相贤泰

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1. 证

   根据链式法则，有

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x})=\frac{\partial^2 f}{\partial x'^2}\cos^2(\theta)+2\frac{\partial^2 f}{\partial x'\partial y'}\cos(\theta)\sin(\theta)+\frac{\partial^2 f}{\partial y'^2}\sin^2(\theta)$$

   同理，有

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial x'^2}\sin^2(\theta)-2\frac{\partial^2 f}{\partial x'\partial y'}\cos(\theta)\sin(\theta)+\frac{\partial^2 f}{\partial y'^2}\cos^2(\theta)$$

   则

$$\nabla^2 f(x,y)=(\cos^2(\theta)+\sin^2(\theta))\frac{\partial^2 f}{\partial x'^2}+(\sin^2(\theta)+\cos^2(\theta))\frac{\partial^2 f}{\partial y'^2}=\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}$$

   即拉普拉斯算子是各向同性的。

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2. 证

   由二维傅里叶变换可知

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

   考虑平移函数

$$g(x,y)=f(x-x_0,y-y_0)$$

   其对应的傅里叶变换为

$$\begin{aligned}G(u,v)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)e^{-j2\pi(ux+vy)}dxdy\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-x_0,y-y_0)e^{-j(ux+vy)}dxdy\end{aligned}$$

   换元得

$$\begin{aligned}G(u,v)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x',y')e^{-j2\pi(u(x'+x_0)+v(y'+y_0))}dx'dy'\\&=e^{-j2\pi(ux_0+vy_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x',y')e^{-j2\pi(ux'+vy')}dx'dy'\\&=e^{-j2\pi(ux_0+vy_0)}F(u,v)\end{aligned}$$

   离散化即为书中表达式。

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3. 解

   1. **乘上** $(-1)^{(x+y)}$:
      $f_1(x,y)=f(x,y)\times(-1)^{x+y}$

      实质是用棋盘格进行图像调制；

2. **对调制图像进行DFT**:
   $F_1(u,v) = \mathrm{DFT}[f_1(x,y)]$
3. **取复共轭**:
   将$F_1(u,v)$写作$A(u,v)e^{j\theta(u,v)}$，则其复共轭为
   $F_1^*(u,v) = A(u,v)e^{-j\theta(u,v)}$，改变了该图像频率的相位信息
4. **傅里叶反变换**:
   $f_2(x,y) = \mathrm{IDFT}[F_1^*(u,v)]$
   这使得相位信息变为原来的2倍，这也使得空域图像得到了反转的效果
5. **再次乘**$(-1)^{(x+y)}$:
   $f_3(x,y) = f_2(x,y) \times (-1)^{x+y}$，对图像进行解调。