

Project Week 05: Advanced VaR and Expected Shortfall

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1 PROBLEM 1

Please see the repo. The *RiskManagement* folder contains all files for tests:

- Cov** – Covariance estimation techniques.
- NonPSD** – Non-PSD fixes for correlation matrices.
- Sim** – Simulation Methods.
- VaR** – VaR calculation methods (all discussed).
- ES** – ES calculation.

1.1 Missing covariance calculations

Two common way yo calculate:

- (i) Only use the days on which all markets are open;
- (ii) Use pairwise calculations. Find the matching rows for each pair, and build the covariance matrix piece by piece.

1.1.1 Skip missing rows-covariance

	x1	x2	x3	x4	x5
x1	2.148513	-1.389470	-0.516466	-0.129327	-1.056814
x2	-1.389470	1.035342	0.339993	0.193888	0.626876
x3	-0.516466	0.339993	0.942388	0.947887	0.051788
x4	-0.129327	0.193888	0.947887	1.113436	-0.204731
x5	-1.056814	0.626876	0.051788	-0.204731	0.592027

1.1.2 Skip missing rows-correlation

	x1	x2	x3	x4	x5
x1	1.000000	-0.931618	-0.362959	-0.083616	-0.937042
x2	-0.931618	1.000000	0.344202	0.180583	0.800698
x3	-0.362959	0.344202	1.000000	0.925357	0.069333
x4	-0.083616	0.180583	0.925357	1.000000	-0.252163
x5	-0.937042	0.800698	0.069333	-0.252163	1.000000

1.1.3 Pairwise-covariance

	x1	x2	x3	x4	x5
x1	1.173986	-0.629631	-0.278932	-0.081448	-0.735140
x2	-0.629631	1.318197	0.018090	0.446047	0.139309
x3	-0.278932	0.018090	0.918102	0.360836	0.258613
x4	-0.081448	0.446047	0.360836	0.894764	-0.235190
x5	-0.735140	0.139309	0.258613	-0.235190	0.522607

1.1.4 Pairwise-correlation

	x1	x2	x3	x4	x5
x1	1.000000	-0.483199	-0.241787	-0.067767	-0.714761
x2	-0.483199	1.000000	0.015446	0.405660	0.178286
x3	-0.241787	0.015446	1.000000	0.488250	0.336248
x4	-0.067767	0.405660	0.488250	1.000000	-0.322136
x5	-0.714761	0.178286	0.336248	-0.322136	1.000000

1.2 EW covariance

$$w_{t-i} = (1 - \lambda)\lambda^{i-1} \quad (1)$$

$$\widehat{w_{t-i}} = \frac{w_{t-i}}{\sum_{j=i}^n w_{t-j}} \quad (2)$$

$$\widehat{cov}(x, y) = \sum_{i=1}^n \widehat{w_{t-i}}(x_{t-i} - \bar{x})(y_{t-i} - \bar{y}) \quad (3)$$

1.2.1 EW covariance $\lambda=0.97$

	x1	x2	x3	x4	x5
x1	0.855911	0.127559	0.186929	0.081415	0.052412
x2	0.127559	1.08735	0.032715	0.112515	-0.432729
x3	0.186929	0.032715	0.744771	0.131065	0.065806
x4	0.081415	0.112515	0.131065	0.86881	0.113836
x5	0.052412	-0.432729	0.065806	0.113836	1.13918

1.2.2 EW correlation $\lambda=0.94$

	x1	x2	x3	x4	x5
x1	1.0	0.109711	0.218511	0.116902	0.059677
x2	0.109711	1.0	-0.046716	0.191773	-0.444896
x3	0.218511	-0.046716	1.0	0.184148	0.089927
x4	0.116902	0.191773	0.184148	1.0	0.122028
x5	0.059677	-0.444896	0.089927	0.122028	1.0

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1.2.3 EW cov w/EW var($\lambda=0.94$) EW correlation($\lambda=0.97$)

	x1	x2	x3	x4	x5
x1	0.855911	0.10584	0.174461	0.100809	0.058928
x2	0.10584	1.08735	-0.04204	0.186396	-0.495153
x3	0.174461	-0.04204	0.744771	0.148129	0.082832
x4	0.100809	0.186396	0.148129	0.86881	0.121399
x5	0.058928	-0.495153	0.082832	0.121399	1.13918

1.3 Non-psd matrices

$$\Lambda = \text{diag}(\lambda_i) \quad (4)$$

$$CS = \Lambda S \quad (5)$$

$$\lambda'_i = \max(\lambda_i, 0) \quad (6)$$

$$t_i = \left[\sum_{j=1}^n s_{i,j}^2 \lambda'_j \right]^{-1} \quad (7)$$

$$T = \text{diag}(t_i) \quad (8)$$

$$B = \sqrt{T} S \sqrt{\Lambda'} \quad (9)$$

$$BB^T = \hat{C} \approx C \quad (10)$$

1.3.1 Near_psd covariance

	x1	x2	x3	x4	x5
x1	1.173986	-0.617989	-0.284559	-0.065152	-0.688287
x2	-0.617989	1.318197	0.017092	0.445696	0.139176
x3	-0.284559	0.017092	0.918102	0.354147	0.246056
x4	-0.065152	0.445696	0.354147	0.894764	-0.218717
x5	-0.688287	0.139176	0.246056	-0.218717	0.522607

1.3.2 Near_psd Correlation

	x1	x2	x3	x4	x5
x1	1.000000	-0.483199	-0.241787	-0.067767	-0.714761
x2	-0.483199	1.000000	0.015446	0.405660	0.178286
x3	-0.241787	0.015446	1.000000	0.488250	0.336248
x4	-0.067767	0.405660	0.488250	1.000000	-0.322136
x5	-0.714761	0.178286	0.336248	-0.322136	1.000000

1.3.3 Higham covariance

	x1	x2	x3	x4	x5
x1	1.173986	-0.623870	-0.294335	-0.057677	-0.693888
x2	-0.623870	1.318197	0.016449	0.448579	0.143703
x3	-0.294335	0.016449	0.918102	0.354067	0.246866
x4	-0.057677	0.448579	0.354067	0.894764	-0.217062
x5	-0.693888	0.143703	0.246866	-0.217062	0.522607

1.3.4 Higham correlation

	x1	x2	x3	x4	x5
x1	1.000000	-0.483199	-0.241787	-0.067767	-0.714761
x2	-0.483199	1.000000	0.015446	0.405660	0.178286
x3	-0.241787	0.015446	1.000000	0.488250	0.336248
x4	-0.067767	0.405660	0.488250	1.000000	-0.322136
x5	-0.714761	0.178286	0.336248	-0.322136	1.000000

1.4 Cholesky factorization

- (i) Column j , start on the diagonal element
- (ii) Subtract the sum of the squares of the values on the root matrix for row j from the value on the input matrix on the diagonal.
- (iii) Update the root matrix at position (j, j) with the square root of 2
- (iv) Moving down the column, row i
 - (a) Calculate the dot product of sub matrix vector $[i, 1 : (j - 1)]$ and $[j, 1 : (j - 1)]$
 - (b) Subtract a. from the (i, j) element of the input matrix.
 - (c) Divide b. by the j diagonal element of the root matrix
 - (d) Store that value in element (i, j) of the root matrix.
- (v) Repeat for the next column.

	x1	x2	x3	x4	x5
x1	1.083506	0.000000	0.000000	0.000000	0.000000e+00
x2	-0.570360	0.996437	0.000000	0.000000	0.000000e+00
x3	-0.262628	-0.133175	0.911807	0.000000	0.000000e+00
x4	-0.060130	0.412871	0.431384	0.731160	0.000000e+00
x5	-0.635240	-0.223938	0.054179	-0.256892	1.053671e-08

1.5 Normal simulation

$$x = F^{-1}(\text{random uni form}) \quad (11)$$

1.5.1 PD input

	x1	x2	x3	x4	x5
x1	0.085367	0.087933	0.042383	0.009032	0.003874
x2	0.087933	0.160844	0.058218	0.012410	0.005335
x3	0.042383	0.058218	0.037386	0.005975	0.002566
x4	0.009032	0.012410	0.005975	0.001695	0.000548
x5	0.003874	0.005335	0.002566	0.000548	0.000314

1.5.2 PSD input

	x1	x2	x3	x4	x5
x1	0.085474	0.117461	0.042377	0.008987	0.003869
x2	0.117461	0.161419	0.058236	0.012350	0.005317
x3	0.042377	0.058236	0.037285	0.005926	0.002564
x4	0.008987	0.012350	0.005926	0.001679	0.000543
x5	0.003869	0.005317	0.002564	0.000543	0.000314

1.5.3 NonPSD input, near_psd fix

	x1	x2	x3	x4	x5
x1	0.085318	0.008679	0.037962	0.008066	0.003476
x2	0.008679	0.160988	0.052052	0.011104	0.004768
x3	0.037962	0.052052	0.037545	0.006033	0.002593
x4	0.008066	0.011104	0.006033	0.001699	0.000553
x5	0.003476	0.004768	0.002593	0.000553	0.000315

1.5.4 NonPSD input Higham fix

	x1	x2	x3	x4	x5
x1	0.084845	0.013741	0.039073	0.008274	0.003577
x2	0.013741	0.160394	0.053686	0.011398	0.004918
x3	0.039073	0.053686	0.037571	0.006248	0.002700
x4	0.008274	0.011398	0.006248	0.001692	0.000572
x5	0.003577	0.004918	0.002700	0.000572	0.000315

1.5.5 PSD Input - PCA simulation

	x1	x2	x3	x4	x5
x1	0.085344	0.117282	0.042516	0.009038	0.003896
x2	0.117282	0.161173	0.058427	0.012420	0.005355
x3	0.042516	0.058427	0.037562	0.006046	0.002595
x4	0.009038	0.012420	0.006046	0.001103	0.000474
x5	0.003896	0.005355	0.002595	0.000474	0.000204

1.6 Returns

1.6.1 Arithmetic returns

$$P_t = P_{t-1}(1 + r_t) \quad (12)$$

1.7 Fit

1.7.1 Fit normal distribution

Date	SPY	AAPL	MSFT	AMZN	NVDA
2022-09-02	-0.010544	-0.013611	-0.016667	-0.002425	-0.020808
2022-09-06	-0.003773	-0.008215	-0.010974	-0.010980	-0.013336
2022-09-07	0.017965	0.009254	0.019111	0.026723	0.018795
2022-09-08	0.006536	-0.009618	0.001666	0.002626	0.020126
2022-09-09	0.015535	0.018840	0.022977	0.026575	0.028377
...
2023-09-18	0.000586	0.016913	-0.003513	-0.002920	0.001503
2023-09-19	-0.002074	0.006181	-0.001246	-0.016788	-0.010144
2023-09-20	-0.009193	-0.019992	-0.023977	-0.017002	-0.029435
2023-09-21	-0.016528	-0.008889	-0.003866	-0.044053	-0.028931
2023-09-22	-0.002249	0.004945	-0.007887	-0.001624	0.014457

1.7.2 Log returns

$$P_t = P_{t-1}e^{r_t} \quad (13)$$

1.8 Fit

1.8.1 Fit normal distribution

Date	SPY	AAPL	MSFT	AMZN	NVDA
2022-09-02	-0.010600	-0.013705	-0.016807	-0.002428	-0.021027
2022-09-06	-0.003780	-0.008249	-0.011035	-0.011040	-0.013426
2022-09-07	0.017806	0.009211	0.018931	0.026372	0.018621
2022-09-08	0.006515	-0.009664	0.001665	0.002623	0.019926
2022-09-09	0.015416	0.018664	0.022717	0.026228	0.027982
...
2023-09-18	0.000586	0.016772	-0.003519	-0.002925	0.001502
2023-09-19	-0.002076	0.006162	-0.001247	-0.016931	-0.010196
2023-09-20	-0.009236	-0.020195	-0.024269	-0.017148	-0.029877
2023-09-21	-0.016666	-0.008929	-0.003873	-0.045053	-0.029357
2023-09-22	-0.002251	0.004932	-0.007918	-0.001625	0.014354

1.9 Fit

1.9.1 Fit normal distribution

mu	sigma
0.046026	0.046545

1.9.2 Fit t distribution

$$ll = \frac{n}{2} \ln(\sigma^2 2\pi) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \quad (14)$$

mu	sigma	nu
0.04594	0.045443	6.336867

1.9.3 Fit t regression

mu	5.951481e-07
sigma	0.048548
nu	4.598303
Alpha	0.042633
B1	0.97501
B2	2.041187
B3	3.154751

1.10 VaR and ES

$$VaR_\alpha(x) = -F_x^{-1}(\alpha) \quad (15)$$

$$ES_\alpha(X) = -\frac{1}{\alpha} \int_{-\infty}^{-VaR(X)} xf(x)dx \quad (16)$$

1.10.1 VaR normal distribution

VaR Absolute	VaR Diff from Mean
0.030535	0.07656

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1.10.2 VaR t distribution

VaR Absolute	VaR Diff from Mean
0.04153	0.08747

1.10.3 VaR simulation

VaR Absolute	VaR Diff from Mean
0.041848	0.087703

1.10.4 ES normal distribution

ES Absolute	ES Diff from Mean
0.049984	0.09601

1.10.5 ES t distribution

ES Absolute	ES Diff from Mean
0.075232	0.121172

1.10.6 VaR Simulation

ES Absolute	ES Diff from Mean
0.076033	0.122302

1.11 Risk with copula

$$C_R(X) = \Phi_R(\Phi^{-1}(F_1(x_1)), \Phi^{-1}(F_2(x_2)), \dots, \Phi^{-1}(F_n(x_n))) \quad (17)$$

Stock	VaR95	ES95	VaR95_Pct	ES95_Pct
A	93.986214	117.630036	0.046993	0.058815
B	108.399648	152.062133	0.036133	0.050687
Total	153.620537	201.552029	0.030724	0.04031

2 PROBLEM 2

2.1 a

Using a normal distribution with an exponentially weighted variance (lambda=0.97):

VaR Absolute	VaR Diff from Mean
0.091169	0.09029
ES Absolute	ES Diff from Mean
0.113349	0.113227

2.2 b

Using a MLE fitted T distribution:

VaR Absolute	VaR Diff from Mean
0.076476	0.076382
ES Absolute	ES Diff from Mean
0.113218	0.113124

2.3 c

Using a Historic Simulation:

VaR Absolute	VaR Diff from Mean
0.078245	0.077245
ES Absolute	ES Diff from Mean
0.120318	0.116927

3 PROBLEM 3

Portfolio Total				
Stock	VaR95	ES95	VaR95_Pct	ES95_Pct
AAPL	317.465211	414.398701	0.036335	0.04743
ABBV	256.645275	322.998933	0.023323	0.029353
ABT	231.944693	290.561932	0.027168	0.034033
ACN	273.995667	363.900022	0.033133	0.044004
ADBE	319.080784	449.219532	0.042452	0.059767
...
WFC	274.43128	389.418363	0.033347	0.047319
WMT	225.073975	359.760232	0.023049	0.036842
XOM	521.057161	697.143923	0.032548	0.043547
ZTS	253.575624	338.527727	0.032181	0.042962
Total	20319.687801	25526.91869	0.023508	0.029532

Portfolio A					Portfolio B				
Stock	VaR95	ES95	VaR95_Pct	ES95_Pct	Stock	VaR95	ES95	VaR95_Pct	ES95_Pct
AAPL	318.067725	414.961803	0.036404	0.047494	MSFT	313.414874	417.455685	0.036027	0.047986
TSLA	143.605392	184.784627	0.069268	0.08913	GOOGL	15.927829	21.121487	0.041911	0.055577
JPM	262.304101	350.090308	0.029187	0.038956	NVDA	552.893586	697.332359	0.065138	0.082155
HD	255.883765	356.703172	0.030034	0.041867	JNJ	174.056358	238.110363	0.017974	0.024588
BAC	243.157025	340.486158	0.032571	0.045608	PG	186.95317	262.422115	0.021653	0.030394
XOM	528.486741	704.143573	0.033012	0.043984	MA	311.821922	436.810054	0.031178	0.043675
AVGO	373.765206	488.643268	0.03668	0.047953	DIS	273.288422	362.11452	0.03752	0.049715
PEP	185.881414	265.840447	0.018874	0.026993	ADBE	320.513471	450.762529	0.042643	0.059972
TMO	318.223993	429.227423	0.032902	0.044379	KO	182.491606	266.387144	0.018779	0.027412
CMCSA	224.301601	310.994165	0.030506	0.042297	NFLX	419.472912	609.319436	0.057858	0.084044
META	339.418763	510.985577	0.057809	0.08703	COST	289.691373	454.652238	0.028933	0.045408
ACN	272.874763	362.809386	0.032997	0.043872	WFC	280.794064	396.417224	0.03412	0.04817
INTC	203.995624	281.737837	0.040869	0.056445	WMT	224.524156	359.048472	0.022993	0.036769
PYPL	250.394701	336.042687	0.056017	0.075178	LLY	370.14537	497.115875	0.026414	0.035474
MRK	245.391332	356.01979	0.018694	0.027122	NKE	349.585221	491.927359	0.042077	0.05921
T	172.119457	257.275642	0.024656	0.036854	LIN	306.317181	409.790311	0.029712	0.039748
LOW	285.545338	384.340329	0.032586	0.04386	UNP	229.293441	301.034104	0.027575	0.036203
NEE	261.290309	374.13793	0.029472	0.042201	UPS	271.098876	355.712783	0.029947	0.039293
AMD	382.362612	513.366232	0.062097	0.083372	MDT	208.939023	277.053776	0.026871	0.035631
INTU	396.708853	498.135647	0.049744	0.062462	ORCL	300.142603	413.897008	0.03012	0.041536
MS	315.120741	422.462691	0.031509	0.042242	RTX	278.975413	382.030816	0.026156	0.035819
AMAT	347.570053	439.973971	0.049904	0.063171	AMGN	201.125524	301.621383	0.019489	0.029227
CVS	211.908503	311.498378	0.024765	0.036404	CAT	345.468973	517.985049	0.031881	0.047801
GS	282.228242	391.755975	0.028405	0.039429	AMT	274.070034	365.525	0.03184	0.042465
BA	429.066013	622.770002	0.044978	0.065283	COP	533.052825	721.630995	0.041802	0.05659
SBUX	369.467779	558.455296	0.034839	0.05266	AXP	391.078446	537.744146	0.036338	0.049966
GE	264.914585	375.219243	0.033805	0.04788	SPGI	268.297617	357.220871	0.030741	0.04093
ISRG	343.041366	484.006313	0.042572	0.060066	BKNG	520.30309	689.207786	0.042716	0.056582
MU	310.692396	399.81835	0.050123	0.064502	ZTS	255.466947	340.393244	0.032421	0.043199
NOW	489.129033	650.303991	0.055566	0.073876	MDLZ	194.951881	269.941047	0.020628	0.028563
ADP	264.439779	364.104274	0.026841	0.036957	GILD	266.411058	415.577382	0.022249	0.034706
PNC	235.434232	312.62768	0.032214	0.042776	GM	309.870421	399.011221	0.045387	0.058443
ADI	369.8875	488.422107	0.035118	0.046372	Total	6757.398177	8491.975425	0.022954	0.028846
SYK	327.940676	451.710593	0.032645	0.044966					
TJX	363.442427	510.857883	0.031917	0.044863					
Total	8041.424336	10099.754012	0.026809	0.033671					

Portfolio C				
Stock	VaR95	ES95	VaR95_Pct	ES95_Pct
AMZN	20.571407	25.655556	0.05235	0.065288
GOOG	16.253115	20.299385	0.042565	0.053162
BRK-B	220.85555	276.849265	0.023191	0.02907
UNH	272.759972	341.653798	0.025525	0.031972
V	317.265121	399.132373	0.029432	0.037027
PFE	219.672371	274.584326	0.027849	0.034811
CSCO	231.056765	289.533934	0.030334	0.038012
CVX	439.866158	555.247149	0.03348	0.042262
ABBV	256.545108	322.912599	0.023314	0.029346
ABT	232.54804	291.081769	0.027238	0.034094
CRM	370.261914	463.572493	0.048457	0.060669
VZ	183.651123	228.613938	0.024538	0.030546
QCOM	326.773642	411.306316	0.047235	0.059454
MCD	211.303099	265.407734	0.020786	0.026108
DHR	310.346374	389.628816	0.034758	0.043637
TXN	292.42086	370.204315	0.030946	0.039177
PM	255.459428	320.288631	0.025815	0.032366
HON	225.816512	284.540446	0.024641	0.031049
BMJ	222.407863	280.316504	0.020071	0.025297
SCHW	315.232433	395.492654	0.037726	0.047331
C	263.283583	328.826962	0.035119	0.043861
BLK	325.606432	408.881596	0.037898	0.047591
IBM	238.576162	300.342998	0.023783	0.029941
PLD	306.328023	382.956324	0.036249	0.045317
TGT	387.538893	483.683956	0.049161	0.061358
DE	395.805117	497.698564	0.035929	0.045178
MMM	194.466523	242.5489	0.03075	0.038353
F	247.78214	309.640829	0.049183	0.061462
LRCX	382.042913	483.417876	0.052934	0.066981
MO	241.579302	302.737514	0.026469	0.033169
LMT	332.138874	419.896663	0.026168	0.033082
TFC	243.059444	303.644665	0.034034	0.042518
Total	5805.020843	7284.144053	0.021497	0.026974