Adv Abstract Algebra Spr2022 midterm

xianzhi wang

Sept 2022

Q 1

Question 1. *Let $H \leq G$ such that $|G : H| = 4$. Prove: if $g \in G$ and g has order 19, then $g \in H$.*

Let G act on $\{Hb\} = \mathcal{R}$, the set of right cosets by right action. We have homomorphism

$$\begin{aligned} \rho : G &\mapsto Sym(\mathcal{R}) \cong S_4 \quad \text{since there are 4 cosets of } H. \\ g &\mapsto g^\rho, \\ g^\rho : \mathcal{R} &\mapsto \mathcal{R} \\ Hb &\mapsto Hbg \end{aligned}$$

Let $g \in G$, and $|g| = 19$. Want to show $g \in H$. Since $\rho(g)$ is in $Sym(\mathcal{R}) \cong S_4$, $|\rho(g)|$ divide $4!$, which is the order of S_4. Also, $|\rho(g)|$ divide $|g| = 19$ since ρ is homomorphism, so $|\rho(g)|$ divide $\gcd(24, 19) = 1$, so $\rho(g) = id$. So g is mapped to the identity permutation on the right cosets of H. Thus,

$$g \in \ker \rho = \bigcap_{g \in G} g^{-1} H g \leq H.$$

Q 2

Question 2. *let a (finite) group G act on Ω and on Δ. the action on Ω is transitive, and $|\Omega| = 22$, $|\Delta| = 10$. Prove that the action of G on Δ is not faithful. (i.e. the kernel of the action on Δ is non-trivial)*

Use O-S, G act on Ω, for $x \in \Omega$, $|\mathcal{O}(x)| = 22$,

$$|\mathcal{O}(x)| \cdot |stab(x)| = |G| \implies 22 \mid |G|$$

Let ρ be denote the homomorphism assoicated with the group action.

$$\begin{aligned} \rho : G &\mapsto Sym(\Delta) \cong S_{10} \\ g &\mapsto g^\rho \end{aligned}$$

We have

$$\begin{aligned} &G/\ker \rho \cong \operatorname{Im} \rho \leq Sym(\Delta) \\ &|\operatorname{Im} \rho| \text{ divide } |G| = 22k \\ &|\operatorname{Im} \rho| \text{ divide } 10! \end{aligned}$$

If G has finite order, then assume for contradiction $|\ker\rho| = 1$, then $|G| = |\operatorname{Im}\rho|$ divide $10!$, but $|G|$ has a factor of 11. Contradiction. So $\ker\rho > 1$.

If G has infinite order, then since $|\operatorname{Im}\rho| \le 10! < \infty$, $|\ker\rho|$ must be infinite.

Q 3

Question 3. *Let $\mathbb{C}^\times$ denote the multiplicative group of all non-zero complex numbers (under the ordinary multiplication). Prove that $\mathbb{C}^\times$ does not have any non-trivial subgroup of finite index.*

Supp that H has finite index in $\mathbb{C}^\times$ and $m = [\mathbb{C}^\times : H]$. Then for any nonzero complex number $z^m \in H$, we have

$$a^m H = (zH)^m = H, \implies z^m \in H.$$

For all $w \in \mathbb{C}$, we can solve $z^m - w = 0$ to write w as z^m for some z, hence $w \in H$, so $\mathbb{C} \subseteq H$, so $H = \mathbb{C}$.

Q 4

Proposition 1. *Let a group G have order $2^2 \cdot 5 \cdot 17$. Show that*

1. *G has a unique Sylow 5-subgroup and a unique Sylow 17-subgroup.*
2. *$\exists$ an element of order $85 = 5 \cdot 17$ in G.*

Q 5

Proposition 2. *Let $Z \trianglelefteq G$ such that $|Z| = 2$ and $|G : Z| = 97$. Show that*

1. *$Z \le Z(G)$*
2. *G is cyclic.*