Sheet 5: A spectral diameter bound

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Let G be a finite undirected d-regular graph on n vertices. Let A denote the adjacency operator, and let $b_0, b_1, \ldots, b_{n-1}$ be an orthonormal eigenbasis with real eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Let $\rho = \rho_0(G)$ be the spectral radius. On this sheet, we give an upper bound on the diameter using ρ.

Of course, $b_i^\top b_i = 1$ $(0 \leq i < n)$. Let us force product the other way round. Let

$$U_i = b_i b_i^\top \ \ (0 \leq i < n). \tag{1}$$

These matrices may look boring at first, but they are not. It turns out that they give an 'orthonormal eigendecomposition' for A.

Lemma 1 *Let A, B be $n \times n$ matrices such that $Av_i = Bv_i$ $(0 \leq i < n)$ for a basis (v_i) for $\mathbb{R}^n$. Then $A = B$.*

Theorem 2 *The following hold for $1 \leq i, j < n$:*
1) $U_i^2 = U_i$;
2) $U_i U_j = 0$ $(i \neq j)$;
3) $AU_i = \lambda_i U_i$;
4) ${}^{n}_{k=1} U_k = I$;
5) ${}^{n}_{k=1} \lambda_k U_k = A$;

Definition 3 *For $x, y \in V(G)$ let the* distance $d(x, y)$ *be the minimal length of a path going from x to y.*

Lemma 4 *Let G be an undirected, connected graph. Then d is a metric on $V(G)$.*

Definition 5 *Let the* diameter *of G be*

$$\mathrm{diam}(G) = \max \{d(x, y) \mid x, y \in V(G)\}.$$

One can give a general lower bound on the diameter as follows.

Lemma 6 *Let G be a d-regular undirected graph on n vertices with $d \geq 3$. Then*

$$\mathrm{diam}(G) \geq \log_{d-1}(n/3). \tag{2}$$

So, for a fixed degree, the diameter is at least logarithmic in the number of vertices.
It turns out that for good expanders, one can get a logarithmic upper bound for the diameter.

Lemma 7 *Let $x, y \in V(G)$. Then*

$$d(x,y) = \min\left\{k \mid X_{\{x\}}^{\top}(A^k)X_{\{y\}} \neq 0\right\}. \tag{3}$$

Theorem 8 *We have*

$$\operatorname{diam}(G) \leq \frac{\log(n)}{\log(d/\rho_0(G))}. \tag{4}$$

Hint: decompose $X_{\{x\}}$ and $X_{\{y\}}$.
solution:

$$diam(G) = \max_{x,y\in V(G)} \min_{\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0} k \tag{5}$$

(since A^k has non-negative entries, so $\neq 0$ is the same as > 0).

$$\begin{aligned} \chi_{\{x\}}^T A^k \chi_{\{y\}} &= \chi_{\{x\}}^T \sum_{i=0}^{n-1} \lambda_i^k U_i \chi_{\{y\}} & (6)\\ &= \sum_{i=0}^{n-1} \lambda_i^k (U_i)_{x,y} & (7)\\ &= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} & (8)\\ & & (9) \end{aligned}$$

since $\lambda_0 = d$, and

$$U_0 = b_0 b_0^T = \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \end{bmatrix} \tag{10}$$

and

$$\begin{aligned} (U_i)_{x,y} &= (b_i b_i^T)_{x,y} & (11)\\ &= (b_i)_x \cdot (b_i)_y & (12)\\ &= x_i \cdot y_i & (13) \end{aligned}$$

where

$$\begin{aligned} x_i &:= (b_i)_x & (14)\\ y_i &:= (b_i)_y & (15)\\ \text{for } i &\in \{0,1,2,\ldots,n-1\} & (16) \end{aligned}$$

is the x-th entry in vector b_i, the y-th entry in vector b_i.

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} = \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} \tag{17}$$

$$= \frac{d^k}{n} + \sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i \tag{18}$$

$$\geq |\frac{d^k}{n}| - |\sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i| > \tag{19}$$

$$\geq \frac{d^k}{n} - \rho^k \sum_{i=1}^{n-1} |x_i||y_i| \tag{20}$$

$$\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=1}^{n-1} |x_i|^2} \sqrt{\sum_{i=1}^{n-1} |y_i|^2} \tag{21}$$

$$\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=0}^{n-1} |x_i|^2} \sqrt{\sum_{i=0}^{n-1} |y_i|^2} \tag{22}$$

$$\tag{23}$$

where we used inverse triangle inequality, and then Cauchy-Shwarz inequality.
since

$$\sum_{i=0}^{n-1} |x_i|^2 = \sum_{i=0}^{n-1} (b_i)_x (b_i)_x = \sum_{i=0}^{n-1} \left(b_i b_i^T\right)_{x,x} = (I)_{x,x} = 1 \tag{24}$$

$$= \frac{d^k}{n} - \rho^k \cdot 1 \cdot 1 \tag{25}$$

$$= \frac{d^k}{n} - \rho^k \tag{26}$$

Thus,

$$\chi_{\{x\}}^T A^k \chi_{\{y\}} \geq \frac{d^k}{n} - \rho^k \; (*) \tag{27}$$

Since max is taken over $x, y \in V(G)$, we want $\frac{d^k}{n} - \rho^k > 0$. So

$$\frac{d^k}{n} > \rho^k \tag{28}$$

$$\frac{d^k}{\rho^k} > n \tag{29}$$

$$\log \left(\frac{d}{\rho}\right)^k > \log n \tag{30}$$

$$k > \frac{\log n}{\log (d/\rho)} \tag{31}$$

Min of k is taken over all $\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0$, so k has $\frac{\log n}{\log(d/\rho)}$ growth.

end solution

When G is bipartite, the above theorem (and also the theorem on the previous sheet) do not give anything reasonable, as $\rho_0(G) = d$. However, there is a similar diameter bound for bipartite graphs where all the other eigenvalues (in absolute value) are bounded away from d.

Theorem 9 *Let G be a finite, connected, undirected d-regular bipartite graph on n vertices. Assume that $|\lambda_i| \leq r$ ($1 \leq i < n-1$). Then G has logarithmic diameter.*