Graph Limits: Graph Limits lower bound

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Professor Miklos Abert fa21

Xianzhi

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Problem 1

If G_n are finite, connected, d-regular, $|G_n| \implies \infty, \implies \liminf \rho_0(G_n) \geq \rho(T_d) = \|A_{T_d}\| = 2\sqrt{d-1}$.
T_d is the infinite d-regular tree.
Let $\omega_{G_n} = \#$ of closed walks of length $2k$ on G_n at a vertex v.
$\omega_{T_d} = \#$ of closed walks of length $2k$ starting at the root of T_d.
$\omega_{G_n} \geq \omega_{T_d}$ since given a walk on T_d, we could have the same walk on G, but G may have cycles that creates more walks.
$\omega_{T_d} = C_k(d-1)^k$ where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.
We use C_k since valid walks cannot have more "return" steps than "away" steps. $(d-1)^k$ is because at each step, there is $d-1$ possible ways to go on, and we go away k steps. C_k has asymptotic $\frac{4^k}{k^{\frac{3}{2}}}$.
so ω_{T_d} has asymptotic

$$\frac{4^k}{k^{3/2}}(d-1)^k = \frac{1}{k^{3/2}}2^{2k}\left(\sqrt{d-1}\right)^{2k} = k^{-3/2}\left(2\sqrt{d-1}\right)^{2k} \tag{1}$$

Now

$$d^{2k} + (n-1)\rho_0^{2k} \geq \sum_{i=0}^{n-1} \lambda_i^{2k} = n\omega_{G_n} \geq n \cdot \omega_{T_d} \tag{2}$$

$$(n-1)\rho_0^{2k} \geq n \cdot C_k(d-1)^k - d^{2k} \tag{3}$$

$$n\rho_0^{2k} \geq n \cdot C_k(d-1)^k - d^{2k} \tag{4}$$

$$\rho_0^{2k} \geq C_k(d-1)^k - \frac{d^{2k}}{n} \tag{5}$$

$$\rho_0 \geq 2\sqrt{d-1}, \text{ as } n \mapsto \infty \tag{6}$$

The i,j-th entry of A^k is the number of path of length k from vertex i to vertex j,

$$\sum_{i=0}^{n-1} \lambda_i^{2k} = trace\left(A^{2k}\right) = n \cdot \omega_{G_n} \tag{7}$$

there are ω_{G_n} at a specific vertex v, and there are n vertices in G_n.