Galois Theory: GAL #05

Due on Mar 18, 2022 at 11:59pm

Prof Matyas Domokos Section 7

Xianzhi

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HW05
Exercise 7.2.5
Exercise 7.2.7
Exercise 7.2.8

Problem 1

Exercise 7.2.5 Let $\gamma = \sqrt{2+\sqrt{2}}$.

1. Show that $\mathbb{Q}(\gamma) : \mathbb{Q}$ is normal with cyclic Galois Group.

2. Show that $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\phi)$ with $\phi^4 = i$.

Soln:
Part A

Let

$$\sqrt{2+\sqrt{2}} = X \tag{1}$$
$$\sqrt{2} = X^2 - 2 \tag{2}$$
$$2 = (X^2-2)^2 = X^4 - 4X^2 + 4 \tag{3}$$

Thus, $\sqrt{2+\sqrt{2}}$ is a root of $X^4 - 4X^2 + 2 =: f$ Since f is irreducible, by Eisenstein ($p = 2$), $\mathbb{Q}(\gamma)$ is degree 4 over $\mathbb{Q}$, we could find the roots of f:

$$f = \left(X + \sqrt{2+\sqrt{2}}\right)\left(X - \sqrt{2+\sqrt{2}}\right)\left(X + \sqrt{2-\sqrt{2}}\right)\left(X - \sqrt{2-\sqrt{2}}\right) \tag{4}$$

Let $\phi \in \Gamma(\mathbb{Q}(\gamma) : \mathbb{Q})$, we know ϕ permutes the roots of f.
If we can find an element ϕ with order strictly greater than 2, then since order of the element need to divide the order of the group, $|\phi|$ must be 4. Since up to isomorphism, group of order 4 is $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we know once there exists $|\phi| = 4$, $\Gamma(\mathbb{Q}(\gamma) : \mathbb{Q})$ must be cyclic.
First, we show $\mathbb{Q}(\gamma)$ is splitting field of f that has no multiple roots, which would imply it is normal. since

$$\sqrt{2+\sqrt{2}} = \gamma \in \mathbb{Q}(\gamma) \implies -\sqrt{2+\sqrt{2}} \in \mathbb{Q}(\gamma) \tag{5}$$
$$2+\sqrt{2} = \gamma^2 \in \mathbb{Q}(\gamma) \tag{6}$$

so $\sqrt{2} \in \mathbb{Q}(\gamma)$. Since

$$\sqrt{2+\sqrt{2}} \cdot \sqrt{2-\sqrt{2}} = \sqrt{2} \in \mathbb{Q}(\gamma) \implies \sqrt{2-\sqrt{2}} \in \mathbb{Q}(\gamma) \tag{7}$$

so $-\sqrt{2-\sqrt{2}} \in \mathbb{Q}(\gamma)$.
Thus, all 4 roots of f are in $\mathbb{Q}(\gamma)$. And these roots are all distinct. so

$$\mathbb{Q}\left(\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, +\sqrt{2-\sqrt{2}}, -\sqrt{2-\sqrt{2}}\right) \subseteq \mathbb{Q}(\gamma) \tag{8}$$
$$\mathbb{Q}\left(\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, +\sqrt{2-\sqrt{2}}, -\sqrt{2-\sqrt{2}}\right) \supseteq \mathbb{Q}(\gamma) \tag{9}$$
$$\text{so } \mathbb{Q}\left(\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, +\sqrt{2-\sqrt{2}}, -\sqrt{2-\sqrt{2}}\right) = \mathbb{Q}(\gamma) \tag{10}$$
$$\tag{11}$$

Thus, $\mathbb{Q}(\gamma)$ is indeed splitting field of f with no multiple roots, so $\mathbb{Q}(\gamma) : \mathbb{Q}$ is Galois extension, and is normal extension. Now we find $\phi \in \Gamma(\mathbb{Q}(\gamma) : \mathbb{Q})$ that has order ≥ 3. Claim:

$$\sqrt{2+\sqrt{2}} \mapsto_\phi \sqrt{2-\sqrt{2}} \tag{12}$$

Problem 1 continued on next page. . .

does the job.
Why does such an automorphism exist? Answer: Galois group acts transitively on the roots of a minimal polynomial
Let

$$X_1 = -\sqrt{2+\sqrt{2}},\ X_2 = \sqrt{2+\sqrt{2}},\ X_3 = -\sqrt{2-\sqrt{2}},\ X_4 = \sqrt{2-\sqrt{2}} \tag{13}$$

Hence

$$\begin{aligned}
\phi\circ\phi\left(\sqrt{2+\sqrt{2}}\right) &= \phi\left(\sqrt{2-\sqrt{2}}\right) && (14)\\
&= \phi\left(\frac{\sqrt{2}}{\sqrt{2+\sqrt{2}}}\right) && (15)\\
&= \frac{\phi(\sqrt{2})}{\phi\left(\sqrt{2+\sqrt{2}}\right)} && (16)\\
&= \frac{-\sqrt{2}}{\sqrt{2-\sqrt{2}}} && (17)\\
&= -\sqrt{2+\sqrt{2}} && (18)
\end{aligned}$$

We also have

$$\begin{aligned}
\phi(2)+\phi(\sqrt{2}) &= \phi(2+\sqrt{2}) && (19)\\
&= \phi\left(\sqrt{2+\sqrt{2}}\cdot\sqrt{2+\sqrt{2}}\right) && (20)\\
&= \phi\left(\sqrt{2+\sqrt{2}}\right)\cdot\phi\left(\sqrt{2+\sqrt{2}}\right) && (21)\\
&= \left(\sqrt{2-\sqrt{2}}\right)\cdot\left(\sqrt{2-\sqrt{2}}\right) && (22)\\
&= 2-\sqrt{2} && (23)
\end{aligned}$$

so $\phi(2)+\phi(\sqrt{2}) = 2-\sqrt{2}$, since $\phi(2) = 2$ implies $\phi(\sqrt{2}) = -\sqrt{2}$.
Since ϕ^2 is not identity automorphism, $|\phi| \geq 3$, and we found the desired ϕ.
We can check ϕ indeed permutes X_1, X_2, X_3, X_4.

$$\begin{aligned}
&X_2 \xrightarrow{\phi} X_4 && (24)\\
&\phi(X_1) = \phi(-\sqrt{2+\sqrt{2}} = -\sqrt{2-\sqrt{2}} = X_3 && (25)\\
&\phi(X_3) = \phi(-\sqrt{2-\sqrt{2}}) = (-1)\cdot(-\sqrt{2+\sqrt{2}}) = \sqrt{2+\sqrt{2}} = X_2 && (26)\\
&\phi(X_4) = -\sqrt{2+\sqrt{2}} = X_1 && (27)\\
&\phi : X_1 \to X_3 \to X_2 \to X_4 \to X_1 && (28)
\end{aligned}$$

Part B

Show $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\phi)$ with $\phi^4 = i$.

Use formula

$$\cos(A) = 1 - 2\sin^2\frac{A}{2} \tag{29}$$

$$1 - 2\sin^2(22.5^\circ) = \cos 45^\circ \tag{30}$$

$$\sin^2(22.5^\circ) = \frac{\sqrt{2}-1}{2\sqrt{2}} \tag{31}$$

$$\sin^2(22.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2} \tag{32}$$

And also,

$$\cos(A) = 2\cos^2\frac{A}{2} - 1 \tag{33}$$

$$\cos 45^\circ = 2\cos^2 22.5^\circ - 1 \tag{34}$$

$$\frac{1}{\sqrt{2}} + 1 = 2\cos^2(22.5^\circ) \tag{35}$$

$$\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} = \cos(22.5^\circ) \tag{36}$$

Problem 2

Exercise 7.2.7 Find the degree of

$$\sqrt[5]{81} + 29\sqrt[5]{9} + 17\sqrt[5]{3} - 16 \tag{37}$$

over $\mathbb{Q}$.

Soln:

Problem 3

Exercise 7.2.8 Find the degree of $\sqrt[5]{81}$ over $\mathbb{Q}(\sqrt[81]{5})$.
Soln: