Introduction to Optical Thomson Scattering Diagnostics

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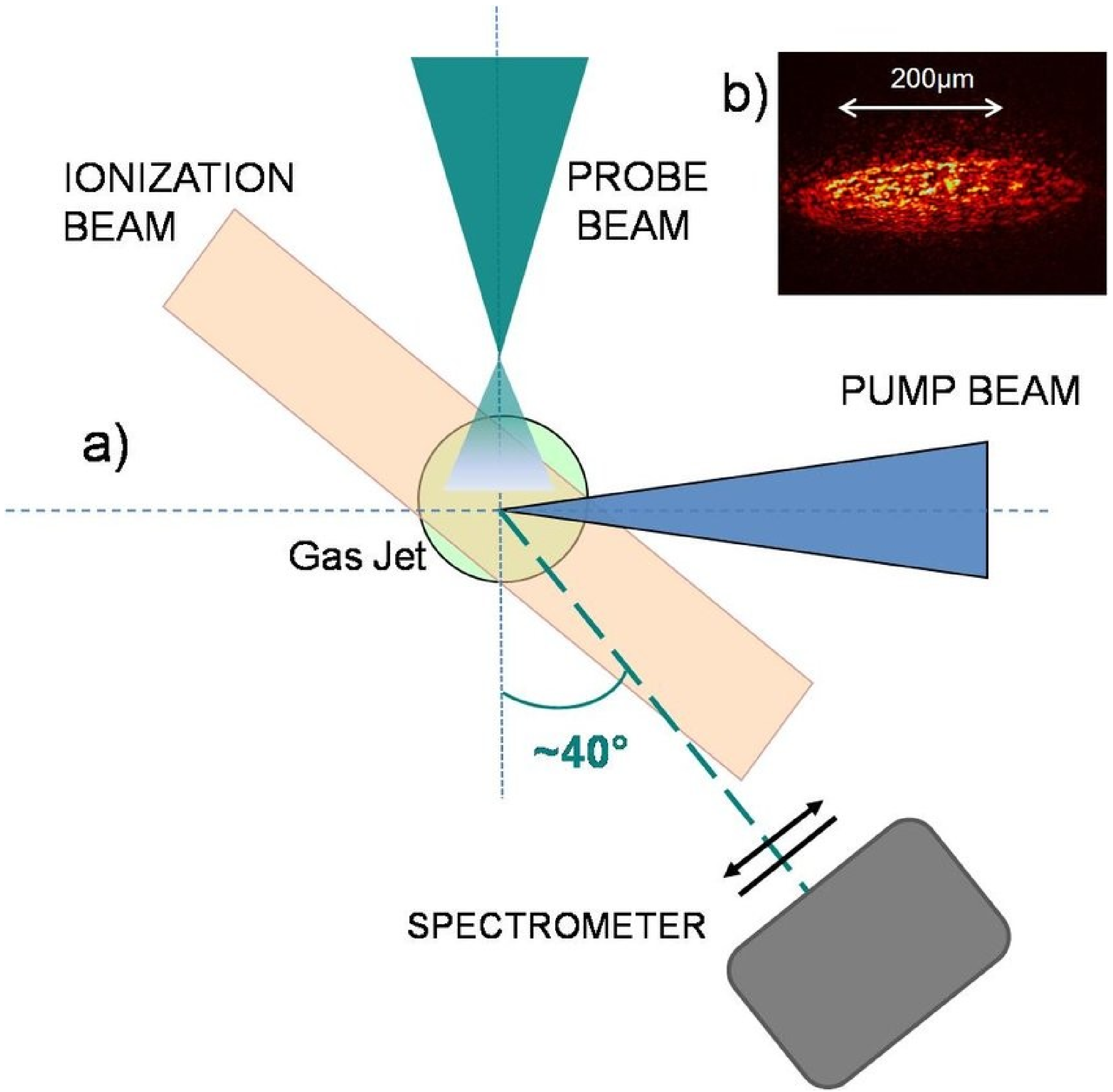
Setup and ideal spectrum



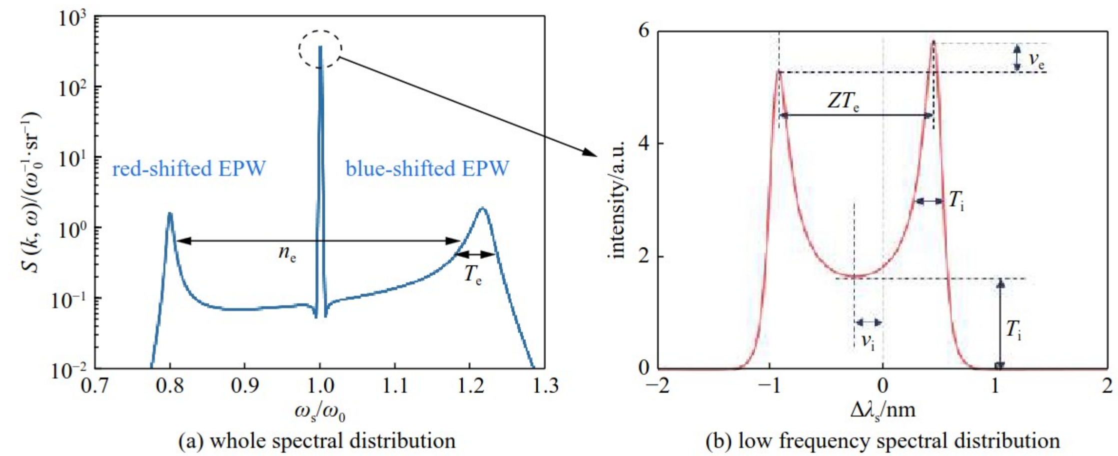
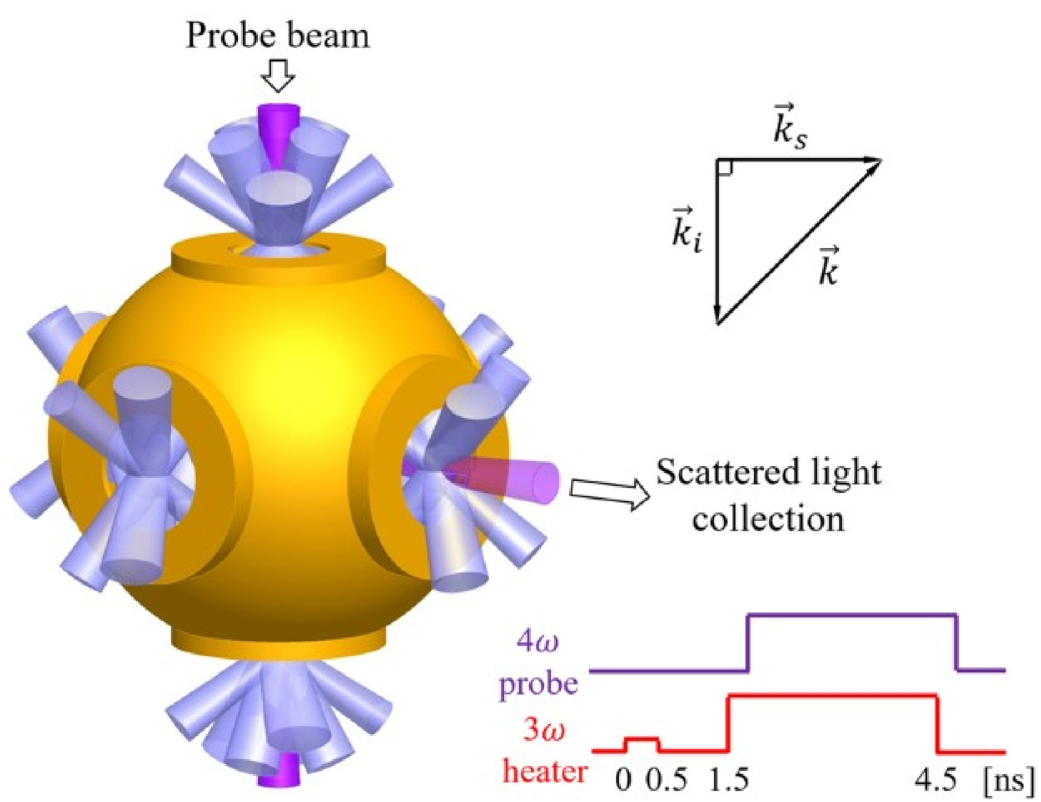
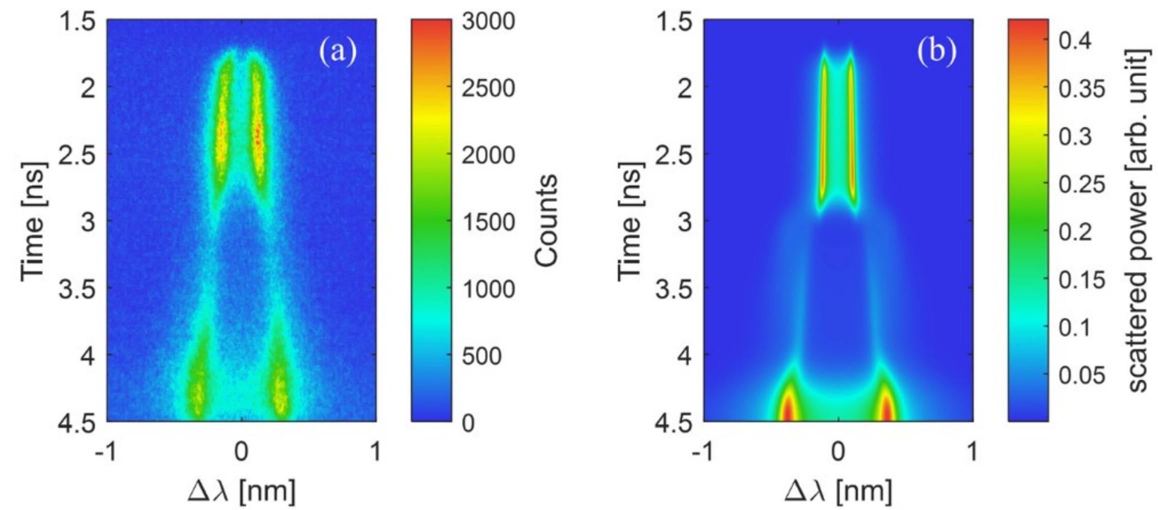


Fig. 1 Optical Thomson scattering (OTS) spectral distribution is closely related with plasma parameters

图 1 光学汤姆逊散射的光谱形貌与等离子体状态参数密切相关

OTS experiments on Shenguang-III





Method of calculating OTS spectrum

OTS spectrum of collisionless plasmas:

$$P_s d\Omega d\omega = \frac{1}{2\pi} I_0 r_e^2 V n_e \left(1 + \frac{2\omega}{\omega_i}\right) sin(\varphi)^2 S\left(\bar{k}, \omega\right) d\Omega d\omega$$

$$S\left(\bar{k}, \omega\right) = \frac{2\pi}{k} \left|1 - \frac{\chi_e}{\epsilon}\right|^2 f_{e0}\left(\frac{\omega}{k}\right) + \frac{2\pi}{k} \left|\frac{\chi_e}{\epsilon}\right|^2 \sum_j \frac{Z_j^2 N_j}{\sum_i Z_i^2 N_i} f_{i0,j}\left(\frac{\omega}{k}\right)$$

Electric susceptibility:

$$\chi_e(k, \omega) = \alpha^2 w\left(\omega \Big/ \left(\sqrt{2} k v_{te}\right)\right)$$

$$\chi_j(k, \omega) = \alpha^2 \frac{Z_j^2 N_j}{\sum_j Z_j N_j} \frac{T_e}{T_j} w\left(\omega \Big/ \left(\sqrt{2} k v_{tj}\right)\right)$$

Maxwell distributions:

$$f_{e0}(v) = \frac{1}{\sqrt{2} v_{te}} exp\left(-\frac{v^2}{2 v_{te}^2}\right)$$

$$f_{i0,j}(v) = \frac{1}{\sqrt{2} v_{t,j}} exp\left(-\frac{v^2}{2 v_{t,j}^2}\right)$$

$$w(x) = 1 - 2x e^{-x^2} \int_0^x dp\, e^{p^2} + i\sqrt{\pi} x e^{-x}$$

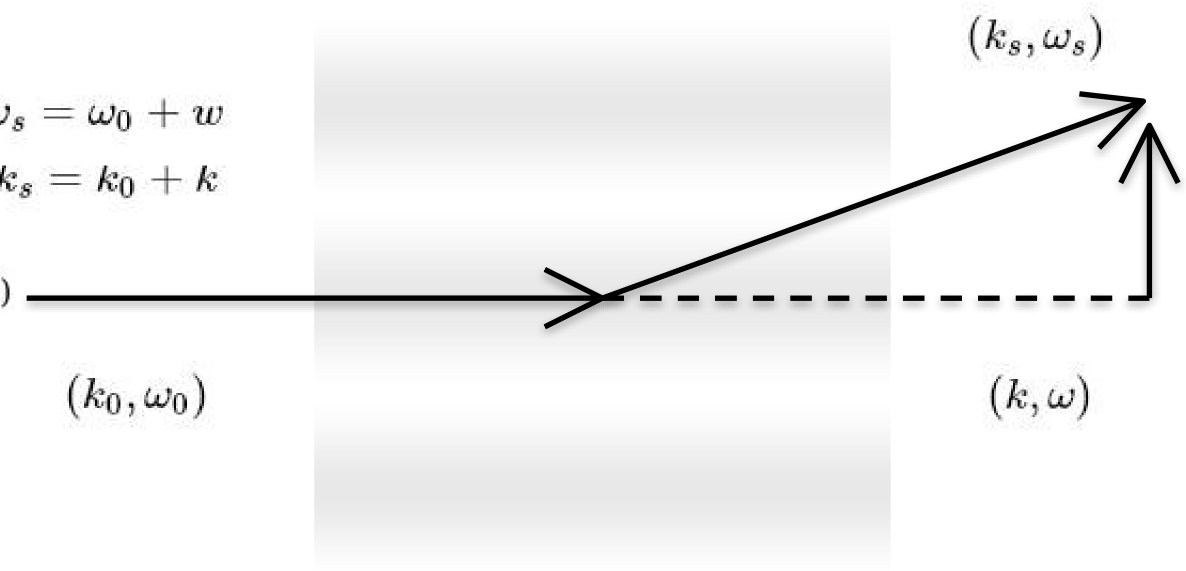
OTS theory: spectrum

$$\frac{d\vec{\beta}}{dt} = -\frac{e\vec{E}_{i0}}{m_e c}\cos\left(\vec{k}_0 \cdot \vec{r} - \omega_0 t\right)$$

$$\vec{E}_t\left(\vec{R}, \omega\right) = \int_{-\infty}^{\infty} dt \left(\int dr \int dv \left(-F_e\left(\vec{r}, \vec{v}, t\right)\frac{e}{cR}\left(\hat{s} \times \left(\hat{s} \times \frac{d\vec{\beta}}{dt}\right)\right)\right)\right) exp(-i\omega t)$$

$$\omega_s = \omega_0 + w$$

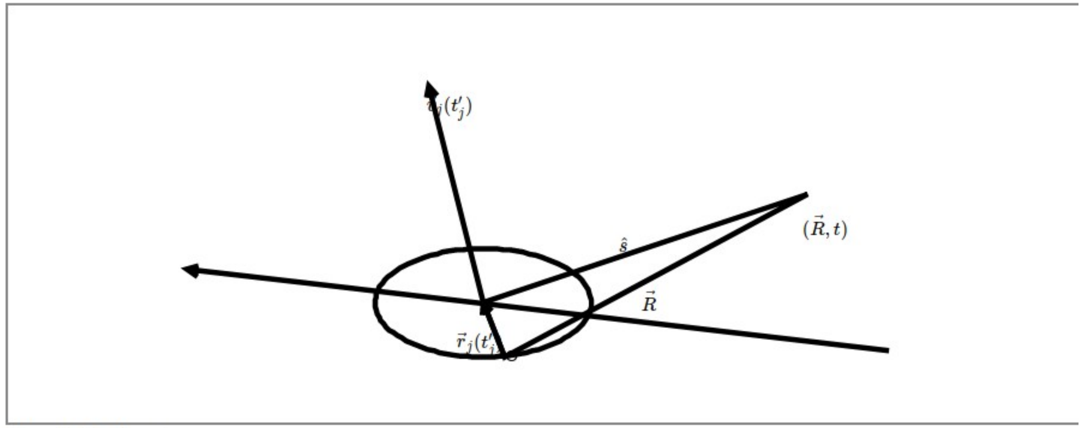
$$k_s = k_0 + k$$



$$P_s\left(\hat{k}, \omega_s\right) d\Omega d\omega_s = \frac{1}{16\pi^2}\frac{e^4}{m_e^2 c^3} d\Omega d\omega_s \left|\hat{s} \times \left(\hat{s} \times \vec{E}_{i0}\right)\right|^2 \frac{1}{T}\left|n_e\left(\vec{k}_0 - \frac{\hat{s}}{c}\left(\omega - \omega_0\right), \omega - \omega_0\right)\right|^2$$

OTS theory: spectrum

Retarded time and scattered field



The retarded time is,

$$t' = t - \frac{\vec{R} - \vec{r}}{c} \approx t - \frac{R}{c} + \frac{\hat{s} \cdot \vec{r}}{c} \tag{1}$$

where $\vec{r}$ is the position of an electron at the retarded time t'. The orbit of an electron is,

$$\vec{r}_j = \vec{r}_{j0} + \vec{v}_j t' \tag{2}$$

As a result, the retarded time is around,

$$t \approx \left(1 - \hat{s} \cdot \vec{\beta}\right) t' + \frac{R}{c} - \frac{\hat{s} \cdot \vec{r}_0}{c} \tag{3}$$

The far field form of the scattered electric field is,

$$\vec{E}_s\left(\vec{R}, t\right) = -\frac{e}{cR}\left(\frac{\hat{s} \times \left(\left(\hat{s} - \vec{\beta}\right) \times \frac{d\vec{\beta}}{dt}\right)}{\left(1 - \hat{s} \cdot \vec{\beta}\right)^3}\right) \tag{4}$$

where β is the velocity of an electron at the retarded time. Then the total scattered field is,

$$\vec{E}_t\left(\vec{R}, t\right) = \int dr \int dv F_e\left(\vec{r}, \vec{v}, t\right) \vec{E}_s\left(\vec{R}, t\right) \tag{5}$$

$$F_e\left(\vec{r}, \vec{v}, t\right) = \int_{-\infty}^{\infty} dt' \sum_{j=1}^{N} \delta\left(\vec{r} - \vec{r}_j\left(t'\right)\right) \delta\left(\vec{v} - \vec{v}_j\left(t'\right)\right) \delta\left(t' - t + \frac{R}{c} - \frac{\hat{s}}{c} \cdot \vec{r}_j\right) \tag{5.1}$$

Spectrum of radiation

The radiation angular distribution is,

$$P_s\left(\vec{R}\right) d\Omega = \frac{cR^2}{4\pi} d\Omega \frac{1}{T} \int_{-\infty}^{\infty} dt \left|\vec{E}_t\left(\vec{R}, t\right)\right|^2 \tag{6}$$

Then the spectrum is,

$$P_s\left(\vec{R}, \omega_s\right) d\Omega d\omega_s = \frac{cR^2}{4\pi} d\Omega d\omega_s \frac{1}{\pi T}\left|\vec{E}_t\left(\vec{R}, \omega_s\right)\right|^2 \tag{7}$$

where,

$$\vec{E}_t\left(\vec{R}, \omega\right) = \int_{-\infty}^{\infty} dt \vec{E}_t\left(\vec{R}, t\right) exp\left(-i\omega t\right) \tag{7.1}$$

Low-temperature plasma

In this situation, the scattered field is,

$$\vec{E}_s\left(\vec{R}, t\right) = -\frac{e}{cR}\left(\hat{s} \times \left(\hat{s} \times \frac{d\vec{\beta}}{dt}\right)\right) \tag{8}$$

and

$$\frac{d\vec{\beta}}{dt} = -\frac{e}{m_e c} \vec{E}_i\left(\vec{r}, t\right) \tag{9}$$

$$\frac{d\vec{\beta}}{dt} = -\frac{e\vec{E}_{i0}}{m_e c} cos\left(\vec{k}_0 \cdot \vec{r} - \omega_0 t\right) \tag{10}$$

Then the Fourier transform of the total scattered field is,

$$\vec{E}_t\left(\vec{R}, \omega\right) = \int_{-\infty}^{\infty} dt \left(\int dr \int dv \left(-F_e\left(\vec{r}, \vec{v}, t\right) \frac{e}{cR}\left(\hat{s} \times \left(\hat{s} \times \frac{d\vec{\beta}}{dt}\right)\right)\right)\right) exp\left(-i\omega t\right) \tag{11}$$

$$\vec{E}_t\left(\vec{R}, \omega\right) = \frac{e^2}{m_e c^2 R}\left(\hat{s} \times \left(\hat{s} \times \vec{E}_{i0}\right)\right) \int_{-\infty}^{\infty} dt \left(\int dr \int dv F_e\left(\vec{r}, \vec{v}, t\right) cos\left(\vec{k}_0 \cdot \vec{r} - \omega_0 t\right)\right) exp\left(-i\omega t\right) \quad (1$$

OTS theory: spectrum

$$\vec{E}_t\left(\vec{R},\omega\right)=\int_{-\infty}^{\infty}dt\left(\int dr\int dv\left(-F_e\left(\vec{r},\vec{v},t\right)\frac{e}{cR}\left(\hat{s}\times\left(\hat{s}\times\frac{d\vec{\beta}}{dt}\right)\right)\right)\right)exp\left(-i\omega t\right) \qquad (11)$$

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\int_{-\infty}^{\infty}dt\left(\int dr\int dvF_e\left(\vec{r},\vec{v},t\right)cos\left(\vec{k}_0\cdot\vec{r}-\omega_0t\right)\right)exp\left(-i\omega t\right) \quad (1$$

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\int_{-\infty}^{\infty}dt'\left(\int dr\int dv\sum_{j=1}^{N}\delta\left(\vec{r}-\vec{r}_j\left(t'\right)\right)\delta\left(\vec{v}-\vec{v}_j\left(t'\right)\right)cos\left(\vec{k}_0\cdot\vec{r}-\right.\right.$$

If we ignored the difference made by velocity of electrons, then,

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\int_{-\infty}^{\infty}dt'\left(\int drn\left(\vec{r},t'\right)cos\left(\vec{k}_0\cdot\vec{r}-\omega_0\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right)\right)ex$$

The density distribution is fluctuating all the time, which can be described as,

$$n\left(\vec{r},t'\right)=\frac{1}{\left(2\pi\right)^4}\iint d\omega_sdk_sn_e\left(\vec{k}_s,\omega_s\right)e^{i\vec{k}_s\cdot\vec{r}-i\omega_st'} \qquad (15)$$

And there are delta functions,

$$A=\iint dt'drcos\left(\vec{k}_0\cdot\vec{r}-\omega_0\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right)e^{i\vec{k}_s\cdot\vec{r}-i\omega_st'}exp\left(-i\omega\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right) \qquad (16)$$

$$A=\frac{1}{2}\iint dt'dr\left(exp\left(i\vec{k}_0\cdot\vec{r}-i\omega_0\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right)+exp\left(-i\vec{k}_0\cdot\vec{r}+i\omega_0\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right)\right)$$

The exponential part in formulas is,

$$\phi_\pm=\pm\left(\vec{k}_0\cdot\vec{r}-\omega_0\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right)\right)+\vec{k}_s\cdot\vec{r}-\omega_st'-\omega\left(t'+\frac{R}{c}-\frac{\hat{s}}{c}\cdot\vec{r}\right) \qquad (18)$$

$$\phi_\pm=\left(\pm\vec{k}_0+\vec{k}_s+\frac{\hat{s}}{c}\left(\pm\omega_0-\omega\right)\right)\cdot\vec{r}+\left(\pm\omega_0-\omega-\omega_s\right)t'+C \qquad (19)$$

Then eq.(14) becomes,

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{1}{2}\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\iint d\omega_sdk_sn_e\left(\vec{k}_s,\omega_s\right)\left(\delta\left(\vec{k}_0+\vec{k}_s+\frac{\hat{s}}{c}\left(\omega_0-\omega\right)\right)\delta\left(\omega_0-\omega-\right.\right.$$

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{1}{2}\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\left(n_e\left(-\vec{k}_0-\frac{\hat{s}}{c}\left(\omega_0-\omega\right),\omega_0-\omega\right)+n_e\left(\vec{k}_0-\frac{\hat{s}}{c}\left(-\omega_0-\omega\right),\right.\right.$$

$$\vec{E}_t\left(\vec{R},\omega\right)=\frac{1}{2}\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)\left(n_e\left(-\vec{k}_0-\frac{\hat{s}}{c}\left(\omega_0-\omega\right),\omega_0-\omega\right)+n_e\left(\vec{k}_0-\frac{\hat{s}}{c}\left(-\omega_0-\omega\right),\right.\right.$$

Because $n(r,t')$ is a real number distribution, we should have $n_e(k,-\omega)=n_e(k,\omega)$ and $n_e(-k,\omega)=n_e(k,\omega)$. And the frequency of density oscillation should be much smaller than the frequency of laser, as a result, $n_e(k,\omega_0)\approx 0$. Then we have,

$$\vec{E}_t\left(\hat{k},\omega\right)=\frac{1}{2}\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)n_e\left(\vec{k}_0-\frac{\hat{s}}{c}\left(\omega-\omega_0\right),\omega-\omega_0\right) \qquad (22)$$

The radiation power of OTS is,

$$P_s\left(\hat{k},\omega_s\right)d\Omega d\omega_s=\frac{cR^2}{4\pi}d\Omega d\omega_s\frac{1}{\pi T}\left|\vec{E}_t\left(\hat{k},\omega_s\right)\right|^2 \qquad (23)$$

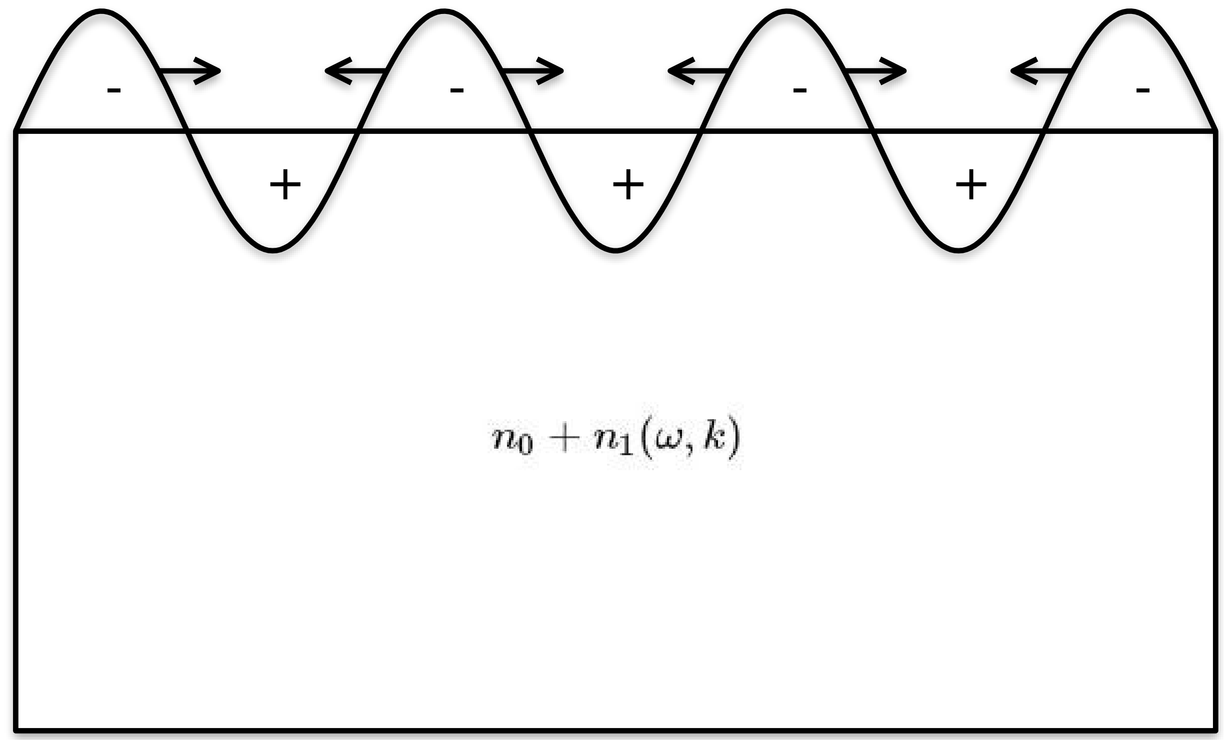
$$P_s\left(\hat{k},\omega_s\right)d\Omega d\omega_s=\frac{cR^2}{4\pi}d\Omega d\omega_s\frac{1}{\pi T}\left|\frac{1}{2}\frac{e^2}{m_ec^2R}\left(\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right)n_e\left(\vec{k}_0-\frac{\hat{s}}{c}\left(\omega-\omega_0\right),\omega-\omega_0\right)\right|^2 \qquad (24)$$

$$P_s\left(\hat{k},\omega_s\right)d\Omega d\omega_s=\frac{1}{16\pi^2}\frac{e^4}{m_e^2c^3}d\Omega d\omega_s\left|\hat{s}\times\left(\hat{s}\times\vec{E}_{i0}\right)\right|^2\frac{1}{T}\left|n_e\left(\vec{k}_0-\frac{\hat{s}}{c}\left(\omega-\omega_0\right),\omega-\omega_0\right)\right|^2 \qquad (25)$$

OTS theory: n_e(k,w)

$$\frac{\partial F_{1q}}{\partial t} + v \cdot \frac{\partial F_{1q}}{\partial r} + \frac{q}{m} E_1 \cdot \frac{\partial F_{0q}}{\partial v} = C$$

$$\vec{\nabla} \cdot \vec{E}_1 = 4\pi \sum_i \left(q_i \int d\bar{v}\, F_{1,i} \right)$$



$$\chi_e(k,\omega) = \left(\int dv \, \frac{\frac{\partial}{\partial v_k}\left(f_e(v)\right)}{\bar{v} \cdot i\vec{k} - i\omega} \right) N_{0e} \frac{e^2}{m_e} \frac{4i\pi}{k}$$

$$\chi_i(k,\omega) = \left(\int dv \, \frac{\frac{\partial}{\partial v_k}\left(f_i(v)\right)}{\bar{v} \cdot i\vec{k} - i\omega} \right) N_{0i} \frac{(Ze)^2}{m_i} \frac{4i\pi}{k}$$

$$n_{1e}(k,\omega) = \frac{\left(1 - \chi_i(k,\omega)\right) \int dv \, \frac{F_{1et}(k,v,0)}{\bar{v} \cdot i\vec{k} - i\omega} - Z\chi_e(k,\omega) \int dv \, \frac{F_{1it}(k,v,0)}{\bar{v} \cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega) - \chi_i(k,\omega)}$$

OTS theory: $n_e(k,w)$

The spectral density function

The Boltzmann equation in this problem is,

$$\frac{\partial F_{0q}}{\partial t} + \vec{v} \cdot \frac{\partial F_{0q}}{\partial \vec{r}} = 0 \tag{1}$$

$$\frac{\partial F_{1q}}{\partial t} + \vec{v} \cdot \frac{\partial F_{1q}}{\partial \vec{r}} + \frac{q}{m} \vec{E}_1 \cdot \frac{\partial F_{0q}}{\partial \vec{v}} = C \tag{2}$$

$$\vec{\nabla} \cdot \vec{E}_1 = 4\pi \sum_i \left(q_i \int d\vec{v}\, F_{1,i} \right) \tag{3}$$

The Fourier transform of the eq.(2) is,

$$-i\omega F_{1q} + \vec{v} \cdot i\vec{k} F_{1q} + \frac{q}{m} \vec{E}_1 \cdot \frac{\partial F_{0q}}{\partial \vec{v}} = C \tag{5}$$

The Fourier transform of the eq.(3) is,

$$i\vec{k} \cdot \vec{E}_1 = 4\pi \sum_i \left(\int d\vec{v}\, q_i F_{1,i} \right) \tag{4}$$

It is reasonable to assume that,

$$\vec{E}_1 = -\frac{4i\pi\hat{k}}{k} \sum_i \left(\int d\vec{v}\, q_i F_{1,i} \right) \tag{6}$$

$$-i\omega F_{1q} + \vec{v} \cdot i\vec{k} F_{1q} - \frac{q}{m} \frac{4i\pi}{k} \sum_i \left(\int d\vec{v}\, q_i F_{1,i} \right) \frac{\partial F_{0q}}{\partial v_k} = C \tag{7}$$

Collisionless plasma

The term C is the collision term. In a collisionless plasma, eq.(7) is,

$$-i\omega F_{1q}(k,\omega,v) + \vec{v} \cdot i\vec{k} F_{1q}(k,\omega,v) - \frac{q}{m} \frac{4i\pi}{k} \rho_1(k,\omega) \frac{\partial}{\partial v_k} (F_{0q}(r,t,v)) = 0 \tag{8}$$

$$F_{1q}(k,\omega,v) = -\frac{-\frac{q}{m}\frac{4i\pi}{k}\rho_1(k,\omega)\frac{\partial}{\partial v_k}(F_{0q}(r,t,v))}{\vec{v} \cdot i\vec{k} - i\omega} \tag{9}$$

$$n_{1q}(k,\omega) = -\left(\int dv \frac{-\frac{\partial}{\partial v_k}(F_{0q}(r,t,v))}{\vec{v} \cdot i\vec{k} - i\omega} \right) \frac{q}{m} \frac{4i\pi}{k} \rho_1(k,\omega) \tag{10}$$

Let's consider the effect of initial distribution. The equation should be,

$$-i\omega F_{1q}(k,\omega,v) - F_{1qt}(k,v,0) + \vec{v} \cdot i\vec{k} F_{1q}(k,\omega,v) - \frac{q}{m} \frac{4i\pi}{k} \rho_1(k,\omega) \frac{\partial}{\partial v_k} (F_{0q}(r,t,v)) = 0 \tag{11}$$

$$n_{1q}(k,\omega) = -\int dv \frac{-F_{1qt}(k,v,0) - \frac{q}{m}\frac{4i\pi}{k}\rho_1(k,\omega)\frac{\partial}{\partial v_k}(F_{0q}(r,t,v))}{\vec{v} \cdot i\vec{k} - i\omega} \tag{12}$$

$$n_{1q}(k,\omega) = \int dv \frac{F_{1qt}(k,v,0)}{\vec{v} \cdot i\vec{k} - i\omega} + \left(\int dv \frac{\frac{\partial}{\partial v_k}(F_{0q}(r,t,v))}{\vec{v} \cdot i\vec{k} - i\omega} \right) \frac{q}{m} \frac{4i\pi}{k} \rho_1(k,\omega) \tag{13}$$

It should be safe to assume that the thermal distribution is same anywhere. Then we should have,

$$F_{0q}(r,t,v) = N_{0q} f_q(v) \tag{14}$$

Then we will have,

$$n_{1q}(k,\omega) = \int dv \frac{F_{1qt}(k,v,0)}{\vec{v} \cdot i\vec{k} - i\omega} + \left(\int dv \frac{\frac{\partial}{\partial v_k}(f_q(v))}{\vec{v} \cdot i\vec{k} - i\omega} \right) N_{0q} \frac{q}{m} \frac{4i\pi}{k} \rho_1(k,\omega) \tag{15}$$

There are ions and electrons in the plasma. They should follow two equations as following,

$$n_{1e}(k,\omega) = \int dv \frac{F_{1et}(k,v,0)}{\vec{v} \cdot i\vec{k} - i\omega} + \left(\int dv \frac{\frac{\partial}{\partial v_k}(f_e(v))}{\vec{v} \cdot i\vec{k} - i\omega} \right) N_{0e} \frac{-e}{m_e} \frac{4i\pi}{k} \rho_1(k,\omega) \tag{16}$$

$$n_{1i}(k,\omega) = \int dv \frac{F_{1it}(k,v,0)}{\vec{v} \cdot i\vec{k} - i\omega} + \left(\int dv \frac{\frac{\partial}{\partial v_k}(f_i(v))}{\vec{v} \cdot i\vec{k} - i\omega} \right) N_{0i} \frac{Ze}{m_i} \frac{4i\pi}{k} \rho_1(k,\omega) \tag{17}$$

OTS theory: $n_e(k,w)$

And,

$$\rho_1(k,\omega) = Zen_{1i}(k,\omega) - en_{1e}(k,\omega) \tag{18}$$

We can define that,

$$\chi_e(k,\omega) = \left(\int dv \, \frac{\frac{\partial}{\partial v_k}(f_e(v))}{\vec{v}\cdot i\vec{k} - i\omega}\right) N_{0e} \frac{e^2}{m_e} \frac{4i\pi}{k} \tag{19}$$

$$\chi_i(k,\omega) = \left(\int dv \, \frac{\frac{\partial}{\partial v_k}(f_i(v))}{\vec{v}\cdot i\vec{k} - i\omega}\right) N_{0i} \frac{(Ze)^2}{m_i} \frac{4i\pi}{k} \tag{20}$$

Then we have,

$$n_{1e}(k,\omega) = \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - \frac{\chi_e(k,\omega)}{e}\rho_1(k,\omega) \tag{21}$$

$$n_{1i}(k,\omega) = \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} + \frac{\chi_i(k,\omega)}{Ze}\rho_1(k,\omega) \tag{22}$$

Or equivalently,

$$n_{1e}(k,\omega) = \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - \chi_e(k,\omega) Zn_{1i}(k,\omega) + \chi_e(k,\omega) n_{1e}(k,\omega) \tag{23}$$

$$n_{1i}(k,\omega) = \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} + \chi_i(k,\omega) n_{1i}(k,\omega) - \frac{\chi_i(k,\omega)}{Z} n_{1e}(k,\omega) \tag{24}$$

We can solve it,

$$n_{1e}(k,\omega) = -\frac{\chi_e(k,\omega) Zn_{1i}(k,\omega) - \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega)} \tag{25}$$

$$n_{1i}(k,\omega) = \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} + \chi_i(k,\omega) n_{1i}(k,\omega) + \frac{\chi_i(k,\omega)}{Z} \frac{\chi_e(k,\omega) Zn_{1i}(k,\omega) - \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega)} \quad (26$$

$$n_{1i}(k,\omega) = -\frac{\chi_i(k,\omega) \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - Z(1-\chi_e(k,\omega)) \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{Z(1 - \chi_e(k,\omega) - \chi_i(k,\omega))} \tag{27}$$

And we have,

$$n_{1e}(k,\omega) = -\frac{-\chi_e(k,\omega) Z \frac{\chi_i(k,\omega) \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - Z(1-\chi_e(k,\omega)) \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{Z(1-\chi_e(k,\omega)-\chi_i(k,\omega))} - \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega)} \tag{28}$$

The final result is,

$$n_{1e}(k,\omega) = \frac{(1-\chi_i(k,\omega)) \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - Z\chi_e(k,\omega) \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega) - \chi_i(k,\omega)} \tag{29}$$

$$n_{1i}(k,\omega) = \frac{(1-\chi_e(k,\omega)) \int dv \, \frac{F_{1it}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega} - \frac{\chi_i(k,\omega)}{Z} \int dv \, \frac{F_{1et}(k,v,0)}{\vec{v}\cdot i\vec{k} - i\omega}}{1 - \chi_e(k,\omega) - \chi_i(k,\omega)} \tag{30}$$

%[31].

Method of calculating OTS spectrum

OTS spectrum of collisionless plasmas:

$$P_s d\Omega d\omega = \frac{1}{2\pi} I_0 r_e^2 V n_e \left(1 + \frac{2\omega}{\omega_i}\right) sin(\varphi)^2 S\left(\bar{k}, \omega\right) d\Omega d\omega$$

$$S\left(\bar{k}, \omega\right) = \frac{2\pi}{k} \left|1 - \frac{\chi_e}{\epsilon}\right|^2 f_{e0}\left(\frac{\omega}{k}\right) + \frac{2\pi}{k} \left|\frac{\chi_e}{\epsilon}\right|^2 \sum_j \frac{Z_j^2 N_j}{\sum_i Z_i^2 N_i} f_{i0,j}\left(\frac{\omega}{k}\right)$$

Electric susceptibility:

$$\chi_e(k, \omega) = \alpha^2 w\left(\omega \middle/ \left(\sqrt{2} k v_{te}\right)\right)$$

$$\chi_j(k, \omega) = \alpha^2 \frac{Z_j^2 N_j}{\sum_j Z_j N_j} \frac{T_e}{T_j} w\left(\omega \middle/ \left(\sqrt{2} k v_{tj}\right)\right)$$

Maxwell distributions:

$$f_{e0}(v) = \frac{1}{\sqrt{2} v_{te}} exp\left(-\frac{v^2}{2 v_{te}^2}\right)$$

$$f_{i0,j}(v) = \frac{1}{\sqrt{2} v_{t,j}} exp\left(-\frac{v^2}{2 v_{t,j}^2}\right)$$

$$w(x) = 1 - 2x e^{-x^2} \int_0^x dp\, e^{p^2} + i\sqrt{\pi} x e^{-x}$$

OTS spectrum of arbitrary distributions

Electric susceptibility:

$$\chi_e(k,\omega) = \alpha^2 w\left(\omega / \left(\sqrt{2} k v_{te}\right)\right)$$

$$\chi_j(k,\omega) = \alpha^2 \frac{Z_j^2 N_j}{\sum_j Z_j N_j} \frac{T_e}{T_j} w\left(\omega / \left(\sqrt{2} k v_{tj}\right)\right)$$

Maxwell distributions:

$$f_{e0}(v) = \frac{1}{\sqrt{2} v_{te}} exp\left(-\frac{v^2}{2v_{te}^2}\right)$$

$$f_{i0,j}(v) = \frac{1}{\sqrt{2} v_{t,j}} exp\left(-\frac{v^2}{2v_{t,j}^2}\right)$$

$$w(x) = 1 - 2xe^{-x^2}\int_0^x dp\, e^{p^2} + i\sqrt{\pi} x e^{-x}$$

Electric susceptibility:

$$\chi_e(k,\omega) = \left(\int dv \frac{\frac{\partial}{\partial v_k}\left(f_e(v)\right)}{\vec{v}\cdot i\vec{k} - i\omega}\right) N_{0e} \frac{e^2}{m_e} \frac{4i\pi}{k}$$

$$\chi_i(k,\omega) = \left(\int dv \frac{\frac{\partial}{\partial v_k}\left(f_i(v)\right)}{\vec{v}\cdot i\vec{k} - i\omega}\right) N_{0i} \frac{(Ze)^2}{m_i} \frac{4i\pi}{k}$$

Example: super-Gaussian distribution

Super-Gaussian distribution

One kind of super-Gaussian distribution is,

$$f_e(v) = Ae^{-\left(\frac{\left(v_x^2+v_y^2+v_z^2\right)m_e}{2kT}\right)^P} \tag{1}$$

The expression of χ_e is,

$$\chi_e(k,\omega) = -2\pi\left(\iint dcos(\theta)\ dv\, v^2 \frac{\frac{\partial}{\partial v_k}(f_e(v))}{vkcos(\theta)-\omega}\right) N_{0e}\frac{e^2}{m_e}\frac{4\pi}{k} \tag{2}$$

The partial differential, $\partial f_e(v)/\partial v_k$, is,

$$\frac{\partial}{\partial v_k}(f_e(v)) = -P\frac{Am_e vcos(\theta)}{kT}\left(\frac{\left(v_x^2+v_y^2+v_z^2\right)m_e}{2kT}\right)^{P-1} e^{-\left(\frac{\left(v_x^2+v_y^2+v_z^2\right)m_e}{2kT}\right)^P} \tag{3}$$

$$\frac{\partial}{\partial v_k}(f_e(v)) = -P\frac{m_e vcos(\theta)}{kT}\left(\frac{v^2}{v_{eT}^2}\right)^{P-1} f_e(v) \tag{4}$$

Then we will have,

$$\chi_e(k,\omega) = \frac{2\pi P}{k_B T}\left(\iint dcos(\theta)\ dv\, \frac{v^2}{k} f_e(v)\left(\frac{v^2}{v_{eT}^2}\right)^{P-1}\left(1+\frac{\frac{\omega}{vk}}{cos(\theta)-\frac{\omega}{vk}}\right)\right) N_{0e}e^2\frac{4\pi}{k} \tag{5}$$

The integral of θ is,

$$\chi_e(k,\omega) = \frac{8\pi^2 P}{k_B T}\frac{N_{0e}e^2}{k^2}\int dv\, v^2 f_e(v)\left(\frac{v^2}{v_{eT}^2}\right)^{P-1}\left(2+\frac{\omega}{vk}log\left(\frac{\omega-vk}{\omega+vk}\right)\right) \tag{6}$$

We can define variables,

$$v_{eT} = \sqrt{\frac{2k_B T}{m_e}} \tag{7}$$

$$x = v/v_{eT} \tag{8}$$

$$\xi = \omega/k/v_{eT} \tag{9}$$

And the equation becomes,

$$\chi_e(k,\omega) = \frac{16\pi^2 P N_{0e}e^2}{m_e k^2}\int_0^\infty dx\, g_P(x)\, x^{2P-2}\left(2x^2 + x\xi log\left(\frac{\xi-x}{\xi+x}\right)\right) \tag{10}$$

Using definition of Debye length,

$$\lambda_D^2 = \frac{\epsilon_0 k_B T}{N_{0e}e^2} \tag{11}$$

$$\chi_e(k,\omega) = 8\pi^2 v_{eT}^2\epsilon_0\frac{P}{k^2\lambda_D^2}\int_0^\infty dx\, g(x)\, x^{2P-2}\left(2x^2 + x\xi log\left(\frac{\xi-x}{\xi+x}\right)\right) \tag{12}$$

Example: super-Gaussian distribution

Electric susceptibility:

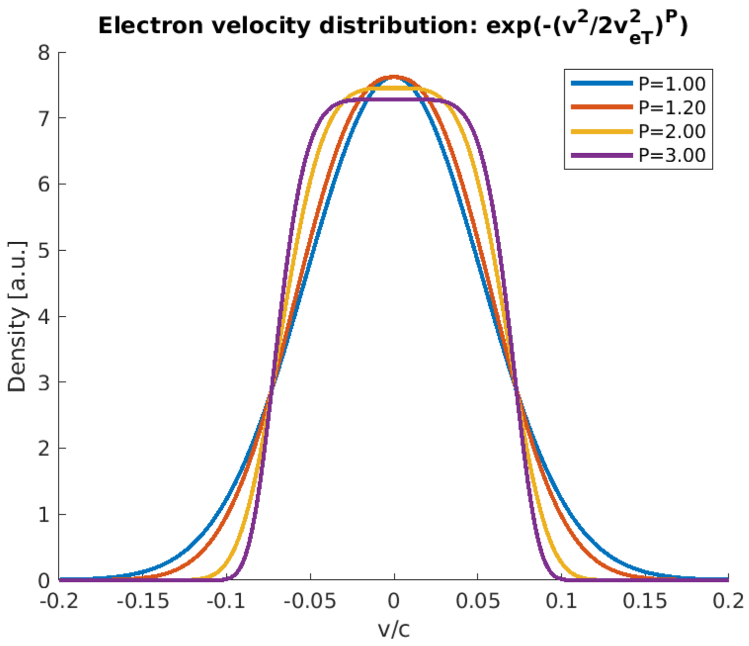
$$\chi_e(k,\omega) = \alpha^2 w\left(\omega \Big/ \left(\sqrt{2} k v_{te}\right)\right)$$

$$\chi_j(k,\omega) = \alpha^2 \frac{Z_j^2 N_j}{\sum_j Z_j N_j} \frac{T_e}{T_j} w\left(\omega \Big/ \left(\sqrt{2} k v_{tj}\right)\right)$$

super-Gaussian distributions:

$$g_P(x) = A e^{-x^{2P}}$$

$$w(x) = \int_0^\infty dp \; g_P(p) p^{2P-2} \left(2p^2 + p x \log\left(\frac{x-p}{x+p}\right)\right)$$



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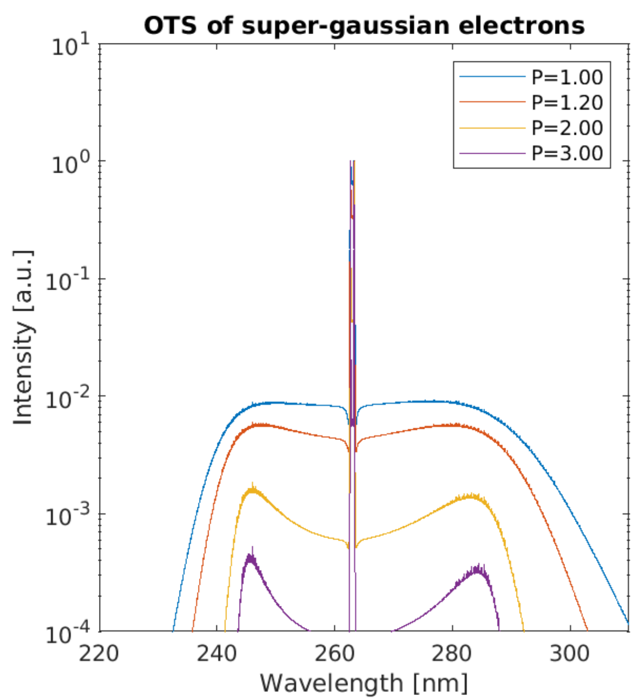

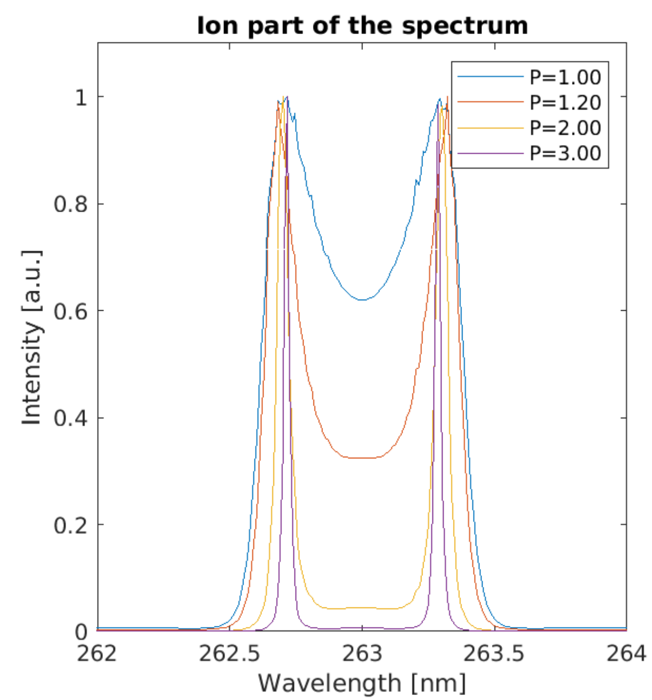

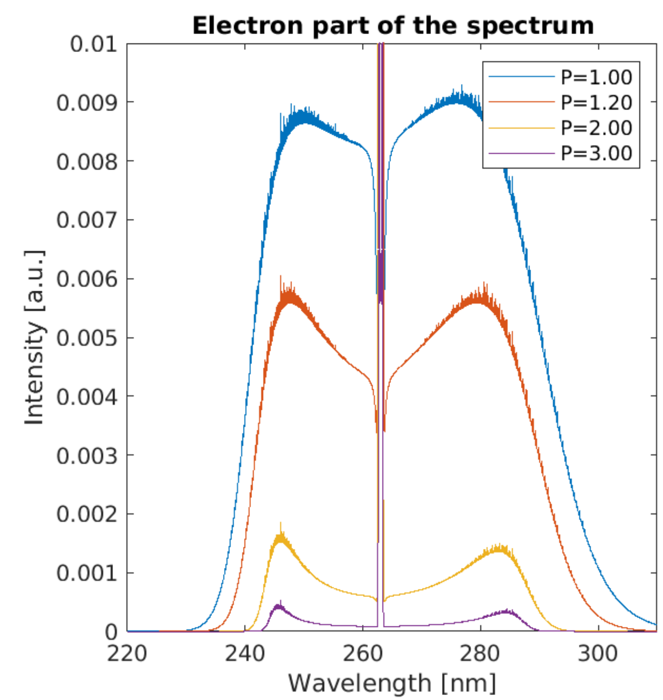

Example: Maxwell distribution

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Example: Maxwell distribution

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