Adv Abstract Algebra Spr2022 midterm

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Q 1

Question 1. *Let $H \leq G$ such that $|G : H| = 4$. Prove: if $g \in G$ and g has order 19, then $g \in H$.*

Let G act on $\{Hb\} = \mathcal{R}$, the set of right cosets by right action. We have homomorphism

$$\begin{aligned} \rho : G &\mapsto Sym(\mathcal{R}) \cong S_4 \quad \text{since there are 4 cosets of } H. \\ g &\mapsto g^\rho, \\ g^\rho : \mathcal{R} &\mapsto \mathcal{R} \\ Hb &\mapsto Hbg \end{aligned}$$

Let $g \in G$, and $|g| = 19$. Want to show $g \in H$. Since $\rho(g)$ is in $Sym(\mathcal{R}) \cong S_4$, $|\rho(g)|$ divide $4!$, which is the order of S_4. Also, $|\rho(g)|$ divide $|g| = 19$ since ρ is homomorphism, so $|\rho(g)|$ divide $\gcd(24, 19) = 1$, so $\rho(g) = id$. So g is mapped to the identity permutation on the right cosets of H. Thus,

$$g \in \ker \rho = \bigcap_{g \in G} g^{-1}Hg \leq H.$$

Q 2

Question 2. *let a (finite) group G act on Ω and on Δ. the action on Ω is transitive, and $|\Omega| = 22$, $|\Delta| = 10$. Prove that the action of G on Δ is not faithful. (i.e. the kernel of the action on Δ is non-trivial)*

Use O-S, G act on Ω, for $x \in \Omega$, $|\mathcal{O}(x)| = 22$,

$$|\mathcal{O}(x)| \cdot |stab(x)| = |G| \implies 22 \mid |G|$$

Let ρ be denote the homomorphism assoicated with the group action.

$$\begin{aligned} \rho : G &\mapsto Sym(\Delta) \cong S_{10} \\ g &\mapsto g^\rho \end{aligned}$$

We have

$$\begin{aligned} &G/\ker \rho \cong \operatorname{Im} \rho \leq Sym(\Delta) \\ &|\operatorname{Im} \rho| \text{ divide } |G| = 22k \\ &|\operatorname{Im} \rho| \text{ divide } 10! \end{aligned}$$

If G has finite order, then assume for contradiction $|\ker\rho| = 1$, then $|G| = |\operatorname{Im}\rho|$ divide $10!$, but $|G|$ has a factor of 11. Contradiction. So $\ker\rho > 1$.

If G has infinite order, then since $|\operatorname{Im}\rho| \leq 10! < \infty$, $|\ker\rho|$ must be infinite.

Q 3

Question 3. *Let $\mathbb{C}^\times$ denote the multiplicative group of all non-zero complex numbers (under the ordinary multiplication). Prove that $\mathbb{C}^\times$ does not have any non-trivial subgroup of finite index.*

Supp that H has finite index in $\mathbb{C}^\times$ and $m = [\mathbb{C}^\times : H]$. Then for any nonzero complex number $z^m \in H$, we have

$$z^m H = (zH)^m = H, \implies z^m \in H.$$

For all $w \in \mathbb{C}$, we can solve $z^m - w = 0$ to write w as z^m for some z, hence $w \in H$, so $\mathbb{C} \subseteq H$, so $H = \mathbb{C}$.

Q 4

Proposition 1. *Let a group G have order $2^2 \cdot 5 \cdot 17$. Show that*

1. *G has a unique Sylow 5-subgroup and a unique Sylow 17-subgroup.*
2. *$\exists$ an element of order $85 = 5 \cdot 17$ in G.*

Soln 1: $|G| = 2^2 \cdot 5 \cdot 17$. Let $n_5 = \#$ of Sylow 5 subgroup. By Sylow's theorem, we have:

$$n_5 \equiv 1 \mod 5 \tag{1}$$
$$n_5 \mid 2^2 \cdot 17 \tag{2}$$

hence, n_5 can be $1, 2, 4, 17, 34,\ 4 \cdot 17 = 68$. Only $1 \equiv 1 \mod 5$ among them. Thus, $n_5 = 1$, so Sylow 5 subgroup is unique.

Let $n_{17} = \#$ of Sylow 17 subgroup. We have

$$n_{17} \equiv 1 \mod 17 \tag{3}$$
$$n_{17} \mid 2 \cdot 2 \cdot 5 \tag{4}$$

Thus, n_{17} can be $1, 2, 5, 4, 10, 20$, and only 1 satisfy

$$n_{17} \equiv 1 \mod 17 \tag{5}$$

among them. So $n_{17} = 1$, and the Sylow 17 subgroup is unique.

Soln 2: Since there exists unique Sylow 5 subgroup $=: H$ and there exists unique Sylow 17 subgroup $=: N$. We know that $H \triangleleft G,\ N \triangleleft G$, so we know that $H \cdot N$ is a group (as long as one of H or N is normal).

$$
\begin{aligned}
&H \cdot N \leq G \\
&|H| = 5 \\
&|N| = 17
\end{aligned}
$$

Thus, since there are of prime order, they are cyclic.

$$\exists h \in H,\ |h| = 5$$

(take the generator for example.)

$$\exists n \in N,\ |n| = 17.$$

Also, $H \cap N = \{1\}$, since for cyclic group of prime order, every element $\neq 1$ has same order, so if $1 \neq x \in H \cap N$, then $x \in H$, x has order 5, but $x \in N$ implies $|x|$ divide $N = 17$, but $|x| = 5 \nmid 17$.

$N \cdot N$ is the internal direct product of N and N,

$$H \cdot N \cong H \times N \cong \mathbb{Z}_5^+ \times \mathbb{Z}_{17}^+ \cong \mathbb{Z}_{5 \cdot 17}^+ = \mathbb{Z}_{85}^+ \tag{6}$$

Since 5 and 17 are coprime, so $\exists$ an element $x \in H \cdot N$ of order 85, since Isomorphism preserve the order, so $x \in H \cdot N \leq G$ implies $x \in G$, and x has order 85.

Q 5

Proposition 2. *Let $Z \trianglelefteq G$ such that $|Z| = 2$ and $|G : Z| = 97$. Show that*

1. *$Z \leq Z(G)$*
2. *G is cyclic.*

soln to 1:
$Z \triangleleft G$ and $|Z| = 2$. 97 is prime. G act on Z by conjugation.

$$G \overset{\rho}{\mapsto} Aut(Z) \tag{7}$$
$$g \mapsto g^\rho \tag{8}$$

$g^\rho \in Aut(Z)$ so $g^\rho : Z \to Z$ is isomorphism from Z to Z,

$$g^\rho : Z \to Z \tag{9}$$
$$z \to g^{-1} z g \in Z, \tag{10}$$

since $Z \triangleleft G$. Recall that (g in the center of G) $g \in Z(G) \iff g^\rho$ is trivial $\iff g^\rho = id$.
WTS: $x^\rho = id\ \forall\ x \in G$
We have

$$x^\rho \in Im\rho \leq Aut(Z) \tag{11}$$
$$|Aut(z)| = 1 \text{ since } |z| = 2 \tag{12}$$

Thus, $|x^\rho|$ divides $|Aut(z)| = 1$.
This implies that $x^\rho = id$, and any $x \to x^\rho : \; x^{-1}zx = z$ for $z \in Z$.
$1 \in Z(G)$ automatically, so $Z \subset Z(G)$.

soln to 2 (not complete):

We have $Z \triangleleft G$ and $|G/Z| = 97$. Since $|Z| = 2$ is prime, Z is cyclic.
G/Z is cyclic? G abelian? See HW01.

$$G \overset{\eta}{\mapsto} G/Z \tag{13}$$

$$g \mapsto gz \tag{14}$$

Consider the following map?

$$f : G \to Z \subset Z(G) \tag{15}$$

$$g \to g^{97} \tag{16}$$

so we have

$$Zg^{97} = (Zg)^{97} = Z, \implies g^{97} \in Z. \tag{17}$$

Do we have

$$G \cong \mathbb{Z}_2 \otimes \mathbb{Z}_{97} \cong \mathbb{Z}_{194}? \tag{18}$$