序号：64

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1. 解:

   令$\sqrt{2X}=U$，$|Y|=V$，

   则有

$$f_U(u)=u\exp(-\frac{u^2}{2}),u>0$$

$$f_V(v)=\sqrt{\frac{\pi}{2}}\exp(-\frac{v^2}{2}),v>0$$

   当$z>0$时，有

$$\begin{aligned}f_Z(z)&=\int_0^\infty\frac{1}{u}f_U(u)f_V(\frac{z}{u})du\\&=\sqrt{\frac{\pi}{2}}\int_0^\infty\exp(-\frac{u^2}{2})\exp(-\frac{z^2}{2u^2})du\\&=\sqrt{\frac{\pi}{2}}\int_0^\infty\exp(-\frac{1}{2}(\mu^2+\frac{z^2}{u^2}))du\end{aligned}$$

   令：$u=\sqrt{z}t$，则有

$$\begin{aligned}f_Z(z)&=\sqrt{\frac{\pi}{2}}\int_0^\infty\exp(-\frac{1}{2}(\mu^2+\frac{z^2}{u^2}))du\\&=\sqrt{\frac{\pi}{2}}\sqrt{z}\int_0^\infty\exp(-\frac{1}{2}z(t^2+\frac{1}{t^2}))dt\\&=\exp(-z)\sqrt{\frac{\pi}{2}}\sqrt{z}\int_0^\infty\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)dt\end{aligned}$$

   因为有

$$\begin{aligned}\int_0^\infty\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)dt&=\int_0^\infty\frac{t^2}{t^2+1}\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)d(t-\frac{1}{t})\\&=\int_0^\infty\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)d(t-\frac{1}{t})-\int_0^\infty\frac{1}{t^2+1}\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)d(t-\frac{1}{t})\\&=\int_{-\infty}^{+\infty}\exp(-\frac{1}{2}zm^2)dm-\int_0^\infty\frac{1}{t^2}\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)dt\\&=\int_{-\infty}^{+\infty}\exp(-\frac{1}{2}zm^2)dm-\int_0^\infty\exp(-\frac{1}{2}z(n-\frac{1}{n})^2)dn\end{aligned}$$

   因此，有

$$\int_0^\infty\exp(-\frac{1}{2}z(t-\frac{1}{t})^2)dt=\frac{1}{2}\int_{-\infty}^{+\infty}\exp(-\frac{1}{2}zm^2)dm=\sqrt{\frac{1}{2z}}\Gamma(\frac{1}{2})=\sqrt{\frac{\pi}{2z}}$$

   则有

$$f_Z(z)=\begin{cases}\exp(-z),z>0\\0,z\le 0\end{cases}$$

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2. 解:

   令$Z=\frac{X_2}{X_1}$，

则有

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty x_1 f_{X_1}(x_1) f_{X_2}(zx_1)dx_1 \\
&= \lambda^2 \int_0^\infty x_1 \exp(-\lambda(z+1)x_1)dx_1 \\
&= \frac{1}{(z+1)^2}\Gamma(2) \\
&= \frac{1}{(z+1)^2}
\end{aligned}
$$

继续令$Y=\frac{1}{1+Z}$，则有

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(Z \ge \frac{1-y}{y}) \\
&= \int_{\frac{1-y}{y}}^\infty \frac{1}{(z+1)^2}dz \\
&= y
\end{aligned}
$$

可知

$$
Y = \frac{X_1}{X_1+X_2}
$$

所以有

$$
f_{\frac{X_1}{X_1+X_2}}(y) = \begin{cases} 1, \ 0 \le y \le 1 \\ 0, \ \text{others} \end{cases}
$$

即

$$
\frac{X_1}{X_1+X_2} \sim U[0,1]
$$

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3. 解:

(a)

$$
\begin{aligned}
f_{X+Y}(u) &= \int_{\mathbb{R}} f(x)f(u-x)dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \exp[-\frac{1}{2}(2x^2-2ux+u^2)]dx \\
&= \frac{1}{2\sqrt{\pi}}\exp(-\frac{u^2}{4})
\end{aligned}
$$

同理,

$$
f_{X-Y}(v) = \frac{1}{2\sqrt{\pi}}\exp(-\frac{v^2}{4})
$$

(b)

$X+Y$和$X-Y$独立

令$U=X+Y$，$V=X-Y$

则有

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{1}{2}f_{X,Y}(\frac{u+v}{2},\frac{u-v}{2}) \\
&= \frac{1}{2}f_X(\frac{u+v}{2})f_Y(\frac{u-v}{2}) \\
&= f_U(u)f_V(v)
\end{aligned}
$$

4. 解:

(a)

$$
\begin{aligned}
f_X(x) &= \int_0^x f(x,y)dy \\
&= 12(1-x)x^2,\ 0<x<1
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \int_y^1 f(x,y)dx \\
&= 12y(1-y)^2,\ 0<y<1
\end{aligned}
$$

$$
\begin{aligned}
f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)} = \frac{2(1-x)}{(1-y)^2},\ 0<y<x \\
f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{2y}{x^2},\ y<x<1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbb{E}(X|Y=y) &= \int_y^1 x f_{X|Y}(x|y)dx \\
&= \frac{2y+1}{3}
\end{aligned}
$$

令$Z=\mathbb{E}(X|Y)$，则有

$$
\begin{aligned}
Z &= \frac{2Y+1}{3} \\
F_Z(z) &= P(Z\leq z) \\
&= P(Y\leq \frac{3z-1}{2}) \\
&= \int_0^{\frac{3z-1}{2}} f_Y(y)dy \\
&= F_Y(\frac{3z-1}{2})
\end{aligned}
$$

因此，有

$$
f_Z(z) = \frac{81(3z-1)(1-z)^2}{4},\ \frac{1}{3}<z<1
$$