Adv Abstract Algebra: AAA #HW02

Due on 2022 at 11:59PM

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Homework set 2

Problem 1

Let Γ be an ordinary graph and denote the set of its vertices by $V(\Gamma)$ and the set of its edges by $E(\Gamma)$. A bijection $\beta : V(\Gamma) \mapsto V(\Gamma)$ is called an *automorphism* of Γ if for all $x, y \in V(\Gamma)$

$$(x, y) \in E(\Gamma) \iff (\beta(x), \beta(y)) \in E(\Gamma) \tag{1}$$

holds. Then $Aut\Gamma := \{\beta | \beta \text{ is an automorphism of } \Gamma\}$ is a group w.r.t. composition. (It is called the *automorphism group of* Γ. It obviously acts on $V(\Gamma)$.)

1. Consider the following graph

Problem 2

Problem 3

Let G be a finite group and k an integer. Prove that $\left\{g^k | g \in G\right\} = G$ if and only if $gcd\left(|G|, k\right) = 1$.

Solution:

$\left\{g^k | g \in G\right\} = G \iff gcd(|G|, k) = 1$.
We show " $\implies$ "
Assume for contradiction $gcd(|G|, k) \neq 1$, so there exist a smallest prime divisor p such that $p \mid |G|$, and $p \mid k$.
By sylow's theorems, there exist $h \in G$ such that $|h| = p$. Thus,

$$|h^k| = \frac{p}{gcd(k,p)} = \frac{p}{p} = 1 \tag{2}$$

$$\implies h^k = e. \tag{3}$$

Now, since $e^k = e$,

$$\left\{g^k \mid g \in G\right\} \tag{4}$$

has cardinality strictly less than G,
so $\left\{g^k | g \in G\right\} \neq G$, contradiction.

We show "$\Leftarrow$" in

$$\left\{g^k | g \in G\right\} = G \iff gcd(|G|, k) = 1 \tag{5}$$

$$\tag{6}$$

Since $g \in G$, we have $g^k \in G \; \forall \; g$.

$$\left\{g^k \mid g \in G\right\} \subset G \tag{7}$$

follows.
Let $y \in G$. Use Euclidean Algorithm $\exists x_1, x_2 \in \mathbb{Z}$.

$$\begin{aligned}
y = y' &= y^{gcd(|G|,k)} && (8)\\
&= y^{x_1|G| + x_2 k} && (9)\\
&= y^{x_1|G|} y^{x_2 k} && (10)\\
&= y^{x_2 k} && (11)\\
&= (y^{x_2})^k && (12)
\end{aligned}$$

Thus, $\forall \; y \in G$, we could express y as y^{x_2} raise to the power k. So

$$\begin{aligned}
&\left\{g^k \mid g \in G\right\} \supseteq G && (13)\\
\implies &\left\{g^k \mid g \in G\right\} = G && (14)
\end{aligned}$$