Galois Theory hw2

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Q1

Question 1. *Show $\pi - \sqrt{\pi}$ is not algebraic over $\mathbb{Q}$.*

First we show $\sqrt{\pi}$ is not algebraic over $\mathbb{Q}$. Assume for contradiction that $\sqrt{\pi}$ is algebraic over $\mathbb{Q}$, so $\exists\ a_n, \cdot, a_0 \in \mathbb{Q}$ not all zero such that

$$a_n(\sqrt{\pi})^n + \cdots + a_1\sqrt{\pi} + a_0 = 0 \quad \text{denote by } p(\sqrt{\pi}) = 0$$

Consider

$$0 = p(\sqrt{\pi})p(-\sqrt{\pi}) = q(\pi) \text{ for some polynomial } q$$

some algebra omitted. Thus, $q(\pi) = 0$, so π is algebraic over $\mathbb{Q}$, contradiction.

Now assume for contradiction $\pi - \sqrt{\pi}$ is algebraic over $\mathbb{Q}$, so $\exists\ b_n \cdots, b_0 \in \mathbb{Q}$ not all zero s.t.

$$b_n(\pi - \sqrt{\pi})^n + \cdots + b_1(\pi - \sqrt{\pi}) + b_0 = 0$$

multiply out this expression, we obtain a not all zero polynomial of $\sqrt{\pi}$ such that it equals 0. A contradiction.

Q2

i is not contained in the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

We know the splitting field of $X^3 - 2$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$, first we show $i \notin \mathbb{Q}(\omega)$. Assume for contradiction $i \in \mathbb{Q}(w) \implies \sqrt{3} \in \mathbb{Q}(\omega)$, which implies

$$\mathbb{Q}(i, \sqrt{3}) \subset \mathbb{Q}(\omega),$$

but the LHS has degree 4 over $\mathbb{Q}$, and RHS has degree 2 over $\mathbb{Q}$.

$i \notin \mathbb{Q}(\omega)$, so $\deg_{\mathbb{Q}(\omega)} i > 1$. Thus, $\mathbb{Q}(\omega, i)$ is degree 4 extension over $\mathbb{Q}$. However, if assume for contradiction $i \in \mathbb{Q}(\omega, \sqrt[3]{2}$, then $\mathbb{Q}(\omega, i) \subset \mathbb{Q}(\omega, \sqrt[3]{2})$ but the former has degree 4, and the latter has degree 6, and $4 \nmid 6$.