Supplementary Proof

1 Proof of Theorem 1

Theorem 1. *Under Assumptions 1, 2, and given constant desired parametric differences* $\Delta_1^{[i,j]}$, $\Delta_2^{[i,j]}$ *for* $(i,j) \in \mathcal{E}$, *the DGVF* $\boldsymbol{\mathfrak{X}}^{[i]}(t, \boldsymbol{\xi}^{[i]})$ *designed by (13) and choosing*

$$\boldsymbol{v} = (-1)^{n+1}(0, \cdots, 0, \dot{w}_2^*, -\dot{w}_1^*)$$

in (9), solve Problem 1 globally.

Proof. We firstly give the definitions, i.e., $K^{[i]} := \mathrm{diag}\{k_1^{[i]}, \cdots, k_n^{[i]}\}$, $\boldsymbol{f}_{\cdot}^{[i]\prime} := (f_1^{[i]\prime}, \cdots, f_n^{[i]\prime})^\top$, $\boldsymbol{g}_{\cdot}^{[i]\prime} := (g_1^{[i]\prime}, \cdots, g_n^{[i]\prime})^\top$.

One has

$$\boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{j}}^{[i]\top}{}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]} = k_j^{[i]}\phi_j^{[i]} - f_j^{[i]\prime}\boldsymbol{f}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]} - g_j^{[i]\prime}\boldsymbol{g}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]}, \tag{1}$$

for $i \in \mathbb{Z}_1^N$. Since $j \in \mathbb{Z}_1^n$, we stack Eq. (1) as

$$\begin{bmatrix} \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]} \\ \vdots \\ \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]} \end{bmatrix} = -K^{[i]}\boldsymbol{\Phi}^{[i]} - \boldsymbol{f}_{\cdot}^{[i]\prime}\boldsymbol{f}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]} - \boldsymbol{g}_{\cdot}^{[i]\prime}\boldsymbol{g}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]} \tag{2}$$

Following the same way, we can obtain

$$\begin{bmatrix} \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]} \\ \vdots \\ \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]} \end{bmatrix} = -c_1^{[i]}(\boldsymbol{w})\begin{bmatrix} f_1^{[i]\prime} \\ \vdots \\ f_n^{[i]\prime} \end{bmatrix} = -c_1^{[i]}(\boldsymbol{w})\boldsymbol{f}_{\cdot}^{[i]\prime}(\boldsymbol{w}), \tag{3}$$

and

$$\begin{bmatrix} \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]} \\ \vdots \\ \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]} \end{bmatrix} = -c_2^{[i]}(\boldsymbol{w})\boldsymbol{g}_{\cdot}^{[i]\prime}(\boldsymbol{w}). \tag{4}$$

With Eqs. (1),(2),(3), the time derivative of $\boldsymbol{\Phi}^{[i]}$ is

$$\begin{aligned}\dot{\boldsymbol{\Phi}}^{[i]} &= \begin{bmatrix} \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]} + k_{c1}\boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]} + k_{c2}\boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{1}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]} \\ \vdots \\ \boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]} + k_{c1}\boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]} + k_{c2}\boldsymbol{\nabla}\boldsymbol{\phi}_{\boldsymbol{n}}^{[i]\top}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]} \end{bmatrix} \\ &= -K^{[i]}\boldsymbol{\Phi}^{[i]} - \boldsymbol{f}_{\cdot}^{[i]\prime}\boldsymbol{f}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]} - \boldsymbol{g}_{\cdot}^{[i]\prime}\boldsymbol{g}_{\cdot}^{[i]\prime\top}K^{[i]}\boldsymbol{\Phi}^{[i]} - k_{c1}c_1^{[i]}(\boldsymbol{w})\boldsymbol{f}_{\cdot}^{[i]\prime}(\boldsymbol{w}) - k_{c2}c_2^{[i]}(\boldsymbol{w})\boldsymbol{g}_{\cdot}^{[i]\prime}(\boldsymbol{w}).\end{aligned} \tag{5}$$

Define

$$\begin{aligned} \mathfrak{F}_1 &:= \mathrm{diag}\{\boldsymbol{f}_{\cdot}^{[\boldsymbol{1}]\prime}, \cdots, \boldsymbol{f}_{\cdot}^{[\boldsymbol{N}]\prime}\} \in \mathbb{R}^{nN\times N} \\ \mathfrak{F}_2 &:= \mathrm{diag}\{\boldsymbol{g}_{\cdot}^{[\boldsymbol{1}]\prime}, \cdots, \boldsymbol{g}_{\cdot}^{[\boldsymbol{N}]\prime}\} \in \mathbb{R}^{nN\times N} \\ K &:= \mathrm{diag}\{K^{[1]}, \cdots, K^{[N]}\} \in \mathbb{R}^{nN\times nN} \\ \boldsymbol{\Phi} &:= (\boldsymbol{\Phi}^{[1]}, \cdots, \boldsymbol{\Phi}^{[N]})^\top \in \mathbb{R}^{nN}. \end{aligned} \tag{6}$$

Thereby, we have $\dot{\boldsymbol{\Phi}}$. Now we define the composite error:

$$\dot{\boldsymbol{e}} = \begin{bmatrix} \dot{\boldsymbol{\Phi}} \\ D^\top \dot{\tilde{\boldsymbol{w}}}_{\boldsymbol{1}}^{[\cdot]} \\ D^\top \dot{\tilde{\boldsymbol{w}}}_{\boldsymbol{2}}^{[\cdot]} \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{\Phi} - \mathfrak{F}_1\mathfrak{F}_1^\top K\boldsymbol{\Phi} - \mathfrak{F}_2\mathfrak{F}_2^\top K\boldsymbol{\Phi} + k_{c1}\mathfrak{F}_1 L\tilde{\boldsymbol{w}}_{\boldsymbol{1}}^{[\cdot]} + k_{c2}\mathfrak{F}_2 L\tilde{\boldsymbol{w}}_{\boldsymbol{2}}^{[\cdot]} \\ D^\top \mathfrak{F}_1^\top K\boldsymbol{\Phi} - k_{c1}D^\top L\tilde{\boldsymbol{w}}_{\boldsymbol{1}}^{[\cdot]} \\ D^\top \mathfrak{F}_2^\top K\boldsymbol{\Phi} - k_{c2}D^\top L\tilde{\boldsymbol{w}}_{\boldsymbol{2}}^{[\cdot]} \end{bmatrix} \tag{7}$$

According to [1, Proof of Theorem 2], we conclude that $\|\boldsymbol{\Phi}^{[i]}(t)\| \to 0$ as $t \to \infty$ for $i \in \mathbb{Z}_1^N$. This further implies that $\|\boldsymbol{e}\|$ converges to 0 globally under arbitrary initial error.

Following [1, Proof of Theorem 2] and with Assumptions 1 and 2, one can also obtain that the differences in virtual coordinates $w_1^{[i]}$ and $w_2^{[i]}$ of neighboring robots converge to the desired values, i.e., $\lim_{t\to\infty} w_1^{[i]}(t) - w_1^{[j]}(t) - \Delta_1^{[i,j]} \to 0, \lim_{t\to\infty} w_2^{[i]}(t) - w_2^{[j]}(t) - \Delta_2^{[i,j]} \to 0$. This implies that $\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]}(\boldsymbol{t}, \boldsymbol{w}_{\boldsymbol{1}}^{[\cdot]})$ and $\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]}(t, \boldsymbol{w}_{\boldsymbol{2}}^{[\cdot]})$ converge to 0 as $t \to \infty$. Combining $\lim_{t\to\infty}\|\boldsymbol{\Phi}^{[i]}(t)\| \to 0$, one has

$$\begin{aligned}
&\lim_{t\to\infty} \dot{w}_1^{[i]}(t) \\
=&\lim_{t\to\infty} [0 \cdots 1\ 0] \left({}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]}(t, \boldsymbol{\xi}^{[i]}) + k_{c1}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{1}}^{[i]}(t, \boldsymbol{w}_{\boldsymbol{1}}^{[\cdot]}) + k_{c2}{}^{\mathbf{cr}}\boldsymbol{\chi}_{\boldsymbol{2}}^{[i]}(t, \boldsymbol{w}_{\boldsymbol{2}}^{[\cdot]}) \right) \\
=&\lim_{t\to\infty} [0 \cdots 1\ 0] \cdot {}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]}(t, \boldsymbol{\xi}^{[i]}) \\
=&\lim_{t\to\infty} [0 \cdots 1\ 0] \cdot (-1)^n \cdot \begin{bmatrix} v_{n+2}f_1^{[i]\prime} - v_{n+1}g_1^{[i]\prime} \\ \vdots \\ v_{n+2}f_n^{[i]\prime} - v_{n+1}g_n^{[i]\prime} \\ v_{n+2} \\ -v_{n+1} \end{bmatrix} \\
=&(-1)^{n+1}(-1)^{n+1}\dot{w}_1^* = \dot{w}_1^*.
\end{aligned}$$

Applying the same calculation to $\lim_{t\to\infty} \dot{w}_2^{[i]}(t)$, one has

$$\begin{aligned}
&\lim_{t\to\infty} \dot{w}_2^{[i]}(t) \\
=&\lim_{t\to\infty} [0 \cdots 0\ 1] \cdot {}^{\mathbf{cn}}\boldsymbol{\chi}^{[i]}(t, \boldsymbol{\xi}^{[i]}) \\
=&(-1)^{n+1}v_{n+1} = (-1)^{n+1}(-1)^{n+1}\dot{w}_2^* = \dot{w}_2^*.
\end{aligned}$$

In light of the above, Theorem 1 is proved to solve problem 1 globally.

References

[1] W. Yao, H. G. de Marina, Z. Sun, and M. Cao, "Guiding vector fields for the distributed motion coordination of mobile robots," *IEEE Transactions on Robotics*, vol. 39, no. 2, pp. 1119–1135, 2023.