Adv Abstract Algebra: AAA #11

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HW11

Problem 1

Which of the following actions is primitive?

1. $$\langle (1,2,3,4,5,6,7,8), (1,2,3,4,5) \rangle \text{ on } \{1,2,3,4,5,6,7,8\}; \tag{1}$$

2. $$\langle (1,2,3,4,5,6,7,8), (2,4,6) \rangle \text{ on } \{1,2,3,4,5,6,7,8\}; \tag{2}$$

Solution:
Part A

Let

$$G := \langle (1,2,3,4,5,6,7,8), (1,2,3,4,5) \rangle \text{ act on } \Omega := \{1,2,3,4,5,6,7,8\}; \tag{3}$$

G acts transitively on Ω, since G has a 8-cycle that can take i to j for any $i, j \in \Omega$ by raise this cycle to some power.
Now, assume this action is not primitive, there are only 2 possibilities,

1. 4 blocks, each block has 2 elements
2. 2 blocks, each has 4 elements

G act on the set of 4 blocks: X, let $a = (12345)$.

$$G \xrightarrow{\varphi} Sym(X) \cong S_4 \tag{4}$$
$$order\ (\varphi(a)) \mid\ order\ (a) = 5 \tag{5}$$
$$order\ (\varphi(a)) \mid\ |S_4| = 4! \tag{6}$$
$$\implies |\varphi(a)| = 1 \tag{7}$$

so $\varphi(a)$ is the identity permutation on X.
Similarly, a is identity permutation on each element (block) in $B_i \in X$. So a is the identity on $\{1,2,\ldots,8\}$. contradiction.

G act on the set of 2 blocks: Y

$$G \xrightarrow{\varphi} Sym(Y) \cong S_2 \tag{8}$$

Similarly, let $a = (12345)$, then $\varphi(a)$ acts as identity permutation on Y, $\varphi(a)$ acts as identity on 4 elements in each block $B_i \in Y$. Then $\varphi(a)$ is id on $\{1,2,\ldots,8\}$. hence contradiction.
Thus: the action is primitive.

Part B

We could verify $\{2,4,6,8\}$ and $\{1,3,5,7\}$ are 2 blocks, since

$$\{1,3,5,7\} \xrightarrow{(12345678)} \{2,4,6,8\} \tag{9}$$

and (246) fix the 2 blocks, since all other elements of the group can be written as generators, those are indeed blocks.

Problem 2

Consider the action of the dihedral group D_n on the vertices. Determine all values of n for which the action is primitive.

Solution:

For $n = p$, a prime, the action is primitive ($|\Delta| \mid |\Omega|$) If n is a composite, $n = d \cdot r$ for some $1 < d < n$, if we mark d vertices that all has $(r-1)$ vertices in between with the same color, then those d vertices form a block.

Problem 3

Let A be a group and consider the *right regular action* of A on A [defined by $\alpha^g := \alpha \cdot g$ for all $\alpha, g \in A$]. When will this action be primitive (in terms of A)?

Solution:

Let A be a group, let $a \in A,\ a \neq e$, then consider the cyclic group $\langle a \rangle \leq A$. If $\langle a \rangle = A$, then if further A is prime order, then this right regular action is primitive. If A has composite order then let $|A| = d \cdot r$, $1 < d < |A|$. We know there exist a cyclic subgroup of order d, generated by a^r, so the cosets of $\langle a^r \rangle$ form a block system, (since cosets either disjoint or coincide completely).

If $\langle a \rangle \lneqq A$, then consider the cosets of $\langle a \rangle$, which forms a block system.

Problem 4

Let G act on Ω and suppose that N is a *minimal normal subgroup* of G. [$1 \neq N \lhd G$ and there is no $1 \neq K \subsetneqq N$ such that $K \lhd G$.] If N is abelian and acts transitively on Ω then G acts primitively on Ω.

1. Hint: N acts regularly on Ω, i.e. $N_\omega = 1$ for all $\omega \in \Omega$. (Remember: in a transitively acting group stabilizers are conjugate to each other.)
2. Using that N acts transitively prove that $G = G_\alpha \cdot N$ for all $\alpha \in \Omega$; here, necessarily, $G_\alpha \cap N = 1$.
3. If $G_\alpha \subsetneqq H \subsetneqq G$ then $G = H \cdot N$ and $H \cap N \neq 1$. In this case $H \cap N \lhd G$ and $1 \subsetneqq H \cap N \subsetneqq N$.

Solution:

Lemma: G abelian, G acts transitively, faithfully on Ω, $\implies$ then this action is regular.

WLOG, we assume G acts faithfully on Ω, since if G is not faithfully acting, take

$$\bar{G} = G/ker(\varphi) \text{ where } G \xrightarrow{\varphi} Sym(\Omega) \tag{10}$$

where φ is the homomorphism.
Then we prove $\bar{G}$ acts primitively on Ω would imply G acts primitively on Ω.
Now $G \xrightarrow{\varphi} Sym(\Omega)$ $ker(\varphi) = \{1\}$.
Then $N \lhd G$.

$$N \xrightarrow{\varphi|_N} Sym(\Omega),\ ker(\varphi|_N) = \{1\}\,. \tag{11}$$

The kernel of φ restrict to N is also $\{1\}$, so N acts faithfully, N is abelian, transitive on Ω, then N acts regularly by lemma. This implies $N_x = 1 \forall x \in \Omega$.
N transitive on Ω implies for any $g \in G$ there exists $n \in N$ such that $x^g = x^n$ for all $x \in \Omega$.

$$\begin{aligned} x^{gn^{-1}} &= x \quad (12)\\ gn^{-1} &\in Stab_x \quad (13)\\ g &\in Stab_x \cdot n \subseteq Stab_x \cdot N \quad (14)\\ \implies G &\subseteq Stab_x \cdot N \quad (15) \end{aligned}$$

since $Stab_x \cdot N \leq G$ (N is normal) which implies $G = Stab_x \cdot N$ for all $x \in \Omega$
so $G = G_x \cdot N$. G_x denotes $Stab_x$ in G, and N_x denotes $Stab_x$ in N.
we have $G_x \cap N = N_x = \{1\}$ since N is regular.
Claim: $G_x <_{max} G$ where ($x \in \Omega$).
Suppose for contradiction $\exists K$ such that

$$G_x \subsetneqq K \subsetneqq G \tag{16}$$

then $K \cap N \lhd G$ (since $G = G_x \cdot N = K \cdot N$) and $N \leq N_G(K \cap N)$, N in normalizer of $K \cap N$ since N is abelian $K \leq N_G(K \cap N)$ since $N \lhd G$. So

$$\begin{aligned} K \cdot N &\leq N_G(K \cap N) \quad (17)\\ \implies G &\leq N_G(K \cap N) \quad (18)\\ \implies G &= N_G(K \cap N) \quad (19) \end{aligned}$$

Now, we consider $K \cap N$. If we show

$$e \subsetneqq K \cap N \subsetneqq N, \tag{20}$$

we obtain the desired contradiction, since N is minimal normal subgroup.
First, for sake of contradiction, if $K \cap N = N$, then $N \leq K$, which implies $G = K \cdot N = K$. Hence contradiction.
Second we show $e \lneqq K \cap N$.
Since $G = N \cdot G_x$, $G_x \lneqq K$, there exists $y \notin G_x, y \in K$ then since y is also in G, we have $y = n \cdot h$ for $n \in N$, $h \in G_x \lneqq K$. $n \neq e$ since $y \notin G_x$. Then $K \ni yh^{-1} = n \in N$, so $n = yh^{-1} \in K \cap N$, and $n \neq e$. Thus, $e \lneqq K \cap N$.

Thus, we showed the claim $G_x <_{max} G$ $(x \in \Omega)$. Use Thm: if G act transitively on Ω, then G act primitively $\Leftrightarrow G_y <_{max} G,\ y \in \Omega$.
(G acts transitively since N acts transitively, $N \lhd G$)

Problem 5

Consider the natural action of S_n on $\Omega = \{1, 2, 3, \ldots, n\}$. Let $2 \leq k \leq \frac{n}{2}$ and define the action of S_n on the set $\mathcal{K} := \{T \mid T \subset \Omega \text{ and } |T| = k\}$ by means of $T^g := \{t^g \mid t \in T\}$. Prove that S_n acts primitively on $\mathcal{K}$ if and only if $k \neq \frac{n}{2}$.

Solution:

S_n acts primitively on $\mathcal{K} \Leftrightarrow k \neq \frac{n}{2}$.
$\Longrightarrow$
To prove this direction, consider the contrapositive: $k = \frac{n}{2} \implies S_n$ acts not primitively.
Indeed, when $k = \frac{n}{2}$, we could form blocks:

$$\{a_1, a_2, \ldots, a_k\} \sim \{b_1, b_2, \ldots b_k\} \tag{21}$$

where those two sets are disjoint set of numbers picked from Ω.
Since they have the same size, knowing $\{a_1, a_2, \ldots, a_k\}$ determines uniquely $\{b_1, b_2, \ldots, b_k\}$, and since elements of S_n are bijecctions, they preserves this equivalence relation $\sim$.

$\Leftarrow$
Since $k \neq \frac{n}{2}$, and $2 \leq k < \frac{n}{2}$, $\implies k < n - k$
We have $|T| = k$, let $T \in \mathcal{K}$,
Claim: $(S_n)_T <_{max} S_n$.
Observe the stabilizer of $T:$ $(S_n)_T \cong S_k \times S_{n-k}$, since we can freely permute the k numbers in T and the $(n-k)$ numbers outside T among themselves.
Let $g \notin (S_n)_T$, Want To Show $H := \langle (S_n)_T, g \rangle = S_n$.
Since g is some cycle,

$$\exists\ \alpha_1 \in T\ s.t.\ \alpha_2 := \alpha_1^g \notin T \tag{22}$$
$$\exists\ \beta_1 \notin T\ s.t.\ \beta_2 := \beta_1^g \notin T \tag{23}$$
$$\tag{24}$$

consider transposition

$$(\alpha_1 \beta_1) = (\alpha_2^{g^{-1}} \beta_2^{g^{-1}}) = g^{-1}(\alpha_2 \beta_2) g \in H = \langle (S_n)_T, g \rangle \tag{25}$$

$(\alpha_2, \beta_2) \in (S_n)_T$. I think for $n \in \mathbb{N}$,