AAA: Homework #09

Due on May 22, 2022 at 11:59PM

BSM Section A

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Problem 1

Suppose that γ is an algebraic integer and n is a positive integer. Prove that $\sqrt[n]{\gamma}$ is an algebraic integer.

Solution:

Since γ is an algebraic integer, $\exists$ monic $f(x) \neq 0$, and $f(x) \in \mathbb{Z}[x]$ such that $f(\gamma) = 0$. Let

$$f = X^m + a_{m-1}X^{m-1} + \ldots + a_1 X + a_0 \tag{1}$$

Consider

$$g = X^{mn} + a_{m-1}X^{n(m-1)} + a_{m-2}X^{n(m-2)} + a_1 X^n + a_0, \tag{2}$$

we see that $\sqrt[n]{\gamma}$ is clearly a root of g, since γ is a root of f. Now, g is monic, $g \in \mathbb{Z}[x], g \neq 0$, since $f \neq 0$.

Problem 2

Let α be an algebraic number, i.e., $g(\alpha) = 0$ for some $0 \neq g(x) \in \mathbb{Q}[x]$. Show that $\alpha = \frac{\beta}{n}$ with some algebraic integer β and positive integer n.

Solution:

Let α be an algebraic number. Thus $\exists 0 \neq g(x) \in \mathbb{Q}[x]$ such that $g(\alpha) = 0$. g is not necessary in $\mathbb{Z}[x]$, but we can multiply by the least common multiple of the denominators of all the rational coefficients to obtain g' such that

$$0 = g'(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0. \tag{3}$$

such that $a_i \in \mathbb{Z}\forall i$. Now we claim that $\beta = a_n \cdot \alpha$ is algebraic integer. First, we multiply both sides of equation 3 by a_n^{n-1}.

$$0 = a_n^n\alpha^n + a_{n-1}a_n^{n-1}\alpha^{n-1} + a_{n-2}a_n^{n-1}\alpha^{n-2} + \ldots + a_1a_n^{n-1}\alpha + a_0a_n^{n-1}. \tag{4}$$

Let $X = a_n \cdot \alpha$, Then

$$X^n + a_{n-1}X^{n-1} + \ldots + a_1a_n^{n-2}X + a_0a_n^{n-1} \tag{5}$$

is a monic polynomial with coefficient in $\mathbb{Z}$, such that β is a root. And this polynomial is non-zero since polynomial g' we started with is non-zero. Hence, $\alpha = \frac{\beta}{a_n}$ as wanted.

Problem 3

Problem 4

Problem 5

Proof. ☐

Problem 18

Problem 19

Problem 6