Galois Theory: GAL Final

Due on May, 2022 at 11:59pm

Prof Matyas Domokos Spr 2022

Xianzhi

2023

Final

Problem 1

$$Gal_{\mathbb{Q}}(h) = \Gamma(L : \mathbb{Q}) \tag{1}$$

where L is the splitting field of h over $\mathbb{Q}$.
We try to find the roots of $h = f \circ g$.
Let $Y = X^2 + X + 2$. so $h = Y^3 - Y + 7$. This is a cubic, so we know it's solvable by radicals, thus, we could obtain a radical sequence, using Cardano's formula, say, $a_1, a_2, \ldots, a_n$, so $Y^3 - Y + 7$ splits as root factors in

$$\mathbb{Q}(a_1, a_2, \ldots, a_n) \tag{2}$$

then $X^2 + X + 2$ has roots

$$\alpha_1, \alpha_2 = \frac{-1 \pm \sqrt{7}i}{2} \tag{3}$$

so we adjoin $\sqrt{7}, i$ to $\mathbb{Q}(a_1, a_2, \ldots, a_n)$

$$L := \mathbb{Q}(a_1, a_2, \ldots, a_n, \sqrt{7}, i) \tag{4}$$

is clearly a radical sequence. and h splits as root factors in L. Thus, apply Galois's Theorem $\Longrightarrow$ the Galois group $Gal_{\mathbb{Q}}(h)$ is solvable.