Adv Abstract Algebra: AAA #Final

Due on May 2022 at 11:59PM

Prof. Peter Hermann All Section

Xianzhi

2023

#Final Exam Spr 2022

Problem 1

Assume that $H \leq G,\ |G:H| = 4,\ L \leq G,\ |L| = 77$, show that $L \leq H$.

Solution:

Let G act on the set of right cosets of H, $\mathcal{R}$ by right action

$$G \xrightarrow{\varphi} Sym\mathcal{R} \cong S_4. \tag{1}$$
$$g \to (Ha \to Ha \cdot g) \tag{2}$$

Consider the Image of L under φ. Since $L \leq G$.

$$\implies \varphi(L) \leq S_4 \tag{3}$$
$$\implies |\varphi(L)| \text{ divide } |S_4| = 4! = 24 \tag{4}$$

so for $\ell \in L$, $\varphi(\ell) \mid order(\ell)$, and $order(\ell) \mid |L|$, so $\varphi(\ell) \mid |L|$ which implies $\varphi(\ell) \mid 77$.

$$\frac{|L|}{|ker\varphi|_L|} = |\text{Im of L under}\varphi| \tag{5}$$

which implies

$$\varphi(L) \mid |L| \tag{6}$$
$$\implies \varphi(L) \mid 77 \tag{7}$$
$$\implies |\varphi(\ell)| \mid gcd(77, 24) = 1 \tag{8}$$
$$\implies L \in ker\varphi \tag{9}$$
$$\implies L \in ker\varphi = \cap_{g\in G} gHg^{-1} \leq H \tag{10}$$
$$\implies L \leq H \tag{11}$$

Problem 2

Let $|G| = 3 \cdot 5 \cdot 59$. Prove:

1. $|Z(G)| > 1$.
2. G is abelian.
3. G is cyclic.

Solution:
Part A

onsider Sylow 59 subgroup

$$n_{59} \equiv 1 \mod 59 \tag{12}$$
$$n_{59} \mid 3 \cdot 5 \tag{13}$$

which implies $n_{59} = 1$.
Hence $\exists$ unique Sylow 59 subgroup we call Q,
$\implies Q \triangleleft G$, consider G act on Q by conjugation.

$$G \xrightarrow{\varphi} Aut(Q)\ (q \in Q) \tag{14}$$
$$g \to\ (q \to gqg^{-1}) \tag{15}$$

WTS: $Q \leq Z(G)$
WTS: $ker\varphi = G$ Let $g \in G$ $\varphi(g) \mid Aut(Q) = 58$ since $|Q| = 59$ is prime, Q is cyclic. Also,

$$\frac{|G|}{|ker\varphi|} = |Im\varphi| \tag{16}$$
$$\varphi(g) \text{ divides } |Im\varphi| \text{ which divides } |G| \tag{17}$$
$$\implies \varphi(g) \mid |G| = 3 \cdot 5 \cdot 59 \tag{18}$$

Problem 3

Let $|G| = 2 \cdot 5 \cdot 7 \cdot 79^3$. Show that G is solvable.

Solution:

Problem 4

Let a solvable group G act faithfully and transitively on the set Ω, where $|\Omega| = 35$.

1. Prove that this action is not primitive.
2. Show that if G is abelian then it must be cyclic.

Solution:
Part A

G has a subnormal chain with abelian factor

$$1 = G_n \lhd \ldots \lhd G_1 \lhd G_0 = G \tag{19}$$
$$G \xrightarrow{\varphi} Sym(\Omega) \cong S_{35} \tag{20}$$

$ker\varphi = 1$ since G is faithful,

$$G \cong \frac{G}{ker\varphi} \cong Im\varphi \leq S_{35} \tag{21}$$

We have $G \leq S_{35},\ G \hookrightarrow S_{35},\ |G| \leq 35!$ G finite
G transitive on Ω:
Primitive $\iff$ $G_y <_{max} G,\ y \in \Omega$ Try to show G_y is not max subgroup in $G, y \in \Omega$?

Transitive, Orbit Stabilizer Theorem $\implies$ $|\Omega|$ divide $|G|$, so $35 \mid |G|$, now

$$|\text{orbit of } y| \cdot |G_y| = |G| \tag{22}$$
$$|\Omega| = 5 \cdot 7 \tag{23}$$

there exist minimal normal subgroup $N \lhd G$, *(comment from instructor) suppose: primitive, faithful, solvable,* $\implies$ $|N|$ *is a prime power, but* $|N| = 35$*, contradiction*

$\exists$ blocks,

1. 7 blocks $|B_i| = 5$
2. 5 blocks $|B_i| = 7$

$$G \to Sym(B_i) \cong S_5 \tag{24}$$

G acts transitively on B_i's, O-S implies $5 \mid |G|$

G abelian, transitive, faithful on Ω implies G acts regularly on Ω.
$Stab_x = 1\ \forall\ x \in \Omega$.
G transitive, $G \to Sym(\Omega)$,

$$1 = |\#\text{ of orbits}| = \frac{1}{|G|}\sum_{g\in G} |\text{fixed elts by } g| \tag{25}$$
$$|G| = \sum_{g\in G} |\text{fixed elts by } g| \tag{26}$$

only identity can fix some elts, hence all Ω.
$e \neq g$ cannot fix any element $\implies$ since action is regular.

Problem 4 continued on next page...

Hence $|G| = 35 + 0$, so $|G| = 35$.
Since G is abelian, use fundamental theorem for abelian group

$$G \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35} \tag{27}$$

is cyclic, since 5,7 coprime.

Problem 5

Let G be a finite group, e the identity of G and a some nonidentity element. Suppose that χ is a character of G such that $(\forall g \in G)$

$$\chi(g) = \begin{cases} 5, & \text{if } g = e, \\ 3, & \text{if } g = a, \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

Determine G.

Solution:

We have $|G| < \infty$, and

$$[\chi, \chi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \tag{29}$$

$$= \frac{1}{|G|}(25 + 9 + 0) \tag{30}$$

$$= \frac{34}{|G|} \tag{31}$$

is an integer. so $|G| \mid 34 = 2 \cdot 17$.

$$[\chi, \mathbb{1}_G] = \frac{1}{|G|}(5 + 3) = \frac{8}{|G|} \text{ is integer} \tag{32}$$

$$\implies |G| \mid 8 \tag{33}$$

$$\implies |G| \mid gcd(8, 2 \cdot 17) = 2 \tag{34}$$

implies G is the unique group of 2 elements, the cyclic group of 2 elements.