RG: RG #02-3

Due on Sept 2021 at 11:59pm

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RG HW 02 Question 3

Problem 1

First we fix a chart

$$\Phi^{-1} : \mathbb{R}^2 \to S^2 \tag{1}$$
$$(\phi, \theta) \to \Phi^{-1}(\phi, \theta) := (R\cos\phi\sin\theta, R\sin\phi\sin\theta, R\cos\theta) \tag{2}$$

where R is the radius of sphere S^2.
Now we calculate the Riemannian Metric induced from $\mathbb{R}^3$.

$$(\phi, \theta) \xrightarrow{\Phi^{-1}} (x, y, z) \tag{3}$$
$$\frac{\partial}{\partial\phi}\left(\Phi^{-1}(\phi,\theta)\right) = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x}\left(\Phi^{-1}(\phi,\theta)\right) + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y}\left(\Phi^{-1}(\phi,\theta)\right) + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}\left(\Phi^{-1}(\phi,\theta)\right) \tag{4}$$
$$= (-R\sin\phi\sin\theta, R\cos\phi\sin\theta, 0) \tag{5}$$

In basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} \tag{6}$$
$$= (R\cos\phi\cos\theta, R\sin\phi\cos\theta, -R\sin\theta) \tag{7}$$
$$g_{\theta,\theta} = \langle\partial_\theta, \partial_\theta\rangle = \left\langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle = R^2\cos^2\phi\cos^2\theta + R^2\sin^2\phi\cos^2\theta + R^2\sin^2\theta = R^2 \tag{8}$$

similarly,

$$g_{\phi\phi} = \langle\partial_\phi, \partial_\phi\rangle = \left\langle\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right\rangle = R^2\sin^2\phi\sin^2\theta + R^2\cos^2\phi\sin^2\theta = R^2\sin^2\theta \tag{9}$$

similarly,

$$g_{\theta\phi} = g_{\phi\theta} = \left\langle\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right\rangle = -R^2\sin\phi\sin\theta\cos\phi\cos\theta + R^2\cos\phi\sin\theta\sin\phi\cos\theta + 0 = 0 \tag{10}$$

Thus,

$$[g_{\phi\theta}] = \begin{bmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} R^2\sin^2\theta & 0 \\ 0 & R^2 \end{bmatrix} \tag{11}$$

ϕ is (encoded 1) x-axis, θ (encoded 2) is y-axis.

$$\begin{bmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} R^2\sin^2\theta & 0 \\ 0 & R^2 \end{bmatrix} =: [g_{\phi\theta}] \tag{12}$$

similarly

$$\begin{bmatrix} g^{\phi\phi} & g^{\phi\theta} \\ g^{\theta\phi} & g^{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{R^2\sin^2\theta} & 0 \\ 0 & \frac{1}{R^2} \end{bmatrix} =: \left[g^{\phi\theta}\right] \tag{13}$$

Now,

$$\Gamma^{\phi}_{\phi\phi} = \sum_{\ell=1}^{2}\frac{1}{2}\left(\partial_\phi g_{\phi\ell} + \partial_\phi g_{\phi\ell} - \partial_\ell g_{\phi\phi}\right)g^{\ell\phi} \tag{14}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial\phi}g_{\phi\phi} + \frac{\partial}{\partial\phi}g_{\phi\phi} - \frac{\partial}{\partial\phi}g_{\phi\phi}\right)g^{\phi\phi} \tag{15}$$
$$+ \frac{1}{2}\left(\frac{\partial}{\partial\phi}g_{\phi\theta} + \frac{\partial}{\partial\phi}g_{\phi\theta} - \frac{\partial}{\partial\theta}g_{\phi\phi}\right)g^{\theta\phi} \tag{16}$$
$$= 0\cdot\frac{1}{R^2\sin^2\theta} + \frac{1}{2}\left(-\frac{\partial}{\partial\theta}(R^2\sin^2\theta)\right)\cdot 0 = 0 \tag{17}$$

similarly,

$$\Gamma^{\phi}_{\theta\theta} = \sum_{\ell=1}^{2} \frac{1}{2}\left(\partial_\theta g_{\theta\ell} + \partial_\theta g_{\theta\ell} - \partial_\ell g_{\theta\theta}\right) g^{\ell\phi} \tag{18}$$

$$= \frac{1}{2}\left(\partial_\theta g_{\theta\phi} + \partial_\theta g_{\theta\phi} - \partial_\phi g_{\theta\theta}\right) g^{\phi\phi} \tag{19}$$

$$+ \frac{1}{2}\left(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}\right) g^{\theta\phi} \tag{20}$$

$$= \frac{1}{2}(0+0-0)\frac{1}{R^2\sin^2\theta} + \frac{1}{2}(0+0-0)\cdot 0 = 0 \tag{21}$$

Similarly,

$$\Gamma^{\theta}_{\theta\theta} = \sum_{\ell=1}^{2} \frac{1}{2}\left(\partial_\theta g_{\phi\ell} + \partial_\phi g_{\theta\ell} - \partial_\ell g_{\theta\phi}\right) g^{\ell\theta} \tag{22}$$

$$= \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\theta g_{\theta\phi} - \partial_\phi g_{\theta\phi}\right) g^{\phi\theta} \tag{23}$$

$$+ \dots \tag{24}$$

$$= \frac{1}{2}(0+0-0)\cdot 0 + \frac{1}{2}(0+0-0)\frac{1}{R^2} = 0 \tag{25}$$

Problem 2

Find the derivative of $f(x) = x^4 + 3x^2 - 2$