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13. 解

(a)

$$Q=\begin{bmatrix}-\lambda & \lambda & 0 & 0 & \cdots & 0\\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots & 0\\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & n\mu & -(n\mu+\lambda) & \lambda & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}$$

(b)

其福柯-普朗克方程为

$$\begin{cases}\frac{dp_0(t)}{dt}=-\lambda p_0(t)+\mu p_1(t)\\ \frac{dp_n(t)}{dt}=\lambda p_{n-1}(t)-(\lambda+n\mu)p_n(t)+(n+1)\mu p_{n+1}(t)(n\geq 1)\end{cases}$$

(c)

均值函数为

$$\begin{aligned}M_\xi(t)&=\sum_{n=1}^{\infty}np_n(t)=\sum_{n=1}^{\infty}ne^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left\{\frac{1}{n!}\left[\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right]^n\right\}\\ &=e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left[\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right]^n\\ &=e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\cdot\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\right]^n\\ &=e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\cdot\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\cdot e^{\frac{\lambda}{\mu}(1-e^{-\mu t})}=\frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)\end{aligned}$$

(d)

$$\lim_{t\to\infty}p_0(t)=\exp(-\frac{\lambda}{\mu})$$

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14. 解

(a)

由题可知

$$p_{i,i+1}(\Delta t)=C_i^1(\lambda\Delta t+o(\Delta t))$$

且

$$p_{i,i-1}(\Delta t)=0$$

因此是典型的生灭过程

(b)

因为有

$$p_{i,i+1}(\Delta t) = C_i^1(\lambda \Delta t + o(\Delta t))$$

因此$\lambda_i = i\lambda$

同理，$\mu_i = 0$

前进方程为

$$\frac{dP(t)}{dt} = P(t)Q$$

后退方程为

$$\frac{dS(t)}{dt} = QS(t)$$

(c)

由后退方程有

$$\frac{dp_{kj}(t)}{dt} = (j-1)\lambda p_{k,j-1}(t) - j\lambda p_{k,j}(t)$$

将解代入检验即可。

$$\mathbb{E}(X(s+t) - X(s)|X(s) = m) = \sum_{n=0}^{\infty} n p_{m,n+m}(t) = \sum_{n=0}^{\infty} n C_{n+m-1}^{n} (e^{-\lambda t})^m (1 - e^{-\lambda t})^n = m(e^{\lambda t} - 1)$$

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15. 解

(a)

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots & 0 \\ 0 & 2\mu & -(2\mu+2\lambda) & 2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & n\mu & -(n\mu+n\lambda) & n\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

相应的微分方程为

$$\frac{dp_n(t)}{dt} = p_n(t)Q$$

(b)

$$\begin{aligned} m_X(t) &= \mathbb{E}(X(t)) \\ &= \sum_n n p_n(t) \end{aligned}$$

根据福柯-普朗克方程有

$$\begin{aligned} \frac{dm_X(t)}{dt} &= \sum_n n \frac{dp_n(t)}{dt} \\ &= \sum_n n[(n-1)\lambda p_{n-1}(t) - n(\lambda+\mu)p_n(t) + (n+1)\mu p_{n+1}(t)] \\ &= (\lambda - \mu) m_X(t) \end{aligned}$$

(c)

根据初始条件

$$m_X(0) = n_0$$

解上述微分方程有

$$m_X(t) = n_0 \exp((\lambda - \mu)t)$$

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16. 解

(a)

状态空间$S = \{0, 1, 2, \dots, m\}$

Q矩阵为

$$Q = \begin{pmatrix} -m\lambda & m\lambda & 0 & 0 & \cdots & 0 \\ \mu & -[\mu + (m-1)\lambda] & (m-1)\lambda & 0 & \cdots & 0 \\ 0 & 2\mu & -[2\mu + (m-2)\lambda] & (m-2)\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & m\mu & -m\mu \end{pmatrix}$$

(b)

$$\frac{dp(\vec{t})}{dt} = p(\vec{t})Q$$

(c)

当$t \to \infty$时，有

$$P_n Q = 0$$

解为

$$p_n = C_m^n (\frac{\lambda}{\mu})^n p_0$$

又因为

$$\sum_n p_n = 1$$

所以

$$p_0 = \frac{\mu^m}{(\lambda + \mu)^m}$$

代入即可得到P_n。