Adv Abstract Algebra: AAA #HW03

Due on 2022 at 11:59PM

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Homework set 3

Problem 1

Suppose that a group G has order 312. Prove that G has a proper normal subgroup.

Solution:

$312 = 2^3 \times 3 \times 13$.

The number of Sylow $p = 2$ subgroup, n, has several possibilities

$$1, 3, 13, 39 \equiv \mod p = 2 \tag{1}$$

and they divide $m = 3 \times 13$.

The number of Sylow $p = 3$ subgroup, n has several possibilities.

$$1, 4, 13 \equiv 1 \mod p = 3 \tag{2}$$
$$\tag{3}$$

and they divide $m = 8 \cdot 13$.

However, the number of Sylow $p = 13$ subgroup is one, since 1 is the only number $\equiv 1 \mod p = 13$ and divide $m = 2^3 \cdot 3$ at the same time.

By Sylow's theorem, (and corollary) G has a proper normal subgroup of order 13.

The unique Sylow 13 subgroup.

Problem 2

Suppose that a group G has order 1960. Prove that G has a proper normal subgroup.

Solution:

Suppose a group has order $1960 = 2^3 \cdot 5 \cdot 7^2$, the number of Sylow $p = 2$ subgroup has several possibilites

$$1, 5, 7 \equiv 1 \mod p = 2 \tag{4}$$

and divide $m = 5 \cdot 7 \cdot 7$.
The number of Sylow $p = 5$ subgroup has several possibilities, for example, $1, 56, 196 \equiv 1 \mod p = 5$ and divide $m = 2^3 \cdot 7^2$.
But the number n of Sylow $p = 7$ subgroup has 2 possibilities

$$1, 8 \equiv 1 \mod p = 7 \tag{5}$$

$$\tag{6}$$

and divide $m = 2^3 \cdot 5 = 40$.
Suppose $n = 8$. (If $n = 1$, then we are done)
Since G acts on $Syl_7(G)$ by conjugation and the action is transitive, G is essentially permuting the $8Syl_7(G)$ subgroups.
Thus, we could define a homomorphism

$$G \xrightarrow{\psi} S_8 \tag{7}$$

(ψ is indeed a homomorphism because of the definition of group action)
$ker\psi$ cannot be the whole group G, since G acts transitively on the $Syl_7(G)$ groups, and we are assuming there is 8 of them, so ψ cannot map everything in G to the identity permutation.

Assuming $ker\psi = \{e\}, \implies \psi$ is one to one, $|G| = |Im\psi|$ need to divide $|S_8|$ since $Im\psi \leq S_8$.
So $|G|$ need to divide $|S_8|$.
But $|G| = 2^3 \cdot 5 \cdot 7^2$. and $|S_8| = 8!$. Contradiction.

Thus $ker\psi$ is not the whole group G and is not just the identity, so $ker\psi \lhd G$ is the proper normal subgroup we seek.

Problem 3

For $A \leq G$, $|G : A|$ finite and A abelian, let $\tau_{G \mapsto A}$ denote the transfer homomorphism from G to A. Let $g \in G$ and $b \in N_G(A)$. Show that $\tau_{G \mapsto A}(g)$ commutes with b.
Hint: If $h_1, \ldots, h_n$ is a set of right coset representatives of A then show that $bh_1, \ldots, bh_n$ is also a set of right coset representatives of A.

Solution:

Let $G = \dot{\coprod}_{i=1}^{n} Ah_i$ and $b \in N_G(A) = \{g \in G : gA = Ag\}$,
then $b^{-1}Ab = A \implies Ab = bA$.
Thus

$$G = bG = b\dot{\coprod}_{i=1}^{n} Ah_i \tag{8}$$
$$= \dot{\coprod}_{i=1}^{n} bAh_i \tag{9}$$
$$= \dot{\coprod}_{i=1}^{n} Abh_i \tag{10}$$
$$\tag{11}$$

since $Ab = bA$ set wise, not element wise.
so if h_i's are coset representatives, then bh_i's are also coset representatives.
Now, let $g \in G$, we have

$$Ah_i \cdot g = Ah_{i^g} \tag{12}$$
$$\implies \exists a_{i,g} \ \forall i \text{ such that} \tag{13}$$
$$h_i \cdot g = a_{i,g} \cdot h_{i^g} \tag{14}$$
$$h_i \cdot g \cdot h_{i^g} = a_{i,g} \tag{15}$$

similarly,

$$Ab \cdot h_i \cdot g = Ab \cdot h_{i^g} \implies \exists a^*_{i,g} \ \forall i \text{ such that} \tag{16}$$
$$b \cdot h_i \cdot g = a^*_{i,g} \cdot b \cdot h_{i^g} \tag{17}$$
$$b \cdot \underbrace{h_i \cdot g \cdot h_{i^g}}_{a_{i,g}} \cdot b^{-1} = a^*_{i,g} \in A \tag{18}$$
$$ba_{i,g}b^{-1} = a^*_{i,g} \ \forall \ i \in \{1, \ldots, n\} \tag{19}$$
$$\implies a_{i,g} = b^{-1}a^*_{i,g}b \ \forall \ i \tag{20}$$

$\{h_i\}, \{bh_i\}$ are both coset representations
the $\tau(g)$ is independent of the representatives we pick, so $\tau(g) = \prod_{i=1}^{n} a_{i,g} = \prod_{i=1}^{n} a^*_{i,g}$. Hence,

$$\tau(g) = \prod_{i=1}^{n} a_{i,g} \tag{21}$$
$$= \prod_{i=1}^{n} \left(b^{-1}a^*_{i,g}b\right) \tag{22}$$
$$= b^{-1}\left(\prod_{i=1}^{n} a^*_{i,g}\right) b \tag{23}$$
$$= b^{-1}\left(\tau(g)\right) b \tag{24}$$

so $b\tau(g) = \tau(g)b$. Thus commute.