

# Galois Theory: GAL #06

Due on Apr 01, 2022 at 11:59pm

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2023

HW06

Apr 01, 2022

Exercise 9.4.1

Exercise 9.4.2

Exercise 9.4.3

## Problem 1

**Exercise 9.4.1** Let  $L$  be the splitting field over  $\mathbb{Q}$  of a cubic polynomial with rational coefficients, and  $\omega$  a primitive cubic root of unity. Show that  $L(\omega)$  is a radical extension of  $\mathbb{Q}$ , by exhibiting explicitly a radical sequence.

(Hint: recall Cardano's Method.)

**Soln:**

Let  $L$  be splitting field over  $\mathbb{Q}$  of a cubic polynomial with radical coefficients  $aX^3 + bX^2 + cX + d$ ,  $a, b, c, d \in \mathbb{Q}$ . WLOG,  $L$  is the same splitting field if the polynomial is monic

$$X^3 + \frac{b}{a}X^2 + \frac{c}{a}X + \frac{d}{a}, \quad (1)$$

so we could assume  $a = 1$  from the beginning.

Also,  $L$  is the same if we shift by a rational amount  $b/3$  of all the roots of this polynomial, because

$$X^3 + bX^2 + cX + d = (X + \frac{b}{3})^3 + (c - \frac{b^2}{3})(X + \frac{b}{3}) + d + \frac{b^3}{9} - \frac{cb}{3} - \frac{b^3}{27} \quad (2)$$

Thus, we could assume our polynomial is of the form:

$$X^3 + pX + q, \quad p, q \in \mathbb{Q}. \quad (3)$$

$L(\omega)$  is radical extension, since

$$\mathbb{Q} \left( \sqrt{-3}, \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right) = L(\omega). \quad (*) \quad (4)$$

LHS of  $(*)$  is a radical sequence, since

$$(\sqrt{-3})^2 \in \mathbb{Q} \quad (5)$$

$$\left( \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^2 \in \mathbb{Q}(\sqrt{-3}) \quad (6)$$

$$u^3 := \left( \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right)^3 \in \mathbb{Q} \left( \sqrt{-3}, \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right) \quad (7)$$

$$v := \frac{-p}{3u} = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad (8)$$

$$\omega := -\frac{1}{2} + \frac{\sqrt{-3}}{2} \quad (9)$$

## Problem 2

**Exercise 9.4.2** Let  $L$  be the splitting field over  $\mathbb{Q}$  of a monic irreducible cubic polynomial  $f$  in  $\mathbb{Q}[x]$ .

1. Show that  $\Gamma(L : \mathbb{Q})$  has order 3 iff the discriminant of  $f$  is the square of a rational number. Recall that the discriminant of  $f$  is

$$\prod_{1 \leq i < j \leq 3} (\alpha_i - \alpha_j)^2, \quad (10)$$

where  $\alpha_i$  are the complex roots of  $f$ .

2. Give an example of a monic cubic polynomial  $f$  with  $|\Gamma(L : \mathbb{Q})| = 3$ .  
*You may want to use the fact that the discriminant of  $X^3 + pX + q \in \mathbb{Q}[X]$  is  $-4p^3 - 27q^2$ .*

**Soln:**

**Part A**

**Part B**

### Problem 3

**Exercise 9.4.3** Let  $L$  be a subfield of  $\mathbb{C}$  such that  $\Gamma(L)$  is the dihedral group  $D_4$  (having 8 elements), and  $L$  a Galois extension of  $\mathbb{Q}$ . Show that  $L$  is a radical extension of  $\mathbb{Q}$ .

*Proof.* Let us denote the dihedral group  $D_4$  this way:

$$D_4 = \langle f, t \mid f^4 = 1 = t^2, ft = tf^3 \rangle \quad (11)$$

We know degree 2 extension is obtained by adjoining an square root from previous homework. Since  $\Gamma(L : \mathbb{Q})$  is  $\cong D_4$ , and is Galois, we can use Galois Correspondence. Normal subgroup corresponds to normal (Galois) extension.

$$\begin{array}{ccccccc} \{e\} & \triangleleft & \langle t \rangle & \triangleleft & \langle t, f^2 \rangle & \triangleleft & D_4 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ L & \supset & E & \supset & F & \supset & \mathbb{Q} \end{array} \quad (12)$$

Where the up-down arrow  $\updownarrow$  indicates the relationship being the field is fixed field of the group. Each of the subgroup has index 2 in the previous one, and it's a subnormal chain. So we have

$$\Gamma(F : \mathbb{Q}) \cong \Gamma(L : \mathbb{Q}) / \Gamma(L : F) = |D_4 / \langle t, f^2 \rangle| = 2 \quad (13)$$

So  $\Gamma(F : \mathbb{Q})$  has order 2, so  $F : \mathbb{Q}$  has degree 2. So  $F = \mathbb{Q}(\alpha)$  where  $\alpha^2 \in \mathbb{Q}$ . Similarly,

$$\Gamma(E : F) \cong \Gamma(L : F) / \Gamma(L : E) = |\langle t, f^2 \rangle / \langle t \rangle| = 2 \quad (14)$$

So  $|\Gamma(E : F)| = 2$  implies that  $E : F$  has degree 2, so  $E = F(\beta)$ , where  $\beta^2 \in F$ .

$\Gamma(L : E) \cong \langle t \rangle$ ,  $|\langle t \rangle| = 2$  implies that  $L : E$  has degree 2.

$\implies L = E(\gamma)$ , where  $\gamma^2 \in E$ .

$\implies L = \mathbb{Q}(\alpha, \beta, \gamma)$ , so  $L$  is a radical extension.

□

**Problem 6**

Evaluate the integrals  $\int_0^1 (1 - x^2)dx$  and  $\int_1^\infty \frac{1}{x^2}dx$ .