Algorithms: COMP3121/3821/9101/9801

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

TOPIC 2: FAST LARGE INTEGER MULTIPLICATION

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

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        X X X X  <- first input integer
      * X X X X  <- second input integer
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- **Can we do it faster than in n^2 many steps??**

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$$\begin{aligned} AB &= A_1 B_1 2^n + (A_1 B_0 + A_0 B_1) 2^{\frac{n}{2}} + A_0 B_0 \\ &= A_1 B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1 B_1 - A_0 B_0) 2^{\frac{n}{2}} + A_0 B_0 \end{aligned}$$

function MULT(A, B)
  if $|A| = |B| = 1$ **then return** AB
  else
    $A_1 \leftarrow \text{MoreSignificantPart}(A);$
    $A_0 \leftarrow \text{LessSignificantPart}(A);$
    $B_1 \leftarrow \text{MoreSignificantPart}(B);$
    $B_0 \leftarrow \text{LessSignificantPart}(B);$
    $U \leftarrow A_0 + A_1;$
    $V \leftarrow B_0 + B_1;$
    $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$
    $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1);$
    $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V});$
    return $W\, 2^n + (Y - X - W)\, 2^{n/2} + X$
  end if
end function

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$$f(n) = c\,n = O(n^{\log_2 3 - \varepsilon}) \quad \text{for any} \quad 0 < \varepsilon < 0.5$$

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- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) < \Theta(n^{1.585})$$

without going through the messy calculations!

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- So,

$$AB = A_2 B_2\, 2^{4k} + (A_2 B_1 + A_1 B_2) 2^{3k} + (A_2 B_0 + A_1 B_1 + A_0 B_2) 2^{2k} + \\ + (A_1 B_0 + A_0 B_1) 2^k + A_0 B_0$$

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$$AB = A_2B_2\,2^{4k} + (A_2B_1 + A_1B_2)2^{3k} + (A_2B_0 + A_1B_1 + A_0B_2)2^{2k} +$$
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$$(A_2 + A_1 + A_0)(B_2 + B_1 + B_0) = \\ A_0B_0 + A_1B_0 + A_2B_0 + A_0B_1 + A_1B_1 + A_2B_1 + A_0B_2 + A_1B_2 + A_2B_2 \quad ???$$

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- Not clear at all how to get $C_0 - C_4$ with 5 multiplications only ...

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P_A(x) &= A_2\, x^2 + A_1\, x + A_0; \\
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- Note that
$$A = A_2\, (2^k)^2 + A_1\, 2^k + A_0 = P_A(2^k);$$
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with only 5 multiplications, we can then obtain the product of numbers A and B simply as

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- Thus, we compute

$$P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$$
$$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$

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- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$\begin{aligned}
P_A(-2) &= A_2(-2)^2 + A_1(-2) + A_0 = 4A_2 - 2A_1 + A_0 \\
P_A(-1) &= A_2(-1)^2 + A_1(-1) + A_0 = A_2 - A_1 + A_0 \\
P_A(0) &= A_2 0^2 + A_1 0 + A_0 = A_0 \\
P_A(1) &= A_2 1^2 + A_1 1 + A_0 = A_2 + A_1 + A_0 \\
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P_A(0) &= A_2 0^2 + A_1 0 + A_0 = A_0 \\
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P_A(2) &= A_2 2^2 + A_1 2 + A_0 = 4A_2 + 2A_1 + A_0.
\end{aligned}$$

- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

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P_B(-1) &= B_2(-1)^2 + B_1(-1) + B_0 = B_2 - B_1 + B_0 \\
P_B(0) &= B_2 0^2 + B_1 0 + B_0 = B_0 \\
P_B(1) &= B_2 1^2 + B_1 1 + B_0 = B_2 + B_1 + B_0 \\
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\end{aligned}$$

- These evaluations involve only additions because $2A = A + A;\ \ 4A = 2A + 2A.$

The Karatsuba trick: slicing into 3 pieces

- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and
$$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$
we can now obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$ with only 5 multiplications of large numbers:

$$\begin{aligned} P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2) \end{aligned}$$

$$\begin{aligned} P_C(-1) &= P_A(-1)P_B(-1) \\ &= (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(0) &= P_A(0)P_B(0) \\ &= A_0 B_0 \end{aligned}$$

$$\begin{aligned} P_C(1) &= P_A(1)P_B(1) \\ &= (A_0 + A_1 + A_2)(B_0 + B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(2) &= P_A(2)P_B(2) \\ &= (A_0 + 2A_1 + 4A_2)(B_0 + 2B_1 + 4B_2) \end{aligned}$$

The Karatsuba trick: slicing into 3 pieces

- Thus, if we represent the product $C(x) = P_A(x)P_B(x)$ in the coefficient form as $C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ we get

$$\begin{aligned}
C_4(-2)^4 + C_3(-2)^3 + C_2(-2)^2 + C_1(-2) + C_0 &= P_C(-2) = P_A(-2)P_B(-2) \\
C_4(-1)^4 + C_3(-1)^3 + C_2(-1)^2 + C_1(-1) + C_0 &= P_C(-1) = P_A(-1)P_B(-1) \\
C_4 0^4 + C_3 0^3 + C_2 0^2 + C_1 \cdot 0 + C_0 &= P_C(0) = P_A(0)P_B(0) \\
C_4 1^4 + C_3 1^3 + C_2 1^2 + C_1 \cdot 1 + C_0 &= P_C(1) = P_A(1)P_B(1) \\
C_4 2^4 + C_3 2^3 + C_2 2^2 + C_1 \cdot 2 + C_0 &= P_C(2) = P_A(2)P_B(2).
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\end{aligned}$$

- Simplifying the left side we obtain

$$\begin{aligned}
16C_4 - 8C_3 + 4C_2 - 2C_1 + C_0 &= P_C(-2)\\
C_4 - C_3 + C_2 - C_1 + C_0 &= P_C(-1)\\
C_0 &= P_C(0)\\
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- With the coefficients C_0, C_1, C_2, C_3, C_4 obtained, we can now form the polynomial $P_C(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4$.

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- We can now compute $P_C(2^k) = C_0 + C_1 2^k + C_2 2^{2k} + C_3 2^{3k} + C_4 2^{4k}$ in linear time, because computing $P_C(2^k)$ involves only binary shifts of the coefficients plus $O(k)$ additions.

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- Thus we have obtained $A \cdot B = P_A(2^k)P_B(2^k) = P_C(2^k)$ with only 5 multiplications!

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- With the coefficients C_0, C_1, C_2, C_3, C_4 obtained, we can now form the polynomial $P_C(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4$.
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- Thus we have obtained $A \cdot B = P_A(2^k)P_B(2^k) = P_C(2^k)$ with only 5 multiplications!
- Here is the complete algorithm:

function MULT(A, B)

obtain A_0, A_1, A_2 and B_0, B_1, B_2 such that $A = A_2\, 2^{2\,k} + A_1\, 2^k + A_0$; $\quad B = B_2\, 2^{2\,k} + B_1\, 2^k + B_0$;

form polynomials $P_A(x) = A_2 x^2 + A_1 x + A_0$; $\quad P_B(x) = B_2 x^2 + B_1 x + B_0$;

$$
\begin{aligned}
P_A(-2) &\leftarrow 4A_2 - 2A_1 + A_0 & \qquad P_B(-2) &\leftarrow 4B_2 - 2B_1 + B_0\\
P_A(-1) &\leftarrow A_2 - A_1 + A_0 & P_B(-1) &\leftarrow B_2 - B_1 + B_0\\
P_A(0) &\leftarrow A_0 & P_B(0) &\leftarrow B_0\\
P_A(1) &\leftarrow A_2 + A_1 + A_0 & P_B(1) &\leftarrow B_2 + B_1 + B_0\\
P_A(2) &\leftarrow 4A_2 + 2A_1 + A_0 & P_B(2) &\leftarrow 4B_2 + 2B_1 + B_0
\end{aligned}
$$

$$
\begin{aligned}
&P_C(-2) \leftarrow \text{MULT}(P_A(-2), P_B(-2)); \qquad && P_C(-1) \leftarrow \text{MULT}(P_A(-1), P_B(-1));\\
&P_C(0) \leftarrow \text{MULT}(P_A(0), P_B(0)); && \\
&P_C(1) \leftarrow \text{MULT}(P_A(1), P_B(1)); && P_C(2) \leftarrow \text{MULT}(P_A(2), P_B(2))
\end{aligned}
$$

$$
\begin{aligned}
C_0 &\leftarrow P_C(0); \qquad C_1 \leftarrow \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12}\\
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form $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$; compute
$P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$

return $P_C(2^k) = A \cdot B$.

end function

- How fast is this algorithm?

The Karatsuba trick: slicing into 3 pieces

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- We have replaced a multiplication of two n bit numbers with 5 multiplications of $n/3$ bit numbers with an overhead of additions, shifts and the similar, all doable in linear time $c\,n$;

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$$T(n) = 5T\left(\frac{n}{3}\right) + c\,n$$

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  we have $a = 5,\ \ b = 3$, so we consider $n^{\log_b a} = n^{\log_3 5} \approx n^{1.465\ldots}$

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- We now apply the Master Theorem:

  we have $a = 5,\;\; b = 3$, so we consider $n^{\log_b a} = n^{\log_3 5} \approx n^{1.465\ldots}$
- Clearly, the first case of the MT applies and we get $T(n) = O(n^{\log_3 5}) < O(n^{1.47})$.

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- Recall that the original Karatsuba algorithm runs in time

$$n^{\log_2 3} \approx n^{1.58} > n^{1.47}.$$

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- Thus, we got a significantly faster algorithm.
- Then why not slice numbers A and B into even larger number of slices? Maybe we can get even faster algorithm?
- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $n+1$ many equal slices...

Generalizing Karatsuba's algorithm

The general case - slicing the input numbers A, B into $n+1$ many slices

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- Slice A, B into $n+1$ pieces each:

$$
\begin{aligned}
A = \quad & A_n 2^{kn} \quad + \quad A_{n-1} 2^{k(n-1)} + \quad \cdots + \quad A_0 \\
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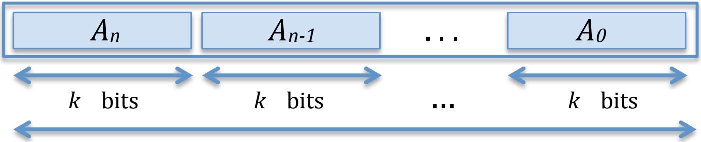

A divided into $n+1$ slices each slice k bits = $(n+1)\,k$ bits in total

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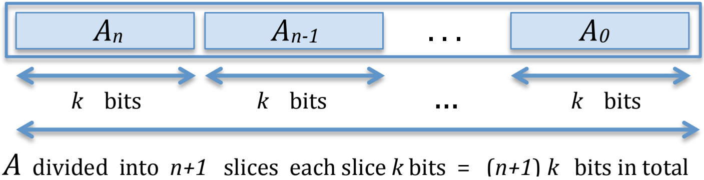

- We form the naturally corresponding polynomials:

$$
\begin{aligned}
P_A(x) &= A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0 \\
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- As before, we have:
  $A = P_A(2^k);\;\; B = P_B(2^k);\;\; AB = P_A(2^k)P_B(2^k) = (P_A(x) \cdot P_B(x))\,\big|_{x=2^k}$

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  $$P_C(x) = \sum_{j=0}^{2n} C_j x^j$$

Generalizing Karatsuba's algorithm

- Example:

$$
\begin{aligned}
(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\
a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
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+(a_0b_1 + a_1b_0)x + a_0b_0
\end{aligned}
$$

- In general: for

$$
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P_A(x) &= A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0 \\
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Generalizing Karatsuba's algorithm

- Example:

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P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \left(\sum_{i+k=j} A_iB_k \right) x^j = \sum_{j=0}^{2n} C_j\, x^j
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- We need to find the coefficients $C_j = \sum_{i+k=j} A_iB_k$ without performing $(n+1)^2$ many multiplications necessary to get all products of the form A_iB_k.

A VERY IMPORTANT DIGRESSION:

If you have two sequences $\vec{A} = (A_0, A_1, \ldots, A_{n-1}, A_n)$ and $\vec{B} = (B_0, B_1, \ldots, B_{m-1}, B_m)$, and if you form the two corresponding polynomials

$$P_A = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$$
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then the sequence $\vec{C} = (C_0, C_1, \ldots, C_{n+m})$ of the coefficients of the product polynomial, with these coefficients given by

$$C_j = \sum_{i+k=j} A_i B_k, \quad \text{for} \quad 0 \le j \le n+m,$$

is **extremely important** and is called the **LINEAR CONVOLUTION** of sequences $\vec{A}$ and $\vec{B}$ and is denoted by $\vec{C} = \vec{A} \star \vec{B}$.

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- Convolutions are bread-and-butter of signal processing, and for that reason it is **extremely important** to find fast ways of multiplying two polynomials of possibly very large degrees.
- In signal processing these degrees can be greater than 1000.
- This is the main reason for us to study methods of fast computation of convolutions (aside of finding products of large integers, which is what we are doing at the moment).

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n+1$ distinct input values $x_0, x_1, \ldots, x_n$:

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- It can be shown that if x_i are all distinct then this matrix is invertible.
- Such a matrix is called *the Vandermonde matrix*.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are all distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_n)$ the coefficients $A_0, A_1, \ldots, A_n$ of the polynomial $P_A(x)$ are uniquely determined:

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  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

Coefficient vs value representation of polynomials- ctd.

- If we fix the inputs $x_0, x_1, \ldots, x_n$ then commuting between a representation of a polynomial $P_A(x)$ via its coefficients and a representation via its values at these points is done via the following two matrix multiplications, with matrices made up from **constants**:

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- Thus, for fixed input values $x_0, \ldots, x_n$ this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

1. Given two polynomials of degree at most n,

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$$P_A(x)P_B(x) \leftrightarrow \{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\}$$

Our strategy to multiply polynomials fast:

1. Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

convert them into value representation at $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_{2n}, P_A(x_{2n}))\}$$

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3. Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

Fast multiplication of polynomials - continued

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$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \cdots + A_0 : \quad -n \leq m \leq n,$$

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can be found in time linear in the number of bits of the input numbers!

Fast multiplication of polynomials - ctd.

- We now perform $2n+1$ **multiplications of large numbers** to obtain

$$P_A(-n)P_B(-n),\ \ldots\ ,\ P_A(-1)P_B(-1),\quad P_A(0)P_B(0),\quad P_A(1)P_B(1),\ \ldots\ ,\ P_A(n)P_B(n)$$

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- For $P_C(x) = P_A(x)P_B(x)$ these products are $2n+1$ many values of $P_C(x)$:

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- Let $C_0, C_1, \ldots, C_{2n}$ be the coefficients of the product polynomial $C(x)$, i.e., let

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- We now have:

$$\begin{aligned}
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C_{2n}(-(n-1))^{2n} + C_{2n-1}(-(n-1))^{2n-1} + \cdots + C_0 &= P_C(-(n-1))\\
&\vdots\\
C_{2n}(n-1)^{2n} + C_{2n-1}(n-1)^{2n-1} + \cdots + C_0 &= P_C(n-1)\\
C_{2n}n^{2n} + C_{2n-1}n^{2n-1} + \cdots + C_0 &= P_C(n)
\end{aligned}$$

Fast multiplication of polynomials - ctd.

- This is just a system of linear equations, that can be solved for $C_0, C_1, \ldots, C_{2n}$:

$$\begin{pmatrix} 1 & -n & (-n)^2 & \ldots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \ldots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \ldots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

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- Thus the coefficients C_i can be obtained in linear time.
- So here is the algorithm we have just described:

function MULT(n, A, B)

if $|A| = |B| = 1$ **then return** AB

else

obtain $n+1$ slices $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ such that

$$A = A_n 2^{n\,k} + A_{n-1} 2^{(n-1)\,k} + \ldots + A_0$$

$$B = B_n 2^{n\,k} + B_{n-1} 2^{(n-1)\,k} + \ldots + B_0$$

form polynomials

$$P_A(x) = A_n x^n + A_{n-1} x^{(n-1)} + \ldots + A_0$$

$$P_B(x) = B_n x^n + B_{n-1} x^{(n-1)} + \ldots + B_0$$

for $m = -n$ to $m = n$ **do**

compute $P_A(m)$ and $P_B(m)$;

$P_C(m) \leftarrow$ MULT$(n, P_A(m) P_B(m))$

end for

compute $C_0,\ C_1, \ldots\ C_{2n}$ via

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -n & (-n)^2 & \ldots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \ldots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \ldots & n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix}.$$

form $P_C(x) = C_{2n} x^{2n} + \ldots + C_0$ and compute $P_C(2^k)$

return $P_C(2^k) = A \cdot B$

end if

end function

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- it is easy to see that the values of the two polynomials we are multiplying have at most $k + s$ bits where s is a constant which depends on n but does NOT depend on k:

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This is because each A_i is smaller than 2^k because each A_k has k bits; thus

$$|P_A(m)| < n^n(|A_n| + |A_{n-1}| + \cdots + |A_0|) < n^n \times n \times 2^k$$

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- Since s is constant, its impact can be neglected.

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- so we get:

$$T(N) = \Theta\left(N^{\log_b a}\right) = \Theta\left(N^{\log_{n+1}(2n+1)}\right)$$

- Note that

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- Thus, we would have to slice the input numbers into $2^{10} = 1024$ pieces!!

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- Consequently, slicing the input numbers in more than just a few slices results in a hopelessly slow algorithm, despite the fact that the asymptotic bounds improve as we increase the number of slices!
- The moral is: **In practice, asymptotic estimates are useless if the size of the constants hidden by the O-notation are not estimated and found to be reasonably small!!!**

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- After we study the FFT we will have a guest lecture by a Dolby engineer to demonstrate to you some cool applications of FFT.