Adv Abstract Algebra: AAA #04

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Problem 1

Let G be a finite group and p a prime. Suppose that $N_p \lhd G$ such that $|G : N_p|$ is some power of p and $|N_p|$ is not divisible by p. (Then N_p is called *a normal p-complement in G.*) Prove that (for the give p) N_p is unique, i.e., a group can have at most one normal p-complement.

Solution:

Take normal p-complement N and N' in G, we show that they are actually equal.

$$G \tag{1}$$
$$N \cdot N' \tag{2}$$
$$N \qquad\qquad N' \tag{3}$$
$$N \cap N' \tag{4}$$

Since the order of a normal p-complement is not divisible by p, and its index in G is some power of p, we write

$$|N| = |N'| = m \tag{5}$$
$$|G| = p^k \cdot m \text{ where } p \not| m. \tag{6}$$

We know that in general, for $H \leq G,\ N \lhd G,\ \implies HN \leq G$.
Thus, $N \cdot N' \leq G$, so $|N \cdot N'|$.

By 2nd iso theorem

$$\frac{N \cdot N'}{N} \cong \frac{N'}{N \cap N'} \tag{7}$$
$$\frac{|N \cdot N'|}{|N|} = |\frac{N \cdot N'}{N}| = |\frac{N'}{N \cap N'}| \tag{8}$$
$$\implies |N \cdot N'| = |\frac{N'}{N \cap N'}| \cdot |N| \tag{9}$$
$$\tag{10}$$

Thus, since $|N'|$ is normal p-complement, $p \not| |N'|$, and since

$$|\frac{N'}{N \cap N'}| = \frac{|N'|}{|N \cap N'|} \tag{11}$$
$$\implies p \not| |\frac{N'}{N \cap N'}|. \tag{12}$$

Also, $p \not| |N|$ implies $p \not| |N \cdot N'|$.

Since we have $N \cdot N' \leq G$, and $|N \cdot N'| \mid |G| = p^k \cdot m$, and $|N| = m \mid |N \cdot N'|$,
We have $|N \cdot N'|| = p^{k_0} \cdot m$ for some $k_0 \in \mathbb{Z}^+ \cup \{0\}$,
but $p \not| |N \cdot N'|$ implies $k_0 = 0$ so $|N \cdot N'| = m$.
Thus, since $|N| = |N \cdot N'|$ and $N \leq N \cdot N'$, we conclude $N = N \cdot N'$. By symmetry, $N = N \cdot N' = N'$. And we are done.

Part A

Show $L \cap N_p$ is a normal p-complement in L.
Let $|G| = p^k \cdot m$ where $p \not| m$.
Take $a \in L \cap N_p,\ \ell \in L$.
Then $\ell a \ell^{-1} \in L$, and $\ell a \ell^{-1} \in N_p$ since N_p is normal.

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So $\ell a \ell^{-1} \in L \cap N_p \implies L \cap N_p \lhd L$. Or just use 2nd Iso Thm. By 2nd Iso Thm,

$$\frac{L}{L \cap N_p} \cong \frac{L \cdot N_p}{N_p} \tag{13}$$

so

$$|L : L \cap N_p| = \frac{|L|}{|L \cap N_p|} = |\frac{L}{L \cap N_p}| = |\frac{L \cdot N_p}{N_p}| \tag{14}$$

since $L \cdot N_p \leq G \implies \frac{L \cdot N_p}{N_p} \leq \frac{G}{N_p}$ (the factor group $\frac{L \cdot N_p}{N_p}$ is subgroup of the factor group $\frac{G}{N_p}$. Thus,

$$|\frac{L \cdot N_p}{N_p}| \mid |\frac{G}{N_p}| = \frac{|G|}{|N_p|} = p^k. \tag{15}$$

so $|L : L \cap N_p|$ divide p^k implies that $|L : L \cap N_p|$ is some power of p.
$|L \cap N_p|$ is not divisible by p since $|N_p|$ is not divisible by p, and $|L \cap N_p|$ is a factor of $|N_p|$. (Since $L \cap N_p \leq N_p$).
Thus, $L \cap N_p$ is indeed a normal p-complement in L.

Now we show $\frac{L \cdot N_p}{L}$ is a normal p-complement in $\frac{G}{L}$.
Show

$$(L \cdot N_p)/L \lhd G/L \tag{16}$$

Let $\ell \in L,\ n \in N_p,\ g \in G$, so

$$\begin{aligned}
&(g + L)(\ell \cdot n + L)(g^{-1} + L) && (17)\\
&= g \cdot \ell \cdot n \cdot g^{-1} + L && (18)\\
&= g \ell g^{-1} g n g^{-1} + L && (19)\\
&= \ell' n' + L && (20)\\
&\implies \in (L \cdot N_p)/L && (21)
\end{aligned}$$

where $\ell' = g \ell g^{-1} \in L$ and $n' = g n g^{-1} \in N_p$ since normality.
so the last inclusion follows. Thus,

$$(L \cdot N_p)/L \lhd G/L \tag{22}$$

By third isomorphism, we have

$$\frac{G/L}{(L \cdot N_p)/L} \cong G/(L \cdot N_p) \tag{23}$$

The reason we could use 3rd Iso Thm because

$$L \lhd G, \tag{24}$$

and $L \cdot N_p \lhd G$ because of the same proof.
for $\ell \in L, n \in N_p, g \in G$, so we have

$$g(\ell \cdot n) g^{-1} = (g \ell g^{-1}) \left(g n g^{-1}\right) = \ell' n' \in L \cdot N_p. \tag{25}$$

Thus,

Problem 2

Let G be a finite group, $L \lhd G$ and p a prime. Suppose that N_p is a normal p-complement in G. Show that $L \cap N_p$ is a normal p-complement in L and $L \cdot N_p/L$ is a normal p-complement in G/L.

Solution:

Problem 3

Let p be a prime and denote the cyclic group of order p by C. Determine the number of all automorphisms of $C \times C$.

Problem 4

For fun: Let p be a prime, $0 < k < t$ integers, and denote the cyclic group of order p^k by A and the cyclic group of order p^t by B. Determine the number of all automorphisms of $A \times B$ and $A \times A$.

Problem 5

From Final: Let a solvable group G act faithfully and transitively on the set Ω, where $|\Omega| = 35$.

1. Prove that this action is not primitive.
2. Show that if G is abelian then it must be cyclic.