A theory of incoherent scattering of radio waves by a plasma
II. Scattering in a magnetic field

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A general expression for the frequency spectrum of radio waves scattered by the random thermal fluctuations of electron density in a plasma in a magnetic field is derived. The derivation is based on the generalized Nyquist noise theorem used in part I. The exact result is then simplified by means of an approximation which amounts to assuming the velocity of light to be infinite. It is shown that this approximation is quite adequate for ionospheric applications of the theory.

Next it is proved, without appealing to any approximation, that the magnetic field can never alter the total scattered signal power; it can only redistribute this power over the spectrum.

Finally, the detailed shape of the frequency spectrum of the scattered signal is examined. Analytic expressions are given for certain limiting cases, but for the cases of most interest, numerical methods must be used. The results of some numerical calculations are shown in figures 1 and 2.

From these results, it can be seen that the magnetic field has a significant effect on the shape of the spectrum only if the incident radio beam is very nearly orthogonal to the magnetic lines of force. For example, for an operating frequency of 40 Mc/s, no significant magnetic effect is observed even when the beam is within 5° of orthogonality. As this angle is decreased further, however, the spectrum rapidly begins to develop spikes at Doppler shifts which are approximate multiples of the ion gyro-frequency. These spikes are quite pronounced when the beam is 2° from orthogonality. At higher operating frequencies, the beam must be proportionally closer to orthogonality to achieve the same effect.

1. Introduction

Following the suggestion of Gordon (1958), a considerable amount of interest in the phenomenon of incoherent scattering of radio waves in the ionosphere has arisen. If a very powerful radio beam whose frequency is well above the penetration frequency is directed into the ionosphere, a very weak but still measurable signal is scattered back from the random thermal fluctuations of electron density. The strength of the scattered signal is directly proportional to the electron density, and thus one can obtain a true height electron density profile, both above and below the F region peak, from measurements of the scattered power. Further information can be derived from the frequency spectrum of the scattered signal.

Measurements of this phenomenon (Bowles 1958, 1959, 1961; VanZandt & Bowles 1960; Pineo, Kraft & Briscoe 1960*a*, *b*) have indicated that Gordon's original explanation of the effect was oversimplified, primarily because he predicted a much larger Doppler spread in the scattered signal than has been observed. It was realized that the ions, which Gordon had ignored, play an important role in determining the Doppler spread, even though it is the electrons which do the actual scattering.

An approximate calculation of the spectrum, taking into account the effect of the ns, was made by Bowles (1959). This work was followed by the more exact calcula- ons of Dougherty & Farley (1960, hereafter referred to as 'I'), Fejer (1960), ad Salpeter (1960*a*, *b*), all of whom obtained similar results from quite different ethods. These results agree very well with the available measurements of the ectrum, and also predict that the total scattered power will be somewhat less an that predicted by Gordon. In the practical case this reduction is by a factor of 2. enau (1960) has also given a derivation of this last result by a method not involving e question of the Doppler broadening.

Except for some qualitative remarks by Bowles, the effect of the earth's magnetic ld has been neglected in the theoretical work of the authors just cited. (Recently aspere (1960) has considered the case for which the magnetic field is important t the influence of the ions is negligible. His results provide some insight into the ect of a magnetic field, but except perhaps at very great heights they are appli- ble in the ionosphere only if the radio wavelength is less than about 1 mm.) It was ntioned in I that the method used there can be applied in principle to the more eral case in which there is a uniform magnetic field present, and this task is the ect of the present paper.

Upon carrying out this work, one finds that, for values of the various parameters propriate to the ionospheric problem, the magnetic field produces a significant ect only if the direction of propagation of the incident radio wave is very nearly rpendicular to the magnetic field. The spectrum becomes more and more 'spikey' as hogonality is approached. In the most common cases the spikes occur at frequen- es corresponding to Doppler shifts which are approximately equal to multiples of e ion gyro-frequency. It is important to note, however, that the magnetic field n never alter the *total* scattered power; only the form of the spectrum can be changed.

These results are derived by using a generalized version of Nyquist's noise eorem in much the same way as was done in I. The fundamental theory required s given there in sufficiently general form to be applicable also to the present case, d so it will not be repeated in detail here. The essential results, however, are mmarized in §2. Nyquist's theorem has to be used under slightly more general nditions than before, as discussed in §3, but this does not affect the procedure. e formal solution of the problem, in the completely general case, is given in §4. is solution is extremely difficult to work with, but fortunately a simplifying proximation (which amounts to assuming the velocity of light to be infinite) is ssible. This approximation is discussed in §5.

In the remaining sections the total cross-section for back-scatter and the fre- uency spectrum of the scattered signal are computed and discussed. To complete he latter calculations one must usually use a digital computer, and this has been lone for a number of cases of practical interest. The limiting case of propagation rthogonal to the magnetic field is examined separately.

For clarity we shall assume throughout that the ions are singly charged and of only ne species. The results for more general cases are quoted, however. Similarly we hall consider only back-scattering, but the generalization to other scattering angles s, of course, trivial.

2. Summary of paper I

The scattering of an electromagnetic wave of angular frequency ω_0 and wave vector $\mathbf{k}_0$ (in the medium) by a plasma confined in a large cubical box of side L was examined. The differential cross-section for back-scattering, σ_b (i.e. the power scattered through 180° per unit incident power per unit solid angle per unit volume per unit frequency range) at the Doppler shifted frequency $\omega_0 \pm \omega$ was found to be

$$\sigma_b(\omega_0 \pm \omega)\,\mathrm{d}\omega = r_e^2 L^3 \langle|\Delta N(\mathbf{k},\omega)|^2\rangle\,\mathrm{d}\omega, \tag{2·1}$$

where r_e is the classical electron radius $e^2/m_e c^2 (= 2{\cdot}8\times10^{-13}\,\mathrm{cm})$, and $\mathbf{k} = 2\mathbf{k}_0$ in the case of back-scattering. The case of scattering in other directions is readily handled by a suitable choice of $\mathbf{k}$ (see, for example, Villars & Weisskopf 1955). The power spectrum $\langle|\Delta N(\mathbf{k},\omega)|^2\rangle$ of the statistical fluctuations of electron density is defined by the relations

$$\Delta N(\mathbf{r},t) = \sum_{\mathbf{k}} \Delta N(\mathbf{k},t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$$

and

$$\langle \Delta N^*(\mathbf{k},t)\,\Delta N(\mathbf{k},t+\tau)\rangle = \int_{-\infty}^{\infty} \langle|\Delta N(\mathbf{k},\omega)|^2\rangle\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega,$$

where $\Delta N(\mathbf{r},t)$ is the deviation of the electron density from its mean value N_0, as a function of position and time. The angular brackets denote the expected value or ensemble average.

To find this spectrum the response to a hypothetical force

$$\mathbf{F} = \mathbf{F}_0\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}_0\cdot\mathbf{r})}$$

applied to the *electrons only* in a plasma was calculated from the Boltzmann equations for the ions and electrons, together with Maxwell's equations. This force produced a mean bulk motion $\mathbf{u}_e$ of the electrons, given by

$$N_0\mathbf{u}_e = \mathbf{Y}'\,.\,\mathbf{F}, \tag{2·2}$$

where the admittance tensor $\mathbf{Y}'$ is

$$\mathbf{Y}' = (\mathbf{Y}_i - \mathbf{\Gamma}/e^2)\,(\mathbf{Y}_i + \mathbf{Y}_e - \mathbf{\Gamma}/e^2)^{-1}\,\mathbf{Y}_e. \tag{2·3}$$

Here $\mathbf{Y}_i$ and $\mathbf{Y}_e$ are admittance tensors for the ions and electrons considered separately. $\mathbf{\Gamma}$ is a conductivity tensor derivable solely from Maxwell's equations and is defined by

$$\Gamma_{pq} = (\mathrm{i}c^2/4\pi\omega)\,[(k^2 - \omega^2/c^2)\,\delta_{pq} - k_p k_q]. \tag{2·4}$$

From Nyquist's theorem the spontaneous bulk motion $\langle|u_e(\mathbf{k},\omega)|^2\rangle$ was then obtained. This was converted to density fluctuations, giving

$$\langle|\Delta N_e(\mathbf{k},\omega)|^2\rangle\,\mathrm{d}\omega = (KTk^2/\pi L^3\omega^2)\,\mathscr{R}(Y'_{zz})\,\mathrm{d}\omega$$

if $\mathbf{k}$ is along the z axis, or, expressed in invariant form,

$$\langle|\Delta N_e(\mathbf{k},\omega)|^2\rangle\,\mathrm{d}\omega = (KT/\pi L^3\omega^2)\,\mathscr{R}(k_i Y'_{ij} k_j)\,\mathrm{d}\omega. \tag{2·5}$$

In the above, K is Boltzmann's constant, T is the absolute temperature, and a summation over i and j is implied. Substituting (2·5) into (2·1) gives the final result

$$\sigma_b(\omega_0 \pm \omega)\,\mathrm{d}\omega = (r_e^2 KT/\pi\omega^2)\,\mathscr{R}(k_i Y'_{ij} k_j)\,\mathrm{d}\omega. \tag{2·6}$$

Note that L does not appear in (2·6), and thus the size of the box is ultimately irrelevant.

3. The application of Nyquist's theorem to the present problem

The generalization of Nyquist's theorem given by Callen & Welton (1951) holds for any linear dissipative system with one degree of freedom. In applying it in I to the problem of finding the thermal density fluctuations in a gas in equilibrium, we regarded each spatial Fourier component as providing three degrees of freedom (one for displacements parallel to the wave vector **k** and two for displacements transverse to it). It was tacitly assumed that Nyquist's theorem could be applied separately to each of these many degrees of freedom. This assumption was certainly plausible because **Y** is actually diagonal if one co-ordinate axis is taken parallel to **k**, and therefore the degrees of freedom are 'normal' and non-interacting.

When a magnetic field is introduced, **Y** ceases to be diagonal, as will be seen later; so one might ask whether the theorem remains valid. A still more general version of Nyquist's theorem appropriate to this case has been given by Callen, Barasch & Jackson (1952) and is as follows: Suppose that a system with several degrees of freedom is such that when a set of generalized forces V_1, V_2, ... is applied to it, corresponding responses I_1, I_2, ... are produced, these responses being defined in such a way that the instantaneous rate of working of the forces on the system is

$$\sum_j V_j(t)\, I_j(t).$$

The responses are further assumed to be linear, in the sense that if $V_j(t)$ is of the form $\mathscr{R}(V_j\, \mathrm{e}^{\mathrm{i}\omega t})$, where V_j is a complex constant, with a similar notation for I_j, then

$$I_j = \sum_k Y_{jk}(\omega)\, V_k, \tag{3·1}$$

where $Y_{jk}(\omega)$ is a generalized admittance tensor. If these conditions are satisfied, the theorem states that, in thermal equilibrium at a temperature T, the responses are stochastic functions of time satisfying

$$\langle |I_j(\omega)\, I_k(\omega)|\rangle\, \mathrm{d}\omega = (KT/2\pi)\, \mathscr{R}[Y_{jk}(\omega) + Y_{kj}(\omega)]\, \mathrm{d}\omega. \tag{3·2}$$

The left-hand side of (3·2) is defined by

$$\tfrac{1}{2}\langle I_j^*(t)\, I_k(t+\tau) + I_k^*(t)\, I_j(t+\tau)\rangle = \int_{-\infty}^{\infty} \langle |I_j(\omega)\, I_k(\omega)|\rangle\, \mathrm{e}^{\mathrm{i}\omega\tau}\, \mathrm{d}\omega$$

and is thus the spectrum function of a certain cross-correlation between $I_j(t)$ and $I_k(t)$ evaluated at times separated by τ. If there is only one degree of freedom, (3·2) reduces to

$$\langle |I(\omega)|^2\rangle\, \mathrm{d}\omega = (KT/\pi)\, \mathscr{R}[Y(\omega)]\, \mathrm{d}\omega,$$

which is the theorem used in I.

The point we wish to make here is that even if $Y_{jk}(\omega)$ is not diagonal, its diagonal terms still give the power spectrum of the variables, i.e. putting $j = k$ in (3·2) gives

$$\langle |I_j(\omega)|^2\rangle\, \mathrm{d}\omega = (KT/\pi)\, \mathscr{R}[Y_{jj}(\omega)]\, \mathrm{d}\omega \tag{3·3}$$

(no summation over j). The additional content of the theorem in this more general form is the interesting significance of the off-diagonal components of **Y**. However, these will not be required here.

It can now be seen that the work of I is formally correct for the case in which there is an applied magnetic field, provided that new admittance tensors, which take account of the magnetic forces, are calculated for use in (2·3). This is done in the appendix of the present paper.

4. General expression for the spectrum function

It is convenient to use two co-ordinate systems throughout the rest of this paper. In the first, we take $\mathbf{B}$ to be along the z axis, and introduce complex co-ordinates defined by the unitary transformation

$$x_1 = (x+\mathrm{i}y)/2^{\frac{1}{2}}, \quad x_{-1} = (x-\mathrm{i}y)/2^{\frac{1}{2}}, \quad x_0 = z, \tag{4·1}$$

where (x, y, z) are ordinary co-ordinates (cf. Bernstein, 1958). In what follows Greek letters will stand for the suffixes $(1, 0, -1)$, and the summation convention applies. The rule for the formation of a scalar product is mentioned in the appendix. By choice of the orientation of the axes, we may, with no loss of generality, take a typical wave vector to be $k_1 = k_{-1} = k \sin\alpha/2^{\frac{1}{2}}$, $k_0 = k\cos\alpha$, where $k = |\mathbf{k}|$, and α is the angle between $\mathbf{k}$ and $\mathbf{B}$. (In some matrix products the complex conjugate $\mathbf{k}^*$ will be written where it is formally required, but the asterisk may in fact be omitted with this particular choice of $\mathbf{k}$.)

The second co-ordinate system is an ordinary Cartesian one in which $\mathbf{k}$ is along the z axis. This will be called the 'x, y, z' system, and will be indicated by Latin suffixes. It is the system used throughout I.

As an example, (2·4) becomes, in the complex system,

$$\Gamma_{\mu\nu} = (\mathrm{i}c^2/4\pi\omega)\,[(k^2-\omega^2/c^2)\,\delta_{\mu\nu} - k_\mu k_\nu^*]. \tag{4·2}$$

As in I, it saves writing to normalize the frequencies by introducing dimensionless variables

$$\theta = \frac{\omega}{k}\left(\frac{m}{2KT}\right)^{\frac{1}{2}} \tag{4·3}$$

and the normalized gyro-frequency

$$\phi = \frac{eB}{mck}\left(\frac{m}{2KT}\right)^{\frac{1}{2}} = (2^{\frac{1}{2}}kR_L)^{-1}, \tag{4·4}$$

where R_L is the Larmor radius $c(mKT)^{\frac{1}{2}}/eB$. These each take a suffix i or e corresponding to ions or electrons. Then the result proved in the appendix is that the admittance for a single species of particle is

$$\mathbf{Y} = (N_0\omega/KTk^2)\,\mathbf{y}, \tag{4·5}$$

where $$y_{\mu\nu} = \frac{1}{2\theta}\int_0^\infty \exp\left(-\mathrm{i}\theta t - \phi^{-2}\sin^2\alpha\sin^2\tfrac{1}{2}\phi t - \tfrac{1}{4}t^2\cos^2\alpha\right) F_{\mu\nu}(\phi, \alpha, t)\,\mathrm{d}t \tag{4·6}$$

and $$F_{\mu\nu} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}(\mu+\nu)\phi t}\left(\delta_{\mu\nu} - \frac{2k_\mu k_{-\nu}\sin\frac{1}{2}\mu\phi t\sin\frac{1}{2}\nu\phi t}{k^2\phi^2\mu\nu}\right), \tag{4·7}$$

it being understood that if one or both of μ, ν are zero, $F_{\mu\nu}$ is given its limiting value as μ and/or ν tends to zero. The explicit expressions so obtained are listed in equations (A 17). For negatively charged particles, the sign of ϕ should be reversed. In

·5), **y** is a 'reduced' dimensionless admittance; it is convenient to define a similar
mbol $\boldsymbol{\gamma}$ by

$$\boldsymbol{\Gamma}/e^2 = (\mathrm{i}N_0\omega/KTk^2)\,\boldsymbol{\gamma}, \tag{4·8}$$

e.

$$\gamma_{\mu\nu} = h^2k^2\left[\left(\frac{c^2k^2}{\omega^2}-1\right)\delta_{\mu\nu}-\frac{c^2}{\omega^2}k_\mu k_\nu^*\right], \tag{4·9}$$

d h is the Debye length $(KT/4\pi N_0 e^2)^{\frac{1}{2}}$. It is readily shown that in this new nota-
on equation (2·6) becomes

$$\sigma_b(\omega_0\pm\omega)\,\mathrm{d}\omega = N_0 r_e^2\,\mathscr{R}(k_\mu^* y'_{\mu\nu}k_\nu/k^2)\,\mathrm{d}\omega/\pi\omega, \tag{4·10}$$

ere

$$\begin{aligned}\mathbf{y}' &= (\mathbf{y}_i-\mathrm{i}\boldsymbol{\gamma})(\mathbf{y}_i+\mathbf{y}_e-\mathrm{i}\boldsymbol{\gamma})^{-1}\mathbf{y}_e\\ &= [\mathbf{y}_e^{-1}+(\mathbf{y}_i-\mathrm{i}\boldsymbol{\gamma})^{-1}]^{-1}.\end{aligned} \tag{4·11}$$

is is the formal result on which the rest of this paper depends.

One may wish to consider the more general case in which the plasma contains veral species of ions, the nth species have a number density $\beta_n N_0$, where N_0 refers the electrons, and an ionic charge $Z_n e$. This may be done by replacing $\mathbf{y}_i$ of ·11) with

$$\sum_n \beta_n Z_n^2\,\mathbf{y}_n. \tag{4·12}$$

re each $\mathbf{y}_n$ is calculated as in (4·6), provided that θ and ϕ are calculated with the propriate mass and charge. The condition of charge neutrality requires that

$$\sum_n \beta_n Z_n = 1. \tag{4·13}$$

Since Nyquist's theorem can be applied only to systems in thermal equilibrium, does not seem to be possible, within the framework of the present theory, to amine properly the case in which the ion and electron temperatures differ, though it was erroneously suggested in I that this could be done. This is not a rious restriction, since it is expected that the plasma in the ionosphere will be thermal equilibrium, except possibly at very great distances from the earth.

Some discussion of the case in which the temperatures are unequal has been ven by Fejer (1961) in a recent extension of his earlier work.

5. Simplification of the result at low frequency

A glance at (4·9) shows that if $\omega \ll ck$, $\boldsymbol{\gamma}$ contains some extremely large terms. urther, we see from (4·6) that $\mathbf{y}_i$ and $\mathbf{y}_e$ apparently become infinite as $\omega\to 0$, and rom (4·10) that even if $\mathbf{y}'$ were finite at $\omega = 0$, the spectrum function would seem to e infinite. Intuitively, one does not expect anything unusual to happen as $\omega\to 0$ it does not in the case $\mathbf{B} = 0$; see I), and we shall shortly see that these infinities are n fact quite spurious. This indicates that the present form of the result is very nconvenient for numerical work, at least for small Doppler shifts.

The way to avoid this difficulty is best seen by using the 'x, y, z' co-ordinates, in hich $\mathbf{k} = (0, 0, k)$. Then $\boldsymbol{\gamma}$ is diagonal, and

$$\gamma_{xx} = \gamma_{yy} = h^2k^2(c^2k^2/\omega^2-1), \quad \gamma_{zz} = -h^2k^2.$$

The important point to notice is that the 'large' terms, proportional to c^2k^2/ω^2, form a singular matrix. Thus $(\mathbf{y}_i - \mathrm{i}\boldsymbol{\gamma})^{-1}$, required in (4·11), is not as small as it appears at first sight. Referring to equation (4·6), the components of $\mathbf{y}$ are seen to be generally of order θ^{-1}, and so are proportional to ω^{-1} as $\omega \to 0$. It is, however, shown in the appendix that, in the 'x, y, z' system, the components in both the third row and the third column are actually all finite as $\omega \to 0$. Bearing these facts in mind, a careful examination of all the types of term appearing in the determinant and the minors of the matrix $\mathbf{y}_i - \mathrm{i}\boldsymbol{\gamma}$ shows that

$$(\mathbf{y}_i - \mathrm{i}\boldsymbol{\gamma})^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (y^i_{zz} + \mathrm{i}h^2k^2)^{-1} \end{pmatrix} + \text{residual terms.} \tag{5·1}$$

Here the 'residual terms' which ought to replace the zeros in the right-hand side are all proportional to ω^2, while the 'residual term' representing the error in the zz component is proportional to ω, the error being specifically of the order of

$$\frac{\omega}{(ck)\,(hk)^2 \cos\alpha}\left(\frac{KT}{m_i c^2}\right)^{\frac{1}{2}}. \tag{5·2}$$

Although we shall be interested in the case $hk \ll 1$, we shall also insist that $KT \ll m_i c^2$ and shall only wish to investigate values of ω much smaller than ck. As we shall see the expression in (5·2) is very small in practice, and the other residual terms are even smaller. We may therefore omit all these residual terms in (5·1), subject to the condition

$$\omega \ll (ck)\,(hk)^2\,(m_i c^2/KT)^{\frac{1}{2}} \cos\alpha,$$

or

$$\cos\alpha \gg (hk)^{-2}\,(KT/m_i c^2)\,2^{\frac{1}{2}}\theta_i. \tag{5·3}$$

In the F region of the ionosphere (5·3) becomes approximately

$$\cos\alpha \gg 2 \times 10^{-5}\theta_i$$

for an incident frequency $f_0 (= \omega_0/2\pi)$ of 40 Mc/s. Since we are interested mainly in values of θ_i of the order of unity or smaller, it can be seen that (5·3) will be satisfied unless α is within about 10^{-4} radians of $\frac{1}{2}\pi$. For higher incident frequencies the dependence of (5·3) on k will make this range even smaller.

In the 'x, y, z' axes, all we need to evaluate is the zz component of $\mathbf{y}'$ as given by (4·11). Let us therefore consider the matrix

$$\mathbf{M} = \mathbf{A}^{-1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix},$$

where $\mathbf{A}$ is non-singular. The minor formed by the first two rows and columns of $\mathbf{M}$ is simply that of the matrix $\mathbf{A}^{-1}$ itself, namely $A_{zz}/\det\mathbf{A} = D$, say. The determinant of $\mathbf{M}$ is just

$$\det(\mathbf{A}^{-1}) + DC = (1 + CA_{zz})/\det\mathbf{A}.$$

Thus the zz component of $\mathbf{M}^{-1}$ is

$$A_{zz}/(1 + CA_{zz}).$$

† The suffixes denoting ions and electrons are written as superscripts where there is the possibility of confusion with co-ordinate suffixes.

Hence, in the approximation just discussed,

$$y'_{zz} = \left(\frac{1}{y^e_{zz}} + \frac{1}{y^i_{zz} + \mathrm{i}h^2k^2}\right)^{-1}. \tag{5·4}$$

We can calculate y^i_{zz} and y^e_{zz} from

$$y_{zz} = \mathbf{k}^*.\mathbf{y}.\mathbf{k}/k^2$$

and the expressions (4·6); however, a further result proved in the appendix is that this simplifies (without any approximations) to

$$y_{zz}(\theta, \alpha, \phi) = i + \theta \int_0^\infty \exp\left(-i\theta t - \phi^{-2}\sin^2\alpha \sin^2 \tfrac{1}{2}\phi t - \tfrac{1}{4}t^2\cos^2\alpha\right) \mathrm{d}t. \tag{5·5}$$

We have now obviated the need to perform matrix calculations numerically, and greatly reduced the number of integrals which have to be evaluated. This procedure began as an attempt to resolve the difficulties arising near $\omega = 0$, but the results are very accurate for all the values of ω of practical interest, except when α is extremely close to $\frac{1}{2}\pi$. It can also be regarded as the result of letting $c \to \infty$, and is therefore identical with the 'longitudinal approximation' introduced by Bernstein (1958). The results could therefore have been obtained more directly by taking this limit at the start, i.e. by replacing Maxwell's equations by the Poisson equation and writing curl $\mathbf{E} = 0$ (this is rigorously correct when there is no magnetic field, since the longitudinal and transverse fields are then uncoupled). The present method, however, reveals (5·3) as a condition under which this is in general correct.

It is possible that (5·4), which is all that we require, may be a good approximation under even less stringent conditions. Indeed, it seems rather unlikely that anything unusual will occur in the extremely small range of α for which (5·3) is not satisfied. This question is not of much practical importance, in any event, because any effects occurring in this small range would probably be nearly impossible to observe, owing to the inevitable angular spread of the incident radio beam.

We should note here that, using different methods, Fejer (1961), Hagfors (1961), and Salpeter (1961*a*, *b*) have arrived at results equivalent to those obtained by combining equations (4·10), (5·4) and (5·5) of the present paper.

6. The total cross-section for back-scatter

Before applying the results just obtained to find the frequency spectrum of the scattered radiation, we will evaluate the total power scattered, in a way parallel to Appendix C of I. From (4·10) the total cross-section for back-scatter is given by

$$\sigma_{\text{total}} = N_0 r_e^2 \mathscr{R} \int_{-\infty}^{\infty} y'_{zz} \frac{\mathrm{d}\omega}{\pi\omega}. \tag{6·1}$$

To evaluate this integral, we use a contour in the complex ω plane consisting of the real axis (indented at the origin to avoid the singularity there) and a large semicircle below it. On this large semicircle $\boldsymbol{\gamma}$ is a finite non-singular matrix (see 4·9). It is also easy to see from (4·6) that as $|\theta| \to \infty$, with $\mathscr{I}(\theta) < 0$, the components of y are of order $|\theta|^{-2}$. Hence from (4·11) the components of $\mathbf{y}'$ are also of order $|\theta|^{-2}$, so that the integrand of (6·1) is of order $|\omega|^{-3}$ on the large semicircle, and the integral along this semicircle therefore tends to zero as the radius tends to infinity.

As in the case of no magnetic field, I, we can rule out the possibility that there are any singularities of the integrand below the real axis, since they would indicate the existence of growing waves, i.e. instabilities, in a plasma in thermal equilibrium. The integral is therefore $-\pi i$ times the residue of the simple pole at $\omega = 0$, or $-\mathrm{i}\lim_{\omega\to 0} y'_{zz}$. Furthermore, the longitudinal approximation, and thus (5·4), is exact (except perhaps at $\alpha \equiv \frac{1}{2}\pi$) in the limit $\omega \to 0$. Thus from (6·1), (5·4), and (5·5) we obtain

$$\sigma_{\text{total}} = N_0 r_e^2 \frac{1+h^2k^2}{2+h^2k^2}. \tag{6·2}$$

Thus the *total* cross-section for back-scatter is *independent* of the magnetic field. It should be emphasized that this result is *exact* and holds for all angles arbitrarily close to $\alpha = \frac{1}{2}\pi$.

It may be helpful to some readers to point out that the argument used in this derivation is tantamount to verifying that the function $y'_{zz}(\omega)$ satisfies the conditions for the Kramers–Kronig relations (see, for instance, Landau & Lifshitz, 1958, § 122).

In the more general case in which there is a mixture of ions, the same argument, modified as indicated by (4·12), yields

$$\sigma_{\text{total}} = N_0 r_e^2 \frac{\sum_n \beta_n Z_n^2 + h^2k^2}{1+\sum_n \beta_n Z_n^2 + h^2k^2}. \tag{6·3}$$

Note that, in view of (4·13), (6·3) indicates that the same total power is scattered from a plasma containing any arbitrary mixture of singly charged ions, even though the frequency spectrum of the scattered signal will be different for different mixtures.

7. General method for evaluating the frequency spectrum

The results of §§ 4 and 5 are summarized by

$$\sigma(\omega_0+\omega)\,\mathrm{d}\omega = N_0 r_e^2 \mathscr{R} \frac{y_e(y_i+\mathrm{i}h^2k^2)}{y_e+y_i+\mathrm{i}h^2k^2}\frac{\mathrm{d}\omega}{\pi\omega}, \tag{7·1}$$

where the suffix zz may now be omitted and the y's are given by (5·5) with the appropriate parameters for ions and electrons. Thus all that remains is to evaluate the integral

$$J(\theta,\alpha,\phi) = \int_0^\infty \exp\left(-\mathrm{i}\theta t - \phi^{-2}\sin^2\alpha \sin^2 \tfrac{1}{2}\phi t - \tfrac{1}{4}t^2\cos^2\alpha\right)\mathrm{d}t. \tag{7·2}$$

This integral had already been thoroughly discussed in the literature on plasma oscillations (for example, Bernstein 1958). There are no simple closed analytic expressions for it, and so one must resort to some sort of approximate or numerical method.

As Bernstein (1958) has pointed out, one can make the following expansion (for any ϕ)

$$\exp\left(-\phi^{-2}\sin^2\alpha\sin^2\tfrac{1}{2}\phi t\right) = \mathrm{e}^{-\eta}\sum_{-\infty}^{\infty} I_n(\eta)\,\mathrm{e}^{\mathrm{i}n\phi t}, \tag{7·3}$$

where $\eta = (\sin^2\alpha)/2\phi^2$ and I_n is the Bessel function of the first kind, of imaginary argument.† Using this expansion, together with the fact that

$$\int_0^\infty \exp\left(-\mathrm{i}\theta t - \tfrac{1}{4}t^2\cos^2\alpha\right)\mathrm{d}t = \frac{1}{\theta}G(\theta/\cos\alpha), \tag{7·4}$$

where

$$G(z) \equiv z\,\mathrm{e}^{-z^2}\left(\pi^{\frac{1}{2}} - 2\mathrm{i}\int_0^z \mathrm{e}^{p^2}\,\mathrm{d}p\right), \tag{7·5}$$

we can integrate (7·2) term by term. The result is

$$J(\theta,\alpha,\phi) = \mathrm{e}^{-\eta}\sum_{-\infty}^{\infty} I_n(\eta)\frac{1}{\theta - n\phi}\,G\left(\frac{\theta - n\phi}{\cos\alpha}\right). \tag{7·6}$$

At this point it is useful to note the following relationships concerning θ, ϕ, and η

$$\frac{\theta_i}{\theta_e} = \frac{\phi_e}{\phi_i} = \left(\frac{\eta_i}{\eta_e}\right)^{\frac{1}{2}} = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}. \tag{7·7}$$

For ionospheric applications we are mainly concerned with the region

$$\theta_i \lesssim 1, \quad \theta_e \ll 1.$$

Representative calculations of ϕ_e and ϕ_i give

$$\left.\begin{aligned}\phi_e &= 2{\cdot}34\lambda_0,\\ \phi_i &= 1{\cdot}36\times 10^{-2}\lambda_0,\end{aligned}\right\} \tag{7·8}$$

where λ_0 is the wavelength in metres of the incident radio wave, the electron gyro-frequency is taken to be 1 Mc/s, the temperature is assumed to be 1500 °K, and the ions to be O^+. Thus ϕ_e may be greater or less than unity, whereas ϕ_i must always be $\ll 1$ for all frequencies above the penetration frequency of the ionosphere. Similarly

$$\eta_i = \sin^2\alpha/2\phi_i^2 \gg 1$$

for all but small α($\eta_i = 48\sin^2\alpha$ for $f_0 = 40$ Mc/s; $4800\sin^2\alpha$ for $f_0 = 400$ Mc/s), but η_e is less than 1, unless f_0 exceeds 1000 Mc/s.

If the Larmor radius is much smaller than the wavelength of interest (i.e. $\eta \ll 1$, $\phi \gg 1$), then a simplification of (7·6) is possible. For large z, $G(z)$ becomes approximately

$$G(z) \approx \pi^{\frac{1}{2}}z\,\mathrm{e}^{-z^2} - \mathrm{i}(1 + \tfrac{1}{2}z^{-2} + \ldots). \tag{7·9}$$

From this and the behaviour of $I_n(\eta)$ for small η, one obtains

$$J(\theta,\alpha,\phi) \approx \frac{1}{\theta}\left(1 - \frac{\sin^2\alpha}{2\phi^2}\right)G\left(\frac{\theta}{\cos\alpha}\right) + O(\phi^{-4}). \tag{7·10}$$

For the case of ionospheric scattering, at frequencies of a few hundred megacycles or less, we see from (7·8) that (7·10) may validly be used for the electrons, but not for the ions.

A similar approximation is sometimes possible for the case of large Larmor radius. When $\cos^2\alpha \gg \phi^2$, $J(\theta,\alpha,\phi)$ is approximately independent of the magnetic field and reduces to

$$J(\theta,\alpha,\phi) \approx (1/\theta)\,G(\theta), \tag{7·11}$$

† Defined by $I_n(z) = \mathrm{i}^{-n}J_n(\mathrm{i}z)$ in the usual notation.

as in I. Since ϕ_i is generally quite small, this result is applicable to the ions for most values of α. We are most interested, of course, in values of α for which (7·11) is not valid, for it is here that we expect the magnetic field effects to be most important.

Thus, in general, one evaluates (7·2) or (7·11) for the ions and (7·2) or (7·10) for the electrons. In this way the spectrum curve is obtained numerically, using (5·5) and (7·1).

Certain special situations, for which the above procedure is either awkward or unnecessary, must be considered separately. For example, at $\theta = 0$ (i.e. $\omega = 0$), the ω in the denominator of (7·1) makes the present form of the result inconvenient, even after the simplification made in § 5. The spectrum function is in fact still finite here, since $\mathscr{R}(y'_{zz}) \to 0$ also, and a special expression for the limiting value of the spectrum function can easily be derived. We shall not give such a derivation here, however, since it is not usually needed in practice.

A somewhat more serious problem occurs as $\alpha \to \frac{1}{2}\pi$, in which case (7·2) converges more and more slowly. Eventually an unreasonably large amount of computer time would be required for the calculation, and the results would become very inaccurate. In the practical cases which we have considered, this difficulty arises in only a very small range of angles. In this range, though, as might be expected, an approximate analytic expression for the scattering spectrum can be obtained. Since the results give some insight into the physics of the scattering, this limiting case of $\alpha \to \frac{1}{2}\pi$ will be considered in the next section.

Analytical work is also of interest for the case $h^2k^2 \gg 1$, and for discussing the effect of the magnetic field on the 'plasma resonance' effect noted in I. The results for these cases are also given in the next section.

8. Evaluation of the spectrum function for certain limiting cases

8·1. *The case* $\alpha \to \frac{1}{2}\pi$

The expansion (7·6) shows that the divergence of (7·2) occurring when $\cos\alpha \to 0$ corresponds to a resonance effect for frequencies which are integral multiples of the ion or electron gyro-frequencies. For all other real frequencies (i.e. $\theta_i \neq n\phi_i, \theta_e \neq n\phi_e$ for any integer n), (7·6) and (7·5) show that the real part of J is exponentially small for small $\cos\alpha$, and since (5·5) can be written

$$y = \mathrm{i} + \theta J, \tag{8·1}$$

it follows that y_i and y_e are almost purely imaginary. Hence σ as given by (7·1) is very small, unless $y_i + y_e + \mathrm{i}h^2k^2$ can be made small. The equation

$$y_i + y_e + \mathrm{i}h^2k^2 = 0 \tag{8·2}$$

is just the dispersion relation for free oscillations of the plasma, and, as mentioned in § 6, its solutions for $\cos\alpha \neq 0$ all lie in the upper half of the complex ω plane. Bernstein (1958) has shown that as $\cos\alpha \to 0$ the phenomenon of Landau damping disappears, so that the solutions of (8·2) descend from the upper half plane to the real axis; let us call them $\omega = \omega_0\ (= 0),\ \pm\omega_1,\ \pm\omega_2, \ldots$ and exclude the case $\omega = 0$ for the present. As these are simple zeros, the fraction appearing in (7·1) has simple oles, and it is easily seen that in the limit the real part becomes a series of delta-unctions located at $\omega_1, \omega_2, \ldots$, with amplitudes proportional to the residues there, i.e.

$$\sigma(\omega_0+\omega) = N_0 r_e^2 \sum_{-\infty}^{\infty} A_m \delta(\omega-\omega_m) + \text{contribution at } \omega = 0, \tag{8·3}$$

here, for $m \neq 0$,

$$A_m = \left.\frac{\mathrm{i}y_e(y_i+\mathrm{i}h^2k^2)}{\omega(\partial/\partial\omega)\,(y_e+y_i+\mathrm{i}h^2k^2)}\right|_{\substack{\omega=\omega_m\\ \alpha=\frac{1}{2}\pi}}$$

$$= \left.\frac{-\mathrm{i}y_e^2}{\theta(\partial/\partial\theta)\,(y_i+y_e)}\right|_{\substack{\omega=\omega_m\\ \alpha=\frac{1}{2}\pi}} \tag{8·4}$$

d the last expression follows from (8·2).

On taking the limit $\alpha \to \frac{1}{2}\pi$ in (7·6) and using (7·9), since $\theta = 0$ is excluded, we find

$$\lim_{\alpha\to\frac{1}{2}\pi} y = i + \theta\,\mathrm{e}^{-\eta} \sum_{-\infty}^{\infty} I_n(\eta)\left[\pi\delta(\theta-n\phi) - \frac{\mathrm{i}}{\theta-n\phi}\right] \tag{8·5}$$

d it may be noted that $\eta \to 1/2\phi^2 = k^2R_L^2$ (cf. 4·4).

As shown by Bernstein, the solutions of (8·2) are not integral multiples of the gyro-equencies, so the delta functions in (8·5) may be omitted in this calculation.

For the ionospheric applications, we need only consider the poles for which $\lesssim 1$, so $\theta_e \ll 1$, and we may also take $\phi_i \ll 1$. Then it is found that, with obvious tation,

$$\theta_{im} = m\phi_i + \epsilon_m, \tag{8·6}$$

here $\epsilon_m \ll \phi_i$, provided $0 < m \ll 1/\phi_i$. For if this is so, (8·5) gives approximately

$$y_i \approx \mathrm{i}\left[1 - \frac{m\phi_i\,\mathrm{e}^{-\eta_i}I_m(\eta_i)}{\epsilon_m}\right] \approx \mathrm{i}\left(1 - \frac{m\phi_i^2}{\pi^{\frac{1}{2}}\epsilon_m}\right) \tag{8·7}$$

d

$$y_e \approx \mathrm{i}[1-\mathrm{e}^{-\eta_e}I_0(\eta_e)], \tag{8·8}$$

here in each case only the dominant term of the summation has been retained, and e result $\mathrm{e}^{-z}I_m(z) \approx (2\pi z)^{-\frac{1}{2}}$ has been applied for the Bessel function of large gument.

Solving (8·2) for ϵ_m yields

$$\epsilon_m = \frac{m\phi_i^2}{\pi^{\frac{1}{2}}}\,\frac{1}{2+h^2k^2-\mathrm{e}^{-\eta_e}I_0(\eta_e)} \tag{8·9}$$

d the assumption that $\epsilon_m \ll \phi_i$ is verified since $m\phi_i \ll 1$. Making similar approxi-ations for $\partial y/\partial\theta$ and using the result (8·9) in (8·4), we obtain finally

$$A_m \approx \frac{\phi_i}{\pi^{\frac{1}{2}}}\left[1 + \frac{1+h^2k^2}{1-\mathrm{e}^{-\eta_e}I_0(\eta_e)}\right]^{-2}. \tag{8·10}$$

To ensure that the poles are sufficiently close to the real axis for the present treat-ment to be relevant (i.e. that the real parts of both y_i and y_e are small at the resonant oints), $\cos^2\alpha$ must be much smaller than both ϵ_m^2 and the resonant value of θ_e^2 [$\approx (m_e/m_i)\, m^2\phi_i^2$].

It may be noted that none of these solutions of the dispersion relation (8·2) corre-ponds to Alfvén waves. Such solutions were eliminated when we made the longitu-dinal approximation. However, this is not likely to be a serious defect since Alfvén waves do not exist on a scale smaller than the 'plasma wavelength' ($= 2\pi c/\omega_p$), and the present problem is necessarily concerned with scales much smaller than that.

A finite scattering cross-section is also concentrated at $\omega = 0$ in the limit $\alpha \to \frac{1}{2}\pi$ so that a term $m = 0$ should be included in (8·3); the coefficient A_0 is not given by (8·4) but can be obtained as follows. We must calculate

$$\int_{\theta_i=-\delta}^{\delta} \sigma(\omega_0+\omega)\,\mathrm{d}\theta_i, \tag{8·11}$$

where $\cos\alpha \ll \delta \ll \phi_i$. We shall evaluate this in essentially the same way that the total cross-section was evaluated in § 6. We use a contour in the complex θ plane which goes along the real axis from $-\delta$ to $+\delta$ (except for an indentation downwards at the origin) and is closed by a semicircle below the axis. As in § 6, there are no poles inside this contour. The integral along the semicircle is not zero in the present case, but it can be calculated easily since, on the semicircle, y_i and y_e are given by

$$y = \mathrm{i}[1-\mathrm{e}^{-\eta} I_0(\eta)]$$

and so are independent of θ. The residue at the origin is the same as in (6·2), and the result is

$$A_0 = \frac{1+h^2k^2}{2+h^2k^2} - \frac{[1-\mathrm{e}^{-\eta_e} I_0(\eta_e)]\,[1+h^2k^2-\mathrm{e}^{-\eta_i} I_0(\eta_i)]}{2+h^2k^2-\mathrm{e}^{-\eta_e} I_0(\eta_e)-\mathrm{e}^{-\eta_i} I_0(\eta_i)}. \tag{8·12}$$

If, as in the above work, we take $\eta_i \gg 1$, $\mathrm{e}^{-\eta_i} I_0(\eta_i)$ may be neglected, and (8·12) simplifies to

$$A_0 = \frac{(1+h^2k^2)^2}{2+h^2k^2}\,\frac{\mathrm{e}^{-\eta_e} I_0(\eta_e)}{2+h^2k^2-\mathrm{e}^{-\eta_e} I_0(\eta_e)}. \tag{8·13}$$

When η_e is small, which is usually the case in the ionosphere ($\eta_e \approx 1$ for operating frequency of about 1000 Mc/s), $\mathrm{e}^{-\eta_e} I_0(\eta_e)$ reduces to $1-\eta_e$, so that

$$A_0 \approx (1+h^2k^2)/(2+h^2k^2).$$

Thus for frequencies well below 1000 Mc/s, practically all the scattered power has no Doppler shift if $\cos\alpha$ is sufficiently small. This is confirmed by (8·10), which shows that in these circumstances A_m is very small for $m \neq 0$.

Conclusions similar to these have also been arrived at by Fejer (1961). He has also examined the less common case of large η_e in considerably more detail than given here, and has shown that the complete envelope of the delta functions is, in this instance, identical to the spectrum obtained when the magnetic field is neglected.

It should be realized that some of the approximations used in this section require α to be within 10^{-3} to 10^{-4} radians of $\frac{1}{2}\pi$. Since this could never be achieved in practice for any significant fraction of the radio beam, these results are not of much practical importance. They are of some theoretical interest, though, because in the limit we have used, $\cos\alpha$ is so small that, for both the ions and the electrons, thermal motion along the lines of force plays no part in determining the thermal density fluctuations. The electrons are now so tightly bound to the lines of force that the electrostatic control by the ions is no longer the main effect. Instead, the strong electric interaction between ions and electrons has the result that the density fluctuations of both gases stay almost constant in time for scales much larger than he electron Larmor radius. This is why all the scattered power is concentrated on he central spike, corresponding to no Doppler shift. For small deviations from rthogonality, of the order of $(m_e/m_i)^{\frac{1}{2}}$ radians or more, electrostatic control by the ons reappears because the electrons are free to move very rapidly along the lines f force; this latter situation is illustrated by the computer calculations presented n §9.

8·2. *The case $h^2k^2 \gg 1$.*

In the above work it was tacitly assumed that h^2k^2 was not very large, and indeed or the ionospheric applications h^2k^2 is usually very small. It is of some interest, owever, to consider the result of letting $h^2k^2 \to \infty$ while keeping α arbitrary. Then 21) reduces to

$$\sigma(\omega_0+\omega)\,\mathrm{d}\omega = N_0 r_e^2 \mathscr{R}(y_e)\frac{\mathrm{d}\omega}{\pi\omega}$$

$$= \frac{N_0 r_e^2}{\pi^{\frac{1}{2}}\cos\alpha}\sum_{-\infty}^{\infty} \mathrm{e}^{-\eta_e} I_m(\eta_e)\exp\left[-\frac{(\theta_e - m\phi_e)^2}{\cos^2\alpha}\right]\mathrm{d}\theta_e. \qquad (8·14)$$

he spectrum now consists of a set of Gaussian functions centred on multiples of the *ectron* gyro-frequency. This result has also been obtained by Laaspere (1960), who nsidered scattering from individual freely spiralling electrons. If we now let s $\alpha \to 0$, (8·14) becomes

$$\sigma(\omega_0+\omega)\,\mathrm{d}\omega = N_0 r_e^2 \sum_{-\infty}^{\infty} \mathrm{e}^{-\eta_e} I_m(\eta_e)\,\delta(\theta_e - m\phi_e)\,\mathrm{d}\theta_e \qquad (8·15)$$

nd we again have a set of delta functions, but this time located at $\theta_e = \pm m\phi_e$. heir envelope generally resembles the Gaussian shape of the spectrum computed r no magnetic field with $h^2k^2 \gg 1$, since for $\eta_e \gg 1$ (virtually required by the ndition $h^2k^2 \gg 1$) we have the asymptotic formula

$$\mathrm{e}^{-\eta_e} I_m(\eta_e) \approx (2\pi\eta_e)^{-\frac{1}{2}}\exp(-m^2/2\eta_e) = \phi_e \pi^{-\frac{1}{2}}\mathrm{e}^{-m^2\phi_e^2} \qquad (8·16)$$

alid for $|m| \ll \eta_e$.

8·3. *The plasma resonance effect*

We have dealt here only with the low-frequency part of the scattering spectrum, hich is usually all that is important. However, it was noted in I that, provided $k^2 \ll 1$, there will be a sharp spike in the spectrum at a Doppler shift of approximately the plasma frequency, ω_p. Provided that the electron gyro-frequency Ω_e is uch smaller than the plasma frequency, there is a similar resonance in the presence of a magnetic field, at a frequency of approximately

$$\omega \approx \omega_p(1+\tfrac{1}{2}\Omega_e^2\omega_p^{-2}\sin^2\alpha). \qquad (8·17)$$

Although we shall not investigate the shape of the resonant peak here, one would expect, by analogy with the results and arguments of I, that if $h^2k^2 \ll 1$, the peak is very sharp, but corresponds to a fraction of only about h^2k^2 of the total power scattered. So our previous conclusion, that the effect is usually unimportant experimentally, is unaltered. This aspect of the problem is discussed in more detail by Salpeter (1961*b*)

9. Numerical results

For various values of α, numerical calculations of the spectrum function were made using the values of temperature, etc., quoted in connexion with equation (7·8). The operating frequency was taken to be 40 Mc/s, and thus ϕ_i was 0·102.

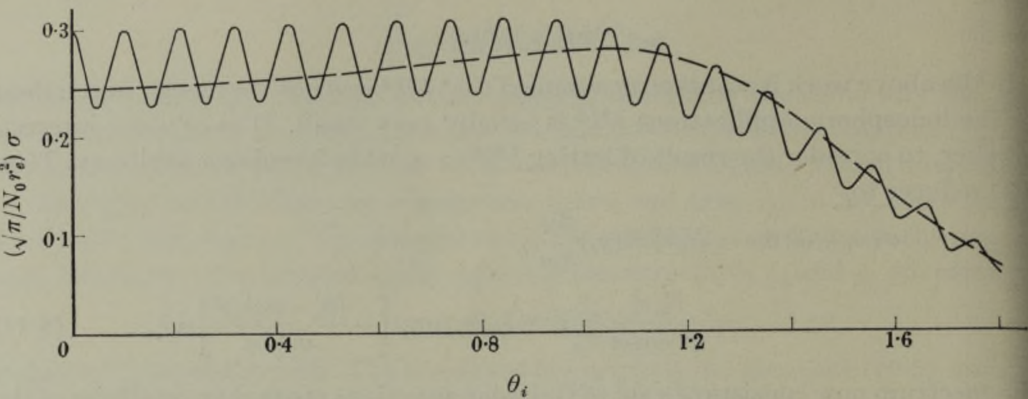

Figure 1. The frequency spectrum for incoherent back-scattering at 40 Mc/s, $\alpha = 87°$, other parameters typical for the F region of the ionosphere.

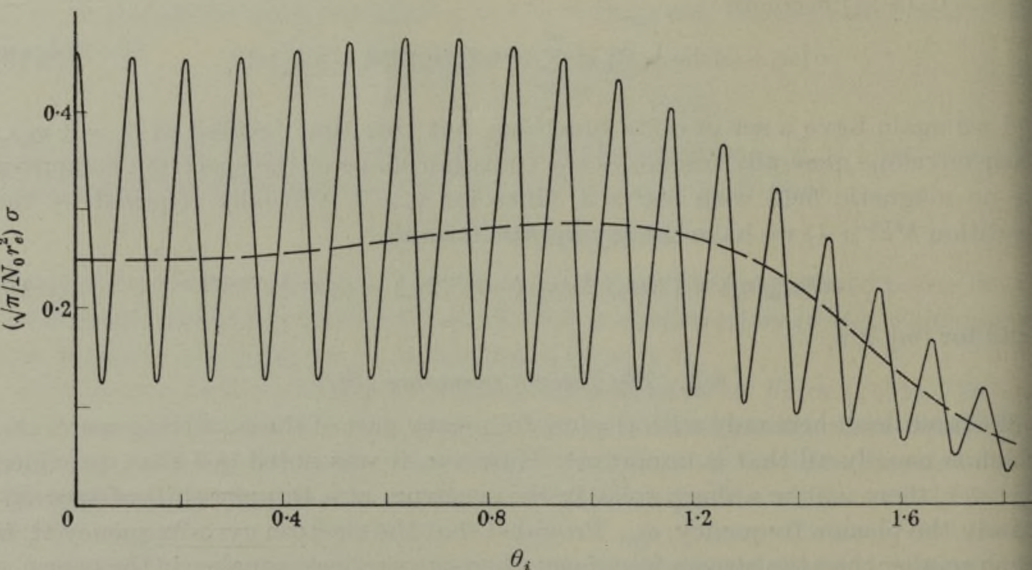

Figure 2. As for figure 1, but with $\alpha = 88°$.

The results of these calculations, for values of α up to 85°, are almost identical to those for $\alpha = 0$ (shown by the dashed curve in figures 1 and 2). For $\alpha = 86°$, very slight undulations appear in the spectrum, but not until $\alpha = 87°$ (the solid curve of figure 1) do these become really noticeable. When $\alpha = 88°$ (the solid curve of figure 2), the peaks can be seen to be well developed. Curves for angles still closer to 90° have not been calculated because of the excessive computer time required. It is apparent, however, that a further increase of α will further sharpen the spikes. Eventually, When $\frac{1}{2}\pi - \alpha$ is of the order of or less than $(m_e/m_i)^{\frac{1}{2}}$ the magnetic field will begin to prevent the electrons from freely following the ions, and the whole spectrum will begin to narrow. Finally, the situation described in § 8·1 will be reached.

The rather limited influence of the magnetic field indicated by these results is roughly what one would have expected from the discussion given in the latter part of § 7. There it was pointed out that y_i is independent of the magnetic field, and thus no gyro-resonance effects are possible, as long as $\cos^2\alpha \gg \phi_i^2$.

This same conclusion can also be reached from a very simple physical argument. Usually one can imagine the scattering to be from irregularities whose motion is controlled primarily by the ions. The component of this motion in the direction of will consist of a periodic part, due to the rotation of the ions around the magnetic lines of force, and a random part, due to the motion of the ions along the lines. The scattered signal will have corresponding periodic and random phase shifts. The periodic component can be measured, to an accuracy of say b radians/s, only if the total phase shift remains quasi-periodic for a time $\tau = 1/b$; that is, if the root mean square random phase shift occurring in a time τ is less than about one radian. This random phase shift will be of the order of

$$\frac{[KT/m_i]^{\frac{1}{2}}\,\tau\cos\alpha}{\lambda_0/4\pi}$$

radians. If this quantity is greater than unity, the periodic effects will be obscured by the random ones, and no gyro-resonance phenomena will be observed. Using (4) and setting $1/\tau$ equal to the ion gyro-frequency, we again obtain the inequality $\cos^2\alpha \gg \phi_i^2$ as a condition for no resonance effects.

What the numerical results of figures 1 and 2 show is that orthogonality must be approached rather more closely than might be expected from the above rough argument. It seems that, for small Debye length, $\cos\alpha$ must be equal to or less than about $\frac{1}{2}\phi_i$, i.e. that $\frac{1}{2}\pi - \alpha$ must be less than about $\frac{1}{2}\phi_i$ radians. In the case we have calculated, this is about 3°. This is, however, a less stringent condition than that required for the power to become concentrated on the central peak, so there does exist a usable range in which the whole series of peaks would be observable, in the way predicted by Bowles (1959).

Since ϕ_i is proportional to $B\lambda_0/(m_i T)^{\frac{1}{2}}$, one can easily deduce the range of angles over which magnetic effects can be observed when the various parameters have values different from those used in the calculation here. For instance, at 400 Mc/s, with the other parameters unchanged, the radio beam would have to be within about 0·3° of orthogonality to the magnetic field lines.

The orthogonality requirement may be a little less stringent at heights where H^+ is the main ion present, because of the $m_i^{-\frac{1}{2}}$ dependence of ϕ_i. However, the decrease of B and possible increase of T with height will tend to prevent any large increase in ϕ_i.

From the two figures shown, we can see that when the magnetic field does alter the spectrum, the resonances do, as expected, appear at Doppler shifts which are approximate (but not exact) multiples of the ion gyro-frequency. For example, the peaks of the first few spikes in figure 2 occur approximately at $\theta_i = 0, 0{\cdot}108, 0{\cdot}215,$

0·323, 0·430, etc., whereas the exact multiples are at $\theta_{i,n} = 0{\cdot}102n$. The actual position of the peaks agrees well with the work in § 8·1, which shows (see (8·6) and (8·9)) that, in the limit of very small cos α, the peaks should occur at approximately

$$\theta_{i,m} = m\phi_i[1+\phi_i/\pi^{\frac{1}{2}}] = 0{\cdot}108\,m. \tag{9·1}$$

10. Conclusions

The main conclusions that we have arrived at concerning the effect of the earth's magnetic field on incoherent scattering from the ionosphere are the following:

(1) The magnetic field does not affect the *total* scattered power in any way. Thus the measurement of electron density by incoherent scattering is not affected by the magnetic field.

(2) When the radio wavelength is large compared to the Debye length, the magnetic field does not affect the *spectrum* of the scattered signal either, unless most of the radio beam is confined to a cone which is within about $\frac{1}{2}\phi_i$ (see (4·4)) radians of orthogonality to the magnetic lines of force, i.e. within about 3° for an operating frequency of 40 Mc/s and for typical values of temperature and magnetic field strength appropriate to the ionosphere.

(3) If the beam *is* within the cone specified above, the spectrum exhibits resonances at Doppler shifts which are approximate (but not exact) multiples of the ion gyro-frequency. The amplitudes of the first several spikes are about equal. Even in this case, as long as the beam is not well within $(m_e/m_i)^{\frac{1}{2}}$ radians (about 0·3° for O^+ ions) of orthogonality, the overall width of the spectrum is not changed appreciably by the magnetic field. Thus this width is still a measure of the plasma temperature. Further, for operating frequency greater than about 1000 Mc/s, the final reduction in the width does not occur.

(4) It should be possible, although difficult, to use an incoherent scatter sounder having a sufficiently large antenna as a mass spectrometer in the ionosphere.

(5) The plasma resonance effect is virtually unaltered by the magnetic field, the only effect being a slight change in the resonant frequency.

(6) If the Debye length is large compared to the radio wavelength, spikes in the spectrum will be observed at Doppler shifts which are exact multiples of the *electron* gyro-frequency, provided that the beam is within roughly ϕ_e radians of orthogonality. Otherwise the magnetic field has no effect. This case has been discussed fully by Laaspere (1960).

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Appendix. Calculation of the admittance tensor for a plasma in a uniform magnetic field

The calculation which follows is adapted from work by Dr O. Buneman (unublished lecture notes, Cambridge University, February 1960).

For the purposes of this derivation, we let (x, y, z) be a co-ordinate system in which he magnetic field is $\mathbf{B} = (0, 0, B)$, and define a complex system by

$$x_{\pm 1} = (x \pm \mathrm{i}y)/2^{\frac{1}{2}}, \quad x_0 = z.$$

should be noted that if $\mathbf{A}$ is a vector which is real in ordinary co-ordinates, and $\mathbf{B}$ is ther a similar real vector or a complex one representing harmonically varying uantities in the usual way, then the scalar product $\mathbf{A}.\mathbf{B}$ is given in the $(1, 0, -1)$ stem by $\mathbf{A}^*.\mathbf{B} = A_1 B_{-1} + A_0 B_0 + A_{-1} B_1 = \sum_\gamma A_{-\gamma} B_\gamma$. We shall always use Greek uffixes for vector and tensor components in the complex system, but will *not* use he summation convention.

As is well known, Boltzmann's equation for a collisionless gas, even when there re magnetic forces, may be written

$$\mathrm{d}f/\mathrm{d}t = 0 \tag{A 1}$$

rovided that the total derivative is interpreted as the rate of change following the ctual orbits of the particles in phase space. We now write the distribution function $(\mathbf{r}, \mathbf{v}, t)$ as

$$f = f_0 + f_1,$$

here f_0 depends solely on the energy $W = \frac{1}{2}mv^2$, and so represents a uniform sotropic distribution, while f_1 represents perturbations. If we linearize the theory in he usual way, (A 1) becomes

$$\left(\frac{\mathrm{d}f_0}{\mathrm{d}t}\right)_{\text{exact orbit}} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}t}\right)_{\text{unperturbed orbit}} = 0. \tag{A 2}$$

he second term need be calculated only along the unperturbed orbit, since this iffers from the exact one by quantities of the first order, while f_1 is itself of the first rder.

The forces acting on the particle at $(\mathbf{r}, \mathbf{v}, t)$ in phase space are the Lorentz force $c^{-1}\mathbf{v} \times \mathbf{B}$ and the 'applied' force $\mathbf{F}(\mathbf{r}, t)$; but since the former does no work, we have

$$\left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)_{\text{exact orbit}} = \mathbf{F}.\mathbf{v}. \tag{A 3}$$

Combining (A 2) and (A 3), we have

$$\left(\frac{\mathrm{d}f_1}{\mathrm{d}t}\right)_{\text{unperturbed orbit}} = -\mathbf{F}(\mathbf{r}, t).\mathbf{v}\frac{\mathrm{d}f_0}{\mathrm{d}W}. \tag{A 4}$$

The interpretation of this equation is that if we are given f_1 throughout phase space at the initial instant t_0, then to find $f_1(\mathbf{r}, \mathbf{v}, t)$ at a later instant we first construct the unperturbed orbit arriving at $(\mathbf{r}, \mathbf{v})$ at time t. We then integrate the right-hand side of (A 4) along this orbit, from time t_0 to t. It is easily shown that the unperturbed orbit is given by

$$x_\gamma(t-t') = x_\gamma(t) - v_\gamma(t)\,\tau_\gamma(t') \tag{A 5}$$

and

$$v_\gamma(t-t') = v_\gamma(t)\,\mathrm{e}^{\mathrm{i}\gamma\Omega t'}, \tag{A 6}$$

where

$$\tau_\gamma(t) = \frac{\mathrm{e}^{\mathrm{i}\gamma\Omega t}-1}{\mathrm{i}\gamma\Omega} \quad (\gamma = 1, 0, -1) \tag{A 7}$$

and Ω is the gyro-frequency eB/mc. In equations such as (A 7), which are apparently indeterminate for $\gamma = 0$, the correct result is always obtained by adopting the convention that γ is to be allowed to tend continuously to 0. As in I, we are to take $\mathbf{F}(\mathbf{r}, t)$ to be of the form

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}\,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}.\mathbf{r})}, \tag{A 8}$$

where $\mathbf{F}$ is constant, $\mathbf{k}$ is a real vector, and ω will eventually be real but must first be supposed to have a very small negative imaginary part. In (A 4), $\mathrm{d}f_0/\mathrm{d}W$ may be evaluated at the (constant) unperturbed value of W, as we are only working to the first order. Integration of the equation then gives

$$f_1(\mathbf{r}, \mathbf{v}, t) = -\frac{\mathrm{d}f_0}{\mathrm{d}W}\int_\infty^0 \sum_\nu F_\nu \exp\left[\mathrm{i}\omega(t-t') - \mathrm{i}\mathbf{k}.\mathbf{r} + \sum_\gamma k_{-\gamma} v_\gamma \tau_\gamma(t')\right] v_{-\nu} \exp\left[-\mathrm{i}\nu\Omega t'\right] \mathrm{d}(-t')$$
$$= f_1 \exp\left[\mathrm{i}(\omega t - \mathbf{k}.\mathbf{r})\right],$$

say, where

$$f_1 = -\frac{\mathrm{d}f_0}{\mathrm{d}W}\sum_\nu F_\nu v_{-\nu}\int_0^\infty \exp\left[-\mathrm{i}\omega t' + \mathrm{i}\sum_\gamma k_{-\gamma} v_\gamma \tau_\gamma(t') - \mathrm{i}\nu\Omega t'\right] \mathrm{d}t'.$$

The mean velocity, $\mathbf{u}$, of the particles [with factor $\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}.\mathbf{r})}$ assumed] is given by

$$N_0 u_\mu = \int v_\mu f_1\,\mathrm{d}^3v = \sum_\nu Y_{\mu\nu} F_\nu,$$

where the 'admittance tensor' is

$$Y_{\mu\nu} = -\int_{\mathbf{v}} \frac{\mathrm{d}f_0}{\mathrm{d}W} v_\mu v_{-\nu} \int_{t=0}^\infty \exp\left[-\mathrm{i}(\omega + \nu\Omega)\,t + \mathrm{i}\sum_\gamma k_{-\gamma} v_\gamma \tau_\gamma(t)\right] \mathrm{d}t\,\mathrm{d}^3v. \tag{A 9}$$

In equation (A 9), d^3v is to be interpreted in ordinary Cartesian co-ordinates. To evaluate the integrals over velocity space, it is convenient to define a vector with components

$$p_\gamma(t) = k_\gamma \tau_{-\gamma}(t) \quad (\textit{no}\ \text{summation}). \tag{A 10}$$

This is a real vector in ordinary components, since $\mathbf{k}$ is real and $\tau_{\pm\gamma}$ are complex conjugates. Then we have to evaluate

$$\int \frac{\mathrm{d}f_0}{\mathrm{d}W} v_\mu v_{-\nu}\,\mathrm{e}^{\mathrm{i}\mathbf{p}.\mathbf{v}}\,\mathrm{d}^3v,$$

i.e.

$$-\frac{\partial^2}{\partial p_{-\mu}\,\partial p_\nu}\int \frac{\mathrm{d}f_0}{\mathrm{d}W}\,\mathrm{e}^{\mathrm{i}\mathbf{p}.\mathbf{v}}\,\mathrm{d}^3v.$$

For the Maxwellian distribution,

$$f_0(\mathbf{v}) = N_0\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \mathrm{e}^{-m\mathbf{v}^2/2KT} \tag{A 11}$$

and the latter integral is readily evaluated in ordinary co-ordinates, the whole expression becoming

$$\frac{\partial^2}{\partial p_{-\mu}\,\partial p_\nu}\left(\frac{N_0}{KT}\,\mathrm{e}^{-\mathbf{p}^2KT/2m}\right).$$

On performing the differentiations and incorporating the result in (A 9), we have

$$Y_{\mu\nu} = \frac{N_0}{m}\int_0^\infty \exp\left[-\mathrm{i}(\omega+\nu\Omega)t - \mathbf{p}^2KT/2m\right]\left(\delta_{\mu\nu} - \frac{KT}{m}p_\mu p_{-\nu}\right)\mathrm{d}t. \qquad \text{(A 12)}$$

Finally, we have to use (A 7) and (A 10) to write $\mathbf{p}$ explicitly. At the same time we shall introduce the normalized frequencies θ and ϕ defined in (4·3) and (4·4) and suitably normalize the variable of integration t (although we use the same symbol). The result is

$$Y_{\mu\nu} = (N_0\omega/KTk^2)\,y_{\mu\nu}, \qquad \text{(A 13)}$$

where

$$y_{\mu\nu} = \frac{1}{2\theta}\int_0^\infty \mathrm{e}^{-\mathrm{i}\theta t+g(t)}\,F_{\mu\nu}\,\mathrm{d}t \qquad \text{(A 14)}$$

$$g(t) = -\phi^{-2}\sin^2\alpha\sin^2\tfrac{1}{2}\phi t - \tfrac{1}{4}t^2\cos^2\alpha \qquad \text{(A 15)}$$

and

$$F_{\mu\nu} = \exp\left[-\tfrac{1}{2}\mathrm{i}(\mu+\nu)\,\phi t\right]\left(\delta_{\mu\nu} - \frac{2k_\mu k_{-\nu}\sin\tfrac{1}{2}\mu\phi t\sin\tfrac{1}{2}\nu\phi t}{k^2\phi^2\mu\nu}\right). \qquad \text{(A 16)}$$

Here α is the angle between $\mathbf{k}$ and $\mathbf{B}$. We can, without loss of generality, choose the original co-ordinate axes so that $k_x = k\sin\alpha$, $k_y = 0$, $k_z = k\cos\alpha$, in which case $k_1 = k_{-1} = k\sin\alpha/2^{\frac{1}{2}}$, $k_0 = k\cos\alpha$. Then, after taking the necessary limits in cases where one or both of μ, ν is zero, the explicit expressions for $F_{\mu\nu}$ are

$$\left.\begin{aligned}
F_{0,0} &= 1 - \tfrac{1}{2}t^2\cos^2\alpha,\\
F_{\pm1,\pm1} &= \mathrm{e}^{\mp\mathrm{i}\phi t}(1-\phi^{-2}\sin^2\alpha\sin^2\tfrac{1}{2}\phi t),\\
F_{\pm1,\mp1} &= -\phi^{-2}\sin^2\alpha\sin^2\tfrac{1}{2}\phi t,\\
F_{\pm1,0} = F_{0,\pm1} &= -\mathrm{e}^{\mp\frac{1}{2}\mathrm{i}\phi t}\,2^{-\frac{1}{2}}\phi^{-1}(\cos\alpha\sin\alpha)\,t\sin\tfrac{1}{2}\phi t.
\end{aligned}\right\} \qquad \text{(A 17)}$$

Note that with this choice of axes, $y_{\mu\nu}$ is symmetric. For particles with negative charge, the sign of ϕ should be reversed if, as is conventional, the gyro-frequency is defined to be positive.

Some properties of the admittance tensor

At a first glance it appears that as $\theta \to 0$ the components of $\mathbf{y}$, as given by (A 14), become infinite. In § 5, it is mentioned that, in ordinary Cartesian co-ordinates in which $\mathbf{k}$ is now taken to be along the z axis, the components of $\mathbf{y}$ in both the row and the column corresponding to the z axis are in fact finite as $\theta \to 0$. Further, a simplified expression for y_{zz} is quoted.

To prove these results, we recall that if $\mathbf{a}$, $\mathbf{b}$ are two unit vectors orthogonal to one another, then the tensor component corresponding to the pair of directions $\mathbf{a}$, $\mathbf{b}$ is $\mathbf{a}^*.\mathbf{y}.\mathbf{b}$ evaluated in the $(1, 0, -1)$ co-ordinates. Thus we need to consider the expressions

$$\sum_\nu y_{\mu\nu}k_\nu \quad \text{and} \quad \sum_\mu\sum_\nu y_{\mu\nu}k_\mu^* k_\nu.$$

Using the appropriate expressions from (A 17), it is readily verified that

$$\sum_\nu F_{0\nu} k_\nu = k \cos\alpha\, \mathrm{e}^{-g}\, \mathrm{d}(t\,\mathrm{e}^{g})/\mathrm{d}t$$

and

$$\sum_\nu F_{\pm 1,\nu} k_\nu = \frac{k \sin\alpha}{2^{\frac{1}{2}}} \mathrm{e}^{-g} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1-\mathrm{e}^{\mp \mathrm{i}\phi t}}{\pm \mathrm{i}\phi} \mathrm{e}^{g}\right).$$

Inserting these into (A 14), an integration by parts becomes possible in each case, the integrated parts vanishing provided $\cos\alpha \neq 0$, yielding

$$\sum_\nu y_{0\nu} k_\nu = \tfrac{1}{2}\mathrm{i}k\cos\alpha \int_0^\infty t\,\mathrm{e}^{-\mathrm{i}\theta t + g(t)}\,\mathrm{d}t \qquad \text{(A 18)}$$

and

$$\sum_\nu y_{\pm 1,\nu} k_\nu = \pm \frac{k \sin\alpha}{2^{\frac{3}{2}}\phi} \int_0^\infty (1-\mathrm{e}^{\mp \mathrm{i}\phi t})\,\mathrm{e}^{-\mathrm{i}\theta t + g(t)}\,\mathrm{d}t. \qquad \text{(A 19)}$$

These are all finite as $\theta \to 0$, and the first of our results follows.

Combining (A 18) and (A 19) by contracting again with $\mathbf{k}$, we obtain

$$y_{zz} = k^{-2} \sum_\mu \sum_\nu y_{\mu\nu} k_\mu^* k_\nu = -\mathrm{i} \int_0^\infty \frac{\mathrm{d}g}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}\theta t + g(t)}\,\mathrm{d}t,$$

which by a further integration by parts, using $g(0) = 0$, becomes

$$y_{zz} = \mathrm{i} + \theta \int_0^\infty \mathrm{e}^{-\mathrm{i}\theta t + g(t)}\,\mathrm{d}t, \qquad \text{(A 20)}$$

the expression quoted in § 5.

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Erratum to paper I. On p. 98 of I, equation (C 2), the denominator of the right-hand side should be KTk^2, not NTk^2.