Galois Theory HW04

Xianzhi

Due March 11, 2022

Exercise 5.3.9

Question 1. *Is the polynomial $X^4 - 2$ irreducible over the field $\mathbb{Q}(\sqrt{3})$?*

Exercise 6.4.6

Question 2. *Let L be the splitting field over $\mathbb{Q}$ of $X^5 - 2$ over $\mathbb{Q}$. Show that the Galois group $G := \Gamma(L : \mathbb{Q})$ has order 20, and G has a normal subgroup N with $|N| = 5$ such that the factor group G/N is cyclic.*

Exercise 6.4.7

Question 3. *Let p be an irreducible polynomial over a subfield K of $\mathbb{C}$, and denote by L the splitting field of p over K. Show that if the Galois group $\Gamma(L : K)$ is abelian (i.e. commutative), then its order equals the degree of p.*

A question from HW02

Question 4. *Show number of automorphisms of a finite degree field extension divides the degree of the field extension.*

Let $K \subset L, L : K$ be a finite degree field extension. Recall

$$\Gamma(L : K) = \{g \in \Gamma(L) : g(x) = x \quad \forall x \in K\}$$
$$\text{WTS:} \quad |\Gamma(L : K)| \mid [L : K].$$

Recall Artin's theorem, let $\Gamma(L : K)$ be the finite subgroup. (Since $|\Gamma(L : K)|$ is bounded by $[L : K] < \infty$.) and

$$M = \{x \in L : \forall g \in \Gamma(L : K) : g(x) = x\}$$

so $K \subset M$, and $[L : M] = |\Gamma(L : K)|$. Thus, consider $K \subset M \subset L$,

$$[L : K] = [L : M][M : K]$$

where $[L : M] = |\Gamma(L : K)|$, so $|\Gamma(L : K)|$ divides $[L : K]$.