Graph Limits: Graph Limits #Fekete

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Problem 1

Let $a_1, a_2, a_3, \ldots$, seq of non-negative real numbers with $a_{i+j} \le a_i + a_j \ \forall\ i, j \ge 1$. Then

$$\lim_{n \mapsto \infty} \frac{a_n}{n} \tag{1}$$

exists.

Soln:

Let $L = \inf_{n \ge 1} \frac{a_n}{n}$, and let $\epsilon > 0$ be arbitrary.
since L is the inf of $\frac{a_n}{n}$, we could choose n_0 such that

$$L \le \frac{a_{n_0}}{n_0} \le L + \epsilon \tag{2}$$

now, for $n \ge n_0$, write n as $n = pn_0 + q$, $p, q \in \mathbb{N}$, $0 \le q < n_0$.

$$\begin{aligned} a_n &= a_{pn_0+q} && (3)\\ &\le \underbrace{a_{n_0} + a_{n_0} + \ldots + a_{n_0}}_{p \text{ terms}} + a_q && (4)\\ &= pa_{n_0} + a_q && (5)\\ & && (6) \end{aligned}$$

Now, divide both sides by n.

$$\begin{aligned} \frac{a_n}{n} &\le \frac{p}{n} a_{n_0} + \frac{a_q}{n} && (7)\\ \limsup_{n \mapsto \infty} \frac{a_n}{n} &\le \frac{a_{n_0}}{n_0} + 0 && (8) \end{aligned}$$

since

$$\begin{aligned} \frac{a_q}{n} &\mapsto 0 \text{ as } n \mapsto \infty && (9)\\ \text{since } a_q &\le \max_{0 \le q < n_0} a_q < \infty && (10) \end{aligned}$$

since

$$\begin{aligned} \limsup_{n \mapsto \infty} \frac{p}{n} &= \limsup_{n \mapsto \infty} \frac{\frac{n-q}{n_0}}{n} && (11)\\ &= \limsup_{n \mapsto \infty} \frac{n-q}{n} \cdot \frac{1}{n_0}. && (12)\\ &= \lim_{n \mapsto \infty} \frac{n-q}{n} \cdot \frac{1}{n_0}. && (13)\\ &= \frac{1}{n_0} && (14) \end{aligned}$$

Thus, $\limsup_{n \mapsto \infty} \frac{a_n}{n} \le \frac{a_{n_0}}{n_0} \le L + \epsilon$
since $\epsilon > 0$ is arbitrary, we could make $\epsilon \mapsto 0$, so

$$\limsup_{n \mapsto \infty} \frac{a_n}{n} \le L = \inf_{n \ge 0} \frac{a_n}{n} \tag{15}$$

In other direction,

$$\limsup_{n \Longrightarrow \infty} \frac{a_n}{n} \ge \liminf_{n \Longrightarrow \infty} \frac{a_n}{n} \ge \inf_{n \ge 0} \frac{a_n}{n} \tag{16}$$

so

$$\lim_{n \implies \infty} \frac{a_n}{n} = \limsup_{n \implies \infty} \frac{a_n}{n} = \inf_{n \geq 0} \frac{a_n}{n}. \tag{17}$$

end of proof.

G finite undirected d-regular graph n vertices,
A: adjacency operator
$b_0, b_1, \ldots, b_{n-1}$ orthonormal eigenbasis
$\lambda_0 \geq \ldots \geq \lambda_{n-1}$
$\rho = \rho_0(G)$ spectral radius.
$b_i^T b_i = 1 (0 \leq i < n)$.
$U_i = b_i b_i^T (0 \leq i < n)$.
orthonormal eigendecomposition for A.