Sheet 5: A spectral diameter bound

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Let G be a finite undirected d-regular graph on n vertices. Let A denote the adjacency operator, and let $b_0, b_1, \ldots, b_{n-1}$ be an orthonormal eigenbasis with real eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Let $\rho = \rho_0(G)$ be the spectral radius. On this sheet, we give an upper bound on the diameter using ρ.

Of course, $b_i^\top b_i = 1$ $(0 \leq i < n)$. Let us force product the other way round. Let

$$U_i = b_i b_i^\top \quad (0 \leq i < n). \tag{1}$$

These matrices may look boring at first, but they are not. It turns out that they give an 'orthonormal eigendecomposition' for A.

Lemma 1 *Let A, B be $n \times n$ matrices such that $Av_i = Bv_i$ $(0 \leq i < n)$ for a basis (v_i) for $\mathbb{R}^n$. Then $A = B$.*

Theorem 2 *The following hold for $1 \leq i, j < n$:*
1) $U_i^2 = U_i$;
2) $U_i U_j = 0$ $(i \neq j)$;
3) $AU_i = \lambda_i U_i$;
4) ${}_{k=1}^{n} U_k = I$;
5) ${}_{k=1}^{n} \lambda_k U_k = A$;

Definition 3 *For $x, y \in V(G)$ let the* distance *$d(x, y)$ be the minimal length of a path going from x to y.*

Lemma 4 *Let G be an undirected, connected graph. Then d is a metric on $V(G)$.*

Proof. We show d is a metric on $V(G)$. where G undirected connected graph.
(1) $d(x, y) \geq 0$, since the length of a path is always positive or zero, when x and y are the same vertex. $x = y \implies d(x, y) = 0$ since the shortest path is of zero length when we are going from a vertex to itself. $d(x, y) = 0 \implies x = y$, the shortest path is zero length, so we never moved, it's the same vertex.
(2) $d(x, y) = d(y, x)$ since traversing the shortest path backwards, the length is the same. (The graph is undirected)
(3) $d(x, z) \leq d(x, y) + d(y, z)\ \forall\ x, y, z \in V(G)$. the shortest path from x to y, then the shortest path from y to z could only be a longer path than the shortest path from x to z.
Since graph G is connected, $d(x, y) < \infty$. ■

Definition 5 *Let the* diameter *of G be*

$$\operatorname{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}.$$

One can give a general lower bound on the diameter as follows.

Lemma 6 *Let G be a d-regular undirected graph on n vertices with $d \geq 3$. Then*

$$\text{diam}(G) \geq \log_{d-1}(n/3). \tag{2}$$

Proof. Let G be d-regular, undirected graph on n vertices $(d \geq 3)$

$$diam(G) \geq \log_{d-1}\left(\frac{n}{3}\right) \tag{3}$$

$$\tag{4}$$

Let r denote $diam(G)$, then showing

$$r \geq \log_{d-1}\left(\frac{n}{3}\right) \tag{5}$$

is the same as showing $(d-1)^r \geq \frac{n}{3}$, which is

$$3(d-1)^r \geq n. \tag{6}$$

$$n \leq 3(d-1)^r \tag{7}$$

we could have an upper bound on n by fixing a vertex v_0 first, then since G is d regular, v_0 has d neighbors, at 1st level, as we expand outwards. At 2nd level, each of those neighbors of v_0 has $d-1$ new neighbors, so we have $d(d-1)$ in total (at max). At 3rd level, $d(d-1)^2$ vertices in total (at max). As we go on, we could iterate at maximum $r-1$ levels, since the diameter of G is r. If our fixed vertex v_0 happens to be at the "center" of the graph, then n might be small compared to the upper bound we obtained in this way.

$$n \leq 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{r-1} \tag{8}$$

$$= 1 + \frac{d(d-1)^r - d}{d-2} \tag{9}$$

$$\leq 3(d-1)^r \tag{10}$$

$$\tag{11}$$

if we show the last $\leq$, then we are done.
Now, we have $3 \leq d$.
Use $X := (d-1)^r$ as short hand.

$$3X - 1 < 3X < dX \tag{12}$$

$$6X - 2 < 2dX \tag{13}$$

$$dX - 2 < 3X(d-2) \tag{14}$$

$$\frac{dX-2}{d-2} < 3X \tag{15}$$

$$\frac{d-2+dX-d}{d-2} < 3X \tag{16}$$

$$1 + \frac{dX}{d-2} - \frac{d}{d-2} < 3X \tag{17}$$

$$1 + \frac{d(d-1)^r}{d-2} - \frac{d}{d-2} < 3(d-1)^r \tag{18}$$

as desired. and we are done.

■

So, for a fixed degree, the diameter is at least logarithmic in the number of vertices.

It turns out that for good expanders, one can get a logarithmic upper bound for the diameter.

Lemma 7 *Let $x, y \in V(G)$. Then*

$$d(x,y) = \min\left\{k \mid X_{\{x\}}^{\top}(A^k)X_{\{y\}} \neq 0\right\}. \tag{19}$$

Theorem 8 *We have*

$$\operatorname{diam}(G) \leq \frac{\log(n)}{\log(d/\rho_0(G))}. \tag{20}$$

Hint: decompose $X_{\{x\}}$ and $X_{\{y\}}$.

solution:

$$diam(G) = \max_{x,y \in V(G)} \min_{\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0} k \tag{21}$$

(since A^k has non-negative entries, so $\neq 0$ is the same as > 0).

$$\begin{aligned}
\chi_{\{x\}}^T A^k \chi_{\{y\}} &= \chi_{\{x\}}^T \sum_{i=0}^{n-1} \lambda_i^k U_i \chi_{\{y\}} && (22)\\
&= \sum_{i=0}^{n-1} \lambda_i^k (U_i)_{x,y} && (23)\\
&= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} && (24)\\
& && (25)
\end{aligned}$$

since $\lambda_0 = d$, and

$$U_0 = b_0 b_0^T = \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \end{bmatrix} \tag{26}$$

and

$$\begin{aligned}
(U_i)_{x,y} &= (b_i b_i^T)_{x,y} && (27)\\
&= (b_i)_x \cdot (b_i)_y && (28)\\
&= x_i \cdot y_i && (29)
\end{aligned}$$

where

$$\begin{aligned}
x_i &:= (b_i)_x && (30)\\
y_i &:= (b_i)_y && (31)\\
\text{for } i &\in \{0, 1, 2, \ldots, n-1\} && (32)
\end{aligned}$$

is the x-th entry in vector b_i, the y-th entry in vector b_i.

$$
\begin{aligned}
\chi_{\{x\}}^T A^k \chi_{\{y\}} &= \lambda_0^k (U_0)_{x,y} + \sum_{i=1}^{n-1} \lambda_i^k (U_i)_{x,y} && (33)\\
&= \frac{d^k}{n} + \sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i && (34)\\
&\geq |\frac{d^k}{n}| - |\sum_{i=1}^{n-1} \lambda_i^k x_i \cdot y_i| > && (35)\\
&\geq \frac{d^k}{n} - \rho^k \sum_{i=1}^{n-1} |x_i||y_i| && (36)\\
&\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=1}^{n-1} |x_i|^2} \sqrt{\sum_{i=1}^{n-1} |y_i|^2} && (37)\\
&\geq \frac{d^k}{n} - \rho^k \sqrt{\sum_{i=0}^{n-1} |x_i|^2} \sqrt{\sum_{i=0}^{n-1} |y_i|^2} && (38)\\
& && (39)
\end{aligned}
$$

where we used inverse triangle inequality, and then Cauchy-Shwarz inequality.
since

$$
\begin{aligned}
\sum_{i=0}^{n-1} |x_i|^2 = \sum_{i=0}^{n-1} (b_i)_x (b_i)_x = \sum_{i=0}^{n-1} \left(b_i b_i^T\right)_{x,x} = (I)_{x,x} = 1 && (40)\\
= \frac{d^k}{n} - \rho^k \cdot 1 \cdot 1 && (41)\\
= \frac{d^k}{n} - \rho^k && (42)
\end{aligned}
$$

Thus,

$$
\chi_{\{x\}}^T A^k \chi_{\{y\}} \geq \frac{d^k}{n} - \rho^k \ (*) \tag{43}
$$

Since max is taken over $x, y \in V(G)$, we want $\frac{d^k}{n} - \rho^k > 0$. So

$$
\begin{aligned}
\frac{d^k}{n} &> \rho^k && (44)\\
\frac{d^k}{\rho^k} &> n && (45)\\
\log \left(\frac{d}{\rho}\right)^k &> \log n && (46)\\
k &> \frac{\log n}{\log (d/\rho)} && (47)
\end{aligned}
$$

Min of k is taken over all $\chi_{\{x\}}^T A^k \chi_{\{y\}} > 0$, so k has $\frac{\log n}{\log(d/\rho)}$ growth.

end solution

When G is bipartite, the above theorem (and also the theorem on the previous sheet) do not give anything reasonable, as $\rho_0(G) = d$. However, there is a similar diameter bound for bipartite graphs where all the other eigenvalues (in absolute value) are bounded away from d.

Theorem 9 *Let G be a finite, connected, undirected d-regular bipartite graph on n vertices. Assume that $|\lambda_i| \leq r$ ($1 \leq i < n-1$). Then G has logarithmic diameter.*