Sheet 4: Spectral radius and randomness

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Let G be a finite undirected d-regular graph on n vertices. Let A denote the adjacency matrix of G, and let $b_0, b_1, \ldots, b_{n-1}$ be an orthonormal eigenbasis for A with real eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$.

Definition 1 *Let the spectal radius of G be*

$$\rho_0(G) = \max\{|\lambda_i| \mid 1 \leq i \leq n-1\}.$$

That is, we exclude the trivial eigenvalue d.

Lemma 2 *If G is connected, then $b_0 = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$.*

Solution 3 *If G is connected, then*

$$b_0 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{1}$$

Since G is a finite, undirected, d-regular graph on n vertices, thus, by results from the previous sheet, d regular and connectedness implies $\lambda_0(G) = d$, where d is also maximal degree.
The associated eigenvector for eigenvalue d is the all 1's vector,

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{2}$$

since every row of the adjacency matrix of a regular graph sum to d. Thus, we just need to normalize the all one's vector to get

$$b_0 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{3}$$

Let S, T be subsets of $V(G)$. Assume now that the edges of G are chosed randomly.

Exercise 4 *What value do we expect for $e(S,T)$ for a random G?*

solution:
Originally, $e(S,T) = \chi_S^T A \chi_T$, but here, the edges between the 2 vertex set, S, T are chosen randomly.
since the graph G is d-regular on n vertices, there must be $\frac{d \cdot n}{2}$ edges in total, and we have $\binom{n}{2}$ possible "slots" to put those edges. And we are doing this randomly. Thus, for each "slot", the possibility that there is an edge there, is

$$\mathbb{P}(\exists \text{ an edge}) = \frac{\frac{d \cdot n}{2}}{\frac{n(n-1)}{2}} = \frac{d}{n-1} \tag{4}$$

Thus, we should have a matrix

$$P = \begin{bmatrix} 0 & & & \frac{d}{n-1} \\ & 0 & & \\ & & \ddots & \\ \frac{d}{n-1} & & & 0 \end{bmatrix} \tag{5}$$

with zero on the diagonal and $\frac{d}{n-1}$ everywhere except the diagonal. $e(S,T) = \chi_S^T P \chi_T$ would take into account that an edge is in place with possibility $\frac{d}{n-1}$.

end solution

It turns out that when the spectral radius is small, the graph mimicks the random behaviour.

Exercise 5 *For $S \subseteq V(G)$ decompose the characteristic function X_S as*

$$X_S = \sum_{i=0}^{n-1} \alpha_i b_i.$$

What is α_0? What is $\sum_{i=0}^{n-1} \alpha_i^2$?

Solution 6 *$S \subseteq V(G)$ decompose the characteristic function χ_s as $\chi_s = \sum_{i=0}^{n-1} \alpha_i b_i$ where*

$$\chi_s(v) = \begin{cases} 1 \text{ if } v \in S \\ 0 \text{ if } v \notin S \end{cases} \tag{6}$$

We could think of χ_s as a column vector, where we have the n vertices, and we put 1 and 0 according to whether v_i is in S or not.
Since χ_s is a vector, it could be expressed as a linear combination of eigenvectors, since those eigenvectors form an orthonormal eigenbasis:

$$b_0, b_1, \ldots, b_{n-1}. \tag{7}$$

Thus, we have

$$\chi_s = \alpha_0 b_0 + \alpha_1 b_1 + \ldots + \alpha_{n-1} b_{n-1} \tag{8}$$

α_0 *is the coefficient corresponding to the eigenvector*

$$b_0 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{9}$$

Also,

$$\begin{aligned}
\langle \chi_s, b_0 \rangle &= \left\langle \sum_i \alpha_i b_i, b_0 \right\rangle && (10) \\
&= \sum_i \alpha_i \langle b_i, b_0 \rangle && (11) \\
&= \sum_i \alpha_i \delta_{i,0} && (12) \\
&= \alpha_0 && (13) \\
\sum_{i=0}^{n-1} \alpha_i^2 &= \alpha_0^2 + \alpha_1^2 + \ldots + \alpha_{n-1}^2 && (14)
\end{aligned}$$

Theorem 7 *For all $S, T \subseteq V(G)$ we have*

$$\left| e(S,T) - \frac{d\,|S|\,|T|}{n} \right| \leq \rho_0(G) \sqrt{|S|\,|T|}.$$

Proof. As suggested in exercises that builds up to this theorem, write

$$\chi_S = \sum_{i=0}^{n-1} \alpha_i b_i \tag{15}$$

$$\chi_T = \sum_{i=0}^{n-1} \beta_i b_i \tag{16}$$

in eigenbasis. The α_i and β_i are coefficients, and

$$\alpha_i = \langle \chi_S, b_i \rangle \tag{17}$$

$$\beta_i = \langle \chi_T, b_i \rangle \tag{18}$$

also, in perticular,

$$\begin{aligned}
\alpha_0 &= \langle \chi_S, b_0 \rangle && (19) \\
&= \left\langle \chi_S, \frac{1}{\sqrt{n}} \mathbf{1} \right\rangle \text{ where } \mathbf{1} \text{ is } n \times 1 \text{ vector of all 1's} && (20) \\
&= \frac{1}{\sqrt{n}} \langle \chi_S, \mathbf{1} \rangle && (21) \\
&= \frac{1}{\sqrt{n}} |S| && (22)
\end{aligned}$$

similarly, $\beta_0 = \frac{1}{\sqrt{n}}|T|$.
Thus

$$
\begin{aligned}
e(S,T) &= \chi_S^T A \chi_T && (23)\\
&= \sum_{i=0}^{n-1} \alpha_i b_i^T A \beta_j b_j && (24)\\
&= \sum_{i=0}^{n-1} \alpha_i \beta_i \lambda_i \text{ since } b_i \text{ form orthonormal basis} && (25)\\
&= \alpha_0 \beta_0 \lambda_0 + \sum_{i=1}^{n-1} \alpha_i \beta_i \lambda_i && (26)\\
&= \frac{|S|}{\sqrt{n}} \frac{|T|}{\sqrt{n}} d + \sum_{i=1}^{n-1} \alpha_i \beta_i \lambda_i. && (27)
\end{aligned}
$$

Thus, we have

$$
|e(S,T) - \frac{d|S||T|}{n}| = |\sum_{i=1}^{n-1} \alpha_i \beta_i \lambda_i|. \tag{28}
$$

Now, LHS of equation we want to show

$$
\begin{aligned}
&= |\sum_{i=1}^{n-1} \alpha_i \beta_i \lambda_i| && (29)\\
&\leq |\sum_{i=1}^{n-1} \alpha_i \beta_i| \rho_0(G) && (30)\\
&\leq |\sum_{i=0}^{n-1} \alpha_i \beta_i| \rho_0(G) && (31)\\
&= |\langle \chi_S, \chi_T \rangle| \rho_0(G) && (32)\\
&\leq \|\chi_S\| \|\chi_T\| \rho_0(G) && (33)\\
&= \sqrt{|S|}\sqrt{|T|} \rho_0(G) && (34)\\
&= \rho_0(G) \sqrt{|S||T|} && (35)\\
&= RHS && (36)
\end{aligned}
$$

observe

$$
\begin{aligned}
&|\langle \chi_S, \chi_T \rangle| && (37)\\
&= |\left\langle \sum_i \alpha_i b_i, \sum_j \beta_j b_j \right\rangle| && (38)\\
&= |\sum_{i,j} \alpha_i \beta_j \langle b_i, b_j \rangle| && (39)\\
&= |\sum_{i,j} \alpha_i \beta_j \delta_{i,j}| && (40)\\
&= |\sum_i \alpha_i \beta_i| && (41)
\end{aligned}
$$

and

$$\|\chi_S\| = \sqrt{\langle \chi_S, \chi_S \rangle} \tag{42}$$

$$= \sqrt{|S|} \tag{43}$$

$$\|\chi_T\| = \sqrt{\langle \chi_T, \chi_T \rangle} \tag{44}$$

$$= \sqrt{|T|} \tag{45}$$

$$\tag{46}$$

■

A subset $S \subseteq V(G)$ is *independent*, if $E(S, S)$ is empy. Using the above theorem, one can easily get an upper bound for the maximal size of an independent subset of G.

Corollary 8 *An independent subset of G has size at most*

$$\frac{\rho_0(G)}{d} n.$$

Proof. From the previous theorem, we have $\forall S, T \subseteq V(G)$,

$$|e(S,T) - \frac{d|S||T|}{n}| \leq \rho_0(G)\sqrt{|S||T|} \tag{47}$$

Let $S \subseteq V(G)$ be independent, we have $e(S, S) = 0$. So the above formula reduce to

$$|e(S,S) - \frac{d|S||S|}{n}| \leq \rho_0(G)\sqrt{|S||S|} \tag{48}$$

$$\implies |\frac{d}{n}||S|^2 \leq \rho_0(G)|S| \tag{49}$$

$$\implies |\frac{d}{n}||S| \leq \rho_0(G) \tag{50}$$

$$|S| \leq \rho_0(G)\frac{n}{d}. \tag{51}$$

Thus, an independent subset S has size at most $\frac{\rho_0(G)}{d} n$.

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Of course, this bound is only meaningful, if $\rho_0(G)$ can go below, say, $d/2$. It is also not clear how small $\rho_0(G)$ can get for large d-regular graphs – we will get a good estimate on this later, using graph limits.