Sheet 3: The Adjacency matrix

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All graphs are finite on this sheet. Let G be a directed graph on the vertex set $\{1, \ldots, n\}$. Let us define the *adjacency matrix* $A = \mathrm{Adj}(G)$ by setting

$$A_{i,j} = \text{number of edges from } i \text{ to } j \text{ in } G.$$

So, we allow multiple edges and even loops in G.

Exercise 1 *Express the following in linear algebra terms, using A:*
1) the degrees of a vertex;
2) the number of edges in G;
3) $e(X, Y)$ for $X, Y \subseteq V(G)$.

One of the main reasons why we look at the adjacency (or neighboring) relation as a matrix is the following correspondance between matrix multiplication and walks in G.

Definition 2 *A* directed walk of length n *in G is a sequence of directed edges $e_1, \ldots, e_n$ such that $e_i^+ = e_{i+1}^-$ ($1 \leq i \leq n-1$). The walk is a* loop (or returning), *if $e_1^- = e_n^+$.*

Note that we redefine the notion of walk here: it is a sequence of edges rather than vertices.

Theorem 3 *For every $k > 0$, $(A^k)_{i,j}$ equals the number of directed walks of length k from i to j.*

Proof. Let's prove by induction.
When $k = 1$, $(A)_{ij}$ is indeed the number of directed walks of length 1, (which is just directed edge) from i to j by definition.
Assume true for $k - 1$.
Denote the ijth entry of A by a_{ij} and ijth entry of A^{k-1} by b_{ij}

$$(A^k)_{i,j} = (A \cdot A^{k-1})_{i,j} = \sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha j} \tag{1}$$

Thus, for fixed α, we multiply together $a_{i\alpha} b_{\alpha j}$, which means multiply number of directed walks of length 1 (which are directed edges) from i to α and the number of directed walks of length $k - 1$ from α to j. After this, we sum over α, which run through 1 to n, and the result $\sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha j}$ is indeed number of directed walks of length k.
Thus, we have proved the theorem using induction. ■

Corollary 4 *For every $k > 0$, the trace $\mathrm{tr}(A^k)$ equals the number of loops of length k in G.*

Proof. Since $(A^k)_{ii}$ equals the number of directed walks of length k from i to i, (i.e., a loop), we could sum over the n entries in the diagonal to obtain all the loops of length k in G. This is exactly $tr(A^k)$.

■

Now assume that G is undirected.

This turns A to be a symmetric real matrix. Using the spectral theorem, it follows that A admits an orthonormal eigenbasis $b_0(G), b_1(G), \ldots, b_{n-1}(G)$ with real eigenvalues $\lambda_0(G) \geq \lambda_1(G) \geq \ldots \geq \lambda_{n-1}(G)$. That is, we have

$$Ab_i = \lambda_i b_i \ \ (0 \leq i < n).$$

Note that the λ_i are well defined, but the b_i are not. Also:

Lemma 5 *The eigenvalues are graph invariants, that is, isomorphic graphs have the same eigenvalues.*

Proof. Isomorphic graphs have the same eigenvalues. Since Isomorphic graphs are structurally the same, they have the exact same adjacency matrix, thus the same eigenvalues. ■

Exercise 6 *Compute λ_i and b_i for the triangle.*

One way to visualize the adjacency matrix as an operator is as follows. Write real numbers on the vertices of G, call this function f. Now A acts by taking all neighbors of the vertex x, add up the f-values there and write it to the position x. This will be the value of Af at x:

$$(Af)(x) = \sum_{(x,y) \in E(G)} f(y).$$

Actually this is how we will *define* the adjacency operator for infinite graphs. Using this image, one can prove.

Theorem 7 *Let G be an undirected graph with maximal degree d. Then $|\lambda_i(G)| \leq d$ ($0 \leq i < n$). When G is d-regular, we have $\lambda_0(G) = d$.*

Hint: take an eigenvector. Find a particular vertex for it..

Exercise 8 *Assume that G is undirected and connected with maximal degree d. Then $\lambda_0(G) = d$ if and only if G is d-regular.*

Lemma 9 *Let G be a d-regular undirected graph. Then the multiplicity of d as an eigenvalue of G equals the number of connected components of G.*

The eigenvalue $-d$ also comes into the picture naturally.

Lemma 10 *Let G be a d-regular undirected, connected graph. Then $\lambda_{n-1} = -d$ if and only if G is bipartite.*

Exercise 11 *Let G be a d-regular undirected graph. What is the multiplicity of $-d$ as an eigenvalue of G?*

Now we look at some simple examples.

Exercise 12 *Compute the eigenvalues and eigenvectors for the cycle of length n.*

Hint: What are the eigenvalues for the directed cycle of length n?

Exercise 13 *Compute the eigenvalues and an orthonormal eigenbasis for the complete graph on $d+1$ points.*

Hint: What happens to the eigenvalues and eigenvectors of A if you add a scalar matrix to A?