Galois Theory: GAL HW

Due on Apr 29, 2022 at 11:59pm

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HW extra
Exercise 15.1.1

Problem 1

Exercise 15.1.1 Show that for a prime p and a positive integer n we have

$$\Phi_{p^n}(x) = 1 + x^{p^{n-1}} + x^{2p^{n-1}} + \ldots + x^{(p-1)p^{n-1}} \tag{1}$$

Soln:

Let n in the formula be p^n

$$X^n - 1 = \prod_{d|n} \Phi_d(X) \tag{2}$$

$$X^{p^n} - 1 = \prod_{d|p^n} \Phi_d(X) \tag{3}$$

$$X^{p^k} - 1 = \Phi_1(X)\Phi_p(X)\Phi_{p^2}(X)\cdots\Phi_{p^k}(X) \tag{4}$$

$$X^{p^{(k+1)}} - 1 = \Phi_1(X)\Phi_p(X)\Phi_{p^2}(X)\cdots\Phi_{p^k}(X)\Phi_{p^{(k+1)}}(X) \tag{5}$$

hence, we deduce

$$\Phi_{p^{(k+1)}}(X) = \frac{X^{p^{(k+1)}} - 1}{X^{p^k} - 1} = \frac{\left(X^{p^k}\right)^p - 1}{X^{p^k} - 1} \tag{6}$$

$$= 1 + \left(X^{p^k}\right) + \left(X^{p^k}\right)^2 + \left(X^{p^k}\right)^3 + \ldots + \left(X^{p^k}\right)^{p-1} \tag{7}$$

and we have

$$X^{p^{(k+1)}} = X^{p^k \cdot p} = \left(X^{p^k}\right)^p \tag{8}$$