Adv Abstract Algebra: AAA #04

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Problem 1

Let G be a finite group and p a prime. Suppose that $N_p \lhd G$ such that $|G : N_p|$ is some power of p and $|N_p|$ is not divisible by p. (Then N_p is called *a normal p-complement in G*.) Prove that (for the give p) N_p is unique, i.e., a group can have at most one normal p-complement.

Solution:

Take normal p-complement N and N' in G, we show that they are actually equal.

$$G \tag{1}$$
$$N \cdot N' \tag{2}$$
$$N \qquad\qquad N' \tag{3}$$
$$N \cap N' \tag{4}$$

Since the order of a normal p-complement is not divisible by p, and its index in G is some power of p, we write

$$|N| = |N'| = m \tag{5}$$
$$|G| = p^k \cdot m \text{ where } p \not| m. \tag{6}$$

We know that in general, for $H \leq G,\ N \lhd G, \implies HN \leq G$.
Thus, $N \cdot N' \leq G$, so $|N \cdot N'|$.

By 2nd iso theorem

$$\frac{N \cdot N'}{N} \cong \frac{N'}{N \cap N'} \tag{7}$$
$$\frac{|N \cdot N'|}{|N|} = |\frac{N \cdot N'}{N}| = |\frac{N'}{N \cap N'}| \tag{8}$$
$$\implies |N \cdot N'| = |\frac{N'}{N \cap N'}| \cdot |N| \tag{9}$$
$$\tag{10}$$

Thus, since $|N'|$ is normal p-complement, $p \not| |N'|$, and since

$$|\frac{N'}{N \cap N'}| = \frac{|N'|}{|N \cap N'|} \tag{11}$$
$$\implies p \not| |\frac{N'}{N \cap N'}|. \tag{12}$$

Also, $p \not| |N|$.
implies $p \not| |N \cdot N'|$.

Problem 2

Let G be a finite group, $L \lhd G$ and p a prime. Suppose that N_p is a normal p-complement in G. Show that $L \cap N_p$ is a normal p-complement in L and $L \cdot N_p/L$ is a normal p-complement in G/L.

Solution:

Problem 3

Let p be a prime and denote the cyclic group of order p by C. Determine the number of all automorphisms of $C \times C$.

Problem 4

For fun: Let p be a prime, $0 < k < t$ integers, and denote the cyclic group of order p^k by A and the cyclic group of order p^t by B. Determine the number of all automorphisms of $A \times B$ and $A \times A$.

Problem 5

From Final: Let a solvable group G act faithfully and transitively on the set Ω, where $|\Omega| = 35$.

1. Prove that this action is not primitive.
2. Show that if G is abelian then it must be cyclic.