Galois Theory: GAL Final

Due on May, 2022 at 11:59pm

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Final

Problem 1

$$Gal_{\mathbb{Q}}(h) = \Gamma(L : \mathbb{Q}) \tag{1}$$

where L is the splitting field of h over $\mathbb{Q}$.
We try to find the roots of $h = f \circ g$.
Let $Y = X^2 + X + 2$. so $h = Y^3 - Y + 7$. This is a cubic, so we know it's solvable by radicals, thus, we could obtain a radical sequence, using Cardano's formula, say, $a_1, a_2, \ldots, a_n$, so $Y^3 - Y + 7$ splits as root factors in

$$\mathbb{Q}(a_1, a_2, \ldots, a_n) \tag{2}$$

then $X^2 + X + 2$ has roots

$$\alpha_1, \alpha_2 = \frac{-1 \pm \sqrt{7}i}{2} \tag{3}$$

so we adjoin $\sqrt{7}, i$ to $\mathbb{Q}(a_1, a_2, \ldots, a_n)$

$$L := \mathbb{Q}(a_1, a_2, \ldots, a_n, \sqrt{7}, i) \tag{4}$$

is clearly a radical sequence. and h splits as root factors in L. Thus, apply Galois's Theorem $\implies$ the Galois group $Gal_{\mathbb{Q}}(h)$ is solvable.

Problem 2

$\alpha \in \mathbb{F}_{p^n}$ of p^n elements.
We need to show

$$\alpha + \alpha^p + \ldots + \alpha^{p^{n-1}} \tag{5}$$
$$\alpha^{1+p+p^2+\ldots+p^{n-1}} \tag{6}$$

are fixed by the Frobenious automorphism

$$\Phi : \alpha \to \alpha^p \tag{7}$$

which would imply that all elements in the Galois Group $\Gamma(\mathbb{F}_{p^n} : \mathbb{F}_p)$ fix those 2 elements, so those 2 elements are in the prime field. (Any finite extension of finite field is Galois.)

$$\left(\alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{p^{n-1}}\right)^p \tag{8}$$
$$= \alpha^p + \alpha^{p^2} + \alpha^{p^3} + \ldots + \alpha^{p^{n-1}} + \alpha^{p^n-1} \cdot \alpha \tag{9}$$
$$= \alpha + \alpha^p + \ldots + \alpha^{p^{n-1}} \tag{10}$$
$$\text{where } \alpha^{p^n-1} = 1 \ \forall \ \alpha \neq 0, \ \alpha \in \mathbb{F}_{p^n}^\times \tag{11}$$

Thus this element is indeed fixed by Φ, so in $\mathbb{F}_p$.

$$\left(\alpha^{1+p+p^2+\ldots+p^{n-1}}\right)^p \tag{12}$$
$$= \alpha^{p+p^2+p^3+\ldots+p^{n-1}+p^n} \tag{13}$$
$$= \alpha^{p+p^2+p^3+\ldots+p^{n-1}+1} \cdot \alpha^{p^n-1} \tag{14}$$
$$= \alpha^{1+p+\ldots+p^{n-1}} \tag{15}$$

Thus, this element is also fixed by Φ so it's in the $\mathbb{F}_p$.