序号：64

姓名：相贤泰

学号：202328019427026

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5. 解

令$Z=Y(1+X)$，则有

$$f_Z(z)=\int_{\mathbb{R}}f(x,\frac{z}{1+x})|\frac{1}{1+x}|dx$$

当$0<z<1$时，

$$\begin{aligned}f_Z(z)&=\int_0^1 f(x,\frac{z}{1+x})|\frac{1}{1+x}|dx\\&=\frac{4\ln 2}{7}(z+1)\end{aligned}$$

当$1\leq z<2$时，

$$\begin{aligned}f_Z(z)&=\int_{z-1}^1 f(x,\frac{z}{1+x})|\frac{1}{1+x}|dx\\&=\frac{4}{7}(z+1)(\ln 2-\ln z)\end{aligned}$$

因此，有

$$f_Z(z)=\begin{cases}\frac{4\ln 2}{7}(z+1),\ 0<z<1\\ \frac{4}{7}(z+1)(\ln 2-\ln z),\ 1\leq z<2\\ 0,\text{其它}\end{cases}$$

---

6. 解

(a)

$$P(Y\leq y)=\int_{\mathbb{R}}P(\frac{X_1+X_2x_3}{\sqrt{1+x_3^2}}\leq y|X_3=x_3)f_{X_3}(x_3)dx_3$$

由正态分布可加性有

$$\frac{X_1+X_2x_3}{\sqrt{1+x_3^2}}=\frac{X_1}{\sqrt{1+x_3^2}}+\frac{X_2x_3}{\sqrt{1+x_3^2}}$$

也服从均值为0，方差为1的正态分布，因此有

$$
\begin{aligned}
P(Y \le y) &= \int_{\mathbb{R}} P(\frac{X_1 + X_2 x_3}{\sqrt{1+x_3^2}} \le y | X_3 = x_3) f_{X_3}(x_3) dx_3 \\
&= \int_{\mathbb{R}} [\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du] f_{X_3}(x_3) dx_3 \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du
\end{aligned}
$$

因而，有

$$
f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})
$$

(b)

可以。

---

7. 解

设分布函数为$f(x)$

(a)

$$
\begin{aligned}
P(\tau = k) &= \int_{-\infty}^{+\infty} P(\xi_k > \xi_1, \xi_2 \le \xi_1, \ldots, \xi_{k-1} \le \xi_1 | \xi_1 = x) f_{\xi_1}(x) dx \\
&= \int_0^1 (1 - F_\xi(x)) F_\xi(x)^{k-2} dF_\xi(x) \\
&= \frac{1}{k(k-1)}
\end{aligned}
$$

因此可知，τ的数学期望不存在

(b)

令$U = \max_{1 \le k \le m}(\xi_i)$，则有

$$
f_U(u) = mF^{m-1}(u) f(u)
$$

那么

$$
\begin{aligned}
P(\sigma = i) &= \int_{-\infty}^{+\infty} P(\xi_i > U, \xi_{m+1} \le U, \ldots, \xi_{i-1} \le U | U = x) f_U(x) dx \\
&= \int_0^1 (1 - F_\xi(x)) F_\xi(x)^{i-m-1} m F_\xi^{m-1}(x) dF_\xi(x) \\
&= \frac{m}{i(i-1)}
\end{aligned}
$$

因此

$$
\begin{aligned}
P(\sigma > n) &= \sum_{n+1}^{\infty} \frac{m}{n(n+1)} \\
&= \frac{m}{n}
\end{aligned}
$$

---

8. 解

(a)

$$
\begin{aligned}
P(\xi_1=x_1,\dots,\xi_n=x_n) &= \int_0^1 P(\xi_1=x_1,\dots,\xi_n=x_n|\eta=p)f_\eta(p)dp \\
&= \int_0^1 P(\xi_1=x_1|\eta=p)\dots P(\xi_n=x_n|\eta)f_\eta(p)dp \\
&= \int_0^1 p^{\sum_i x_1+\ldots+x_n}(1-p)^{n-\sum_i x_1+\ldots+x_n}dp \\
&= \frac{\Gamma(1+\sum_i x_1+\ldots+x_n)\Gamma(n+1-\sum_i x_1+\ldots+x_n)}{\Gamma(n+2)} \\
&= \frac{1}{(n+1)C_n^{\sum_i x_1+\ldots+x_n}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\text{当 } k=0,\dots,n \text{ 时} \\
&P(S_n=k) = \int_0^1 P(\xi_1+\ldots+\xi_n=k|\eta=p)f_\eta(p)dp \\
&\qquad = \int_0^1 C_n^k p^k(1-p)^{n-k}dp \\
&\qquad = \frac{1}{n+1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f_{\eta|S_n}(u|S_n=x) &= \lim_{h\to 0}\frac{P(u\le\eta\le u+h|S_n=x)}{h} \\
&= \frac{P(S_n=x|\eta=u)}{P(S_n=x)} \\
&= (n+1)C_n^x u^x(1-u)^{n-x}
\end{aligned}
$$

(d)

相同。

$$
\begin{aligned}
f_{\eta|(\xi_1,\dots,\xi_n)}(v|x_1,\dots,x_n) &= \lim_{h\to 0}\frac{P(v\le\eta\le v+h|\xi_1=x_1,\dots\xi_n=x_n)}{h} \\
&= \frac{P(\xi_1=x_1|\eta=v)\dots P(\xi_n=x_n|\eta=v)}{P(\xi_1=x_1,\dots\xi_n=x_n)} \\
&= (n+1)C_n^x v^x(1-v)^{n-x}
\end{aligned}
$$

所以有，

$$
\begin{aligned}
P(\eta\le p|S_n=x) &= \int_{-\infty}^p f_{\eta|S_n}(u|S_n=x)du \\
&= \int_{-\infty}^p (n+1)C_n^x u^x(1-u)^{n-x}du \\
&= \int_{-\infty}^p (n+1)C_n^x v^x(1-v)^{n-x}dv \\
&= P(\eta\le p|\xi_1=x_1,\dots,\xi_n=x_n)
\end{aligned}
$$