GNSS 卫星精密定轨之轨道积分

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1 基本原理

根据牛顿第二定律，卫星的加速度 $\vec{a}$ 可以表示为时间 t 、位置 $\vec{r}$ 、速度 $\vec{v}$ 以及力模型参数 $\vec{p}$ 的函数，如下：

$\vec{a} = f\left(t, \vec{r}, \vec{v}, \vec{p}\right)$（1-1）

根据 t_0 时刻的卫星位置、速度以及力模型参数 $\left(\vec{r}_0, \vec{v}_0, \vec{p}_0\right)$，对（3-1）式积分，即可确定 t 时刻的卫星位置、速度 $\left(\vec{r}, \vec{v}\right)$。由于 $\left(\vec{r}_0, \vec{v}_0, \vec{p}_0\right)$ 的不准确性，当存在外部观测量（如距离、距离变率等）时，需要对其进行修正。通常，在一定长度的积分弧段内，认为力模型参数 $\vec{p}$ 不变，即 $\vec{p} = \vec{p}_0$。

（1-1）式对应的一阶微分方程如下：

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \vec{v} \\ \dfrac{d\vec{v}}{dt} = \vec{a} \\ \dfrac{d\vec{p}}{dt} = 0 \end{cases} \text{（1-2）}$$

令 $\vec{x} = \left(\vec{r}, \vec{v}, \vec{p}\right)$，得到

$\dot{\vec{x}} = \left(\vec{v}\ \vec{a}\ 0\right)^T = \vec{F}\left(t, \vec{x}\right)$（1-3）

（1-3）式即为卫星轨道运动方程。根据卫星初始状态 $\vec{x}_0$，可以积分确定卫星当前状态 $\vec{x}$。

由（1-3）式可以得到卫星当前状态 $\vec{x}$ 与初始状态 $\vec{x}_0$ 的关系如下：

$$\begin{cases} \dfrac{\partial \dot{\vec{x}}}{\partial \vec{x}_0} = d\left(\dfrac{\partial \vec{x}}{\partial \vec{x}_0}\right) \Big/ dt \\ \dfrac{\partial \dot{\vec{x}}}{\partial \vec{x}_0} = \dfrac{\partial \vec{F}\left(t, \vec{x}\right)}{\partial \vec{x}_0} = \dfrac{\partial \vec{F}\left(t, \vec{x}\right)}{\partial \vec{x}} \dfrac{\partial \vec{x}}{\partial \vec{x}_0} \end{cases} \Rightarrow d\left(\dfrac{\partial \vec{x}}{\partial \vec{x}_0}\right) \Big/ dt = \dfrac{\partial \vec{F}\left(t, \vec{x}\right)}{\partial \vec{x}} \dfrac{\partial \vec{x}}{\partial \vec{x}_0} \quad （1\text{-}4）$$

其中，$\dfrac{\partial \vec{F}\left(t, \vec{x}\right)}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial \vec{v}}{\partial \vec{r}} & \dfrac{\partial \vec{v}}{\partial \vec{v}} & \dfrac{\partial \vec{v}}{\partial \vec{p}} \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \\ \dfrac{\partial 0}{\partial \vec{r}} & \dfrac{\partial 0}{\partial \vec{v}} & \dfrac{\partial 0}{\partial \vec{p}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \\ 0 & 0 & 0 \end{bmatrix}$。

令 $\dfrac{\partial \vec{x}}{\partial \vec{x}_0} = X$，$\dfrac{\partial \vec{F}\left(t, \vec{x}\right)}{\partial \vec{x}} = A$，简化（1-4）式，得到：

$$\dot{X} = AX \quad （1\text{-}5）$$

（1-5）式解的形式可以表示如下：

$$X\left(t\right) = \Phi\left(t, t_0\right) X\left(t_0\right) \quad （1\text{-}6）$$

将（1-6）式代入（1-5）式，得到：

$$\dot{\Phi}\left(t, t_0\right) = A\Phi\left(t, t_0\right) \quad （1\text{-}7）$$

其中，$\Phi\left(t, t_0\right) = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{p}}{\partial \vec{r}_0} & \dfrac{\partial \vec{p}}{\partial \vec{v}_0} & \dfrac{\partial \vec{p}}{\partial \vec{p}_0} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ 0_{n\times 3} & 0_{n\times 3} & I_{n\times n} \end{bmatrix}$，称为转移矩阵。

（1-7）式即为卫星轨道变分方程。根据初始转移矩阵 $\Phi\left(t_0, t_0\right)$，可以积分确定当前转移矩阵 $\Phi\left(t, t_0\right)$。

综上所述，可以得到卫星轨道的运动方程和变分方程分别如下：

运动方程：$\dot{\vec{x}}(t)=\vec{F}(t,\vec{x})$（1-8）

变分方程：$\dot{\Phi}(t,t_0)=A(t,x)\Phi(t,t_0)$（1-9）

联合运动方程与变分方程，得到：

$$\begin{pmatrix}\dot{\vec{x}}(t)\\ \dot{\Phi}(t,t_0)\end{pmatrix}=\begin{pmatrix}\vec{F}(t,\vec{x})\\ A(t,\vec{x})\Phi(t,t_0)\end{pmatrix}$$（1-10）

其中，

$\vec{x}(t)$为t时刻的卫星位置、速度及力模型参数；

$\vec{F}(t,\vec{x})$为t时刻的卫星速度、加速度及力模型参数对时间的微分（$\frac{\partial\vec{p}}{\partial t}=0$表明在积分弧段内力模型参数不变）；

$\Phi(t,t_0)$为t时刻的转移矩阵，包括状态转移矩阵和参数敏感矩阵；

$A(t,\vec{x})$为t时刻的卫星加速度对卫星位置、速度及力模型参数的微分。

（1-10）式的具体形式为：

$$\begin{pmatrix}\dot{\vec{r}}\\ \dot{\vec{v}}\end{pmatrix}=\begin{pmatrix}\vec{v}\\ \vec{a}\end{pmatrix}$$（1-11）

$$\begin{pmatrix}\frac{d\left(\frac{\partial\vec{r}}{\partial\vec{r}_0}\right)}{dt} & \frac{d\left(\frac{\partial\vec{r}}{\partial\vec{v}_0}\right)}{dt} & \frac{d\left(\frac{\partial\vec{r}}{\partial\vec{p}_0}\right)}{dt}\\ \frac{d\left(\frac{\partial\vec{v}}{\partial\vec{r}_0}\right)}{dt} & \frac{d\left(\frac{\partial\vec{v}}{\partial\vec{v}_0}\right)}{dt} & \frac{d\left(\frac{\partial\vec{v}}{\partial\vec{p}_0}\right)}{dt}\end{pmatrix}=\begin{pmatrix}\frac{\partial\vec{v}}{\partial\vec{r}} & \frac{\partial\vec{v}}{\partial\vec{v}} & \frac{\partial\vec{v}}{\partial\vec{p}}\\ \frac{\partial\vec{a}}{\partial\vec{r}} & \frac{\partial\vec{a}}{\partial\vec{v}} & \frac{\partial\vec{a}}{\partial\vec{p}}\end{pmatrix}\begin{pmatrix}\frac{\partial\vec{r}}{\partial\vec{r}_0} & \frac{\partial\vec{r}}{\partial\vec{v}_0} & \frac{\partial\vec{r}}{\partial\vec{p}_0}\\ \frac{\partial\vec{v}}{\partial\vec{r}_0} & \frac{\partial\vec{v}}{\partial\vec{v}_0} & \frac{\partial\vec{v}}{\partial\vec{p}_0}\\ 0 & 0 & I\end{pmatrix}$$（1-12）

初始状态的不准确性包括初始位置、速度以及力模型参数的不准确性，仅对运动方程（1-11）式积分，将会产生位置和速度误差。为修正初始状态的误差，需要建立当前状态与初始状态的联系（当前状态对初始状态的偏导数），而当前状态又可以与外部观测建立联系（观测值对当前状态的偏导数），从而可以利用外部观测修正初始状态的误差。

因此，当初始状态准确时，通过积分运动方程（1-11）式即可得到高精度的位置和速度（仅含积分误差），不需要积分变分方程（1-12）式；当初始状态不准确时，需要对运动方程（1-11）式和变分方程（1-12）式同时积分，利用运动方程积分得到的位置和速度以及变分方程积分得到的转移矩阵，建立外部观测与初始状态的关系，从而实现对初始状态的修正。

对变分方程，由于 $\frac{\partial \vec{v}}{\partial \vec{r}}=0, \frac{\partial \vec{v}}{\partial \vec{v}}=I, \frac{\partial \vec{v}}{\partial \vec{p}}=0$（对于 GNSS 卫星，速度与力模型参数无关），得到：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{v}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{p}_0}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{v}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{p}_0}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \quad (1\text{-}13)$$

状态转移矩阵微分方程：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{v}_0}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{v}_0}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} \end{pmatrix} \quad (1\text{-}14)$$

参数敏感矩阵微分方程：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{p}_0}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{p}_0}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ I \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \quad (1\text{-}15)$$

2 力模型

对 GNSS 卫星精密定轨而言，所受到的力主要包括地球引力（包括球形引力和非球形引力）、三体引力（主要为日月引力）、太阳辐射压、地球辐射压、发动机推进力、潮汐效应（固体潮、海潮、极潮）、相对论效应、经验力等。

表 2-1 GNSS 卫星精密定轨力模型

力模型	量级（m/s^2）	模型
地球引力	0.5	EGM2008 模型（12 阶 12 次）
三体引力	10^{-6}	月球、太阳 JPL DE 405 星历
太阳光压	10^{-8}	CODE 9 参数/5 参数模型
固体潮	10^{-9}	IERS 2010
海潮	10^{-10}	IERS 2010
极潮		IERS 2010
相对论效应	10^{-10}	IERS 2010
经验力		

2.1 地球引力

地球引力位表达式如下：

$$V(r,\varphi,\lambda)=\frac{GM_E}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right) \quad (2\text{-}1)$$

其中，(r,φ,λ) 分别为计算点的地心向径、纬度和经度，a_e 为地球赤道半径，$\bar{P}_{nm}(\sin\varphi)$ 为完全正规化的缔合勒让德多项式，$\bar{C}_{nm}$ 和 $\bar{S}_{nm}$ 为球谐系数。

ICRS 直角坐标系下的地球引力：

$$
\begin{aligned}
\vec{f}_{ICRS} &= C_{ITRS}^{ICRS}\,\vec{f}_{ITRS} \\
&= C_{ITRS}^{ICRS}\frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \\
&= C_{ITRS}^{ICRS}\begin{pmatrix}\dfrac{\partial V_{ITRS}}{\partial x_{ITRS}} & \dfrac{\partial V_{ITRS}}{\partial y_{ITRS}} & \dfrac{\partial V_{ITRS}}{\partial z_{ITRS}}\end{pmatrix}^{T} \\
&= C_{ITRS}^{ICRS}\begin{pmatrix}
\dfrac{\partial V_{ITRS}}{\partial r_{ITRS}}\dfrac{\partial r_{ITRS}}{\partial x_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}}\dfrac{\partial \varphi_{ITRS}}{\partial x_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}}\dfrac{\partial \lambda_{ITRS}}{\partial x_{ITRS}} \\
\dfrac{\partial V_{ITRS}}{\partial r_{ITRS}}\dfrac{\partial r_{ITRS}}{\partial y_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}}\dfrac{\partial \varphi_{ITRS}}{\partial y_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}}\dfrac{\partial \lambda_{ITRS}}{\partial y_{ITRS}} \\
\dfrac{\partial V_{ITRS}}{\partial r_{ITRS}}\dfrac{\partial r_{ITRS}}{\partial z_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}}\dfrac{\partial \varphi_{ITRS}}{\partial z_{ITRS}}+\dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}}\dfrac{\partial \lambda_{ITRS}}{\partial z_{ITRS}}
\end{pmatrix} \\
&= C_{ITRS}^{ICRS}\begin{pmatrix}
\dfrac{\partial r_{ITRS}}{\partial x_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial x_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial x_{ITRS}} \\
\dfrac{\partial r_{ITRS}}{\partial y_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial y_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial y_{ITRS}} \\
\dfrac{\partial r_{ITRS}}{\partial z_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial z_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial z_{ITRS}}
\end{pmatrix}\begin{pmatrix}
\dfrac{\partial V_{ITRS}}{\partial r_{ITRS}} \\
\dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}} \\
\dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}}
\end{pmatrix} \qquad \text{（2-2）} \\
&= C_{ITRS}^{ICRS}\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^{T}\frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}
\end{aligned}
$$

ICRS 直角坐标系下的地球引力对 ICRS 直角坐标系下的卫星位置的偏导数：

$$
\begin{aligned}
\frac{\partial \vec{f}_{ICRS}}{\partial \vec{r}_{ICRS}} &= \frac{\partial\left[C_{ITRS}^{ICRS}\left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^{T}\dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial \vec{r}_{ICRS}} \\
&= C_{ITRS}^{ICRS}\frac{\partial\left[\left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^{T}\dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial \vec{r}_{ITRS}}\frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}} \qquad \text{（2-3）} \\
&= C_{ITRS}^{ICRS}\frac{\partial\left[\left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^{T}\dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial(r,\varphi,\lambda)_{ITRS}}\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}}
\end{aligned}
$$

其中，

$$\frac{\partial\left[\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial(r,\varphi,\lambda)_{ITRS}} = \frac{\partial\begin{pmatrix} \frac{\partial V}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x} \\ \frac{\partial V}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y} \\ \frac{\partial V}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z} \end{pmatrix}_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}$$

$$\begin{aligned}\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x}\right)_{ITRS}}{\partial r_{ITRS}} &= \frac{\partial^2 V}{\partial r^2}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial x}\right)}{\partial r} \\ &+\frac{\partial^2 V}{\partial \varphi \partial r}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial x}\right)}{\partial r} \\ &+\frac{\partial^2 V}{\partial \lambda \partial r}\frac{\partial \lambda}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial x}\right)}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x}\right)_{ITRS}}{\partial \varphi_{ITRS}} &= \frac{\partial^2 V}{\partial r \partial \varphi}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial x}\right)}{\partial \varphi} \\ &+\frac{\partial^2 V}{\partial \varphi^2}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial x}\right)}{\partial \varphi} \\ &+\frac{\partial^2 V}{\partial \lambda \partial \varphi}\frac{\partial \lambda}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial x}\right)}{\partial \varphi}\end{aligned}$$

$$\begin{aligned}\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x}\right)_{ITRS}}{\partial \lambda_{ITRS}} &= \frac{\partial^2 V}{\partial r \partial \lambda}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial x}\right)}{\partial \lambda} \\ &+\frac{\partial^2 V}{\partial \varphi \partial \lambda}\frac{\partial \varphi}{\partial x}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial x}\right)}{\partial \lambda} \\ &+\frac{\partial^2 V}{\partial \lambda^2}\frac{\partial \lambda}{\partial x}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial x}\right)}{\partial \lambda}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial r_{ITRS}} &= \frac{\partial^2 V}{\partial r^2}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial y}\right)}{\partial r} \\
&+\frac{\partial^2 V}{\partial \varphi \partial r}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial y}\right)}{\partial r} \\
&+\frac{\partial^2 V}{\partial \lambda \partial r}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial y}\right)}{\partial r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial \varphi_{ITRS}} &= \frac{\partial^2 V}{\partial r \partial \varphi}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial y}\right)}{\partial \varphi} \\
&+\frac{\partial^2 V}{\partial \varphi^2}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial y}\right)}{\partial \varphi} \\
&+\frac{\partial^2 V}{\partial \lambda \partial \varphi}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial y}\right)}{\partial \varphi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial \lambda_{ITRS}} &= \frac{\partial^2 V}{\partial r \partial \lambda}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial y}\right)}{\partial \lambda} \\
&+\frac{\partial^2 V}{\partial \varphi \partial \lambda}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial y}\right)}{\partial \lambda} \\
&+\frac{\partial^2 V}{\partial \lambda^2}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial y}\right)}{\partial \lambda}
\end{aligned}
$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z}\right)_{ITRS}}{\partial r_{ITRS}}=\frac{\partial^2 V}{\partial r^2}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial z}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial \varphi\partial r}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial z}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial \lambda\partial r}\frac{\partial \lambda}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial z}\right)}{\partial r}$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z}\right)_{ITRS}}{\partial \varphi_{ITRS}}=\frac{\partial^2 V}{\partial r\partial \varphi}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial z}\right)}{\partial \varphi}$$

$$+\frac{\partial^2 V}{\partial \varphi^2}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial z}\right)}{\partial \varphi}$$

$$+\frac{\partial^2 V}{\partial \lambda\partial \varphi}\frac{\partial \lambda}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial z}\right)}{\partial \varphi}$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z}\right)_{ITRS}}{\partial \lambda_{ITRS}}=\frac{\partial^2 V}{\partial r\partial \lambda}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial z}\right)}{\partial z}$$

$$+\frac{\partial^2 V}{\partial \varphi\partial \lambda}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\frac{\partial \varphi}{\partial z}\right)}{\partial \lambda}$$

$$+\frac{\partial^2 V}{\partial \lambda^2}\frac{\partial \lambda}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\frac{\partial \lambda}{\partial z}\right)}{\partial \lambda}$$

由 $\vec{r}_{ICRS}=C_{ITRS}^{ICRS}\vec{r}_{ITRS}$，得到：

$$\frac{d\vec{r}_{ICRS}}{dt}=\frac{d\left(C_{ITRS}^{ICRS}\vec{r}_{ITRS}\right)}{dt}=\frac{dC_{ITRS}^{ICRS}}{dt}\vec{r}_{ITRS}+C_{ITRS}^{ICRS}\frac{d\vec{r}_{ITRS}}{dt} \quad (2\text{-}4)$$

由 $\frac{dC_{ITRS}^{ICRS}}{dt}\cong 0$，得到：

$$\frac{d\vec{r}_{ICRS}}{dt} \cong C_{ITRS}^{ICRS} \frac{d\vec{r}_{ITRS}}{dt} \Rightarrow \frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}} \cong \left(C_{ITRS}^{ICRS}\right)^{-1} \quad (2\text{-}5)$$

将（2-5）式代入（2-3）式，得到：

$$\frac{\partial \vec{f}_{ICRS}}{\partial \vec{r}_{ICRS}} = C_{ITRS}^{ICRS} \frac{\partial\left[\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial(r,\varphi,\lambda)_{ITRS}} \frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \left(C_{ITRS}^{ICRS}\right)^{-1} \quad (2\text{-}6)$$

ICRS 直角坐标系下的地球引力对球谐系数的偏导数：

$$\begin{aligned}\frac{\partial \vec{f}_{ICRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)} &= \frac{\partial\left[C_{ITRS}^{ICRS}\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)} \\ &= C_{ITRS}^{ICRS} \frac{\partial\left[\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\end{aligned} \quad (2\text{-}7)$$

其中，

$$\frac{\partial\left[\left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)} = \frac{\partial\begin{pmatrix} \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x} \\ \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y} \\ \frac{\partial V}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z} \end{pmatrix}_{ITRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x}\right)_{ITRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)} = \frac{\partial\left(\frac{\partial V}{\partial r}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial r}{\partial x} + \frac{\partial\left(\frac{\partial V}{\partial \varphi}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial x} + \frac{\partial\left(\frac{\partial V}{\partial \lambda}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial x}$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)} = \frac{\partial\left(\frac{\partial V}{\partial r}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial r}{\partial y} + \frac{\partial\left(\frac{\partial V}{\partial \varphi}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial y} + \frac{\partial\left(\frac{\partial V}{\partial \lambda}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial y}$$

$$\frac{\partial\left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z}+\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial z}\right)_{ITRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}=\frac{\partial\left(\frac{\partial V}{\partial r}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial r}{\partial z}+\frac{\partial\left(\frac{\partial V}{\partial \varphi}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial z}+\frac{\partial\left(\frac{\partial V}{\partial \lambda}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial z}$$

由（2-2）、（2-6）和（2-7）式可知，为确定地球引力及其相关偏导数，需要确定 C_{ITRS}^{ICRS}、$\left(C_{ITRS}^{ICRS}\right)^{-1}$、$\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}$、$\frac{\partial\left[\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}\right]}{\partial(r,\varphi,\lambda)_{ITRS}}$、$\frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}$、$\frac{\partial^2 V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}{}^2}$ 以及 $\frac{\partial\left[\frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}$。

ITRS 球坐标与 ITRS 直角坐标的相互转换：

$$\begin{pmatrix}x\\y\\z\end{pmatrix}_{ITRS}=\begin{pmatrix}r\cos\varphi\cos\lambda\\r\cos\varphi\sin\lambda\\r\sin\varphi\end{pmatrix},\quad\begin{pmatrix}r\\\varphi\\\lambda\end{pmatrix}_{ITRS}=\begin{pmatrix}\sqrt{x^2+y^2+z^2}\\\arctan\frac{z}{\sqrt{x^2+y^2}}\\\arctan\frac{y}{x}\end{pmatrix}\quad(2\text{-}8)$$

ITRS 球坐标对 ITRS 直角坐标的偏导数：

$$\begin{aligned}\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}&=\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}\\&=\begin{pmatrix}\cos\varphi\cos\lambda&\cos\varphi\sin\lambda&\sin\varphi\\-\frac{1}{r}\sin\varphi\cos\lambda&-\frac{1}{r}\sin\varphi\sin\lambda&\frac{1}{r}\cos\varphi\\-\frac{1}{r\cos\varphi}\sin\lambda&\frac{1}{r\cos\varphi}\cos\lambda&0\end{pmatrix}\quad(2\text{-}9)\end{aligned}$$

$\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}$ 对 ITRS 球坐标的偏导数：

$$\frac{\partial}{\partial r}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{r^2}\sin\varphi\cos\lambda & \frac{1}{r^2}\sin\varphi\sin\lambda & -\frac{1}{r^2}\cos\varphi \\ \frac{1}{r^2\cos\varphi}\sin\lambda & -\frac{1}{r^2\cos\varphi}\cos\lambda & 0 \end{pmatrix} \quad (2\text{-}10)$$

$$\frac{\partial}{\partial \varphi}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\frac{1}{r}\cos\varphi\cos\lambda & -\frac{1}{r}\cos\varphi\sin\lambda & -\frac{1}{r}\sin\varphi \\ -\frac{\sin\varphi}{r\cos^2\varphi}\sin\lambda & \frac{\sin\varphi}{r\cos^2\varphi}\cos\lambda & 0 \end{pmatrix} \quad (2\text{-}11)$$

$$\frac{\partial}{\partial \lambda}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\cos\varphi\sin\lambda & \cos\varphi\cos\lambda & 0 \\ \frac{1}{r}\sin\varphi\sin\lambda & -\frac{1}{r}\sin\varphi\cos\lambda & 0 \\ -\frac{1}{r\cos\varphi}\cos\lambda & -\frac{1}{r\cos\varphi}\sin\lambda & 0 \end{pmatrix} \quad (2\text{-}12)$$

引力位对 ITRS 球坐标的一阶偏导数：

$$\begin{cases} \dfrac{\partial V}{\partial r} = -\dfrac{GM}{r^2}\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi)\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right) \\ \dfrac{\partial V}{\partial \varphi} = \dfrac{GM}{r}\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \frac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right) \\ \dfrac{\partial V}{\partial \lambda} = \dfrac{GM}{r}\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} m\left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi)\cdot\left(-\bar{C}_{nm}\sin(m\lambda)+\bar{S}_{nm}\cos(m\lambda)\right) \end{cases} \quad (2\text{-}13)$$

引力位对 ITRS 球坐标的二阶偏导数：

$$
\begin{cases}
\dfrac{\partial^2 V}{\partial r^2}=\dfrac{GM}{r^3}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}(n+1)(n+2)\left(\dfrac{a}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right)\\
\dfrac{\partial^2 V}{\partial r\partial\varphi}=-\dfrac{GM}{r^2}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}(n+1)\left(\dfrac{a}{r}\right)^n\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right)\\
\dfrac{\partial^2 V}{\partial r\partial\lambda}=-\dfrac{GM}{r^2}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}(n+1)m\left(\dfrac{a}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(-\bar{C}_{nm}\sin(m\lambda)+\bar{S}_{nm}\cos(m\lambda)\right)\\
\dfrac{\partial^2 V}{\partial\varphi^2}=\dfrac{GM}{r}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}\left(\dfrac{a}{r}\right)^n\dfrac{d^2\bar{P}_{nm}(\sin\varphi)}{d\varphi^2}\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right)\\
\dfrac{\partial^2 V}{\partial\varphi\partial\lambda}=\dfrac{GM}{r}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}m\left(\dfrac{a}{r}\right)^n\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}\cdot\left(-\bar{C}_{nm}\sin(m\lambda)+\bar{S}_{nm}\cos(m\lambda)\right)\\
\dfrac{\partial^2 V}{\partial\lambda^2}=\dfrac{GM}{r}\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{n}m^2\left(\dfrac{a}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(-\bar{C}_{nm}\cos(m\lambda)-\bar{S}_{nm}\sin(m\lambda)\right)
\end{cases}
\tag{2-14}
$$

$\dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}$对球谐系数的偏导数：

$$
\begin{cases}
\dfrac{\partial\left[\dfrac{\partial V}{\partial r}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}=-\dfrac{GM}{r^2}(n+1)\left(\dfrac{a}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(\cos(m\lambda),\sin(m\lambda)\right)\\
\dfrac{\partial\left[\dfrac{\partial V}{\partial\varphi}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}=\dfrac{GM}{r}\left(\dfrac{a}{r}\right)^n\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}\cdot\left(\cos(m\lambda),\sin(m\lambda)\right)\\
\dfrac{\partial\left[\dfrac{\partial V}{\partial\lambda}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}=\dfrac{GM}{r}m\left(\dfrac{a}{r}\right)^n\bar{P}_{nm}(\sin\varphi)\cdot\left(-\sin(m\lambda),\cos(m\lambda)\right)
\end{cases}
\tag{2-15}
$$

确定以上偏导数的关键在于确定完全正规化的缔合勒让德多项式$\bar{P}_{nm}(\sin\varphi)$及其对地心纬度的一阶导数$\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}$和二阶导数$\dfrac{d^2\bar{P}_{nm}(\sin\varphi)}{d\varphi^2}$。

$\bar{P}_{nm}(\sin\varphi)$的递推公式：

$$\begin{cases}\bar{P}_{0,0}(\sin\varphi)=1.0\\ \bar{P}_{n,n}(\sin\varphi)=\sqrt{\dfrac{(1+\delta_{1n})(2n+1)}{2n}}\cos\varphi\bar{P}_{n-1,n-1}(\sin\varphi),n\geq 1\\ \bar{P}_{n,m}(\sin\varphi)=g_{n,m}\bar{P}_{n-1,m}(\sin\varphi)-h_{n,m}\bar{P}_{n-2,m}(\sin\varphi),n\geq m+1\end{cases} \quad (2\text{-}16)$$

其中，$g_{n,m}=\sqrt{\dfrac{(2n+1)(2n-1)}{(n+m)(n-m)}}$，$h_{n,m}=\sqrt{\dfrac{(2n+1)(n-m-1)(n+m-1)}{(2n-3)(n+m)(n-m)}}$。

$\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}$的递推公式：

$$\begin{cases}\dfrac{d\bar{P}_{n,m}(\sin\varphi)}{d\varphi}=\sqrt{\dfrac{(2-\delta_{0m})(n-m)(n+m+1)}{2}}\bar{P}_{n,m+1}(\sin\varphi)-m\tan\varphi\bar{P}_{n,m}(\sin\varphi)\\ \dfrac{d\bar{P}_{0,0}(\sin\varphi)}{d\varphi}=-m\tan\varphi\bar{P}_{0,0}(\sin\varphi)\end{cases} \quad (2\text{-}17)$$

$\dfrac{d^2\bar{P}_{nm}(\sin\varphi)}{d\varphi^2}$的递推公式：

$$\begin{cases}\dfrac{d^2\bar{P}_{n,m}(\sin\varphi)}{d\varphi^2}=\left[m^2\sec^2\varphi-n(n+1)\right]\bar{P}_{n,m}(\sin\varphi)+\tan\varphi\dfrac{d\bar{P}_{n,m}(\sin\varphi)}{d\varphi}\\ \dfrac{d^2\bar{P}_{0,0}(\sin\varphi)}{d\varphi^2}=\tan\varphi\dfrac{d\bar{P}_{n,m}(\sin\varphi)}{d\varphi}\end{cases} \quad (2\text{-}18)$$

IERS Coventions 2010 建议采用 EGM2008 模型。EGM2008 模型系数对应的GM 和 a_e分别取为 398600.4415km^3/s^{2}和 6378136.3m（TT 时），模型阶次截断至 12 阶 12 次。

低阶系数的时变改正：

$$\bar{C}_{n0}(t)=\bar{C}_{n0}(t_0)+d\bar{C}_{n0}/dt\times(t-t_0) \quad (2\text{-}19)$$

其中，t_0 为 J2000.0，$\bar{C}_{n0}(t_0)$和$d\bar{C}_{n0}/dt$的值由表 6.2 查取。

$$\begin{cases}\bar{C}_{21}(t)=\sqrt{3}\bar{x}_p(t)\bar{C}_{20}-\bar{x}_p(t)\bar{C}_{22}+\bar{y}_p(t)\bar{S}_{22}\\ \bar{S}_{21}(t)=-\sqrt{3}\bar{y}_p(t)\bar{C}_{20}-\bar{y}_p(t)\bar{C}_{22}-\bar{x}_p(t)\bar{S}_{22}\end{cases} \quad (2\text{-}20)$$

其中，$\bar{x}_p(t)$ 和 $\bar{y}_p(t)$ 表示 IERS 协议平均地球极，单位为弧度，计算公式如下：

$$\begin{cases} \bar{x}_p(t) = \sum_{i=0}^{3}(t-t_0)^i \times \bar{x}_p^i \\ \bar{y}_p(t) = \sum_{i=0}^{3}(t-t_0)^i \times \bar{y}_p^i \end{cases} \quad (2\text{-}21)$$

其中，t_0 为 J2000.0，$\bar{x}_p^i$ 和 $\bar{y}_p^i$ 的值由表 7.7 查取。

2）固体潮

采用两步法，将 Love 数分为两个部分：与频率无关的部分以及与频率有关的部分。

1）考虑与频率无关的部分

$$\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} = \frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)e^{-im\lambda_j} \quad (2\text{-}22)$$

其中，k_{nm} 为 Love 数，R_e 为地球赤道半径，$GM_\oplus$ 为地球引力参数，Φ_j 为太阳或月亮的地心纬度，λ_j 为太阳或月亮的地心经度（从格林尼治起算）。

由欧拉公式可知，$e^{-im\lambda_j} = \cos\left(-m\lambda_j\right) + i\sin\left(-m\lambda_j\right)$

$$\begin{aligned} \Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} &= \frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\left(\cos\left(-m\lambda_j\right)+i\sin\left(-m\lambda_j\right)\right) \\ &= \frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\cos\left(-m\lambda_j\right) \\ &\quad + i\frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\sin\left(-m\lambda_j\right) \\ &= \frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) \\ &\quad - i\frac{k_{nm}}{2n+1}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{aligned} \quad (2\text{-}23)$$

$$\begin{cases}\Delta\bar{C}_{nm}=\dfrac{k_{nm}}{2n+1}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right)\\ \Delta\bar{S}_{nm}=\dfrac{k_{nm}}{2n+1}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{n+1}\bar{P}_{nm}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right)\end{cases}\tag{2-24}$$

考虑由 2 阶和 3 阶潮汐引起的 2 阶和 3 阶球谐系数变化：

$$\begin{cases}\Delta\bar{C}_{2m}=\dfrac{k_{2m}}{5}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right)\\ \Delta\bar{S}_{2m}=\dfrac{k_{2m}}{5}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right)\end{cases}\tag{2-25}$$

$$\begin{cases}\Delta\bar{C}_{3m}=\dfrac{k_{3m}}{7}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{4}\bar{P}_{3m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right)\\ \Delta\bar{S}_{3m}=\dfrac{k_{3m}}{7}\displaystyle\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^{4}\bar{P}_{3m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right)\end{cases}\tag{2-26}$$

另外，考虑由 2 阶潮汐引起的 4 阶球谐系数变化：

$$\begin{aligned}\Delta\bar{C}_{4m}-i\Delta\bar{S}_{4m}&=\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)e^{-im\lambda_j},\left(m=0,1,2\right)\\ &=\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\left(\cos\left(-m\lambda_j\right)+i\sin\left(-m\lambda_j\right)\right)\\ &=\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(-m\lambda_j\right)+\\ &\quad i\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(-m\lambda_j\right)\\ &=\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right)-\\ &\quad i\frac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\left(\frac{R_e}{r_j}\right)^{3}\bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right)\end{aligned}\tag{2-27}$$

$$\begin{cases} \Delta\bar{C}_{4m} = \dfrac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^3 \bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) \\ \Delta\bar{S}_{4m} = \dfrac{k_{2m}^{+}}{5}\sum_{j=2}^{3}\dfrac{GM_j}{GM_\oplus}\left(\dfrac{R_e}{r_j}\right)^3 \bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{cases} \quad (2\text{-}28)$$

其中，k_{2m}、k_{3m}、k_{2m}^{+} 的值由表 6.3 查取。

2）考虑与频率有关的部分

考虑不同频率的长周期潮汐成分对 $\Delta\bar{C}_{20}$ 的影响：

$$\mathrm{Re}\sum_{f(2,0)}\left(A_0\delta k_f H_f\right)e^{i\theta_f} = \sum_{f(2,0)}\left[\left(A_0 H_f \delta k_f^R\right)\cos\theta_f - \left(A_0 H_f \delta k_f^I\right)\sin\theta_f\right] \quad (2\text{-}29)$$

$$\Delta\bar{C}_{20} = \sum_{f(2,0)}\left[\left(A_0 H_f \delta k_f^R\right)\cos\theta_f - \left(A_0 H_f \delta k_f^I\right)\sin\theta_f\right] \quad (2\text{-}30)$$

其中，$A_0 H_f \delta k_f^R$、$A_0 H_f \delta k_f^I$ 的值由表 6.5b 查取。

考虑周日潮汐成分对 $\left(\Delta\bar{C}_{21} - i\Delta\bar{S}_{21}\right)$ 以及半日潮汐成分对 $\left(\Delta\bar{C}_{22} - i\Delta\bar{S}_{22}\right)$ 的影响：

$$\begin{aligned} \Delta\bar{C}_{2m} - i\Delta\bar{S}_{2m} &= \eta_m \sum_{f(2,m)}\left(A_m\delta k_f H_f\right)e^{i\theta_f},\left(m=1,2\right) \\ &= \eta_m \sum_{f(2,m)}\left(A_m\delta k_f^R H_f + iA_m\delta k_f^I H_f\right)\left(\cos\theta_f + i\sin\theta_f\right) \\ &= \eta_m \sum_{f(2,m)}\left(A_m\delta k_f^R H_f\right)\cos\theta_f + \\ &\quad i\eta_m \sum_{f(2,m)}\left(A_m\delta k_f^R H_f\right)\sin\theta_f + \left(A_m\delta k_f^I H_f\right)\cos\theta_f + \\ &\quad i^2\eta_m \sum_{f(2,m)}\left(A_m\delta k_f^I H_f\right)\sin\theta_f \\ &= \eta_m \sum_{f(2,m)}\left(A_m\delta k_f^R H_f\right)\cos\theta_f + \left(A_m\delta k_f^I H_f\right)\sin\theta_f + \\ &\quad i\eta_m \sum_{f(2,m)}\left(A_m\delta k_f^R H_f\right)\sin\theta_f + \left(A_m\delta k_f^I H_f\right)\cos\theta_f \end{aligned} \quad (2\text{-}31)$$

其中，

$$A_0 = \frac{1}{R_e\sqrt{4\pi}} = 4.4288\times10^{-8}m^{-1},\quad A_m = \frac{(-1)^m}{R_e\sqrt{8\pi}} = (-1)^m\left(3.1274\times10^{-8}\right)m^{-1},\left(m\neq0\right);$$

$\eta_1 = -i$， $\eta_2 = 1$；

δk_f 为 $k_f - k_{2m}$，加上海潮负荷的影响；

δk_f^R 为 δk_f 的实部， δk_f^I 为 δk_f 的虚部；

H_f 为振幅，

$\theta_f = \bar{n} \cdot \bar{\beta} = \sum_{i=1}^{6} n_i \beta_i$ 或者 $\theta_f = m\left(\theta_g + \pi\right) - \bar{N} \cdot \bar{F} = m\left(\theta_g + \pi\right) - \sum_{j=1}^{5} N_j F_j$，

其中， $\bar{\beta}$ 为 6 个 Doodson 基本参数 $\beta_i, \left(\tau, s, h, p, N', p_s\right)$， $\bar{n}$ 为 6 个 Doodson 基本参数的乘数 n_i， $\bar{F}$ 为 5 个章动理论基本参数 F_j（Delaunay 变量， l, l', F, D, Ω ）， $\bar{N}$ 为 5 个章动理论基本参数的乘数 N_j， θ_g 为 GMST（格林尼治平恒星时）的角度表示。

当 m=1 时， $\eta_1 = -i$，从而有：

$$\begin{aligned}
\Delta\bar{C}_{21} - i\Delta\bar{S}_{21} &= \eta_1 \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \cos\theta_f + \left(A_1 \delta k_f^I H_f\right) \sin\theta_f \\
&\quad + i\eta_1 \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \sin\theta_f + \left(A_1 \delta k_f^I H_f\right) \cos\theta_f \\
&= \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \sin\theta_f + \left(A_1 \delta k_f^I H_f\right) \cos\theta_f \\
&\quad - i \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \cos\theta_f + \left(A_1 \delta k_f^I H_f\right) \sin\theta_f
\end{aligned} \tag{2-32}$$

$$\begin{cases}
\Delta\bar{C}_{21} = \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \sin\theta_f + \left(A_1 \delta k_f^I H_f\right) \cos\theta_f \\
\Delta\bar{S}_{21} = \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right) \cos\theta_f + \left(A_1 \delta k_f^I H_f\right) \sin\theta_f
\end{cases} \tag{2-33}$$

其中， $A_1 \delta k_f^R H_f$ 、 $A_1 \delta k_f^I H_f$ 的值由表 6.5a 查取。

当 m=2 时， $\eta_2 = 1$，从而有：

$$
\begin{aligned}
\Delta\bar{C}_{22} - i\Delta\bar{S}_{22} &= \eta_2 \sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\cos\theta_f + \left(A_2\delta k_f^I H_f\right)\sin\theta_f \\
&\quad + i\eta_2 \sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\sin\theta_f + \left(A_2\delta k_f^I H_f\right)\cos\theta_f \\
&= \sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\cos\theta_f + \left(A_2\delta k_f^I H_f\right)\sin\theta_f \\
&\quad + i\sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\sin\theta_f + \left(A_2\delta k_f^I H_f\right)\cos\theta_f
\end{aligned}
\quad (2\text{-}34)
$$

$$
\begin{cases}
\Delta\bar{C}_{22} = \sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\cos\theta_f + \left(A_2\delta k_f^I H_f\right)\sin\theta_f \\
\Delta\bar{S}_{22} = -\sum_{f(2,2)} \left(A_2\delta k_f^R H_f\right)\sin\theta_f + \left(A_2\delta k_f^I H_f\right)\cos\theta_f
\end{cases}
\quad (2\text{-}35)
$$

其中，$A_2\delta k_f^R H_f$ 的值由表 6.5c 查取。表 6.5c 中只给出了实部部分的改正，因此，$A_2\delta k_f^I H_f$ 为 0。

3）考虑永久性潮汐部分

由 2 阶带谐项 C_{20} 潮汐引起的引力位，对其取时间平均，结果并不为 0。在构建引力位模型时，如果 C_{20} 包含了潮汐中与时间无关的部分，则称其为“zero tide”模型，如果不包含，则称其为“conventional tide free”模型。对于“zero tide”模型，由于其 C_{20} 已经包含了潮汐中与时间无关的部分，因此，在进行潮汐改正时不应重复包含潮汐中与时间无关的部分，从步骤（1）中移去该部分影响的改正公式如下：

$$\Delta\bar{C}_{20}^{zt} = \Delta\bar{C}_{20} - \Delta\bar{C}_{20}^{perm} \quad (2\text{-}36)$$

其中，$\Delta\bar{C}_{20}$ 为步骤（1）中的潮汐改正，$\Delta\bar{C}_{20}^{perm} = A_0 H_0 k_{20} = \left(4.4228\times10^{-8}\right)\left(-0.31460\right)k_{20}$ 为与时间无关的部分。

在 EGM2008 模型中，“zero-tide”和“tide-free”在 C_{20} 上的差异为 -4.1736×10^{-9}。

3）海潮

海潮的动态效应会使正规化的 Stokes 系数产生周期性变化：

$$
\begin{aligned}
\left[\Delta\bar{C}_{nm}-i\Delta\bar{S}_{nm}\right](t) &= \sum_{f}\sum_{+}^{-}\left(\mathbb{C}^{\pm}_{f,nm}\mp i\mathbb{S}^{\pm}_{f,nm}\right)e^{\pm i\theta_f(t)}\\
&= \sum_{f}\sum_{+}^{-}\left(\mathbb{C}^{\pm}_{f,nm}\mp i\mathbb{S}^{\pm}_{f,nm}\right)\left(\cos\left(\pm\theta_f\right)+i\sin\left(\pm\theta_f\right)\right)\\
&= \sum_{f}\begin{matrix}\left(\mathbb{C}^{+}_{f,nm}-i\mathbb{S}^{+}_{f,nm}\right)\left(\cos\left(+\theta_f\right)+i\sin\left(+\theta_f\right)\right)+\\ \left(\mathbb{C}^{-}_{f,nm}+i\mathbb{S}^{-}_{f,nm}\right)\left(\cos\left(-\theta_f\right)+i\sin\left(-\theta_f\right)\right)\end{matrix}\\
&= \sum_{f}\begin{matrix}\left(\mathbb{C}^{+}_{f,nm}-i\mathbb{S}^{+}_{f,nm}\right)\left(\cos\theta_f+i\sin\theta_f\right)+\\ \left(\mathbb{C}^{-}_{f,nm}+i\mathbb{S}^{-}_{f,nm}\right)\left(\cos\theta_f-i\sin\theta_f\right)\end{matrix}\\
&= \sum_{f}\begin{matrix}\mathbb{C}^{+}_{f,nm}\cos\theta_f+\mathbb{S}^{+}_{f,nm}\sin\theta_f+i\mathbb{C}^{+}_{f,nm}\sin\theta_f-i\mathbb{S}^{+}_{f,nm}\cos\theta_f+\\ \mathbb{C}^{-}_{f,nm}\cos\theta_f+\mathbb{S}^{-}_{f,nm}\sin\theta_f-i\mathbb{C}^{-}_{f,nm}\sin\theta_f+i\mathbb{S}^{-}_{f,nm}\cos\theta_f\end{matrix}\\
&= \sum_{f}\begin{matrix}\mathbb{C}^{+}_{f,nm}\cos\theta_f+\mathbb{S}^{+}_{f,nm}\sin\theta_f+i\mathbb{C}^{+}_{f,nm}\sin\theta_f-i\mathbb{S}^{+}_{f,nm}\cos\theta_f+\\ \mathbb{C}^{-}_{f,nm}\cos\theta_f+\mathbb{S}^{-}_{f,nm}\sin\theta_f-i\mathbb{C}^{-}_{f,nm}\sin\theta_f+i\mathbb{S}^{-}_{f,nm}\cos\theta_f\end{matrix}\\
&= \sum_{f}\begin{matrix}\left(\left(\mathbb{C}^{+}_{f,nm}+\mathbb{C}^{-}_{f,nm}\right)\cos\theta_f+\left(\mathbb{S}^{+}_{f,nm}+\mathbb{S}^{-}_{f,nm}\right)\sin\theta_f\right)-\\ i\left(\left(\mathbb{S}^{+}_{f,nm}-\mathbb{S}^{-}_{f,nm}\right)\cos\theta_f-\left(\mathbb{C}^{+}_{f,nm}-\mathbb{C}^{-}_{f,nm}\right)\sin\theta_f\right)\end{matrix}
\end{aligned}
\quad (2\text{-}37)
$$

$$
\begin{cases}\Delta\bar{C}_{nm}=\left(\mathbb{C}^{+}_{f,nm}+\mathbb{C}^{-}_{f,nm}\right)\cos\theta_f+\left(\mathbb{S}^{+}_{f,nm}+\mathbb{S}^{-}_{f,nm}\right)\sin\theta_f\\ \Delta\bar{S}_{nm}=\left(\mathbb{S}^{+}_{f,nm}-\mathbb{S}^{-}_{f,nm}\right)\cos\theta_f-\left(\mathbb{C}^{+}_{f,nm}-\mathbb{C}^{-}_{f,nm}\right)\sin\theta_f\end{cases}\quad (2\text{-}38)
$$

其中，$\mathbb{C}^{\pm}_{f,nm}$和$\mathbb{S}^{\pm}_{f,nm}$为潮汐频率 f 上的引力位球谐振幅，IERS 2010 推荐采用 FES 2004 海潮模型，具体数值参见 fes2004_Cnm-Snm.dat 文件，$\theta_f\left(t\right)$与固体潮中的计算公式相同。

Doodson 数 A 与 Doodson 乘数 $\bar{n}$ 的关系如下：

$$A=n_1\left(n_2+5\right)\left(n_3+5\right)\cdot\left(n_4+5\right)\left(n_5+5\right)\left(n_6+5\right)\quad (2\text{-}39)$$

IERS Conventions 1996：

$$\Delta\bar{C}_{nm}-i\Delta\bar{S}_{nm}=F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C^{\pm}_{s,nm}\mp iS^{\pm}_{s,nm}\right)e^{\pm i\theta_f}$$

$$
\begin{aligned}
\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} &= F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm} \mp iS_{s,nm}^{\pm}\right)e^{\pm i\theta_f} \\
&= F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm} \mp iS_{s,nm}^{\pm}\right)\left(\cos\left(\pm\theta_f\right) + i\sin\left(\pm\theta_f\right)\right) \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} - iS_{s,nm}^{+}\right)\left(\cos\left(+\theta_f\right) + i\sin\left(+\theta_f\right)\right) + \\ \left(C_{s,nm}^{-} + iS_{s,nm}^{-}\right)\left(\cos\left(-\theta_f\right) + i\sin\left(-\theta_f\right)\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} - iS_{s,nm}^{+}\right)\left(\cos\theta_f + i\sin\theta_f\right) + \\ \left(C_{s,nm}^{-} + iS_{s,nm}^{-}\right)\left(\cos\theta_f - i\sin\theta_f\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+}\cos\theta_f - iS_{s,nm}^{+}\cos\theta_f + iC_{s,nm}^{+}\sin\theta_f + S_{s,nm}^{+}\sin\theta_f\right) + \\ \left(C_{s,nm}^{-}\cos\theta_f + iS_{s,nm}^{-}\cos\theta_f - iC_{s,nm}^{-}\sin\theta_f + S_{s,nm}^{-}\sin\theta_f\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} + C_{s,nm}^{-}\right)\cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-}\right)\sin\theta_f - \\ i\left(\left(S_{s,nm}^{+} - S_{s,nm}^{-}\right)\cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-}\right)\sin\theta_f\right)\end{matrix}
\end{aligned}
$$

$$
\begin{cases}
\Delta\bar{C}_{nm} = F_{nm}\sum_{s(n,m)}\left(C_{s,nm}^{+} + C_{s,nm}^{-}\right)\cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-}\right)\sin\theta_f \\
\Delta\bar{S}_{nm} = F_{nm}\sum_{s(n,m)}\left(S_{s,nm}^{+} - S_{s,nm}^{-}\right)\cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-}\right)\sin\theta_f
\end{cases}
$$

$$
F_{nm} = \frac{4\pi G\rho_w}{g}\sqrt{\frac{(n+m)!}{(n-m)!(2n+1)(2-\delta_{0m})}}\left(\frac{1+k_n'}{2n+1}\right)
$$

IERS Conventions 2003:

$$
\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} = F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm} \mp iS_{s,nm}^{\pm}\right)e^{\pm i\theta_f}
$$

$$
\begin{aligned}
\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} &= F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm} \mp iS_{s,nm}^{\pm}\right)e^{\pm i\theta_f} \\
&= F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm} \mp iS_{s,nm}^{\pm}\right)\left(\cos\left(\pm\theta_f\right) + i\sin\left(\pm\theta_f\right)\right) \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} - iS_{s,nm}^{+}\right)\left(\cos\left(+\theta_f\right) + i\sin\left(+\theta_f\right)\right) + \\ \left(C_{s,nm}^{-} + iS_{s,nm}^{-}\right)\left(\cos\left(-\theta_f\right) + i\sin\left(-\theta_f\right)\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} - iS_{s,nm}^{+}\right)\left(\cos\theta_f + i\sin\theta_f\right) + \\ \left(C_{s,nm}^{-} + iS_{s,nm}^{-}\right)\left(\cos\theta_f - i\sin\theta_f\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+}\cos\theta_f - iS_{s,nm}^{+}\cos\theta_f + iC_{s,nm}^{+}\sin\theta_f + S_{s,nm}^{+}\sin\theta_f\right) + \\ \left(C_{s,nm}^{-}\cos\theta_f + iS_{s,nm}^{-}\cos\theta_f - iC_{s,nm}^{-}\sin\theta_f + S_{s,nm}^{-}\sin\theta_f\right)\end{matrix} \\
&= F_{nm}\sum_{s(n,m)}\begin{matrix}\left(C_{s,nm}^{+} + C_{s,nm}^{-}\right)\cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-}\right)\sin\theta_f - \\ i\left(\left(S_{s,nm}^{+} - S_{s,nm}^{-}\right)\cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-}\right)\sin\theta_f\right)\end{matrix}
\end{aligned}
$$

$$
\begin{cases}
\Delta\bar{C}_{nm} = F_{nm}\sum_{s(n,m)}\left(C_{s,nm}^{+} + C_{s,nm}^{-}\right)\cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-}\right)\sin\theta_f \\
\Delta\bar{S}_{nm} = F_{nm}\sum_{s(n,m)}\left(S_{s,nm}^{+} - S_{s,nm}^{-}\right)\cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-}\right)\sin\theta_f
\end{cases}
$$

$$
F_{nm} = \frac{4\pi G\rho_w}{g}\sqrt{\frac{(n+m)!}{(n-m)!(2n+1)(2-\delta_{0m})}}\left(\frac{1+k_n'}{2n+1}\right)
$$

IERS Conventions 2010：

$$
\left[\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm}\right](t) = \sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm} \mp i\mathbb{S}_{f,nm}^{\pm}\right)e^{\pm i\theta_f(t)} \quad \text{（6-15）}
$$

$$
\begin{aligned}
\left[\Delta\bar{C}_{nm}-i\Delta\bar{S}_{nm}\right]&=\sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm}\mp i\mathbb{S}_{f,nm}^{\pm}\right)e^{\pm i\theta_f}\\
&=\sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm}\mp i\mathbb{S}_{f,nm}^{\pm}\right)\left(\cos\left(\pm\theta_f\right)+i\sin\left(\pm\theta_f\right)\right)\\
&=\sum_{f}\begin{matrix}\left(\mathbb{C}_{f,nm}^{+}-i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\left(+\theta_f\right)+i\sin\left(+\theta_f\right)\right)+\\ \left(\mathbb{C}_{f,nm}^{-}+i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\left(-\theta_f\right)+i\sin\left(-\theta_f\right)\right)\end{matrix}\\
&=\sum_{f}\begin{matrix}\left(\mathbb{C}_{f,nm}^{+}-i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\theta_f+i\sin\theta_f\right)+\\ \left(\mathbb{C}_{f,nm}^{-}+i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\theta_f-i\sin\theta_f\right)\end{matrix}\\
&=\sum_{f}\begin{matrix}\mathbb{C}_{f,nm}^{+}\cos\theta_f+\mathbb{S}_{f,nm}^{+}\sin\theta_f+i\mathbb{C}_{f,nm}^{+}\sin\theta_f-i\mathbb{S}_{f,nm}^{+}\cos\theta_f+\\ \mathbb{C}_{f,nm}^{-}\cos\theta_f+\mathbb{S}_{f,nm}^{-}\sin\theta_f-i\mathbb{C}_{f,nm}^{-}\sin\theta_f+i\mathbb{S}_{f,nm}^{-}\cos\theta_f\end{matrix}\\
&=\sum_{f}\begin{matrix}\mathbb{C}_{f,nm}^{+}\cos\theta_f+\mathbb{S}_{f,nm}^{+}\sin\theta_f+i\mathbb{C}_{f,nm}^{+}\sin\theta_f-i\mathbb{S}_{f,nm}^{+}\cos\theta_f+\\ \mathbb{C}_{f,nm}^{-}\cos\theta_f+\mathbb{S}_{f,nm}^{-}\sin\theta_f-i\mathbb{C}_{f,nm}^{-}\sin\theta_f+i\mathbb{S}_{f,nm}^{-}\cos\theta_f\end{matrix}\\
&=\sum_{f}\begin{matrix}\left(\left(\mathbb{C}_{f,nm}^{+}+\mathbb{C}_{f,nm}^{-}\right)\cos\theta_f+\left(\mathbb{S}_{f,nm}^{+}+\mathbb{S}_{f,nm}^{-}\right)\sin\theta_f\right)-\\ i\left(\left(\mathbb{S}_{f,nm}^{+}-\mathbb{S}_{f,nm}^{-}\right)\cos\theta_f-\left(\mathbb{C}_{f,nm}^{+}-\mathbb{C}_{f,nm}^{-}\right)\sin\theta_f\right)\end{matrix}
\end{aligned}
$$

$$
\begin{cases}\Delta\bar{C}_{nm}=\left(\mathbb{C}_{f,nm}^{+}+\mathbb{C}_{f,nm}^{-}\right)\cos\theta_f+\left(\mathbb{S}_{f,nm}^{+}+\mathbb{S}_{f,nm}^{-}\right)\sin\theta_f\\ \Delta\bar{S}_{nm}=\left(\mathbb{S}_{f,nm}^{+}-\mathbb{S}_{f,nm}^{-}\right)\cos\theta_f-\left(\mathbb{C}_{f,nm}^{+}-\mathbb{C}_{f,nm}^{-}\right)\sin\theta_f\end{cases}
$$

$$
\begin{cases}\mathbb{C}_{f,nm}^{\pm}=\dfrac{4\pi G\rho_w}{g}\left(\dfrac{1+k_n'}{2n+1}\right)\hat{\bar{C}}_{f,nm}^{\pm}\sin\left(\varepsilon_{f,nm}^{\pm}+\chi_f\right)\\ \mathbb{S}_{f,nm}^{\pm}=\dfrac{4\pi G\rho_w}{g}\left(\dfrac{1+k_n'}{2n+1}\right)\hat{\bar{C}}_{f,nm}^{\pm}\cos\left(\varepsilon_{f,nm}^{\pm}+\chi_f\right)\end{cases}
$$

其中，χ_f 的值由表 6.6 确定。$\hat{\bar{C}}_{f,nm}^{\pm}$ 和 $\varepsilon_{f,nm}^{\pm}$ 分别为频率 f 上 n 阶 m 次的振幅和相位。

4）极潮

极潮包括两部分：固体地球极潮以及海洋极潮。

固体地球极潮对 C_{21} 和 S_{21} 的影响如下：

$$
\begin{cases}\Delta\bar{C}_{21}=-1.333\times10^{-9}\left(m_1+0.0115m_2\right)\\ \Delta\bar{S}_{21}=-1.333\times10^{-9}\left(m_2-0.0115m_1\right)\end{cases}\quad（2-40）
$$

其中，$m_1 = x_p - \bar{x}_p, m_2 = -\left(y_p - \bar{y}_p\right)$，$\left(x_p, y_p\right)$为极移参数，$\left(\bar{x}_p, \bar{y}_p\right)$为 IERS（2010）平均地球极，与地球引力中的计算公式相同。

海洋极潮对正规化的地球引力位系数的影响如下：

$$\begin{bmatrix} \Delta\bar{C}_{nm} \\ \Delta\bar{S}_{nm} \end{bmatrix} = R_n \left\{ \begin{bmatrix} \bar{A}_{nm}^R \\ \bar{B}_{nm}^R \end{bmatrix} \left(m_1\gamma_2^R + m_2\gamma_2^I\right) + \begin{bmatrix} \bar{A}_{nm}^I \\ \bar{B}_{nm}^I \end{bmatrix} \left(m_2\gamma_2^R - m_1\gamma_2^I\right) \right\} \quad (2\text{-}41)$$

式中，

$$R_n = \frac{\Omega^2 a_E^4}{GM} \frac{4\pi G\rho_w}{g_e} \left(\frac{1+k_n'}{2n+1}\right),$$

Ω, a_E, GM, g_e和G 的值由表 1.1 查取，

ρ_w=$1025kgm^{-3}$为海水密度，k_n'为负荷形变系数（$k_2' = -0.3075$，$k_3' = -0.195$，$k_4' = -0.132$，$k_5' = -0.1032$，$k_6' = -0.0892$），

$\gamma = \gamma_2^R + i\gamma_2^I = \left(1 + k_2 - h_2\right) = 0.6870 + i0.0036$，

$\left(m_1, m_2\right)$与固体地球极潮中的计算公式相同，

系数 $\bar{A}_{nm} = \bar{A}_{nm}^R + i\bar{A}_{nm}^I$ 和 $\bar{B}_{nm} = \bar{B}_{nm}^R + i\bar{B}_{nm}^I$ 参见 desaiscopolecoef.txt 文件。

$\left(n, m\right) = \left(2,1\right)$的系数为海洋极潮的主项，由上述计算公式可以得到其表达式如下：

$$\begin{cases} \Delta\bar{C}_{21} = -2.1778\times10^{-10}\left(m_1 - 0.01724m_2\right) \\ \Delta\bar{S}_{21} = -1.7232\times10^{-10}\left(m_2 - 0.03365m_1\right) \end{cases} \quad (2\text{-}42)$$

5）三体引力

两个质点 M 和 m 的相互作用的运动方程为：

$$\begin{cases} M\vec{a}_M = GMm\dfrac{\vec{r}_{Mm}}{r_{Mm}^3} \\ m\vec{a}_m = GMm\dfrac{\vec{r}_{mM}}{r_{mM}^3} \end{cases} \quad (2\text{-}43)$$

其中，r 为向量 $\vec{r}$ 的长度；下标 Mm 表示向量从点质量 M 指向 m；单独的下标 M 或 m 表示向量指向质点 M 或 m。

引入附加的质点 $m_j, j=1,2,\ldots$，类似地，可以得到 m_j 作用在 M 和 m 上的引力方程，求和得到总引力为：

$$\begin{cases} M\vec{a}_M = GMm\dfrac{\vec{r}_{Mm}}{r_{Mm}^3} + \sum\limits_j GMm_j\dfrac{\vec{r}_{Mm_j}}{r_{Mm_j}^3} \\ m\vec{a}_m = GMm\dfrac{\vec{r}_{mM}}{r_{mM}^3} + \sum\limits_j Gmm_j\dfrac{\vec{r}_{mm_j}}{r_{mm_j}^3} \end{cases} \quad (2\text{-}44)$$

以上两式分别除以 $-M$ 和 m 后相加，得到：

$$\vec{a}_m - \vec{a}_M = -G\left(M+m\right)\frac{\vec{r}_{Mm}}{r_{mM}^3} + \sum_j Gm_j\cdot\left[\frac{\vec{r}_{mm_j}}{r_{mm_j}^3} - \frac{\vec{r}_{Mm_j}}{r_{Mm_j}^3}\right] \quad (2\text{-}45)$$

令 $\vec{r} = \vec{r}_m - \vec{r}_M$，即令质点 M 为原点，从而 $\vec{r}_{mm_j} = -\left(\vec{r}_m - \vec{r}_{m_j}\right)$，忽略质量 m，得到：

$$\vec{a} = -G\left(M\right)\frac{\vec{r}}{r^3} - \sum_j Gm_j\cdot\left[\frac{\vec{r}_m - \vec{r}_{m_j}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^3} + \frac{\vec{r}_{m_j}}{r_{m_j}^3}\right] \quad (2\text{-}46)$$

式中，右边第一项为地球中心引力，第二项为三体中心引力，即：

$$\vec{f}_m = -\sum_j Gm_j\cdot\left[\frac{\vec{r}_m - \vec{r}_{m_j}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^3} + \frac{\vec{r}_{m_j}}{\vec{r}_{m_j}^3}\right] \quad (2\text{-}47)$$

其中，Gm_j 为三体的引力常数。

三体中心引力的计算依赖于三体位置的计算，三体位置的计算主要有两种方法：1. 解析公式，2. 行星星历。

三体质点引力对卫星位置的偏导数：

$$
\begin{aligned}
\frac{\partial \vec{a}_{m,j}}{\partial \vec{r}_m} &= -Gm_j \cdot \frac{\partial\left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^3} + \dfrac{\vec{r}_{m_j}}{r_{m_j}^3}\right]}{\partial \vec{r}_m} \\
&= -Gm_j \cdot \frac{\partial\left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^3}\right]}{\partial \vec{r}_m} \\
&= -Gm_j \cdot \left[\frac{I_{3\times3}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^3} - 3\frac{\vec{r}_m - \vec{r}_{m_j}}{\left|\vec{r}_m - \vec{r}_{m_j}\right|^5} I_{3\times3}\right]
\end{aligned}
\tag{2-48}
$$

对月球而言，由于月地距离很近，需要另外考虑其对地球的非球形引力部分的作用。表达式如下：

$$
\triangle \vec{a} = \frac{3}{2} C_{20} \frac{GM_l}{r_l^2} \left(\frac{R}{r_l}\right)^2 \begin{bmatrix} \left(5\sin^2\phi_l - 1\right)\cos\phi_l \cos\lambda_l \\ \left(5\sin^2\phi_l - 1\right)\cos\phi_l \sin\lambda_l \\ \left(5\sin^2\phi_l - 3\right)\sin\phi_l \end{bmatrix} \tag{2-49}
$$

其中，(ϕ_l, λ_l) 分别为月球在 ICRS 坐标系下的纬度和经度。

6）太阳辐射压

ECOM 模型：

$$
\vec{a} = D(u)\vec{e}_D + Y(u)\vec{e}_Y + B(u)\vec{e}_B \tag{2-50}
$$

其中，

$\vec{e}_D \doteq \dfrac{\vec{r}_s - \vec{r}}{\left|\vec{r}_s - \vec{r}\right|}$，$\vec{e}_Y \doteq -\dfrac{\vec{e}_r \times \vec{e}_D}{\left|\vec{e}_r \times \vec{e}_D\right|}$，$\vec{e}_B \doteq \vec{e}_D \times \vec{e}_Y$，$\vec{r}_s$ 和 $\vec{r}$ 分别为太阳和卫星在 ICRS 坐标系下的位置矢量。

（1）当采用 9 参数模型时

$$
\begin{cases}
D(u) = D_0 + D_c \cos u + D_s \sin u \\
Y(u) = Y_0 + Y_c \cos u + Y_s \sin u \\
B(u) = B_0 + B_c \cos u + B_s \sin u
\end{cases}
\tag{2-51}
$$

（2）当采用 5 参数模型时

$$\begin{cases} D(u) = D_0 \\ Y(u) = Y_0 \\ B(u) = B_0 + B_c \cos u + B_s \sin u \end{cases} \qquad (2\text{-}52)$$

太阳辐射压对卫星位置的偏导数：

$$\begin{aligned} \frac{\partial \vec{a}}{\partial \vec{r}} &= \frac{\partial \left(D(u)\vec{e}_D + Y(u)\vec{e}_Y + B(u)\vec{e}_B \right)}{\partial \vec{r}} \\ &= D(u)\frac{\partial \vec{e}_D}{\partial \vec{r}} + Y(u)\frac{\partial \vec{e}_Y}{\partial \vec{r}} + B(u)\frac{\partial \vec{e}_B}{\partial \vec{r}} \end{aligned} \qquad (2\text{-}53)$$

其中，

$$\begin{aligned} \frac{\partial \vec{e}_D}{\partial \vec{r}} &= \frac{\partial \left(\dfrac{\vec{r}_s - \vec{r}}{|\vec{r}_s - \vec{r}|} \right)}{\partial \vec{r}} \\ &= \frac{1}{|\vec{r}_s - \vec{r}|}\frac{\partial (\vec{r}_s - \vec{r})}{\partial \vec{r}} + (\vec{r}_s - \vec{r})\frac{\partial \left(|\vec{r}_s - \vec{r}|^{-1} \right)}{\partial \vec{r}}, \\ &= \frac{1}{|\vec{r}_s - \vec{r}|}\cdot(-I_{3\times3}) + (\vec{r}_s - \vec{r})\frac{-(\vec{r}_s - \vec{r})^T}{|\vec{r}_s - \vec{r}|^2} \end{aligned}$$

$$\frac{\partial \vec{e}_Y}{\partial \vec{r}} = -\frac{\partial \dfrac{\vec{e}_r \times \vec{e}_D}{|\vec{e}_r \times \vec{e}_D|}}{\partial \vec{r}} = 0_{3\times3},$$

$$\begin{aligned} \frac{\partial \vec{e}_B}{\partial \vec{r}} &= \frac{\partial (\vec{e}_D \times \vec{e}_Y)}{\partial \vec{r}} \\ &= -\frac{\partial (\vec{e}_Y \times \vec{e}_D)}{\partial \vec{r}} \\ &= -\left(\frac{\partial (\vec{e}_Y \times)}{\partial \vec{r}}\vec{e}_D + (\vec{e}_Y \times)\frac{\partial \vec{e}_D}{\partial \vec{r}} \right)^{\circ} \\ &= -(\vec{e}_Y \times)\frac{\partial \vec{e}_D}{\partial \vec{r}} \end{aligned}$$

太阳辐射压对力模型参数的偏导数：

（1）当采用 9 参数时

$$\frac{\partial \vec{a}}{\partial\left(D_0, D_c, D_s\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(D_0, D_c, D_s\right)}=\frac{\partial D(u)}{\partial\left(D_0, D_c, D_s\right)}\vec{e}_D$$

$$\frac{\partial \vec{a}}{\partial\left(Y_0, Y_c, Y_s\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(Y_0, Y_c, Y_s\right)}=\frac{\partial Y(u)}{\partial\left(Y_0, Y_c, Y_s\right)}\vec{e}_Y \quad (2\text{-}54)$$

$$\frac{\partial \vec{a}}{\partial\left(B_0, B_c, B_s\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(B_0, B_c, B_s\right)}=\frac{\partial B(u)}{\partial\left(B_0, B_c, B_s\right)}\vec{e}_B$$

（2）当采用 5 参数时

$$\frac{\partial \vec{a}}{\partial\left(D_0\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(D_0\right)}=\frac{\partial D(u)}{\partial\left(D_0\right)}\vec{e}_D$$

$$\frac{\partial \vec{a}}{\partial\left(Y_0\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(Y_0\right)}=\frac{\partial Y(u)}{\partial\left(Y_0\right)}\vec{e}_Y \quad (2\text{-}55)$$

$$\frac{\partial \vec{a}}{\partial\left(B_0, B_c, B_s\right)}=\frac{\partial\left(D(u)\vec{e}_D+Y(u)\vec{e}_Y+B(u)\vec{e}_B\right)}{\partial\left(B_0, B_c, B_s\right)}=\frac{\partial B(u)}{\partial\left(B_0, B_c, B_s\right)}\vec{e}_B$$

7）地球辐射压

暂不考虑。

8）相对论效应

在 GCRS 坐标系下，人造地球卫星运动加速度的相对论改正：

$$\Delta\ddot{\vec{r}} = \frac{GM_E}{c^2 r^3}\left\{\left[2(\beta+\gamma)\frac{GM_E}{r} - \gamma\dot{\vec{r}}\cdot\dot{\vec{r}}\right]\vec{r} + 2(1+\gamma)(\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}\right\} +$$

$$(1+\gamma)\frac{GM_E}{c^2 r^3}\left[\frac{3}{r^2}(\vec{r}\times\dot{\vec{r}})(\vec{r}\cdot\vec{J}) + (\dot{\vec{r}}\times\vec{J})\right] + \tag{2-56}$$

$$\left\{(1+2\gamma)\left[\dot{\vec{R}}\times\left(\frac{-GM_S\vec{R}}{c^2 R^3}\right)\right]\times\dot{\vec{r}}\right\}$$

其中，c 为光速，β 和 γ 为 PPN（parameterized post-Newtonian）参数，在广义相对论中等于 1，$\vec{r}$ 为卫星相对于地球的位置，$\vec{R}$ 为地球相对于太阳的位置，$\vec{J}$ 为单位质量的地）球角动量（$\left|\vec{J}\right| \cong 9.8\times10^8\, m^2/s$），$GM_E$ 和 GM_S 分别为地球和太阳的引力系数。

忽略（2-55）式中的第 2 和第 3 项，并代入 β 和 γ，得到：

$$\Delta\vec{a} = \frac{GM_E}{c^2 r^3}\left[\left(4\frac{GM_E}{r} - \vec{v}\cdot\vec{r}\right)\vec{r} + 4(\vec{r}\cdot\vec{v})\vec{v}\right] \tag{2-57}$$

相对论改正对卫星位置的偏导数：

$$\begin{aligned}
\frac{\partial\Delta\vec{a}}{\partial\vec{r}} &= \frac{\partial\dfrac{GM_E}{c^2 r^3}\left[\left(4\dfrac{GM_E}{r} - \vec{v}\cdot\vec{r}\right)\vec{r} + 4(\vec{r}\cdot\vec{v})\vec{v}\right]}{\partial\vec{r}} \\
&= \frac{\partial\dfrac{GM_E}{c^2 r^3}}{\partial\vec{r}}\left[\left(4\frac{GM_E}{r} - \vec{v}\cdot\vec{r}\right)\vec{r} + 4(\vec{r}\cdot\vec{v})\vec{v}\right] + \\
&\quad \frac{GM_E}{c^2 r^3}\frac{\partial\left[\left(4\dfrac{GM_E}{r} - \vec{v}\cdot\vec{r}\right)\vec{r} + 4(\vec{r}\cdot\vec{v})\vec{v}\right]}{\partial\vec{r}} \\
&= -3\frac{GM_E}{c^2 r^5} I_{3\times3}\left[\left(4\frac{GM_E}{r} - \vec{v}\cdot\vec{r}\right)\vec{r} + 4(\vec{r}\cdot\vec{v})\vec{v}\right] + \\
&\quad \frac{GM_E}{c^2 r^3}\left[\begin{array}{l}\left(-4\dfrac{GM_E}{r} I_{3\times3} - \vec{v}\cdot I_{3\times3}\right)\vec{r} + \\ \left(4\dfrac{GM_E}{r} - \vec{v}\cdot\vec{r}\right) + 4(I_{3\times3}\cdot\vec{v})\vec{v}\end{array}\right]
\end{aligned} \tag{2-58}$$

相对论改正对卫星速度的偏导数：

$$\frac{\partial\Delta\vec{a}}{\partial\vec{v}}=\frac{\partial\frac{GM_E}{c^2r^3}\left[\left(4\frac{GM_E}{r}-\vec{v}\cdot\vec{r}\right)\vec{r}+4\left(\vec{r}\cdot\vec{v}\right)\vec{v}\right]}{\partial\vec{v}}$$

$$=\frac{GM_E}{c^2r^3}\left(\frac{\partial\left[\left(4\frac{GM_E}{r}-\vec{v}\cdot\vec{r}\right)\vec{r}\right]}{\partial\vec{v}}+\frac{\partial\left[4\left(\vec{r}\cdot\vec{v}\right)\vec{v}\right]}{\partial\vec{v}}\right) \quad (2\text{-}59)$$

$$=\frac{GM_E}{c^2r^3}\left(-I_{3\times3}\cdot r^2+8\vec{r}\cdot I_{3\times3}\right)$$

9）经验力

暂不考虑。

3 积分器

卫星轨道积分问题如下：

$$\begin{pmatrix}\dot{x}(t)\\ \dot{\Phi}(t,t_0)\end{pmatrix}=\begin{pmatrix}F(t,x)\\ A(t)\Phi(t,t_0)\end{pmatrix},x(t_0),\Phi(t_0,t_0) \quad (3\text{-}1)$$

已知t_0时刻的卫星位置$r(t_0)$、速度$v(t_0)$以及力模型参数$p(t_0)$。对 GNSS 卫星轨道积分而言，在卫星加速度的计算中不包含与卫星速度有关的力，力模型参数主要为太阳光压参数，采用 CODE 9 参数模型，则初始状态包括 3 个位置参数+3 个速度参数+9 个太阳光压参数。由于积分方程右边的卫星加速度计算比较复杂，因此，通常采用多步积分法。

参考 ESA 分析中心所采用的方案，以 8 阶 Embedded-Runge-Kutta 变步长（初始步长 75s）单步积分法作为起步，以 8 阶 Adams-Bashforth-Moulton 定步长（步长 300s）多步积分法作为后续。

步骤如下：

1）以t_0时刻为起始时刻，采用 8 阶 Embedded-Runge-Kutta 变步长单步积分器向后积分 8*300/75=32 个历元，得到相应历元的卫星位置、速度、力模型参数以及转移矩阵$x_i,\frac{\partial x_i}{\partial x_0},i=1,32$；

积分公式：

$$\begin{cases} \hat{y}_{n+1} = y_n + h\sum_{i=0}^{s=10} \hat{c}_i \cdot k_i \\ y_{n+1} = y_n + h\sum_{i=0}^{s=12} c_i \cdot k_i \end{cases} \quad (3\text{-}2)$$

其中，

$$\begin{cases} k_0 = h_n f\left(t_n, y_n\right) \\ k_i = h_n f\left(t_n + a_i h_n, \hat{y}_n + \sum_{j=1}^{i-1} b_{ij} K_j\right), \left(i = 1, 2, \ldots, 12\right) \end{cases}$$

误差估计：

$$TE = \left|\hat{y}_{n+1} - y_{n+1}\right| = \frac{41}{840} \cdot h \cdot \left|k_0 + k_{10} - k_{11} - k_{12}\right| \quad (3\text{-}3)$$

收敛条件：

$$A = \sum_{components} \left(\frac{TE_j}{tol_j}\right)^2 < 1 \quad (3\text{-}4)$$

步长调整：

$$h_{new} = h \cdot \left(\frac{0.0025}{A}\right)^{\frac{1}{16}} \quad (3\text{-}5)$$

系数参考 Fehlberg（1968）第 65 页。

2）根据 Embedded-Runge-Kutta 变步长单步积分器积分得到的前 8 个历元的值，启动 8 阶 Adams-Bashforth-Moulton 定步长多步积分器，积分得到后续各个历元的卫星位置、速度、力模型参数以及转移矩阵 $x_i, \frac{\partial x_i}{\partial x_0}, i = 1, \ldots$。

Adams-Bashforth 预测公式：

$$y_{n+1}^p = y_n + \frac{h}{3628800} \sum_{j=0}^{k=8} \beta_j^p \cdot f_{n-j} \quad (3\text{-}6)$$

其中，$\beta_j^p=\sum_{m=j}^{k=8}(-1)^j C_m^j \gamma_m^p$，$\gamma_m^p=\int_0^1 C_{s+m-1}^m ds$，$\begin{cases}\gamma_0^p=1\\ \gamma_m^p=1-\sum_{j=1}^{m}\frac{1}{j+1}\gamma_{m-j}^p,\ m\geq 1\end{cases}$。

Adams-Moulton 校正公式：

$$y_{n+1}^c=y_n+\frac{h}{3628800}\left(\beta_0^c\cdot f_{n+1}^p+\sum_{j=1}^{k=8}\beta_j^c\cdot f_{n+1-j}\right) \quad (3\text{-}7)$$

其中，$\beta_j^c=\sum_{m=j}^{k}(-1)^j C_m^j \gamma_m^c$，$\gamma_m^c=\int_{-1}^0 C_{s+m-1}^m ds$，$\begin{cases}\gamma_0^c=1\\ \gamma_m^c=-\sum_{j=1}^{m}\frac{1}{j+1}\gamma_{m-j}^c,\ m\geq 1\end{cases}$。

表 3-1 Adams-Bashforth/Adams-Moulton 系数

j	Adams-Bashforth 系数 β_j^p	Adams-Moulton 系数 β_j^c
0	14097247	1070017
1	-43125206	4467094
2	95476786	-4604594
3	-139855262	5595358
4	137968480	-5033120
5	-91172642	3146338
6	38833486	-1291214
7	-9664106	312874
8	1070017	-33953

4 轨道拟合

在 GNSS 卫星轨道积分中，初始位置和速度由精密星历或广播星历给出，初始力模型参数由经验值给出或全部置零。

在轨道拟合过程中，首先，对运动方程和变分方程积分，得到卫星当前位置和速度以及其与初始位置、速度和力模型参数的偏导数关系；然后，将卫星当前时刻的位置和

速度与精密星历中的位置和速度作差，形成虚拟观测值；最后，建立虚拟观测方程，利用最小二乘求解得到初始位置、速度和力模型参数的改正值。

t 时刻的轨道拟合虚拟观测方程：

$$r_{oi}\left(t\right)-r_{sp3}\left(t\right)=\frac{\partial r}{\partial p_0}\left(t\right)\cdot dp_0 \quad （4-1）$$