# Galois Theory hw1

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## 2.1.8

**Question 1.** *Let $u$ be one of the complex roots of the polynomial $x^3 - 2x + 2$. Present $u^7$ and $1/(u-1)$ as a $\mathbb{Q}$-linear combination of $1, u, u^2$.*

Since $u$ is a root of $x^3 - 2x + 2$, we have $u^3 = 2u - 2$, so

$$\begin{aligned} u^7 &= u \cdot u^3 \cdot u^3 \\ &= -8u^2 + 12u - 8 \end{aligned}$$

Write

$$\begin{aligned} 0 = u^3 - 2u + 2 &= (u^2 + u - 1)(u - 1) + 1 \\ \frac{1}{u-1} &= \frac{u^2 + u - 1}{(u-1)(u^2 + u - 1)} = \cdots \end{aligned}$$

## 2.1.9

**Proposition 1.** *A quadratic field extension can always be obtained by adjoining the square root of some element of the base field.*

$K \subset L$, with $\dim_k(L) = 2$. Take $\alpha \in L, \alpha \notin K$. Thus, $(1, \alpha)$ is a linearly independent set over $K$. Since $\dim_k(L) = 2$, $(1, \alpha)$ is a basis for $L$. $\alpha^2 \in L$ can be expressed as a linear combination of 1 and $\alpha$, with coefficients in $K$.

$$\alpha^2 = -b\alpha - c \tag{1}$$

$$\implies \alpha^2 + b\alpha + c = 0, b, c \in K. \tag{2}$$

Thus, $\alpha$ is a root of $f(x) = x^2 + bx + c$. Since $\alpha \notin K$, this $f(x)$ is irreducible over $K$. Let $\beta = \sqrt{b^2 - 4c}$, then $L = K(\beta)$, and $\beta^2 \in K$.

## 3.2.10

The trick:

$$[(\mathbb{Q}(\alpha))(\beta) : \mathbb{Q}] = \underbrace{[(\mathbb{Q}(\alpha))(\beta) : \mathbb{Q}(\alpha)]}_{=\deg_{\mathbb{Q}(\alpha)}(\beta) \le \deg_{\mathbb{Q}}(\beta) = 3}[\mathbb{Q}(\alpha) : \mathbb{Q}]$$

Enlarge

$$[(\mathbb{Q}(\alpha))(\beta) : \mathbb{Q}(\alpha)] = \deg_{\mathbb{Q}(\alpha)}(\beta) \le \deg_{\mathbb{Q}}(\beta) = 3$$