Sheet 5: A spectral diameter bound

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Let G be a finite undirected d-regular graph on n vertices. Let A denote the adjacency operator, and let $b_0, b_1, \ldots, b_{n-1}$ be an orthonormal eigenbasis with real eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Let $\rho = \rho_0(G)$ be the spectral radius. On this sheet, we give an upper bound on the diameter using ρ.

Of course, $b_i^\top b_i = 1$ $(0 \leq i < n)$. Let us force product the other way round. Let

$$U_i = b_i b_i^\top \;\; (0 \leq i < n). \tag{1}$$

These matrices may look boring at first, but they are not. It turns out that they give an 'orthonormal eigendecomposition' for A.

Lemma 1 *Let A, B be $n \times n$ matrices such that $Av_i = Bv_i$ $(0 \leq i < n)$ for a basis (v_i) for $\mathbb{R}^n$. Then $A = B$.*

Proof. We show $A = B$ by showing $Ax = Bx$ for all $x \in \mathbb{R}^n$. For all vector x, decompose $x = \sum_{i=0}^{n-1} \alpha_i v_i$,

$$Ax = A(\sum_i \alpha_i v_i) \tag{2}$$

$$= \sum_i \alpha_i A(v_i) \tag{3}$$

$$= \sum_i \alpha_i B v_i \tag{4}$$

$$= B(\sum_i \alpha_i v_i) \tag{5}$$

$$= B(x) \tag{6}$$

for all $x \in \mathbb{R}^n$.

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Theorem 2 *The following hold for $1 \leq i, j < n$:*
1) $U_i^2 = U_i$;
2) $U_i U_j = 0$ $(i \neq j)$;
3) $AU_i = \lambda_i U_i$;
4) ${}_{k=1}^{n} U_k = I$;
5) ${}_{k=1}^{n} \lambda_k U_k = A$;

Proof. 1)

$$U_i^2 = (b_i b_i^T)^2 = b_i b_i^T b_i b_i^T = b_i b_i^T = U_i \tag{7}$$

2)

$$b_i b_i^T b_j b_j^T = b_i \cdot 0 \cdot b_j^T = 0 \; (i \neq j) \tag{8}$$

3)

$$AU_i = Ab_i b_i^T = \lambda_i b_i b_i^T = \lambda_i U_i \tag{9}$$

since $b_0, b_1, \ldots, b_{n-1}$ is eigenbasis.
4)

$$\sum_{k=1}^{n} U_k = U_1 + U_2 + \ldots + U_n \tag{10}$$

$$= b_1 b_1^T + b_2 b^{2T} + \ldots + b_n b_n^T \tag{11}$$

$$= \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix} \cdot \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix} = I \tag{12}$$

since it's the product of a matrix with orthonormal columns with its transpose.
5)

$$\sum_{k=1}^{n} \lambda_k U_k = A \tag{13}$$

$$\sum_{k=1}^{n} \lambda_k U_k = \sum_k \lambda_k b_k b_k^T \tag{14}$$

$$= \sum_k A b_k b_k^T \tag{15}$$

$$= A \sum_k b_k b_k^T \tag{16}$$

$$= A \sum_k U_k = AI = A \tag{17}$$

■

Definition 3 *For $x, y \in V(G)$ let the* distance $d(x, y)$ *be the minimal length of a path going from x to y.*

Lemma 4 *Let G be an undirected, connected graph. Then d is a metric on $V(G)$.*

Proof. We show d is a metric on $V(G)$. where G undirected connected graph.
(1) $d(x, y) \geq 0$, since the length of a path is always positive or zero, when x and y are the same vertex. $x = y \implies d(x, y) = 0$ since the shortest path is of zero length when we are going from a vertex to itself. $d(x, y) = 0 \implies x = y$, the shortest path is zero length, so we never moved, it's the same vertex.

(2) $d(x,y) = d(y,x)$ since traversing the shortest path backwards, the length is the same. (The graph is undirected)
(3) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall$ $x, y, z \in V(G)$. the shortest path from x to y, then the shortest path from y to z could only be a longer path than the shortest path from x to z.
Since graph G is connected, $d(x,y) < \infty$. ■

Definition 5 *Let the* diameter *of G be*

$$\mathrm{diam}(G) = \max \{d(x,y) \mid x, y \in V(G)\}.$$

One can give a general lower bound on the diameter as follows.

Lemma 6 *Let G be a d-regular undirected graph on n vertices with $d \geq 3$. Then*

$$\mathrm{diam}(G) \geq \log_{d-1}(n/3). \tag{18}$$

Proof. Let G be d-regular, undirected graph on n vertices $(d \geq 3)$

$$diam(G) \geq \log_{d-1}\left(\frac{n}{3}\right) \tag{19}$$

$$\tag{20}$$

Let r denote $diam(G)$, then showing

$$r \geq \log_{d-1}\left(\frac{n}{3}\right) \tag{21}$$

is the same as showing $(d-1)^r \geq \frac{n}{3}$, which is

$$3(d-1)^r \geq n. \tag{22}$$

$$n \leq 3(d-1)^r \tag{23}$$

we could have an upper bound on n by fixing a vertex v_0 first, then since G is d regular, v_0 has d neighbors, at 1st level, as we expand outwards. At 2nd level, each of those neighbors of v_0 has $d-1$ new neighbors, so we have $d(d-1)$ in total (at max). At 3rd level, $d(d-1)^2$ vertices in total (at max). As we go on, we could iterate at maximum $r-1$ levels, since the diameter of G is r. If our fixed vertex v_0 happens to be at the "center" of the graph, then n might be small compared to the upper bound we obtained in this way.

$$n \leq 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{r-1} \tag{24}$$

$$= 1 + \frac{d(d-1)^r - d}{d-2} \tag{25}$$

$$\leq 3(d-1)^r \tag{26}$$

$$\tag{27}$$

if we show the last $\leq$, then we are done.
Now, we have $3 \leq d$.

Use $X := (d-1)^r$ as short hand.

$$3X - 1 < 3X < dX \tag{28}$$
$$6X - 2 < 2dX \tag{29}$$
$$dX - 2 < 3X(d-2) \tag{30}$$
$$\frac{dX-2}{d-2} < 3X \tag{31}$$
$$\frac{d-2+dX-d}{d-2} < 3X \tag{32}$$
$$1 + \frac{dX}{d-2} - \frac{d}{d-2} < 3X \tag{33}$$
$$1 + \frac{d(d-1)^r}{d-2} - \frac{d}{d-2} < 3(d-1)^r \tag{34}$$

as desired. and we are done.

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So, for a fixed degree, the diameter is at least logarithmic in the number of vertices.

It turns out that for good expanders, one can get a logarithmic upper bound for the diameter.

Lemma 7 *Let $x, y \in V(G)$. Then*

$$d(x,y) = \min\left\{k \mid X_{\{x\}}^{\top}(A^k)X_{\{y\}} \neq 0\right\}. \tag{35}$$

Theorem 8 *We have*

$$\operatorname{diam}(G) \leq \frac{\log(n)}{\log(d/\rho_0(G))}. \tag{36}$$

Hint: decompose $X_{\{x\}}$ and $X_{\{y\}}$.

solution:

$$diam(G) = \max_{x,y\in V(G)} \min_{\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0} k \tag{37}$$

(since A^k has non-negative entries, so $\neq 0$ is the same as > 0).

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} = \chi_{\{x\}}^T \sum_{i=0}^{n-1} \lambda_i^k U_i \chi_{\{y\}} \tag{38}$$
$$= \sum_{i=0}^{n-1} \lambda_i^k (U_i)_{x,y} \tag{39}$$
$$= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} \tag{40}$$
$$\tag{41}$$

since $\lambda_0 = d$, and

$$U_0 = b_0 b_0^T = \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \end{bmatrix} \tag{42}$$

and

$$\begin{aligned} (U_i)_{x,y} &= (b_i b_i^T)_{x,y} && (43) \\ &= (b_i)_x \cdot (b_i)_y && (44) \\ &= x_i \cdot y_i && (45) \end{aligned}$$

where

$$\begin{aligned} x_i &:= (b_i)_x && (46) \\ y_i &:= (b_i)_y && (47) \\ \text{for } i &\in \{0, 1, 2, \ldots, n-1\} && (48) \end{aligned}$$

is the x-th entry in vector b_i, the y-th entry in vector b_i.

$$\begin{aligned} \chi_{\{x\}}^T A^k \chi_{\{y\}} &= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} && (49) \\ &= \frac{d^k}{n} + \sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i && (50) \\ &\geq |\frac{d^k}{n}| - |\sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i| > && (51) \\ &\geq \frac{d^k}{n} - \rho^k \sum_{i=1}^{n-1} |x_i||y_i| && (52) \\ &\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=1}^{n-1} |x_i|^2} \sqrt{\sum_{i=1}^{n-1} |y_i|^2} && (53) \\ &\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=0}^{n-1} |x_i|^2} \sqrt{\sum_{i=0}^{n-1} |y_i|^2} && (54) \\ & && (55) \end{aligned}$$

where we used inverse triangle inequality, and then Cauchy-Shwarz inequality.
since

$$\begin{aligned} \sum_{i=0}^{n-1} |x_i|^2 = \sum_{i=0}^{n-1} (b_i)_x (b_i)_x = \sum_{i=0}^{n-1} \left(b_i b_i^T\right)_{x,x} = (I)_{x,x} = 1 && (56) \\ = \frac{d^k}{n} - \rho^k \cdot 1 \cdot 1 && (57) \\ = \frac{d^k}{n} - \rho^k && (58) \end{aligned}$$

Thus,

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} \geq \frac{d^k}{n} - \rho^k \ (*) \tag{59}$$

Since max is taken over $x, y \in V(G)$, we want $\frac{d^k}{n} - \rho^k > 0$. So

$$\frac{d^k}{n} > \rho^k \tag{60}$$

$$\frac{d^k}{\rho^k} > n \tag{61}$$

$$\log\left(\frac{d}{\rho}\right)^k > \log n \tag{62}$$

$$k > \frac{\log n}{\log\left(d/\rho\right)} \tag{63}$$

Min of k is taken over all $\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0$, so k has $\frac{\log n}{\log(d/\rho)}$ growth.

end solution

When G is bipartite, the above theorem (and also the theorem on the previous sheet) do not give anything reasonable, as $\rho_0(G) = d$. However, there is a similar diameter bound for bipartite graphs where all the other eigenvalues (in absolute value) are bounded away from d.

Theorem 9 *Let G be a finite, connected, undirected d-regular bipartite graph on n vertices. Assume that $|\lambda_i| \leq r$ ($1 \leq i < n-1$). Then G has logarithmic diameter.*