Adv Abstract Algebra: AAA #HW05

Due on 2022 at 11:59PM

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Homework Set 5

Problem 1

Prove that none of $(\mathbb{Q},+)$, $(\mathbb{Q}\setminus\{0\},\cdot)$ is finitely generated.

Solution:

Assume for contradiction

$$G := (\mathbb{Q}\setminus\{0\},\cdot) \tag{1}$$

is finitely generated. Then

$$G = \langle \frac{r_1}{s_1}, \frac{r_2}{s_2}, \dots, \frac{r_n}{s_n} \rangle \tag{2}$$

where r_i and s_i are coprime.

Now, take prime number $p > \max\{r_i, s_j \mid 1 \le i \le n, 1 \le j \le n\}$ since there are infinitely many primes numbers, we can take such p, then $\frac{1}{p}$ cannot be expressed using the generators. Because: A general element of G is of the form:

$$\frac{r_1^{i_1} \dots r_n^{i_n}}{s_1^{i_1} \dots s_n^{i_n}} \tag{3}$$

and if

$$\frac{1}{p} = \frac{r_1^{i_1} \dots r_n^{i_n}}{s_1^{i_1} \dots s_n^{i_n}}, \tag{4}$$

$$p = \frac{s_1^{i_1} \dots s_n^{i_n}}{r_1^{i_1} \dots r_n^{i_n}} \tag{5}$$

then it contradicts p is prime.

Assume for contradiction $G := (\mathbb{Q},+)$ is finitely generated, then $G = \langle \frac{r_1}{s_1}, \frac{r_2}{s_2}, \dots \frac{r_n}{s_n} \rangle$ where r_i and s_i are coprime.

Again we take a prime number $p > \max\{r_i, s_j \mid 1 \le i \le n, 1 \le j \le n\}$ and $\frac{1}{p}$ cannot be expressed using generators. Because: a general element is of the form $(k_1, k_2 \in \mathbb{Z})$.

$$\frac{k_2}{k_1 \cdot s_1 \cdot s_2 \dots s_n} \tag{6}$$

so we have a contradiction if

$$\frac{1}{p} = \frac{k_2}{k_1 \cdot s_1 \cdot s_2 \dots s_n} \tag{7}$$

we could assume k_2 and $k_1 \cdot s_1 \dots s_n$ are coprime. Then $p = k_1 \cdot s_1 \cdot s_2 \dots s_n$ which is a contradiction, since p cannot have strictly smaller prime factors.

Problem 2

Let A be abelian.

1. Prove that A is finitely generated if and only if there exist finitely many subgroups A_i such that

$$A = A_0 \geq A_1 \geq A_2 \geq \ldots \geq A_n \geq A_{n+1} = 1 \tag{8}$$
$$\tag{9}$$

   and all the factor groups A_i/A_{i+1} are cyclic.

2. Let $B \leq A$ and assume that A is finitely generated. Show that B is finitely generated.

Solution:

Problem 3