Sheet 3: The Adjacency matrix

Xianzhi Wang

fall 2021

All graphs are finite on this sheet. Let G be a directed graph on the vertex set $\{1, \ldots, n\}$. Let us define the *adjacency matrix* $A = \mathrm{Adj}(G)$ by setting

$$A_{i,j} = \text{number of edges from } i \text{ to } j \text{ in } G.$$

So, we allow multiple edges and even loops in G.

Exercise 1 *Express the following in linear algebra terms, using A:*
1) the degrees of a vertex;
2) the number of edges in G;
3) $e(X, Y)$ for $X, Y \subseteq V(G)$.

One of the main reasons why we look at the adjacency (or neighboring) relation as a matrix is the following correspondance between matrix multiplication and walks in G.

Definition 2 *A* directed walk of length n *in G is a sequence of directed edges $e_1, \ldots, e_n$ such that $e_i^+ = e_{i+1}^-$ ($1 \le i \le n-1$). The walk is a* loop (or returning), *if $e_1^- = e_n^+$.*

Note that we redefine the notion of walk here: it is a sequence of edges rather than vertices.

Theorem 3 *For every $k > 0$, $(A^k)_{i,j}$ equals the number of directed walks of length k from i to j.*

Proof. Let's prove by induction.
When $k = 1$, $(A)_{ij}$ is indeed the number of directed walks of length 1, (which is just directed edge) from i to j by definition.
Assume true for $k - 1$.
Denote the ijth entry of A by a_{ij} and ijth entry of A^{k-1} by b_{ij}

$$(A^k)_{i,j} = (A \cdot A^{k-1})_{i,j} = \sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha j} \tag{1}$$

Thus, for fixed α, we multiply together $a_{i\alpha} b_{\alpha j}$, which means multiply number of directed walks of length 1 (which are directed edges) from i to α and the number of directed walks of length $k-1$ from α to j. After this, we sum over α, which run through 1 to n, and the result $\sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha j}$ is indeed number of directed walks of length k.
Thus, we have proved the theorem using induction. ■

Corollary 4 *For every $k > 0$, the trace $\mathrm{tr}(A^k)$ equals the number of loops of length k in G.*

Proof. Since $(A^k)_{ii}$ equals the number of directed walks of length k from i to i, (i.e., a loop), we could sum over the n entries in the diagonal to obtain all the loops of length k in G. This is exactly $tr(A^k)$.

■

Now assume that G is undirected.

This turns A to be a symmetric real matrix. Using the spectral theorem, it follows that A admits an orthonormal eigenbasis $b_0(G), b_1(G), \ldots, b_{n-1}(G)$ with real eigenvalues $\lambda_0(G) \geq \lambda_1(G) \geq \ldots \geq \lambda_{n-1}(G)$. That is, we have

$$Ab_i = \lambda_i b_i \ \ (0 \leq i < n).$$

Note that the λ_i are well defined, but the b_i are not. Also:

Lemma 5 *The eigenvalues are graph invariants, that is, isomorphic graphs have the same eigenvalues.*

Proof. Isomorphic graphs have the same eigenvalues. Since Isomorphic graphs are structurally the same, they have the exact same adjacency matrix, thus the same eigenvalues. ■

Exercise 6 *Compute λ_i and b_i for the triangle.*

One way to visualize the adjacency matrix as an operator is as follows. Write real numbers on the vertices of G, call this function f. Now A acts by taking all neighbors of the vertex x, add up the f-values there and write it to the position x. This will be the value of Af at x:

$$(Af)(x) = \sum_{(x,y)\in E(G)} f(y).$$

Actually this is how we will *define* the adjacency operator for infinite graphs. Using this image, one can prove.

Theorem 7 *Let G be an undirected graph with maximal degree d. Then $|\lambda_i(G)| \leq d$ ($0 \leq i < n$). When G is d-regular, we have $\lambda_0(G) = d$.*

Hint: take an eigenvector. Find a particular vertex for it..

Proof. Let G be undirected graph with max degree d.
First we show $|\lambda_k| := |\lambda_k(G)| \leq d$ for $k \in \{0, \ldots, n-1\}$. Let λ_k be an eigenvalue of adjacency matrix A_G, and let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

be a corresponding eigenvector. Choose x_i the max component $\{x_1, \ldots, x_n\}$. WLOG $x_i > 0$. We have

$$\begin{aligned} |\lambda_k x_i| &= A \cdot x & (3)\\ &= |\sum_{j=1}^{n} a_{ij} x_j| & (4)\\ &\leq |\sum_{j=1}^{n} a_{ij} x_i| & (5)\\ &\leq |dx_i| & (6)\\ &= |x_i| d & (7) \end{aligned}$$

since summing over row gives the degree, which is less than the degree bound. Thus, we have $|\lambda_k||x_i| \leq |x_i|d$ so $|\lambda_k| \leq d$ as desired.
Now, when G is d-regular $\implies \lambda_0(G) = d$.
We need an eigenvector that has d as its eigenvalue.
Consider the vector

$$x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{8}$$

with n copies of 1. Since G is d-regular, every row of A_G sum to d. Thus,

$$A_G \cdot x = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} \tag{9}$$

Thus, d is an eigenvalue. Since all eigenvalues are $\leq d$, d is $\lambda_0(G)$, the biggest.

■

Exercise 8 *Assume that G is undirected and connected with maximal degree d. Then $\lambda_0(G) = d$ if and only if G is d-regular.*

Proof. G is d regular $\implies \lambda_0(G) = d$. We already proved this in the previous theorem.
Let G undirected, connected, max degree is d. $\lambda_0(G) = d \implies G$ is d-regular. Let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{10}$$

be an eigenvector for eigenvalue d. Again, WLOG, pick a max component $x_i >$ among the entries $x_i, \dots, x_n$. With $A_G = [a_{ij}]$ we have

$$|dx_i| = |\sum_{j=1}^{n} a_{ij} x_j| \tag{11}$$

$$\leq |\sum_{j=1}^{n} a_{ij} x_i| \tag{12}$$

$$\leq |dx_i| \tag{13}$$

meaning we have equal signs above. so x is a vector of constants of value x_i. Thus, we could rescale to have

$$x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{14}$$

so $\sum_{j=1}^{n} a_{ij} = d$, thus, vertex i, corresponding the row

$$\begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix} \tag{15}$$

has degree d, and since G connected, we could apply

$$x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{16}$$

to other row i to obtain v_i has degree d. Thus, G is d-regular.

■

Lemma 9 *Let G be a d-regular undirected graph. Then the multiplicity of d as an eigenvalue of G equals the number of connected components of G.*

The eigenvalue $-d$ also comes into the picture naturally.

Proof. Let G be d-regular, undirected, graph that has n vertices. $\implies$ multiplicity of d as an eigenvalue of $G = \#$ of connected components of G. Let G have i connected components C_i with $i \in \{1, \ldots, k\}$ for some k. We define vectors X^i corresponding to C_i in this way:

$$X^i_j = \begin{cases} 1 \text{ if } v_j \in C_i \\ 0 \text{ else} \end{cases} \tag{17}$$

$$j \in \{1, \ldots, n\} \tag{18}$$

The adjacency matrix of G, A_G looks like this:

$$A_G \cdot X_2 = \begin{bmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & C_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = d \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \tag{19}$$

since d-regular row sum to d. We could see that X^i are eigenvectors of eigenvalue d. And x^i's are orthogonal to each other. Thus, the multiplicity of d is at least the number of connected components of G.

We want to show that those X^i actually span the eigenbasis of d, so there are no more eigenvectors of d that we didn't already take into account.

So let y be an eigenvector of d, we want to express it as a linear combination of X^i's.

$$i\text{-th row of } A_G \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = i\text{-th row of } d \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \tag{20}$$

We assume $|y_i|$ is the max component in vector y, and $y_i \in C_i$.

$$\sum_{(i,j) \in \text{Edge Set}} y_j = dy_i \tag{21}$$

LHS has d summands, since G is d-regular. Also,

$$|y_j| \leq |y_i| \forall j \tag{22}$$

so in fact all y_j are y_i. So those underlying vertices v_j are all connected to the v_i, so they are in component C_i. Thus, for vector y, the "block" corresponding to C_i could be expressed as a linear combination of X^i, like αX^i.

■

Lemma 10 *Let G be a d-regular undirected, connected graph. Then $\lambda_{n-1} = -d$ if and only if G is bipartite.*

Proof. We know from Ex 8 that since G is d-regular, undirected, connected, d is an eigenvalue. Want To Show: G bipartite $\implies$ $-d$ is also eigenvalue. (Then $\lambda_{n-1} = -d$ since $|\lambda_i(G)| \leq d$ eigenvalue cannot get smaller than $-d$.)
Since G is bipartite, A_G could be expressed as:

$$\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} \tag{23}$$

for some matrix M that is $k \times (n-k)$. Now let

$$\begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{24}$$

be an eigenvector of eigenvalue d.
Therefore, we get

$$d \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} MY \\ M^T X \end{bmatrix} \tag{25}$$

Thus:

$$dX = MY \tag{26}$$

$$dY = M^T X \tag{27}$$

Claim: $-d$ is eigenvalue with eigenvector

$$\begin{bmatrix} X \\ -Y \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ -x_{k+1} \\ \vdots \\ -x_n \end{bmatrix} \tag{28}$$

so we have

$$\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} \begin{bmatrix} X \\ -Y \end{bmatrix} = \begin{bmatrix} M(-Y) \\ M^T X \end{bmatrix} = \begin{bmatrix} -MY \\ M^T X \end{bmatrix} = \begin{bmatrix} -dX \\ dY \end{bmatrix} = \begin{bmatrix} -dX \\ -d(-Y) \end{bmatrix} = -d \begin{bmatrix} X \\ -Y \end{bmatrix} = -d \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ -x_{k+1} \\ \vdots \\ -x_n \end{bmatrix} \tag{29}$$

Thus, $-d$ is eigenvalue as wanted.
$\lambda_{n-1} = -d$, $-d$ is also eigenvalue $\implies$ G bipartite.
■

Exercise 11 *Let G be a d-regular undirected graph. What is the multiplicity of $-d$ as an eigenvalue of G?*

Now we look at some simple examples.

Exercise 12 *Compute the eigenvalues and eigenvectors for the cycle of length n.*

Hint: What are the eigenvalues for the directed cycle of length n?

Exercise 13 *Compute the eigenvalues and an orthonormal eigenbasis for the complete graph on $d+1$ points.*

Hint: What happens to the eigenvalues and eigenvectors of A if you add a scalar matrix to A?