Boundary Majorana CFT and Bosonization

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Introduction

Majorana BCFT

CFT on a Complex Plane $\Rightarrow$ CFT with Boundaries (BCFT)

Here we want to study the behaviour of 2D Free Majorana Fermion (Majorana CFT) on a Manifold with boundaries:

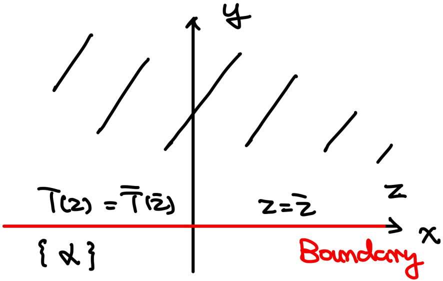
- Boundary Conditions, Partition Function and Boundary States
- Majorana CFT is a $c = 1/2$ CFT, we might guess it is related to Ising Model (which also has $c = 1/2$ but is a bosonic minimal model). But how are they related in a BCFT setup?

Preliminaries

Boundary Conformal Field Theory

- How to Induce a Parent CFT on a full plane to a CFT on a Manifold with boundaries? (e.g. UHP) suitable conditions for the boundary?
  - **Gluing Conditions:** $T_{01}(x, 0) = 0 \quad \Rightarrow \quad T(z) = \bar{T}(\bar{z})$ at $z = \bar{z}$

  Gluing condition is not enough to characterize the boundary conditions, we can have different boundary conditions $\{\alpha\}$ satisfying the same gluing condition.



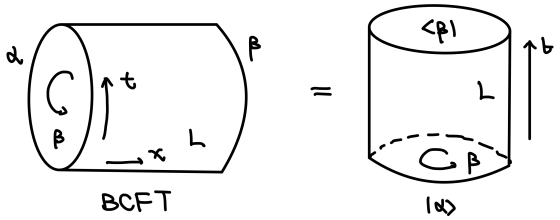
- Question: how to characterize the boundary conditions? Boundary States!

Boundary States Formalism

Boundary States are state of the Parent CFT on a full plane that characterize the boundary conditions of the BCFT. which defined to satisfy two conditions:

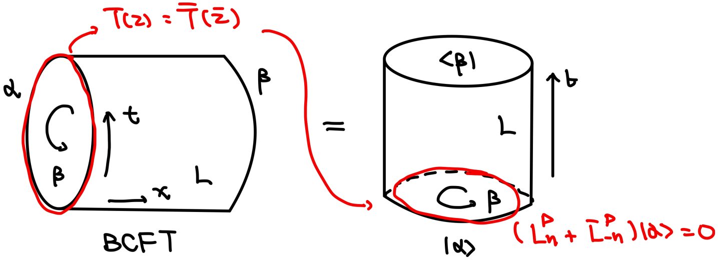
- **(1) Cardy's Condition**
  BCFT partition function with α, β boundary conditions on two edges
  $\sim$
  Parent CFT amplitude between Boundary State $|\alpha\rangle, |\beta\rangle$

$$\mathrm{Tr}_{\mathcal{H}_{\alpha\beta}} e^{-\beta H_{\text{open}}} = \langle \alpha | e^{-L H_{\text{close}}} | \beta \rangle \tag{1}$$



Boundary States Formalism

- **(2)** Being "consistent with" the **Gluing Conditions**



$$T(z) = \bar{T}(\bar{z}) \quad \Rightarrow \quad \left(L_n^P - \bar{L}_{-n}^P\right)|\alpha\rangle = 0 \tag{2}$$

Boundary States thus is believed to characterize the boundary conditions.

Majorana CFT

Majorana CFT is the 2 Dimensional theory of free massless real fermion:

$$S = \frac{1}{2\pi} \int d^2 z \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right) \tag{3}$$

- **Neveu-Schwarz (NS) Boundary Condition**:

$$\psi(e^{2\pi i} z) = \psi(z), \tag{4}$$

- **Ramond (R) Boundary Condition**:

$$\psi(e^{2\pi i} z) = -\psi(z), \tag{5}$$

Majorana CFT Quantization

- **Mode Expansion:**

$$\psi(z) = \sum_k b_k z^{-k-1/2} \quad \bar{\psi}(\bar{z}) = \sum_k \bar{b}_k \bar{z}^{-k-1/2} \tag{6}$$

where $k \in \mathbb{Z} + 1/2$ for NS sector and $k \in \mathbb{Z}$ for R sector.

$$\{b_k, b_l\} = \delta_{k+l,0}, \quad \{\bar{b}_k, \bar{b}_l\} = \delta_{k+l,0}, \quad \{b_k, \bar{b}_l\} = 0. \tag{7}$$

- **Hamiltonian:**

$$H = L_0 + \bar{L}_0, \qquad L_0 = \begin{cases} \sum_{k>0} k\, b_{-k} b_k, & \text{NS } (k \in \mathbb{Z} + \frac{1}{2}), \\ \sum_{k>0} k\, b_{-k} b_k + \frac{1}{16}, & \text{R } (k \in \mathbb{Z}), \end{cases} \tag{8}$$

- **Hilbert Space:** $b_{-k}, k > 0$ act on Ground State (?)

Ground States and R Sector Zero Mode

There's something special about the Ground States in two sectors:

- **NS Sector**
  Ground state $|0\rangle_{NS}$ is unique, defined by:

$$b_k|0\rangle_{NS} = 0, \quad \bar{b}_k|0\rangle_{NS} = 0, \quad k > 0. \tag{9}$$

- **R Sector**

$$\{b_0, b_0\} = 1, \quad \{\bar{b}_0, \bar{b}_0\} = 1, \quad \{b_0, \bar{b}_0\} = 0. \tag{10}$$

2d clifford algebra $\Rightarrow$ 2-dim representation $\Rightarrow$ ground state is doubly degenerate.

$$|+\rangle_R \quad \text{and} \quad |-\rangle_R \tag{11}$$

Fermion Number Operator and Partition Functions

We want to define a operator that measure the even and odd of fermion number of a state:

- **NS Sector:** $(-1)^F = (-1)^{F_{nz}} = (-1)^{\sum_{k>0} b_{-k}b_k + \sum_{k>0} \bar{b}_{-k}\bar{b}_k}$
- **R Sector:** we can define $(-1)^F = -2i\, b_0\bar{b}_0\ (-1)^{F_{nz}}$

Partition Function:

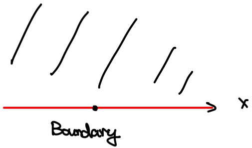
$$Z_+ = \mathrm{Tr}_{\mathcal{H}}\left[e^{-\beta H^C}\right] \quad \text{and} \quad Z_- = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\beta H^C}\right] \tag{12}$$

Majorana CFT with Boundaries

Majorana BCFT

$T(z) = \bar{T}(\bar{z})$ leads to:

- **Free** $(+)$ **Boundary Condition:**
  $\psi(x) = \bar{\psi}(x)$
- **Fixed** $(-)$ **Boundary Condition:**
  $\psi(x) = -\bar{\psi}(x)$

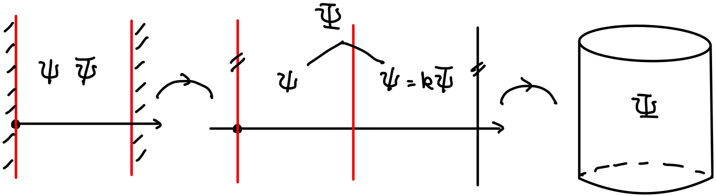


Then we want to study the Majorana CFT on a Stip to calculate the Partition Function
Doubling Trick: **Majorana BCFT on a Strip** $\psi, \bar{\psi} \sim$ **Chiral Majorana CFT on a Cylinder** Ψ

Boundary conditions after doubling trick:

- **Same BCs at two edges** $\rightarrow$ Neveu-Schwarz (NS) BC of Ψ
- **Different BCs at two edges** $\rightarrow$ Ramond (R) BC of Ψ

Majorana BCFT Quantization



- **Hamiltonian:**

$$H = \sum_{k>0} k b_{-k} b_k + E_0 \tag{13}$$

- **Hilbert Space:**
  - **Same BCs at two edges:** NS Sector similar to full Majorana CFT
  - **Different BCs at two edges:** R Sector with only one ground state!!!

Majorana BCFT Partition Functions

Partition Functions for fermionic theory

$$Z_- = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\beta H}\right] \quad Z_+ = \mathrm{Tr}_{\mathcal{H}}\left[e^{-\beta H}\right] \tag{14}$$

Something Bizzar Here:

- Is $(-1)^F$ well defined in a Chiral Majorana CFT? NO!!
- Partition Function is irrelevant with boundary conditions?? Only depends on whether two boundaries are the same or not? Boundary condition can not be characterized by partition functions and thus boundary states?? NO!!

Boundary Fermion

Resolution of above problems: **Add a Boundary Free Fermion to free boundary conditions** $(+)$

$$S_\xi = \frac{i}{4}\int dt\ \xi(t)\frac{d}{dt}\xi(t) \quad S = S_{\text{Majorana}} + S_\xi \tag{15}$$

Under Quantization:

$$\{\xi,\xi\} = 2 \quad \xi^2 = 2, \quad \{\xi, b_k\} = 0\ \forall\ k \tag{16}$$

- **two fixed boundary conditions:** nothing changes
- **fixed-free boundary conditions:** ξ and b_0 forms a 2d clifford algebra $\Rightarrow$ ground state degeneracy & well defined fermion number operator $-i\sqrt{2}b_0\xi(-1)^{F_{nz}}$
- **two free boundary conditions:** ξ_1, ξ_2 forms a 2d clifford algebra $\Rightarrow$ ground state degeneracy & well defined fermion number operator $-i\xi_1\xi_2(-1)^{F_{nz}}$

Majorana BCFT Partition Functions with Boundary Fermion

Partition Function	q character	$\tilde{q}$ character
$Z_{(--),-}$	$\chi_0(q) - \chi_{1/2}(q)$	$\sqrt{2}\,\chi_{1/16}(\tilde{q})$
$Z_{(--),+}$	$\chi_0(q) + \chi_{1/2}(q)$	$\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})$
$Z_{(++),-}$	0	0
$Z_{(++),+}$	$2\big(\chi_0(q) + \chi_{1/2}(q)\big)$	$2\big(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})\big)$
$Z_{(-+),-}$	0	0
$Z_{(-+),+}$	$2\,\chi_{1/16}(q)$	$\sqrt{2}\big(\chi_0(\tilde{q}) - \chi_{1/2}(\tilde{q})\big)$

Spin Cardy's Condition

For a Fermion Theory, we have two kinds of partiton functions for a given boundary condition. We need to generalize Cardy's Condition to **Spin Cardy's Condition**.

- **Spin Cardy's Condition:** for a boundary condition we define two boundary states as $|\alpha, NS\rangle$ and $|\alpha, R\rangle$ satisfying:

$$\langle\alpha, NS| \, e^{-L_1 H_{\text{closed}}} \, |\beta, NS\rangle \; = \; \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left(e^{-L_2 H_{\text{open}}}\right) = Z_{(\alpha\beta),+}, \tag{17}$$

$$\langle\alpha, R| \, e^{-L_1 H_{\text{closed}}} \, |\beta, \mathrm{R}\rangle \; = \; \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left((-1)^F e^{-L_2 H_{\text{open}}}\right) = Z_{(\alpha\beta),-}. \tag{18}$$

Majorana BCFT Boundary States

4 boundary states satisfying Gluing Conditon and Spin Cardy's Condition are:

$$|-,NS\rangle = |-,NS\rangle\rangle \tag{19}$$

$$|-,R\rangle = \sqrt[4]{2}|-,R\rangle\rangle, \tag{20}$$

$$|+,NS\rangle = \sqrt{2}|+,NS\rangle\rangle, \tag{21}$$

$$|+,R\rangle = 0|+,R\rangle\rangle = 0. \tag{22}$$

where:

$$|\pm,NS\rangle\rangle = \prod_{k\in\mathbb{N}_+-1/2} e^{i\pm b_{-k}\bar{b}_{-k}}|0\rangle_{\mathrm{NS}} \quad |\pm,R\rangle\rangle = \prod_{k\in\mathbb{N}_+} e^{i\pm b_{-k}\bar{b}_{-k}}|\pm\rangle_{\mathrm{R}} \tag{23}$$

Partition Functions

Open Channel	q character	$\tilde{q}$ character	Closed Channel
$Z_{(--),-}$	$\chi_0(q) - \chi_{1/2}(q)$	$\sqrt{2}\,\chi_{1/16}(\tilde{q})$	$\langle -, R \vert e^{-LH_{\text{closed}}} \vert -, R\rangle$
$Z_{(--),+}$	$\chi_0(q) + \chi_{1/2}(q)$	$\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})$	$\langle -, NS \vert e^{-LH_{\text{closed}}} \vert -, NS\rangle$
$Z_{(++),-}$	0	0	$\langle +, R \vert e^{-LH_{\text{closed}}} \vert +, R\rangle$
$Z_{(++),+}$	$2\big(\chi_0(q) + \chi_{1/2}(q)\big)$	$2\big(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})\big)$	$\langle +, NS \vert e^{-LH_{\text{closed}}} \vert +, NS\rangle$
$Z_{(-+),-}$	0	0	$\langle -, R \vert e^{-LH_{\text{closed}}} \vert +, R\rangle$
$Z_{(-+),+}$	$2\,\chi_{1/16}(q)$	$\sqrt{2}\big(\chi_0(\tilde{q}) - \chi_{1/2}(\tilde{q})\big)$	$\langle -, NS \vert e^{-LH_{\text{closed}}} \vert +, NS\rangle$

Bosonization

Naive Idea of Bosonization

Main Idea: A fermion theory, if we restrict that all fermions exist in pairs then it behaves like a bosonic theory.

Mathematically Realization: Project the Hilbert Space with $P = (1 + (-1)^F)/2$

BCFT: Not only have to project the Hilbert Space, but also the Boundary Conditions must be "bosonized". Boundary State provide a perfect platform for us to do this for we can project the Boundary States directly:

Idea of Bosonizing Boundary States

- **Step 1:** Project the Boundary States with P projector
- **Step 2:** Combine the NS states and R states to satisfy the normal Cardy's Condition of BCFT.

Bosonization of Boundary State

Project the Boundary States with P projector:

$$|\pm, NS\rangle = P\,|\pm, NS\rangle \quad |-, R\rangle = P\,|-, R\rangle \quad P\,|+, R\rangle = 0 \tag{24}$$

Then we find states that satisfy Cardy's Condition and Assume that the Bosonic Theory is Diagonal RCFT:

$$|f\rangle = \frac{1}{\sqrt{2}}|+, NS\rangle \oplus 0 \tag{25}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|-, NS\rangle \oplus |-, R\rangle\right) \tag{26}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|-, NS\rangle \oplus -|-, R\rangle\right) \tag{27}$$

Bosonized Partition Functions

We then Calculate the Partition Functions with the Bosonized Boundary States:

q character	$\tilde{q}$ character	Closed Channel
$\chi_0(q)$	$\frac{1}{2}\big(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})\big) + \frac{1}{\sqrt{2}}\chi_{1/16}(\tilde{q})$	$\langle + \vert e^{-LH_{\text{closed}}} \vert + \rangle$
$\chi_0(q)$	$\frac{1}{2}\big(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})\big) + \frac{1}{\sqrt{2}}\chi_{1/16}(\tilde{q})$	$\langle - \vert e^{-LH_{\text{closed}}} \vert - \rangle$
$\chi_{1/2}(q)$	$\frac{1}{2}\big(\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})\big) + \frac{1}{\sqrt{2}}\chi_{1/16}(\tilde{q})$	$\langle - \vert e^{-LH_{\text{closed}}} \vert + \rangle$
$\chi_0(q) + \chi_{1/2}(q)$	$\chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q})$	$\langle f \vert e^{-LH_{\text{closed}}} \vert f \rangle$
$\chi_{1/16}(q)$	$\frac{1}{\sqrt{2}}\big(\chi_0(\tilde{q}) - \chi_{1/2}(\tilde{q})\big)$	$\langle + \vert e^{-LH_{\text{closed}}} \vert f \rangle$
$\chi_{1/16}(q)$	$\frac{1}{\sqrt{2}}\big(\chi_0(\tilde{q}) - \chi_{1/2}(\tilde{q})\big)$	$\langle - \vert e^{-LH_{\text{closed}}} \vert f \rangle$

Exactly the Ising BCFT Partition Functions!!!

What else can we ask?

- **Non-Diagonal CFT:** in the bosonization procedure, we use an assumption that the bosonized theory is a diagonal CFT, what if there are other opentions?

- **Bosonization in Closed Channel:** can we do this procedure in the closed channel directly? certainly we can find that:

$$Z_{++}^{Ising} = \frac{1}{2}(Z_{(--),-} + Z_{(--),+}) \tag{28}$$

$$Z_{+-}^{Ising} = \frac{1}{2}(Z_{(--),-} - Z_{(--),+}) \tag{29}$$

$$Z_{ff}^{Ising} = \frac{1}{2}Z_{(++),+} \tag{30}$$

But what is the physical meaning?

Thank You !!!! Questions?