Galois Theory: GAL #10

Due on May 6, 2022 at 11:59pm

Prof Matyas Domokos Section 12 & 15

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2023

HW10
Exercise 12.4.12
Exercise 12.4.13
Exercise 15.1.2

Problem 1

Exercise 12.4.12 Prove that $X^4 - 10X^2 + 1$ is irreducible over $\mathbb{Q}$, but it is reducible in $(\mathbb{Z}/p\mathbb{Z})[X]$ for any prime p.

Soln:

Proof. We claim the minimum polynomial is $M_{\mathbb{Q}}(\sqrt{2}+\sqrt{3}) = X^4 - 10X^2 + 1$. Observe that

$$
\begin{aligned}
&(\sqrt{2}+\sqrt{3})^4 - 10(\sqrt{2}+\sqrt{3})^2 + 1 && (1)\\
=&(5+2\sqrt{6})^2 - 10(5+2\sqrt{6}) + 1 && (2)\\
=&0 && (3)
\end{aligned}
$$

Thus, $(\sqrt{2}+\sqrt{3})$ is a root of $X^4 - 10X^2 + 1$. We know $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ is a degree 4 extension over $\mathbb{Q}$, so $X^4 - 10X^2 + 1$ is the minimal polynomial over $\mathbb{Q}$, hence irreducible.

□

Problem 2

Exercise 12.4.13 **Soln:**

Problem 3

Exercise 15.1.2 Soln: