generally, when we define a subset of $\mathbb{R}^n$ by constraint equations, the result is not necessarily a manifold; we get an object called an algebraic variety which may have singular points. In this case the group structure forbids this: if there were a singularity at g_1, then there must be a singularity at any other group element g_2, since the action of multiplication by $g_2 g_1^{-1}$ is smooth. But varieties cannot be singular everywhere, so the group must be smooth everywhere.

3.2 $*$ Lie Algebras

We begin with some fundamental definitions.

* A Lie algebra $\mathfrak{g}$ is a vector space with a bracket operation $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying antisymmetry, linearity, and the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Its dimension is the dimension of $\mathfrak{g}$ as a vector space. The Lie algebra is real if it is over $\mathbb{R}$ and complex if it is over $\mathbb{C}$.

* For any vector space V, given an associative linear product $\star \colon V \times V \to V$, the commutator $[X, Y] = X \star Y - Y \star X$ is a Lie bracket. For example, for the vector space of matrices such a product is matrix multiplication.

* For a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with basis T^a, we define

$$[T^a, T^b] = f^{ab}{}_c T^c$$

where the $f^{ab}{}_c \in \mathbb{F}$ are called the structure constants. They are antisymmetric in a and b by the antisymmetry of the bracket, while the Jacobi identity gives

$$f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0.$$

Here one lower index is free, another is contracted, and the rest are cyclically permuted.

- A Lie algebra isomorphism is a map $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ which is an isomorphism of vector spaces and preserves the bracket. A Lie algebra homomorphism is defined similarly.
- A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ which satisfies $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An ideal of $\mathfrak{g}$ is a subalgebra which satisfies $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. We now give some examples of ideals.
  - The empty set and the entirety of $\mathfrak{g}$ are trivially ideals; all others are 'proper' ideals.
  - The derived algebra $\mathfrak{i} = \mathrm{span}([\mathfrak{g}, \mathfrak{g}])$ is the set of linear combinations of brackets in $\mathfrak{g}$. A Lie algebra is perfect if $\mathfrak{i} = \mathfrak{g}$.
  - The center $\mathfrak{z}(\mathfrak{g})$ is the set of elements in $\mathfrak{g}$ which have vanishing bracket with all of $\mathfrak{g}$.
  - If $\mathfrak{h}$ and $\mathfrak{k}$ are ideals, so are $\mathfrak{h} + \mathfrak{k}$, $\mathfrak{h} \cap \mathfrak{k}$, and $[\mathfrak{h}, \mathfrak{k}]$.
  - The kernel of any Lie algebra homomorphism is an ideal; this is really the motivation behind the definition. They are analogous to normal subgroups of groups.
- We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ if this holds considering the Lie algebras as vector spaces, and $[\mathfrak{h}, \mathfrak{k}] = 0$.
- An abelian Lie algebra is one where the bracket identically vanishes, i.e. $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}$.
- We say $\mathfrak{g}$ is simple if it is nonabelian and has no proper ideals. The 'nonabelian' requirement is a little strange, but turns out to be convenient because many theorems fail in the abelian case. All simple Lie algebras are perfect.
- We say $\mathfrak{g}$ is semi-simple if it has no abelian ideals; then semi-simple Lie algebras are direct sums of simple ones. Thus all semi-simple Lie algebras are perfect.

We now formally relate Lie groups and Lie algebras, for the case of matrix Lie groups.

- Given an n-dimensional matrix Lie group G, the tangent space $\mathfrak{g} = T_e(G)$ is an n-dimensional vector space. Given coordinates $\boldsymbol{\theta}$ on G, the tangent space is spanned by the vectors

  $$X^i = \left.\frac{\partial g(\boldsymbol{\theta})}{\partial \theta^i}\right|_{\boldsymbol{\theta}=0}.$$

  The bracket operation is simply the matrix commutator, which we know satisfies the axioms.
- The nontrivial step is to show closure under the bracket. Consider two paths

  $$g_i(t) = 1 + X_i t + W_i t^2 + O(t^3)$$

  which satisfy

  $$g_1(t) g_2(t) = 1 + (X_1 + X_2)t + (X_1 X_2 + W_1 + W_2)t^2 + O(t^3)$$

with a similar expression for $g_2(t)g_1(t)$. Then $h(t) = g_1^{-1}(t)g_2^{-1}(t)g_1(t)g_2(t)$ obeys

$$g_1(t)g_2(t) = g_2(t)g_1(t)h(t)$$

and hence we have

$$h(t) = 1 + [X_1, X_2]t^2 + O(t^3).$$

Then $h(\sqrt{t})$ is a curve with tangent vector $[X_1, X_2]$, as desired. We see the Lie bracket is the infinitesimal version of the commutator of group elements.

We now give some examples of matrix Lie algebras. Note that all these examples are real Lie algebras even though the matrices are complex; the dimension are real dimensions. A complex Lie algebra would instead give us a complex Lie group, which lives on a complex manifold; we will not consider such objects here.

- For $GL(n, \mathbb{F})$, the Lie algebra is $\mathrm{Mat}(n, \mathbb{F})$ since the determinant is continuous. Restricting to $SL(n)$ restricts the Lie algebra to contain matrices with zero trace, because

$$\det \exp X = \exp \operatorname{tr} X.$$

  This provides an easy way to compute $\dim GL(n, \mathbb{F})$, as it is equal to $\dim \mathrm{Mat}(n, \mathbb{F})$.

- For $O(n)$, we have $R^T R = 1$ and setting $R = 1 + X$ gives $X^T = -X$. Note that the Lie algebra of $SO(n)$ is the same, because antisymmetric matrices are automatically traceless.

- Similarly, for $U(n)$, we get $X^\dagger = -X$. We restrict to $SU(n)$ by requiring $\operatorname{tr} X = 0$, which counts as a single constraint since the trace of an anti-Hermitian matrix is imaginary.

- We consider the structure of $\mathfrak{su}(2)$ in detail. One basis is

$$T^a = -\frac{i}{2}\sigma_a.$$

  This differs from the angular momentum by a factor of i, since it is anti-Hermitian. Now

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc}\sigma_c$$

  which implies that in our basis, $f^{ab}{}_c = \epsilon_{abc}$.

- Similarly, we may define the following basis for $\mathfrak{so}(3)$,

$$T^1 = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}, \quad T^2 = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad T^3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

  where has the properties

$$(T^a)_{bc} = -\epsilon_{abc}, \quad [T^a, T^b] = \epsilon_{abc} T^c$$

  which establishes that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

Next we define some maps on the Lie group and Lie algebra.

- Left translation L_h is a diffeomorphism of G with corresponds to multiplying by h on the left,
$$L_h g = hg.$$
Taking the differential gives a map $L_h^*\colon T_g \to T_{hg}$.
- For matrix Lie groups, L_h^* is implemented by matrix multiplication. That is, for $X \in \mathfrak{g}$,
$$L_h^*(X) = hX \in T_h.$$
To prove this explicitly, note that a curve $g(t) = I + tX + O(t^2)$ maps by L_h to the curve $h(t) = h + thX + O(t^2)$ which passes through h with tangent vector hX.
- Given $X \in \mathfrak{g}$, we may define the left-invariant vector field V by
$$V(g) = L_g^*(X).$$
Conversely, if a manifold has a Lie group structure, this means it has a global field of frames, which is a strong constraint; it means that S^2 cannot be a Lie group.
- An integral curve $g(t)$ of the left-invariant vector field V associated with $X \in \mathfrak{g}$ satisfies
$$\frac{dg(t)}{dt} = V(g(t)) = L_{g(t)}^*(X) = g(t)X, \quad g(t) = \exp(tX)$$
where the exponential for a matrix Lie group is defined as a series.
- The curve $\exp(tX)$ defines a one-parameter subgroup with
$$\exp(t_1 X)\exp(t_2 X) = \exp(t_2 X)\exp(t_1 X) = \exp((t_1 + t_2)X)$$
by expanding out the series. Note that the subgroup might be isomorphic to either $\mathbb{R}$ or $U(1)$.
- Setting $t = 1$, we have a map $\exp\colon \mathfrak{g} \to G$ which is bijective in a neighborhood of the identity. It is not injective if G has a $U(1)$ subgroup, and it is not surjective if G is not connected. Somewhat more subtly, the exponential map is not necessarily surjective if G is not compact.
- Writing $g_X = \exp(X)$, the Baker-Campbell-Hausdorff theorem states
$$g_X g_Y = g_Z, \quad Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots.$$
Thus the group multiplication law is encoded in the Lie bracket.

Example. The group $SU(2)$ is a double cover of $SO(3)$ by the map
$$d\colon SU(2) \to SO(3), \quad d(A)_{ij} = \frac{1}{2}\operatorname{tr}(\sigma_i A \sigma_j A^\dagger)$$
where $\ker d = \{\pm I\}$, as can be checked by using the identity
$$\sum_i (\sigma_i)_{\alpha\beta}(\sigma_i)_{\delta\gamma} = 2\delta_{\alpha\gamma}\delta_{\delta\beta} - \delta_{\alpha\beta}\delta_{\delta\gamma}.$$
The inverse of this map is
$$A = \pm\frac{I_2 + \sigma_i R_{ij}\sigma_j}{2\sqrt{1 + \operatorname{tr} R}}.$$

3.3 $*$ Representations

We now define representations for Lie group and Lie algebras. We will be casual with notation, writing equations that only work for matrix Lie groups.

- A representation $D(g)$ of a Lie group is a homomorphism $D\colon G \to \mathrm{Mat}_n(\mathbb{F})$, or alternatively an action of G on V by linear transformations. A representation of a Lie algebra is a linear map $d\colon \mathfrak{g} \to \mathrm{Mat}_n(\mathbb{F})$ which preserves the bracket. Here, n is the dimension of the representation, which may be different from the dimension m of the group and algebra.

- Every representation D of G corresponds to a representation d of $\mathfrak{g}$ by
$$d(X) = \frac{d}{dt} D(g(t)) \bigg|_{t=0}$$
for a curve $g(t)$ passing through the identity at $t = 0$ with tangent vector X. That is,
$$D(1_m + tX + O(t^2)) = 1_n + td(X) + O(t^2).$$

- Linearity is easy to see, but checking the bracket is preserved is a bit trickier. Consider curves $g_1(t)$ and $g_2(t)$ as defined above. Then as before,
$$h(t) = g_1^{-1}(t)g_2^{-1}(t)g_1(t)g_2(t) = 1_m + t^2[X_1, X_2] + O(t^3).$$
Taking the representation of both sides,
$$D(h(t)) = 1_n + t^2 d([X_1, X_2]) + O(t^3).$$
On the other hand, we can also write
$$D(h(t)) = D(g_1^{-1}(t))D(g_2^{-1}(t))D(g_1(t))D(g_2(t)) = 1_n + t^2[d(X_1), d(X_2)] + O(t^3)$$
by the same logic, giving the result.

- Similarly, given a representation $d(X)$, we can define $D(g) = \exp d(X)$ where $g = \exp X$. Since the exponential map is not generally bijective, this may not be a representation of G, but it is at least a representation 'locally'. To verify it is a representation, note that if $g_1 = \exp(X_1)$ and $g_2 = \exp(X_2)$,
$$\begin{aligned} D(g_1 g_2) &= \exp\left(d\left(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \ldots\right)\right) \\ &= \exp\left(d(X_1) + d(X_2) + \frac{1}{2}[d(X_1), d(X_2)] + \ldots\right) \\ &= \exp(d(X_1))\exp(d(X_2)) = D(g_1)D(g_2) \end{aligned}$$
where we applied the Baker-Campbell-Hausdorff theorem twice.

- The representation space of a representation of a Lie algebra $\mathfrak{g}$ is also called a $\mathfrak{g}$-module. In general, for a ring R, an R-module is simply a vector space where the scalars are elements of R. A Lie algebra is a ring where the operation is the bracket, and 'scalar multiplication' by X in the $\mathfrak{g}$-module is application of $d(X)$.

We now give some basic examples of Lie group and Lie algebra representations.

- Starting with the trivial representation $D(g) = I$, we get the trivial representation $d(X) = 0$.
- Starting from the fundamental representation $D(g) = g$, for an n-dimensional matrix Lie group, we have the fundamental representation $d(X) = X$, which is also n-dimensional.
- The adjoint representation of G has representation space $\mathfrak{g}$, and

$$D(g)X = (\operatorname{Ad} g)X = gXg^{-1}.$$

To show the action of $\operatorname{Ad} g$ is closed, note that if X is the tangent vector of a curve $h(t)$, then $(\operatorname{Ad} g)X$ is the tangent vector of a curve $gh(t)g^{-1}$.

- The adjoint representation of G corresponds to the adjoint representation of $\mathfrak{g}$,

$$d(X) = \mathrm{ad}_X, \quad \mathrm{ad}_X(Y) = [X, Y].$$

whose dimension is also $\dim \mathfrak{g}$. Concretely, expand in a basis T^a of $\mathfrak{g}$, giving

$$\mathrm{ad}_X(Y) = [X, Y] = X_a Y_b [T^a, T^b] = X_a Y_b f^{ab}{}_c T^c.$$

Therefore, we have

$$[\mathrm{ad}_X(Y)]_c = (X_a f^{ab}{}_c) Y_b, \quad (\mathrm{ad}_X)^b_c = X_a f^{ab}{}_c.$$

- If we didn't know how the adjoint representation was derived, we would have to check that

$$[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}$$

where $[\cdot, \cdot]$ means the commutator on the left and the Lie bracket on the right. This follows directly from the Jacobi identity, so one might say the point of the Jacobi identity for an abstract Lie algebra is to ensure that the adjoint representation still exists.

- The kernel of the adjoint representation is the center of $\mathfrak{g}$, so it is faithful if $\mathfrak{g}$ is semi-simple.

3.4 Integration

In this section, we define a measure for integration over a Lie group.

- Previously, we showed the orthogonality of characters for a finite group. We would like to do the same for a continuous group, but this requires replacing the sum $\sum_g$ with an integration measure $\int d\mu(g)$.
- In the case of a finite group, the crucial step was being able to 'shift the sum',

$$A = \sum_g D^\dagger(g) X D(g) \text{ satisfies } D^\dagger(g) A D(g) = A.$$

For a Lie group, we have

$$D^\dagger(g) \left(\int d\mu(g')\, D^\dagger(g') X D(g') \right) D(g) = \int d\mu(g')\, D^\dagger(g'g) X D(g'g) = \int d\mu(g'g^{-1})\, D^\dagger(g') X D(g').$$

Then the analogous requirement is $d\mu(g) = d\mu(g')$ for any two group elements.

- More concretely, suppose we take a small patch of the group manifold. The patch can be moved around by multiplication by a group element, and we demand this leaves the measure of the patch invariant.
- The 'volume' of a group is $\int d\mu(g)$. Unlike the cardinality of a group, it has no canonical normalization; it is finite when the group is compact.
- Expanding in terms of coordinates, we are requiring

$$d\mu(g) = dx^1 \ldots dx^n \rho(x^1, \ldots, x^n), \quad dx^1 \ldots dx^n \rho(x^1, \ldots, x^n) = dx'^1 \ldots dx'^n \rho(x'^1, \ldots, x'^n)$$

  where the primed quantities are defined by group multiplication.
- As a first example, consider $SO(2)$. If we parametrize by θ, then the group multiplication law $R(\theta)R(\theta') = R(\theta + \theta')$ means that a segment of length $\delta\theta$ is mapped to a segment of length $\delta\theta$. Then we have $\rho(\theta) = \rho(\theta')$ for any two angles, so $d\mu(\theta) = d\theta$. In this case, orthogonality of characters recovers Fourier series.
- Next, consider the restricted Lorentz group $SO(1,1)$. If we parametrize by rapidity φ, we have $L(\varphi)L(\varphi') = L(\varphi + \varphi')$ so $d\mu(\varphi) = d\varphi$. On the other hand, if we parametrize by velocity, we have to calculate. We have

$$L(u)L(v) = L(v'), \quad v' = \frac{v+u}{1+vu}.$$

  Now consider the segment $[v, v + dv]$. A direct computation gives

$$dv' = \frac{1-u^2}{(1+uv)^2} dv, \quad \rho(v) = \frac{1-u^2}{(1+uv)^2} \rho\left(\frac{v+u}{1+uv}\right).$$

  Finally, setting $v = 0$ gives $\rho(u) = \rho(0)/(1-u^2)$. We could also have found this by changing variables from rapidity, picking up a Jacobian. In both cases, the group volume is infinite.
- The analogue of the great orthogonality theorem is called the Peter-Weyl theorem, and it applies to all compact Lie groups. It contains as special cases many of the orthogonality and completeness results we've used.

As an application, we consider the extended example of $SO(3)$.

- We recall that elements $SO(3)$ can be parametrized as $R(\mathbf{n}, \psi)$ where ψ is the rotation angle. Hence $SO(3)$ is a three-dimensional ball with opposite points on the boundary identified. The equivalence classes are rotations with the same ψ.
- Now consider computing the character of the spin–j representation of $SO(3)$ on the equivalence class with angle ψ. It is convenient to choose $\mathbf{n} = \mathbf{z}$, so that

$$R(\mathbf{z}, \psi)|jm\rangle = e^{i\psi J_3}|jm\rangle = e^{im\psi}|jm\rangle.$$

  Then the character is

$$\chi(j, \psi) = \sum_{m=-j}^{j} e^{im\psi} = \frac{\sin(j+1/2)\psi}{\sin(\psi/2)}.$$

- Choosing coordinates (θ, φ, ψ), the measure on $SO(3)$ has the form

$$d\mu(g) = d\Omega d\psi f(\psi), \quad d\Omega = d\theta d\varphi \, \sin\theta$$

by rotational invariance. For small ψ, we expect $f(\psi) \propto \psi^2$ since the group is 'locally Euclidean'.

- The trick is to consider rotations next the identity, expanded as

$$R(\delta, \epsilon, \sigma) = I + \begin{pmatrix} 0 & -\delta & \sigma \\ \delta & 0 & -\epsilon \\ -\sigma & \epsilon & 0 \end{pmatrix} = I + A.$$

By direct multiplication, we have $R(\delta, \epsilon, \sigma) R(\delta', \epsilon', \sigma') \approx R(\delta + \delta', \epsilon + \epsilon', \sigma + \sigma')$ up to second order terms. Then the measure is $d\delta d\epsilon d\sigma$.

- Next, we transport this result across the group by multiplying it with a finite rotation and seeing how it changes (θ, φ, ψ). We have

$$R(\mathbf{n}, \psi') = R(\mathbf{z}, \psi) R(\delta, \epsilon, \sigma), \quad R(\mathbf{z}, \psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A long calculation shows that

$$n_1 = \frac{\epsilon \sin\psi + \sigma(1 + \cos\psi)}{2\sin\psi}, \quad n_2 = (-\sigma \sin\psi + \epsilon(1 + \cos\psi))(2\sin\psi), \quad n_3 = 1, \quad \psi' = \psi + \delta$$

to first order.

- Switching back to our original parametrization, we have

$$d\Omega d\psi = d\epsilon d\sigma \left(\frac{\partial(n_1, n_2)}{\partial(\epsilon, \sigma)} \right) d\delta$$

where the factor in parentheses is a Jacobian, and we only need to evaluate a 2×2 Jacobian because δ only affects ψ. We thus find

$$d\mu(g) = d\Omega d\psi \, \sin^2(\psi/2)$$

which indeed is proportional to ψ^2 for small ψ. Thus the integral of a class function $F(g)$ is

$$\int d\mu(g) \, F(g) = \int_0^\pi d\psi \, \sin^2(\psi/2) F(\psi).$$

- Using this result, we can check character orthogonality,

$$\int d\mu(g) \, \chi(k, \psi)^* \chi(j, \psi) = \frac{\pi}{2} \delta_{jk}.$$

We can also run this argument in reverse; character orthogonality can be used to find the measure. We can also decompose tensor products of representations, where the characters multiply, reproducing the usual Clebsch-Gordan decomposition.

Note. The general procedure works as follows. For a matrix Lie group, consider some parametrization $A(\xi)$ where the coordinates are ξ and $A \in G$. Take a basis J_α of $\mathfrak{g}$, and let

$$A^{-1}\frac{\partial A}{\partial \xi_i} = J_\alpha \widetilde{A}(\xi)_i^\alpha.$$

Then the weight function for the measure is

$$\rho(\xi) = \det \widetilde{A}(\xi).$$

To understand this, note that the vectors $\partial A/\partial \xi_i$ form a parallelopiped at $A(\xi)$ and left multiplying by A^{-1} moves it to the identity. Here its volume can be compared to the parallelopiped formed by the Lie algebra elements J_α, with the conversion factor being the Jacobian of the transformation between them. We have essentially done this procedure above.

Finally, we turn to the final example of $SU(2)$.

- As we've seen, $SU(2)$ is geometrically the sphere S^3 by the parametrization

  $$U = t + i\mathbf{x}\cdot\boldsymbol{\sigma}, \quad t^2 + x^2 = 1.$$

  By symmetry, the measure on $SU(2)$ is just the rotationally symmetric measure on the sphere.
- To find the measure on the sphere, we define $t = \cos\zeta$ and parametrize $\mathbf{x}/|\mathbf{x}|$ by the usual spherical coordinates. For a fixed value of t, the $\mathbf{x}$ coordinates trace out a sphere of radius $\sin\zeta$. Then the measure is $\sin^2\zeta\, d\Omega d\zeta$.
- To compare this with our earlier result, note that $\psi = 2\zeta$ is the angle of rotation, so the integral of a class function of $SU(2)$ would be

  $$\int_0^{2\pi} d\psi\, \sin^2(\psi/2) F(\psi).$$

  This is identical to our result for $SO(3)$, which makes sense since the two are locally isomorphic, but has double the integration range since $SU(2)$ double covers $SO(3)$.
- We can now apply a 'reality check' to the spin-j representation of $SU(2)$, using

  $$\eta^{(j)} = \frac{1}{|G|}\sum_g \chi^{(j)}(g^2) = \frac{\int d\mu(g)\,\chi(j, 2\psi)}{\int d\mu(g)}.$$

  Carrying out the integral shows that the integer spin representations are real and the half-integer spin representations are pseudoreal.
- Finally, there is a local isomorphism between $SO(4)$ and $SU(2)\times SU(2)$ given by

  $$W \to W' = U^\dagger W V$$

  where W, U, and V are in $SU(2)$, and W and W' are regarded geometrically as points in S^3. Then two elements of $SU(2)$ yield a rotation of the sphere, and it can be shown that any small rotation of the sphere can be written this way.

4 Examples of Representations

4.1 $*$ Representations of $\mathfrak{su}(2)$

Now we find the representations of $\mathfrak{su}(2)$. We begin with some remarks about complexification.

- The standard basis of the Lie algebra is

$$T^a = -\frac{1}{2} i\sigma_a, \quad [T^a, T^b] = f^{ab}{}_c T^c, \quad f^{ab}{}_c = \epsilon_{abc}.$$

  It can also be regarded as the fundamental representation. Note that the representation is complex, because it contains complex matrices like T^z, but the Lie algebra $\mathfrak{su}(2)$ is real. Finally, the representation is real in the physical sense since it is similar to its conjugate.

- In general, we will care about complex representations of real Lie algebras $\mathfrak{g}$ since Hilbert spaces are complex vector spaces. However, it is much simpler to find and classify complex representations of their complexifications $\mathfrak{g}_{\mathbb{C}}$.

- Every complex representation d of $\mathfrak{g}$ extends to a complex representation $d_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ by

$$d_{\mathbb{C}}(X + iY) = d(X) + id(Y).$$

  Conversely, given a representation $d_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ we may define a representation d of $\mathfrak{g}$ by restriction.

- Given d and $d_{\mathbb{C}}$ as defined above, it can be shown that $d_{\mathbb{C}}$ is an irrep of $\mathfrak{g}_{\mathbb{C}}$ if and only if d is an irrep of $\mathfrak{g}$. Thus to classify irreps of $\mathfrak{g}$ it is completely equivalent to classify irreps of $\mathfrak{g}_{\mathbb{C}}$.

- Given a complex Lie algebra, there are multiple ways to restrict to a real Lie algebra, i.e. to take a 'real form'. For example, the complexification of $\mathfrak{su}(n)$ is the set of $n \times n$ traceless complex matrices, which is $\mathfrak{sl}(n, \mathbb{C})$. But this has $\mathfrak{sl}(n, \mathbb{R})$ as a real form, so the representation theory of $\mathfrak{sl}(n, \mathbb{R})$ is the same as that of $\mathfrak{su}(n)$.

We now turn to finding the irreps of $\mathfrak{su}(2)_{\mathbb{C}}$. From this point on we'll suppress the $\mathbb{C}$ subscript, always implicitly working with a complexified Lie algebra.

- It is convenient to work in the Cartan-Weyl basis

$$H = 2iT^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad E_+ = iT^1 + T^2 = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad E_- = iT^1 - T^2 = \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

  The matrices $E_\pm$ are called the raising and lowering operators.

- The commutation relations are

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = H$$

  which can be rewritten as

$$\mathrm{ad}_H(E_\pm) = \pm 2E_\pm, \quad \mathrm{ad}_H(H) = 0.$$

  Therefore, the generators $\{H, E_\pm\}$ are the eigenvectors of $\mathrm{ad}_H \colon \mathfrak{su}(2) \to \mathfrak{su}(2)$. The eigenvalues $\{0, \pm 2\}$ are known as the roots of $\mathfrak{su}(2)$.

- Now consider a finite-dimensional representation R with representation space V and suppose that $R(H)$ is diagonalizable, with
$$R(H)v_\lambda = \lambda v_\lambda.$$
This will always be the case in quantum mechanics, where $R(H)$ is Hermitian. The eigenvalues v_λ are the weights of the representation R, and they are raised and lowered by 2 by $E_\pm$.

- There exists a highest weight Λ with
$$R(H)v_\Lambda = \Lambda v_\Lambda, \quad R(E_+)v_\Lambda = 0.$$
If the representation is irreducible, then v_Λ must be unique, and we must be able to reach all of V by applying E_- repeatedly. Therefore the other eigenvectors are
$$v_{\Lambda-2n} = (R(E_-))^n v_n.$$

- To see when this chain must terminate, note that
$$R(E_+)v_{\Lambda-2n} = R(E_+)R(E_-)v_{\Lambda-2n+2} = (R(E_-)R(E_+) + (\Lambda - 2n + 2))v_{\Lambda-2n+2}.$$
Therefore we have
$$R(E_+)v_{\Lambda-2n} = r_n v_{\Lambda-2n+2}, \quad r_n = r_{n-1} + \Lambda - 2n + 2, \quad r_0 = 0$$
which, by induction, gives the solution
$$r_n = (\Lambda + 1 - n)n.$$
Now, let $\Lambda - 2n$ be the lowest weight, so $R(E_-)v_{\Lambda-2N} = 0$. Then we must have $r_{N+1} = 0$, which implies $\Lambda = N$.

- We have thus shown that finite dimensional irreps R_Λ of $\mathfrak{su}(2)$ are labeled by their highest weight $\Lambda \in \mathbb{Z}$, with weights $S_\Lambda = \{-\Lambda, -\Lambda + 2, \ldots, \Lambda - 2, \Lambda\}$ and $\dim R_\Lambda = \Lambda + 1$.

- Decomplexifying back down to $\mathfrak{su}(2)_\mathbb{R}$, R_0 is the trivial representation, R_1 is the fundamental representation, and R_2 is the adjoint representation.

- We can also decomplexify to $\mathfrak{sl}(2)_\mathbb{R}$. This is easier, as H, E_+, and E_- are all traceless and real and hence already form a basis for $\mathfrak{sl}(2)_\mathbb{R}$.

We now relate our results to the familiar theory of angular momentum in quantum mechanics.

- To establish notation, in quantum mechanics we have
$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_3|jm\rangle = m|jm\rangle, \quad J^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+).$$
Then $J_3 = R(H)/2$, since its eigenvalues are half-integer rather than integer, and $J_\pm = R(E_\pm)$. The spin value j is $\Lambda/2$. We call J^2 a 'quadratic Casimir'. It is not part of the Lie algebra, but useful for classifying the irreps.

- Our irreps $R_\Lambda(X)$ exponentiate to representations of $SU(2)$ by

$$D_\Lambda(A) = \exp R_\Lambda(X), \quad A = \exp X.$$

All of the irreps give representations of $SU(2)$, which in turn yield irreps of $SO(3)$ if $D_\Lambda(I) = D_\Lambda(-I)$. Now note that $-I = \exp(i\pi H)$ and

$$D_\Lambda(-I) = \exp(i\pi R_\Lambda(H)).$$

Plugging in the eigenvalues of $R_\Lambda(H)$, the right-hand side is I for $\Lambda \in 2\mathbb{Z}$ and $-I$ otherwise. Then the former give representations of $SO(3)$, and the latter yield projective representations.

Note. The Casimir operator is not an element of the Lie algebra. Formally, define the universal enveloping algebra (UEA) of a Lie algebra $\mathfrak{g}$ to be the algebra generated by elements of $\mathfrak{g}$ subject to the relation $T^aT^b - T^bT^a = f^{ab}{}_cT^c$. Then Casimir operators are elements of the UEA which commute with all other elements. Representations may be extended from $\mathfrak{g}$ to the UEA in the obvious way, and by Schur's lemma, Casimir operators are represented in irreps by multiples of the identity. The quadratic Casimir is only one of many examples, and we use it to index irreps.

New representations can be built from our irreps.

- Given a representation R of a real Lie algebra $\mathfrak{g}$, the conjugate representation $\overline{X}$ satisfies

$$\overline{R}(X) = R(X)^*.$$

In this case, there is only one irrep for each dimension, so each irrep is its own conjugate.

- The contragradient representation, or dual representation R^* is

$$R^*(X) = -R(X)^T.$$

If R comes from a representation $D(g)$ of a Lie group, the contragradient representation is $D^*(g) = D(g^{-1})^T$. This is just the expression for how covectors transform, so the contragradient representation can be thought of as the analogue of D on the dual space. We won't run into it much because for $SU(n)$, it coincides with the conjugate representation.

- The direct sum of two Lie algebra representations R_1 and R_2 is defined as

$$(R_1 \oplus R_2)(X) = R_1(X) \oplus R_2(X)$$

and their tensor product is defined as

$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X).$$

The reason for this definition is that we want the tensor product representation to be the usual tensor product for Lie group representations, and taking the logarithm of a product yields a sum,

$$\exp(tX_1) \otimes \exp(tX_2) = (1 + tX_1) \otimes (1 + tX_2) + O(t^2) = 1 + t(X_1 \otimes I_2 + I_1 \otimes X_2) + O(t^2).$$

For example, when we consider two particles with spin, the Hilbert space is the tensor product space, but the angular momentum observable is $\mathbf{J}_1 + \mathbf{J}_2$ (with implicit identities), not $\mathbf{J}_1\mathbf{J}_2$.

- It can be shown that the tensor product of finite-dimensional irreps of a simple Lie algebra $\mathfrak{g}$ is always fully reducible. As an example, we explicitly decompose $R_\Lambda \otimes R_{\Lambda'}$. It is useful to again work in the Cartan-Weyl basis, where, for example,

$$(R_\Lambda \otimes R_{\Lambda'})(H) = R_\Lambda(H) \otimes I_2 + I_1 \otimes R_{\Lambda'}(H).$$

  Then we know the weight set is

$$S_{\Lambda,\Lambda'} = \{\lambda + \lambda' | \lambda \in S_\Lambda, \lambda' \in S_{\lambda'}\}$$

  from which we conclude

$$R_\Lambda \otimes R_{\Lambda'} = R_{\Lambda+\Lambda'} \oplus R_{\Lambda+\Lambda'-2} \oplus \cdots \oplus R_{|\Lambda-\Lambda'|}.$$

- Note that the Cartan-Weyl basis is not unique; as we've seen, it corresponds to picking out J_z, and we can just as well pick J_x. But the decomposition of a representation into irreps is unique; changing the basis just changes the preferred basis within each irrep.

4.2 The Heisenberg Algebra

As a second example, we consider the infinite-dimensional algebra of creation and annihilation operators for a bosonic field.

- The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

  and the equal-time canonical commutators are

$$[\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}).$$

- For simplicity, suppose the fields live on the unit circle. Then the modes are the Fourier components,

$$\phi(\mathbf{x}) = a_0 + \sum_{n \neq 0} \frac{1}{n} a_n e^{2\pi i n x}$$

  and applying the equal-time commutators gives

$$[a_n, a_m] = n\delta_{n,-m}.$$

  This is known as the Heisenberg algebra. It splits into three subalgebras: a_0, the creation operators $\{a_n, n \geq 1\}$, and the annihilation operators $\{a_n, n \leq -1\}$ where $a_n^\dagger = a_{-n}$.

- It is convenient to rescale the operators as

$$b_n = \frac{1}{\sqrt{n}} a_{-n}.$$

  Then for every $n > 0$, the operators b_n and $b_n^\dagger$ form a copy of the algebra of creation and annihilation operators for a single harmonic oscillator. Using this observation, we can construct the Fock space starting from a vacuum, assumed to be unique.

This example has a number of common features with the previous example.

- In both cases, we construct a maximal set of commuting operators, $\{H\}$ and $\{a_0\}$, called the Cartan subalgebra. This is an analogue of a complete set of commuting operators in quantum mechanics, but we do not allow operators that are not in the algebra, such as L^2 or $a_n^\dagger a_n$.
- The rest of the algebra splits into two subalgebras, consisting of raising and lowering operators, which are related by conjugation. They are also called creation and annihilation operators, or step operators.
- The representations can be understood by starting from a 'highest weight' or 'vacuum' vector, which is annihilated by the annihilation operators, and applying creation operators. Acting with these operators modify the quantum numbers of the state, i.e. the eigenvalues of the operators in the Cartan subalgebra, also called the weights.
- The dimension of the Cartan subalgebra is called the rank of the Lie algebra; it is the number of quantum numbers we have to work with. So far we have only seen rank one, but in the next section we consider an example with rank two.

4.3 Representations of $SO(n)$

In this section we heuristically explore some of the representations of $SO(3)$, with some of our results generalizing to $SO(n)$.

- The group $SO(n)$ has the vector representation $V^i \to R^{ij}V^j$. Tensors transform in tensor powers of the vector representation, and we may break them into irreps.
- Note that antisymmetric rank 2 tensors transform to antisymmetric tensors. The same goes from symmetric tensors, so the rank 2 tensor representation decomposes as $3+3 = 3+6$. However, the trace is also invariant under rotations (using $R^T = R^{-1}$), so we have the further decomposition $6 = 5+1$, where the 5 contains symmetric traceless tensors.
- This decomposition can be expressed in index notation. For the rank 2 tensor T^{ij}, we have

  $$A^{ij} = T^{ij} - T^{ji}, \quad S^{ij} = T^{ij} + T^{ji}, \quad \tilde{S}^{ij} = S^{ij} - \delta^{ij} S^{kk}/N.$$

  Then the A^{ij} are the antisymmetric representation, in the sense that the vector space spanned by the A^{ij} (for all 9 values of (i,j)) is the antisymmetric irrep.
- The exact same reasoning holds for the rank 2 tensor in $SO(n)$.
- The fact that the elements of $SO(n)$ are rotations means that δ is an 'invariant symbol', i.e.

  $$\delta^{ij} R^{ik} R^{jl} = \delta^{kl}.$$

  Similarly, we generally have

  $$\epsilon^{ij\dots n} R^{ip} R^{jq} \cdots R^{ns} = \epsilon^{pq\dots s} \det R$$

  so that ϵ is an invariant symbol. We call these objects symbols instead of tensors because they are defined by their components.

- In general, contracting tensors with invariant symbols yields other tensors. For example, the 'dual tensor' of A^{ij} is $B^k = \epsilon^{ijk} A^{ij}$. It is a vector, since

$$B^k \to \epsilon^{ijk} R^{ip} R^{jq} A^{pq} = \epsilon^{ijk'} R^{ip} R^{jq} R^{k'r} R^{kr} A^{pq} = \epsilon^{pqr} R^{kr} A^{pq} = R^{kr} B^r.$$

  Moreover, contraction with invariant tensors projects representations to subrepresentations. For example, for T^{ij}, contraction with δ^{ij} gives the 1 and contraction with ϵ^{ijk} gives the 3. Also note that contraction with ϵ always removes symmetric parts.

- Contraction with ϵ yields strong constraints on the irreps of $SO(3)$ that appear in the tensor product representations, because we can always trade two antisymmetric indices for a single index. We claim the only new irrep for rank n is the traceless totally symmetric rank n tensor.

  We have already shown this for $n \leq 2$. Now consider T^{ijk}. It splits into the representations $T^{[ij]k}$ and $T^{\{ij\}k}$, and the former is just the rank 2 antisymmetric tensor. Now note that

$$3T^{\{ij\}k} = (T^{\{ij\}k} + T^{\{jk\}i} + T^{\{ki\}j}) + (T^{\{ij\}k} - T^{\{jk\}i}) + (T^{\{ij\}k} - T^{\{ki\}j})$$

  where the first term is totally symmetric, and the other terms are antisymmetric in ki and kj respectively. Thus the only new representation is totally symmetric. Finally, given a totally symmetric tensor S^{ijk} we can always remove all of its traces,

$$\tilde{S}^{ijk} = S^{ijk} - \frac{1}{N+2}(\delta^{ij} S^{hhk} + \delta^{ik} S^{hhj} + \delta^{jk} S^{hhi})$$

  leaving only the new irrep, as claimed. A similar proof holds for general n.

- We now count the dimension of these representations. A totally symmetric rank n tensor has

$$\binom{n+2}{2}$$

  degrees of freedom. By symmetry, there is only one independent trace to remove, which is a totally symmetric rank $n-2$ tensor, so the dimension is

$$\binom{n+2}{2} - \binom{n}{2} = 2n+1.$$

  These are, in fact, all of the irreps of $SO(3)$.

- A similar argument holds for $SO(2)$. In this case, the dimension counting works out as $(n+1) - (n-1) = 2$, so all of the irreps are two-dimensional. This is not in contradiction with the fact that the irreps of $U(1) \cong SO(2)$ are all one-dimensional, because we are only considering real representations.

- For $SO(4)$, contraction with ϵ does not decrease the rank, while for $SO(5)$ and higher contraction increases the rank. Hence the tricks above fail, and the irreps can have a more complex symmetry structure. They can be enumerated using Young tableux.

We now make some extra remarks.

- Consider the representation of totally antisymmetric rank n tensors of $SO(2n)$. This representation is mapped to itself by duality; hence it breaks into two irreps, containing 'self-dual' and 'anti self-dual' tensors.

- Consider an irrep of a group G. If we restrict to a subgroup $H \subset G$, then the irrep will generally not be an irrep in H. For example, the vector representation of $SO(4)$ splits into $4 = 3 + 1$ when restricting to $SO(3)$, corresponding to the splitting of space and time. The antisymmetric rank 2 tensor splits into $6 = 3 + 3$, corresponding to the splitting of $F^{\mu\nu}$ into $\mathbf{E}$ and $\mathbf{B}$. The symmetric rank 2 tensor splits into $9 = 5 + 3 + 1$.

- The adjoint representation of a Lie group G has representation space $\mathfrak{g}$, where $g \in G$ acts by conjugation. For $SO(n)$, the adjoint representation is simply the antisymmetric tensor representation, as
$$A^{ij} \to R^{ip} R^{jq} A^{pq} = (RAR^{-1})^{ij}.$$
For $SO(3)$ only, the adjoint representation coincides with the vector representation. That is, angular momentum (as a physical quantity) is a vector only in three dimensions.

- Since we are working with $SO(n)$ rather than $O(n)$, we have neglected the difference between vectors and axial vectors, and so on. Since the metric is Euclidean, we haven't bother with raising or lowering indices.

Example. Decomposing $P^{ijk} = S^{ij}T^k$ where S is symmetric and traceless, in $SO(3)$. First, construct the symmetric tensor
$$U^{ijk} = S^{ij}T^k + S^{jk}T^i + S^{ki}T^j.$$
This is a totally symmetric rank 3 tensor, so it decomposes into a traceless part and a trace; the trace yields the vector representation. The other degrees of freedom are in the antisymmetric part
$$V^{i\ell} = S^{ij}T^k \epsilon^{jk\ell}.$$
The tensor V is neither symmetric nor antisymmetric, but it is traceless. Then it decomposes into a symmetric traceless and antisymmetric part; the latter is just the vector we have already found. Hence we have shown $5 \times 3 = 7 + 5 + 3$.

More generally, suppose we multiply two symmetric traceless tensors, with j and j' indices. Then we can construct a symmetric traceless tensor with $j + j'$ indices, as shown above. The remainder of the degrees of freedom are given by contracting with ϵ^{ijk}, which leaves $j + j' - 1$ indices. We can then take out the symmetric traceless part again and repeat the procedure, so
$$j \times j' = (j + j') + (j + j' - 1) + \ldots + |j - j'|.$$
This is simply the usual Clebsch-Gordan decomposition. Note that we have switched notation so that j is really an irrep of dimension $2j + 1$. However, we'll usually stick to the convention where the name indicates the dimension.

4.4 Representations of $SU(n)$

Next, we extend our results by analogy to $SU(n)$.

- The fundamental representation transforms as
$$\psi^i \to \psi'^i = U^{ij}\psi^j.$$
By analogy with our earlier work, we consider tensors $\varphi^{i_1 \ldots i_n}$.

- For example, the totally symmetric tensor φ^{ijk} of $SU(3)$ has 10 degrees of freedom (corresponding to the baryon decuplet). At this point one might take out the trace, but the trace does not transform correctly,
$$\delta^{ij}\varphi^{ijk} \to (\delta^{ij}U^{if}U^{jg})U^{kh}\varphi^{fgh}.$$
The quantity in parentheses is U^TU, which was the identity for $SO(n)$, but nothing here; we need an extra complex conjugation.

- To address this, we introduce the conjugate/antifundamental representation $\psi_i = \psi^{i*}$. Introducing upper and lower indices, we have
$$\psi^i \to U^i{}_j\,\psi^j, \quad \psi_i \to \psi_j (U^\dagger)^j{}_i.$$
We are allowed to contract upper and lower indices together; for instance, $\psi^i\psi_i$ is a scalar.

- Formally, for $SO(n)$, the fundamental representation tensored with itself contains the trivial representation, but in $SU(n)$, we must tensor the fundamental and antifundamental representation. Physically, a color singlet meson is made of a quark and an antiquark.

- We can think of a conjugate representation as living in the dual space of the original representation; our transformation rule for ψ_i is simply the transformation rule for a bra.

- As a result, we can consider tensors with arbitrary mixed rank (r, s). When both r and s are nonzero, we can subtract out traces with δ. Finally, since the elements of $SU(n)$ have determinant one,
$$\epsilon_{i_1 \ldots i_N} U_1^{i_1} \cdots U_N^{i_N} = \epsilon^{i_1 \ldots i_N} U_{i_1}^1 \cdots U_{i_N}^N = 1$$
yielding two more invariant symbols, which allow raising and lowering of indices.

- As an example, the totally symmetric tensor φ^{ijk} considered earlier is irreducible. However, a tensor T_k^{ij} breaks into four irreps. As usual, the symmetric and antisymmetric parts S_k^{ij} and A_k^{ij} (in the upper two indices) form subrepresentations. However, both of these have traces, which form copies of the fundamental representation. The dimensions are
$$N^3 = \left(\frac{1}{2}N^2(N+1) - N\right) + \left(\frac{1}{2}N^2(N-1) - N\right) + N + N.$$
The naming conventions are somewhat more complex. We will usually stick with naming by dimension, adding a star for representations with mostly lower indices.

- When we work with the Lie algebra, we will have several types of indices. The generators are named T^a, where a has $\dim G$ possible values. If we work in an irrep of dimension d, then each of the generators is a $d \times d$ matrix, so we let p and q index over the irrep, e.g.
$$\delta\varphi^p = i\theta^a (T^a)^p{}_q \varphi^q.$$
The φ^p can also be written as traceless tensors with definite symmetry properties, as we did above; in that case the indices on the ranges i and j range from 1 to N. Note that there is no meaning to the upstairs or downstairs placement of the other types of indices.

- The adjoint representation turns out to be the antisymmetric $(1, 1)$ tensor representation φ^i_j, which indeed has dimension $N^2 - 1$.

Example. The irreps of $SU(2)$. We can use ϵ^{ij} to raise any downstairs indices, so all the irreps have solely upstairs indices. Furthermore, the irreps must be totally symmetric, because any antisymmetric part can be projected out by contraction with ϵ_{ij}, reducing it to a lower rank tensor. Hence the irreps are totally symmetric tensors with n upper indices, which have dimension $n+1$.

We can go further and use this to find the representation matrices $D^s(R)$. The fundamental representation is a two-element spinor, which is multiplied by $D^{1/2}$. Higher irreps are symmetrized tensor powers of that spinor, and a rank n tensor transforms with a factor of $D^{1/2}$ on each index. Hence it is not surprising that the d-functions $d^j(\beta)_{mm'}$ are polynomials in $\cos(\beta/2)$ and $\sin(\beta/2)$.

Note that the antifundamental representation does not appear here; it is similar to the fundamental representation, which turns out to be pseudoreal. Explicitly, we have

$$\sigma_2 \sigma_a^* \sigma_2 = -\sigma_a$$

so the change of basis matrix is σ_2 itself. Since σ_2 is antisymmetric, the representation is pseudoreal.

Note. Consider a tensor product $j \otimes j'$. The resulting $j + j'$ irrep is symmetric, because its highest J_z state is symmetric and symmetry is preserved by lowering. Then the $j + j' - 1$ irrep is antisymmetric, since its highest J_z is determined by orthogonality. This pattern continues, with symmetry alternating between irreps.

Note. Naively, one might say that $U(N) = SU(N) \times U(1)$. However, this is incorrect because in general, N elements of the form $e^{i\theta} I$ are in $SU(N)$. The actual relationship is

$$U(N) = (SU(N)/\mathbb{Z}_N) \times U(1).$$

When one says the Standard Model gauge group is $SU(3) \times SU(2) \times U(1)$, one is really talking about the Lie algebra; there are several possibilities for the Lie group and the correct one is unknown.

Next, we turn to the representations of $SU(3)$.

- In this case, we have the invariant symbols ϵ^{ijk} and ϵ_{ijk}, and the fundamental and antifundamental representations are not equivalent.
- We claim that all irreps are traceless (m,n) tensors that are totally symmetric in both the upper and lower indices. The claim is obvious for rank $r = m + n = 1$. For $r = 2$, the antisymmetric tensors φ^{ij} and φ_{ij} are equivalent to vectors, so they give nothing new, and we may subtract out the trace of φ^i_j.
- For the case $r = 3$, consider the tensor φ^{ijk}. The antisymmetric part $\varphi^{[ij]k}$ can be reduced to rank $r = 2$. The symmetric part $\varphi^{(ij)k}$ can be split into a totally symmetric part and antisymmetric tensors in ki and kj, as shown for $SO(3)$, so the new irrep is totally symmetric.
- Next, we find the dimension of the (m,n) irrep. Without accounting for the traceless condition, the dimension is
$$\binom{n+2}{2}\binom{m+2}{2}.$$
Now, the trace of a totally symmetric (m,n) tensor is a totally symmetric $(m-1, n-1)$ tensor, giving
$$\binom{n+2}{2}\binom{m+2}{2} - \binom{n+1}{2}\binom{m+1}{2} = \frac{1}{2}(m+1)(n+1)(m+n+2).$$

- A few of the low-dimensional irreps are

$$(1,0)=3, \quad (0,1)=3^*, \quad (1,1)=8, \quad (2,0)=6, \quad (3,0)=10, \quad (2,1)=15, \quad (2,2)=27.$$

  Many of these numbers play a role in the Eightfold Way.

Next, we infer the multiplication rule for irreps of $SU(3)$ by example.

- Note that $(1,0)\times(0,1)$ is a general $(1,1)$ tensor. Subtracting its trace gives

$$(1,0)\times(0,1)=(1,1)+(0,0), \quad 3\times 3^*=8+1.$$

- Next, $(1,0)\times(1,0)$ is a general $(2,0)$ tensor, so

$$(1,0)\times(1,0)=(2,0)+(0,1), \quad 3\times 3=6+3^*$$

  where the two terms on the right are simply the symmetric and antisymmetric parts.

- Next, consider $(1,0)\times(2,0)$, i.e. the tensor $\psi^i\varphi^{jk}$. Consider the antisymmetric and symmetric parts in ij. The antisymmetric part gives a $(1,1)$ tensor which is automatically traceless. The symmetric part is symmetric in all three indices, so

$$(1,0)\times(2,0)=(3,0)+(1,1), \quad 3\times 6=10+8.$$

  Therefore $3\times 3\times 3=10+8+8+1$, which we will see applied in the Eightfold Way.

- Finally, consider $(1,1)\times(1,1)$. The tensor $\psi^i_j\chi^k_\ell$ has two distinct traces,

$$\psi^i_j\chi^j_\ell, \quad \psi^i_j\chi^k_i.$$

  These yield two copies of $(1,1)$ plus a copy of $(0,0)$, since their traces are equal. Now consider the traceless part $T^{ik}_{j\ell}$. It can be shown that $T^{[ik]}_{j\ell}$ is automatically symmetric in j and ℓ, providing a $(0,3)$. Similarly antisymmetrizing $j\ell$ gives a $(3,0)$ and the symmetric remainder is $(2,2)$, for

$$(1,1)\times(1,1)=(2,2)+(3,0)+(0,3)+(1,1)+(1,1)+(0,0).$$

  This computation contains all the ideas necessary for the general case.

Example. Consider the three-dimensional harmonic oscillator. Such a system has an $SU(3)$ symmetry; the generators take the form

$$Q_\alpha = a_k^\dagger [T_\alpha]_{k\ell} a_\ell, \quad T_\alpha = \frac{\lambda_\alpha}{2}.$$

The creation operators transform in the fundamental representation while the lowering operators transform in the antifundamental. Note that the $SU(2)$ rotational symmetry is a subset; it is generated by

$$L_3 = 2Q_2, \quad L_1 = 2Q_7, \quad L_2 = -2Q_5.$$

The $SU(3)$ symmetry explains additional degeneracy that the $SU(2)$ symmetry does not. For example, all six $n=2$ states are degenerate, even though they split up as $5+1$ under $SU(2)$.

Example. A more complicated example. Consider two three-dimensional harmonic oscillators. If they are uncoupled, we have an $SU(3) \times SU(3)$ symmetry. Now let

$$H^{\text{int}} = \lambda a_k^\dagger b_k^\dagger a_\ell b_\ell.$$

This interaction is designed so it commutes with

$$Q_\alpha = a_k^\dagger [T_\alpha]_{k\ell} a_\ell - b_k^\dagger [T_\alpha^*]_{k\ell} b_\ell$$

so an $SU(3)$ symmetry remains. Here, the $a^\dagger$'s transform in the 3 and the $b^\dagger$'s transform in the $\overline{3}$. Then we have $[Q_\alpha, a_\ell b_\ell] = 0$ because removing a red quark and red antiquark conserves color. The Hamiltonian commutes with the number operators, so the energy eigenstates have definite number. In particular, the highest weight state of an (n, m) irrep is

$$(a_1^\dagger)^n (b_3^\dagger)^m (\mathbf{a}^\dagger \cdot \mathbf{b}^\dagger)^k |0\rangle$$

for any k, and lowering generates degenerate states.

4.5 The Symplectic Groups

The symplectic groups are the least familiar of the matrix Lie groups.

- The real symplectic group $Sp(2n, \mathbb{R})$ contains real $2n \times 2n$ matrices which satisfy

  $$R^T J R = J, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

  That is, they are matrices that preserve the quadratic form J, like how the elements of $O(p, q)$ preserve $\text{diag}(I_p, -I_q)$.

- To count parameters, note that the left-hand side is automatically antisymmetric, so the number of constraints is $(2n)(2n-1)/2$. Then

  $$\dim Sp(2n, \mathbb{R}) = 4n^2 - n(2n-1) = n(2n+1).$$

- Directly from the definition, we see $\det R = \pm 1$. However, it turns out that we automatically have $\det R = 1$. For example, for $n = 1$, the symplectic condition is simply $\det R = 1$, so $Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$.

- The symplectic groups appear in Hamiltonian mechanics. Combining the canonical positions and momenta into a single vector Z, Hamilton's equations are

  $$\frac{dZ_a}{dt} = J_{ab} \frac{\partial H}{\partial Z_b}$$

  and the form of the equations are preserved under any symplectic transformation $Z \to RZ$.

- Similarly, one may define the complex symplectic groups $Sp(2n, \mathbb{C})$, where the definition still contains a transpose rather than a conjugate transpose, $R^T U R = U$.

- Finally, we define the compact symplectic group

$$USp(2n) = Sp(2n, \mathbb{C}) \cap U(2n).$$

This group is sometimes also called $Sp(2n)$ or $Sp(n)$. To count parameters, we write $U = I + iH$ where H is Hermitian, which gives $H^T J + JH = 0$. Then the general form of H is

$$H = \begin{pmatrix} P & W^* \\ W & -P^T \end{pmatrix}, \quad P^\dagger = P, \quad W^T = W, \quad \dim USp(2n) = n(2n+1).$$

Note that H is generally traceless, so $USp(2n) \subset SU(2n)$. In particular, $USp(2) \cong SU(2)$.

- The compact symplectic groups will appear in the Cartan classification; they also are involved when writing down Lagrangians for Majorana spinors.

It is also useful to write an explicit basis for the algebra $\mathfrak{usp}(2n)$.

- Using tensor product notation, e.g. $J = I \otimes i\sigma_2$, we claim that a basis for $\mathfrak{usp}(2n)$ is given by

$$iA \otimes I, \quad S_1 \otimes \sigma_1, \quad S_2 \otimes \sigma_2, \quad S_3 \otimes \sigma_3$$

where A is real antisymmetric and the S_i are real symmetric. This can be seen by directly comparing with our general form above.

- Using this basis, it is easy to check that the algebra closes. For example,

$$[iA \otimes I, S_a \otimes \sigma_a] = i[A, S_a] \otimes \sigma_a = iS'_a \otimes \sigma_a$$

since the commutator of an antisymmetric and symmetric matrix is symmetric. We also have

$$[S_1 \otimes \sigma_1, S'_1 \otimes \sigma_1] = [S_1, S'_1] \otimes I = i(-iA) \otimes I$$

as well as

$$[S_1 \otimes \sigma_1, S_2 \otimes \sigma_2] = iS_1 S_2 \otimes \sigma_3 + iS_2 S_1 \otimes \sigma_3 = iS_3 \otimes \sigma_3.$$

Similarly, it is straightforward to check that the symplectic condition $JHJ = H^T$ is satisfied.

- As an example, note that linear combinations of the generators $I \otimes iA$ and $\sigma_3 \otimes S$ yield $H = \mathrm{diag}(P, -P^T)$ for Hermitian p. This is the $U(n)$ subgroup of $USp(2n)$.

5 Physical Applications

5.1 Isospin

We give a historical account of the development of isospin and some of its successes.

- In 1932, the neutron was discovered and found to have a mass very similar to that of the proton. It was immediately proposed that the neutron and proton form a doublet under an $SU(2)$ symmetry of the strong interaction, called isospin, with the symmetry broken by electromagnetic effects.

- We will refer to both the quantum numbers s and m as the 'spin'. Similarly we refer to both I and I_3 as the isospin.

- In 1935, Yukawa proposed that the nuclear force could be mediated by the exchange of mesons. In 1947, the charged pions $\pi^{\pm}$ were discovered which participated in the processes

$$p \to n + \pi^+, \quad n \to p + \pi^-.$$

  Applying isospin addition, the isospin of the charged pions can be either 0 or 1. Since the charged pions have nearly the same mass, we suppose they are part of an isospin triplet, leading to the prediction of a third pion π^0 which was found in 1950.

- The Gell-Mann Nishijima formula is the empirical result

$$Q = I_3 + \frac{Y}{2}$$

  where Y is the hypercharge, an operator lying outside of $SU(2)$ conserved by strong interactions. For nucleons, $Y = 1$, while for pions $Y = 0$.

- As an example, the deuteron is a bound state of the proton and neutron, and can be produced in the processes

$$p + p \to d + \pi^+, \quad p + n \to d + \pi^0.$$

  Then the isospin of the deuteron is either 0 or 1. In the case of isospin 1, applying the isospin raising and lowering operators implies the existence of p-p and n-n bound states, which are not observed. Hence the deuteron has isospin 0. Since the deuteron has zero orbital angular momentum, it must thus have spin 1 to make the full wavefunction antisymmetric.

- Isospin can also make quantitative predictions. The amplitudes for these two processes are proportional to the Clebsch-Gordan coefficients

$$\left\langle 1, 1 \middle| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle = 1, \quad \left\langle 1, 0 \middle| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}}$$

  respectively. The cross section for the first process is twice as great, as confirmed in 1953.

- As another example, it was found that the cross section for

$$\pi^+ + p \to \pi^+ + p$$

  had a sharp peak, which could be interpreted as the formation of a short-lived particle, or 'resonance', called the N^*. By adding I values, the resonance could have $I = 1/2$ or $I = 3/2$, but since it has $I_3 = 3/2$ it must have $I = 3/2$. Then isospin predicts three additional resonances, which were shortly found. Again, the Clebsch-Gordan coefficients provide simple relationships between the cross sections.

- Finally, we turn to the couplings between the pions and nucleons. The couplings allowed by isospin are the two shown above, plus $p \to \pi^0 + p$ and $n \to \pi^0 + n$. Using the same Clebsch-Gordan symbols as above we find

$$g_{p,\pi^0 p} = g, \quad g_{p,\pi^+ n} = -\sqrt{2}g, \quad g_{n,\pi^- p} = \sqrt{2}g, \quad g_{n,\pi^0 n} = -g.$$

  Therefore the cross sections for pp and nn scattering are equal, $\sigma \propto g^2$. For pn scattering, we can have either an intermediate π^0 or intermediate charged pion (by the u-channel), giving $\sigma \propto (2-1)^2 g^2 = g^2$. Thus all nucleons are interchangable under the strong force as expected.

- Note that there aren't two separate contributions for an intermediate π^+ or intermediate π^-. These two possibilities form a single Feynman diagram, where the virtual particle can have either positive or negative energy.

- Finally, the couplings above can also be found by building an isospin-scalar Lagrangian. The proton and neutron are combined into fields N^i and N_i, while the pions form a traceless tensor ϕ^i_j. Hence the only possible term is $\Delta\mathcal{L} = N_i \phi^i_j N^j$, which gives the couplings above.

We can also formalize isospin using creation and annihilation operators.

- For the proton and neutron, we define

$$|p, \alpha\rangle = a^\dagger_{1/2,\alpha}|0\rangle, \quad |n, \alpha\rangle = a^\dagger_{-1/2,\alpha}|0\rangle$$

  where the $\pm 1/2$ stands for the isospin and α stands for everything else about the state. Since nucleons are fermions, these creation operators anticommute.

- We can also write the isospin generators in terms of these operators,

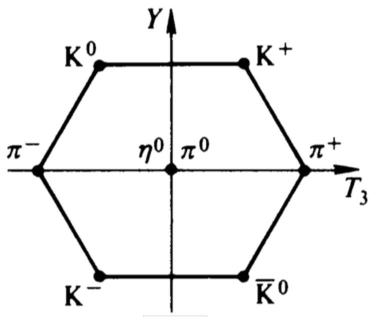
$$T_a = \frac{1}{2} a^\dagger_{m',\alpha} (\sigma_a)_{m'm} a_{m,\alpha}, \quad T_a |m, \alpha\rangle = (J_a^{1/2})_{m'm} |m', \alpha\rangle = \frac{1}{2} (\sigma_a)_{m'm} |m', \alpha\rangle$$

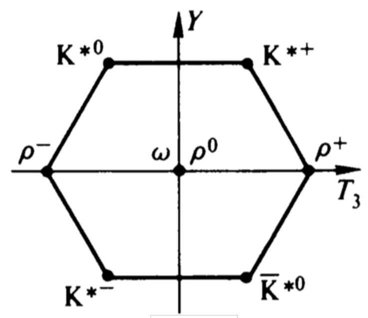
  where there are implicit sums over m, m', and α.

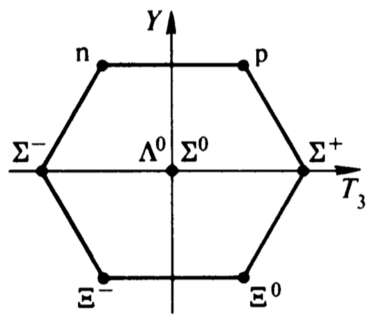
- Note that $|0\rangle$ is an isospin singlet while the $|m, \alpha\rangle$ states form an isospin doublet. Then the operators $a^\dagger_{\pm 1/2,\alpha}$ are an isospin 1/2 tensor operator, e.g.

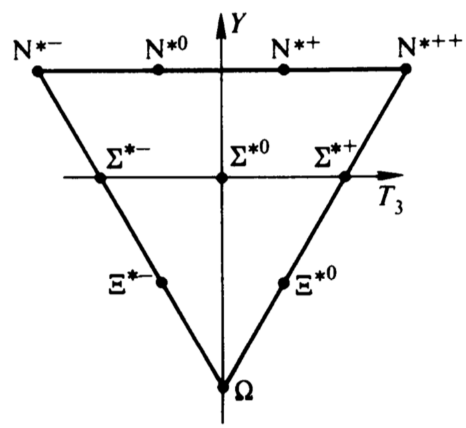
$$[T_a, a^\dagger_{m,\alpha}] = (J_a^{1/2})_{m'm} a^\dagger_{m',\alpha}.$$

  For more species of particles, we simply augment $a^\dagger$ with an extra index and sum over it. Note that whether the creation operators commute or anticommute depends on the particle, but the tensor operator relation above always uses commutators.









5.2 $*$ The Eightfold Way

Next, we introduce the Eightfold Way historically.

- In the early 1950s, a number of new particles were discovered, in particular four pseudoscalar K mesons. Since the three pions were known, there was a total of seven pseudoscalar mesons, which were proposed to form an irrep (since all particles in an irrep automatically have the same parity). Gell-Mann proposed that the irrep was the $(1,1)$ of $SU(3)$, thus predicting an eighth particle, the η^0. (There is also another octet containing vector mesons.)
- Part of the confusion was that the $SU(3)$ symmetry was much more badly broken than isospin; the masses of the K mesons were over three times the masses of the pions, though they were still significantly lighter than any baryons.
- In addition, experimentalists found an isospin triplet of Σ baryons, an isospin doublet of Ξ baryons, and an isospin singlet Δ baryon, which Gell-Mann proposed fit with the neutron and proton in another $(1,1)$ of $SU(3)$.
- Finally, a number of short-lived hadron resonances were known, including the four N^* particles above. They were proposed to form the $(3,0)$ of $SU(3)$, predicting a tenth resonance, the Ω.
- The triality of the (m,n) irrep of $SU(3)$ is defined as $(m-n)(\bmod 3)$. Note that all of the irreps listed above have zero triality. Physically, all of these representations are built from the 3 (containing quarks) and the $\overline{3}$ (containing antiquarks). Thus triality zero ensures that observed

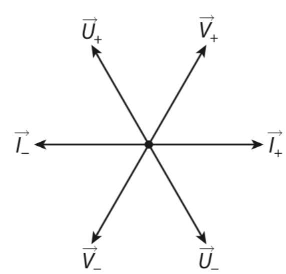
mesons and baryons have integer charge and baryon number. Confinement in QCD ensures that all observable free particles have zero triality.

- By restricting to $SU(2) \subset SU(3)$, an $SU(3)$ irrep breaks into isospin irreps, e.g. $3 \to 2+1$, so we can recover isospin from the Eightfold Way. The isospin and hypercharge obey
$$I_3 = \frac{1}{2}\,\mathrm{diag}(1,-1,0), \quad Y = \frac{1}{3}\,\mathrm{diag}(1,1,-2)$$
so the up and down quark have hypercharge $1/3$, and the strange quark has hypercharge $-2/3$. Then hypercharge generates a $U(1)$ subgroup and we can think of restricting $SU(3)$ to $SU(2) \times U(1)$, so that every isospin irrep is labeled by a hypercharge.

- We write I_{3Y} to indicate an isospin I irrep with hypercharge Y. Then we have
$$3 \to 2_1 + 1_{-2}, \quad 3^* \to 2_{-1} + 1_2.$$
For example, decomposing both sides of $3 \times 3^* = 8 + 1$ yields
$$8 \to 3_0 + 1_0 + 2_3 + 2_{-3}$$
which reproduces the structure of the meson and baryon octets. This is experimentally useful, since isospin is a much more accurate symmetry than the $SU(3)$.

- Finally, assigning the quarks charges of $(2/3, -1/3, 2/3)$ and isospin $(1/2, -1/2, 0)$ recovers the Gell-Mann Nishijima formula. Physically, a formula like this had to work because electromagnetic interactions preserve isospin and hypercharge, so the electromagnetic field has to couple to some combination of I_3 and Y, and we call this combination the charge.

5.3 Roots and Weights for $\mathfrak{su}(3)$



In this section, we investigate the structure of the Lie algebra $\mathfrak{su}(3)$.

- We define the Gell-Mann matrices so that
$$\lambda_1 = \begin{pmatrix} \sigma_x & \\ & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \sigma_y & \\ & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} \sigma_z & \\ & 0 \end{pmatrix}.$$
The pairs λ_4 and λ_5 are analogous to σ_x and σ_y but act on the first and third column, while the pair λ_6 and λ_7 does the same on the second and third column. Finally we have
$$\lambda_8 = \frac{1}{\sqrt{3}}\,\mathrm{diag}(1,1,-2).$$
The matrices are normalized so that $\mathrm{tr}\,\lambda_a\lambda_b = 2\delta_{ab}$.

- For concreteness, we work in the fundamental representation of $\mathfrak{su}(3)$, where $T_a = \lambda_a/2$. All of our results will hold in general, since they depend only on the structure constants of $\mathfrak{su}(3)$.
- The Lie algebra contains three copies of the $\mathfrak{su}(2)$ algebra, defining

$$I_\pm = T_1 \pm iT_2, \quad U_\pm = T_6 \pm iT_7, \quad V_\pm = T_4 \pm iT_5.$$

These are the raising and lowering operators for

$$[I_+, I_-] = 2I_3, \quad [U_+, U_-] = 2U_3, \quad [V_+, V_-] = 2V_3.$$

There are also nontrivial commutators between distinct raising and lowering operators.

- We identify (the z-component of) isospin with T_3 and hypercharge Y with $(2/\sqrt{3})T_8$. Then the step operators change the isospin and hypercharge by

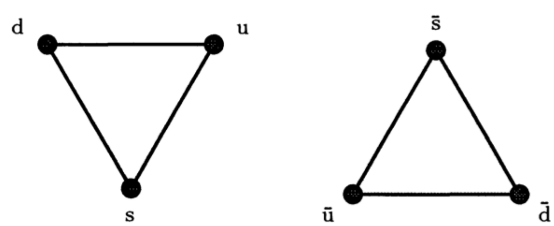
$$I_\pm\colon\ \pm(1,0), \quad U_\pm\colon\ \pm(-1/2,1), \quad V_\pm\colon\ \pm(1/2,1).$$

These vectors are known as the roots of $\mathfrak{su}(3)$.

- The raising and lowering operators have a simple interpretation in the quark model. Here, I_+ replaces a down quark with an up quark, U_+ replaces a strange quark with a down quark, and V_+ replaces a strange quark with an up quark.
- Reverting to the 'mathematical' normalization (i.e. T_8 instead of Y), the root vectors are

$$\pm(1,0), \quad \pm(-1/2,\sqrt{3}/2), \quad \pm(1/2,\sqrt{3}/2).$$

Then all of the roots have equal length, and they form a regular hexagon.



The root diagram gives us some insight into the structure of the Lie algebra and its representations.

- Given a representation, we may simultaneously diagonalize T_3 and T_8. Then the weights are the set of vectors of eigenvalues. Since the raising and lowering operators modify weights by roots, the weights must form part of a hexagonal lattice.
- The roots give us some commutators 'for free'. For example, $[U_+, I_-]$ must be proportional to a step operator with root $U_+ + I_-$, but there is no such root, so the commutator is zero. Similarly, $[U_+, I_+]$ must be proportional to V_+.
- In general, it turns out all irreps of $\mathfrak{su}(3)$ have weight sets that are hexagons built around an equilateral triangle core, with the degeneracy increasing by one every time we go inward by one hexagonal layer.

- The simplest example is the fundamental representation 3 and the antifundamental representation $\overline{3}$, occupied by the quarks and antiquarks, shown above.
- Note that the treatment is completely symmetric between I, U, and V. Previously, we used isospin and Clebsch-Gordan coefficients to relate cross sections involving particles in the same isospin irrep. Similarly we can use "U-spin" to relate particles in the same U-spin irrep.

Note. The Schwinger model. We can get intuition for the algebra for $\mathfrak{su}(n)$ in general using something like the quark model. First, consider two uncoupled harmonic oscillators of equal frequency, creation operators $a_i^\dagger$, and number operators N_i. Alternatively, these could be the two components of a two-dimensional harmonic oscillator. Then the operators

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = N_1 - N_2$$

form an $\mathfrak{su}(2)$ algebra. The states $|n, m\rangle$ decompose into $\mathfrak{su}(2)$ irreps; acting with J_- on the highest weight vector $|n, 0\rangle$ gives the $n+1$-dimensional irrep. This is the Schwinger model of angular momentum. We may physically interpret the state $|n, m\rangle$ as containing $n+m$ identical spin 1/2 particles, symmetrized, with n in the spin up state and m in the spin down state. In the case of isospin, the excitations of the oscillators are up and down quarks, which have isospin $\pm 1/2$. More generally, the Schwinger model works for $\mathfrak{su}(n)$, where the raising and lowering operators create one of n particles and destroy another; the quark model is just the case $n = 3$.

Next, we apply our results to find mass splittings for the four hadron octets/decuplets.

- We begin with the pseudoscalar meson octet, using the wavefunction technique. First, we construct the meson states in terms of the quark states. For example,

  $$|\pi^+\rangle = |u\overline{d}\rangle$$

  which means by isospin lowering that

  $$|\pi^0\rangle = \frac{1}{\sqrt{2}} \left(|u\overline{u}\rangle + |d\overline{d}\rangle\right).$$

  We can then infer $|\eta\rangle$ by orthogonality. None of these manipulations require quarks to actually exist; we can simply think of them at this stage as a useful notational device.
- The meson wavefunctions thus fit into a traceless tensor Φ^i_j so that

  $$\Phi = \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \overline{K}^0 & -2\eta/\sqrt{6} \end{pmatrix}.$$

  Under this notation, we have

  $$\Phi^i_j |i\overline{j}\rangle = \pi^0 |\pi^0\rangle + \ldots + \eta |\eta\rangle.$$

  Here, η is a wavefunction for the η particle that accounts for everything besides flavor, i.e. it contains position degrees of freedom and would contain spin if we weren't dealing with scalars.

- The utility of this notation is that we can write down Lagrangians or Hamiltonians that are $SU(8)$ scalars by just properly contracting all the indices. We are interested in mass terms which are bilinear in Φ, and

$$8 \times 8 = 27 + 10 + 10^* + 8 + 8 + 1.$$

  One has to be careful because matrix and tensor notation conflict. With tensors, the 1 gives

$$\langle \hat{H}_0 \rangle = \overline{\Phi}^i_j \Phi^j_i.$$

  where the bar denotes a bra wavefunction. In matrix notation, taking the bar means taking the adjoint, giving

$$\langle \hat{H}_0 \rangle = \Phi^{\dagger}{}^i_j \Phi^j_i = \Phi^*{}^j_i \Phi^j_i = \sum_{ij} |\Phi^j_i|^2$$

  where, since we're no longer working with tensors, the indices no longer line up. All the masses are the same, as expected. If we worked only with matrices, we could also have constructed the 1 by constructing a valid scalar expression from $\Phi^\dagger$ and Φ, which here would be $\operatorname{tr} \Phi^\dagger \Phi$.

- Next, we break $SU(3)$ while preserving isospin. The situation is simplified because the mesons are each others' antiparticles; since antiparticles have the same mass as the corresponding particles, the perturbing Hamiltonian $\hat{H}'$ must be symmetric under $\Phi \to \Phi^\dagger$. Thus it must contain one of the symmetric pieces of 8×8, leaving only the 27 and 8.

- We *guess* that the 27 does not contribute. Then the $\hat{H}'$ wavefunction H' is in the 8, so it is again a traceless matrix in flavor space, which must commute with isospin and hypercharge. Thus we must have

$$\langle \hat{H}' \rangle = \operatorname{tr}((\Phi^\dagger \Phi + \Phi \Phi^\dagger) H') = \lambda \operatorname{tr}((\Phi^\dagger \Phi + \Phi \Phi^\dagger) T^8).$$

  Since $\hat{H}_0$ and $\hat{H}'$ are described by two parameters and we have three distinct masses (four isospin triplets, but two related by $\widehat{C}$), we expect to get one nontrivial relation.

- For convenience, we may shift H' by the identity so only the H'^3_3 is nonzero. Switching to matrix notation, we have

$$\langle \hat{H}' \rangle \propto \sum_i |\Phi^i_3|^2 + |\Phi^3_i|^2 = |K^-|^2 + |\overline{K}^0|^2 + |K^0|^2 + |K^+|^2 + \frac{4}{3}|\eta|^2$$

  from which we conclude

$$4 m_K = 3 m_\eta + m_\pi.$$

  This is reasonably accurate, but becomes much more accurate if we square all the masses. This is fair, since both expressions hold to first order in the perturbation, and the squared version can be justified to be more accurate by chiral perturbation theory.

- There's an easier way to come to this conclusion: if the entire Hamiltonian is determined by two parameters, those parameters are essentially the up/down quark mass and the strange quark mass. Then the mass splittings are entirely due to the amount of strange quark content in each meson. This isn't true in general but it's a nice shortcut.

Now we consider the other octets and decuplets, which present more challenges.

- In the vector meson octet, the analogous formula is wrong, because the mesons we observe don't have the naive quark content shown above; mesons with the same quantum numbers can mix, and the isospin singlet ω mixes with the ϕ so that one of the physical states are

$$|s\bar{s}\rangle, \quad \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right).$$

  This makes sense because the mixing is induced by the $SU(3)$-breaking perturbation.

- Thus, we have to account for why the η doesn't mix with the η'. The reason is that the $SU(3)$ singlet η' is made much heavier by axial anomaly effects.

- Next, consider the baryon octet. The reasoning is similar, with

$$\Phi = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6} \end{pmatrix}.$$

  The antiparticles sit in a different octet, so we can't use symmetry. Then we have two terms,

$$\langle \hat{H}' \rangle = \lambda_1 \operatorname{tr} \Phi^\dagger \Phi T^8 + \lambda_2 \operatorname{tr} \Phi \Phi^\dagger T^8 \sim \sum_i \lambda_1 |\Phi_3^i|^2 + \lambda_2 |\Phi_i^3|^2.$$

  Since the Hamiltonian has three terms and there are four masses, we again have a relation,

$$2(m_p + m_\Xi) = 3m_\Lambda + m_\Sigma.$$

  This is the Gell-Mann Okubo formula.

- Finally, we consider the baryon decuplet, also called the 'hadron resonances'. Given the above successes we again assume H' is in an 8. But since $10 \times \overline{10} \times 8$ has only one factor of 1, there is only one term. It must be proportional to the hypercharge, so the splittings are uniform,

$$m_{\Sigma^*} - m_\Delta = m_{\Xi^*} - m_{\Sigma^*} = m_\Omega - m_{\Xi^*}.$$

  This was used by Gell-Mann to predict the mass of the Ω.

- Given the quark model, the part of the Hamiltonian that is bilinear in the quarks must be in $3 \times \overline{3} = 8 + 1$, justifying the assumption that H' is in an 8.

- One final example is the computation of the baryon octet magnetic moments. The magnetic moment operator must be proportional to the charge Q, so it is in 8 by the Gell-Mann Nishijima formula. Thus there are two allowed terms, so all of the magnetic moments can be written in terms of the proton and neutron magnetic moments.

6 The Cartan Classification

6.1 $*$ The Cartan-Weyl Basis

We now introduce the Cartan-Weyl basis. In this section, all Lie algebras are implicitly complex.

- We say $X \in \mathfrak{g}$ is ad-diagonalizable if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra containing only ad-diagonalizable elements. They are not unique, but their dimension $r = \dim \mathfrak{h}$ is, and is called the rank of $\mathfrak{g}$. Physically, the rank is the number of independent quantum numbers.
- The adjoint maps of the generators H^i of a Cartan subalgebra $\mathfrak{g}$ commute, as
$$[H^i, H^j] = 0, \quad [\mathrm{ad}_{H^i}, \mathrm{ad}_{H^j}] = \mathrm{ad}_{[H^i, H^j]} = 0.$$
Then they are simultaneously diagonalizable, and the rest of the Lie algebra is spanned by simultaneous eigenvectors E^α, called step operators, which satisfy
$$[H^i, E^\alpha] = \alpha^i E^\alpha$$
where α is an r-dimensional complex vector called a root. Note that α is nonzero, because elements with $\alpha = 0$ would be in the Cartan subalgebra. A basis consisting of the H^i and E^α is called a Cartan-Weyl basis of $\mathfrak{g}$.
- The set of roots Φ of $\mathfrak{g}$ is called the root space. We can think of each root α as an element of the dual space $\mathfrak{h}^*$, so that $\alpha(H)$ is the eigenvalue of E^α under ad_H,
$$[H, E^\alpha] = \alpha^i e_i E^\alpha = \alpha(H) E^\alpha, \quad H = H^i e_i.$$
Here the $e_i \in \mathbb{C}$ are the components of H.

Example. As we've already seen, for $\mathfrak{su}(2)$ we have $r = 1$, where we may take $\mathfrak{g}_0$ to be spanned by $H = 2iT^3 = \mathrm{diag}(1, -1)$. For $\mathfrak{su}(n)$ note that we have the commuting operators
$$(H^i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha(i+1)}\delta_{\beta(i+1)}.$$
To show that this is indeed a Cartan subalgebra, note that the general element in the span of the H^i has the form $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with the λ_i summing to zero. Then if $\mathrm{ad}_H X = \mu X$,
$$(\lambda_\ell - \lambda_m) X_{\ell m} = \mu X_{\ell m}$$
with no summation. The solutions are of the form
$$X = E^{(r,s)}, \quad E^{(r,s)}_{\ell m} = \delta_{\ell r}\delta_{ms}, \quad \mu = \lambda_r - \lambda_s$$
with $r \neq s$. Together, the H^i and the $E^{(r,s)}$ span the algebra, and all of the roots of the $E^{(r,s)}$ are nonzero. Hence the H^i are a Cartan subalgebra and $\mathfrak{su}(n)$ has rank $n-1$.

Note. In the special case of $\mathfrak{su}(n)$, the ad-diagonalizable elements are precisely the diagonal elements. In general, for a matrix Lie algebra, a good guess for a Cartan subalgebra is the elements that are diagonal or almost diagonal, as we'll see in more detail below. The E^α typically have only a few elements on the off-diagonal nonzero.

To make further progress, we introduce the Killing form.

- The Killing form is defined as
$$\kappa(X, Y) = \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y).$$
It is symmetric by the cyclic property of the trace, which is a trace for linear operators on $\mathfrak{g}$, and bilinear, so it is an inner product.

- Explicitly, working in a basis $\{T^a\}$, we have
$$[X, [Y, Z]] = X_a Y_b Z_c [T^a, [T^b, T^c]] = X_a Y_b Z_c f^{ad}{}_e f^{bc}{}_d T^e.$$
Finally, taking the trace contracts the indices e and c together, so
$$\kappa(X, Y) = X_a Y_b f^{ad}{}_c f^{bc}{}_d = \kappa^{ab} X_a Y_b$$
where the κ^{ab} are the components of the Killing form.

- The Killing form is invariant under the adjoint action of $\mathfrak{g}$,
$$\kappa([Z, X], Y) + \kappa(X, [Z, Y]) = 0.$$
To show this, we simply expand the definitions,
$$\kappa([Z, X], Y) = \operatorname{tr}(\operatorname{ad}_{[Z,X]} \operatorname{ad}_Y) = \operatorname{tr}(\operatorname{ad}_Z \operatorname{ad}_X \operatorname{ad}_Y - \operatorname{ad}_X \operatorname{ad}_Z \operatorname{ad}_Y).$$
This cancels with the other term using the cyclic property of the trace. Intuitively, this is just the infinitesimal version of the adjoint action of G on $\mathfrak{g}$, i.e. conjugation by a group element, and the two terms come from the product rule. This indicates that the Killing form is 'the same' everywhere in the group.

- For a simple Lie algebra, it can be shown that the properties of symmetry, linearity, and invariance under the adjoint action determine the inner product up to scalar multiples.

- A real Lie algebra is of compact type if there is a basis where the Killing form is negative definite. It can be shown that every finite-dimensional complex semi-simple Lie algebra has a real form of compact type.

- Cartan's criterion states the Killing form is nondegenerate if and only if $\mathfrak{g}$ is semi-simple.

  We'll show only the forward direction. Suppose $\mathfrak{g}$ has an abelian ideal $\mathfrak{i}$ so that the ideal has basis $\{T^i\}$ and the rest of the Lie algebra has basis $\{T^a\}$. Then $[T^i, T^j] = 0$ since $\mathfrak{i}$ is abelian, and $[T^i, T^a] \in \mathfrak{i}$ since $\mathfrak{i}$ is an ideal.

  Now consider $\operatorname{ad}_X \operatorname{ad}_Y Z$ where $Y \in \mathfrak{i}$. If $Z \in \mathfrak{i}$, this is automatically zero. If $Z \notin \mathfrak{i}$, then $\operatorname{ad}_Y Z \in \mathfrak{i}$, and hence $\operatorname{ad}_X \operatorname{ad}_Y Z \in \mathfrak{i}$. Hence $\operatorname{tr} \operatorname{ad}_X \operatorname{ad}_Y = 0$, so $\kappa(X, Y) = 0$ for any $Y \in \mathfrak{i}$, so κ is degenerate.

- As a partial converse, if $\mathfrak{g}$ is simple, the the Killing form is nondegenerate. To prove this, note that if the Killing form were degenerate, then the set of elements Y so that $\kappa(X, Y) = 0$ for all $X \in \mathfrak{g}$ forms a proper ideal $\mathfrak{i}$, as
$$\kappa(X, [Z, Y]) = -\kappa([Z, X], Y) = 0$$
for any $Z \in \mathfrak{g}$, so $[Z, Y] \in \mathfrak{i}$.

Example. Consider a simple matrix Lie algebra. Then $\operatorname{tr} XY$ satisfies the properties of the Killing form, so it is proportional to the Killing form; this makes computations much easier.

One might wonder why $\operatorname{tr} X \operatorname{tr} Y$ wouldn't work as well. This quantity vanishes identically, since $\operatorname{tr} E^\alpha = 0$ since there are no diagonal elements, while $\operatorname{tr} H^i = 0$ because the eigenvalues of $\mathfrak{su}(2)$ representations sum to zero. Note that the latter statement fails when there are abelian ideals.

Note. The Killing form defines a metric on $T_e G$, and we may extend this to a metric on G by left and right-translation. Adding an extra minus sign, if $\mathfrak{g}$ is of compact type, then the metric on G is positive definite, and it can be shown that G is compact. Geodesics corresponding to this metric are one-parameter subgroups of G.

Note. The Killing form is used to raise and lower all indices in $\mathfrak{g}$, e.g. we may define $f^{abc} = \kappa^{cd} f^{ab}{}_d$. Now for a Lie algebra of compact type, suppose we choose a basis T^a where $\kappa^{ab} = -\delta^{ab}$. Then

$$0 = \kappa(T^a, [T^b, T^c]) + \kappa([T^b, T^a], T^c) = f^{bca} + f^{bac}$$

so in this basis, f^{abc} is totally antisymmetric. We've often chosen our bases to satisfy this implicitly, such as in $\mathfrak{su}(2)$, where we had $f^{abc} = \epsilon^{abc}$. As another example, the quadratic Casimir, which we simply defined as $\sum_a (T_a)^2$, only has this form in this basis; more generally it should be $\kappa^{ab} T_a T_b$.

Next, we find some more properties of the root system of a semi-simple Lie algebra.

- The roots span all of $\mathfrak{h}^*$. This is just because if some direction were not represented, we would have a Cartan subalgebra element that commuted with everything in the group, giving an abelian ideal.

- The roots are nondegenerate, so there is exactly one step operator E^α for each root α. We'll prove this fact below and simply take it as given here.

- Note that $\operatorname{ad}_{E_\alpha}$ raises roots by α, where we regard the Cartan subalgebra itself as associated with the zero root. Then we must have

  $$\kappa(H^i, E^\alpha) = 0, \quad \kappa(E^\alpha, E^\beta) = 0 \text{ if } \alpha + \beta \neq 0$$

  because only $\operatorname{ad}_{E^\alpha} \operatorname{ad}_{E^{-\alpha}}$ and $\operatorname{ad}_{H^i} \operatorname{ad}_{H^j}$ can take roots to themselves.

- To prove these results more formally, we can use the invariance of the Killing form. Since

  $$\alpha(H')\kappa(H, E^\alpha) = \kappa(H, [H', E^\alpha]) = -\kappa([H, H'], E^\alpha) = 0$$

  for any H', we must have $\kappa(H, E^\alpha) = 0$. Similarly

  $$(\alpha(H') + \beta(H'))\kappa(E^\alpha, E^\beta) = \kappa([H', E^\alpha], E^\beta) + \kappa(E^\alpha, [H', E^\beta]) = 0$$

  so if $\alpha + \beta \neq 0$, then $\kappa(E^\alpha, E^\beta) = 0$.

- Next, we can get constraints from the nondegeneracy of the Killing form.

  - If α is a root, so is $-\alpha$, with $\kappa(E^\alpha, E^{-\alpha}) \neq 0$, because otherwise $\kappa(E^\alpha, \cdot) = 0$.
  - The Killing form is nondegenerate on $\mathfrak{h}$, as if $\kappa(H^i, H^j) = 0$ for all j, then $\kappa(H^i, \cdot) = 0$.

- Since the Killing form is a nondegenerate inner product on $\mathfrak{h}$, we may use it to correspond elements of $\mathfrak{h}$ and $\mathfrak{h}^*$ and hence define a nondegenerate inner product on $\mathfrak{h}^*$. In components,
$$\kappa(H, H') = \kappa^{ij} e_i e'_j, \quad H = H^i e_i, \quad H' = H^i e'_i$$
and the dual element H^α of α is defined by
$$\kappa(H^\alpha, H) = \alpha(H), \quad H^\alpha = (\kappa^{-1})_{ij} \alpha^j H^i.$$
We define an inner product on the H^α, and hence an inner product on the roots, by
$$(\alpha, \beta) = \kappa(H^\alpha, H^\beta) = (\kappa^{-1})_{ij} \alpha^i \beta^j.$$
The inverse here is just the result of lowering indices, analogous to how the metric and inverse metric are related in differential geometry.

Next, we work out more of the algebra in the Cartan-Weyl basis.

- By the Jacobi identity, we have
$$[H^i, [E^\alpha, E^\beta]] = -[E^\alpha, [E^\beta, H^i]] - [E^\beta, [H^i, E^\alpha]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta].$$
Therefore, $[E^\alpha, E^\beta]$ is proportional to $E^{\alpha+\beta}$ if $\alpha + \beta$ is a root, as anticipated above.
- For the case $\alpha + \beta = 0$, note that
$$\kappa([E^\alpha, E^{-\alpha}], H) = \kappa(E^\alpha, [E^{-\alpha}, H]) = \alpha(H)\kappa(E^\alpha, E^{-\alpha}) = \kappa(H^\alpha, H)\kappa(E^\alpha, E^{-\alpha})$$
where we used the invariance of the Killing form. By the results we found from nondegeneracy of the Killing form, we may conclude
$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{\kappa(E^\alpha, E^{-\alpha})}.$$
- In summary, the algebra for the step operators takes the form
$$[E^\alpha, E^\beta] = \begin{cases} \kappa(E^\alpha, E^{-\alpha}) H^\alpha & \alpha + \beta = 0, \\ N_{\alpha,\beta} E^{\alpha+\beta} & \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$
Here, the $N_{\alpha,\beta}$ are unknown complex numbers. Finally, we have
$$[H^\alpha, E^\beta] = (\kappa^{-1})_{ij} \alpha^i [H^j, E^\beta] = (\kappa^{-1})_{ij} \alpha^i \beta^j E^\beta = (\alpha, \beta) E^\beta.$$
- To simplify these relations, we rescale all of our operators, as
$$e^\alpha = \sqrt{\frac{2}{(\alpha, \alpha)\kappa(E^\alpha, E^{-\alpha})}} E^\alpha, \quad h^\alpha = \frac{2}{(\alpha, \alpha)} H^\alpha.$$
Here we've implicitly assumed that $(\alpha, \alpha) \neq 0$, which we will show below. Our algebra simplifies to the final form
$$[h^\alpha, h^\beta] = 0, \quad [h^\alpha, e^\beta] = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} e^\beta, \quad [e^\alpha, e^\beta] = \begin{cases} h^\alpha & \alpha + \beta = 0, \\ n_{\alpha,\beta} e^{\alpha+\beta} & \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

6.2 $*$ The Root Space

Our normalization above makes it easy to identify $\mathfrak{sl}(2)$ subalgebras of $\mathfrak{g}$.

- For any root α, the set $\{h^\alpha, e^\alpha, e^{-\alpha}\}$ obeys

$$[h^\alpha, e^{\pm\alpha}] = \pm 2 e^{\pm\alpha}, \quad [e^\alpha, e^{-\alpha}] = h^\alpha$$

  which are exactly the commutation relations for $\mathfrak{sl}(2)$. We call this subalgebra $\mathfrak{sl}(2)_\alpha$.

- Define the α-string passing through β as the set

$$S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi \,|\, n \in \mathbb{Z}\}$$

  and define a corresponding subspace of $\mathfrak{g}$,

$$V_{\alpha,\beta} = \mathrm{span}(\{e^\delta \,|\, \delta \in S_{\alpha,\beta}\}).$$

- Next, consider the action of $\mathfrak{sl}(2)_\alpha$ on $V_{\alpha,\beta}$. We have

$$[h^\alpha, e^{\beta+n\alpha}] = \frac{2(\alpha, \beta + n\alpha)}{(\alpha,\alpha)} e^{\beta+n\alpha} = \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n\right) e^{\beta+n\alpha}$$

  and

$$[e^{\pm\alpha}, e^{\beta+n\alpha}] \propto \begin{cases} e^{\beta+(n\pm1)\alpha} & \text{if } \beta + (n\pm 1)\alpha \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

  Therefore, $V_{\alpha,\beta}$ is a representation of $\mathfrak{sl}(2)_\alpha$.

- Now we can prove the nondegeneracy of the roots. Let $V_{\alpha,\alpha}$ contain all raising and lowering operators with roots $n\alpha$, plus the single operator h^α for the 'zero root'. By similar reasoning to above, it is a representation of $\mathfrak{sl}(2)_\alpha$, where all the weights are even integers.

  Since there is only one element with weight zero, $V_{\alpha,\alpha}$ is an odd-dimensional $\mathfrak{sl}(2)_\alpha$ irrep. But we also know $\mathfrak{sl}(2)_\alpha$ is a subrepresentation of $V_{\alpha,\alpha}$, so $\mathfrak{sl}(2)_\alpha = V_{\alpha,\alpha}$. Then the roots $\pm\alpha$ are nondegenerate, and furthermore no other integer multiples of α can be roots.

- Note that roots of the form $\beta = \pm\alpha/2$ above are allowed by $\mathfrak{sl}(2)_\alpha$ representation theory. But if β is a root, then 2β is a root, a contradiction. Therefore if α is a root, the only nonzero multiple of it that is also a root is $-\alpha$.

- Returning to the general case $V_{\alpha,\beta}$ note that the weights are

$$S = \left\{ \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n \,\middle|\, \beta + n\alpha \in S_{\alpha,\beta} \right\}.$$

  Since the weights are nondegenerate and evenly spaced, the representation is a finite-dimensional irrep. Therefore we must have $S = \{\Lambda, -\Lambda + 2, \ldots, \Lambda - 2, \Lambda\}$ for some integer Λ.

- If the range of n is $n_- \le n \le n_+$, we have

$$-\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_-, \quad \Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+.$$

  Adding these equations gives

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-) \in \mathbb{Z}.$$

The result above gives a strong constraint as the roots, as we now show.

- First, we return to the original definition of the Killing form. Since $[H^i, E^\alpha] = \alpha^i E^\alpha$,
$$\kappa^{ij} = \kappa(H^i, H^j) = \operatorname{tr}(\operatorname{ad}_{H_i} \operatorname{ad}_{H_j}) = \sum_{\delta \in \Phi} \alpha^i \alpha^j.$$
This is essentially a completeness relation, as
$$(\alpha, \beta) = \alpha^i \beta^j (\kappa^{-1})_{ij} = \alpha_i \beta_j \kappa^{ij} = \sum_{\delta \in \Phi} \alpha_i \delta^i \delta^j \beta_j = \sum_{\delta \in \Phi} (\alpha, \delta)(\beta, \delta).$$
- Now, we know the ratio $(\alpha, \beta)/(\alpha, \alpha)$ is real from our work above. But then
$$\frac{(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} = \sum_{\delta \in \Phi} \frac{(\alpha, \delta)}{(\alpha, \alpha)} \frac{(\beta, \delta)}{(\beta, \beta)}$$
which tells us that (β, β) is real, and hence (α, β) is real. Moreover, we have
$$(\alpha, \alpha) = \sum_{\delta \in \Phi} (\alpha, \delta)^2 > 0$$
by nondegeneracy. Therefore, if we restrict the root space $\mathfrak{h}^*$ to the real span of the roots $\mathfrak{h}^*_{\mathbb{R}}$, we have a Euclidean inner product. This is important, since so far every structure introduced has been complex.
- The real span of the roots has the same dimension, as a real vector space, as $\mathfrak{h}^*$ does as a complex vector space. To see this, choose a basis of roots $\alpha_{(i)} \in \mathfrak{h}^*$. Then for any $\beta \in \Phi$,
$$\beta = \sum_i \beta^i \alpha_{(i)}, \quad (\beta, \alpha_{(j)}) = \sum_i \beta^i (\alpha_{(i)}, \alpha_{(j)}).$$
Since the inner product is nondegenerate, combining these equations for all j gives $\beta^i \in \mathbb{R}$, so $\Phi \subset \operatorname{span}_{\mathbb{R}}(\{\alpha_{(i)}\})$, which is an r-dimensional real vector space.
- We can thus define the length of a root $|\alpha| = \sqrt{(\alpha, \alpha)}$, as well as angles ϕ between roots in the standard way. Now we apply our earlier constraints, for
$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \frac{2|\beta|}{|\alpha|} \cos\phi \in \mathbb{Z}, \quad \frac{2(\beta, \alpha)}{(\beta, \beta)} = \frac{2|\alpha|}{|\beta|} \cos\phi \in \mathbb{Z}.$$
Multiplying these equations gives
$$4\cos^2\phi \in \mathbb{Z}$$
which implies the angles between roots must be 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, or their supplements.
- These conditions are geometrically intuitive. The constraint $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$ says that the states in the α-string through β have α values that are half-integers or integers, in accordance with $\mathfrak{sl}(2)$ representation theory. The other constraint says the same for the β-string through α. Then the constraint is trivial when α and β are orthogonal, but very restrictive otherwise.

Next, we define simple and positive roots.

- We divide Φ into two halves $\Phi_\pm$, called the positive and negative roots, by drawing an arbitrary hyperplane through the origin that does not intersect any root. Note that the $\Phi_\pm$ are each closed under addition. We call step operators associated with positive roots raising operators, and with negative roots lowering operators.
- A simple root is a positive root which cannot be written as a positive linear combination of positive roots. Geometrically, these are typically the positive roots closest to the hyperplane.
- If α and β are simple roots, then $\alpha - \beta$ is not a root. To see this, note that if $\alpha - \beta$ were a positive root, α could not be simple, while if it were a negative root, β could not be simple.
- If α and β are distinct simple roots, then the α-string through β has

$$n_- = 0, \quad n_+ = -\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

Since we know $n_+ \geq 0$, we have $(\alpha, \beta) \leq 0$.

- The simple roots are linearly independent. Denoting the simple roots by $\alpha_{(i)}$, let

$$\lambda = \sum_{c_i \geq 0} c_i \alpha_{(i)} + \sum_{c_i < 0} c_i \alpha_{(i)} \equiv \lambda_+ - \lambda_-.$$

Then we have, assuming the c_i are not all zero,

$$(\lambda, \lambda) = (\lambda_+ - \lambda_-, \lambda_+ - \lambda_-) > -2(\lambda_+, \lambda_-) = 2 \sum_{c_i \geq 0} \sum_{c_j < 0} c_i c_j (\alpha_{(i)}, \alpha_{(j)}) > 0$$

because $(\alpha, \beta) \leq 0$ for simple roots, so $\lambda \neq 0$ as desired.

- Every positive root can be written as a linear combination of the simple roots with nonnegative integer coefficients. This can be shown recursively: if α is positive and simple, we're done. Otherwise $\alpha = \alpha_1 + \alpha_2$ and we may repeat the procedure until we get the desired decomposition.
- As a result, all roots can be written as an integer combination of the simple roots. Then the simple roots are a basis for $\mathfrak{h}_{\mathbb{R}}^*$, so there are r of them.

6.3 $*$ The Cartan Matrix

The content of the simple roots can be encoded in the Cartan matrix.

- Define the elements of the Cartan matrix by

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})} \in \mathbb{Z}.$$

We note that $A^{ii} = 2$, and $A^{ij} = 0$ if and only if $A^{ji} = 0$. Since the inner product of roots is negative, $A^{ij} \leq 0$ for $i \neq j$. Intuitively, $-A^{ij}$ is the number of times $\alpha_{(j)}$ can be added to $\alpha_{(i)}$ and yield a root; this interpretation also works for the diagonal elements.

- For each simple root $\alpha_{(i)}$, we have an $\mathfrak{su}(2)$ subalgebra with operators $h^i \equiv h^{\alpha_{(i)}}$, $e^i_\pm \equiv e^{\alpha_{(i)}}_\pm$,
$$[h^i, e^i_\pm] = \pm 2 e^i_\pm, \quad [e^i_+, e^i_-] = h^i.$$
These $3r$ operators together generate all of $\mathfrak{g}$ by brackets. A Cartan-Weyl basis chosen so that these $3r$ operators are normalized in this way is called a Chevalley basis.

- In a Chevalley basis, the algebra is
$$[h^i, h^j] = 0, \quad [h^i, e^j_\pm] = \pm A^{ji} e^j_\pm, \quad [e^i_+, e^j_-] = \delta_{ij} h^i.$$
We also have the Serre relations
$$(\mathrm{ad}_{e^i_\pm})^{1-A^{ji}} e^j_\pm = 0$$
by the intuitive interpretation of A^{ij} above.

- A finite-dimensional, semi-simple complex Lie algebra is uniquely determined by its Cartan matrix. To reconstruct $\mathfrak{g}$, we start with the simple roots and construct all $\alpha^{(i)}$ strings through $\alpha^{(j)}$, with the Cartan matrix telling us the length of all root strings. We then repeat this procedure until no more new roots appear.

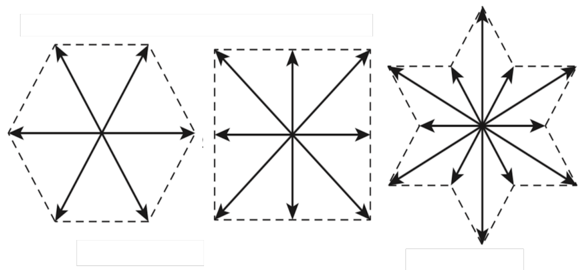
- The Cartan matrix satisfies $\det A > 0$. To see this, note that the inner product in the basis of simple roots is
$$(\lambda, \mu) = (\alpha_{(i)}, \alpha_{(j)}) \lambda^i \mu^j, \quad \lambda = \sum_i \lambda^i \alpha_{(i)}, \quad \mu = \sum_i \mu^i \alpha_{(i)}.$$
Since the inner product is positive definite, the matrix with entries $(\alpha_{(i)}, \alpha_{(j)})$ is positive definite and hence has positive determinant. The Cartan matrix is the product of this matrix with a diagonal matrix with entries $2/(\alpha_{(i)}, \alpha_{(i)})$ which also has positive determinant.

- If $\mathfrak{g}$ is simple, then the Cartan matrix is not reducible, i.e. there is no reordering of the simple roots that makes A block-diagonal. Essentially, if the Cartan matrix were reducible, then the step operators generated by one of the blocks of simple roots (along with the corresponding Cartan subalgebra elements) would form a proper ideal of $\mathfrak{g}$.

The above constraints very strongly restrict the form of the Cartan matrix.

- By our previous identities, we have
$$A^{ij} A^{ji} = 4 \cos^2 \phi \in \mathbb{Z}$$
which implies that $A^{ij} A^{ji} \in \{0, 1, 2, 3\}$. There are only a few possibilities. Taking the $\alpha^{(i)}$ root to be not shorter without loss of generality, we have:
  - $A^{ij} = A^{ji} = 0$. The simple roots are perpendicular with indefinite ratio of lengths; no other roots are attained by adding them.
  - $A^{ij} = A^{ji} = -1$. The simple roots have equal length and angle 120°, and one additional root is attained by adding them.
  - $A^{ij} = -2$, $A^{ji} = -1$. Then $|\alpha_{(i)}| = \sqrt{2} |\alpha_{(j)}|$ and the angle is 135°.
  - $A^{ij} = -3$, $A^{ji} = -1$. Then $|\alpha_{(i)}| = \sqrt{3} |\alpha_{(j)}|$ and the angle is 150°.

The roots generated by these three latter possibilities are shown below.



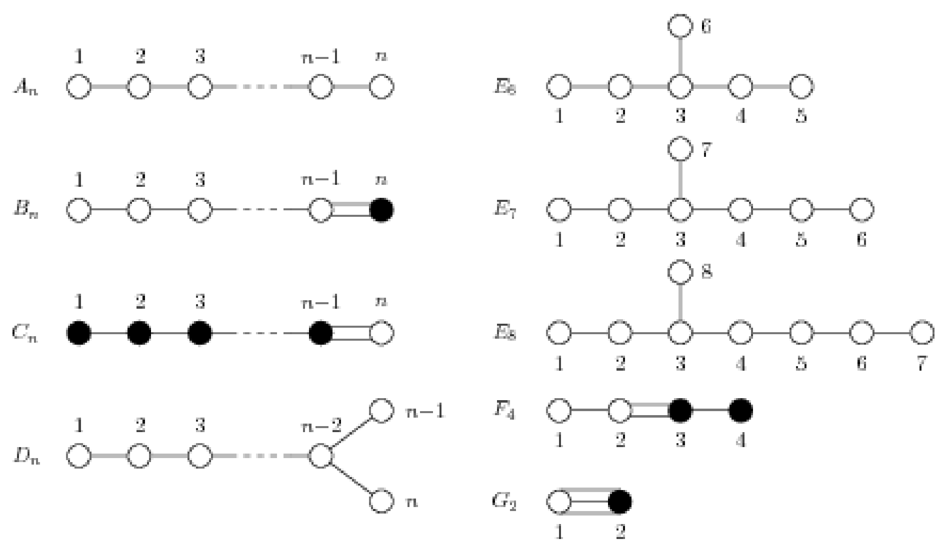
There are no more possibilities, by the det $A > 0$ constraint.

- One useful fact is that if α and β are roots, then so is the 'Weyl reflection' of β in the hyperplane normal to α. This can be shown by casework with our results on root strings. The set of Weyl reflections form the Weyl group.

- If $\mathfrak{g}$ is simple, it can be shown that the simple roots can only have two distinct lengths. In fact, one can show there are only two distinct lengths among all the roots. The proof is easy if we ignore the case of perpendicular roots; we can deal with those by Weyl reflection.

- The information in the Cartan matrix can be written in a Dynkin diagram. We draw a node for every simple root $\alpha_{(i)}$, then connect nodes i and j with $\max(|A^{ij}|, |A^{ji}|)$ lines. If the roots have different lengths, we draw an arrow pointing from the longer root to the shorter, or shade the shorter roots black. Note that Dynkin diagrams must be connected, by simplicity.

- The set of all Dynkin diagrams is shown below, classifying all simple complex Lie algebras.



Apart from the five exceptional cases, the four infinite families are known in physics as

$$A_n = \mathfrak{su}(n+1) \cong \mathfrak{sl}(n+1), \quad B_n = \mathfrak{so}(2n+1), \quad C_n = \mathfrak{sp}(2n), \quad D_n = \mathfrak{so}(2n).$$

In all cases the subscript indicates the number of roots. Note that D_1 isn't counted since $\mathfrak{so}(2)$ is not simple. Also, D_2 technically does not belong since it is disconnected.

- Generally, any angle besides 90° provides a strong constraint since it generates new roots, so most angles must be 90°. After that, angles of 120° are nice, as they form regular polyhedra in higher dimensions. An angle of 150° makes such a strange pattern that it only appears in the rank 2 Lie algebra G_2, whose root system is shown above.

- Looking at the Dynkin diagrams, we can read off some low-dimensional coincidences.
  - We have $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ from $A_1 \cong B_1$.
  - We have $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ from $D_2 \cong A_1 \oplus A_1$.
  - We have $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$ from $B_2 \cong C_2$.
  - We have $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ from $A_3 \cong D_3$.

  Thus the representation theory of the Lorentz group is only 'nice' in four dimensions.
- Another useful application is that we can 'cut' Dynkin diagrams to find subalgebras. For example, cutting off the long root of B_n shows that $\mathfrak{su}(n) \subset \mathfrak{so}(2n+1)$. (Later we will see a natural embedding of $\mathfrak{su}(n)$ in $\mathfrak{so}(2n)$.) Cutting off the short root of $\mathfrak{sp}(2n)$ gives $\mathfrak{su}(n) \subset \mathfrak{sp}(2n)$ as we observed earlier.

Note. A systematic algorithm for reconstructing all roots from the simple roots.

- For a general root $\beta = \sum_i \alpha_{(i)} k_i$, let $k = \sum_i k_i$. Here the Cartan subalgebra has $k = 0$ and the simple roots have $k = 1$.
- For each simple root $\alpha_{(j)}$, we draw a box with entries

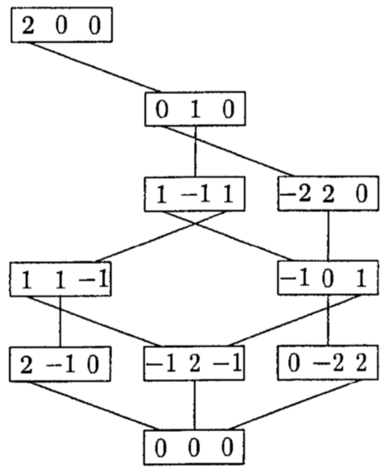
$$c_i = A_{ji} = -(n_+ + n_-).$$

  That is, we just write row j of the Cartan matrix. Hence if $c_i < 0$ then we can add the root $\alpha_{(i)}$, and hence we can construct the $k = 2$ roots.
- For each $k = 2$ root, we again draw a box; its entries c_i are the sum of the root it came from and the simple root added to it. We can then continue systematically upward in k, since at every point we know the value of n_-, until the procedure terminates. We then reflect to get all the negative roots.
- A slightly faster method is to start from the simple roots and draw all possible root strings all the way to the end. This will produce some roots with new negative c_i, which sit at the bottom of new root strings. We then repeat this process until termination.

Example. The algebra C_3. The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

and the resulting positive roots are shown below.



Note. Reconstructing the algebra from the roots. For a simple root α, if the α-string through β has length $2s+1$, then the action of the $E^{\pm\alpha}$ operators on this subspace by commutator is just like that of the angular momentum raising and lowering operators for spin s, with the same constants up to phases. This gives the action of E^{α} on everything, and we can find the action of $E^{\alpha+\beta}$ on everything by the Jacobi identity, and so on. The 'up to phases' is because we don't have enough freedom to take Cordan-Shortley phases everywhere.

Note. The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The classification of Lie algebras is related to this, because $SO(n)$ describes linear transformations that preserve the length of a vector in $\mathbb{R}$, $SU(n)$ does the same for $\mathbb{C}$, $Sp(n)$ does the same for $\mathbb{H}$. There is no analogue for $\mathbb{O}$ because its multiplication law is not associative, so we don't get a group. But all of the other Lie groups, including the exceptional groups, F_4, and G_2, are related to $\mathbb{O}$ in some way.

6.4 $*$ Representations and Weights

In this section we apply our results to representations of $\mathfrak{g}$.

- We consider an N-dimensional representation R, and assume that the $R(H^i)$ are diagonalizable. Since the $R(H^i)$ all commute, they can be simultaneously diagonalized, with
$$R(H^i)v = \lambda^i v, \quad v \in V_\lambda$$
where $\lambda \in \mathfrak{h}^*$ is a weight of R and the set of weights is S_R.

- Roots are the weights of the adjoint representation $R(X) = \mathrm{ad}_X$. But unlike roots, weights can be degenerate in general, with multiplicity $m_\lambda = \dim V_\lambda \geq 1$. Note that it doesn't make sense to associate a weight with an element of $\mathfrak{g}$. This only made sense for roots because the representation space was the algebra itself.

- As with roots, step operators raise and lower the weights,
$$R(H^i)R(E^\alpha)v = R(E^\alpha)R(H^i)v + [R(H^i), R(E^\alpha)]v = (\lambda^i + \alpha^i)R(E^\alpha)v.$$
Then if $v \in V_\lambda$, then $R(E^\alpha)v \in V_{\lambda+\alpha}$.

- Next, we consider the action of the $\mathfrak{sl}(2)_\alpha$ generators $\{R(h^\alpha), R(e^\alpha), R(e^{-\alpha})\}$ on V. This makes V into the representation space for a representation R_α of $\mathfrak{sl}(2)$. The $\mathfrak{sl}(2)$ weights are

$$R(h^\alpha)v = \frac{2}{(\alpha,\alpha)}(\kappa^{-1})_{ij}\alpha^i R(H^j)v = \frac{2}{(\alpha,\alpha)}(\kappa^{-1})_{ij}\alpha^i \lambda^j v = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}v.$$

  Since the $\mathfrak{sl}(2)$ weights must be integers, we have

$$\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}, \quad \lambda \in S_R, \quad \alpha \in \Phi.$$

  This is just the same condition we found for roots; the only difference is that we don't also have the same constraint with α and λ swapped.

- To understand this constraint geometrically, note that all the roots lie in the root lattice

$$\mathcal{L}[\mathfrak{g}] = \mathrm{span}_{\mathbb{Z}}\{\alpha_{(1)}, \ldots, \alpha_{(r)}\}.$$

  Now define the simple coroots and coroot lattice

$$\alpha^\vee_{(i)} = \frac{2\alpha_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})}, \quad \mathcal{L}^\vee[\mathfrak{g}] = \mathrm{span}_{\mathbb{Z}}\{\alpha^\vee_{(1)}, \ldots, \alpha^\vee_{(r)}\}.$$

- Then the weight lattice is the dual of the co-root lattice,

$$\mathcal{L}_W[\mathfrak{g}] = \mathcal{L}^\vee[\mathfrak{g}]^* = \{\lambda \in \mathfrak{h}^* \mid (\lambda,\mu) \in \mathbb{Z} \text{ for all } \mu \in \mathcal{L}^\vee[\mathfrak{g}]\}$$

  Consider the basis $\{\alpha^\vee_{(i)}\}$ of $\mathcal{L}^\vee[\mathfrak{g}]$. The weight lattice has the dual basis $\{w_{(i)}\}$ where

$$(\alpha^\vee_{(i)}, w_{(j)}) = \delta_{ij}.$$

  This basis is called the Dynkin basis of the weight space, and its elements are called the fundamental weights of $\mathfrak{g}$.

- Now consider the expansion

$$w_{(i)} = \sum_j B_{ij}\alpha_{(j)}.$$

  Taking the inner product of both sides with $\alpha^\vee_{(k)}$ and relabeling indices,

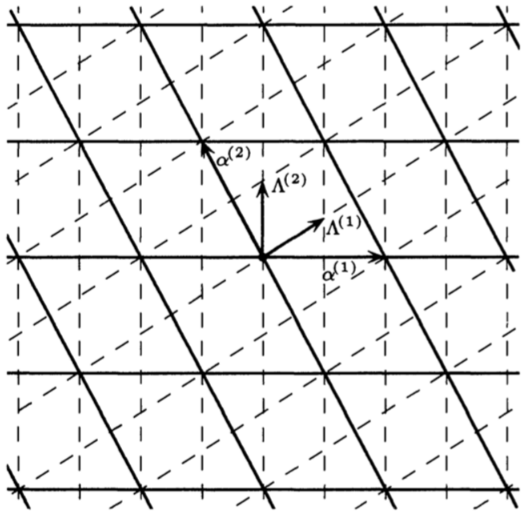
$$\sum_k \frac{2(\alpha_{(i)}, \alpha_{(k)})}{(\alpha_{(i)}, \alpha_{(i)})} B_{jk} = \delta^i_k$$

  which shows that $B = A^{-1}$ where A is the Cartan matrix, so

$$\alpha_{(i)} = \sum_j A^{ij} w_{(j)}.$$

  Thus starting from the Cartan matrix we can read off the fundamental weights.

Example. The root and weight lattices of A_2 are shown below.



The relation between the two can be read off from the Cartan matrix,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Note that the root lattice is contained in the weight lattice. This must hold in general because the simple roots are weights of the adjoint representation.

The fundamental weights allow us to extract the weight set from the highest weight alone.

- Any weight can be expanded as
$$\lambda = \lambda^i w_{(i)}$$
and the integers λ^i are called the Dynkin labels of λ. We've already seen these; in our graphical calculation above, the numbers in the boxes were the Dynkin labels of roots/weights.
- Every finite-dimensional representation has a highest weight
$$\Lambda = \Lambda^i w_{(i)}$$
so that all $v \in V_\Lambda$ are annihilated by all raising operators,
$$R(E^\alpha) v_\Lambda = 0, \quad \alpha \in \Phi_+.$$
The Dynkin labels of the highest weight are called the Dynkin labels of the representation.
- Starting from the highest weight, we can find more weights by applying the lowering operators, $R(E^{-\alpha})$ for $\alpha \in \Phi^+$. The reasoning is just like how we used simple roots to construct all the roots, but in reverse; starting with any positive Dynkin label, we can go downward. We can get every weight this way if the representation is an irrep; note that the adjoint representation itself is an irrep when the algebra is simple.
- Note that we never have to go 'upward'. Suppose we had a state of the form
$$E_1 E_2 \dots E_n |\mu\rangle$$

where $|\mu\rangle$ is the highest weight state and the E_i are raising and lowering operators. If any of the E_i are raising operators, we may commute it all the way to the right, picking up extra terms as we go, until it annihilates $|\mu\rangle$. We can repeat this procedure until all operators are lowering operators. We also know all such operators can be written in terms of the lowering operators of simple roots

- The general principle is that if $\lambda = \sum_i \lambda^i w_{(i)}$ is a weight, then we also have the weights

$$\lambda - m^i \alpha_{(i)}, \quad 0 \leq m_i \leq \lambda^i$$

because representations of $\mathfrak{sl}(2)_{\alpha_{(i)}}$ must have weights symmetric about zero.

- The weight set of a tensor product is the set of sums of weights, which gives a visual method for decomposing tensor products. We get one factor for free, since one of the highest weights is the product of the individual highest weights; for example, $(1,2) \times (5,2)$ contains a $(6,4)$.

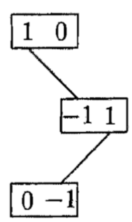
Example. The fundamental representation of $A_2 \cong \mathfrak{su}(3)$ has Dynkin labels $(1,0)$, so $\Lambda = w_{(1)}$, and

$$\Lambda - \alpha_{(1)} = w_{(1)} - (2w_{(1)} - w_{(2)}) = -w_{(1)} + w_{(2)}$$

is also a weight. This yields the new weight

$$(\Lambda - \alpha_{(1)}) - \alpha_{(2)} = -w_{(2)}$$

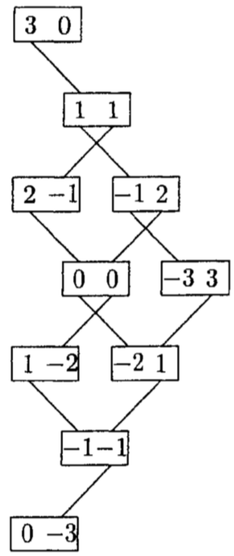
and this weight produces no further weights. Alternatively, we can use a diagram.



Thus the representation has three weights, which form a small upside-down triangle. More generally, the Dynkin labels agree with the labels we assigned earlier based on the ranks of symmetric traceless tensors. Then $R_{(1,1)}$ is still the adjoint representation, $R_{(3,0)}$ is still the 10, and so on.

Note. The main difference with working with weights rather than roots is that the weights can be degenerate. Clearly, any weight that can be reached in only one way by lowering operators is nondegenerate. Also note that a degenerate weight doesn't create more degeneracy 'further down' unless it sits at the top of a new $\mathfrak{su}(2)$ representation. Another trick is that if a possibly degenerate weight can be related to a nondegenerate weight by Weyl reflection, it is nondegenerate.

Example. The $(3,0)$ representation of $\mathfrak{su}(3)$. We get the diagram shown below.



First, note that there is a 'spine' on the right leading from the highest to the lowest weight. In general for the (n, m) irrep, we can reach it by lowering with α^1 n times, lowering with α^2 $n+m$ times, then lowering with α^1 m times, yielding the lowest weight $(-m, -n)$. There is also the possibility of degeneracy; by the remark above, we only need to check the degeneracy of the $\boxed{0\ 0}$ state; we can't use Weyl reflection since it is at the origin.

Let E_1 and E_2 be the lowering operators for $\alpha_{(1)}$ and $\alpha_{(2)}$. Then

$$E_2E_1E_1 - E_1E_2E_1 = [E_2, E_1]E_1 = E_1[E_2, E_1]$$

where the second step follows because $-2\alpha^1 - \alpha^2$ is not a root. Now act with both sides on the highest weight $|\mu\rangle$. On the right-hand side we have

$$E_1[E_2, E_1]|\mu\rangle = E_1E_2E_1|\mu\rangle$$

because $E_2|\mu\rangle = 0$. This shows that the two possibly degenerate states are proportional.

Note. There is a fully general method for determining whether two states are degenerate. Consider two states of the form

$$|A\rangle = E_{a_1} \dots E_{a_n}|\mu\rangle, \quad |B\rangle = E_{b_1} \dots E_{b_n}|\mu\rangle.$$

Then the states are linearly independent if and only if

$$\langle A|B\rangle\langle B|A\rangle \neq \langle A|A\rangle\langle B|B\rangle$$

where all of these inner products can be computed systematically using the algebra.

Note. Taking the conjugate of a representation just flips the sign on every weight, because the Cartan subalgebra elements $R(H^i)$ become $-R(H^i)^* = -R(H^i)$. For $\mathfrak{su}(3)$, the (n, m) representation has lowest weight $(-m, -n)$, which implies the conjugate representation is (n, m). For example, the antifundamental representation has $\Lambda = w_{(2)}$ and its weights form a triangle. We see that a representation is real if its weights are symmetric about the origin.

Note. In general, we call the representations with exactly one nonzero Dynkin label, which is equal to one, the fundamental representations. For example, $\mathfrak{su}(4)$ has rank three, and hence has three 'fundamental' representations. One is its 'usual' fundamental representation in terms of matrices, one is its conjugate, and the third is something completely different; it is not even four-dimensional. All representations can be found by multiplying fundamental representations, as the representation with Dynkin labels Λ^i is contained in $\otimes_i V_i^{\otimes \Lambda^i}$ where V_i is the i^{th} fundamental.

Example. Consider $B_2 \cong C_2$, where $B_2 \cong \mathfrak{so}(5)$ and $C_2 \cong \mathfrak{sp}(4)$. Then there are two fundamental representations with dimensions 5 and 4; they correspond to the fundamental matrix representations of $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$. The adjoint representation has Dynkin labels $(0, 2)$.

Note. If we use an explicit representation, rather than just the abstract algebra, everything simplifies. We defined the Killing form and worked to show that it gives a natural inner product on the roots. But we could also work in the adjoint representation and take the inner product on the Hilbert space. The action of X_a in the adjoint representation is

$$X_a|X_b\rangle = |[X_a, X_b]\rangle$$

so it's easy to see how the E^α function as both weights and roots,

$$H^i|E^\alpha\rangle = \alpha^i|E^\alpha\rangle, \quad [H^i, E^\alpha] = \alpha^i E^\alpha.$$

Moreover, we have $H_\alpha^\dagger = H_\alpha$ since its eigenvalues are real, while taking the adjoint of the above commutation relation gives $E^{\alpha\dagger} = E^{-\alpha}$.

6.5 Examples of Roots

Often, the roots are a subset of the differences between weights of the fundamental representation; we'll use this as a shortcut to find roots. Since we deal with only matrix Lie algebras, we use the Killing form $\text{tr}(XY)$.

Example. The roots of $\mathfrak{su}(3)$. An orthonormal basis for the Cartan subalgebra is

$$H^1 = \text{diag}(1, -1, 0)/\sqrt{2}, \quad H^2 = \text{diag}(1, 1, -2)/\sqrt{6}.$$

In the fundamental representation, we consider the three vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, which have the weights

$$w^1 = (1, 1/\sqrt{3})/\sqrt{2}, \quad w^2 = (-1, 1/\sqrt{3})/\sqrt{2}, \quad w^3 = (0, -2/\sqrt{3})/\sqrt{2}.$$

These three weights form an equilateral triangle. Then the roots are the vertices of a hexagon. Alternatively, we could just identify the roots by computing commutators of Gell-Mann matrices.

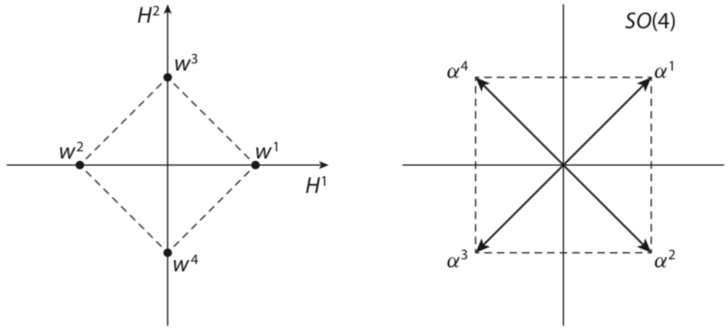
Example. The roots of $\mathfrak{so}(4)$. In this case one basis for the Cartan subalgebra is J_{12} and J_{34}, which can be simultaneously diagonalized to

$$H^1 = \text{diag}(1, -1, 0, 0), \quad H^2 = (0, 0, 1, -1)$$

by switching to the 'polar' basis $(x_1, x_2, x_3, x_4) \to x_1 \pm ix_2, x_3 \pm ix_4$. We can then read off the weights of the fundamental representation,

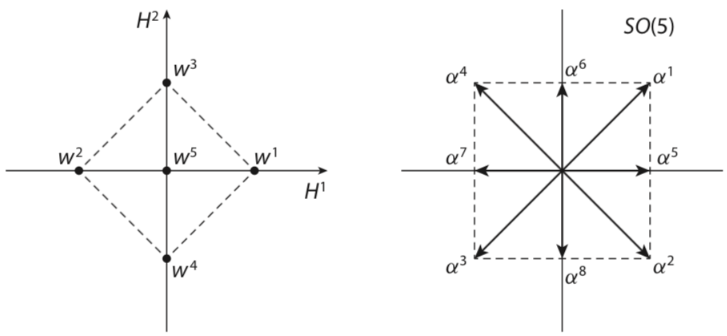
$$w^1 = (1, 0), \quad w^2 = (-1, 0), \quad w^3 = (0, 1), \quad w^4 = (0, -1)$$

which form the vertices of a square. The roots also form a square in this case, tilted by 45° relative to the weights. The catch is that $w^1 - w^3$ is a root, but $w^1 - w^2$ is not, as one can see by counting the total number of roots.



This result also shows that $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, as we've seen.

Example. The roots of $\mathfrak{so}(5)$. There are still two Cartan generators; the only difference is that we now have a zero weight from the vector $(0,0,0,0,1)$.



We thus recover a familiar root pattern.

Example. The roots of $\mathfrak{so}(6)$. Following the pattern of $\mathfrak{so}(4)$, the roots are $e_1 \pm e_2$, $-e_1 \pm e_2$, and cyclic permutations thereof. The pattern for $\mathfrak{so}(2n)$ is identical, and the pattern for $\mathfrak{so}(2n+1)$ is just slightly more complicated, with the addition of the roots $\pm e_i$. The simple roots are

$$\mathfrak{so}(2n)\colon e^1 - e^2, \ldots, e^{n-1} - e^n, e^{n-1} + e^n, \quad \mathfrak{so}(2n+1)\colon e^1 - e^2, \ldots, e^{n-1} - e^n, e^n$$

from which one can read off the Dynkin diagram.

Example. The roots of $\mathfrak{su}(n)$. In this case, the counting is simple, since every difference of weights in the fundamental is a root: this is because $\dim \mathfrak{su}(n) = n^2$, so there are $n^2 - n$ roots, and there are $n(n-1)$ possible differences of weights. In the case of $\mathfrak{su}(4)$, we have

$$H^1 = \mathrm{diag}(1,-1,0,0)/\sqrt{2}, \quad H^2 = \mathrm{diag}(1,1,-2,0)/\sqrt{6}, \quad H^3 = \mathrm{diag}(1,1,1,-3)/2\sqrt{3}.$$

The weights are the vertices of a tetrahedron, so the roots are the sides of a tetrahedron. In particular, adjacent sides make an angle of 60°, while nonadjacent sides make an angle of 90°, entirely in accordance with the Dynkin diagram. For $\mathfrak{su}(n)$, the pattern continues, with a 'higher-dimension tetrahedron'.

Example. The roots of $\mathfrak{sp}(2n)$. Earlier, we showed that $\mathfrak{sp}(2n)$ had basis

$$iA \otimes I, \quad S_i \otimes \sigma_i$$

from which we identify a Cartan subalgebra

$$H^i = u^i \otimes \sigma_3, \quad u^i_{jk} = \delta^i_j \delta^i_k.$$

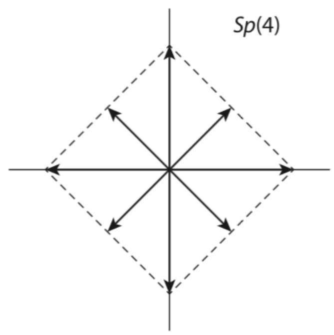
For example, for $\mathfrak{sp}(4)$ we have

$$H^1 = \mathrm{diag}(1, 0, -1, 0), \quad H^2 = \mathrm{diag}(0, 1, 0, -1)$$

from which we read off the weights

$$w^1 = (1, 0), \quad w^2 = (0, 1), \quad w^3 = (-1, 0), \quad w^4 = (0, -1)$$

These are the same weights as for $\mathfrak{so}(4)$, but $\mathfrak{sp}(4)$ has higher dimension, so in this case all of the differences are roots.



Since this is just the root diagram of $\mathfrak{so}(5)$ tilted, $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$.

6.6 Dynkin Diagrams

In this section, we give a quick outline of the Cartan classification.

- First, we can establish the existence of the infinite families A_n, B_n, C_n, and D_n and their Dynkin diagrams by our work in the previous section, reading off angles between the simple roots. We also discovered G_2 earlier by classifying all rank 2 Lie algebras.
- Given simple roots α_i, we define the unit vectors $u_i = \alpha_i/|\alpha_i|$. In this section, we work solely with roots normalized to unit length. Then

$$2u_i \cdot u_j = -\sqrt{\zeta_{ij}}, \quad \zeta_{ij} \in \{0, 1, 2, 3\} = \text{number of lines connecting } i \text{ and } j.$$

  Note that if u_i and u_j are connected at all, then $u_i \cdot u_j \leq -1/2$.
- First, there are no Dynkin diagrams with loops. Summing over the k roots in the loop,

$$\left(\sum_i u_i\right)^2 = k + \sum_{i \neq j} u_i \cdot u_j \leq 0$$

  but the left-hand side must be positive.

- Next, the number of lines coming out of a circle cannot be more than three. Let u be directly connected to the roots $w_1, \ldots, w_k$. The roots cannot be directly connected to each other otherwise, since there are loops, so they must be orthogonal. Now note that

$$\sum_i (u \cdot w_i)^2 = \frac{1}{4} \sum_i \zeta_{ui} \geq 1.$$

But the left-hand side is the length of u when projected down to the subspace spanned by the w_i, so it must be less than one, a contradiction.

- Next, shrinking a chain of circles each connected with a single line gives a valid Dynkin diagram. To see this, let $u_1, \ldots, u_k$ be such a chain. Then we can replace these vectors with the vector $u = \sum_i u_i$, which is properly normalized since

$$u^2 = \sum_{i=1}^{k} u_i^2 + 2 \sum_{i=1}^{k-1} u_i \cdot u_{i+1} = k - 2\frac{k-1}{2} = 1.$$

Any other root w can be directly connected to the chain only once, or else there will be loops. If w connects to u_i, then $w \cdot u_i = w \cdot u$. Hence everything about the Dynkin diagram stays the same when we collapse the u_i to u.

- Suppose a Dynkin diagram contains a 'fork', i.e. there is a root connected to three other roots. Then all of the connections must be by single lines, and moreover, each of the three roots can only continue in a linear chain made of single lines. We'll return to this case below.

- Our results above show that G_2 is the only Dynkin diagram with a triple line. Next, consider a double line; we can connect single line chains on both ends. Suppose there are n and m of them, u_i and v_i, with the last ones connected by the double line, so $u_m \cdot v_m = -1/\sqrt{2}$.

We define $u = \sum_i i u_i$ and $v = \sum_j j v_j$. Then we have

$$u^2 = \frac{n(n+1)}{2}, \quad v^2 = \frac{m(m+1)}{2}, \quad u \cdot v = -\frac{nm}{\sqrt{2}}.$$

Then the Cauchy-Schwartz inequality gives $(m-1)(n-1) \leq 2$. The cases $m = 1$ or $n = 1$ give the infinite families B_n and C_n. The only other case is $m = n = 2$, which gives F_4.

- Finally, we classify Dynkin diagrams with a fork, with n, m, and p roots in each chain, including the central root. Similar considerations show that

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{p} > 1.$$

Then if $n = m = 1$, then p is arbitrary, recovering A_n. If all of the numbers are at least 3, the inequality is violated, so otherwise one of them must be two, say $p = 2$. Then

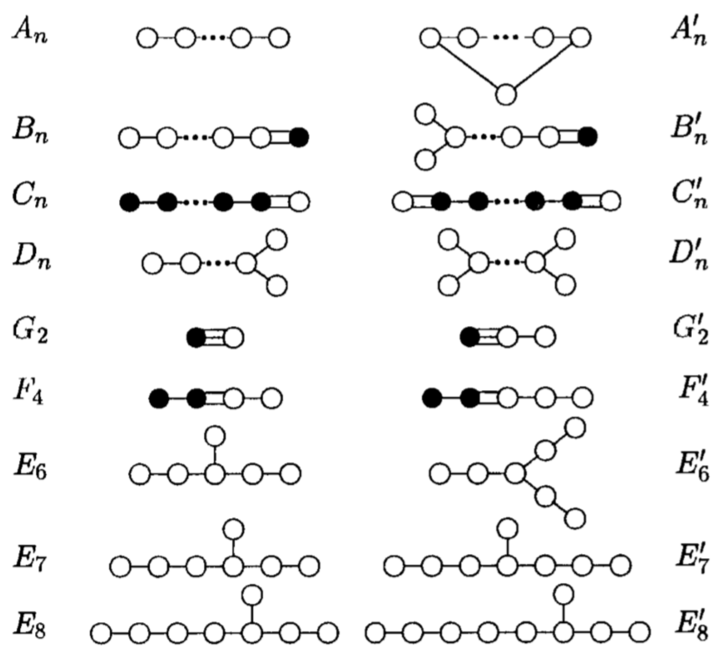
$$\frac{1}{n} + \frac{1}{m} > \frac{1}{2}.$$

In the case $n = 2$, m is arbitrary, recovering D_n. Finally, the only other solutions are $(3,3)$, giving E_6, $(3,4)$, giving E_7, and $(3,5)$, giving E_8.

- Incidentally, the same inequality occurs when classifying the platonic solids; then E_6 corresponds to the tetrahedron, E_7 to the cube, and E_8 to the icosahedron.

As an application, we consider regular subalgebras.

- A regular subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra whose Cartan generators are linear combinations of the Cartan generators of A. A regular maximal subalgebra has the same rank as $\mathfrak{g}$.
- In physics, we generally care about regular subalgebras rather than general subalgebras, because if G is a symmetry group, the conserved charges are the Cartan generators. Thus the conserved charges in the subalgebras we care about are typically built from the original ones.
- We can construct non-maximal regular subalgebras by deleting a circle from a Dynkin diagram; the circle itself provides a $\mathfrak{u}(1)$ factor. For example, applying this to $SU(5)$ gives $SU(3) \times SU(2) \times U(1) \subset SU(5)$.
- To construct regular maximal subalgebras, we use a trick. Given simple roots α^i, we may formally add the lowest root α^0. This root has appropriate angles with the others, because $\alpha^0 - \alpha^i$ is not a root for any i. However, the augmented root system is linearly dependent.
- Working case by case, one can construct the extended Dynkin diagrams below.



  There are some degenerate cases here; for instance A'_1 would need a quadruple line.

- Regular maximal subalgebras can thus be constructed by removing a circle from the extended Dynkin diagram. For example, we find $\mathfrak{so}(2n) \subset \mathfrak{so}(2n+1)$, which we observed earlier, and $\mathfrak{so}(2k) \oplus \mathfrak{so}(2n-2k) \subset \mathfrak{so}(2n)$.

7 More Representations

7.1 The Galilean Group

First, we give an example of group contraction.

- Consider the rotation algebra $\mathfrak{so}(3)$ and let $z = L\zeta$, so $\partial_z = L^{-1}\partial_\zeta$. Then if we take L to infinity while keeping ζ of order one, to lowest order in L we have

$$J_z = -i(x\partial_y - y\partial_x), \quad J_x = -i(y\partial_z - z\partial_y) \to i(L\zeta)\partial_y, \quad J_y \to -i(L\zeta)\partial_x.$$

  That is, J_x and J_y limit to translations P_y and P_x, where $P_i = i\partial_i$.

- Intuitively, imagine the original vector fields as describing symmetries of the sphere. Then taking z to be large corresponds to 'reverting to the flat Earth', zooming in on the flat patch near the North pole. The resulting symmetry group is $ISO(2)$ or $E(2)$, containing symmetries of two-dimensional Euclidean space.

- Similar reasoning works for $\mathfrak{so}(n)$. Letting a Latin index denote a value from 1 to $n-1$, the J_{ij} are unaffected while J_{in} contracts to the translation P_i, giving $E(n-1)$.

- Next, we consider the Lorentz algebra. We start with the generators

$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad [J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - (\rho \leftrightarrow \sigma)).$$

  We then contract by setting $x_0 = ct$ with $c \to \infty$. Then J_{ij} is unmodified and we have

$$J_{0i} \to -K_i = ctP_i$$

  where K_i is a Galilean boost. Now, as before, we forget about the time component, setting it to a $O(1)$ number. Then we can replace K_i with the translation P_i, and

$$[P_i, P_j] = \frac{1}{(ct)^2}(-iJ_{ij}) \to 0$$

  as expected.

- There are some annoying signs to keep track of above. In the Galilean case, all transformations are passive, so K_i translates backwards, giving the minus sign in $J_{0i} \to -K_i = ctP_i$. On the other hand, the Lorentz rotation J_{ij} corresponds to the Galilean rotation J_{ij}, with no extra minus sign, because it picks up a minus sign from lowering the x^μ index in $J_{\mu\nu}$.

- The Galilean translations and rotations satisfy

$$[J_i, P_j] = i\epsilon_{ijk}P_k, \quad J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$$

  which tells us that displacement transforms as a vector. Up to some possible signs, everything is mathematically the same as in $\mathfrak{so}(4)$, and we arrive at $E(3)$.

- Similarly, one can see how the de Sitter $SO(4,1)$ would contract to the Poincare group; the $SO(3,1)$ subgroup gives the Lorentz group, while the remaining four operators give spacetime translations.

Alternatively, we can start from the Poincare group and contract to the full Galilean group. This time, we start with the Galilean group and 'extend', or deform to the Poincare group.

- Unlike before, we keep the time coordinate, so the boost K_i and translation P_i are not equivalent. We also pick up the time translation $H = i\partial_t$. We can explicitly work out
$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, P_j] = i\epsilon_{ijk}P_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k$$
which show that angular momentum, displacement, and velocity are vectors.

- The remaining commutation relations are, by direct calculation,
$$[P_i, H] = 0, \quad [J_i, H] = 0, \quad [P_i, P_j] = 0, \quad [K_i, H] = iP_k, \quad [K_i, P_j] = 0, \quad [K_i, K_j] = 0$$
which all make sense; boosts are the only elements that don't commute with H, since only they involve time. In words, doing a boost earlier or later differs by a translation.

- To return to the Poincare algebra, we extend the boost to
$$K_i = -i\left(t\partial_x + \frac{1}{c^2}x\partial_t\right)$$
introducing a new dimensionful quantity c. The commutation relations that change are
$$[K_i, K_j] = -\frac{1}{c^2}i\epsilon_{ijk}J_k, \quad [K_i, P_j] = i\delta_{ij}\frac{1}{c^2}H.$$
Absorbing a few factors of c into the K_i and H gives back the usual Poincare algebra. The resulting commutators of $P_\mu = i\partial_\mu = (H, P_i)$ are
$$[J_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu).$$

- We can also extend the Galilean algebra to the quantum Galilean algebra, or Bargmann algebra. We introduce the dimensionful quantity $\hbar$, and multiply both P_i and H with it. We then think of H as representing the physical energy.

- There's another tricky sign here, since $-P_i = P^i$ represents the physical momentum. To see this, note that the quantum commutation relation $[x, p_x] = i$ implies that $p_x = -i\partial_x = -P_i$.

- Next, consider the commutation relation $[K_i, P_j] = 0$. Naively, it means that the commutator is zero, but it just as well could mean that the commutator is a nonzero quantity that commutes with everything else in the algebra; from the perspective of the commutators, this is perfectly consistent. Thus we can set
$$[K_i, P_j] = iM\delta_{ij}$$
where M is a real c-number and has dimensions of mass.

- To show that M can be regarded as a mass, note that
$$[K_i, P^2/2M] = \frac{1}{2M}(2iM\delta_{ij})P_j = iP_i = [K_i, H]$$
and everything else commutes with both $P^2/2M$ and H. Therefore it is consistent to say
$$H = \frac{P^2}{2M}$$
up to the freedom of adding another c-number. Expanding in components immediately gives the Schrodinger equation.

- Equivalently, we can directly modify the boost generator to

$$K_x = -it\partial_x - Mx$$

which immediately yields $[K_i, P_j] = iM\delta_{ij}$ as before.

- The mass also comes from the contraction of the Poincare algebra, as

$$[K_x, P_x] = \frac{i}{c^2}H$$

and expanding out the relativistic energy $H = \gamma Mc^2$ gives $[K_x, P_x] \to iM$.

The central extension is essential for understanding the Galilean invariance of nonrelativistic quantum mechanics.

- Consider the de Broglie relations $p = \hbar k$ and $E = \hbar\omega$ for a quantum particle of mass M. Under a Galilean boost of velocity u,

$$p' = p + Mu, \quad E' = E + up + \frac{1}{2}Mu^2.$$

On the other hand, the wavenumber and frequency of a plane wave transform as

$$k' = k, \quad \omega' = \omega + uk$$

where the uk term is from the Doppler shift, and these results are totally different.

- The resolution is that Galilean invariance only requires that $|\psi'(x', t')|^2 = |\psi(x, t)|^2$, and allows the addition of an arbitrary phase. To find this phase, let the original wavefunction be a plane wave $\psi(x, t) = e^{i(px-Et)/\hbar}$. Then the transformed wavefunction should have the same form to satisfy the de Broglie relations,

$$\psi'(x', t') = e^{i(p'x'-E't')/\hbar} = e^{iM(ux-u^2t/2)/\hbar}\psi(x, t).$$

- For an infinitesimal boost, we thus have

$$\psi'(x, t) = (1 - iuK_x)\psi(x, t)$$

where K_x has the extra Mx term. Hence the central extension gives the right extra phases for Galilean boosts of wavefunctions, and it descends all the way down from the relativistic Mc^2 rest energy.

- Mathematically, in Lagrangian mechanics, the Galilean algebra can be represented by differential operators on the configuration space. We have lifted this action to one on wavefunctions; the central extension appears because we are now working with a projective representation. The same happens if we switch to Lagrangian to Hamiltonian mechanics, promoting our operators to functions on phase space by Noether's theorem. All the same equations hold, with commutators replaced with Poisson brackets.

7.2 The Lorentz Group

We now review the structure of the Lorentz algebra and Lorentz group.

- The rotations J_i and boosts K_i satisfy

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

  The first two state that angular momentum and boost velocity are vectors. To see that J_k must appear on the right-hand side of the third equation, and not K_k, note that under parity $\mathbf{J} \to \mathbf{J}$ but $\mathbf{K} \to -\mathbf{K}$. Then the equation $[K_x, K_y] = -iK_z$ cannot be parity invariant.

- In terms of differential operators, we have, e.g.

$$K_x = -i(t\partial_x + x\partial_t), \quad J_z = -i(x\partial_y - y\partial_x).$$

  These provide a more compact way to compute the commutation relations compared to matrices.

- As we saw earlier from Dynkin diagrams, the complex Lie algebra $\mathfrak{so}(4)$ decomposes as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The Lorentz algebra $\mathfrak{so}(3,1)$ is identical; both it and $\mathfrak{so}(4)_\mathbb{R}$ are real forms of $\mathfrak{so}(4)$, with their boost generators differing by a factor of i.

- For the Lorentz algebra, the two subalgebras are

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i), \quad [J_i^+, J_j^-] = 0, \quad [J_i^\pm, J_j^\pm] = i\epsilon_{ijk}J_k^\pm.$$

  Note that these two factors are swapped by parity.

- Our earlier observations about the representation theory of $SO(4)$ carry over. For example, symmetric and antisymmetric tensors decompose in the exact same way. We can still define self-dual and anti self-dual tensors; for example, the electromagnetic field $F^{\mu\nu}$ breaks into the self-dual and anti self-dual fields $\mathbf{E} \pm i\mathbf{B}$. If we restrict the symmetry group to $SO(3)$, then any combination of these representations is a representation, recovering the vectors $\mathbf{E}$ and $\mathbf{B}$.

- The Lorentz generators themselves can be thought of in the same way, because we can package them into an antisymmetric rank 2 tensor

$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad J_{ij} = \epsilon_{ijk}J_3, \quad J_{0i} = -K_i.$$

  Then $J_{\mu\nu}$ decomposes into the $J_i^\pm$, as we saw above. Restricting the symmetry group to $SO(3)$ gives the vectors $\mathbf{J}$ and $\mathbf{K}$.

- The group $SO(3,1)$ has four connected components, which can be reached by time reversal and parity. For general $SO(n,m)$ there are still four connected components, but parity only yields a different connected component if n is odd; otherwise we can get to it by flipping one spatial coordinate.

7.3 Tensor Methods

We'll now introduce tensor methods for finding finite-dimensional irreps of the classical groups, starting with the simplest example of $GL(n, \mathbb{C})$.

- For brevity, let $g \in \mathrm{Mat}(n, \mathbb{C})$ denote the representation of a group element $g \in GL(n, \mathbb{C})$ in the fundamental representation. Then there are four closely related irreps of $GL(n, \mathbb{C})$,

$$\{g\}, \quad \{g^*\}, \quad \{g^{-1T}\}, \quad \{g^{-1\dagger}\}.$$

These are called the fundamental representations of $GL(n, \mathbb{C})$. They are inequivalent, as

$$g^* = SgS^{-1}$$

cannot hold for $g = \alpha I$ for complex α, with similar logic for the others.

- For simplicity, we focus on $\{g\}$ and the dual/contragradient representation $\{g^{-1T}\}$, which act on vector spaces V and $\tilde{V}$ respectively. Vectors in these spaces transform as

$$x^a \to x'^a = g^a{}_b x^b, \quad y_a \to y'_a = y_b (g^{-1})^b{}_a.$$

Here, the transpose swaps the order of the indices on g but maintains their vertical positions; we will always denote the fundamental representation with an upper index and the dual representation with a lower index; note that the contraction $x^a y_a$ is invariant.

- For the other representations, we would need more types of indices to keep our notation explicit. For example, we may use dotted indices,

$$z_{\dot{a}} \to z'_{\dot{a}} = (g^*)_{\dot{a}}{}^{\dot{b}} z_{\dot{b}} = z_{\dot{b}} (g^\dagger)^{\dot{b}}{}_{\dot{a}}, \quad w^{\dot{a}} \to w'^{\dot{a}} = (g^{-1\dagger})^{\dot{a}}{}_{\dot{b}} w^{\dot{b}}$$

for $\{g^*\}$ and $\{g^{-1\dagger}\}$ respectively. The contraction $w^{\dot{a}} z_{\dot{a}}$ is invariant. For simplicity, we'll ignore these representations and return to them later.

- We define the tensor product space of tensors of type (i, j) by

$$T^i_j = V^{\otimes i} \otimes \tilde{V}^{\otimes j}.$$

We can define a map $T^i_j \to T^{i-1}_{j-1}$ by contracting any pair of upper and lower indices.

- Specializing to T^1_1, the tensor $\delta^a{}_b$ is invariant; its components stay the same under any $GL(n, \mathbb{C})$ transformation. One can show this the only elementary invariant tensor; all other invariant tensors can be built out of it, e.g. by tensor product.

- Using the invariant tensor $\delta^a{}_b$, we can decompose T^1_1 into irreps as

$$t^a{}_b = \hat{t}^a{}_b + \delta^a{}_b \frac{t^c{}_c}{n}$$

where $\hat{t}^a{}_b$ is the traceless part of $t^a{}_b$. We can think of the second term as the image of $t^a{}_b$ under the projection $\delta^b{}_a \delta^{a'}{}_{b'}/n$.

- Generally, let I be an arbitrary invariant tensor that is also a projection operator; that is, I is a tensor of type (i,i) which acts by contracting half of its indices, and $I^2 = I$. Then for any tensor t we can write
$$t = I(t) + (1-I)(t)$$
and we claim that if we repeat this process on each term until it terminates, we will have the maximal decomposition, i.e. t will be decomposed into irreps.

- To see this, let P be a projection operator onto an irrep, so
$$gPt = Pgt.$$
Since t is general, this is equivalent to $P = gPg^{-1}$, so P is an invariant tensor. So all projection operators will be accounted for in our decomposition.

- Next, we consider T^2_0. Neither of the naive candidates $\delta_a^{a'}\delta_b^{b'}$ and $\delta_a^{b'}\delta_b^{a'}$ are projection operators. We could find projection operators by brute force, but a better way is to recall that the projection operators in the symmetric group S_2 are the symmetrizer and antisymmetrizer, and S_2 acts on T^2_0, commuting with the action of $GL(n,\mathbb{C})$. Thus these projection operators indeed produce $GL(n,\mathbb{C})$ irreps, explicitly
$$t^{ab} = t^{(ab)} + t^{[ab]}.$$
This is a full decomposition, as there is no trace to remove.

- Now consider the general case T^i_j. We take a Young tableau λ from S_i for the upper indices and a Young tableau σ from S_j for the lower indices. Performing the resulting projection maps $t^{\{a\}}_{\{b\}}$ to $t^{\lambda\{a\}}_{\sigma\{b\}}$, where we use multi-index notation. Now we haven't taken care of the traces yet, because they deal with contractions between upper and lower indices, while we've only treated these sets separately.

- To remove the traces in practice, we work recursively. For T^2_2, we may write
$$t^{ac}_{bd} = r^{ac}_{bd} + \delta^a_b(s_1)^c_d + \delta^a_d(s_2)^c_b + \delta^c_b(s_3)^a_d + \delta^c_d(s_4)^a_b + \delta^a_b\delta^c_d u_1 + \delta^a_d\delta^c_d u_2$$
where r, the s_i, and the u_i are all traceless. We determine the u_i by contracting both sides with two delta functions, giving
$$n(nu_1 + u_2) = t^{ac}_{ac}, \quad n(u_1 + nu_2)t^{ac}_{ca}.$$
Next, we contract both sides of the original equation with one delta function, determining the s_i. This procedure ensures we don't 'double count' any traces. It works independently of the Young tableaux, which simply set some of the s_i and u_i equal to each other.

- There are no more available invariant tensors, so we've arrived at an irrep. One can show that the irrep associated with Young tableaux (λ,σ) is equivalent to that of (λ',σ') if and only if λ and λ' have the same Young diagram, and σ and σ' have the same Young diagram.

- Finally, restoring the $\{g^*\}$ and $\{{g^{-1}}^\dagger\}$ representations, the inequivalent irreps of $GL(n,\mathbb{C})$ generated by the four fundamental representations are indexed by four Young diagrams; they are fully traceless amongst the dotted indices and amongst the undotted indices.

Note. In physics, every Lie group is regarded as a subgroup of $GL(n, \mathbb{C})$ for some n, and we have a fundamental, n-dimensional representation by taking matrices. In general, all representations sufficiently closely related to this matrix representation are also called fundamental representations, and generally there are 4, 2, or 1. These are a subset of the fundamental representations in mathematics, i.e. irreps with exactly one nonzero Dynkin label, which is equal to one. It can be shown that every (finite-dimensional) representation can be built out of these irreps. This is often also true for the physicist's fundamental representations, e.g. in the case of $GL(n, \mathbb{C})$, but not always. Note that the mathematician's fundamental representations need not have dimension n.

The analysis for other classical linear groups is similar, but with more invariant tensors.

Example. The unitary group $U(n)$. In this case, the four fundamental representations collapse to just two, $\{g\}$ and $\{g^*\}$, which are usually called the fundamental and antifundamental. Another way of seeing this is that we have a new invariant tensor,

$$g^a{}_b \delta^b_{\dot{d}} (u^\dagger)^{\dot{d}}{}_{\dot{c}} = \delta^a_{\dot{c}}$$

along with $\delta^{\dot{a}}_b$. These invariant tensors relate dotted and undotted indices; hence we can remove all dots before performing the usual procedure. This is just like how we can raise all indices using the metric tensor in relativity, and by the same reasoning we call these new invariant tensors metrics. The results for $GL(n, \mathbb{C})$ go through unchanged, except that we eliminate the dotted indices.

Example. We may also consider indefinite metrics. Suppose that the matrix of $\delta^{\dot{a}}_b$ has n_+ positive eigenvalues and n_- negative eigenvalues. Then demanding it is an invariant tensor yields the unitary group $U(n_+, n_-)$, and the finite-dimensional irreps are just the same as that of $U(n_+ + n_-)$.

However, making the metric indefinite does have important effects. The resulting group is not compact, so there are no finite-dimensional unitary representations. On the other hand, for definite metric, the group is compact, so there are no infinite-dimensional representations.

Example. The special linear group $SL(n, \mathbb{C})$. In this case we pick up another invariant tensor, the Levi-Civita tensor. Specifically, we have four Levi-Civitas with n indices, which are all upper/lower and dotted/undotted. By itself, the Levi-Civita doesn't further decompose the irreps, but it does set irreps equal; for example, the fully antisymmetric T^n_0 irrep is now equivalent to the trivial irrep.

In general we can replace m antisymmetric upper/lower indices with $n - m$ antisymmetric lower/upper indices. For a Young diagram λ with column lengths m_i, define $\tilde{\lambda}$ to have column lengths $n - m_i$. Then $(\lambda, \mu; \tau, \kappa)$ is equivalent to $(\lambda', \mu'; \tau', \kappa')$ if the Young diagrams obtained by combining the columns of λ and $\tilde{\mu}$, and of τ and $\tilde{\kappa}$, are the same as their primed counterparts.

Example. In the special case $SU(m_+, m_-)$, the same final statement is true, except that we only have undotted indices.

Example. In the more special case $SL(2, \mathbb{C})$, the Levi-Civita ϵ^{ab} is more useful. Since it relates one upper index to one lower index, the fundamental representations collapse to just two, $\{g\}$ and $\{g^*\}$, and all indices can be taken to be upper indices. Moreover, by the logic above, we can replace any two antisymmetric indices with nothing. Then the irreps are indexed by two integers (i, k) and consist of tensors with i symmetric undotted indices and k symmetric dotted indices. This is as one would expect, since $SL(2, \mathbb{C})$ is the double cover of $SO(3, 1)$.

Example. Next, we consider some even more special cases.

- In the case $SU(2)$, two of the above simplifications combine, and we only have a single type of index. The irreps are indexed by a single integer.
- In the case $SU(3)$, we only have undotted indices, and we can replace any two antisymmetric upper/lower indices with a single lower/upper index. The irreps are indexed by two integers (i, j) and consist of tensors with i symmetric upper indices and j symmetric lower indices.
- In the case $SU(4)$, there are three fundamental representations: 4, $\overline{4}$, and 6. However, the physicists' fundamentals are sufficient because the 6 is the antisymmetric part of 4×4.

Example. When we restrict to $GL(n, \mathbb{R})$, we lose all dotted indices, retaining only $\{g\}$ and $\{g^{-1^T}\}$. Now consider $O(n_+, n_-)$ defined similarly to $U(n_+, n_-)$. The new invariant tensor relates upper and lower indices; it is a metric tensor ξ^{ab} in the usual sense, and we can raise all indices; there is only one fundamental representation. We can also use the metric tensor to remove traces between upper indices (not between one upper and one lower index, as we'd done before). Thus the irreps are traceless tensors with only upper indices with definite symmetry. The invariant tensor δ^a_b is rendered obsolete, since it can be built out of the metric, $\xi^{ab}\xi_{bc} = \delta^a_c$.

The addition of the metric splits up $GL(n, \mathbb{R})$ irreps. For example, t^a_b is a $GL(n, \mathbb{R})$ irrep, but we can raise the index with the metric, then split t^{ab} into symmetric and antisymmetric parts.

Example. The special case of $O(2)$. Consider the first two indices in $t^{ab\cdots}$. We can take out the antisymmetric part using the Levi-Civita, and the trace using the metric. These two irreps are not equivalent, as they differ by how they transform under reflection; they are the scalar and pseudoscalar.

By repeating this procedure, we find rank n symmetric traceless tensors, for all $n > 1$. Imagine starting with a symmetric rank n tensor. Then the trace is a symmetric rank $n-2$ tensor, so the symmetric traceless rank n tensor has

$$(n+1) - (n-1) = 2$$

degrees of freedom. All of these two-dimensional irreps are inequivalent, corresponding to 'angular momentum' eigenvalue n.

Example. Finally, we specialize to $SO(n)$. Since there's only one kind of index now, the Levi-Civita can be used to break apart irreps, specifically those from tensors of rank $n/2$, into self-dual and anti self-dual components. We'll consider a few low-dimensional cases.

- In the case $SO(2)$, the one-dimensional irreps of $O(2)$ become equivalent, since one can multiply by ϵ^a_a where an index is lowered using the metric. The two-dimensional irreps decompose, by 'taking out the trace' using ϵ^a_b.
- In the case $SO(3)$, we can symmetrize all the indices by the same logic as for $SU(3)$. By similar counting as for $O(2)$, we confirm that the symmetric traceless tensors indeed have the expected integer dimensions. Irreps of $O(3)$ do not break apart, but do become equivalent, identifying the pseudovector and vector, and pseudoscalar and scalar.
- In the case $SO(3,1)$, the situation is similar to $SO(2)$, in that irreps of $O(3,1)$ do break apart; for example, $F^{\mu\nu}$ decomposes into $\mathbf{E} \pm i\mathbf{B}$. These are self-dual and anti self-dual components.

- Consider Spin(8), the double cover of $SO(8)$. The Lie algebra $\mathfrak{so}(8)$ has threefold symmetry, called triality, so there are three fundamental representations of dimension 8. Only one is a physicist's fundamental; the other two are spinors and hence cannot be built from the first.

We've already seen many of these features in our earlier look at tensor methods.

7.4 Using Tensor Methods

In this section, we give some practical applications of tensor methods.

Note. We can think of tensors by themselves, but in the context of quantum mechanics, they can be interpreted as wavefunctions. For example, consider the state

$$|\psi\rangle = \psi^i |i\rangle$$

where the $|i\rangle$ transform in the fundamental representation of $SU(3)$, say

$$|i\rangle \to U_i^j |j\rangle.$$

Then we have

$$|\psi\rangle \to U_i^j \psi^i |j\rangle = U_j^i \psi^j |i\rangle$$

so we may alternatively transform the wavefunction as

$$\psi^i \to U_j^i \psi^j$$

while keeping the states the same. We use upper and lower indices to distinguish the fundamental and antifundamental. Note that the transformations for the states and wavefunction are similar but not the same; it's like how a translation to the right is realized on the wavefunction with a minus sign, $\psi(x) \to \psi(x-a)$. Now for two particles,

$$|\psi\rangle = \psi^{ij} |ij\rangle$$

the wavefunction ψ^{ij} transforms in a tensor product representation, and we can decompose it using the methods above. The bra transforms in the contragradient representation, which is just the conjugate representation here; thus upstairs and downstairs indices are switched. A inner product $\langle\phi|\psi\rangle$ can be computed by contracting all the indices of the respective tensors; it is zero when the indices don't match up, reflecting the fact that different irreps are orthogonal. Similarly, tensor operators can be expanded in terms of wavefunctions, e.g. $\hat{A} = A^i \hat{O}_i$.

Note. Note that the $(1,0)$ irrep of $SU(3)$ corresponds to a $(1,0)$ tensor, while the $(0,1)$ irrep corresponds to a $(0,1)$ tensor. We also know that the only new irrep that appears in $(1,0)^{\otimes n} \otimes (0,1)^{\otimes m}$ is the (n,m) irrep. But in the tensor picture, the only new irrep is the symmetric traceless tensor with m upper indices and n lower indices. Hence the two labeling schemes of Dynkin labels and tensor ranks coincide, and this reasoning generalizes.

Note. Tensors explain the 'triality' symmetry of the $SU(3)$ representations. For example, in

$$3 \times 8 = 15 + \overline{6} + 3, \quad (1,0) \times (1,1) = (2,1) + (0,2) + (1,0)$$

all of the (n,m) irreps on the right-hand side have the same value of $(n-m)$ mod 3. This holds because all of the invariant tensors in the problem have type (n,m) where $n-m$ is divisible by 3.

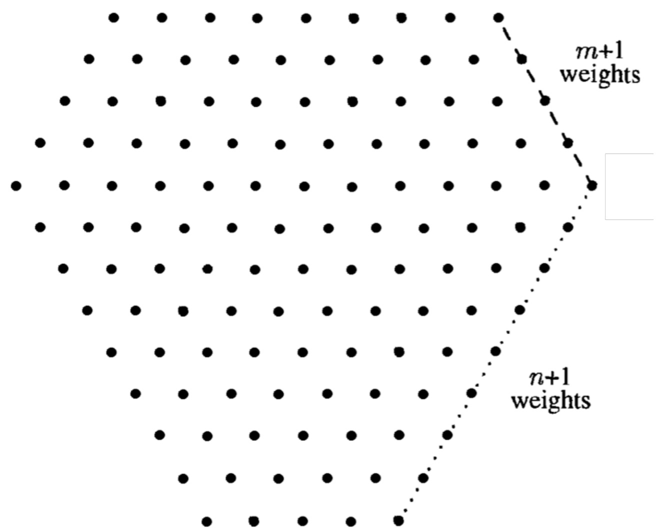
Example. Computing the dimension of the (n, m) irrep of $SU(3)$. Note that

$$B(n, m) = \frac{(n+2)(n+1)}{2} \frac{(m+2)(m+1)}{2}$$

is the number of symmetric (n, m) tensors. Therefore the number of traceless symmetric tensors is

$$\dim(n, m) = B(n, m) - B(n-1, m-1) = \frac{(n+1)(m+1)(n+m+2)}{2}.$$

Example. The weights of the (n, m) irrep of $SU(3)$ are shown below.



We start at the top-right and initially can move leftward or down-right. Then the states on the upper and upper-right edges are nondegenerate, so all the outer states are nondegenerate by Weyl reflection. In general, the degeneracy increases by one every time we move in a layer, until we reach a triangular layer, at which point it remains constant.

Each of the states shown here can be associated with a tensor, i.e. viewing the tensor as the state's wavefunction. For example, when we move along the path shown, the tensors change as

$$v^{11111111}_{3333} \to v^{11111111}_{2222} \to v^{33333333}_{2222}.$$

Now consider the states one layer inward. These contain tensors with one index in common between the top and bottom. To count the number of states in this layer, we find the number of symmetric tensors of this form (where the symmetry allows us to neglect index order) and subtract the number of traces, arriving at a degeneracy of 2. The same logic holds for all layers. The casework changes when we hit a triangular layer, because from that point on one of m or n is zero.

Note. Suppose we're consider the matrix elements $\langle u|W|v\rangle$, where W, $\langle u|$, and $|v\rangle$ transform in irreps R_W, R_u, and R_v. By the Wigner-Eckart theorem, all such matrix elements are specified by only a few numbers, specifically the number of factors of 1 in $R_W \otimes R_u \otimes R_v$. We've seen examples of this in $SU(2)$, where we found the other matrix elements by raising and lowering, but with tensors we can easily find a general expression by contracting indices.

Example. Suppose everything transforms in the 8 of $SU(3)$. There are two ways to contract the tensors W^i_j, $\overline{u}^i_j$, and v^i_j to yield a 1, either 'clockwise' or 'anticlockwise'. (Note that by convention

u stands for the wavefunction of $|u\rangle$, so $\overline{u}$ is the bra wavefunction.) In this case all of these tensors can be written as matrices, giving

$$\langle u|W|v\rangle = \lambda_1 \operatorname{tr} u^\dagger W v + \lambda_2 \operatorname{tr} u^\dagger v W.$$

Here, we replaced $\overline{u}$ with the matrix $u^\dagger$ because taking the bra conjugates and exchanges upper and lower indices, and the latter is just a transposition in matrix notation.

Example. Suppose u and v are 10's and the W is an 8. Then there is only one contraction,

$$\langle u|W|v\rangle = \lambda \overline{u}_{ijk} W^k_\ell v^{ij\ell}.$$

This is more economical in $SU(3)$ than $SU(2)$, because the tensors tend to have lower ranks; the dimension of the irrep grows quadratically in the rank rather than linearly. However, such methods also work for $SU(2)$, where they can supply explicit formulas for the Clebsch-Gordan coefficients.

7.5 Using Young Tableaux

Finally, we apply Young tableaux to our tensor methods, warming up with $SU(3)$.

- As we've seen, the (n, m) irrep has n upper indices and m lower indices, where everything is traceless and symmetric. To use Young tableaux, we convert every lower index into two antisymmetric upper indices by raising with ϵ^{ijk}.
- The symmetry of the resulting tensor is associated with a Young tableau. For example, the $(1, 1)$ irrep becomes

  $$\begin{array}{|c|c|}\hline k & i \\ \hline \ell & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

  There are two constraints: the k and ℓ indices must be antisymmetric, enforced by the final antisymmetrization step, and the original tensor must have been traceless,

  $$\epsilon_{ik\ell} u^{ik\ell} = 0,$$

  which is enforced by the initial symmetrization. Note that we don't need k and ℓ to be symmetric, and this indeed isn't enforced by the Young tableau.
- Therefore, given a rank n tensor with upper indices, we can project out an irrep by applying a Young tableau with n boxes. Columns with more than 3 boxes automatically give zero. Columns with 3 boxes mean an ϵ^{ijk} factors out of the tensor, so they yield the same irrep as a Young tableau without them.
- There is a simple algorithm to compute the product of the irreps α and β corresponding to tableaux A and B.

  1. Write a's in the first row of B and b's in the second row.
  2. Add the a boxes to A anywhere, as long as a valid tableau is formed and no two a's are in the same column.
  3. Add the b boxes to B similarly. Read the result in Hebrew order and cross it out if there are ever more b's than a's, to prevent double-counting.

We won't prove this, though it's apparent this is just counting every way of combining all the indices, maintaining the symmetry properties found above.

- Note that in this system, triality is enforced by just conserving the number of boxes.

Next, we move on to $SU(n)$.

- Let the weights of the defining representation be $\{\nu^1, \ldots, \nu^n\}$. As shown earlier, these form the vertices of a tetrahedron in $n-1$-dimensional space.

- The roots are simply the differences of these weights. For convenience, we define positive roots to have the form $\nu^i - \nu^j$ for $i < j$. Then the simple roots are

$$\alpha^i = \nu^i - \nu^{i+1}, \quad i = 1, 2, \ldots, n-1.$$

  We can then check the angles are in accordance with the Dynkin diagram.

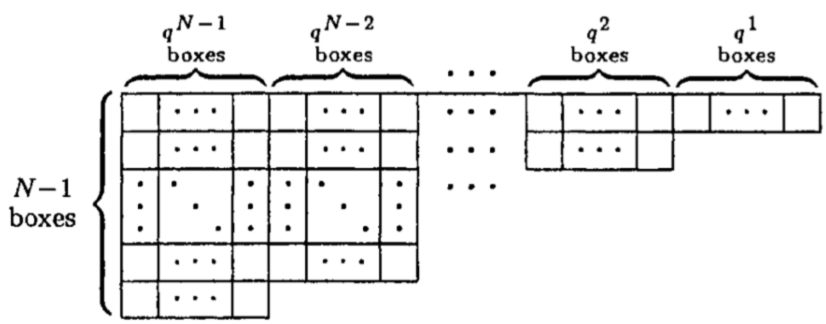
- Taking the dual basis, the fundamental weights are

$$\mu^i = \sum_{j=1}^{i} \nu^j, \quad i = 1, 2, \ldots, n-1.$$

  In particular, the highest weight of the defining representation is $\mu^1 = \nu^1$, as we can then lower with each simple root to get the other ν^i. Then the second-highest weight is ν^2, followed by ν^3, and so on.

- Now consider the antisymmetric tensor product of m copies of the defining representation. Then the highest weight is $\nu^1 + \ldots + \nu^m$ because the indices must be distinct, so we have the m^{th} fundamental representation! In particular, just like in $SU(3)$, we can build everything out of tensor products of only the defining representation.

- General irreps of $SU(n)$ can thus be identified by a Young tableau. The tableau



  represents the irrep with Dynkin indices q^k. We specify a Young tableau with the notation $[\ell_1, \ell_2, \ldots]$ where ℓ_i is the length of column i, so the i^{th} fundamental is specified by $[i]$. The adjoint representation has one defining index and one lowered defining index, so it is $[n-1, 1]$.

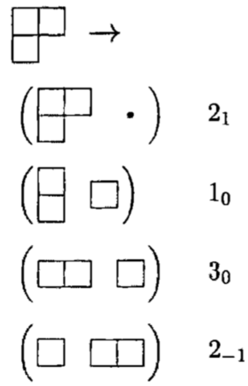
- Consider the conjugate of the defining representation $[1]$. Its lowest weight is ν^n, but since all the ν^i sum to zero, this is equal to $-\mu^{n-1}$. Then the conjugate representation is $[n-1]$. Similarly, the conjugate of $[j]$ is $[n-j]$, and hence the conjugate of any Young tableau can be found by rotating it by 180° and interpreting the top edge as a new bottom edge.

- The algorithm for products above also works, with more letters. To avoid overcounting, there should be at least as many a's as b's, at least as many b's as c's, and so on, at every point. Triality generalizes to N-ality, the number of boxes mod N.
- There is a useful formula for the dimension of an $SU(n)$ irrep, which can be derived from the Weyl character formula. We place an n in the top-left box, then place factors in the other boxes, adding one when we move right and subtracting one when we move down. The product of these factors divided by the product of the hook lengths is the dimension.

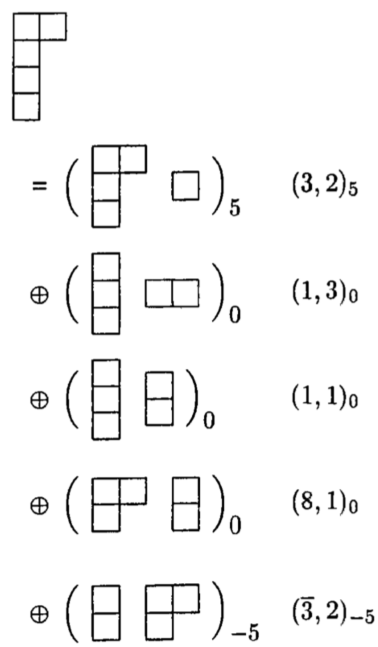
Finally, Young tableaux can be used to decompose representations when restricting to subgroups.

- First, we consider the subgroup
$$SU(n) \times SU(m) \times U(1) \subset SU(n+m).$$
In terms of indices, we imagine the indices can go from 1 to $n+m$, the $SU(n)$ part acts on the indices from 1 to n only, and the $U(1)$ generator is $\mathrm{diag}(m, \ldots, m, -n, \ldots, -n)$ to ensure tracelessness.
- The fundamental decomposes as
$$\square = (\square\,,\ \cdot)_M + (\cdot\ ,\ \square)_N$$
where the first element in each pair is the $SU(n)$ representation, and the subscript indicates the $U(1)$ charge.
- More generally, to decompose a general irrep C we consider the Young tableau. If it has k boxes, then we can only split into a pair of irreps A and B with n and m boxes so that $n+m=k$, and the $U(1)$ charge is $nM - mN$.
- Now, we need to account for the symmetry of C. Let A' and B' be $SU(n+m)$ irreps with the same tableau as A and B. Then if C doesn't appear in $A' \otimes B'$, then $A \otimes B$ surely can't appear in the decomposition of C. In fact, in general the number of times $A \otimes B$ appears in C is the number of times C appears in $A' \otimes B'$.
- Another important situation is the embedding
$$SU(n) \times SU(m) \subset SU(nm)$$
which occurs when we work with tensor product spaces. The best notation here is to have two types of indices: an index i for $SU(n)$ and an index α for $SU(m)$, so that indices in $SU(nm)$ are composites $i\alpha$. Then each factor acts on its index, leaving the other alone.
- Consider an irrep D of $SU(nm)$ with K boxes. Then its tensor has K indices of the $SU(n)$ type and K indices of the $SU(m)$ type, so it decomposes into (D_1, D_2) where D_1 and D_2 both have K boxes, up to eliminating full columns.
- The difference from the previous part is that the Young tableau for D describes the S_K symmetry associated with permuting the composite indices $i\alpha$. Thus, to see if we can recover this symmetry, we regard D_1 and D_2 as Young tableaux for S_K irreps, multiply them as S_K irreps, and look for a factor of D. We don't have an algorithm for this, because we only know how to multiply $SU(n)$ irreps diagrammatically.

Example. The case of $SU(3) \times SU(2) \times U(1) \subset SU(5)$. The adjoint decomposes as

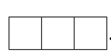


Example. The case of $SU(2) \times U(1) \subset SU(3)$, where the '$SU(1)$' factor is trivial. If we're working with flavor $SU(3)$, these components are isospin and hypercharge. The adjoint decomposes as

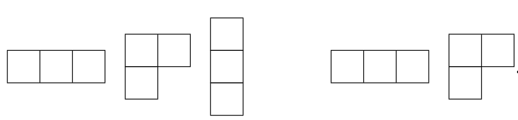


Here the trivial '$SU(1)$' irreps are all one-dimensional, so we don't mark them. However, the fact that $SU(1)$ has only one index value constrains the Young tableau to have a single row.

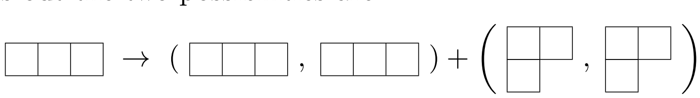
Example. The case of $SU(3) \times SU(2) \subset SU(6)$, where we interpret the factors as quark flavor and quark spin. For the lowest energy baryons, we need total symmetry between the quarks, so we must decompose

$$56 =$$ .

There are three irreps of $SU(3)$ with three boxes, and two of $SU(2)$,



Then it turns out the two possibilities are



$$= (10,4) + (8,2).$$

These are the baryon decuplet, which is totally symmetric in both flavor and spin space, and the baryon octet, which has mixed symmetry in both.

Note. Note that the adjoint A obeys the property

$$R \subset A \otimes R$$

for any nontrivial representation R. This is clear from the Young tableaux method for $SU(n)$. More generally, consider the map

$$A \otimes R \to R, \quad x \otimes v \to x(v).$$

It can be checked this is a map of representations. Since the image is R, R is a quotient of $A \otimes R$.

7.6 Spinor Representations

Next, we turn to $SO(n)$ and spinor representations. We begin with $SO(2n+1)$.

- We label the generators as $M_{ab} = -M_{ba}$, where in the defining representation

  $$[M_{ab}]_{xy} = -i(\delta_{ax}\delta_{by} - \delta_{bx}\delta_{ay}).$$

  Then a basis for the Cartan subalgebra is $H_j = M_{2j-1,2j}$, for $j = 1, \ldots, n$.

- We've already heuristically found that the roots are

  $$e^j \pm e^k, \quad -e^j \pm e^k, \quad \pm e^j.$$

  In terms of the defining representation, they are

  $$E_{\eta e^j} = \frac{1}{\sqrt{2}}(M_{2j-1,2n+1} + i\eta M_{2j,2n+1})$$

  and

  $$E_{\eta e^j + \eta' e^k} = \frac{1}{2}(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k})$$

  as can be checked by direct commutation.

- Under a suitable definition of positivity, the simple roots are

  $$\alpha^j = e^j - e^{j+1} \text{ for } j = 1, \ldots, n-1, \quad \alpha^n = e^n.$$

  Then the fundamental weights are

  $$\mu^j = \sum_{k=1}^{j} e^k \text{ for } j = 1, \ldots, n-1, \quad \mu^n = \frac{1}{2}\sum_{k=1}^{n} e^k.$$

  The representation corresponding to μ^n is the spinor representation. By Weyl reflection in the roots e_j, we get the 2^n weights $(\pm e^1 \pm \ldots \pm e^n)/2$, all of which are nondegenerate since the highest weight was. By some choice of positivity, each of these weights could be the highest weight, and lowering any of them just gives another one, so these are all the weights.

- It is convenient to regard this space as a tensor product of n two-dimensional spaces,

$$|\pm e^1/2 \pm \ldots \pm e^n/2\rangle = |\pm e^1/2\rangle \otimes \ldots \otimes |\pm e^n/2\rangle$$

and we define σ_a^j to be the Pauli matrix σ_a acting on slot j, so

$$H_j = \frac{1}{2}\sigma_3^j, \quad H_j^2 = \frac{1}{4}.$$

Since any generator could have been a Cartan generator, $M_{ab}^2 = 1/4$ for any generator.

- Since any state can only be raised once, $(E_{e^j})^2 = 0$, which implies

$$\{M_{2j-1,2n+1}, M_{2j,2n+1}\} = 0$$

and again, since the Cartan generators are arbitrary, this means

$$\{M_{j\ell}, M_{k\ell}\} = 0, \quad j \neq k \neq \ell \neq j.$$

We can find the roots directly, but we'll just write down the answer,

$$M_{2j-1,2n+1} = \frac{1}{2}\sigma_3^1 \ldots \sigma_3^{j-1}\sigma_1^j, \quad M_{2j,2n+1} = \frac{1}{2}\sigma_3^1 \ldots \sigma_3^{j-1}\sigma_2^j$$

for some suitable phase convention. The anticommutators work because $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, and

$$E_{\pm e^j} = \frac{1}{2}\sigma_3^1 \ldots \sigma_3^{j-1}\sigma_\pm^j.$$

We can then construct everything else by the commutation relations.

- The spinor representation is either real or pseudoreal, as we can see from its weights. In the case $n = 1$, the equivalence to its conjugate representation is by $R = \sigma_2$,

$$\sigma_a = -\sigma_2\sigma_a^*\sigma_2.$$

Demanding the same for the generators $M_{2j-1,2n+1}$ and $M_{2j,2n+1}$ yields

$$R = \sigma_2\sigma_1\sigma_2\sigma_1 \ldots$$

with n alternating factors. Therefore, the spinor representations are real for $SO(8n+1)$ and $SO(8n+7)$, and pseudoreal for $SO(8n+3)$ and $SO(8n+5)$.

Next, we apply similar reasoning for $SO(2n+2)$.

- In this case the simple roots are

$$\alpha^j = e^j - e^{j+1} \text{ for } j = 1, \ldots, n, \quad \alpha^{n+1} = e^n + e^{n+1}$$

so another root appears. The last two fundamental weights are

$$\mu^n = \frac{1}{2}(e^1 + \ldots + e^n - e^{n+1}), \quad \mu^{n+1} = \frac{1}{2}(e^1 + \ldots + e^n + e^{n+1})$$

and correspond to two spinor representations D^n and D^{n+1}.

- Note that all of the roots have the form $\pm e^j \pm e^k$. Then by similar logic to the case of $SO(2n+1)$, the spinor representations have weights

$$\frac{1}{2}\sum_{j=1}^{n+1} \eta_j e^j, \quad \eta_j = \pm 1$$

but with the additional restriction that the η_j multiply to -1 for D^n and 1 for D^{n+1}. Note that both of these representations have dimension 2^n. For even n, the spinor representations are each others' conjugates, and hence complex.

- To construct the representations more explicitly, restrict to $SO(2n+1)$ generated by M_{jk}, with $j, k \leq 2n+1$. Then we lose the last Cartan generator $H_{n+1} = M_{2n+1,2n+1}$, and both of the spinor representations reduce to the one found above.

- Thus, using the same notation, all we have to do is construct H_{n+1}. It is

$$H_{n+1} = \frac{1}{2}\sigma_3^1 \dots \sigma_3^n \times \begin{cases} -1 & D^n, \\ +1 & D^{n+1}. \end{cases}$$

All the other missing generators can be found by commutation.

- Finally, we determine reality and pseudoreality. We can define R exactly as we did for $SO(2n+1)$, and by the same logic find that the spinors are real for $SO(8n)$ and pseudoreal for $SO(8n+4)$. Note that these results are modified when we work in indefinite signature.

Example. Note that $\mathfrak{su}(4) \cong \mathfrak{so}(6)$. In the former, we know that $4 \times 4 = 6 + 10$, the fundamental representations are 4, $\overline{4}$, and 6, and the defining representation is 4. In the context of $\mathfrak{so}(6)$, the 4 and $\overline{4}$ are spinors while the 6 is the defining representation; we clearly can't construct everything using the 6 alone. Here, the 10 is the self-dual antisymmetric rank 3 tensor.

Example. In the case of $\mathfrak{so}(4)$, the arguments don't apply because the algebra is not simple; it is instead $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. On the other hand, the result is similar: there are two pseudo-real spinor representations, which are simply the fundamental representations of each $\mathfrak{su}(2)$ factor. Physically, these are the Weyl spinors.

7.7 Clifford Algebras (TODO)

8 Relativistic Fields

8.1 Representations of E_2

In this section we introduce a method for finding the representations of the noncompact Euclidean group, as practice for the more complicated Poincare group.

- The Euclidean group E_n is the group of linear transformations of n-dimensional Euclidean space connected to the identity that leave the length of all vectors invariant. It is generated by translations and rotations.

- In the special case of E_2, a general group element $g(\mathbf{b}, \theta)$ is a rotation by θ followed by a translation by $\mathbf{b}$. The multiplication law is
$$g(\mathbf{b}_2, \theta_2)g(\mathbf{b}_1, \theta_1) = g(\theta_1 + \theta_2, R(\theta_2)\mathbf{b}_1 + \mathbf{b}_2)$$
and inverses are given by
$$g(\mathbf{b}, \theta)^{-1} = g(-R(-\theta)\mathbf{b}, -\theta).$$

- Both translations and rotations can be written as linear transformations on $\mathbb{R}^3$, where the third component of the vector is always unity,
$$g(\mathbf{b}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & b^1 \\ \sin\theta & \cos\theta & b^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix}.$$
This allows us to deduce commutation relations from the usual matrix algebra.

- The rotations and translations are generated by
$$J = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}, \quad P_1 = \begin{pmatrix} & & i \\ & & \\ & & \end{pmatrix}, \quad P_2 = \begin{pmatrix} & & \\ & & i \\ & & \end{pmatrix}, \quad R(\theta) = e^{-i\theta J}, \quad T(\mathbf{b}) = e^{-i\mathbf{b}\cdot\mathbf{P}}.$$
Note that P_1 and P_2 are not Hermitian. The Lie algebra is
$$[P_1, P_2] = 0, \quad [J, P_k] = i\epsilon^{km}P_m$$
where $\epsilon^{12} = 1$. Physically, the latter relation states that the translation $\mathbf{b}$ is a vector; exponentiating it gives
$$e^{-i\theta J}P_k e^{i\theta J} = P_m R(\theta)_{mk}, \quad e^{-i\theta J}T(\mathbf{b})e^{i\theta J} = T(R(\theta)\mathbf{b})$$
and the latter is just the group multiplication law in another form.

- The translations T_2 form a normal subgroup of E_2, and E_2/T_2 contains pure rotations, so it is isomorphic to $SO(2)$. Since E_2 has an abelian normal subgroup, it is not semi-simple.

- We are interested in unitary representations, but since E_2 is not compact, it has no finite-dimensional faithful unitary representations. The non-faithful finite-dimensional unitary irreps are just those of $SO(2)$, mapping $g(\mathbf{b}, \theta)$ to the phase $e^{im\theta}$.

Next, we construct the faithful unitary irreps of E_2 with the usual method, starting with a reference vector and applying raising and lowering operators.

- We define the raising and lowering operators

$$P_\pm = P_1 \pm iP_2, \quad [P_+, P_-] = 0, \quad [J, P_\pm] = \pm P_\pm.$$

Next, we define the squared momentum operator

$$P^2 = P_1^2 + P_2^2 = P_+P_- = P_-P_+, \quad [P^2, J] = [P^2, P_\pm] = 0.$$

Therefore, P^2 is a Casimir operator and has a single eigenvalue for each irrep.

- In a unitary representation, J, P_1, and P_2 are Hermitian operators; to save space we write the operator associated with a generator with the same symbol. Then $P_+^\dagger = P_-$, so P^2 is positive definite. We thus write its eigenvalue as $p^2 > 0$.

- Since P^2 and J commute, we can simultaneously diagonalize the two, with normalized eigenvectors

$$P^2|pm\rangle = p^2|pm\rangle, \quad J|pm\rangle = m|pm\rangle, \quad m \in \mathbb{Z}$$

where the eigenvalues of J come from our knowledge of the irreps of $SO(2)$. In principle there could be multiple vectors in the same irrep with the same p^2 and m, but we'll see below this isn't the case.

- In the case where $p^2 = 0$, we have $P_\pm|0m\rangle = 0$, so the representation is one-dimensional. This simply reproduces the non-faithful irreps we found above.

- For $p^2 > 0$, we take the phase convention

$$|p, m \pm 1\rangle = (\pm i/p)P_\pm|pm\rangle.$$

Starting with any reference vector $|pm_0\rangle$, repeated application of $P^\pm$ gives all integer values of m precisely once. The matrix elements are

$$\langle pm'|J|pm\rangle = m\delta_{m'm}, \quad \langle pm'|P_\pm|pm\rangle = \mp ip\delta_{m',m\pm 1}.$$

- Finally, we explicitly find the matrix elements of finite transformations. We claim

$$D^p(\mathbf{b}, \theta)_{m'm} = e^{i(m-m')\phi} J_{m-m'}(pb)e^{-im\theta}$$

where (b, ϕ) are the polar coordinates of $\mathbf{b}$ and J_n is the Bessel function of the first kind. The $e^{-im\theta}$ factor is from the rotation part. The translation can be decomposed as

$$T(b, \phi) = R(\phi)T(b, 0)R(\phi)^{-1}, \quad \langle pm'|T(b, \phi)|pm\rangle = e^{i(m-m')\phi}\langle pm'|T(b, 0)|pm\rangle.$$

Therefore, it suffices to show that

$$\langle pm'|T(b, 0)|pm\rangle = J_{m-m'}(pb).$$

We have $T(b, 0) = e^{-ibP_1} = e^{-ib(P_+ + P_-)/2}$. This can be expanded in a double series, which collapses to a single series by orthogonality, giving the defining series for the Bessel function. This gives a hint of why Bessel functions emerge in situations with cylindrical symmetry, for the same reason spherical harmonics emerge in situations with rotational symmetry.

We call the $|pm\rangle$ basis found above the angular momentum basis. Now we work in the 'plane-wave' basis using the method of induced representations.

- Our method works for groups with abelian normal subgroups. The idea is to use the generators of this subgroup as a starting point, i.e. working with eigenvectors of $\mathbf{P}$ instead of P^2 and J.
- We take the reference vector $\mathbf{p}_0 = (p, 0)$ and consider a ket $|\mathbf{p}_0\rangle$ with $\mathbf{P}|\mathbf{p}_0\rangle = \mathbf{p}_0|\mathbf{p}_0\rangle$. The only operation which yields new kets is the rotation $R(\theta) = e^{-i\theta J}$, and we have

  $$P_k R(\theta)|\mathbf{p}_0\rangle = R(\theta)(R(\theta)^{-1}P_k R(\theta))|\mathbf{p}_0\rangle = R(\theta)P_\ell|\mathbf{p}_0\rangle R(-\theta)_{\ell k} = p_k R(\theta)|\mathbf{p}_0\rangle, \quad p_k = R(\theta)_{k\ell}p_{0\ell}.$$

  That is, $R(\theta)|\mathbf{p}_0\rangle$ is also an eigenvector of $\mathbf{P}$ with the rotated momentum $\mathbf{p}$, and we write

  $$|p, \theta\rangle = R(\theta)|\mathbf{p}_0\rangle.$$

  Since $R(2\pi) = 1$, this shows that $|\mathbf{p}_0\rangle$ is the only vector with eigenvalue $\mathbf{p}_0$ is the irrep. Thus the set of $|p, \theta\rangle$ defined above form an irrep of E_2.
- Stepping back, we used the fact that the translation subgroup is abelian to label states with momentum, and we used its normality to show that rotations take states of definite momentum to other states of definite momentum. Then the rest of the procedure is simply generating all the kets we can to get an irrep. In more general situations, there can be group generators that commute with $\mathbf{P}$, yielding multiple kets with the same momentum.
- We choose to normalize the vectors so that

  $$\langle p, \theta'|p, \theta\rangle = 2\pi\delta(\theta' - \theta).$$

  We don't need to normalize over p, since p is constant due to the Casimir operator P^2.
- Next, we find the relationship between the angular momentum and plane wave basis. Dropping the p index, consider the state

  $$|\widetilde{m}\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{im\phi}|\phi\rangle.$$

  By construction, we have

  $$R(\theta)|\widetilde{m}\rangle = e^{-im\theta}|\widetilde{m}\rangle$$

  by shifting the integration variable, so $|\widetilde{m}\rangle$ is proportional to $|m\rangle$. To fix the phases, note

  $$P_\pm|\widetilde{m}\rangle = |\widetilde{m} \pm 1\rangle, \quad |m\rangle = i^m|\widetilde{m}\rangle, \quad \langle\phi|m\rangle = e^{im(\phi+\pi/2)}.$$

  The inner product allows us to switch between bases both ways, e.g.

  $$|\phi\rangle = \sum_m |m\rangle e^{-im(\phi+\pi/2)}.$$
- As an application, note that under a translation, we have

  $$T(\mathbf{b})|m\rangle = \int \frac{d\phi}{2\pi} e^{im(\phi+\pi/2)} e^{-ipb\cos(\theta-\phi)}|\phi\rangle$$

where $\mathbf{b}$ has polar angle θ. Then we immediately have

$$\langle m'|T(\mathbf{b})|m\rangle = \int \frac{d\phi}{2\pi}\, e^{i(m-m')(\phi+\pi/2)} e^{-ipb\cos(\theta-\phi)} = e^{i(m-m')\theta} \int \frac{d\psi}{2\pi}\, e^{i(m-m')\psi - ipb\sin\psi}$$

where $\psi = \pi/2 + \phi - \theta$, which recovers our earlier result using the integral representation of the Bessel function. We can also show that the position-space wavefunctions $\langle \mathbf{r}|m\rangle$ are Bessel functions times $e^{im\theta}$. Further investigation recovers many of the standard properties of Bessel functions, such as their differential equation, recursion formulas, and addition theorems.

Note. As we've seen, E_2 is the group contraction of $SO(3)$ with

$$J_y/R \to P_x, \quad -J_x/R \to P_y, \quad J_z \to J$$

in the limit $R \to \infty$. Then we can recover the representations of E_2 from those of $SO(3)$. This sounds strange since the former has infinite-dimensional representations, but note that for the spin j representation,

$$\langle m'|J_x \pm iJ_y|m\rangle = \delta_{m',m\pm 1}\sqrt{j(j+1) - m(m\pm 1)}.$$

Then when we replace the J_i with P_i, the inner product is proportional to $1/R$, so the representation is automatically one-dimensional unless j grows with R. Specifically, if we pick $j = pR$, then

$$\langle m'|P_x \pm iP_y|m\rangle = p\delta_{m',m\pm 1}$$

which exactly matches the E_2 representation with squared momentum p^2. As an application, it is possible to express Bessel functions in terms of limits of d-functions.

8.2 Representations of E_3

We now apply the same methods to E_3.

- The group E_3 is generated by rotations and translations, and the Lie algebra is

$$[P_i, P_j] = 0, \quad [J_i, J_j] = i\epsilon_{ijk}J_k, \quad [P_i, J_j] = i\epsilon_{ijk}P_k$$

  which says that both $\mathbf{P}$ and $\mathbf{J}$ are vectors. The translations T_3 form a normal subgroup.

- The general group element can be written with the Euler angle parametrization

$$g = T(\mathbf{b})R(\alpha,\beta,\gamma) = e^{-i\mathbf{b}\cdot\mathbf{P}} e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}.$$

  Alternatively, since conjugation by rotations rotates a translation vector, we have

$$g = R(\phi,\theta,0)T(\mathbf{b}_0)R(\alpha',\beta',\gamma')$$

  where $\mathbf{b}$ has spherical coordinates (b,ϕ,θ) and $\mathbf{b}_0$ has spherical coordinates $(b,0,0)$.

- The group has two Casimir operators. We can construct three rotational scalars, P^2, $\mathbf{J}\cdot\mathbf{P}$ and J^2. Then we need to check the commutation relations with $\mathbf{P}$. We have

$$[\mathbf{J}\cdot\mathbf{P}, P_j] = [J_i, P_j]P_i = -i\epsilon_{ijk}P_kP_i = 0$$

  and clearly $[P^2, \mathbf{P}] = 0$, so the Casimir operators are P^2 and $\mathbf{J}\cdot\mathbf{P}$. To work in the angular momentum basis, we diagonalize $\{P^2, \mathbf{J}\cdot\mathbf{P}, J^2, J_z\}$. To work in the plane wave basis, we diagonalize $\{P^2, \mathbf{J}\cdot\mathbf{P}, \mathbf{P}\}$.

- In the plane wave basis, the eigenvalues are $\{p^2, \lambda p, \mathbf{p}\}$ and we write a ket with these eigenvalues as $|p, \lambda, \hat{\mathbf{p}}\rangle$ where $\hat{\mathbf{p}} = \mathbf{p}/p$ and λ is the helicity. As before, we start with the standard vector $\hat{\mathbf{p}}_0 = \hat{\mathbf{e}}_z$.

- Define the little group to be the set of group elements in the factor group that leave the standard vector $\hat{\mathbf{p}}_0$ invariant. In this case, the factor group is $E_3/T_3 \cong SO(3)$ and the little group is $SO(2)$, generated by $R_3(\phi) = e^{-i\phi J_3}$.

- By the same reasoning as before, we may start with an irrep of the little group with the standard momentum $\mathbf{p}_0$ and apply group operations to construct ('induce') an irrep of E_3. The irreps of the little group are all one-dimensional, and they are indexed by the eigenvalue of J_3, which must be an integer. Thus the little group irrep is specified by $\lambda \in \mathbb{Z}$.

- The rest of the E_3 irrep is generated by rotations,

$$|p, \lambda, \hat{\mathbf{p}}\rangle = R(\phi, \theta, 0)|p, \lambda, \hat{\mathbf{p}}_0\rangle, \quad \hat{\mathbf{p}} = (\theta, \phi).$$

  Suppressing the p and λ indices, group elements act on these vectors as

$$T(\mathbf{b})\hat{\mathbf{p}} = e^{-i\mathbf{b}\cdot\mathbf{p}}|\hat{\mathbf{p}}\rangle, \quad R(\alpha, \beta, \gamma)|\hat{\mathbf{p}}\rangle = e^{-i\lambda\psi}|\hat{\mathbf{p}}'\rangle, \quad \hat{\mathbf{p}}' = (\theta', \phi')$$

  where ψ is defined by

$$R(\alpha, \beta, \gamma)R(\phi, \theta, 0) = R(\phi', \theta', \psi).$$

  To show this, note that

$$R(\alpha, \beta, \gamma)|\hat{\mathbf{p}}\rangle = R(\alpha, \beta, \gamma)R(\phi, \theta, 0)|\hat{\mathbf{p}}_0\rangle = R(\phi', \theta', \psi)|\hat{\mathbf{p}}_0\rangle = R(\phi', \theta', 0)R(0, 0, \psi)|\hat{\mathbf{p}}_0\rangle = e^{-i\lambda\psi}|\hat{\mathbf{p}}_0\rangle.$$

- We normalize the vectors by

$$\langle\hat{\mathbf{p}}'|\hat{\mathbf{p}}\rangle = 4\pi\delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi)$$

  since this cancels the invariant measure on the factor group $\sin\theta d\theta d\phi/4\pi$.

- Working in the angular momentum basis, we have vectors $|p, \lambda, j, m\rangle$ corresponding to eigenvalues $\{p^2, \lambda p, j(j+1), m\}$ of $\{P^2, \mathbf{J}\cdot\mathbf{P}, J^2, J_3\}$. We can then construct raising and lowering operators $J_\pm$ which raise and lower m. Since the translations don't commute with J^2, they can change j. It turns out that j takes on all positive integer values, making the representation infinite-dimensional as expected.

- The radial dependence of the angular momentum basis states takes the form of spherical Bessel functions. By similar reasoning to before, we can recover many of their properties.

8.3 The Poincare Group

Finally, we turn to the Poincare group, our original goal. We use the $(-+++)$ metric convention.

- The Poincare group is the analogue of the Euclidean group for Minkowski space. Its elements take the form

$$g(b, \Lambda) = T(b)\Lambda, \quad x^\mu \to \Lambda^\mu{}_\nu x^\nu + b^\mu$$

  where Λ is a Lorentz transformation connected to the identity. By similar reasoning to before, the translations form a normal subgroup and the factor group is the Lorentz group.

- A general Lorentz transformation can be written as

$$\Lambda = R(\alpha, \beta, 0) L_3(\xi) R(\phi, \theta, \psi)^{-1}$$

where $L_3(\xi)$ is a boost of rapidity ξ along the z-axis. This is the same idea as before: a boost can be conjugated by rotations to change its direction to the z-axis.

- To work with the generators, we have to be careful with index placement. An infinitesimal translation by δb is

$$T(\delta b) = I - i\delta b^\mu P_\mu, \quad T(b) = \exp(-ib^\mu P_\mu).$$

The translation generators P_μ are related to the four-momentum by raising an index, so $P^0 = -P_0$ and $P^i = P_i$. By the multiplication law, P_μ indeed transforms as a covector.

- The Lorentz generators are defined as

$$\Lambda(\delta\omega) = I - \frac{i}{2}\delta\omega^{\mu\nu} J_{\mu\nu}$$

where $\delta\omega$ is an antisymmetric tensor that parametrizes the transformation as

$$\delta\omega^{23} \text{ is angle of rotation about } x, \quad \delta\omega^{10} \text{ is boost along } x.$$

We can also split $J_{\mu\nu}$ into rotation and boost generators as

$$J_{ij} = \epsilon^{ijk} J_k, \quad J_{i0} = K_i.$$

One may show $J_{\mu\nu}$ transforms as a tensor, so the J_i and K_i transform as spatial vectors.

- In this notation, the commutation relations are

$$[P_\mu, J_{\lambda\sigma}] = i(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda}), \quad [J_{\mu\nu}, J_{\lambda\sigma}] = i(J_{\lambda\nu} g_{\mu\sigma} - J_{\sigma\nu} g_{\mu\lambda} + J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda}).$$

These can be shown by explicit matrix calculation, where P_μ is a matrix in a five-dimensional space. They can also be derived by taking the infinitesimal version of the transformation laws of P_μ and $J_{\mu\nu}$. That is, the commutators really just say P_μ is a vector and $J_{\mu\nu}$ is a tensor.

- Breaking everything into space and time components, we have, for the translations,

$$[P^0, J_i] = 0, \quad [P_i, J_j] = i\epsilon^{ijk} P_k, \quad [P_i, K_j] = i\delta_{ij} P^0, \quad [P^0, K_i] = iP_i$$

and for the Lorentz subgroup

$$[J_i, J_j] = i\epsilon^{ijk} J_k, \quad [K_i, J_j] = i\epsilon^{ijk} K_k, \quad [K_i, K_j] = -i\epsilon^{ijk} J_k.$$

We see that P_i, J_i, and K_i are all vectors; the new features are the minus sign in the $[K_i, K_j]$ commutation relation, which indicates that the groups is noncompact, and the commutation relations of translations with boosts.

Note. We spend some time on the representations of the Lorentz group. As we've seen, we get two copies of $\mathfrak{su}(2)$ by defining

$$M_i = \frac{J_i + iK_i}{2}, \quad N_i = \frac{J_i - iK_i}{2}.$$

This allows us to easily find the finite-dimensional representations of the Lorentz group from those of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, though they won't be unitary. Such representations are extremely important, since many physical observables transform in them, including position, momentum, and fields. Nonunitarity is acceptable, because these objects do not correspond directly to quantum states. Later, we will show in detail how quantum states in an infinite-dimensional, unitary Poincare irrep give rise to quantum fields in a finite-dimensional, nonunitary Lorentz irrep.

Restricting to $SO(3)$, each Lorentz irrep breaks into $SO(3)$ irreps attained by adding spins of u and v. Hence we can also parametrize irreps and the vectors in them by the minimum and maximum spin

$$j_0 = |u - v|, \quad j_1 = u + v$$

and the usual spin and z-component, j and m. Now we turn to the infinite-dimensional, unitary representations using this parametrization. Acting with the J_i does not change the value of j, but acting with the K_i does. Since the K_i form a vector, we have a set of irreducible tensor operators $\{K_-/\sqrt{2}, K_3, -K_+/\sqrt{2}\}$, so by the Wigner-Eckart theorem

$$\langle j'm'|K_3|jm\rangle = A_j^{j'} \langle j'm'|10jm\rangle, \quad \langle j'm'|K_\pm|jm\rangle = \mp\sqrt{2} A_j^{j'} \langle j'm'|1 \pm 1jm\rangle.$$

The Clebsch-Gordan coefficient is only nonzero when $|j' - j| \leq 1$, so only a few of the $A_j^{j'}$ coefficients matter. One can use the commutation relations to show the general solution depends on an arbitrary complex number ν, and imposing unitarity gives two classes of irreps,

$$\text{principal series: } \nu = -iw, j_0 = 0, 1/2, 1, \ldots, \quad \text{complementary series: } -1 \leq \nu \leq 1, j_0 = 0$$

where w is real and j_1 is infinite. If we had instead required finite-dimensionality here, we would recover our earlier results.

Example. The finite-dimensional irreps can be labeled by the quantum numbers u and v, where the Casimir operators M^2 and N^2 have values $u(u+1)$ and $v(v+1)$. For example, we have

$$\text{scalar: } (0,0), \quad \text{four-vector: } (1/2,1/2), \quad \text{Weyl spinor: } (1/2,0), (0,1/2).$$

A rank two tensor transforms in $(1/2,1/2) \otimes (1/2,1/2)$ which decomposes as

$$\text{symmetric: } (1,1), \quad \text{trace: } (0,0), \quad \text{antisymmetric: } (1,0), (0,1).$$

Note that the antisymmetric electromagnetic field $F_{\mu\nu}$ decomposes under spatial $SO(3)$ into two vectors, which can be arbitrary linear combinations of $\mathbf{E}$ and $\mathbf{B}$. Under the full Lorentz group, the two are forced to be $\mathbf{E} \pm i\mathbf{B}$, the combinations that don't mix under boosts.

8.4 Representations of the Poincare Group

Next, we apply the method of induced representations to the Poincare group, yielding Wigner's classification.

- We use the abelian subgroup of translations, and note there is a Casimir operator

  $$C_1 = -P_\mu P^\mu = P_0^2 - P^2.$$

  We let the value of the operator be c_1. For a momentum p^μ, there are several cases:

  $$\text{trivial: } c_1 = p^\mu = 0, \quad \text{timelike: } c_1 > 0, \quad \text{lightlike: } c_1 = 0, \, p^\mu \neq 0, \quad \text{spacelike: } c_1 < 0.$$

  In the trivial case, the little group is the Lorenz group, but translations do nothing; this recovers the Lorentz irreps found above.

- For the timelike case $c_1 = M^2$, we choose the standard momentum $p_t^\mu = (M, \mathbf{0})$. The little group is $SO(3)$, so we write the states with the standard momentum as $|\mathbf{0}\lambda\rangle$ where

$$P^\mu|\mathbf{0}\lambda\rangle = p_t^\mu|\mathbf{0}\lambda\rangle, \quad J^2|\mathbf{0}\lambda\rangle = s(s+1)|\mathbf{0}\lambda\rangle, \quad J_3|\mathbf{0}\lambda\rangle = \lambda|\mathbf{0}\lambda\rangle.$$

  Next, we must construct the rest of the states by boosts and rotations.

- A general Lorentz transformation can be written as $\Lambda = R(\alpha, \beta, 0)L_3(\xi)R(\phi, \theta, \psi)^{-1}$ where the rightmost factor does nothing since it is in the little group. We thus define

$$|p\hat{\mathbf{z}}\,\lambda\rangle = L_3(\xi)|\mathbf{0}\lambda\rangle, \quad p = M\sinh\xi$$

  and

$$|\mathbf{p}\lambda\rangle = R(\alpha, \beta, 0)|p\hat{\mathbf{z}}\,\lambda\rangle = H(p)|\mathbf{0}\lambda\rangle.$$

  Then the set of $|\mathbf{p}\lambda\rangle$ form an irrep of the Poincare group.

- By the same proof as in E_2, the $|\mathbf{p}\lambda\rangle$ states indeed have momentum $\mathbf{p}$, in the sense that

$$T(b)|\mathbf{p}\lambda\rangle = e^{-ib^\mu p_\mu}|\mathbf{p}\lambda\rangle.$$

  For the Lorentz transformations, the argument is similar to the one we used for E_3. We have

$$\Lambda|\mathbf{p}\lambda\rangle = \Lambda H(p)|\mathbf{0}\lambda\rangle = H(\Lambda p)R(\Lambda, p)|\mathbf{0}\lambda\rangle, \quad R(\Lambda, p) = H^{-1}(\Lambda p)\Lambda H(p).$$

  Note that $R(\Lambda, p)$ fixes p_t, since it maps $p_t \mapsto p \mapsto \Lambda p \mapsto p_t$. Thus it is in the little group, and we know how it acts,

$$\Lambda|\mathbf{p}\lambda\rangle = D^s(R(\Lambda, p))_{\lambda'\lambda}|\Lambda\mathbf{p}\,\lambda'\rangle$$

  where the D^s are the spin s representation matrices, and $\Lambda\mathbf{p}$ is the spatial part of Λp.

- We now give interpretations for all the parameters above. The irrep is labeled by s and M, where s indicates the spin and M indicates the mass, as seen from the relativistic dispersion relation. We've just shown that $\mathbf{p}$ is the spatial momentum, and by the same argument as for E_3, λ may be identified with the helicity, i.e. the eigenvalue of $\mathbf{J} \cdot \mathbf{P}/|\mathbf{p}|$. The difference is that the helicity is no longer a Casimir invariant, as it can be changed by boosts.

- There is a second Casimir invariant related to the spin s. As we've seen, it can't be J^2, since that doesn't commute with boosts. Another guess is $J_{\mu\nu}J^{\mu\nu}$, since that's also quadratic in $J_{\mu\nu}$, but it doesn't commute with translations.

- Instead, we define the Pauli-Lubanski vector

$$W^\lambda = \epsilon^{\lambda\mu\nu\sigma}J_{\mu\nu}P_\sigma/2.$$

  It has the properties

$$W^\lambda P_\lambda = 0, \quad [W^\lambda, P^\mu] = 0, \quad [W^\lambda, J^{\mu\nu}] = i(W^\mu g^{\lambda\nu} - W^\nu g^{\mu\lambda}), \quad [W^\lambda, W^\sigma] = i\epsilon^{\lambda\sigma\mu\nu}W_\mu P_\nu.$$

  The proofs are straightforward, mostly using the antisymmetry of ϵ. The second and third properties say W^λ is a translationally invariant four-vector. The second Casimir operator is

$$C_2 = W^\lambda W_\lambda.$$

  This is because C_2 is a Lorentz scalar, and it is translationally invariant since W^λ is. Also, note that W^0 is $\mathbf{J} \cdot \mathbf{P}$, i.e. W^λ essentially completes the helicity to a four-vector.

- To understand the meaning of C_2, we return to the timelike case. In the rest frame, i.e. for the states with momentum p_t^μ, we may replace P_σ with $p_{\sigma t}$ to give
$$W^0 = 0, \quad W^i = MJ^i$$
so C_2 reduces to $M^2 J^2$ and hence gives the spin parameter s. Thus we recover the situation with E_2 and E_3 where the irrep is fully specified by the values of the Casimir operators.

- In general, it can be shown that in the rest frame, the W^μ span the algebra of the little group; in the timelike case, the commutators of the W^μ reduce to the $\mathfrak{so}(3)$ algebra.

- Next, we turn to the more complicated light-like case, with standard momentum
$$p_\ell^\mu = (\omega_0, 0, 0, \omega_0).$$
In this case the generators of the little group are
$$W^0 = W^3 = \omega_0 J_3, \quad W_1 = \omega_0(J_1 + K_2), \quad W_2 = \omega_0(J_2 - K_1)$$
which have commutation relations
$$[W^1, W^2] = 0, \quad [W^2, J_3] = iW^1, \quad [W^1, J_3] = -iW^2.$$
This is exactly the algebra of E_2, and the little group is indeed E_2.

- We know that E_2 has infinite-dimensional irreps, but none of these appear in nature – such a particle would have infinitely many internal states. There are also one-dimensional irreps indexed by λ, the eigenvalue of J_3, giving states
$$P^\mu|\mathbf{p}_\ell\lambda\rangle = p_\ell^\mu|\mathbf{p}_\ell\lambda\rangle, \quad J_3|\mathbf{p}_\ell\lambda\rangle = \lambda|\mathbf{p}_\ell\lambda\rangle, \quad W_1|\mathbf{p}_\ell\lambda\rangle = W_2|\mathbf{p}_\ell\lambda\rangle = 0.$$

- We then construct the states $|\mathbf{p}\lambda\rangle$ in the same way as before; one can show
$$T(b)|\mathbf{p}\lambda\rangle = e^{-ib^\mu p_\mu}|\mathbf{p}\lambda\rangle, \quad \Lambda|\mathbf{p}\lambda\rangle = e^{-i\lambda\theta(\Lambda,p)}|\Lambda\mathbf{p}\,\lambda\rangle$$
where the phase is
$$e^{-i\lambda\theta(\Lambda,p)} = \langle\mathbf{p}_\ell\lambda|H^{-1}(\Lambda p)\Lambda H(p)|\mathbf{p}_\ell\lambda\rangle.$$
Unlike the timelike case, the helicity λ is now Poincare invariant.

- Which values of λ appear in nature? CPT flips λ, so a relativistic theory must have pairs $\pm\lambda$. For instance, the photon has $\lambda = \pm 1$ and the graviton has $\lambda = \pm 2$. We shouldn't strictly say the photon has spin 1, because spin is a property of $SO(3)$ irreps, not E_2 irreps.

- We can also consider double-valued representations, giving $\lambda = -1/2$ to describe massless neutrinos and $\lambda = 1/2$ to describe massless antineutrinos. (Note that we only have double-valued representations, even though the universal cover of $SO(2)$ is $\mathbb{R}$, because this is part of the Lorentz group which has only a double cover.)

- The spacelike case is rather exotic. Taking the standard momentum $p_s^\mu = (0, 0, 0, Q)$, the little group is $SO(2, 1)$, which is noncompact. Then all the unitary little group irreps are infinite-dimensional. These Poincare irreps do not appear in nature.

- Finally, we normalize the states to be compatible with the Lorentz invariant integration measure, $\text{đ}\mathbf{p}/2p^0$. Then we must have, in the spacelike and lightlike cases,
$$\langle\mathbf{p}'\lambda'|\mathbf{p}\lambda\rangle = 2p^0 \delta\!\!\!/(\mathbf{p} - \mathbf{p}')\delta_\lambda^{\lambda'}.$$

8.5 Relativistic Field Equations

We now connect the transformation properties of fields and particles. We warm up with the case of fields transforming under rotations. Though index placement is not important here, we maintain it since it'll be needed later; for matrices, the upper index is always the first index. For clarity, we always distinguish abstract operators and their representations.

- Under a rotation R, position states transform as

$$|\mathbf{x}\rangle \to U(R)|\mathbf{x}\rangle = |R\mathbf{x}\rangle.$$

  Slightly more generally, if the state space also involves spin 1/2, then

$$U(R)|\mathbf{x}, \sigma\rangle = D^{1/2}(R)^{\lambda}_{\sigma}|R\mathbf{x}, \lambda\rangle$$

  where generally $D^j(R)$ is the representation matrix for spin j.

- Wavefunctions are defined by $\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle$, so the state $|\psi'\rangle = U(R)|\psi\rangle$ has wavefunction

$$\psi'(\mathbf{x}) = \psi(R^{-1}\mathbf{x}).$$

  The inverse here can be understood as the usual active/passive inverse. Similarly, for spinors,

$$\psi'^{\lambda}(\mathbf{x}) = D^{1/2}(R)^{\lambda}_{\sigma}\psi^{\sigma}(R^{-1}\mathbf{x}).$$

  This generalizes immediately to the wavefunction of a spin j particle.

- Next, we consider the transformation of observables. The position operator satisfies $X^i|\mathbf{x}\rangle = x^i|\mathbf{x}\rangle$, and using the transformation of $|\mathbf{x}\rangle$ gives

$$U(R)X^iU(R)^{-1} = (R^{-1})^i_j X^j.$$

  Using the orthogonality of R, we have

$$U(R)X_iU(R)^{-1} = X_j R^j_i.$$

  This is an example of an irreducible tensor operator.

- Finally, we consider second quantized fields, which represent local observables. For a spin 1/2 field $\Psi^{\sigma}(\mathbf{x})$ and a one-particle state $|\psi\rangle$, we have

$$\langle 0|\Psi^{\sigma}(\mathbf{x})|\psi\rangle = \psi^{\sigma}(\mathbf{x})$$

  where $\psi^{\sigma}(\mathbf{x})$ is the wavefunction. Fiddling around with the above properties gives

$$U(R)\Psi^{\sigma}(\mathbf{x})U(R)^{-1} = D^{1/2}(R^{-1})^{\sigma}_{\lambda}\Psi^{\lambda}(R\mathbf{x}).$$

  The factor of $D^{1/2}$ is the same as for tensor operators, while the position argument is changed since rotating, creating a particle, and then rotating back creates a particle in a different place. A field of general spin behaves similarly.

- We can also try to move the factor of $D^{1/2}$ to the right. Since $U(R)$ and $D^{1/2}(R)$ are unitary,

$$U(R)\Psi^\sigma(\mathbf{x})U(R)^{-1} = \Psi^\lambda(R\mathbf{x})D(R)^{*\lambda}_{\sigma}.$$

  The complex conjugate and index placement is unappealing, but we can remove it by taking the adjoint of both sides. Note that $D(R)$ is a set of numbers, not an operator on the Hilbert space, so it gets conjugated rather than adjointed, for

$$U(R){\Psi^\dagger}_\sigma(\mathbf{x})U(R)^{-1} = {\Psi^\dagger}_\lambda(R\mathbf{x})D(R)^{\lambda}_{\sigma}.$$

  In some, but not all cases, Ψ is Hermitian, so we get the same result as for tensor operators.

Next, we move to the relativistic case.

- In general, the Hilbert space carries an infinite-dimensional unitary representation $U(\Lambda, \mathbf{b})$ of the Poincare group. We identify individual particles with irreps of the Poincare group; this is sensible, as taking a particle and moving it around in some way should keep it the 'same' particle. For concreteness, we consider a particle with mass m and spin s.
- The set of one-particle states of that type is $\{|\mathbf{p}\lambda\rangle\}$, while multiple-particle states are built from this irrep by the Fock construction. We define creation operators, which satisfy

$$|\mathbf{p}\lambda\rangle = a^\dagger(\mathbf{p}\lambda)|0\rangle.$$

  Using our previous results, under Lorentz transformations the creation operators obey

$$U(\Lambda)a^\dagger(\mathbf{p}\lambda)U(\Lambda^{-1}) = a^\dagger(\Lambda\mathbf{p}\,\lambda')D^s(R(\Lambda, p))^{\lambda'}_{\lambda}.$$

  Taking the adjoint, the annihilation operators transform as

$$U(\Lambda)a(\mathbf{p}\lambda)U(\Lambda^{-1}) = D^s(R(\Lambda, p)^{-1})^{\lambda}_{\lambda'}a(\Lambda\mathbf{p}\,\lambda').$$

  Here, $R(\Lambda, p)$ is as defined earlier, the D^s are the spin s representation matrices of $SO(3)$, and p is defined to have spatial part $\mathbf{p}$, $p^0 > 0$, and $p^2 = -m^2$.
- As in the nonrelativistic case, we may define the wavefunction of a one-particle state by projecting onto the $|\mathbf{p}\lambda\rangle$ states. Then the wavefunction should transform with a factor of D^s, but this formalism is not manifestly Lorentz invariant.
- Instead, we prefer to let the wavefunction transform under a finite-dimensional representation of the Lorentz group $D(\Lambda)$ as

$$\psi'^\alpha(x) = D(\Lambda)^\alpha_\beta\psi^\beta(\Lambda^{-1}x)$$

  where the α and λ indices are related as follows: for a one-particle state

$$\langle x\alpha|\psi\rangle = \psi^\alpha(x), \quad \langle x\alpha|\mathbf{p}\lambda\rangle = u^\alpha(\mathbf{p}\lambda)e^{ipx}.$$

  Switching to the set of states $|\mathbf{x}\alpha\rangle$ severely enlarges the Hilbert space; it contains particles with the wrong mass, negative energy, and the wrong spin, as a generic Lorentz representation contains multiple spins. Hence we'll have to project out an appropriate subset later.

- Once we have defined wavefunctions, we can define relativistic field operators as
$$\langle 0|\Psi^\alpha(\mathbf{x})|\psi\rangle = \psi^\alpha(\mathbf{x})$$
which by the same proof transform as
$$U(\Lambda)\Psi^\alpha(x)U(\Lambda^{-1}) = D(\Lambda^{-1})^\alpha_\beta \Psi^\beta(\Lambda x).$$
The nomenclature is a bit confusing, as both the states and the observables can be called fields, since they both depend on position. For example, the wavefunction for a Dirac fermion $\psi^\alpha(x)$ is called the Dirac field. We will distinguish operator fields by capital letters.

We restrict to the Poincare irrep using the relativistic wave equations that the field satisfies. These constraints will ensure that the field only creates particles in the desired Poincare irrep.

- The general form of such an equation is
$$\Pi(m, -i\partial)^\alpha_\beta \Psi^\beta(x) = 0$$
which in Fourier space becomes
$$\Pi(m,p)\Phi^\alpha(p) = 0$$
where $\Phi(p)$ is the Fourier transform of $\Psi(x)$, with identical transformation properties. For example, the Klein-Gordan field has $\Pi(m,p) = p^2 + m^2$.
- We demand the wave equation be relativistically invariant, so
$$D(\Lambda)\Pi(m,p)D(\Lambda^{-1}) = \Pi(m, \Lambda p).$$
In other words, if $\Pi(m,p)\Phi^\alpha(p) = 0$, then $\Pi(m,\Lambda p)\Phi(\Lambda p) = 0$, so different observers agree on whether the equation of motion is satisfied.
- The equation of motion must impose the mass shell condition
$$(p^2 + m^2)\Phi(p) = 0.$$
We then parametrize the on-shell degree of freedom by $\Phi(p) = \delta(p^2 + m^2)\widetilde{\Phi}(p)$. This ensures that we only get particles with mass m. The Klein-Gordan equation clearly does this; more subtly, the Dirac equation does too, since the Dirac operator squares to $p^2 + m^2$.
- Note that the mass shell condition has two possible energies for each momentum; that is, it doesn't rule out negative energies. This is related to the prediction of antiparticles in quantum field theory, though we won't go into the details here.
- Finally, $\Pi(m,p)$ must act like a projection matrix that selects out states with spin s. This is necessary, for instance, to remove the spin 0 part of a vector field A^μ to describe a massive spin 1 particle. We also won't go into the details here.
- With this setup, the general quantum field solution has the form
$$\Psi^\alpha(x) = \sum_\lambda \int \widetilde{dp}\, b(\mathbf{p}\lambda)u^\alpha(\mathbf{p}\lambda)e^{ipx} + \text{negative energy term}$$
where $\widetilde{dp}$ is the invariant measure and the $b(\mathbf{p}\lambda)$ are some unknown operators; plugging in the definition of the quantum field shows they are simply the annihilation operators $a(\mathbf{p}\lambda)$.

In summary, particles live in a unitary Poincare irrep, and the states in these irreps correspond to plane wave solutions $u^{\alpha}(\mathbf{p}\lambda)e^{ipx}$ for a quantum field, which transforms in a nonunitary Lorentz representation. The field is not strictly necessary, but it brings the features of locality and causality to the foreground.

9 Gauge Theories

9.1 $*$ Yang-Mills Theory

In this section, we construct the Yang-Mills Lagrangian. We warm up with a $U(1)$ gauge group.

- The gauge potential is a_μ with gauge transformations $a_\mu \to a_\mu + \partial_\mu \alpha$, and the gauge-invariant field strength is $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The Lagrangian is
$$\mathcal{L} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu}.$$
For convenience, we define $A_\mu = -ia_\mu$ and $F_{\mu\nu} = -if_{\mu\nu}$. This is useful because the Lie algebra $\mathfrak{u}(1)$ may be identified with the imaginary axis $i\mathbb{R}$, so A_μ and $F_{\mu\nu}$ naturally live in it.
- Now consider adding a complex scalar field ϕ with Lagrangian
$$\mathcal{L}_\phi = \partial_\mu \phi^* \partial^\mu \phi - W(\phi^* \phi).$$
This is invariant under the $U(1)$ global symmetry
$$\phi \to g\phi, \quad \phi^* \to g^{-1}\phi^*, \quad g = e^{i\delta} \in U(1).$$
Now we consider an infinitesimal global symmetry $g = \exp(\epsilon X) \approx 1 + \epsilon X$, where $X \in \mathfrak{u}(1)$ is a pure imaginary number. Then we have
$$\delta_X \phi = \epsilon X \phi, \quad \delta_X \phi^* = -\epsilon X \phi^*, \quad \delta_X \mathcal{L}_\phi = 0.$$
To gauge the symmetry, we promote g to $g(x)$.
- The Lagrangian is not invariant under the gauged symmetry because
$$\delta_X(\partial_\mu \phi) = \partial_\mu \delta_X \phi = \epsilon \partial_\mu (X\phi) = \epsilon(\phi \partial_\mu X + X \partial_\mu \phi).$$
We get an extra term that does not cancel out.
- We restore gauge invariance by promoting the partial derivative to a covariant derivative,
$$D_\mu = \partial_\mu + A_\mu.$$
We let the gauge field also transform as $\delta_X A_\mu = -\epsilon \partial_\mu X$. Then direct calculation gives
$$\delta_X(D_\mu \phi) = \epsilon X D_\mu \phi.$$
Then by the same logic as in the global case, $\delta_X \mathcal{L}_\phi = 0$ as desired.

We have constructed the theory of scalar QED above. We now consider a general Lie group G.

- For simplicity, we consider a set of scalar fields ϕ which transform in some representation D of G, with representation space $V \cong \mathbb{C}^N$. We begin with the Lagrangian
$$\mathcal{L}_\phi = (\partial_\mu \phi, \partial^\mu \phi) - W((\phi, \phi))$$
where we use the standard inner product on $\mathbb{C}^N$.

- Assuming that D is a unitary representation, $\mathcal{L}_\phi$ is invariant under the global symmetry
$$\phi \to D(g)\phi, \quad D(g)^\dagger = D(g)^{-1}$$
which is infinitesimally
$$D(g) = \exp(\epsilon R(X)) \approx 1 + \epsilon R(X), \quad R(X)^\dagger = -R(X), \quad \delta_X \phi = \epsilon R(X)\phi$$
where $R(X)$ is the representation of $\mathfrak{g}$ corresponding to D.
- Next, we gauge the symmetry by allowing X to depend on x and using the covariant derivative
$$D_\mu \phi = \partial_\mu \phi + R(A_\mu)\phi$$
where A_μ is a $\mathfrak{g}$-valued vector field, which transforms as
$$\delta_X A_\mu = -\epsilon D_\mu X = -\epsilon \partial_\mu X - \epsilon [A_\mu, X]$$
where the second term is new; it is the Lie bracket, which vanished in the $\mathfrak{u}(1)$ case.
- By direct calculation, we may verify
$$\delta_X(D_\mu \phi) = \epsilon R(X) D_\mu \phi$$
where we use the fact that R is linear, so it commutes with derivatives, and that R is a representation, so $R([X, A_\mu]) = [R(X), R(A_\mu)]$. Then we have
$$\delta_X[(D_\mu \phi, D^\mu \phi)] = \epsilon \left((R(X) D_\mu \phi, D^\mu \phi) + (D_\mu \phi, R(X) D^\mu \phi)\right) = 0$$
since $R(X)$ is anti-Hermitian.
- Next, we need a kinetic term for the gauge field. We define the field strength
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathfrak{g}.$$
Then by direct calculation, we have
$$\delta_X F_{\mu\nu} = \epsilon [X, F_{\mu\nu}]$$
where we use standard properties and the Jacobi identity.
- We then use the Killing form to define a kinetic term,
$$\mathcal{L}_A = \frac{1}{g^2} \kappa(F_{\mu\nu}, F^{\mu\nu}).$$
This term is gauge-invariant by the invariance of the Killing form,
$$\delta_X \mathcal{L}_A \propto \kappa([X, F_{\mu\nu}], F^{\mu\nu}) + \kappa(F_{\mu\nu}, [X, F^{\mu\nu}]) = 0.$$
- To get a sensible theory, we need the Killing form to be negative definite, i.e. that $\mathfrak{g}$ is of compact type; this also requires $\mathfrak{g}$ to be simple. That is, we can choose a basis of $\mathfrak{g}$ where
$$\kappa^{ab} = \kappa(T^a, T^b) = -\kappa \delta^{ab}, \quad \kappa > 0$$
so that in components,
$$\mathcal{L}_A = -\frac{\kappa}{g^2} F_{\mu\nu a} F^{\mu\nu a}, \quad F_{\mu\nu} = F_{\mu\nu a} T^a.$$

- In summary, a gauge theory is specified by a simple compact gauge group G, where $A_\mu \in \mathfrak{g}$ (i.e. the gauge field transforms in the adjoint representation), and a set of matter fields that transform in representations of $\mathfrak{g}$. In nature, the matter fields are always fermions that transform in the fundamental representation.

Note. Some more explicit expressions in the case where G is a matrix Lie group, so that $A_\mu \in \mathfrak{g}$ and $g \in G$ are matrices. The gauge field transforms as

$$A_\mu \to A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}$$

which yields the same $\delta_X A_\mu$ by letting $g = \exp(\epsilon X)$.

A field in the fundamental representation transforms as $\phi \to g\phi$, so its covariant derivative is simply $D^F_\mu \phi = \partial_\mu \phi + A_\mu \phi$. On the other hand, a matrix-valued field in the adjoint representation transforms as $\phi \to g\phi g^{-1}$, so its covariant derivative is $D^A_\mu \psi = \partial_\mu \psi + [A_\mu, \psi]$. These are both special cases of our general expression above.

The field strength tensor is the commutator of covariant derivatives,

$$F_{\mu\nu} = [D^A_\mu, D^A_\nu]$$

as can be checked by acting with both sides on ψ. Directly expanding shows that this matches our earlier definition. Moreover, $F_{\mu\nu}$ itself transforms in the adjoint representation, which implies that $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant. This is equivalent to what we found earlier, since $\mathrm{tr}(XY)$ and $\kappa(X, Y)$ are proportional for a simple Lie algebra.

9.2 $SU(5)$ Grand Unification

We begin with a brief review of the gauge groups and matter content of the Standard Model.

- The Standard Model has gauge group $SU(3) \times SU(2) \times U(1)$. The gauge bosons in each of these factors carry color, weak isospin, and hypercharge Y. The fermions transform in representations of this group; we will indicate representations of $SU(3)$ and $SU(2)$ by their dimensions are usual, and representations of $U(1)$ by half their hypercharge, $Y/2$.

- The structure is the same in each generation, so we focus on the first, containing the quarks u and d, the electron e^-, and the neutrino ν_e. The quarks and electron are Dirac fermions, so they contain left-handed and right-handed Weyl fermions, while the massless neutrino is only left-handed.
  - The left-handed quarks transform in $(3, 2, 1/6)$. Counting quarks of different colors as different particles, there are six; the weak force exchanges the up and down quarks.
  - The right-handed up quarks transform in $(3, 1, 2/3)$.
  - The right-handed down quarks transform in $(3, 1, -1/3)$.
  - The left-handed leptons transform in $(1, 2, -1/2)$.
  - The right-handed electron transforms in $(1, 1, -1)$. It has to transform trivially under $SU(2)$ since there's nothing left; there is no right-handed neutrino.
  - All of the antiparticles transform in the corresponding conjugate representation.

- To build a grand unified theory, we would like to combine these representations together. Since gauge transformations commute with Lorentz transformations, they can only work within one Lorentz irrep, so it is convenient to only work with left-handed Weyl spinors. Since the conjugate of a left-handed Weyl spinor is a right-handed Weyl spinor, we have

$$u \text{ and } d \text{ quarks}\colon (3, 2, 1/6), \quad u^c \text{ quark}\colon (\overline{3}, 1, -2/3), \quad d^c \text{ quark}\colon (\overline{3}, 1, 1/3)$$

  and

$$e \text{ and } \nu_e \text{ leptons}\colon (1, 2, -1/2), \quad e^c \text{ lepton}\colon (1, 1, 1)$$

  where the c superscript denotes charge conjugation, so u^c is an anti-up quark.

- The $SU(2) \times U(1)$ symmetry is spontaneously broken down to the $U(1)$ of electromagnetism; the photon is a mixture of the $U(1)$ generator and one of the $SU(2)$ generators, say the third. The explicit relation is given by a modern version of the Gell-Mann Nishijima formula,

$$Q = T^3 + \frac{1}{2}Y.$$

  Using this relation, we can reverse engineer the values of Y above, where T^3 is the third generator of $SU(2)_L$. The Y here is totally different from the hypercharge in the Gell-Mann Nishijima formula, $Q = I^3 + Y/2$ where I^3 is the third component of isospin, and was normalized so the two formulas look the same.

- The naive Dirac mass terms don't work, because left-handed and right-handed quarks and electrons transform differently under $SU(2)$. The resolution is to introduce a new complex scalar field φ transforming in $(1, 2, -1/2)$, called the Higgs field; the hypercharge is chosen so that φ times the Dirac mass term is allowed.

- As a result, the Higgs field contains two uncharged particles, a particle with charge $+1$, and a particle with charge -1. During symmetry breaking, three of these particles combine with the broken $SU(2)$ and $U(1)$ gauge bosons to produce the $W^\pm$ and Z, while the remaining degree of freedom acquires a vev, becoming 'the' Higgs field.

We now turn to grand unification under $SU(5)$.

- One clue that the Standard Model gauge group should be part of a larger one is that the hypercharges of the 15 Weyl fields above add up to zero. This is what we would expect if hypercharge were one of the generators of a larger gauge group, as it should be traceless.

- The smallest group that contains the Standard Model gauge group is $SU(5)$, generated by traceless anti-Hermitian 5×5 matrices. We identify $SU(3)$ with generators in the upper-left 3×3 block, $SU(2)$ with generators in the bottom-right 2×2 block, and $U(1)$ with generator

$$\frac{1}{2}Y = \operatorname{diag}(-1/3, -1/3, -1/3, 1/2, 1/2).$$

  There are 12 more gauge bosons, which we'll get back to later.

- First, consider the fundamental representation ψ^μ of $SU(5)$. This irrep decomposes as

$$5 \to (3, 1, -1/3) + (1, 2, 1/2)$$

which implies that

$$\overline{5} \to (\overline{3}, 1, 1/3) + (1, 2, -1/2).$$

This perfectly accommodates the anti-down quark and the leptons.

- The remaining 10 particles fit into the antisymmetric tensor representation $\psi^{\mu\nu}$. To understand this representation, it's easiest to break the symmetry first; then we want the antisymmetric part of $(3, 1, -1/3) + (1, 2, 1/2)$ times itself. We have

$$(3, 1, -1/3) \times (3, 1, -1/3) = (\overline{3}, 1, -2/3) + (6, 1, -1/3)$$

and

$$(3, 1, -1/3) \times (1, 2, 1/2) = (3, 2, 1/6), \quad (1, 2, 1/2) \times (1, 2, 1/2) = (1, 1, 1) + (1, 3, 1).$$

Taking the antisymmetric part eliminates the $(6, 1, -1/3)$ and $(1, 3, 1)$. It also ensures we only get one copy of $(3, 2, 1/6)$ though we would naively have two. Then

$$10 \to (\overline{3}, 1, -2/3) + (3, 2, 1/6) + (1, 1, 1)$$

which is exactly the anti-up quark, the quarks, and the anti-electron.

- Therefore, the matter content of the Standard Model fits into the $\overline{5} + 10$ of $SU(5)$. Taking the reasoning in reverse, this explains many features of the Standard Model; it forces charge to be quantized, and it ensures the proton and electron charge are exactly opposite.

Next, we give a brief taste of dynamics in the $SU(5)$ GUT.

- We write mass terms using the ψ_μ and $\psi^{\mu\nu}$ fields. We let

$$\mu, \nu = 1, 2, 3, 4, 5, \quad \alpha = 1, 2, 3, \quad i = 4, 5.$$

There are spinor indices everywhere, which we suppress. In terms of the usual particles, $\psi_\alpha = \overline{d}$, $\psi_i = (\nu, e)$, $\psi^{\alpha\beta} = \overline{u}$, $\psi^{\alpha i} = (d, u)$, and $\psi^{ij} = \overline{e}$.

- Heuristically, a Dirac mass term for the up quark looks like $u^c C u$ where C is charge conjugation. But we run into the same problem as in the Standard Model: it's impossible to add in mass terms which are $SU(5)$ scalars. Instead we introduce a Higgs field φ^μ transforming in the 5, with symmetry breaking so that φ^4 acquires a vev, and mass term

$$\epsilon_{\mu\nu\rho\sigma\tau}\psi^{\mu\nu} C \psi^{\rho\sigma}\varphi^\tau \to \epsilon^{\mu\nu\rho\sigma 4}\psi^{\mu\nu} C \psi^{\rho\sigma}\varphi^4 \sim \psi^{12} C \psi^{35} + \text{permutations} \sim u^c C u.$$

We choose φ^4 to have the vev since it is electrically neutral.

- For the down quark and electron, we introduce the mass term

$$\psi_\mu C \psi^{\mu\nu}\varphi_\nu \to \psi_\mu C \psi^{\mu 4} \sim d^c C d + e^c C e$$

with no mass term for the neutrino since $\psi^{44} = 0$, as desired.

- The gauge bosons are in the adjoint representation and hence can be written as components of a traceless tensor A^μ_ν. Then the 12 gauge bosons we haven't accounted for, called the X and Y bosons, mix quarks and leptons because they exist in the same $SU(5)$ irrep.

- For example, we have

$$\psi^{\alpha 4} = d, \quad A^5_\alpha \psi^{\alpha 4} \sim \psi^{54} = e^+.$$

Similarly, we can convert an up quark into an anti-up quark,

$$\psi^{5\alpha} = u, \quad A^5_\alpha \psi^{5\alpha} \sim \psi^{\alpha\beta} = u^c.$$

Therefore, proton decay can occur by emission and reabsorption of an A^5_α boson,

$$p = u + u + d \to u^c + e^+ + d = \pi^0 + e^+.$$

This process occurs very slowly due to high mass of the A^5_α, at the GUT scale.

- Proton decay can also be described below the GUT scale by an effective field theory. In the Standard Model, we have the 'accidental' global symmetries of quark rotation and lepton rotation, yielding conservation of baryon and lepton number B and L. Then to add proton decay, we simply write down terms that don't obey these global symmetries.

- Since we are far below the GUT scale, any terms we add should be scalars under the Standard Model gauge group. Then a qqq term is unacceptable since it has nonzero hypercharge, but a dimension 6 $qqq\ell$ term turns out to be allowed. This term describes proton decay, and we can relate its rate to the rates of other exotic processes just as we did for isospin.

- We notice that in the proton decay process, B and L change, but $B - L$ is conserved. To see why, note that there are only two mass terms but three irreps in play. Therefore we can construct a global symmetry of the Lagrangian, i.e. a conserved quantity X. We have

$$X(10) + X(10) + X(5_\varphi) = X(\overline{5}) + X(10) + X(\overline{5}_\varphi) = 0$$

which implies

$$X(10) = 1, \quad X(5_\varphi) = -2, \quad X(5^*) = -3.$$

However, this symmetry is broken by the Higgs vev, φ_4. Note that φ_4 has hypercharge $Y/2 = -1/2$ and $X = 2$, with some sign flips since the index is lowered. Then neither X nor Y is conserved, but $X + 4(Y/2)$ is, and this is equal to $B - L$, which is conserved in the $SU(5)$ GUT. In more complicated GUTs, it might not be.

Note. The $SU(5)$ GUT comes with a variety of complications.

- Though the strong, weak, and electromagnetic coupling constants do get close at a high energy scale, they don't converge exactly. This might be fixed by supersymmetric particles.

- Experiments have placed a stringent upper bound on the proton decay rate, beyond what would be natural for the $SU(5)$ GUT.

- There are two Higgs fields; the first transforms in the adjoint H^μ_ν and acquires the vev $\mathrm{diag}(2, 2, 2, -3, -3)$, breaking $SU(5)$ to $SU(3) \times SU(2) \times U(1)$ and giving mass to the gauge bosons. The fermion masses come from the second Higgs field φ^μ as discussed above.

- Our mass term sets $m_d = m_e$ at the GUT scale. This is modified by RG flow, which gives

$$\frac{m_b}{m_\tau} \approx \frac{m_s}{m_\mu} \approx \frac{m_d}{m_e} \approx 3.$$

  This is acceptably accurate for the last two generations, but totally wrong for the first generation. One can fix this by adding a third Higgs field, but this starts to make the theory ugly and complicated.

- Since the GUT scale is so high, we still have a hierarchy problem, called the gauge hierarchy problem, due to the distance between the GUT scale and the electroweak scale, i.e. the fact that the Standard Model Higgs is so light compared to the GUT Higgs, which is essentially as bad as the original hierarchy problem.

Note. The electroweak unification is not the same as grand unification! Grand unification combines three forces into one simple gauge group, where they necessarily have one coupling constant. The electroweak theory instead describes the breaking of $SU(2)_L \times U(1)_Y$ to $U(1)_{\text{EM}}$. Since the unbroken group is a product group, there are still two independent coupling constants above the electroweak scale. The feature they have in common is a Higgs symmetry breaking.

Note. More about picking the Higgs fields. The general principle is that for a particle in an irrep R, whose antiparticle is in an irrep R', the Higgs field or its conjugate must be in $R \otimes R'$. For the down quark and lepton we have

$$\overline{5} \times 10 = [4] \times [2] = [1] + [4, 2] = 5 + 45$$

while for the up quark we have

$$10 \times 10 = [2] \times [2] = [4] + [3, 1] + [2, 2] = \overline{5} + \overline{45} + \overline{50}.$$

For simplicity, we'd like to minimize the number of distinct Higgs fields, so we need to take either 5 or 45. To avoid having to add a third Higgs field, we'd like our Higgs to contain the Standard Model Higgs which breaks $SU(2) \times U(1)$ to $U(1)$. Both 5 and 45 work for this purpose and we took 5 above for simplicity.

9.3 $SO(10)$ Grand Unification (TODO)