Adv Abstract Algebra hw 1

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1 HW 1

exercise 1(a) First we prove AB is a subgroup of $G \implies BA = AB$.
Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be arbitrary. We show that for $b_1a_2 \in BA$, b_1a_2 is in AB. Since AB is a subgroup of G, it is closed under binary operation, so there exists $a_3 \in A$ and $b_3 \in B$ such that

$$a_1b_1a_2b_2 = a_3b_3 \tag{1}$$
$$a_1b_1a_2 = a_3b_3b_2^{-1} \tag{2}$$
$$b_1a_2 = a_1^{-1}a_3b_3b_2^{-1} \tag{3}$$

so $b_1a_2 \in AB$. Thus, $BA \subseteq AB$.

Now, since AB is a subgroup, it's closed under taking inverse. Let $a_1b_1 \in AB$, there exists $a_2b_2 \in AB$ such that

$$a_1b_1a_2b_2 = e \tag{4}$$
$$a_1b_1 = b_2^{-1}a_2^{-1} \in BA, \tag{5}$$

since A, B are themselves subgroups. So $AB \subseteq BA$. Thus, $AB = BA$.

Now, we prove $BA = AB \implies AB$ is a subgroup of G.

Assume $BA = AB$. Again let $a_1, a_2, a_3, \cdots \in A$ and $b_1, b_2, b_3, \cdots \in B$. We have

$$a_1b_1a_2b_2 = a_1b_1b_3a_3 \text{ for some} b_3, a_3 \tag{6}$$
$$= a_1(b_1b_3)a_3 \tag{7}$$
$$= b_4a_4a_3 \tag{8}$$
$$= b_4(a_4a_3) \tag{9}$$
$$= a_5b_5 \tag{10}$$

Thus, subset AB is closed under the binary group operation.

Since $AB = BA$, for a_1b_1 we have $b_1^{-1}a_1^{-1} = a_2b_2$ for some a_2, b_2. Thus,

$$a_1b_1a_2b_2 = e \tag{11}$$
$$a_2b_2a_1b_1 = e \tag{12}$$

Also, $e = ee \in G$ is also the identity in AB. Associativity follows from the associativity of G. Thus, AB is a subgroup of G.