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4. 解

由X和$N(t)$独立可知

$$\mathbb{E}(X(-1)^{N(t)}) = \mathbb{E}(X)\mathbb{E}((-1)^{N(t)}) = 0$$

$$\begin{aligned}
R(t_1, t_2) &= \mathbb{E}(X^2(-1)^{N(t_1)+N(t_2)}) \\
&= \frac{a^2}{2}\mathbb{E}((-1)^{N(t_1)+N(t_2)}) \\
&= \frac{a^2}{2}\mathbb{E}[(-1)^{2N(t_1)}]\mathbb{E}[(-1)^{N(t_2)-N(t_1)}] \\
&= \frac{a^2}{2}\sum_{n=0}^{\infty}(-1)^n P(N(t_2) - N(t_1) = n) \\
&= \frac{a^2}{2}\exp(-\lambda(t_2 - t_1))\sum_{n=0}^{\infty}\frac{[-\lambda(t_2 - t_1)]^n}{n!} \\
&= \frac{a^2}{2}\exp(-2\lambda(t_2 - t_1))
\end{aligned}$$

---

5. 解

(1)

利用微元法，有

$$\begin{aligned}
f_{S_2,S_5}(t_2, t_5) &= \lim_{h\to 0}\frac{P(t_2 < S_2 \le t_2 + h, t_5 < S_5 \le t_5 + h)}{h^2} \\
&= \lim_{h\to 0}\frac{P(N(t_2) = 1, N(t_2 + h) - N(t_2) = 1, N(t_5 - t_2 - h) = 2, N(t_5 + h) - N(t_5) = 1)}{h^2} \\
&= \lim_{h\to 0}\frac{\lambda t_2 \lambda^2 (t_5 - t_2 - h)^2 \lambda^2 h^2}{2h^2}\exp(-\lambda t_2 - 2\lambda h - \lambda(t_5 - t_2 - h)) \\
&= \frac{\lambda^5 t_2 (t_5 - t_2)^2}{2}\exp(-\lambda t_5)
\end{aligned}$$

(2)

因为S_1满足指数分布，那么有

$$\begin{aligned}
P(S_1 \le s | N(t) \ge 1) &= \frac{P(S_1 \le s, S_1 \le t)}{P(S_1 \le t)} \\
&= \frac{1 - \exp(-\lambda s)}{1 - \exp(-\lambda t)}
\end{aligned}$$

因此有

$$f_{(S_1|N(t)\ge 1)} = \frac{\lambda\exp(-\lambda s)}{1 - \exp(-\lambda t)}, 0 \le s \le t$$

所以

$$\mathbb{E}(S_1 | N(t) \ge 1) = \int_{0\le s\le t}\frac{\lambda s\exp(-\lambda s)}{1 - \exp(-\lambda t)}ds = \frac{1}{\lambda} + \frac{1 - t\exp(-\lambda t)}{1 - \exp(-\lambda t)}$$

(3)

$$
\begin{aligned}
f_{S_1,S_2|N(t)=1}(t_1,t_2) &= \lim_{h\to 0}\frac{P(t_1 < S_1 \le t_1+h, t_2 < S_2 \le t_2+h, N(t)=1)}{h^2 P(N(t)=1)} \\
&= \lim_{h\to 0}\frac{P(N(t_1)=0, N(t_1+h)-N(t_1)=1, N(t_2)-N(t_1+h)=0, N(t_2+h)-N(t_2)=1)}{h^2 P(N(t)=1)} \\
&= \frac{\lambda^2 \exp(-\lambda t_2)}{\lambda t \exp(-\lambda t)} \\
&= \lambda \exp(-\lambda(t_2 - t))/t
\end{aligned}
$$

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6. 解

(1)

$$
\begin{aligned}
\mathbb{E}(TN(T/a)) &= \mathbb{E}_T[\mathbb{E}(TN(T/a)|T=t)] \\
&= \int_0^\infty \lambda^2 \frac{t^2}{a}\exp(-\lambda t)dt \\
&= \frac{2}{a\lambda}
\end{aligned}
$$

(2)

同理

$$
\begin{aligned}
\mathbb{E}([TN(T/a)]^2) &= \mathbb{E}_T[\mathbb{E}[(TN(T/a)]^2|T=t)] \\
&= \lambda\int_0^\infty (\frac{\lambda}{a}t^3 + \frac{\lambda^2}{a^2}t^4)\exp(-\lambda t)dt \\
&= \frac{\Gamma(3)}{a\lambda^2} + \frac{\Gamma(4)}{a^2\lambda^2} \\
&= \frac{6a+24}{a^2\lambda^2}
\end{aligned}
$$