Galois Theory: GAL #07

Due on Apr 08, 2022 at 11:59pm

Prof Matyas Domokos Section 11

Xianzhi

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HW07
Apr 08, 2022
Exercise 11.4.2
Exercise 11.4.6
Exercise 11.4.8

Problem 1

Exercise 11.4.2 Show that $f \in K[X]$ (where K is a subfield of $\mathbb{C}$) has a root in a radical extension of K $\iff$ f has an irreducible factor p in $K[X]$ such that $Gal_K(p)$ is solvable.

Soln:

"$\Leftarrow$"

Assume f has irreducible factor p such that $Gal_K(p)$ is solvable. Then apply Galois's theorem, $\exists$ a radical extension L of K contain all roots of p, so L must contain at least one root of p, call it α. Since p is a factor of f, α is also a root of f. Thus, f has a root α in radical extension L.

"$\Rightarrow$"

Assume $f \in K[X]$ has a root α in a radical extension L of K. Thus, $L = K(\beta_1, \beta_2, \cdots, \beta_m)$ with $\beta_1, \cdots, \beta_m$ a radical sequence.

Let $p = m_K^\alpha$, then p is automatically an irreducible factor of f, since α is a root of f.

We want to show all the roots of p are in some radical extension of K, but the radical extension we have, L, is not normal, so we modify it. Since

$$\beta_i^{n_i} \in K(\beta_1, \ldots, \beta_{i-1}) \tag{1}$$

The sequence β_i has corresponding sequence $n_i \in \mathbb{N}$, let

$$L' = K(\beta_1, \zeta_{n_1}, \beta_2, \zeta_{n_2}, \ldots, \beta_m, \zeta_{n_m}) \tag{2}$$

where ζ_{n_i} is a primitive n_ith root of unity.

Since we obtain L' by adjoin β_i and ζ_{n_i}, we join all the roots of the polynomial

$$X^{n_i} - \beta_i^{n_i} \tag{3}$$

at each step, so at each step we obtain a splitting field, and since we are in $\mathbb{C}$, L' is a splitting field, hence a normal extension of K, thus, since $\alpha \in L \subset L'$, all the roots of $p = m_K^\alpha$ are in L', so $\exists$ radical extension L' containing all the roots of $p \implies Gal_K(p)$ is solvable by Galois's thm.

Comment from instructor: The way you try to extend L to get a normal extension of K is not correct, see the following exercise: a normal extension of a normal extension is not necessarily normal. Instead, we gave a proof in class that the normal closure of a radical extension is normal.

Problem 2

Exercise 11.4.6 Suppose that $L : K$ and $M : L$ are normal extensions. Does it follow that $M : K$ is a normal extension?

Soln:

We observe that degree 2 extensions are normal, since by a previous exercise, degree 2 extension is obtained by adding "a square root," so we add the other root too.

$$L = K(\alpha), \alpha \notin K, \alpha^2 \in K, \alpha \in L. \tag{4}$$
$$[L : K] = 2 \tag{5}$$

Take irreducible polynomial that has α as a root, assume β is another root.

$$X^2 + aX + b \quad a, b \in K \tag{6}$$

then

$$\alpha + \beta = -a \text{ by Vieta's theorem} \tag{7}$$
$$\alpha\beta = b \tag{8}$$

so if $\alpha \in L$, then $\beta = -a - \alpha \in L$.

Thus consider

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[2]{3}) \subseteq \mathbb{Q}(\sqrt[4]{3}) \tag{9}$$

and we claim that both extensions are degree 2 extension, and thus normal.

$$[\mathbb{Q}(\sqrt[2]{3}) : \mathbb{Q}] = 2 \text{ since } m_{\mathbb{Q}}^{\sqrt[2]{3}} = X^2 - 3 \tag{10}$$
$$[\mathbb{Q}(\sqrt[4]{3}) : \mathbb{Q}] = 4 \text{ since } m_{\mathbb{Q}}^{\sqrt[4]{3}} = X^4 - 3 \tag{11}$$
$$\tag{12}$$

and both polynomial are irreducible by Eisenstein. Now, by Tower Law,

$$[\mathbb{Q}(\sqrt[4]{3}) : \mathbb{Q}(\sqrt[2]{3})] = 2 \tag{13}$$

But $\mathbb{Q}(\sqrt[4]{3}) : \mathbb{Q}$ is not normal. $X^4 - 3$ has (non-real) complex roots $i\sqrt[4]{3}, -i\sqrt[4]{3}$ not in $\mathbb{Q}(\sqrt[4]{3})$. And we are done.

Problem 3

Exercise 11.4.8 Find a degree 6 irreducible polynomial $f \in \mathbb{Q}[X]$ whose Galois group is isomorphic to S_3.

Soln:

X^6+3 is a degree 6 irreducible polynomial $f \in \mathbb{Q}[X]$ Let L be splitting field of X^6+3 over $\mathbb{Q}$, then

$$\Gamma(L:\mathbb{Q}) \cong S_3 \tag{14}$$

We claim

$$L = \mathbb{Q}(\sqrt[6]{-3}, \zeta), \tag{15}$$

where ζ is a primitive 6th root of unity.

$$\zeta = \frac{1}{2} + \frac{\sqrt{3}}{2} i \tag{16}$$

since $(\sqrt[6]{-3})^3 = \sqrt[2]{-3} = i\sqrt[2]{3}$ and

$$\zeta = \frac{1}{2} + \frac{1}{2}(\sqrt[6]{-3})^3 \in \mathbb{Q}(\sqrt[6]{-3}) \tag{17}$$

Thus, $L = \mathbb{Q}(\sqrt[6]{-3})$.
$[L:\mathbb{Q}] = 6$ since $m_{\mathbb{Q}}\sqrt[6]{-3} = X^6+3$ is irreducible by Eisenstein. Then, since L is splitting field over $\mathbb{Q}$, $L:\mathbb{Q}$ is Galois extension. so $|\Gamma(L:\mathbb{Q})| = 6$.
Let $a := \sqrt[6]{-3}$
Then $\phi \in \Gamma(L:\mathbb{Q})$ need to take a to some other root $\zeta^{k_\phi} a$ for $0 \le k_\phi \le 5$.
Up to isomorphism, there are only 2 group of order 6, $\mathbb{Z}_6$ and $D_3 \cong S_3$.
Thus, suffice to show $\Gamma(L:\mathbb{Q})$ is not abelian.
Suffice to show $\Gamma(L:\mathbb{Q})$ has a subgroup that is not normal.

Consider

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{-3}) \subseteq \mathbb{Q}(\sqrt[6]{-3}) \tag{18}$$

$$[\mathbb{Q}(\sqrt[3]{-3}):\mathbb{Q}] = 3 \text{ since } m_{\mathbb{Q}}\sqrt[3]{-3} = X^3+3 \tag{19}$$

is irreducible by Eisenstein. But $\mathbb{Q}(\sqrt[3]{-3})$ is not normal over $\mathbb{Q}$, since X^3+3 has roots $\omega \cdot \sqrt[3]{-3}$ and $\omega^2 \cdot \sqrt[3]{-3}$ for $\omega = (-1+i\sqrt{3})/2$.

$$\omega \cdot \sqrt[3]{-3} \notin \mathbb{Q}(\sqrt[3]{-3}) \iff \omega \notin \mathbb{Q}(\sqrt[3]{-3}) \tag{20}$$

since $[\mathbb{Q}(\omega):\mathbb{Q}] = 2$, and ω is root of X^2+X+1.
$2 \nmid 3$ so $\omega \notin \mathbb{Q}(\sqrt[3]{-3})$
$\implies \mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{-3})$ not normal, by Galois correspondence,

$$L := \mathbb{Q}(\sqrt[6]{-3}) \tag{21}$$

$$\Gamma(L:\mathbb{Q}(\sqrt[3]{-3})) \text{ not normal in } \Gamma(L:\mathbb{Q}) \tag{22}$$

so $\Gamma(L:\mathbb{Q}) \cong S_3$. And we are done.