Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, including damped and driven oscillators in detail. Also see chapter 10 of Kleppner and Kolenkow. For more on normal modes, see any book on waves, such as *Vibrations and Waves* by A.P. French. Aim to solve or attempt at least **40/92** points.

1 Small Oscillations

First we'll consider some situations involving small oscillations.

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a linear restoring term, the motion is approximately simple harmonic for small displacements.

Idea 2

For complicated setups, it is often easier to find an expression for the energy rather than the force. In particular, if the configuration of your system is described by a parameter q, and the energy takes the form

$$E = \frac{1}{2} m_{\text{eff}} \dot{q}^2 + \frac{1}{2} k_{\text{eff}} q^2$$

then the oscillation frequency is always $\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$. Note that q need not have units of position, while m_{eff} need not have units of mass, and so on.

This idea is justified by Lagrangian mechanics, which you can read about in chapter 6 of Morin. As an example of this reasoning, for a particle in a potential $U(x)$ with a minimum at $x = x_{\text{min}}$ we have $k_{\text{eff}} = U''(x_{\text{min}})$. Then we immediately have

$$\omega = \sqrt{\frac{U''(x_{\text{min}})}{m}}.$$

This allows us to bypass forces entirely, which can be quite confusing.

[2] **Problem 1** (KK 4.13)**.** The Lennard-Jones potential

$$U = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right)$$

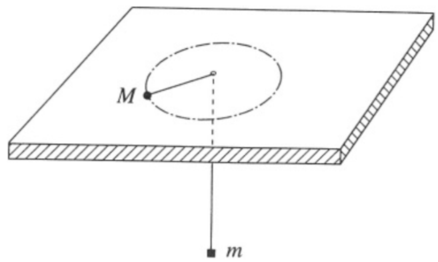
is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard-Jones interaction.

[2] **Problem 2.** Some small oscillations questions about the buoyant force. For these questions, all you need to know is that the buoyant force is equal to the weight of the water displaced; see chapter 15 of Halliday for more details. We'll return to this in much more detail in **M8**.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Assuming a face of the glacier always remains parallel to the water surface, find the frequency of small oscillations.

(b) A ball of radius R floats in water with half its volume submerged. Find the frequency of small oscillations.

[3] **Problem 3** (Grad)**.** A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The motion is frictionless. Show that the orbit is stable with respect to small changes in the radius, and find the frequency of small oscillations.

[3] **Problem 4.** USAPhO 2009, problem A3.

[3] **Problem 5.** USAPhO 1998, problem A2. Note that due to an oversight by the exam writers, part (a) can't be solved explicitly in a reasonable time, so don't bother trying!

[4] **Problem 6.** USAPhO 2010, problem B1.

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions.

[2] **Problem 7.** In this problem we cover some fundamental facts about springs.

(a) Show that when a spring is cut in half, its spring constant doubles.

(b) If a spring with spring constant k_1 and relaxed length ℓ_1 is placed in series with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring.

(c) Do the same for the springs attached in parallel.

(d) Show that if a mass m is hung on a vertical spring, the resulting system behaves exactly the same as a horizontal spring system, except that the relaxed length of the spring is increased by mg/k.

All of these facts are very important, and will be used many times below.

[2] **Problem 8** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

[2] **Problem 9.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. If the projectile is then picked up and thrown, describe the shape of the resulting trajectory geometrically.

[2] **Problem 10.** A mass M oscillates on a spring with spring constant k and mass m. When unstretched, the spring has uniform density. Show that when $m \ll M$, the oscillation frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}$$

in the case of small oscillations.

[5] **Problem 11.** Generalize the previous problem to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses. This is a challenging problem that requires almost all the techniques we've seen so far, so feel free to ask for more hints.)

[3] **Problem 12** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

[3] **Problem 13.** USAPhO 2015, problem A3.

3 Damped and Driven Oscillations

Idea 3

To analyze oscillations, it is often easiest to replace the real sinusoid $A\cos(\omega t + \phi)$ with the complex exponential $\alpha e^{i\omega t}$, where α is a complex number. The real sinusoid is simply the real part of this complex exponential, where A is the magnitude of α and ϕ is its phase.

For more guidance on the problems below, see sections 4.3 and 4.4 of Morin.

[3] **Problem 14.** Analyzing a damped harmonic oscillator.

(a) Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$. Show that Newton's second law can be written as

$$m\ddot{x} + b\dot{x} + kx = 0$$

and that by guessing $x(t) = Ae^{i\alpha t}$, this equation reduces to

$$-m\alpha^2 + ib\alpha + k = 0.$$

(b) There are two roots of this quadratic, $\alpha_\pm$, and the general solution can thus be written as

$$x(t) = A_+ e^{i\alpha_+ t} + A_- e^{-i\alpha_- t}.$$

Using the quadratic formula, find $\alpha_\pm$.

(c) When the roots are complex, the oscillator actually oscillates, and we say the system is underdamped. When the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

[4] **Problem 15.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator, which experiences force $F = -bv - kx + F_0 e^{i\omega t}$. Show that Newton's second law can be written as

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

Show that $A_0 e^{i\omega t}$ is a solution to this equation for some A_0.

(b) The general solution takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\alpha_+ t} + A_- e^{-i\alpha_- t}.$$

The $A_\pm$ are set by initial conditions. After a long time they will decay away, leaving

$$x(t) \approx A_0 e^{i\omega t}.$$

Recalling that the physical position is just the real part, we actually have

$$x(t) \approx |A_0| \cos(\omega t + \phi), \quad \phi = \arg A_0.$$

Evaluate $|A_0|$ and ϕ.

(c) Find the driving frequency ω that maximizes the amplitude $|A_0|$.

(d) Find the driving frequency ω that maximizes the amplitude of the velocity.

(e) Sketch ϕ as a function of ω. Can you intuitively see why ϕ takes the values it does, for ω small, $\omega \approx \sqrt{k/m}$, and ω large?

You should have found that the answers for (c) and (d) above are different; this means that it's really ambiguous what we mean when we say driving is "at resonance". In practice, it doesn't matter, because strong resonance is only noticeable when the damping is weak, and in that case the answers are both approximately equal to $\sqrt{k/m}$.

[3] **Problem 16** (KK 10.9)**.** The quality factor of an oscillator is defined as $Q = m\omega_0/b$. It measures how weak the damping is, but also how sharp the resonance is.

(a) Show that for a lightly damped oscillator near resonance,

$$Q \approx \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}}.$$

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonance frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

(c) Estimate Q for a guitar string.

If you want more information, see pages 424 through 428 of Kleppner and Kolenkow.

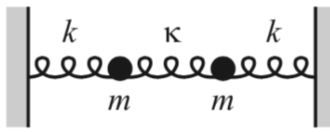
[5] **Problem 17.** GPhO 2016, problem 1.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin. The general motion of the system is a superposition of these normal modes.

[3] **Problem 18** (Morin 4.10). Three springs and two equal masses lie between two walls, as shown.



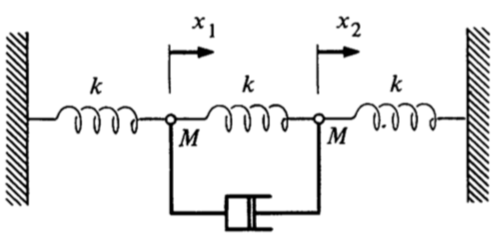
The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like.

[3] **Problem 19.** Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$ and sketch these functions.

[3] **Problem 20** (KK 10.11). Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1 = x_2 = \frac{v_0}{2\omega}\sin\omega t$$

and evaluate ω.

[4] **Problem 21** (Morin 4.12). N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) Do the same for general N. (Hint: it's easiest to do this by promoting the displacements to complex numbers. Do this to your answers to (a) and (b), making the normal modes as symmetric as possible, and try to identify a pattern.)

[4] **Problem 22.** In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy T and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) It can be shown that in general, we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial L}{\partial \theta_i}, \quad L = T - V.$$

This is called the Euler-Lagrange equation. Show that it is equivalent to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}$$

in this particular case. This is a generalization of the equation $dp/dt = F$, but in terms of angles rather than positions. Evaluate these equations.

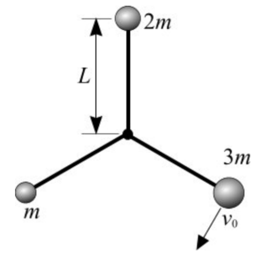
(c) Write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L} K \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where K is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(d) Find the normal modes and their frequencies, using the general method in section 4.5 of Morin.

[5] **Problem 23** (Physics Cup 2012)**.** Three masses are connected to weightless rods of equal length L as shown, and put on a frictionless table.



The rods are connected to each other so that they may freely rotate with respect to each other, i.e. the angles between the rods can change. Initially the angle between the rods is 120° and the system is motionless. The heaviest mass is hit so that it instantaneous attains a velocity v_0 perpendicular to the rod to which it is fixed. Determine the accelerations of all three point masses immediately afterward.

5 Adiabatic Change

Occasionally, tricky Olympiad problems will ask you to analyze slow change.

[4] **Problem 24.** A mass m oscillates on a spring with spring constant k with amplitude A. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k/2$. Find the new amplitude of oscillation by considering how the energy changes over time.

Such questions may be solved directly, but can often be instantly solved using the following theorem.

Idea 5: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p\, dx$$

is conserved.

As you will see in **X1** using quantum statistical mechanics, the conservation of the adiabatic invariant for a single classical particle implies the conservation of the entropy for an adiabatic process in thermodynamics! The two meanings of "adiabatic" are actually one and the same.

[2] **Problem 25.** Solve problem 24 using the adiabatic theorem. Give a physical interpretation of the adiabatic invariant.

[3] **Problem 26.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein relied in the affirmative. Reproduce Einstein's analysis.

[2] **Problem 27** (Grad). A superball is bouncing vertically up and down. It has a velocity v_0 when it strikes the ground, and every collision is perfectly elastic. The acceleration due to gravity is slowly halved over a very long period of time. Find the new value of v_0.

As covered in **T1**, the adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force.

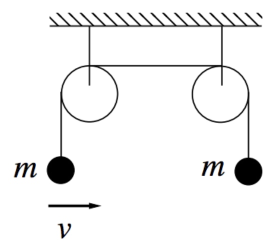
[4] **Problem 28.** A block of mass M and velocity v to the right approaches a stationary ball of mass $m \ll M$. There is a wall a distance L to the right of the ball.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by analyzing each collision. (Note that it does not suffice to simply use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

(c) Using the adiabatic invariant, infer the value of γ for a one-dimensional monatomic gas.

[4] **Problem 29** (***F*** $=$ ***ma*** 2018). Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.