Adv Abs Alg: AAA #HW05

Due on 2022 at 11:59PM

Prof. Peter Hermann Spr22

Xianzhi

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Homework Set 5

Problem 1

Prove that none of $(\mathbb{Q}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$ is finitely generated.
Solution:
Assume for contradiction

$$G := (\mathbb{Q} \setminus \{0\}, \cdot) \tag{1}$$

is finitely generated. Then

$$G = \langle \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n} \rangle \tag{2}$$

where r_i and s_i are coprime.
Now, take prime number $p > \max\{r_i, s_j \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ since there are infinitely many primes numbers, we can take such p, then $\frac{1}{p}$ cannot be expressed using the generators. Because: A general element of G is of the form:

$$\frac{r_1^{i_1} \ldots r_n^{i_n}}{s_1^{i_1} \ldots s_n^{i_n}} \tag{3}$$

and if

$$\frac{1}{p} = \frac{r_1^{i_1} \ldots r_n^{i_n}}{s_1^{i_1} \ldots s_n^{i_n}}, \tag{4}$$

$$p = \frac{s_1^{i_1} \ldots s_n^{i_n}}{r_1^{i_1} \ldots r_n^{i_n}} \tag{5}$$

then it contradicts p is prime.

Assume for contradiction $G := (\mathbb{Q}, +)$ is finitely generated, then $G = \langle \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots \frac{r_n}{s_n} \rangle$ where r_i and s_i are coprime.
Again we take a prime number $p > \max\{r_i, s_j \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ and $\frac{1}{p}$ cannot be expressed using generators. Because: a general element is of the form $(k_1, k_2 \in \mathbb{Z})$.

$$\frac{k_2}{k_1 \cdot s_1 \cdot s_2 \ldots s_n} \tag{6}$$

so we have a contradiction if

$$\frac{1}{p} = \frac{k_2}{k_1 \cdot s_1 \cdot s_2 \ldots s_n} \tag{7}$$

we could assume k_2 and $k_1 \cdot s_1 \ldots s_n$ are coprime. Then $p = k_1 \cdot s_1 \cdot s_2 \ldots s_n$ which is a contradiction, since p cannot have strictly smaller prime factors.

Problem 2

Let A be abelian.

1. Prove that A is finitely generated if and only if there exist finitely many subgroups A_i such that
$$A = A_0 \geq A_1 \geq A_2 \geq \ldots \geq A_n \geq A_{n+1} = 1 \tag{8}$$
$$\tag{9}$$
and all the factor groups A_i/A_{i+1} are cyclic.

2. Let $B \leq A$ and assume that A is finitely generated. Show that B is finitely generated.

Solution:
Part A
" $\Longrightarrow$ "
A is finitely generated, so
$$A = \langle a_1, a_2, \ldots, a_k \rangle \tag{10}$$
let
$$A = A_0 = \langle a_1, \ldots, a_k \rangle \tag{11}$$
$$A_1 = \langle a_1, \ldots, a_{k-1} \rangle \tag{12}$$
$$\vdots \tag{13}$$
$$A_{k-1} = \langle a_1 \rangle \tag{14}$$
$$A_k = \{e\} \tag{15}$$
so this shows existance.
We define φ from the cyclic group $\langle a_k \rangle$ to factor group:
$$\langle a_k \rangle \xrightarrow{\varphi} A_0/A_1 \tag{16}$$
$$a_k^n \xrightarrow{\varphi} A_1 \cdot a_k^n \tag{17}$$
$$\tag{18}$$
In general
$$\langle a_{k-i} \rangle \xrightarrow{\varphi} A_i/A_{i+1} \tag{19}$$
$$(a_{k-i})^n \rightarrow A_{i+1} \cdot (a_{k-i})^n \tag{20}$$
Since A is abelian, all subgroup is normal.
This is well defined, since if we take an element in $A_0/A_1 : A_1 \cdot x$, where $x \in A_0$, then
$$x = a_1^{n_1} \cdot a_2^{n_2} \cdot \ldots \cdot a_k^{n_k} \tag{21}$$
so
$$A_1 \cdot x = A_1 a_k^{n_k} \tag{22}$$
so coset representative is indeed of the form a_k raised to some power. This also shows φ is onto. So $Im\varphi = A_0/A_1$.
φ is homomorphism:
$$\varphi(a_k^n)\varphi(a_k^m) = (A_1 \cdot a_k^n)(A_1 \cdot a_k^m) \tag{23}$$
$$= A_1 \cdot a_k^{n+m} \tag{24}$$
$$\varphi(a_k^n \cdot a_k^m) = A_1 a_k^{n+m} \tag{25}$$

Problem 2 continued on next page. . .

Thus, since image of a cyclic group under homomorphism is cyclic, we conclude A_0/A_1 is cyclic.
The same proof shows A_i/A_{i+1} is cyclic.

Part B

"$\Leftarrow$"

$$\{e\} = A_k \leq A_{k-1} \leq \ldots \leq A_1 \leq A_0 = A \tag{26}$$

since all A_i/A_{i+1} are cyclic, let a_{i+1} denote some generator

$$A_i/A_{i+1} = \langle A_{i+1} \cdot a_{i+1} \rangle \tag{27}$$

so an arbitrary coset is of the form

$$A_{i+1} \cdot a_{i+1}^n \tag{28}$$

let $x \in A$. Then x is in some A_1 coset.

$$x \in A_1 \cdot a_1^{n_1} \tag{29}$$

Thus $x = x_1 \cdot a_1^{n_1}$ for some $x_1 \in A_1$. Again

$$x_1 \in A_2 \cdot a_2^{n_2} \tag{30}$$

implies $x_1 = x_2 \cdot a_2^{n_2}$ for some $x_2 \in A_2$ and so on.
Thus, $x \in A_k \cdot a_{k-1}^{n_{k-1}} \cdot \ldots \cdot a_1^{n_1}$ but $A_k = \{e\}$, so

$$x = a_{k-1}^{n_{k-1}} \cdot \ldots \cdot a_1^{n_1} \tag{31}$$

so A is finitely generated. And we are done.

Part C

A is finitely generated $\implies$

$$\exists A = A_0 \geq \ldots \geq A_k = \{e\} \tag{32}$$

and A_i/A_{i+1} is cyclic.
Let $B \leq A$. If $B = A$ or $B = \{e\}$, then we are done. So assume B is a nontrivial proper subgroup. Let

$$B_0 = B \cap A_0 \tag{33}$$
$$B_1 = B \cap A_1 \tag{34}$$
$$B_i = B \cap A_i \tag{35}$$

in general. Thus, B_i are chain of subgroups

$$B = B_0 \geq B_1 \geq \ldots \geq B_k = \{e\} \tag{36}$$

If we show B_i/B_{i+1} is cyclic, we are done. To simplify notation, we consider B_1/B_2. Write

$$\frac{(B \cap A_1)}{(B \cap A_2)} = \frac{(B \cap A_1)}{\underbrace{A_2}_{N} \cap \underbrace{(B \cap A_1)}_{H}} \tag{37}$$

where the equal sign is because $A_2 \subseteq A_1$. Recall iso thm (we can use it since every subgroup is normal)

$$\frac{H}{H \cap N} \cong \frac{H \cdot N}{N} \tag{38}$$

Thus

$$B_1/B_2 = \frac{\underbrace{B \cap A_1}_{H}}{\underbrace{A_2}_{N} \cap \underbrace{B \cap A_1}_{H}} \cong \frac{\underbrace{B \cap A_1}_{H} \cdot \underbrace{A_2}_{N}}{A_2} \tag{39}$$

since $M := (B \cap A_1) \cdot A_2 \subseteq A_1$, and M is a subgroup, and $A_2 \leq M$, implies M is a subgroup of A_1,

$$\implies M/A_2 \leq A_1/A_2 \tag{40}$$

since subgroup of cyclic group is cyclic, M/A_2 is cyclic.
Hence, B_1/B_2 is cyclic. B_i/B_{i+1} is exactly the same.

Problem 3

Let $n \geq 2$ be an integer and $A = C_1 \times C_2 \times \ldots \times C_n$, the direct product of n infinite cyclic groups. Show that (for any $a \in A$) $|A : \langle a \rangle|$ is infinite.
This completes the proof of the following theorem: $C \leq G$, C cyclic, $|G : C|$ finite, G torsion-free $\implies$ G is cyclic.

Solution:

Let

$$A = C_1 \times C_2 \times \ldots \times C_n \tag{41}$$

where C_i are infinite cyclic groups.
then $A \cong \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z} = \mathbb{Z}^n$
take $a \in A$.
Then a could be identified as

$$(a_1, a_2, \ldots, a_n) \text{ for } a_i \in \mathbb{Z} \tag{42}$$

using additive notation, then

$$\begin{align} \langle a \rangle &= \{k \cdot a \mid k \in \mathbb{Z}\} \tag{43} \\ &= \{(ka_1, ka_2, \ldots, ka_n) \mid k \in \mathbb{Z}\} \tag{44} \end{align}$$

we could see this has infinite index in $\mathbb{Z}^d$, since it is just a "line" in the integer lattice $\mathbb{Z}^d$, and we could have infinitely shifted copy of it, which are cosets.

$$(z_1, z_2, \ldots, z_n) + (ka_1, ka_2, \ldots, ka_n) \tag{45}$$

for $z_i \in \mathbb{Z},\ k \in \mathbb{Z}$. Thus, we have infinitely many distinctive cosets, so $\langle a \rangle$ has infinite index.

Problem 4

For fun: Prove the following characterization of the infinite cyclic group.
Let G be an infinite group; then
G is cyclic $\iff$ $|G : H|$ is finite for all subgroups $1 \neq H \leq G$.

Solution:

G is infinite group.
$\Rightarrow$
Assume G cyclic. Then $G \cong \mathbb{Z}$.
Let $H \neq e$ be a subgroup of G, since subgroup of cyclic group is cyclic, H is cyclic.
Then by the same isomorphism that $G \xrightarrow{\cong} \mathbb{Z}$, H is isomorphic to $m\mathbb{Z}$ for some $m \in \mathbb{N}$.
since cyclic groups in $\mathbb{Z}$ are of the form $m\mathbb{Z}$.
so since $|\mathbb{Z} : m\mathbb{Z}| = m < \infty$
H also has finite index.
$\Leftarrow$
Assume G is not torsion-free, so $\exists 1 \neq h \in G$, such that h has order n for some $n \in \mathbb{N}$.
consider $\langle h \rangle \leq G$.
the order of the cyclic group $\langle h \rangle$ is also n.
and by assumption $|G : \langle h \rangle| = m < \infty$, but

$$\frac{|G|}{|\langle h \rangle|} = |G : \langle h \rangle| = m \tag{46}$$

so $|G| = n \cdot m < \infty$. Contradiction.
so G is torsion free.
Now, let $1 \neq g \in G$. Consider $\langle g \rangle$. by assumption, $|G : \langle g \rangle|$ is finite.
$\langle g \rangle \leq G$, $\langle g \rangle$ cyclic, G torsion free all satisfied, $\implies$ G is cyclic by theorem in class.