Adv Abstract Algebra: AAA #HW03

Due on 2022 at 11:59PM

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Homework set 3

Problem 1

Suppose that a group G has order 312. Prove that G has a proper normal subgroup.

Solution:

$312 = 2^3 \times 3 \times 13$.

The number of Sylow $p = 2$ subgroup, n, has several possibilities

$$1, 3, 13, 39 \equiv \mod p = 2 \tag{1}$$

and they divide $m = 3 \times 13$.

The number of Sylow $p = 3$ subgroup, n has several possibilities.

$$1, 4, 13 \equiv 1 \mod p = 3 \tag{2}$$

$$\tag{3}$$

and they divide $m = 8 \cdot 13$.

However, the number of Sylow $p = 13$ subgroup is one, since 1 is the only number $\equiv 1 \mod p = 13$ and divide $m = 2^3 \cdot 3$ at the same time.

By Sylow's theorem, (and corollary) G has a proper normal subgroup of order 13.

The unique Sylow 13 subgroup.

Problem 2

Suppose that a group G has order 1960. Prove that G has a proper normal subgroup.

Solution:

Suppose a group has order $1960 = 2^3 \cdot 5 \cdot 7^2$, the number of Sylow $p = 2$ subgroup has several possibilites

$$1, 5, 7 \equiv 1 \mod p = 2 \tag{4}$$

and divide $m = 5 \cdot 7 \cdot 7$.
The number of Sylow $p = 5$ subgroup has several possibilities, for example, $1, 56, 196 \equiv 1 \mod p = 5$ and divide $m = 2^3 \cdot 7^2$.
But the number n of Sylow $p = 7$ subgroup has 2 possibilities

$$1, 8 \equiv 1 \mod p = 7 \tag{5}$$

$$\tag{6}$$

and divide $m = 2^3 \cdot 5 = 40$.
Suppose $n = 8$. (If $n = 1$, then we are done)
Since G acts on $Syl_7(G)$ by conjugation and the action is transitive, G is essentially permuting the $8 Syl_7(G)$ subgroups.
Thus, we could define a homomorphism

$$G \xrightarrow{\psi} S_8 \tag{7}$$

(ψ is indeed a homomorphism because of the definition of group action)
$ker\psi$ cannot be the whole group G, since G acts transitively on the $Syl_7(G)$ groups, and we are assuming there is 8 of them, so ψ cannot map everything in G to the identity permutation.

Problem 3

For $A \leq G$, $|G : A|$ finite and A abelian, let $\tau_{G \mapsto A}$ denote the transfer homomorphism from G to A. Let $g \in G$ and $b \in N_G(A)$. Show that $\tau_{G \mapsto A}(g)$ commutes with b.
Hint: If $h_1, \dots, h_n$ is a set of right coset representatives of A then show that $bh_1, \dots, bh_n$ is also a set of right coset representatives of A.