GNSS 卫星精密定轨之轨道积分

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1 基本原理

根据牛顿第二定律，卫星的加速度 $\vec{a}$ 可以表示为时间 t、位置 $\vec{r}$、速度 $\vec{v}$ 以及力模型参数 $\vec{p}$ 的函数，如下：

$$\vec{a} = f\left(t, \vec{r}, \vec{v}, \vec{p}\right) \quad \text{（1-1）}$$

根据 t_0 时刻的卫星位置、速度以及力模型参数 $\left(\vec{r}_0, \vec{v}_0, \vec{p}_0\right)$，对（3-1）式积分，即可确定 t 时刻的卫星位置、速度 $\left(\vec{r}, \vec{v}\right)$。由于 $\left(\vec{r}_0, \vec{v}_0, \vec{p}_0\right)$ 的不准确性，当存在外部观测量（如距离、距离变率等）时，需要对其进行修正。通常，在一定长度的积分弧段内，认为力模型参数 $\vec{p}$ 不变，即 $\vec{p} = \vec{p}_0$。

（1-1）式对应的一阶微分方程如下：

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \vec{v} \\[2mm] \dfrac{d\vec{v}}{dt} = \vec{a} \quad \text{（1-2）} \\[2mm] \dfrac{d\vec{p}}{dt} = 0 \end{cases}$$

令 $\vec{x} = \left(\vec{r}, \vec{v}, \vec{p}\right)$，得到

$$\dot{\vec{x}} = \left(\vec{v} \ \ \vec{a} \ \ 0\right)^T = \vec{F}\left(t, \vec{x}\right) \quad \text{（1-3）}$$

（1-3）式即为卫星轨道运动方程。根据卫星初始状态 $\vec{x}_0$，可以积分确定卫星当前状态 $\vec{x}$。

由（1-3）式可以得到卫星当前状态 $\vec{x}$ 与初始状态 $\vec{x}_0$ 的关系如下：

$$\begin{cases} \dfrac{\partial \vec{x}}{\partial \vec{x}_0} = d\left(\dfrac{\partial \vec{x}}{\partial \vec{x}_0}\right)\bigg/ dt \\ \dfrac{\partial \dot{\vec{x}}}{\partial \vec{x}_0} = \dfrac{\partial \vec{F}(t,\vec{x})}{\partial \vec{x}_0} = \dfrac{\partial \vec{F}(t,\vec{x})}{\partial \vec{x}} \dfrac{\partial \vec{x}}{\partial \vec{x}_0} \end{cases} \Rightarrow d\left(\dfrac{\partial \vec{x}}{\partial \vec{x}_0}\right)\bigg/ dt = \dfrac{\partial \vec{F}(t,\vec{x})}{\partial \vec{x}} \dfrac{\partial \vec{x}}{\partial \vec{x}_0} \quad (1\text{-}4)$$

其中，$\dfrac{\partial \vec{F}(t,\vec{x})}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial \vec{v}}{\partial \vec{r}} & \dfrac{\partial \vec{v}}{\partial \vec{v}} & \dfrac{\partial \vec{v}}{\partial \vec{p}} \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \\ \dfrac{\partial 0}{\partial \vec{r}} & \dfrac{\partial 0}{\partial \vec{v}} & \dfrac{\partial 0}{\partial \vec{p}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \\ 0 & 0 & 0 \end{bmatrix}$。

令 $\dfrac{\partial \vec{x}}{\partial \vec{x}_0} = X$，$\dfrac{\partial \vec{F}(t,\vec{x})}{\partial \vec{x}} = A$，简化（1-4）式，得到：

$$\dot{X} = AX \quad （1\text{-}5）$$

（1-5）式解的形式可以表示如下：

$$X(t) = \Phi(t,t_0) X(t_0) \quad （1\text{-}6）$$

将（1-6）式代入（1-5）式，得到：

$$\dot{\Phi}(t,t_0) = A\Phi(t,t_0) \quad （1\text{-}7）$$

其中，$\Phi(t,t_0) = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{p}}{\partial \vec{r}_0} & \dfrac{\partial \vec{p}}{\partial \vec{v}_0} & \dfrac{\partial \vec{p}}{\partial \vec{p}_0} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ 0_{n\times 3} & 0_{n\times 3} & I_{n\times n} \end{bmatrix}$，称为转移矩阵。

（1-7）式即为卫星轨道变分方程。根据初始转移矩阵 $\Phi(t_0,t_0)$，可以积分确定当前转移矩阵 $\Phi(t,t_0)$。

综上所述，可以得到卫星轨道的运动方程和变分方程分别如下：

运动方程：$\vec{x}(t) = \vec{F}(t, \vec{x})$ （1-8）

变分方程：$\dot{\Phi}(t, t_0) = A(t, x)\Phi(t, t_0)$ （1-9）

联合运动方程与变分方程，得到：

$$\begin{pmatrix} \vec{x}(t) \\ \dot{\Phi}(t, t_0) \end{pmatrix} = \begin{pmatrix} \vec{F}(t, \vec{x}) \\ A(t, \vec{x})\Phi(t, t_0) \end{pmatrix}$$ （1-10）

其中，

$\vec{x}(t)$ 为 t 时刻的卫星位置、速度及力模型参数；

$\vec{F}(t, \vec{x})$ 为 t 时刻的卫星速度、加速度及力模型参数对时间的微分（$\frac{\partial \vec{p}}{\partial t} = 0$ 表明在积分弧段内力模型参数不变）；

$\Phi(t, t_0)$ 为 t 时刻的转移矩阵，包括状态转移矩阵和参数敏感矩阵；

$A(t, \vec{x})$ 为 t 时刻的卫星加速度对卫星位置、速度及力模型参数的微分。

（1-10）式的具体形式为：

$$\begin{pmatrix} \vec{r} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ \vec{a} \end{pmatrix}$$ （1-11）

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{v}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{p}_0}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{r}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{v}_0}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{p}_0}\right)}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \vec{v}}{\partial \vec{r}} & \dfrac{\partial \vec{v}}{\partial \vec{v}} & \dfrac{\partial \vec{v}}{\partial \vec{p}} \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r}_0} & \dfrac{\partial \vec{r}}{\partial \vec{v}_0} & \dfrac{\partial \vec{r}}{\partial \vec{p}_0} \\ \dfrac{\partial \vec{v}}{\partial \vec{r}_0} & \dfrac{\partial \vec{v}}{\partial \vec{v}_0} & \dfrac{\partial \vec{v}}{\partial \vec{p}_0} \\ 0 & 0 & I \end{pmatrix}$$ （1-12）

初始状态的不准确性包括初始位置、速度以及力模型参数的不准确性，仅对运动方程（1-11）式积分，将会产生位置和速度误差。为修正初始状态的误差，需要建立当前状态与初始状态的联系（当前状态对初始状态的偏导数），而当前状态又可以与外部观测建立联系（观测值对当前状态的偏导数），从而可以利用外部观测修正初始状态的误差。

因此，当初始状态准确时，通过积分运动方程（1-11）式即可得到高精度的位置和速度（仅含积分误差），不需要积分变分方程（1-12）式；当初始状态不准确时，需要对运动方程（1-11）式和变分方程（1-12）式同时积分，利用运动方程积分得到的位置和速度以及变分方程积分得到的转移矩阵，建立外部观测与初始状态的关系，从而实现对初始状态的修正。

对变分方程，由于 $\frac{\partial \vec{v}}{\partial \vec{r}} = 0, \frac{\partial \vec{v}}{\partial \vec{v}} = I, \frac{\partial \vec{v}}{\partial \vec{p}} = 0$（对于 GNSS 卫星，速度与力模型参数无关），得到：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{r_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{v_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{p_0}}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{r_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{v_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{p_0}}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r_0}} & \dfrac{\partial \vec{r}}{\partial \vec{v_0}} & \dfrac{\partial \vec{r}}{\partial \vec{p_0}} \\ \dfrac{\partial \vec{v}}{\partial \vec{r_0}} & \dfrac{\partial \vec{v}}{\partial \vec{v_0}} & \dfrac{\partial \vec{v}}{\partial \vec{p_0}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \quad （1\text{-}13）$$

状态转移矩阵微分方程：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{r_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{v_0}}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{r_0}}\right)}{dt} & \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{v_0}}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{r_0}} & \dfrac{\partial \vec{r}}{\partial \vec{v_0}} \\ \dfrac{\partial \vec{v}}{\partial \vec{r_0}} & \dfrac{\partial \vec{v}}{\partial \vec{v_0}} \end{pmatrix} \quad （1\text{-}14）$$

参数敏感矩阵微分方程：

$$\begin{pmatrix} \dfrac{d\left(\dfrac{\partial \vec{r}}{\partial \vec{p_0}}\right)}{dt} \\ \dfrac{d\left(\dfrac{\partial \vec{v}}{\partial \vec{p_0}}\right)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} & \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{p_0}} \\ \dfrac{\partial \vec{v}}{\partial \vec{p_0}} \\ I \end{pmatrix} = \begin{pmatrix} 0 & I \\ \dfrac{\partial \vec{a}}{\partial \vec{r}} & \dfrac{\partial \vec{a}}{\partial \vec{v}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{r}}{\partial \vec{p_0}} \\ \dfrac{\partial \vec{v}}{\partial \vec{p_0}} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{\partial \vec{a}}{\partial \vec{p}} \end{pmatrix} \quad （1\text{-}15）$$

2 力模型

对 GNSS 卫星精密定轨而言，所受到的力主要包括地球引力（包括球形引力和非球形引力）、三体引力（主要为日月引力）、太阳辐射压、地球辐射压、发动机推进力、潮汐效应（固体潮、海潮、极潮）、相对论效应、经验力等。

表 2-1 GNSS 卫星精密定轨力模型

力模型	量级（m/s²）	模型
地球引力	0.5	EGM2008 模型（12 阶 12 次）
三体引力	10^{-6}	月球、太阳 JPL DE 405 星历
太阳光压	10^{-8}	CODE 9 参数/5 参数模型
固体潮	10^{-9}	IERS 2010
海潮	10^{-10}	IERS 2010
极潮	10^{-11}	IERS 2010
相对论效应	10^{-10}	IERS 2010
经验力		

2.1 地球引力

地球引力位表达式如下：

$$V(r,\varphi,\lambda) = \frac{GM_E}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(\bar{C}_{nm}\cos(m\lambda) + \bar{S}_{nm}\sin(m\lambda)\right) \quad （2\text{-}1）$$

其中，(r,φ,λ) 分别为计算点的地心向径、纬度和经度，a_e 为地球赤道半径，$\bar{P}_{nm}(\sin\varphi)$ 为完全正规化的缔合勒让德多项式，$\bar{C}_{nm}$ 和 $\bar{S}_{nm}$ 为球谐系数。

ICRS 直角坐标系下的地球引力：

$$\vec{f}_{ICRS} = C_{ITRS}^{ICRS} \vec{f}_{ITRS}$$

$$= C_{ITRS}^{ICRS} \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}}$$

$$= C_{ITRS}^{ICRS} \left(\frac{\partial V_{ITRS}}{\partial x_{ITRS}} \quad \frac{\partial V_{ITRS}}{\partial y_{ITRS}} \quad \frac{\partial V_{ITRS}}{\partial z_{ITRS}} \right)^T$$

$$= C_{ITRS}^{ICRS} \begin{pmatrix} \dfrac{\partial V_{ITRS}}{\partial r_{ITRS}} \dfrac{\partial r_{ITRS}}{\partial x_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}} \dfrac{\partial \varphi_{ITRS}}{\partial x_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}} \dfrac{\partial \lambda_{ITRS}}{\partial x_{ITRS}} \\[2mm] \dfrac{\partial V_{ITRS}}{\partial r_{ITRS}} \dfrac{\partial r_{ITRS}}{\partial y_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}} \dfrac{\partial \varphi_{ITRS}}{\partial y_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}} \dfrac{\partial \lambda_{ITRS}}{\partial y_{ITRS}} \\[2mm] \dfrac{\partial V_{ITRS}}{\partial r_{ITRS}} \dfrac{\partial r_{ITRS}}{\partial z_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}} \dfrac{\partial \varphi_{ITRS}}{\partial z_{ITRS}} + \dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}} \dfrac{\partial \lambda_{ITRS}}{\partial z_{ITRS}} \end{pmatrix}$$

$$= C_{ITRS}^{ICRS} \begin{pmatrix} \dfrac{\partial r_{ITRS}}{\partial x_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial x_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial x_{ITRS}} \\[2mm] \dfrac{\partial r_{ITRS}}{\partial y_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial y_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial y_{ITRS}} \\[2mm] \dfrac{\partial r_{ITRS}}{\partial z_{ITRS}} & \dfrac{\partial \varphi_{ITRS}}{\partial z_{ITRS}} & \dfrac{\partial \lambda_{ITRS}}{\partial z_{ITRS}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial V_{ITRS}}{\partial r_{ITRS}} \\[2mm] \dfrac{\partial V_{ITRS}}{\partial \varphi_{ITRS}} \\[2mm] \dfrac{\partial V_{ITRS}}{\partial \lambda_{ITRS}} \end{pmatrix}$$

（2-2）

$$= C_{ITRS}^{ICRS} \left(\frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \right)^T \frac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}}$$

ICRS 直角坐标系下的地球引力对 ICRS 直角坐标系下的卫星位置的偏导数：

$$\frac{\partial \vec{f}_{ICRS}}{\partial \vec{r}_{ICRS}} = \frac{\partial \left[C_{ITRS}^{ICRS} \left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \right)^T \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}} \right]}{\partial \vec{r}_{ICRS}}$$

$$= C_{ITRS}^{ICRS} \frac{\partial \left[\left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \right)^T \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}} \right]}{\partial \vec{r}_{ITRS}} \frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}}$$

（2-3）

$$= C_{ITRS}^{ICRS} \frac{\partial \left[\left(\dfrac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \right)^T \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial(r,\varphi,\lambda)_{ITRS}} \right]}{\partial(r,\varphi,\lambda)_{ITRS}} \frac{\partial(r,\varphi,\lambda)_{ITRS}}{\partial(x,y,z)_{ITRS}} \frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}}$$

其中，

$$\frac{\partial\left[\left(\dfrac{\partial\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(x,y,z\right)_{ITRS}}\right)^{T}\dfrac{\partial V\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}\right]}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}=\frac{\partial\left(\begin{array}{c}\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial x}\\[2mm]\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial y}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial y}\\[2mm]\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial z}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial z}\end{array}\right)_{ITRS}}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial x}\right)_{ITRS}}{\partial r_{ITRS}}=\frac{\partial^{2}V}{\partial r^{2}}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial x}\right)}{\partial r}$$

$$+\frac{\partial^{2}V}{\partial\varphi\partial r}\frac{\partial\varphi}{\partial x}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial x}\right)}{\partial r}$$

$$+\frac{\partial^{2}V}{\partial\lambda\partial r}\frac{\partial\lambda}{\partial x}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial x}\right)}{\partial r}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial x}\right)_{ITRS}}{\partial\varphi_{ITRS}}=\frac{\partial^{2}V}{\partial r\partial\varphi}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial x}\right)}{\partial\varphi}$$

$$+\frac{\partial^{2}V}{\partial\varphi^{2}}\frac{\partial\varphi}{\partial x}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial x}\right)}{\partial\varphi}$$

$$+\frac{\partial^{2}V}{\partial\lambda\partial\varphi}\frac{\partial\lambda}{\partial x}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial x}\right)}{\partial\varphi}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial x}\right)_{ITRS}}{\partial\lambda_{ITRS}}=\frac{\partial^{2}V}{\partial r\partial\lambda}\frac{\partial r}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial x}\right)}{\partial\lambda}$$

$$+\frac{\partial^{2}V}{\partial\varphi\partial\lambda}\frac{\partial\varphi}{\partial x}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial x}\right)}{\partial\lambda}$$

$$+\frac{\partial^{2}V}{\partial\lambda^{2}}\frac{\partial\lambda}{\partial x}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial x}\right)}{\partial\lambda}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial y}+\dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial r_{ITRS}}=\frac{\partial^2 V}{\partial r^2}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial y}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial \varphi \partial r}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\dfrac{\partial \varphi}{\partial y}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial \lambda \partial r}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\dfrac{\partial \lambda}{\partial y}\right)}{\partial r}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial y}+\dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial \varphi_{ITRS}}=\frac{\partial^2 V}{\partial r \partial \varphi}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial y}\right)}{\partial \varphi}$$

$$+\frac{\partial^2 V}{\partial \varphi^2}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\dfrac{\partial \varphi}{\partial y}\right)}{\partial \varphi}$$

$$+\frac{\partial^2 V}{\partial \lambda \partial \varphi}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\dfrac{\partial \lambda}{\partial y}\right)}{\partial \varphi}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial y}+\dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial y}\right)_{ITRS}}{\partial \lambda_{ITRS}}=\frac{\partial^2 V}{\partial r \partial \lambda}\frac{\partial r}{\partial y}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial y}\right)}{\partial \lambda}$$

$$+\frac{\partial^2 V}{\partial \varphi \partial \lambda}\frac{\partial \varphi}{\partial y}+\frac{\partial V}{\partial \varphi}\frac{\partial\left(\dfrac{\partial \varphi}{\partial y}\right)}{\partial \lambda}$$

$$+\frac{\partial^2 V}{\partial \lambda^2}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \lambda}\frac{\partial\left(\dfrac{\partial \lambda}{\partial y}\right)}{\partial \lambda}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial z}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial z}\right)_{ITRS}}{\partial r_{ITRS}}=\frac{\partial^2 V}{\partial r^2}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial z}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial\varphi\partial r}\frac{\partial\varphi}{\partial z}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial z}\right)}{\partial r}$$

$$+\frac{\partial^2 V}{\partial\lambda\partial r}\frac{\partial\lambda}{\partial z}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial z}\right)}{\partial r}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial z}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial z}\right)_{ITRS}}{\partial\varphi_{ITRS}}=\frac{\partial^2 V}{\partial r\partial\varphi}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial z}\right)}{\partial\varphi}$$

$$+\frac{\partial^2 V}{\partial\varphi^2}\frac{\partial\varphi}{\partial z}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial z}\right)}{\partial\varphi}$$

$$+\frac{\partial^2 V}{\partial\lambda\partial\varphi}\frac{\partial\lambda}{\partial z}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial z}\right)}{\partial\varphi}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial V}{\partial\varphi}\dfrac{\partial\varphi}{\partial z}+\dfrac{\partial V}{\partial\lambda}\dfrac{\partial\lambda}{\partial z}\right)_{ITRS}}{\partial\lambda_{ITRS}}=\frac{\partial^2 V}{\partial r\partial\lambda}\frac{\partial r}{\partial z}+\frac{\partial V}{\partial r}\frac{\partial\left(\dfrac{\partial r}{\partial z}\right)}{\partial z}$$

$$+\frac{\partial^2 V}{\partial\varphi\partial\lambda}\frac{\partial\varphi}{\partial z}+\frac{\partial V}{\partial\varphi}\frac{\partial\left(\dfrac{\partial\varphi}{\partial z}\right)}{\partial\lambda}$$

$$+\frac{\partial^2 V}{\partial\lambda^2}\frac{\partial\lambda}{\partial z}+\frac{\partial V}{\partial\lambda}\frac{\partial\left(\dfrac{\partial\lambda}{\partial z}\right)}{\partial\lambda}$$

由 $\vec{r}_{ICRS}=C_{ITRS}^{ICRS}\vec{r}_{ITRS}$ ，得到：

$$\frac{d\vec{r}_{ICRS}}{dt}=\frac{d\left(C_{ITRS}^{ICRS}\vec{r}_{ITRS}\right)}{dt}=\frac{dC_{ITRS}^{ICRS}}{dt}\vec{r}_{ITRS}+C_{ITRS}^{ICRS}\frac{d\vec{r}_{ITRS}}{dt}\quad（2\text{-}4）$$

由 $\dfrac{dC_{ITRS}^{ICRS}}{dt}\cong 0$ ，得到：

$$\frac{d\vec{r}_{ICRS}}{dt} \cong C_{ITRS}^{ICRS} \frac{d\vec{r}_{ITRS}}{dt} \Rightarrow \frac{\partial \vec{r}_{ITRS}}{\partial \vec{r}_{ICRS}} \cong \left(C_{ITRS}^{ICRS}\right)^{-1} \quad（2\text{-}5）$$

将（2-5）式代入（2-3）式，得到：

$$\frac{\partial \vec{f}_{ICRS}}{\partial \vec{r}_{ICRS}} = C_{ITRS}^{ICRS} \frac{\partial \left[\left(\dfrac{\partial (r,\varphi,\lambda)_{ITRS}}{\partial (x,y,z)_{ITRS}} \right)^{T} \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial (r,\varphi,\lambda)_{ITRS}} \right]}{\partial (r,\varphi,\lambda)_{ITRS}} \frac{\partial (r,\varphi,\lambda)_{ITRS}}{\partial (x,y,z)_{ITRS}} \left(C_{ITRS}^{ICRS}\right)^{-1} \quad（2\text{-}6）$$

ICRS 直角坐标系下的地球引力对球谐系数的偏导数：

$$\begin{aligned}
\frac{\partial \vec{f}_{ICRS}}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)} &= \frac{\partial \left[C_{ITRS}^{ICRS} \left(\dfrac{\partial (r,\varphi,\lambda)_{ITRS}}{\partial (x,y,z)_{ITRS}} \right)^{T} \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial (r,\varphi,\lambda)_{ITRS}} \right]}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)} \\
&= C_{ITRS}^{ICRS} \frac{\partial \left[\left(\dfrac{\partial (r,\varphi,\lambda)_{ITRS}}{\partial (x,y,z)_{ITRS}} \right)^{T} \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial (r,\varphi,\lambda)_{ITRS}} \right]}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}
\end{aligned} \quad（2\text{-}7）$$

其中，

$$\frac{\partial \left[\left(\dfrac{\partial (r,\varphi,\lambda)_{ITRS}}{\partial (x,y,z)_{ITRS}} \right)^{T} \dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial (r,\varphi,\lambda)_{ITRS}} \right]}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = \frac{\partial \begin{pmatrix} \dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial x} \\[2mm] \dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} + \dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial y} \\[2mm] \dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z} + \dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial z} + \dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial z} \end{pmatrix}_{ITRS}}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}$$

$$\frac{\partial \left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial x} \right)_{ITRS}}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = \frac{\partial \left(\dfrac{\partial V}{\partial r} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial r}{\partial x} + \frac{\partial \left(\dfrac{\partial V}{\partial \varphi} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial x} + \frac{\partial \left(\dfrac{\partial V}{\partial \lambda} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial x}$$

$$\frac{\partial \left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} + \dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial y} \right)_{ITRS}}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = \frac{\partial \left(\dfrac{\partial V}{\partial r} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial r}{\partial y} + \frac{\partial \left(\dfrac{\partial V}{\partial \varphi} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial y} + \frac{\partial \left(\dfrac{\partial V}{\partial \lambda} \right)}{\partial \left(\bar{C}_{nm}, \bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial y}$$

$$\frac{\partial\left(\dfrac{\partial V}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial V}{\partial \varphi}\dfrac{\partial \varphi}{\partial z}+\dfrac{\partial V}{\partial \lambda}\dfrac{\partial \lambda}{\partial z}\right)_{ITRS}}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}=\frac{\partial\left(\dfrac{\partial V}{\partial r}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial r}{\partial z}+\frac{\partial\left(\dfrac{\partial V}{\partial \varphi}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \varphi}{\partial z}+\frac{\partial\left(\dfrac{\partial V}{\partial \lambda}\right)}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}\frac{\partial \lambda}{\partial z}$$

由（2-2）、（2-6）和（2-7）式可知，为确定地球引力及其相关偏导数，需要确定

$$C_{ITRS}^{ICRS}、\left(C_{ITRS}^{ICRS}\right)^{-1}、\frac{\partial\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(x,y,z\right)_{ITRS}}、\frac{\partial\left[\dfrac{\partial\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(x,y,z\right)_{ITRS}}\right]}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}、\frac{\partial V\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}、\frac{\partial^2 V\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(r,\varphi,\lambda\right)_{ITRS}^{2}}$$

以及 $\dfrac{\partial\left[\dfrac{\partial V\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(r,\varphi,\lambda\right)_{ITRS}}\right]}{\partial\left(\bar{C}_{nm},\bar{S}_{nm}\right)}$ 。

ITRS 球坐标与 ITRS 直角坐标的相互转换：

$$\begin{pmatrix}x\\y\\z\end{pmatrix}_{ITRS}=\begin{pmatrix}r\cos\varphi\cos\lambda\\r\cos\varphi\sin\lambda\\r\sin\varphi\end{pmatrix},\quad\begin{pmatrix}r\\\varphi\\\lambda\end{pmatrix}_{ITRS}=\begin{pmatrix}\sqrt{x^2+y^2+z^2}\\\arctan\dfrac{z}{\sqrt{x^2+y^2}}\\\arctan\dfrac{y}{x}\end{pmatrix}\quad(2\text{-}8)$$

ITRS 球坐标对 ITRS 直角坐标的偏导数：

$$\frac{\partial\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(x,y,z\right)_{ITRS}}=\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}$$

$$=\begin{pmatrix}\cos\varphi\cos\lambda&\cos\varphi\sin\lambda&\sin\varphi\\-\dfrac{1}{r}\sin\varphi\cos\lambda&-\dfrac{1}{r}\sin\varphi\sin\lambda&\dfrac{1}{r}\cos\varphi\\-\dfrac{1}{r\cos\varphi}\sin\lambda&\dfrac{1}{r\cos\varphi}\cos\lambda&0\end{pmatrix}\quad(2\text{-}9)$$

$\dfrac{\partial\left(r,\varphi,\lambda\right)_{ITRS}}{\partial\left(x,y,z\right)_{ITRS}}$ 对 ITRS 球坐标的偏导数：

$$\frac{\partial}{\partial r}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{1}{r^2}\sin\varphi\cos\lambda & \dfrac{1}{r^2}\sin\varphi\sin\lambda & -\dfrac{1}{r^2}\cos\varphi \\ \dfrac{1}{r^2\cos\varphi}\sin\lambda & -\dfrac{1}{r^2\cos\varphi}\cos\lambda & 0 \end{pmatrix} \quad （2\text{-}10）$$

$$\frac{\partial}{\partial \varphi}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\dfrac{1}{r}\cos\varphi\cos\lambda & -\dfrac{1}{r}\cos\varphi\sin\lambda & -\dfrac{1}{r}\sin\varphi \\ -\dfrac{\sin\varphi}{r\cos^2\varphi}\sin\lambda & \dfrac{\sin\varphi}{r\cos^2\varphi}\cos\lambda & 0 \end{pmatrix} \quad （2\text{-}11）$$

$$\frac{\partial}{\partial \lambda}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\cos\varphi\sin\lambda & \cos\varphi\cos\lambda & 0 \\ \dfrac{1}{r}\sin\varphi\sin\lambda & -\dfrac{1}{r}\sin\varphi\cos\lambda & 0 \\ -\dfrac{1}{r\cos\varphi}\cos\lambda & -\dfrac{1}{r\cos\varphi}\sin\lambda & 0 \end{pmatrix} \quad （2\text{-}12）$$

引力位对 ITRS 球坐标的一阶偏导数：

$$\begin{cases} \dfrac{\partial V}{\partial r} = -\dfrac{GM}{r^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\left(\dfrac{a}{r}\right)^{n}\bar{P}_{nm}(\sin\varphi)\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right) \\[3mm] \dfrac{\partial V}{\partial \varphi} = \dfrac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\dfrac{a}{r}\right)^{n}\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}\cdot\left(\bar{C}_{nm}\cos(m\lambda)+\bar{S}_{nm}\sin(m\lambda)\right) \\[3mm] \dfrac{\partial V}{\partial \lambda} = \dfrac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}m\left(\dfrac{a}{r}\right)^{n}\bar{P}_{nm}(\sin\varphi)\cdot\left(-\bar{C}_{nm}\sin(m\lambda)+\bar{S}_{nm}\cos(m\lambda)\right) \end{cases} \quad （2\text{-}13）$$

引力位对 ITRS 球坐标的二阶偏导数：

$$\begin{cases} \dfrac{\partial^2 V}{\partial r^2} = \dfrac{GM}{r^3} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n+1)(n+2)\left(\dfrac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(\bar{C}_{nm}\cos(m\lambda) + \bar{S}_{nm}\sin(m\lambda)\right) \\[3mm] \dfrac{\partial^2 V}{\partial r \partial \varphi} = -\dfrac{GM}{r^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n+1)\left(\dfrac{a}{r}\right)^n \dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi} \cdot \left(\bar{C}_{nm}\cos(m\lambda) + \bar{S}_{nm}\sin(m\lambda)\right) \\[3mm] \dfrac{\partial^2 V}{\partial r \partial \lambda} = -\dfrac{GM}{r^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n+1)m\left(\dfrac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(-\bar{C}_{nm}\sin(m\lambda) + \bar{S}_{nm}\cos(m\lambda)\right) \\[3mm] \dfrac{\partial^2 V}{\partial \varphi^2} = \dfrac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\dfrac{a}{r}\right)^n \dfrac{d^2\bar{P}_{nm}(\sin\varphi)}{d\varphi^2} \cdot \left(\bar{C}_{nm}\cos(m\lambda) + \bar{S}_{nm}\sin(m\lambda)\right) \\[3mm] \dfrac{\partial^2 V}{\partial \varphi \partial \lambda} = \dfrac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} m\left(\dfrac{a}{r}\right)^n \dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi} \cdot \left(-\bar{C}_{nm}\sin(m\lambda) + \bar{S}_{nm}\cos(m\lambda)\right) \\[3mm] \dfrac{\partial^2 V}{\partial \lambda^2} = \dfrac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} m^2\left(\dfrac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(-\bar{C}_{nm}\cos(m\lambda) - \bar{S}_{nm}\sin(m\lambda)\right) \end{cases}$$

（2-14）

$\dfrac{\partial V(r,\varphi,\lambda)_{ITRS}}{\partial (r,\varphi,\lambda)_{ITRS}}$ 对球谐系数的偏导数：

$$\begin{cases} \dfrac{\partial\left[\dfrac{\partial V}{\partial r}\right]}{\partial\left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = -\dfrac{GM}{r^2}(n+1)\left(\dfrac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(\cos(m\lambda), \sin(m\lambda)\right) \\[4mm] \dfrac{\partial\left[\dfrac{\partial V}{\partial \varphi}\right]}{\partial\left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = \dfrac{GM}{r}\left(\dfrac{a}{r}\right)^n \dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi} \cdot \left(\cos(m\lambda), \sin(m\lambda)\right) \\[4mm] \dfrac{\partial\left[\dfrac{\partial V}{\partial \lambda}\right]}{\partial\left(\bar{C}_{nm}, \bar{S}_{nm}\right)} = \dfrac{GM}{r}m\left(\dfrac{a}{r}\right)^n \bar{P}_{nm}(\sin\varphi) \cdot \left(-\sin(m\lambda), \cos(m\lambda)\right) \end{cases}$$

（2-15）

确定以上偏导数的关键在于确定完全正规化的缔合勒让德多项式 $\bar{P}_{nm}(\sin\varphi)$ 及其对

地心纬度的一阶导数 $\dfrac{d\bar{P}_{nm}(\sin\varphi)}{d\varphi}$ 和二阶导数 $\dfrac{d^2\bar{P}_{nm}(\sin\varphi)}{d\varphi^2}$ 。

$\bar{P}_{nm}(\sin\varphi)$ 的递推公式：

$$\begin{cases} \overline{P}_{0,0}\left(\sin\varphi\right)=1.0 \\ \overline{P}_{n,n}\left(\sin\varphi\right)=\sqrt{\dfrac{\left(1+\delta_{1n}\right)\left(2n+1\right)}{2n}}\cos\varphi\,\overline{P}_{n-1,n-1}\left(\sin\varphi\right),n\geq1 \quad (2\text{-}16) \\ \overline{P}_{n,m}\left(\sin\varphi\right)=g_{n,m}\overline{P}_{n-1,m}\left(\sin\varphi\right)-h_{n,m}\overline{P}_{n-2,m}\left(\sin\varphi\right),n\geq m+1 \end{cases}$$

其中，$g_{n,m}=\sqrt{\dfrac{\left(2n+1\right)\left(2n-1\right)}{\left(n+m\right)\left(n-m\right)}}$，$h_{n,m}=\sqrt{\dfrac{\left(2n+1\right)\left(n-m-1\right)\left(n+m-1\right)}{\left(2n-3\right)\left(n+m\right)\left(n-m\right)}}$。

$\dfrac{d\overline{P}_{nm}\left(\sin\varphi\right)}{d\varphi}$ 的递推公式：

$$\begin{cases} \dfrac{d\overline{P}_{n,m}\left(\sin\varphi\right)}{d\varphi}=\sqrt{\dfrac{\left(2-\delta_{0m}\right)\left(n-m\right)\left(n+m+1\right)}{2}}\overline{P}_{n,m+1}\left(\sin\varphi\right)-m\tan\varphi\,\overline{P}_{n,m}\left(\sin\varphi\right) \\ \dfrac{d\overline{P}_{0,0}\left(\sin\varphi\right)}{d\varphi}=-m\tan\varphi\,\overline{P}_{0,0}\left(\sin\varphi\right) \end{cases}\quad(2\text{-}17)$$

$\dfrac{d^{2}\overline{P}_{nm}\left(\sin\varphi\right)}{d\varphi^{2}}$ 的递推公式：

$$\begin{cases} \dfrac{d^{2}\overline{P}_{n,m}\left(\sin\varphi\right)}{d\varphi^{2}}=\left[m^{2}\sec^{2}\varphi-n\left(n+1\right)\right]\overline{P}_{n,m}\left(\sin\varphi\right)+\tan\varphi\dfrac{d\overline{P}_{n,m}\left(\sin\varphi\right)}{d\varphi} \\ \dfrac{d^{2}\overline{P}_{0,0}\left(\sin\varphi\right)}{d\varphi^{2}}=\tan\varphi\dfrac{d\overline{P}_{n,m}\left(\sin\varphi\right)}{d\varphi} \end{cases}\quad(2\text{-}18)$$

IERS Coventions 2010 建议采用 EGM2008 模型。EGM2008 模型系数对应的 *GM* 和 a_e 分别取为 398600.4415km³/s² 和 6378136.3m（TT 时），模型阶次截断至 12 阶 12 次。

低阶系数的时变改正：

$$\overline{C}_{n0}\left(t\right)=\overline{C}_{n0}\left(t_{0}\right)+d\overline{C}_{n0}\big/dt\times\left(t-t_{0}\right)\quad(2\text{-}19)$$

其中，t_0 为 J2000.0，$\overline{C}_{n0}\left(t_0\right)$ 和 $d\overline{C}_{n0}\big/dt$ 的值由表 6.2 查取。

$$\begin{cases} \overline{C}_{21}\left(t\right)=\quad\sqrt{3}\,\overline{x}_{p}\left(t\right)\overline{C}_{20}-\overline{x}_{p}\left(t\right)\overline{C}_{22}+\overline{y}_{p}\left(t\right)\overline{S}_{22} \\ \overline{S}_{21}\left(t\right)=-\sqrt{3}\,\overline{y}_{p}\left(t\right)\overline{C}_{20}-\overline{y}_{p}\left(t\right)\overline{C}_{22}-\overline{x}_{p}\left(t\right)\overline{S}_{22} \end{cases}\quad(2\text{-}20)$$

其中， $\overline{x}_p(t)$ 和 $\overline{y}_p(t)$ 表示 IERS 协议平均地球极，单位为弧度，计算公式如下：

$$
\begin{cases}
\overline{x}_p(t) = \sum_{i=0}^{3} (t-t_0)^i \times \overline{x}_p^i \\
\overline{y}_p(t) = \sum_{i=0}^{3} (t-t_0)^i \times \overline{y}_p^i
\end{cases}
\quad （2\text{-}21）
$$

其中， t_0 为 J2000.0， $\overline{x}_p^i$ 和 $\overline{y}_p^i$ 的值由表 7.7 查取。

2）固体潮

采用两步法，将 Love 数分为两个部分：与频率无关的部分以及与频率有关的部分。

1）考虑与频率无关的部分

$$
\Delta\overline{C}_{nm} - i\Delta\overline{S}_{nm} = \frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j) e^{-im\lambda_j} \quad （2\text{-}22）
$$

其中， k_{nm} 为 Love 数， R_e 为地球赤道半径， $GM_\oplus$ 为地球引力参数， Φ_j 为太阳或月亮的地心纬度， λ_j 为太阳或月亮的地心经度（从格林尼治起算）。

由欧拉公式可知， $e^{-im\lambda_j} = \cos(-m\lambda_j) + i\sin(-m\lambda_j)$

$$
\begin{aligned}
\Delta\overline{C}_{nm} - i\Delta\overline{S}_{nm} &= \frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j)\left(\cos(-m\lambda_j) + i\sin(-m\lambda_j)\right) \\
&= \frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j)\cos(-m\lambda_j) \\
&\quad + i\frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j)\sin(-m\lambda_j) \quad （2\text{-}23） \\
&= \frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j)\cos(m\lambda_j) \\
&\quad - i\frac{k_{nm}}{2n+1} \sum_{j=2}^{3} \frac{GM_j}{GM_\oplus} \left(\frac{R_e}{r_j}\right)^{n+1} \overline{P}_{nm}(\sin\Phi_j)\sin(m\lambda_j)
\end{aligned}
$$

$$\begin{cases} \Delta\bar{C}_{nm} = \dfrac{k_{nm}}{2n+1} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{n+1} \bar{P}_{nm}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) \\ \Delta\bar{S}_{nm} = \dfrac{k_{nm}}{2n+1} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{n+1} \bar{P}_{nm}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{cases} \tag{2-24}$$

考虑由 2 阶和 3 阶潮汐引起的 2 阶和 3 阶球谐系数变化：

$$\begin{cases} \Delta\bar{C}_{2m} = \dfrac{k_{2m}}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) \\ \Delta\bar{S}_{2m} = \dfrac{k_{2m}}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{cases} \tag{2-25}$$

$$\begin{cases} \Delta\bar{C}_{3m} = \dfrac{k_{3m}}{7} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{4} \bar{P}_{3m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) \\ \Delta\bar{S}_{3m} = \dfrac{k_{3m}}{7} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{4} \bar{P}_{3m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{cases} \tag{2-26}$$

另外，考虑由 2 阶潮汐引起的 4 阶球谐系数变化：

$$\begin{aligned} \Delta\bar{C}_{4m} - i\Delta\bar{S}_{4m} &= \dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right) e^{-im\lambda_j}, \left(m=0,1,2\right) \\ &= \dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\left(\cos\left(-m\lambda_j\right)+i\sin\left(-m\lambda_j\right)\right) \\ &= \dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(-m\lambda_j\right) + \\ &\quad\; i\dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(-m\lambda_j\right) \\ &= \dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\cos\left(m\lambda_j\right) - \\ &\quad\; i\dfrac{k_{2m}^+}{5} \sum_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^{3} \bar{P}_{2m}\left(\sin\Phi_j\right)\sin\left(m\lambda_j\right) \end{aligned} \tag{2-27}$$

$$\begin{cases} \Delta \bar{C}_{4m} = \dfrac{k_{2m}^+}{5} \sum\limits_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^3 \bar{P}_{2m} \left(\sin \Phi_j \right) \cos \left(m\lambda_j \right) \\ \Delta \bar{S}_{4m} = \dfrac{k_{2m}^+}{5} \sum\limits_{j=2}^{3} \dfrac{GM_j}{GM_\oplus} \left(\dfrac{R_e}{r_j} \right)^3 \bar{P}_{2m} \left(\sin \Phi_j \right) \sin \left(m\lambda_j \right) \end{cases} \quad （2\text{-}28）$$

其中，k_{2m}、k_{3m}、k_{2m}^+ 的值由表 6.3 查取。

2）考虑与频率有关的部分

考虑不同频率的长周期潮汐成分对 $\Delta \bar{C}_{20}$ 的影响：

$$\mathrm{Re} \sum\nolimits_{f(2,0)} \left(A_0 \delta k_f H_f \right) e^{i\theta_f} = \sum\nolimits_{f(2,0)} \left[\left(A_0 H_f \delta k_f^R \right) \cos \theta_f - \left(A_0 H_f \delta k_f^I \right) \sin \theta_f \right] \quad （2\text{-}29）$$

$$\Delta \bar{C}_{20} = \sum\nolimits_{f(2,0)} \left[\left(A_0 H_f \delta k_f^R \right) \cos \theta_f - \left(A_0 H_f \delta k_f^I \right) \sin \theta_f \right] \quad （2\text{-}30）$$

其中，$A_0 H_f \delta k_f^R$、$A_0 H_f \delta k_f^I$ 的值由表 6.5b 查取。

考虑周日潮汐成分对 $\left(\Delta \bar{C}_{21} - i\Delta \bar{S}_{21} \right)$ 以及半日潮汐成分对 $\left(\Delta \bar{C}_{22} - i\Delta \bar{S}_{22} \right)$ 的影响：

$$\begin{aligned} \Delta \bar{C}_{2m} - i\Delta \bar{S}_{2m} &= \eta_m \sum_{f(2,m)} \left(A_m \delta k_f H_f \right) e^{i\theta_f}, \left(m = 1, 2 \right) \\ &= \eta_m \sum_{f(2,m)} \left(A_m \delta k_f^R H_f + i A_m \delta k_f^I H_f \right) \left(\cos \theta_f + i \sin \theta_f \right) \\ &= \eta_m \sum_{f(2,m)} \left(A_m \delta k_f^R H_f \right) \cos \theta_f + \\ &\quad i\eta_m \sum_{f(2,m)} \left(A_m \delta k_f^R H_f \right) \sin \theta_f + \left(A_m \delta k_f^I H_f \right) \cos \theta_f + \quad （2\text{-}31） \\ &\quad i^2 \eta_m \sum_{f(2,m)} \left(A_m \delta k_f^I H_f \right) \sin \theta_f \\ &= \eta_m \sum_{f(2,m)} \left(A_m \delta k_f^R H_f \right) \cos \theta_f + \left(A_m \delta k_f^I H_f \right) \sin \theta_f + \\ &\quad i\eta_m \sum_{f(2,m)} \left(A_m \delta k_f^R H_f \right) \sin \theta_f + \left(A_m \delta k_f^I H_f \right) \cos \theta_f \end{aligned}$$

其中，

$$A_0 = \frac{1}{R_e \sqrt{4\pi}} = 4.4288 \times 10^{-8} m^{-1}, \quad A_m = \frac{(-1)^m}{R_e \sqrt{8\pi}} = (-1)^m \left(3.1274 \times 10^{-8} \right) m^{-1}, \left(m \neq 0 \right);$$

$\eta_1 = -i$ ， $\eta_2 = 1$ ；

δk_f 为 $k_f - k_{2m}$ ，加上海潮负荷的影响；

δk_f^R 为 δk_f 的实部， δk_f^I 为 δk_f 的虚部；

H_f 为振幅，

$$\theta_f = \bar{n} \cdot \bar{\beta} = \sum_{i=1}^{6} n_i \beta_i \text{ 或者 } \theta_f = m\left(\theta_g + \pi\right) - \bar{N} \cdot \bar{F} = m\left(\theta_g + \pi\right) - \sum_{j=1}^{5} N_j F_j ，$$

其中， $\bar{\beta}$ 为 6 个 Doodson 基本参数 $\beta_i, \left(\tau, s, h, p, N', p_s\right)$ ， $\bar{n}$ 为 6 个 Doodson 基本参数的乘数 n_i ， $\bar{F}$ 为 5 个章动理论基本参数 F_j （Delaunay 变量， l, l', F, D, Ω ）， $\bar{N}$ 为 5 个章动理论基本参数的乘数 N_j ， θ_g 为 GMST（格林尼治平恒星时）的角度表示。

当 m=1 时， $\eta_1 = -i$ ，从而有：

$$\begin{aligned}
\Delta \bar{C}_{21} - i\Delta \bar{S}_{21} &= \eta_1 \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\cos\theta_f + \left(A_1 \delta k_f^I H_f\right)\sin\theta_f \\
&\quad + i\eta_1 \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\sin\theta_f + \left(A_1 \delta k_f^I H_f\right)\cos\theta_f \\
&= \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\sin\theta_f + \left(A_1 \delta k_f^I H_f\right)\cos\theta_f \\
&\quad - i\sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\cos\theta_f + \left(A_1 \delta k_f^I H_f\right)\sin\theta_f
\end{aligned} \quad （2\text{-}32）$$

$$\begin{cases}
\Delta \bar{C}_{21} = \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\sin\theta_f + \left(A_1 \delta k_f^I H_f\right)\cos\theta_f \\
\Delta \bar{S}_{21} = \sum_{f(2,1)} \left(A_1 \delta k_f^R H_f\right)\cos\theta_f + \left(A_1 \delta k_f^I H_f\right)\sin\theta_f
\end{cases} \quad （2\text{-}33）$$

其中， $A_1 \delta k_f^R H_f$ 、 $A_1 \delta k_f^I H_f$ 的值由表 6.5a 查取。

当 m=2 时， $\eta_2 = 1$ ，从而有：

$$\Delta \bar{C}_{22} - i\Delta \bar{S}_{22} = \eta_2 \sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \cos \theta_f + \left(A_2 \delta k_f^I H_f \right) \sin \theta_f$$

$$+ i\eta_2 \sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \sin \theta_f + \left(A_2 \delta k_f^I H_f \right) \cos \theta_f$$

$$= \sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \cos \theta_f + \left(A_2 \delta k_f^I H_f \right) \sin \theta_f \qquad （2\text{-}34）$$

$$+ i \sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \sin \theta_f + \left(A_2 \delta k_f^I H_f \right) \cos \theta_f$$

$$\begin{cases} \Delta \bar{C}_{22} = \sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \cos \theta_f + \left(A_2 \delta k_f^I H_f \right) \sin \theta_f \\ \Delta \bar{S}_{22} = -\sum_{f(2,2)} \left(A_2 \delta k_f^R H_f \right) \sin \theta_f + \left(A_2 \delta k_f^I H_f \right) \cos \theta_f \end{cases} \qquad （2\text{-}35）$$

其中，$A_2 \delta k_f^R H_f$ 的值由表 6.5c 查取。表 6.5c 中只给出了实部部分的改正，因此，$A_2 \delta k_f^I H_f$ 为 0。

3）考虑永久性潮汐部分

由 2 阶带谐项 C_{20} 潮汐引起的引力位，对其取时间平均，结果并不为 0。在构建引力位模型时，如果 C_{20} 包含了潮汐中与时间无关的部分，则称其为"zero tide"模型，如果不包含，则称其为"conventional tide free"模型。对于"zero tide"模型，由于其 C_{20} 已经包含了潮汐中与时间无关的部分，因此，在进行潮汐改正时不应重复包含潮汐中与时间无关的部分，从步骤（1）中移去该部分影响的改正公式如下：

$$\Delta \bar{C}_{20}^{zt} = \Delta \bar{C}_{20} - \Delta \bar{C}_{20}^{perm} \qquad （2\text{-}36）$$

其中，$\Delta \bar{C}_{20}$ 为步骤（1）中的潮汐改正，$\Delta \bar{C}_{20}^{perm} = A_0 H_0 k_{20} = \left(4.4228 \times 10^{-8} \right) \left(-0.31460 \right) k_{20}$ 为与时间无关的部分。

在 EGM2008 模型中，"zero-tide"和"tide-free"在 C_{20} 上的差异为 -4.1736×10^{-9}。

3）海潮

海潮的动态效应会使正规化的 Stokes 系数产生周期性变化：

$$\left[\Delta\bar{C}_{nm}-i\Delta\bar{S}_{nm}\right](t)=\sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm}\mp i\mathbb{S}_{f,nm}^{\pm}\right)e^{\pm i\theta_{f}(t)}$$

$$=\sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm}\mp i\mathbb{S}_{f,nm}^{\pm}\right)\left(\cos\left(\pm\theta_{f}\right)+i\sin\left(\pm\theta_{f}\right)\right)$$

$$=\sum_{f}\frac{\left(\mathbb{C}_{f,nm}^{+}-i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\left(+\theta_{f}\right)+i\sin\left(+\theta_{f}\right)\right)+}{\left(\mathbb{C}_{f,nm}^{-}+i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\left(-\theta_{f}\right)+i\sin\left(-\theta_{f}\right)\right)}$$

$$=\sum_{f}\frac{\left(\mathbb{C}_{f,nm}^{+}-i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\theta_{f}+i\sin\theta_{f}\right)+}{\left(\mathbb{C}_{f,nm}^{-}+i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\theta_{f}-i\sin\theta_{f}\right)}$$

$$=\sum_{f}\frac{\mathbb{C}_{f,nm}^{+}\cos\theta_{f}+\mathbb{S}_{f,nm}^{+}\sin\theta_{f}+i\mathbb{C}_{f,nm}^{+}\sin\theta_{f}-i\mathbb{S}_{f,nm}^{+}\cos\theta_{f}+}{\mathbb{C}_{f,nm}^{-}\cos\theta_{f}+\mathbb{S}_{f,nm}^{-}\sin\theta_{f}-i\mathbb{C}_{f,nm}^{-}\sin\theta_{f}+i\mathbb{S}_{f,nm}^{-}\cos\theta_{f}}$$

$$=\sum_{f}\frac{\mathbb{C}_{f,nm}^{+}\cos\theta_{f}+\mathbb{S}_{f,nm}^{+}\sin\theta_{f}+i\mathbb{C}_{f,nm}^{+}\sin\theta_{f}-i\mathbb{S}_{f,nm}^{+}\cos\theta_{f}+}{\mathbb{C}_{f,nm}^{-}\cos\theta_{f}+\mathbb{S}_{f,nm}^{-}\sin\theta_{f}-i\mathbb{C}_{f,nm}^{-}\sin\theta_{f}+i\mathbb{S}_{f,nm}^{-}\cos\theta_{f}}$$

$$=\sum_{f}\frac{\left(\left(\mathbb{C}_{f,nm}^{+}+\mathbb{C}_{f,nm}^{-}\right)\cos\theta_{f}+\left(\mathbb{S}_{f,nm}^{+}+\mathbb{S}_{f,nm}^{-}\right)\sin\theta_{f}\right)-}{i\left(\left(\mathbb{S}_{f,nm}^{+}-\mathbb{S}_{f,nm}^{-}\right)\cos\theta_{f}-\left(\mathbb{C}_{f,nm}^{+}-\mathbb{C}_{f,nm}^{-}\right)\sin\theta_{f}\right)} \quad （2\text{-}37）$$

$$\begin{cases}\Delta\bar{C}_{nm}=\left(\mathbb{C}_{f,nm}^{+}+\mathbb{C}_{f,nm}^{-}\right)\cos\theta_{f}+\left(\mathbb{S}_{f,nm}^{+}+\mathbb{S}_{f,nm}^{-}\right)\sin\theta_{f}\\ \Delta\bar{S}_{nm}=\left(\mathbb{S}_{f,nm}^{+}-\mathbb{S}_{f,nm}^{-}\right)\cos\theta_{f}-\left(\mathbb{C}_{f,nm}^{+}-\mathbb{C}_{f,nm}^{-}\right)\sin\theta_{f}\end{cases} \quad （2\text{-}38）$$

其中，$\mathbb{C}_{f,nm}^{\pm}$ 和 $\mathbb{S}_{f,nm}^{\pm}$ 为潮汐频率 f 上的引力位球谐振幅，IERS 2010 推荐采用 FES 2004 海潮模型，具体数值参见 fes2004_Cnm-Snm.dat 文件，$\theta_{f}(t)$ 与固体潮中的计算公式相同。

Doodson 数 A 与 Doodson 乘数 $\bar{n}$ 的关系如下：

$$A=n_{1}\left(n_{2}+5\right)\left(n_{3}+5\right)\cdot\left(n_{4}+5\right)\left(n_{5}+5\right)\left(n_{6}+5\right) \quad （2\text{-}39）$$

IERS Conventions 1996：

$$\Delta\bar{C}_{nm}-i\Delta\bar{S}_{nm}=F_{nm}\sum_{s(n,m)}\sum_{+}^{-}\left(C_{s,nm}^{\pm}\mp iS_{s,nm}^{\pm}\right)e^{\pm i\theta_{f}}$$

$$\Delta \bar{C}_{nm} - i\Delta \bar{S}_{nm} = F_{nm} \sum_{s(n,m)} \sum_{+}^{-} \left(C_{s,nm}^{\pm} \mp i S_{s,nm}^{\pm} \right) e^{\pm i\theta_f}$$

$$= F_{nm} \sum_{s(n,m)} \sum_{+}^{-} \left(C_{s,nm}^{\pm} \mp i S_{s,nm}^{\pm} \right) \left(\cos\left(\pm\theta_f\right) + i\sin\left(\pm\theta_f\right) \right)$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C_{s,nm}^{+} - i S_{s,nm}^{+} \right) \left(\cos\left(+\theta_f\right) + i\sin\left(+\theta_f\right) \right) + \\ \left(C_{s,nm}^{-} + i S_{s,nm}^{-} \right) \left(\cos\left(-\theta_f\right) + i\sin\left(-\theta_f\right) \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C_{s,nm}^{+} - i S_{s,nm}^{+} \right) \left(\cos\theta_f + i\sin\theta_f \right) + \\ \left(C_{s,nm}^{-} + i S_{s,nm}^{-} \right) \left(\cos\theta_f - i\sin\theta_f \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C_{s,nm}^{+} \cos\theta_f - i S_{s,nm}^{+} \cos\theta_f + i C_{s,nm}^{+} \sin\theta_f + S_{s,nm}^{+} \sin\theta_f \right) + \\ \left(C_{s,nm}^{-} \cos\theta_f + i S_{s,nm}^{-} \cos\theta_f - i C_{s,nm}^{-} \sin\theta_f + S_{s,nm}^{-} \sin\theta_f \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C_{s,nm}^{+} + C_{s,nm}^{-} \right) \cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-} \right) \sin\theta_f - \\ i \left(\left(S_{s,nm}^{+} - S_{s,nm}^{-} \right) \cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-} \right) \sin\theta_f \right) \end{array}$$

$$\begin{cases} \Delta \bar{C}_{nm} = F_{nm} \sum_{s(n,m)} \left(C_{s,nm}^{+} + C_{s,nm}^{-} \right) \cos\theta_f + \left(S_{s,nm}^{+} + S_{s,nm}^{-} \right) \sin\theta_f \\ \Delta \bar{S}_{nm} = F_{nm} \sum_{s(n,m)} \left(S_{s,nm}^{+} - S_{s,nm}^{-} \right) \cos\theta_f - \left(C_{s,nm}^{+} - C_{s,nm}^{-} \right) \sin\theta_f \end{cases}$$

$$F_{nm} = \frac{4\pi G \rho_w}{g} \sqrt{\frac{(n+m)!}{(n-m)!(2n+1)(2-\delta_{0m})}} \left(\frac{1+k_n'}{2n+1} \right)$$

IERS Conventions 2003:

$$\Delta \bar{C}_{nm} - i\Delta \bar{S}_{nm} = F_{nm} \sum_{s(n,m)} \sum_{+}^{-} \left(C_{s,nm}^{\pm} \mp i S_{s,nm}^{\pm} \right) e^{\pm i\theta_f}$$

$$\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} = F_{nm} \sum_{s(n,m)} \sum_{+}^{-} \left(C^{\pm}_{s,nm} \mp iS^{\pm}_{s,nm} \right) e^{\pm i\theta_f}$$

$$= F_{nm} \sum_{s(n,m)} \sum_{+}^{-} \left(C^{\pm}_{s,nm} \mp iS^{\pm}_{s,nm} \right) \left(\cos\left(\pm\theta_f\right) + i\sin\left(\pm\theta_f\right) \right)$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C^{+}_{s,nm} - iS^{+}_{s,nm} \right)\left(\cos\left(+\theta_f\right) + i\sin\left(+\theta_f\right) \right) + \\ \left(C^{-}_{s,nm} + iS^{-}_{s,nm} \right)\left(\cos\left(-\theta_f\right) + i\sin\left(-\theta_f\right) \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C^{+}_{s,nm} - iS^{+}_{s,nm} \right)\left(\cos\theta_f + i\sin\theta_f \right) + \\ \left(C^{-}_{s,nm} + iS^{-}_{s,nm} \right)\left(\cos\theta_f - i\sin\theta_f \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C^{+}_{s,nm}\cos\theta_f - iS^{+}_{s,nm}\cos\theta_f + iC^{+}_{s,nm}\sin\theta_f + S^{+}_{s,nm}\sin\theta_f \right) + \\ \left(C^{-}_{s,nm}\cos\theta_f + iS^{-}_{s,nm}\cos\theta_f - iC^{-}_{s,nm}\sin\theta_f + S^{-}_{s,nm}\sin\theta_f \right) \end{array}$$

$$= F_{nm} \sum_{s(n,m)} \begin{array}{l} \left(C^{+}_{s,nm} + C^{-}_{s,nm} \right)\cos\theta_f + \left(S^{+}_{s,nm} + S^{-}_{s,nm} \right)\sin\theta_f - \\ i\left(\left(S^{+}_{s,nm} - S^{-}_{s,nm} \right)\cos\theta_f - \left(C^{+}_{s,nm} - C^{-}_{s,nm} \right)\sin\theta_f \right) \end{array}$$

$$\begin{cases} \Delta\bar{C}_{nm} = F_{nm} \displaystyle\sum_{s(n,m)} \left(C^{+}_{s,nm} + C^{-}_{s,nm} \right)\cos\theta_f + \left(S^{+}_{s,nm} + S^{-}_{s,nm} \right)\sin\theta_f \\ \Delta\bar{S}_{nm} = F_{nm} \displaystyle\sum_{s(n,m)} \left(S^{+}_{s,nm} - S^{-}_{s,nm} \right)\cos\theta_f - \left(C^{+}_{s,nm} - C^{-}_{s,nm} \right)\sin\theta_f \end{cases}$$

$$F_{nm} = \frac{4\pi G\rho_w}{g} \sqrt{\frac{(n+m)!}{(n-m)!(2n+1)(2-\delta_{0m})}} \left(\frac{1+k'_n}{2n+1} \right)$$

IERS Conventions 2010:

$$\left[\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm} \right](t) = \sum_f \sum_{+}^{-} \left(\mathbb{C}^{\pm}_{f,nm} \mp i\mathbb{S}^{\pm}_{f,nm} \right) e^{\pm i\theta_f(t)}$$

$$\left[\Delta\bar{C}_{nm} - i\Delta\bar{S}_{nm}\right] = \sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm} \mp i\mathbb{S}_{f,nm}^{\pm}\right)e^{\pm i\theta_f}$$

$$= \sum_{f}\sum_{+}^{-}\left(\mathbb{C}_{f,nm}^{\pm} \mp i\mathbb{S}_{f,nm}^{\pm}\right)\left(\cos\left(\pm\theta_f\right) + i\sin\left(\pm\theta_f\right)\right)$$

$$= \sum_{f}\begin{array}{l}\left(\mathbb{C}_{f,nm}^{+} - i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\left(+\theta_f\right) + i\sin\left(+\theta_f\right)\right) + \\ \left(\mathbb{C}_{f,nm}^{-} + i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\left(-\theta_f\right) + i\sin\left(-\theta_f\right)\right)\end{array}$$

$$= \sum_{f}\begin{array}{l}\left(\mathbb{C}_{f,nm}^{+} - i\mathbb{S}_{f,nm}^{+}\right)\left(\cos\theta_f + i\sin\theta_f\right) + \\ \left(\mathbb{C}_{f,nm}^{-} + i\mathbb{S}_{f,nm}^{-}\right)\left(\cos\theta_f - i\sin\theta_f\right)\end{array}$$

$$= \sum_{f}\begin{array}{l}\mathbb{C}_{f,nm}^{+}\cos\theta_f + \mathbb{S}_{f,nm}^{+}\sin\theta_f + i\mathbb{C}_{f,nm}^{+}\sin\theta_f - i\mathbb{S}_{f,nm}^{+}\cos\theta_f + \\ \mathbb{C}_{f,nm}^{-}\cos\theta_f + \mathbb{S}_{f,nm}^{-}\sin\theta_f - i\mathbb{C}_{f,nm}^{-}\sin\theta_f + i\mathbb{S}_{f,nm}^{-}\cos\theta_f\end{array}$$

$$= \sum_{f}\begin{array}{l}\mathbb{C}_{f,nm}^{+}\cos\theta_f + \mathbb{S}_{f,nm}^{+}\sin\theta_f + i\mathbb{C}_{f,nm}^{+}\sin\theta_f - i\mathbb{S}_{f,nm}^{+}\cos\theta_f + \\ \mathbb{C}_{f,nm}^{-}\cos\theta_f + \mathbb{S}_{f,nm}^{-}\sin\theta_f - i\mathbb{C}_{f,nm}^{-}\sin\theta_f + i\mathbb{S}_{f,nm}^{-}\cos\theta_f\end{array}$$

$$= \sum_{f}\begin{array}{l}\left(\left(\mathbb{C}_{f,nm}^{+} + \mathbb{C}_{f,nm}^{-}\right)\cos\theta_f + \left(\mathbb{S}_{f,nm}^{+} + \mathbb{S}_{f,nm}^{-}\right)\sin\theta_f\right) - \\ i\left(\left(\mathbb{S}_{f,nm}^{+} - \mathbb{S}_{f,nm}^{-}\right)\cos\theta_f - \left(\mathbb{C}_{f,nm}^{+} - \mathbb{C}_{f,nm}^{-}\right)\sin\theta_f\right)\end{array}$$

$$\begin{cases}\Delta\bar{C}_{nm} = \left(\mathbb{C}_{f,nm}^{+} + \mathbb{C}_{f,nm}^{-}\right)\cos\theta_f + \left(\mathbb{S}_{f,nm}^{+} + \mathbb{S}_{f,nm}^{-}\right)\sin\theta_f \\ \Delta\bar{S}_{nm} = \left(\mathbb{S}_{f,nm}^{+} - \mathbb{S}_{f,nm}^{-}\right)\cos\theta_f - \left(\mathbb{C}_{f,nm}^{+} - \mathbb{C}_{f,nm}^{-}\right)\sin\theta_f\end{cases}$$

$$\begin{cases}\mathbb{C}_{f,nm}^{\pm} = \dfrac{4\pi G\rho_w}{g}\left(\dfrac{1+k_n'}{2n+1}\right)\hat{\bar{C}}_{f,nm}^{\pm}\sin\left(\varepsilon_{f,nm}^{\pm} + \chi_f\right) \\ \mathbb{S}_{f,nm}^{\pm} = \dfrac{4\pi G\rho_w}{g}\left(\dfrac{1+k_n'}{2n+1}\right)\hat{\bar{C}}_{f,nm}^{\pm}\cos\left(\varepsilon_{f,nm}^{\pm} + \chi_f\right)\end{cases}$$

其中，χ_f 的值由表 6.6 确定。$\hat{\bar{C}}_{f,nm}^{\pm}$ 和 $\varepsilon_{f,nm}^{\pm}$ 分别为频率 f 上 n 阶 m 次的振幅和相位。

4）极潮

极潮包括两部分：固体地球极潮以及海洋极潮。

固体地球极潮对 C_{21} 和 S_{21} 的影响如下：

$$\begin{cases}\Delta\bar{C}_{21} = -1.333\times10^{-9}\left(m_1 + 0.0115m_2\right) \\ \Delta\bar{S}_{21} = -1.333\times10^{-9}\left(m_2 - 0.0115m_1\right)\end{cases} \qquad （2\text{-}40）$$

其中，$m_1 = x_p - \bar{x}_p$，$m_2 = -\left(y_p - \bar{y}_p\right)$，$\left(x_p, y_p\right)$ 为极移参数，$\left(\bar{x}_p, \bar{y}_p\right)$ 为 IERS（2010）平均地球极，与地球引力中的计算公式相同。

海洋极潮对正规化的地球引力位系数的影响如下：

$$\begin{bmatrix} \Delta \bar{C}_{nm} \\ \Delta \bar{S}_{nm} \end{bmatrix} = R_n \left\{ \begin{bmatrix} \bar{A}_{nm}^R \\ \bar{B}_{nm}^R \end{bmatrix} \left(m_1 \gamma_2^R + m_2 \gamma_2^I\right) + \begin{bmatrix} \bar{A}_{nm}^I \\ \bar{B}_{nm}^I \end{bmatrix} \left(m_2 \gamma_2^R - m_1 \gamma_2^I\right) \right\} \quad （2\text{-}41）$$

式中，

$$R_n = \frac{\Omega^2 a_E^4}{GM} \frac{4\pi G \rho_w}{g_e} \left(\frac{1 + k_n'}{2n + 1}\right),$$

Ω, a_E, GM, g_e 和 G 的值由表 1.1 查取，

$\rho_w = 1025 \, kgm^{-3}$ 为海水密度，k_n' 为负荷形变系数（$k_2' = -0.3075$，$k_3' = -0.195$，$k_4' = -0.132$，$k_5' = -0.1032$，$k_6' = -0.0892$），

$\gamma = \gamma_2^R + i\gamma_2^I = \left(1 + k_2 - h_2\right) = 0.6870 + i0.0036$，

$\left(m_1, m_2\right)$ 与固体地球极潮中的计算公式相同，

系数 $\bar{A}_{nm} = \bar{A}_{nm}^R + i\bar{A}_{nm}^I$ 和 $\bar{B}_{nm} = \bar{B}_{nm}^R + i\bar{B}_{nm}^I$ 参见 desaiscopolecoef.txt 文件。

$\left(n, m\right) = \left(2, 1\right)$ 的系数为海洋极潮的主项，由上述计算公式可以得到其表达式如下：

$$\begin{cases} \Delta \bar{C}_{21} = -2.1778 \times 10^{-10} \left(m_1 - 0.01724 m_2\right) \\ \Delta \bar{S}_{21} = -1.7232 \times 10^{-10} \left(m_2 - 0.03365 m_1\right) \end{cases} \quad （2\text{-}42）$$

IERS Conventions 1996：

$$\Delta V = -\left(\Omega^2 R_e^2 / 2\right) \sin 2\theta \left(x_p \cos \lambda - y_p \sin \lambda\right)$$

$k_2 = 0.2977$

$$\Delta \bar{C}_{21} = -1.290 \times 10^{-9} \left(x_p \right)$$

$$\Delta \bar{S}_{21} = 1.290 \times 10^{-9} \left(y_p \right)$$

$$k_2 = 0.3111 - 0.0035i$$

$$\Delta \bar{C}_{21} = -1.348 \times 10^{-9} \left(x_p + 0.0112 y_p \right)$$

$$\Delta \bar{S}_{21} = 1.348 \times 10^{-9} \left(y_p - 0.0112 x_p \right)$$

IERS Conventions 2003：

$$\Delta V \left(r, \theta, \lambda \right) = -\frac{\Omega^2 r^2}{2} \sin 2\theta \left(m_1 \cos \lambda + m_2 \sin \lambda \right)$$

$$= -\frac{\Omega^2 r^2}{2} \sin 2\theta \, \mathbf{Re} \left[\left(m_1 - im_2 \right) e^{i\lambda} \right]$$

$$-\frac{\Omega^2 r^2}{2} \sin 2\theta \, \mathbf{Re} \left[k_2 \left(m_1 - im_2 \right) e^{i\lambda} \right]$$

$$k_2 = 0.3077 + 0.0036i$$

$$\Delta \bar{C}_{21} = -1.333 \times 10^{-9} \left(m_1 - 0.0115 m_2 \right)$$

$$\Delta \bar{S}_{21} = -1.333 \times 10^{-9} \left(m_2 + 0.0115 m_1 \right)$$ （符号错误！）

IERS Conventions 2010：

$$\Delta V \left(r, \theta, \lambda \right) = -\frac{\Omega^2 r^2}{2} \sin 2\theta \left(m_1 \cos \lambda + m_2 \sin \lambda \right)$$

$$= -\frac{\Omega^2 r^2}{2} \sin 2\theta \, \mathbf{Re} \left[\left(m_1 - im_2 \right) e^{i\lambda} \right]$$

$$-\frac{\Omega^2 r^2}{2} \sin 2\theta \, \mathbf{Re} \left[k_2 \left(m_1 - im_2 \right) e^{i\lambda} \right]$$

$$V \left(r, \varphi, \lambda \right) = \frac{GM_E}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r} \right)^n \bar{P}_{nm} \left(\sin \varphi \right) \cdot \left(\bar{C}_{nm} \cos \left(m\lambda \right) + \bar{S}_{nm} \sin \left(m\lambda \right) \right)$$

$$\Delta V_{(2,1)} = \frac{GM_E}{r} \left(\frac{a_e}{r} \right)^2 \bar{P}_{21} \left(\cos \theta \right) \cdot \left(\Delta \bar{C}_{21} \cos \lambda + \Delta \bar{S}_{21} \sin \lambda \right)$$

$$\frac{GM_E}{r}\left(\frac{a_e}{r}\right)^2 \bar{P}_{21}(\cos\theta) \cdot \left(\Delta\bar{C}_{21}\cos\lambda + \Delta\bar{S}_{21}\sin\lambda\right) = -\frac{\Omega^2 r^2}{2}\sin 2\theta \, \mathbf{Re}\left[k_2(m_1 - im_2)e^{i\lambda}\right]$$

$$\left(\Delta\bar{C}_{21}\cos\lambda + \Delta\bar{S}_{21}\sin\lambda\right) = \frac{-\dfrac{\Omega^2 r^2}{2}\sin 2\theta}{\dfrac{GM_E}{r}\left(\dfrac{a_e}{r}\right)^2 \bar{P}_{21}(\cos\theta)} \mathbf{Re}\left[k_2(m_1 - im_2)e^{i\lambda}\right]$$

$$= \frac{-\Omega^2 r^5 \sin\theta\cos\theta}{GM_E a_e{}^2 \bar{P}_{21}(\cos\theta)} \mathbf{Re}\left[\left(R_{k_2} + iI_{k_2}\right)(m_1 - im_2)(\cos\lambda + i\sin\lambda)\right]$$

$$= \frac{-\Omega^2 r^5 \sin\theta\cos\theta}{GM_E a_e{}^2 \bar{P}_{21}(\cos\theta)}\left[R_{k_2}\left(m_1\cos\lambda + m_2\sin\lambda\right) - I_{k_2}\left(m_1\sin\lambda - m_2\cos\lambda\right)\right]$$

$$= \frac{-\Omega^2 r^5 \sin\theta\cos\theta}{GM_E a_e{}^2 \bar{P}_{21}(\cos\theta)}\left[\left(R_{k_2}m_1 + I_{k_2}m_2\right)\cos\lambda + \left(R_{k_2}m_2 - I_{k_2}m_1\right)\sin\lambda\right]$$

$$\Delta\bar{C}_{21} = \frac{-\Omega^2 r^5 \sin\theta\cos\theta}{GM_E a_e{}^2 \bar{P}_{21}(\cos\theta)}\left(R_{k_2}m_1 + I_{k_2}m_2\right) = \frac{-\Omega^2 r^3}{GM_E\sqrt{15}}\left(R_{k_2}m_1 + I_{k_2}m_2\right)$$

$$\Delta\bar{S}_{21} = \frac{-\Omega^2 r^5 \sin\theta\cos\theta}{GM_E a_e{}^2 \bar{P}_{21}(\cos\theta)}\left(R_{k_2}m_2 - I_{k_2}m_1\right) = \frac{-\Omega^2 r^3}{GM_E\sqrt{15}}\left(R_{k_2}m_2 - I_{k_2}m_1\right)$$

$$k_2 = 0.3077 + 0.0036i$$

$$\Delta\bar{C}_{21} = -1.333\times10^{-9}\left(m_1 + 0.0115m_2\right)$$
$$\Delta\bar{S}_{21} = -1.333\times10^{-9}\left(m_2 - 0.0115m_1\right)$$

$$m_1 = x_p - \bar{x}_p,\ m_2 = -\left(y_p - \bar{y}_p\right)$$

5）三体引力

两个质点 M 和 m 的相互作用的运动方程为：

$$\begin{cases} M\vec{a}_M = GMm\dfrac{\vec{r}_{Mm}}{r_{Mm}^3} \\[3mm] m\vec{a}_m = GMm\dfrac{\vec{r}_{mM}}{r_{mM}^3} \end{cases} \quad（2\text{-}43）$$

其中，r 为向量 $\vec{r}$ 的长度；下标 Mm 表示向量从点质量 M 指向 m；单独的下标 M 或 m 表示向量指向质点 M 或 m。

引入附加的质点 m_j, $j=1,2,\ldots$，类似地，可以得到 m_j 作用在 M 和 m 上的引力方程，求和得到总引力为：

$$\begin{cases} M\vec{a}_M = GMm\dfrac{\vec{r}_{Mm}}{r_{Mm}^3} + \sum_j GMm_j\dfrac{\vec{r}_{Mm_j}}{r_{Mm_j}^3} \\ m\vec{a}_m = GMm\dfrac{\vec{r}_{mM}}{r_{mM}^3} + \sum_j Gmm_j\dfrac{\vec{r}_{mm_j}}{r_{mm_j}^3} \end{cases}$$ （2-44）

以上两式分别除以 $-M$ 和 m 后相加，得到：

$$\vec{a}_m - \vec{a}_M = -G(M+m)\dfrac{\vec{r}_{Mm}}{r_{mM}^3} + \sum_j Gm_j \cdot \left[\dfrac{\vec{r}_{mm_j}}{r_{mm_j}^3} - \dfrac{\vec{r}_{Mm_j}}{r_{Mm_j}^3} \right]$$ （2-45）

令 $\vec{r} = \vec{r}_m - \vec{r}_M$，即令质点 M 为原点，从而 $\vec{r}_{mm_j} = -\left(\vec{r}_m - \vec{r}_{m_j} \right)$，忽略质量 m，得到：

$$\vec{a} = -G(M)\dfrac{\vec{r}}{r^3} - \sum_j Gm_j \cdot \left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^3} + \dfrac{\vec{r}_{m_j}}{r_{m_j}^3} \right]$$ （2-46）

式中，右边第一项为地球中心引力，第二项为三体中心引力，即：

$$\vec{f}_m = -\sum_j Gm_j \cdot \left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^3} + \dfrac{\vec{r}_{m_j}}{\vec{r}_{m_j}^3} \right]$$ （2-47）

其中，Gm_j 为三体的引力常数。

三体中心引力的计算依赖于三体位置的计算，三体位置的计算主要有两种方法：1. 解析公式，2. 行星星历。

三体质点引力对卫星位置的偏导数：

$$\frac{\partial \vec{a}_{m,j}}{\partial \vec{r}_m} = -Gm_j \cdot \frac{\partial \left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^3} + \dfrac{\vec{r}_{m_j}}{r_{m_j}^3} \right]}{\partial \vec{r}_m}$$

$$= -Gm_j \cdot \frac{\partial \left[\dfrac{\vec{r}_m - \vec{r}_{m_j}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^3} \right]}{\partial \vec{r}_m} \qquad (2\text{-}48)$$

$$= -Gm_j \cdot \left[\frac{I_{3\times3}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^3} - 3 \frac{\vec{r}_m - \vec{r}_{m_j}}{\left| \vec{r}_m - \vec{r}_{m_j} \right|^5} I_{3\times3} \right]$$

对月球而言，由于月地距离很近，需要另外考虑其对地球的非球形引力部分的作用。表达式如下：

$$\triangle \vec{a} = \frac{3}{2} C_{20} \frac{GM_l}{r_l^2} \left(\frac{R}{r_l} \right)^2 \begin{bmatrix} \left(5\sin^2\phi_l - 1\right)\cos\phi_l \cos\lambda_l \\ \left(5\sin^2\phi_l - 1\right)\cos\phi_l \sin\lambda_l \\ \left(5\sin^2\phi_l - 3\right)\sin\phi_l \end{bmatrix} \qquad (2\text{-}49)$$

其中，$\left(\phi_l, \lambda_l\right)$ 分别为月球在 ICRS 坐标系下的纬度和经度。
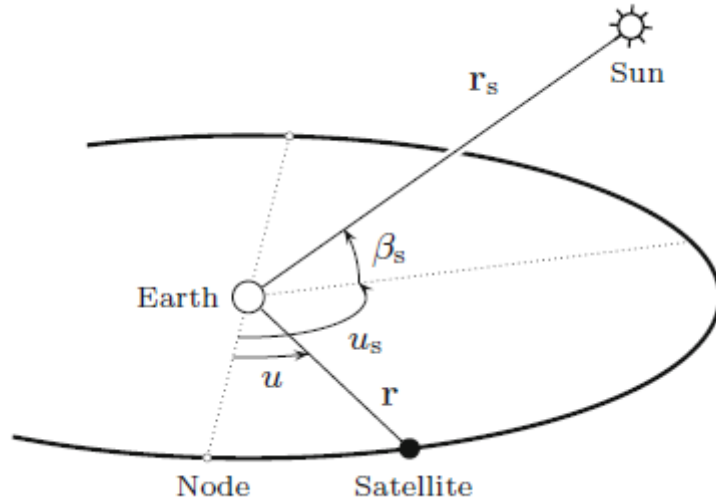
6）太阳辐射压

图 2-1 卫星-地心-太阳几何关系

ECOM 模型：

$$\vec{a} = D(u)\vec{e}_D + Y(u)\vec{e}_Y + B(u)\vec{e}_B \quad （2\text{-}50）$$

其中，

u 为卫星的升交点角距，

$$\vec{e}_D \doteq \frac{\vec{r}_s - \vec{r}}{|\vec{r}_s - \vec{r}|}, \quad \vec{e}_Y \doteq -\frac{\vec{e}_r \times \vec{e}_D}{|\vec{e}_r \times \vec{e}_D|}, \quad \vec{e}_B \doteq \vec{e}_D \times \vec{e}_Y,$$

$\vec{r}_s$ 和 $\vec{r}$ 分别为太阳和卫星在 ICRS 坐标系下的位置矢量，$\vec{v}$ 为卫星在 ICRS 坐标系下的速度矢量。

（1）当采用 9 参数模型时

$$\begin{cases} D(u) = D_0 + D_c \cos u + D_s \sin u \\ Y(u) = Y_0 + Y_c \cos u + Y_s \sin u \\ B(u) = B_0 + B_c \cos u + B_s \sin u \end{cases} \quad （2\text{-}51）$$

（2）当采用 5 参数模型时

$$\begin{cases} D(u) = D_0 \\ Y(u) = Y_0 \\ B(u) = B_0 + B_c \cos u + B_s \sin u \end{cases} \quad （2\text{-}52）$$

太阳辐射压对卫星位置的偏导数：

$$\begin{aligned} \frac{\partial \vec{a}}{\partial \vec{r}} &= \frac{\partial \left(D(u)\vec{e}_D + Y(u)\vec{e}_Y + B(u)\vec{e}_B \right)}{\partial \vec{r}} \quad （2\text{-}53） \\ &= D(u)\frac{\partial \vec{e}_D}{\partial \vec{r}} + Y(u)\frac{\partial \vec{e}_Y}{\partial \vec{r}} + B(u)\frac{\partial \vec{e}_B}{\partial \vec{r}} \end{aligned}$$

其中，

$$\begin{aligned} \frac{\partial \vec{e}_D}{\partial \vec{r}} &= \frac{\partial \left(\dfrac{\vec{r}_s - \vec{r}}{|\vec{r}_s - \vec{r}|} \right)}{\partial \vec{r}} \\ &= \frac{1}{|\vec{r}_s - \vec{r}|} \frac{\partial (\vec{r}_s - \vec{r})}{\partial \vec{r}} + (\vec{r}_s - \vec{r})\frac{\partial \left(|\vec{r}_s - \vec{r}|^{-1} \right)}{\partial \vec{r}}, \\ &= \frac{1}{|\vec{r}_s - \vec{r}|} \cdot (-I_{3\times3}) + (\vec{r}_s - \vec{r})\frac{-(\vec{r}_s - \vec{r})^T}{|\vec{r}_s - \vec{r}|^2} \end{aligned}$$

$$\frac{\partial \vec{e}_Y}{\partial \vec{r}} = -\frac{\partial \dfrac{\vec{e}_r \times \vec{e}_D}{|\vec{e}_r \times \vec{e}_D|}}{\partial \vec{r}} = 0_{3 \times 3} \, ,$$

$$
\begin{aligned}
\frac{\partial \vec{e}_B}{\partial \vec{r}} &= \frac{\partial \left(\vec{e}_D \times \vec{e}_Y \right)}{\partial \vec{r}} \\
&= -\frac{\partial \left(\vec{e}_Y \times \vec{e}_D \right)}{\partial \vec{r}} \\
&= -\left(\frac{\partial \left(\vec{e}_Y \times \right)}{\partial \vec{r}} \vec{e}_D + \left(\vec{e}_Y \times \right) \frac{\partial \vec{e}_D}{\partial \vec{r}} \right)^{\circ} \\
&= -\left(\vec{e}_Y \times \right) \frac{\partial \vec{e}_D}{\partial \vec{r}}
\end{aligned}
$$

太阳辐射压对力模型参数的偏导数：

（1）当采用 9 参数时

$$
\begin{aligned}
\frac{\partial \vec{a}}{\partial \left(D_0, D_c, D_s \right)} &= \frac{\partial \left(D(u) \vec{e}_D + Y(u) \vec{e}_Y + B(u) \vec{e}_B \right)}{\partial \left(D_0, D_c, D_s \right)} \\
&= \frac{\partial D(u)}{\partial \left(D_0, D_c, D_s \right)} \vec{e}_D \\
\frac{\partial \vec{a}}{\partial \left(Y_0, Y_c, Y_s \right)} &= \frac{\partial \left(D(u) \vec{e}_D + Y(u) \vec{e}_Y + B(u) \vec{e}_B \right)}{\partial \left(Y_0, Y_c, Y_s \right)} \\
&= \frac{\partial Y(u)}{\partial \left(Y_0, Y_c, Y_s \right)} \vec{e}_Y \\
\frac{\partial \vec{a}}{\partial \left(B_0, B_c, B_s \right)} &= \frac{\partial \left(D(u) \vec{e}_D + Y(u) \vec{e}_Y + B(u) \vec{e}_B \right)}{\partial \left(B_0, B_c, B_s \right)} \\
&= \frac{\partial B(u)}{\partial \left(B_0, B_c, B_s \right)} \vec{e}_B
\end{aligned}
\qquad (2\text{-}54)
$$

（2）当采用 5 参数时

$$\frac{\partial \vec{a}}{\partial \left(D_0\right)} = \frac{\partial \left(D\left(u\right)\vec{e}_D + Y\left(u\right)\vec{e}_Y + B\left(u\right)\vec{e}_B\right)}{\partial \left(D_0\right)}$$

$$= \frac{\partial D\left(u\right)}{\partial \left(D_0\right)} \vec{e}_D$$

$$\frac{\partial \vec{a}}{\partial \left(Y_0\right)} = \frac{\partial \left(D\left(u\right)\vec{e}_D + Y\left(u\right)\vec{e}_Y + B\left(u\right)\vec{e}_B\right)}{\partial \left(Y_0\right)} \qquad （2\text{-}55）$$

$$= \frac{\partial Y\left(u\right)}{\partial \left(Y_0\right)} \vec{e}_Y$$

$$\frac{\partial \vec{a}}{\partial \left(B_0, B_c, B_s\right)} = \frac{\partial \left(D\left(u\right)\vec{e}_D + Y\left(u\right)\vec{e}_Y + B\left(u\right)\vec{e}_B\right)}{\partial \left(B_0, B_c, B_s\right)}$$

$$= \frac{\partial B\left(u\right)}{\partial \left(B_0, B_c, B_s\right)} \vec{e}_B$$

7）地球辐射压

暂不考虑。

8）相对论效应

在 GCRS 坐标系下，人造地球卫星运动加速度的相对论改正：

$$\Delta \ddot{\vec{r}} = \frac{GM_E}{c^2 r^3} \left\{ \left[2\left(\beta + \gamma\right)\frac{GM_E}{r} - \gamma \dot{\vec{r}} \cdot \dot{\vec{r}} \right]\vec{r} + 2\left(1 + \gamma\right)\left(\vec{r} \cdot \dot{\vec{r}}\right)\dot{\vec{r}} \right\} +$$

$$\left(1 + \gamma\right)\frac{GM_E}{c^2 r^3} \left[\frac{3}{r^2}\left(\vec{r} \times \dot{\vec{r}}\right)\left(\vec{r} \cdot \vec{J}\right) + \left(\dot{\vec{r}} \times \vec{J}\right) \right] + \qquad （2\text{-}56）$$

$$\left\{ \left(1 + 2\gamma\right)\left[\dot{\vec{R}} \times \left(\frac{-GM_S \vec{R}}{c^2 R^3} \right) \right] \times \dot{\vec{r}} \right\}$$

其中，c 为光速，β 和 γ 为 PPN（parameterized post-Newtonian）参数，在广义相对论中

等于 1，$\vec{r}$ 为卫星相对于地球的位置，$\vec{R}$ 为地球相对于太阳的位置，$\vec{J}$ 为单位质量的地）

球角动量（$\left|\vec{J}\right| \cong 9.8 \times 10^8 \, m^2/s$），$GM_E$ 和 GM_S 分别为地球和太阳的引力系数。

忽略（2-56）式中的第 2 和第 3 项，并代入 β 和 γ，得到：

$$\Delta \vec{a} = \frac{GM_E}{c^2 r^3} \left[\left(4\frac{GM_E}{r} - \vec{v} \cdot \vec{r} \right)\vec{r} + 4\left(\vec{r} \cdot \vec{v}\right)\vec{v} \right] \quad （2\text{-}57）$$

相对论改正对卫星位置的偏导数：

$$\frac{\partial \Delta \vec{a}}{\partial \vec{r}} = \frac{\partial \dfrac{GM_E}{c^2 r^3} \left[\left(4\dfrac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} + 4(\vec{r} \cdot \vec{v})\vec{v} \right]}{\partial \vec{r}}$$

$$= \frac{\partial \dfrac{GM_E}{c^2 r^3}}{\partial \vec{r}} \left[\left(4\frac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} + 4(\vec{r} \cdot \vec{v})\vec{v} \right] +$$

$$\frac{GM_E}{c^2 r^3} \frac{\partial \left[\left(4\dfrac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} + 4(\vec{r} \cdot \vec{v})\vec{v} \right]}{\partial \vec{r}} \qquad （2\text{-}58）$$

$$= -3\frac{GM_E}{c^2 r^5} I_{3\times3} \left[\left(4\frac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} + 4(\vec{r} \cdot \vec{v})\vec{v} \right] +$$

$$\frac{GM_E}{c^2 r^3} \left[\begin{array}{l} \left(-4\dfrac{GM_E}{r} I_{3\times3} - \vec{v} \cdot I_{3\times3} \right) \vec{r} + \\ \left(4\dfrac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) + 4(I_{3\times3} \cdot \vec{v})\vec{v} \end{array} \right]$$

相对论改正对卫星速度的偏导数：

$$\frac{\partial \Delta \vec{a}}{\partial \vec{v}} = \frac{\partial \dfrac{GM_E}{c^2 r^3} \left[\left(4\dfrac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} + 4(\vec{r} \cdot \vec{v})\vec{v} \right]}{\partial \vec{v}}$$

$$= \frac{GM_E}{c^2 r^3} \left(\frac{\partial \left[\left(4\dfrac{GM_E}{r} - \vec{v} \cdot \vec{r} \right) \vec{r} \right]}{\partial \vec{v}} + \frac{\partial \left[4(\vec{r} \cdot \vec{v})\vec{v} \right]}{\partial \vec{v}} \right) \qquad （2\text{-}59）$$

$$= \frac{GM_E}{c^2 r^3} \left(-I_{3\times3} \cdot r^2 + 8\vec{r} \cdot I_{3\times3} \right)$$

9）经验力

暂不考虑。

3 积分器

卫星轨道积分问题如下：

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\Phi}(t, t_0) \end{pmatrix} = \begin{pmatrix} F(t, x) \\ A(t)\Phi(t, t_0) \end{pmatrix}, x(t_0), \Phi(t_0, t_0) \qquad （3\text{-}1）$$

已知 t_0 时刻的卫星位置 $r(t_0)$、速度 $v(t_0)$ 以及力模型参数 $p(t_0)$。对 GNSS 卫星轨道积分而言，在卫星加速度的计算中不包含与卫星速度有关的力，力模型参数主要为太阳光压参数，采用 CODE 9 参数模型，则初始状态包括 3 个位置参数+3 个速度参数+9 个太阳光压参数。由于积分方程右边的卫星加速度计算比较复杂，因此，通常采用多步积分法。

参考 ESA 分析中心所采用的方案，以 8 阶 Embedded-Runge-Kutta 变步长（初始步长 75s）单步积分法作为起步，以 8 阶 Adams-Bashforth-Moulton 定步长（步长 300s）多步积分法作为后续。

步骤如下：

1）以 t_0 时刻为起始时刻，采用 8 阶 Embedded-Runge-Kutta 变步长单步积分器向后积分 8*300/75=32 个历元，得到相应历元的卫星位置、速度、力模型参数以及转移矩阵 $x_i, \dfrac{\partial x_i}{\partial x_0}, i = 1, 32$；

积分公式：

$$\begin{cases} \hat{y}_{n+1} = y_n + h\sum_{i=0}^{s=10} \hat{c}_i \cdot k_i \\ y_{n+1} = y_n + h\sum_{i=0}^{s=12} c_i \cdot k_i \end{cases} \quad （3-2）$$

其中，

$$\begin{cases} k_0 = h_n f(t_n, y_n) \\ k_i = h_n f\left(t_n + a_i h_n, \hat{y}_n + \sum_{j=1}^{i-1} b_{ij} K_j\right), (i = 1, 2, \ldots, 12) \end{cases}$$

误差估计：

$$TE = \left| \hat{y}_{n+1} - y_{n+1} \right| = \frac{41}{840} \cdot h \cdot \left| k_0 + k_{10} - k_{11} - k_{12} \right| \quad （3-3）$$

收敛条件：

$$A = \sum_{components} \left(\frac{TE_j}{tol_j} \right)^2 < 1 \quad （3-4）$$

步长调整：

$$h_{new} = h \cdot \left(\frac{0.0025}{A} \right)^{\frac{1}{16}} \quad （3\text{-}5）$$

系数参考 Fehlberg（1968）第 65 页。

2）根据 Embedded-Runge-Kutta 变步长单步积分器积分得到的前 8 个历元的值，启动 8 阶 Adams-Bashforth-Moulton 定步长多步积分器，积分得到后续各个历元的卫星位置、速度、力模型参数以及转移矩阵 $x_i, \frac{\partial x_i}{\partial x_0}, i = 1, \ldots$ 。

Adams-Bashforth 预测公式：

$$y_{n+1}^p = y_n + \frac{h}{3628800} \sum_{j=0}^{k=8} \beta_j^p \cdot f_{n-j} \quad （3\text{-}6）$$

其中，$\beta_j^p = \sum_{m=j}^{k=8} (-1)^j C_m^j \gamma_m^p$ ， $\gamma_m^p = \int_0^1 C_{s+m-1}^m ds$ ， $\begin{cases} \gamma_0^p = 1 \\ \gamma_m^p = 1 - \sum_{j=1}^m \frac{1}{j+1} \gamma_{m-j}^p, & m \geq 1 \end{cases}$ 。

Adams-Moulton 校正公式：

$$y_{n+1}^c = y_n + \frac{h}{3628800} \left(\beta_0^c \cdot f_{n+1}^p + \sum_{j=1}^{k=8} \beta_j^c \cdot f_{n+1-j} \right) \quad （3\text{-}7）$$

其中，$\beta_j^c = \sum_{m=j}^{k} (-1)^j C_m^j \gamma_m^c$ ， $\gamma_m^c = \int_{-1}^0 C_{s+m-1}^m ds$ ， $\begin{cases} \gamma_0^c = 1 \\ \gamma_m^c = -\sum_{j=1}^m \frac{1}{j+1} \gamma_{m-j}^c, & m \geq 1 \end{cases}$ 。

表 3-1 Adams-Bashforth/Adams-Moulton 系数

j	Adams-Bashforth 系数 β_j^p	Adams-Moulton 系数 β_j^c
0	14097247	1070017
1	-43125206	4467094
2	95476786	-4604594

3	-139855262	5595358
4	137968480	-5033120
5	-91172642	3146338
6	38833486	-1291214
7	-9664106	312874
8	1070017	-33953

4 轨道拟合

在 GNSS 卫星轨道积分中，初始位置和速度由精密星历或广播星历给出，初始力模型参数由经验值给出或全部置零。

在轨道拟合过程中，首先，对运动方程和变分方程积分，得到卫星当前位置和速度以及其与初始位置、速度和力模型参数的偏导数关系；然后，将卫星当前时刻的位置和速度与精密星历中的位置和速度作差，形成虚拟观测值；最后，建立虚拟观测方程，利用最小二乘求解得到初始位置、速度和力模型参数的改正值。

t 时刻的轨道拟合虚拟观测方程：

$$r_{oi}\left(t\right)-r_{sp3}\left(t\right)=\frac{\partial r}{\partial p_0}\left(t\right)\cdot dp_0 \quad（4-1）$$