

Sheet 5: A spectral diameter bound

Xianzhi Wang

fall 2021

Let G be a finite undirected d -regular graph on n vertices. Let A denote the adjacency operator, and let b_0, b_1, \dots, b_{n-1} be an orthonormal eigenbasis with real eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$. Let $\rho = \rho_0(G)$ be the spectral radius. On this sheet, we give an upper bound on the diameter using ρ .

Of course, $b_i^\top b_i = 1$ ($0 \leq i < n$). Let us force product the other way round. Let

$$U_i = b_i b_i^\top \quad (0 \leq i < n). \quad (1)$$

These matrices may look boring at first, but they are not. It turns out that they give an ‘orthonormal eigendecomposition’ for A .

Lemma 1 *Let A, B be $n \times n$ matrices such that $Av_i = Bv_i$ ($0 \leq i < n$) for a basis (v_i) for \mathbb{R}^n . Then $A = B$.*

Proof. We show $A = B$ by showing $Ax = Bx$ for all $x \in \mathbb{R}^n$. For all vector x , decompose $x = \sum_{i=0}^{n-1} \alpha_i v_i$,

$$Ax = A\left(\sum_i \alpha_i v_i\right) \quad (2)$$

$$= \sum_i \alpha_i A(v_i) \quad (3)$$

$$= \sum_i \alpha_i Bv_i \quad (4)$$

$$= B\left(\sum_i \alpha_i v_i\right) \quad (5)$$

$$= B(x) \quad (6)$$

for all $x \in \mathbb{R}^n$.

■

Theorem 2 *The following hold for $1 \leq i, j < n$:*

- 1) $U_i^2 = U_i$;
- 2) $U_i U_j = 0$ ($i \neq j$);
- 3) $AU_i = \lambda_i U_i$;
- 4) $\sum_{k=1}^n U_k = I$;
- 5) $\sum_{k=1}^n \lambda_k U_k = A$;

Proof. 1)

$$U_i^2 = (b_i b_i^T)^2 = b_i b_i^T b_i b_i^T = b_i b_i^T = U_i \quad (7)$$

2)

$$b_i b_i^T b_j b_j^T = b_i \cdot 0 \cdot b_j^T = 0 \quad (i \neq j) \quad (8)$$

3)

$$AU_i = Ab_i b_i^T = \lambda_i b_i b_i^T = \lambda_i U_i \quad (9)$$

since b_0, b_1, \dots, b_{n-1} is eigenbasis.

4)

$$\sum_{k=1}^n U_k = U_1 + U_2 + \dots + U_n \quad (10)$$

$$= b_1 b_1^T + b_2 b_2^T + \dots + b_n b_n^T \quad (11)$$

$$= \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \cdot \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix} = I \quad (12)$$

since it's the product of a matrix with orthonormal columns with its transpose.

5)

$$\sum_{k=1}^n \lambda_k U_k = A \quad (13)$$

$$\sum_{k=1}^n \lambda_k U_k = \sum_k \lambda_k b_k b_k^T \quad (14)$$

$$= \sum_k A b_k b_k^T \quad (15)$$

$$= A \sum_k b_k b_k^T \quad (16)$$

$$= A \sum_k U_k = AI = A \quad (17)$$

■

Definition 3 For $x, y \in V(G)$ let the distance $d(x, y)$ be the minimal length of a path going from x to y .

Lemma 4 Let G be an undirected, connected graph. Then d is a metric on $V(G)$.

Proof. We show d is a metric on $V(G)$. where G undirected connected graph.

(1) $d(x, y) \geq 0$, since the length of a path is always positive or zero, when x and y are the same vertex. $x = y \implies d(x, y) = 0$ since the shortest path is of zero length when we are going from a vertex to itself. $d(x, y) = 0 \implies x = y$, the shortest path is zero length, so we never moved, it's the same vertex.

(2) $d(x, y) = d(y, x)$ since traversing the shortest path backwards, the length is the same. (The graph is undirected)

(3) $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in V(G)$. the shortest path from x to y , then the shortest path from y to z could only be a longer path than the shortest path from x to z .

Since graph G is connected, $d(x, y) < \infty$. ■

Definition 5 Let the diameter of G be

$$\text{diam}(G) = \max \{d(x, y) \mid x, y \in V(G)\}.$$

One can give a general lower bound on the diameter as follows.

Lemma 6 Let G be a d -regular undirected graph on n vertices with $d \geq 3$. Then

$$\text{diam}(G) \geq \log_{d-1}(n/3). \quad (18)$$

Proof. Let G be d -regular, undirected graph on n vertices ($d \geq 3$)

$$\text{diam}(G) \geq \log_{d-1}\left(\frac{n}{3}\right) \quad (19)$$

$$(20)$$

Let r denote $\text{diam}(G)$, then showing

$$r \geq \log_{d-1}\left(\frac{n}{3}\right) \quad (21)$$

is the same as showing $(d-1)^r \geq \frac{n}{3}$, which is

$$3(d-1)^r \geq n. \quad (22)$$

$$n \leq 3(d-1)^r \quad (23)$$

we could have an upper bound on n by fixing a vertex v_0 first, then since G is d regular, v_0 has d neighbors, at 1st level, as we expand outwards. At 2nd level, each of those neighbors of v_0 has $d-1$ new neighbors, so we have $d(d-1)$ in total (at max). At 3rd level, $d(d-1)^2$ vertices in total (at max). As we go on, we could iterate at maximum $r-1$ levels, since the diameter of G is r . If our fixed vertex v_0 happens to be at the “center” of the graph, then n might be small compared to the upper bound we obtained in this way.

$$n \leq 1 + d + d(d-1) + d(d-1)^2 + \dots + d(d-1)^{r-1} \quad (24)$$

$$= 1 + \frac{d(d-1)^r - d}{d-2} \quad (25)$$

$$\leq 3(d-1)^r \quad (26)$$

$$(27)$$

if we show the last \leq , then we are done.

Now, we have $3 \leq d$.

Use $X := (d-1)^r$ as short hand.

$$3X - 1 < 3X < dX \quad (28)$$

$$6X - 2 < 2dX \quad (29)$$

$$dX - 2 < 3X(d-2) \quad (30)$$

$$\frac{dX - 2}{d-2} < 3X \quad (31)$$

$$\frac{d-2 + dX - d}{d-2} < 3X \quad (32)$$

$$1 + \frac{dX}{d-2} - \frac{d}{d-2} < 3X \quad (33)$$

$$1 + \frac{d(d-1)^r}{d-2} - \frac{d}{d-2} < 3(d-1)^r \quad (34)$$

as desired. and we are done.

■

So, for a fixed degree, the diameter is at least logarithmic in the number of vertices.

It turns out that for good expanders, one can get a logarithmic upper bound for the diameter.

Lemma 7 *Let $x, y \in V(G)$. Then*

$$d(x, y) = \min \{k \mid X_{\{x\}}^\top (A^k) X_{\{y\}} \neq 0\}. \quad (35)$$

Theorem 8 *We have*

$$\text{diam}(G) \leq \frac{\log(n)}{\log(d/\rho_0(G))}. \quad (36)$$

Hint: decompose $X_{\{x\}}$ and $X_{\{y\}}$.

solution:

$$\text{diam}(G) = \max_{x, y \in V(G)} \min_{\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0} k \quad (37)$$

(since A^k has non-negative entries, so $\neq 0$ is the same as > 0).

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} = \chi_{\{x\}}^T \sum_{i=0}^{n-1} \lambda_i^k U_i \chi_{\{y\}} \quad (38)$$

$$= \sum_{i=0}^{n-1} \lambda_i^k (U_i)_{x,y} \quad (39)$$

$$= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} \quad (40)$$

$$(41)$$

since $\lambda_0 = d$, and

$$U_0 = b_0 b_0^T = \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \end{bmatrix} \quad (42)$$

and

$$(U_i)_{x,y} = (b_i b_i^T)_{x,y} \quad (43)$$

$$= (b_i)_x \cdot (b_i)_y \quad (44)$$

$$= x_i \cdot y_i \quad (45)$$

where

$$x_i := (b_i)_x \quad (46)$$

$$y_i := (b_i)_y \quad (47)$$

$$\text{for } i \in \{0, 1, 2, \dots, n-1\} \quad (48)$$

is the x -th entry in vector b_i , the y -th entry in vector b_i .

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} = \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} \quad (49)$$

$$= \frac{d^k}{n} + \sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i \quad (50)$$

$$\geq \left| \frac{d^k}{n} \right| - \left| \sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i \right| > \quad (51)$$

$$\geq \frac{d^k}{n} - \rho^k \sum_{i=1}^{n-1} |x_i| |y_i| \quad (52)$$

$$\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=1}^{n-1} |x_i|^2} \sqrt{\sum_{i=1}^{n-1} |y_i|^2} \quad (53)$$

$$\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=0}^{n-1} |x_i|^2} \sqrt{\sum_{i=0}^{n-1} |y_i|^2} \quad (54)$$

$$(55)$$

where we used inverse triangle inequality, and then Cauchy-Schwarz inequality.
since

$$\sum_{i=0}^{n-1} |x_i|^2 = \sum_{i=0}^{n-1} (b_i)_x (b_i)_x = \sum_{i=0}^{n-1} (b_i b_i^T)_{x,x} = (I)_{x,x} = 1 \quad (56)$$

$$= \frac{d^k}{n} - \rho^k \cdot 1 \cdot 1 \quad (57)$$

$$= \frac{d^k}{n} - \rho^k \quad (58)$$

Thus,

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} \geq \frac{d^k}{n} - \rho^k (*) \quad (59)$$

Since max is taken over $x, y \in V(G)$, we want $\frac{d^k}{n} - \rho^k > 0$. So

$$\frac{d^k}{n} > \rho^k \quad (60)$$

$$\frac{d^k}{\rho^k} > n \quad (61)$$

$$\log \left(\frac{d}{\rho} \right)^k > \log n \quad (62)$$

$$k > \frac{\log n}{\log (d/\rho)} \quad (63)$$

Min of k is taken over all $\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0$, so k has $\frac{\log n}{\log (d/\rho)}$ growth.

end solution

When G is bipartite, the above theorem (and also the theorem on the previous sheet) do not give anything reasonable, as $\rho_0(G) = d$. However, there is a similar diameter bound for bipartite graphs where all the other eigenvalues (in absolute value) are bounded away from d .

Theorem 9 *Let G be a finite, connected, undirected d -regular bipartite graph on n vertices. Assume that $|\lambda_i| \leq r$ ($1 \leq i < n - 1$). Then G has logarithmic diameter.*