AAA: AAA #06

Due on 2022 at 11:59PM

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HW06

Problem 1

Without using any reference to the Cayley Graphs prove that free groups are torsion free.

Solution:

prove free groups are torsion free.
Let $W \in F(X)$ be a non-identity reduced word, so $W = X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}$ We observe

$$\left(X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}\right) \cdot \left(X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}\right) (\ldots) \ldots \tag{1}$$

The potential place cancelation can happen is between

$$X_{i_m}^{k_m} X_{i_1}^{k_1}, \tag{2}$$

If there is no cancelation here, then it's clear we raise W^n to the power n for all $n \in \mathbb{N}$ will not get the empty word, so W has infinite order.
On the other hand, if there is cancelation between

$$X_{i_m}^{k_m} \; X_{i_1}^{k_1}, \tag{3}$$

we can write $W = a^{-1}ba$ where there is no cancelation between a, b or a^{-1}, b or b and b. We know we can write W in this way since W is assumed not to be the identity.
If

$$\left(X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}\right) \cdot \left(X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}\right) = 1 \tag{4}$$

then it is clear that at some point during cancelation, we get

$$\left(X_{i_1}^{k_1} \cdots X_{i_\ell}^{k_\ell}\right) \left(X_{i_{\ell'}}^{k_{\ell'}} \cdots X_{i_m}^{k_m}\right) \tag{5}$$

where $1 \leq \ell, \ell' \leq m$, then the fact we can keep on performing cancelation means W is of the form $a^{-1}a$, so $a^{-1}aa^{-1}a = 1$, so W is not reduced. Contradiction.
W is also identity, which is another contradiction.

Thus, $W = a^{-1}ba$, and $W^n = a^{-1}b^n a$, and since there is no cancelation between b and b. W has infinite order.

Problem 2

Let $X \subset G$ and suppose $G = \langle X \rangle$. Denote the corresponding (directed) Cayley graph by $\Gamma = \Gamma(G; X)$ such that $V(\Gamma) = G,\ E(\Gamma) = \cup_{x \in X} E_x(\Gamma)$, where for each $x \in X$, we have $E_x(\Gamma) = \{(a, xa) | a \in G\}$ (the set of all directed edges having color x). A bijective mapping $\alpha : V(\Gamma) \mapsto V(\Gamma)$ is called an *automorphism* of Γ provided

$$\forall a, b \in V(\Gamma) \text{ and } x \in X \tag{6}$$

$$(\bar{a}, b) \in E_x(\Gamma) \Leftrightarrow (\bar{a^\alpha}, b^\alpha) \in E_x(\Gamma). \tag{7}$$

Prove that

$$Aut\Gamma = G^* = \{g^* \mid g \in G\}, \tag{8}$$

where g^* denotes multiplication by the element g on the right.

Solution:

Since a bijective mapping $\alpha : V(\Gamma) \to V(\Gamma)$ is automorphism if it preserve edge, direction and label: $\forall a, b \in V(\Gamma), x \in X$, then

$$(a, b) \in E_x(\Gamma) \Leftrightarrow (a^\alpha, b^\alpha) \in E_x(\Gamma) \tag{9}$$

so multiplication by element G on the right clearly satisfy this, let $(a, xa) \in E_x(\Gamma) = \{(a, xa) | a \in G\}$

$$(a, xa) \in E_x(\Gamma) \Leftrightarrow (ag, x(ag)) = (a \cdot g, (xa) \cdot g) \in E_x(\Gamma) \tag{10}$$

so $Aut\Gamma \supset G^*$.

Now let $\phi \in Aut\Gamma$, we have (since ϕ preserve this for all $a, b \in V(\Gamma)$)

$$(1, X \cdot 1) \in E_x(\Gamma) \Leftrightarrow (\phi(1), \phi(x \cdot 1)) \in E_x(\Gamma) \tag{11}$$

since $E_x(\Gamma) = \{(a, xa) | a \in G\}$ could only be $(\phi(1), x \cdot \phi(1)) \in E_x(\Gamma)$ Thus, for all generators $\phi(x) = x \cdot \phi(1)$, and we are right multiplying by $g = \phi(1)$. so for arbitrary $a \in V(\Gamma)$,

$$(a, xa) \in E_x(\Gamma) \Leftrightarrow (\phi(a), \phi(x \cdot a)) \in E_x(\Gamma) \tag{12}$$

$$= (\phi(a), x\phi(a)) \in E_x(\Gamma) \tag{13}$$

similarly $\phi(x^{-1}) = x^{-1}\phi(1)$, for all generators, so since $a = X_{i_1}^{k_1} \cdots X_{i_m}^{k_m}$, we could perform this finitely many times to have $\phi(a) = X_{i_1}^{k_1} \cdots X_{i_m}^{k_m} \phi(1)$ so

$$(\phi(a), \phi(x \cdot a)) = (X_{i_1}^{k_1} \cdots X_{i_m}^{k_m} \phi(1), X \cdot X_{i_1}^{k_1} \cdots X_{i_m}^{k_m} \phi(1)) \in E_x(\Gamma) \tag{14}$$

Thus, ϕ is indeed right multiply by element $g = \phi(1)$.

Problem 3

Let $F(X)$ denote the free group with X being the set of free generators.

1. Determine the center of $F(X)$ when $|X| > 1$.
2. What is $Z(F(X))$ when $|X| = 1$?

Part A

center of $F(X)$ when $|X| > 1$. We know $1 \in Z(F(X))$. Now assume $z \neq 1$, and $z \in Z(F(X))$. so $ZW = WZ$ (assume Z and W are reduced.) so

$$X_{i_1}^{k_1} \cdots X_{i_m}^{k_m} \cdot X_{j_1}^{\ell_1} \cdots X_{j_m}^{\ell_m} = X_{j_1}^{\ell_1} \cdots X_{j_m}^{\ell_m} \cdots X_{i_1}^{k_1} \cdots X_{i_m}^{k_m} \tag{15}$$

$$ZW = WZ \tag{16}$$

$$\tag{17}$$

since W can be any word in $F(X)$, it is possible we pick an W such that there is no cancelation between

$$X_{i_m}^{k_m} \cdot X_{j_1}^{\ell_1} \tag{18}$$

and between

$$X_{j_m}^{\ell_m} \cdot X_{i_1}^{k_1} \tag{19}$$

so ZW and WZ are clearly different words. So cannot equal. *It is somewhat simpler to choose* $w := x_i (\forall i)$, *etc* so $Z(F(X)) = \{1\}$.

Part B

$Z(F(X)) = F(X)$ when $|X|$, so $F(X)$ is generated by a single generator X, Thus, fix any $Z \in F(X)$. $Z = X^k$ then for any other element $W = X^\ell \in F(X)$ (assume reduced),

$$Z \cdot W = X^k \cdot X^\ell = X^{k+\ell} = X^{\ell+k} = X^\ell \cdot X^k = W \cdot Z \tag{20}$$

so $Z \in Z(F(X))$,
so $F(X) \subset Z(F(X))$
since $Z(F(X)) \subset F(X)$
$F(x) = Z(F(X))$.

Problem 4

For Fun: Let $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ denote the free group of rank n, i.e., generated by n free generators. Find $a, b, c \in F_2$ such that

$$|F_2 : \langle a, b, c \rangle| = 2 \tag{21}$$

Solution:

Find $a, b, c \in F_2$ such that $|F_2 : \langle a, b, c \rangle| = 2$

$$\begin{aligned} f : x &\to 0 + 2\mathbb{Z} && (22) \\ y &\to 1 + 2\mathbb{Z} && (23) \end{aligned}$$

f is a map that

$$\begin{aligned} x &\to 0 + 2\mathbb{Z} && (24) \\ y &\to 1 + 2\mathbb{Z} && (25) \end{aligned}$$

$F_2 = \langle x, y \rangle$ is the free group generated by x, y. Universal property gives a surjective homomorphism φ, and by 1st iso thm,

$$\begin{aligned} F_2/(ker\varphi) &\cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 && (26) \\ \implies |F_2 : ker\varphi| &= 2 && (27) \end{aligned}$$

claim:

$$\begin{aligned} ker\varphi &= \left\{ x^{k_1} y^{\ell_1} x^{k_2} y^{\ell_2} \cdots x^{k_m} y^{\ell_m} \mid \sum_i \ell_i = even \right\} && (28) \\ &= \left\langle x, y^2, yxy \right\rangle && (29) \end{aligned}$$

$x \in ker\varphi$ since $\varphi(x) = f(x) = 0 + 2\mathbb{Z} \implies x \in ker\varphi$
$y^2 \in ker\varphi$ since $\varphi(y^2) = \varphi(y)\varphi(y) = f(y)f(y) = (1 + 2\mathbb{Z}) + (1 + 2\mathbb{Z}) = 0 + 2\mathbb{Z}$
$yxy \in ker\varphi$ similarly.