Graph Limits: Graph Limits #Fekete

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Professor Miklos Abert fa21

Xianzhi

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Problem 1

Let $a_1, a_2, a_3, \ldots,$ seq of non-negative real numbers with $a_{i+j} \leq a_i + a_j \ \forall \ i, j \geq 1$. Then

$$\lim_{n \mapsto \infty} \frac{a_n}{n} \tag{1}$$

exists.

Soln:

Let $L = \inf_{n \geq 1} \frac{a_n}{n}$, and let $\epsilon > 0$ be arbitrary.
since L is the inf of $\frac{a_n}{n}$, we could choose n_0 such that

$$L \leq \frac{a_{n_0}}{n_0} \leq L + \epsilon \tag{2}$$

now, for $n \geq n_0$, write n as $n = pn_0 + q$, $p, q \in \mathbb{N}, \ 0 \leq q < n_0$.

$$\begin{align} a_n &= a_{pn_0+q} \tag{3} \\ &\leq \underbrace{a_{n_0} + a_{n_0} + \ldots + a_{n_0}}_{p \text{ terms}} + a_q \tag{4} \\ &= pa_{n_0} + a_q \tag{5} \\ & \tag{6} \end{align}$$

Now, divide both sides by n.

$$\begin{align} \frac{a_n}{n} &\leq \frac{p}{n} a_{n_0} + \frac{a_q}{n} \tag{7} \\ \limsup_{n \mapsto \infty} \frac{a_n}{n} &\leq \frac{a_{n_0}}{n_0} + 0 \tag{8} \end{align}$$

since

$$\begin{align} \frac{a_q}{n} &\mapsto 0 \text{ as } n \mapsto \infty \tag{9} \\ \text{since } a_q &\leq \max_{0 \leq q < n_0} a_q < \infty \tag{10} \end{align}$$

since

$$\begin{align} \limsup_{n \mapsto \infty} \frac{p}{n} &= \limsup_{n \mapsto \infty} \frac{\frac{n-q}{n_0}}{n} \tag{11} \\ &= \limsup_{n \mapsto \infty} \frac{n-q}{n} \cdot \frac{1}{n_0}. \tag{12} \\ &= \lim_{n \mapsto \infty} \frac{n-q}{n} \cdot \frac{1}{n_0}. \tag{13} \\ &= \frac{1}{n_0} \tag{14} \end{align}$$

Thus, $\limsup_{n \mapsto \infty} \frac{a_n}{n} \leq \frac{a_{n_0}}{n_0} \leq L + \epsilon$
since $\epsilon > 0$ is arbitrary, we could make $\epsilon \mapsto 0$, so

$$\limsup_{n \mapsto \infty} \tag{15}$$