Galois Theory: GAL #06

Due on Apr 01, 2022 at 11:59pm

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HW06
Apr 01, 2022
Exercise 9.4.1
Exercise 9.4.2
Exercise 9.4.3

Problem 1

Exercise 9.4.1 Let L be the splitting field over $\mathbb{Q}$ of a cubic polynomial with rational coefficients, and ω a primitive cubic root of unity. Show that $L(\omega)$ is a radical extension of $\mathbb{Q}$, by exhibiting explicitly a radical sequence.
(Hint: recall Cardano's Method.)

Soln:

Let L be splitting field over $\mathbb{Q}$ of a cubic polynomial with radical coefficients $aX^3+bX^2+cX+d, a, b, c, d \in \mathbb{Q}$. WLOG, L is the same splitting field if the polynomial is monic

$$X^3 + \frac{b}{a}X^2 + \frac{c}{a}X + \frac{d}{a}, \tag{1}$$

so we could assume $a = 1$ from the beginning.
Also, L is the same if we shift by a rational amount $b/3$ of all the roots of this polynomial, because

Problem 2

Exercise 9.4.2 Let L be the splitting field over $\mathbb{Q}$ of a monic irreducible cubic polynomial f in $\mathbb{Q}[x]$.

1. Show that $\Gamma(L : \mathbb{Q})$ has order 3 iff the discriminant of f is the square of a rational number. Recall that the discriminant of f is

$$\prod_{1 \leq i < j \leq 3} (\alpha_i - \alpha_j)^2, \tag{2}$$

   where α_i are the complex roots of f.

2. Give an example of a monic cubic polynomial f with $|\Gamma(L : \mathbb{Q})| = 3$.
   You may want to use the fact that the discriminant of $X^3 + pX + q \in \mathbb{Q}[X]$ *is* $-4p^3 - 27q^2$.

Soln:
Part A

Part B

Problem 3

Exercise 9.4.3 Let L be a subfield of $\mathbb{C}$ such that $\Gamma(L)$ is the dihedral group D_4 (having 8 elements), and L a Galois extension of $\mathbb{Q}$. Show that L is a radical extension of $\mathbb{Q}$.

Proof. We know degree 2 extension is obtained by adjoining an square root from previous homework. Since $\Gamma(L:\mathbb{Q})$ is $\cong D_4$, and is Galois, we can use Galois Correspondence. Normal subgroup corresponds to normal (Galois) extension.

$$\begin{array}{ccccccc} \{e\} & \lhd & \langle t\rangle & \lhd & \langle t, f^2\rangle & \lhd & D_4 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ L & \supset & E & \supset & F & \supset & \mathbb{Q} \end{array} \tag{3}$$

Where the up-down arrow $\updownarrow$ indicates the relationship being the field is fixed field of the group. Each of the subgroup has index 2 in the previous one, and it's a subnormal chain. So we have

$$\Gamma(F:\mathbb{Q}) \cong \Gamma(L:\mathbb{Q})/\Gamma(L:F) = |D_4/\langle t, f^2\rangle| = 2 \tag{4}$$

So $\Gamma(F:\mathbb{Q})$ has order 2, so $F:\mathbb{Q}$ has degree 2. So $F = \mathbb{Q}(\alpha)$ where $\alpha^2 \in \mathbb{Q}$. Similarly,

$$\Gamma(E:F) \cong \Gamma(L:F)/\Gamma(L:E) = |\langle t, f^2\rangle/\langle t\rangle| = 2 \tag{5}$$

So $|\Gamma(E:F)| = 2$ implies that $E:F$ has degree 2, so $E = F(\beta)$, where $\beta^2 \in F$.
$\Gamma(L:E) \cong \langle t\rangle$, $|\langle t\rangle| = 2$ implies that $L:E$ has degree 2.
$\implies$ $L = E(\gamma)$, where $\gamma^2 \in E$.
$\implies$ $L = \mathbb{Q}(\alpha, \beta, \gamma)$, so L is a radical extension.

□

Problem 6

Evaluate the integrals $\int_0^1 (1 - x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.